CHAPTER - III

LOW FREQUENCY SCATTERING OF ELASTIC WAVES BY GRIFFITH CRACKS IN ORTHOTROPIC ELASTIC MEDIUM

PAPER 7 : Diffraction of elastic waves by two parallel rigid strips embedded in an orthotropic medium.

PAPER 8 : Interaction of elastic waves with two coplanar Griffith cracks in an orthotropic medium.

PAPER 9 : Diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic medium.

PAPER 10 : Elastic wave scattering from four coplanar Griffith cracks in an orthotropic medium.

DIFFRACTION OF ELASTIC WAVES BY TWO PARALLEL RIGID STRIPS EMBEDDED IN AN INFINITE ORTHOTROPIC MEDIUM

---

1. INTRODUCTION

Recently, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced composites, the study of interaction of elastic waves with cracks or inclusions in composite materials has gained much importance. The problems involving inclusions in isotropic medium have been studied by many authors. Palaiya and Majumder (1981) considered the problem of a single strip at a bimaterial interface. Forced vertical vibration of a single strip was treated by Wickham (1977). Jain and Kanwal (1972b)have solved the problem of two rigid strips embedded in an isotropic elastic medium. Recently Mandal and Ghosh (1992b) have treated the problem of vertical vibration of two rigid strips on the surface of a semi-infinite medium. The problem involving single Griffith crack in orthotropic medium was investigated by Kassir and Bandyopadhyay (1983), Shindo et al. (1986), De and Patra (1990). Shindo et al. (1991) have investigated the impact response of symmetric edge cracks in an orthotropic strip. But perhaps, due to mathematical complexity, elastodynamic problems involving two or more Griffith cracks or strips in anisotropic materials have not yet received much attention.

In our problem, the interaction of normally incident time harmonic elastic waves with two rigid strips embedded in an infinite orthotropic medium has been considered. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. From the solution of the integral equation we have found out the normal stress and vertical displacement at points in the plane of the strips. Finally choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal (1972b). To display the influence of the material orthotropy numerical values of stress intensity factors and vertical displacement have been plotted against dimensionless frequency and distance respectively for several orthotropic materials.

2. FORMULATION OF THE PROBLEM

Let us consider the diffraction of normally incident longitudinal wave by two symmetric coplanar and parallel rigid strips embedded in an infinite orthotropic elastic medium and the strips occupy the region $b \le |x_1| \le a$, $x_2 = 0$, $-\infty < x_3 < \infty$. Let E_i, μ_{ij} and ν_{ij} $(i,j=1,2,3)$ denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the x_1, x_2, x_3 directions which coincide with the axes of material orthotropy. Normalizing all lengths with respect to 'a' and putting $x_1/a=x$, $x_2/a=y$, $x_3/a=z$, $b/a=c$, the rigid strips are defined by $c\le|x|\le 1$, $y=0$, $|z|<\infty$(Fig.1). Let a time harmonic wave given by $u=0$ and $v=v_o\exp[i(ky-\omega t)]$ where $k=a\omega/c_s\sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$ and v_o is a constant, travelling in the direction of positive y-axis be incident normally on the strips.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$
\begin{aligned}
\tau_{yy}/\mu_{12} &= c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y} \\
\tau_{xy}/\mu_{12} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{aligned}
\tag{1}
$$

where $c_{ij}(i,j=1,2)$ are nondimensional parameters related to the elastic constants by the relations

$$
\begin{aligned}
c_{11} &= E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1) \\
c_{22} &= E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1) = c_{11}E_2/E_1 \\
c_{12} &= \nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11}
\end{aligned}
\tag{2}
$$

for generalized plane stress, and by

$$c_{11}=(E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32})$$

$$c_{22}=(E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31})$$

$$c_{12}=E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12}$$

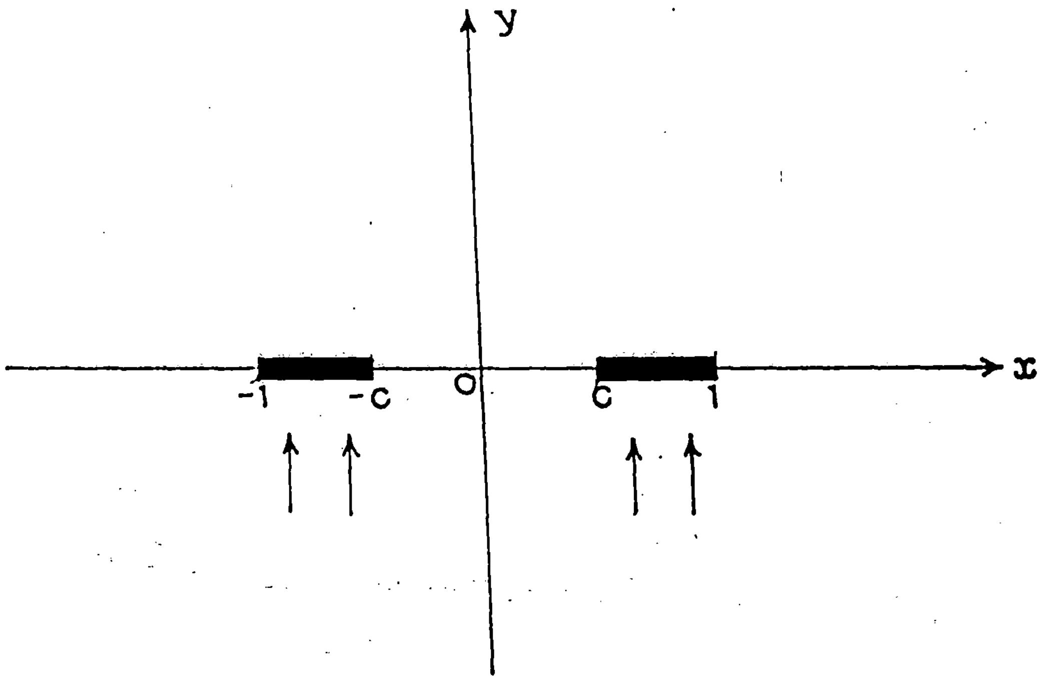

Fig. 1. Geometry of the strips.

$$=E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32} \tag{3}$$

for plane strain. The constants E_i and ν_{ij} satisfy the Maxwell's relation $\nu_{ij}/E_i = \nu_{ji}/E_j$.

The equations of motion for orthotropic material, in terms of displacements are

$$\begin{aligned} c_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1+c_{12})\frac{\partial^2 v}{\partial x\partial y} &= \frac{a^2}{c_s^2}\frac{\partial^2 u}{\partial t^2} \\ c_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1+c_{12})\frac{\partial^2 u}{\partial x\partial y} &= \frac{a^2}{c_s^2}\frac{\partial^2 v}{\partial t^2} \end{aligned} \tag{4}$$

Therefore, substituting $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ our problem reduces to the solution of the equations

$$c_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1+c_{12})\frac{\partial^2 v}{\partial x\partial y} + \frac{a^2\omega^2}{c_s^2}u = 0 \tag{5}$$

and

$$c_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1+c_{12})\frac{\partial^2 u}{\partial x\partial y} + \frac{a^2\omega^2}{c_s^2}v = 0$$

subject to the boundary conditions

$$v(x,0) = -v_0 \quad , \quad c\le|x|\le 1 \tag{6}$$

$$\tau_{yy}(x,0) = 0 \quad , \quad |x|<c,\ |x|>1 \tag{7}$$

$$u(x,0) = 0 \quad , \quad |x|<\infty. \tag{8}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of equations (5) are taken as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)\right]\sin\xi x \, d\xi \quad , \quad y>0 \tag{9}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)\right]\cos\xi x \, d\xi \tag{10}$$

where $$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \quad , \quad i=1,2 \quad , \quad k_s^2 = \frac{a^2\omega^2}{c_s^2} \tag{11}$$

and $A_i(\xi)(i=1,2)$ are the unknowns to be solved, γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0 \tag{12}$$

From the boundary condition (8) it is found that

$$A_2(\xi) = -A_1(\xi).$$

Therefore displacements u, v and stresses τ_{yy}, τ_{xy} finally can be written as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin\xi x \, d\xi \quad , \quad y>0 \tag{13}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1\exp(-\gamma_1|y|) - \alpha_2\exp(-\gamma_2|y|)\right]A_1(\xi)\cos\xi x \, d\xi \tag{14}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \right.$$

$$\left. - \left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right] A_1(\xi)\cos\xi x \; d\xi \quad , \quad y>0 \tag{15}$$

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty \left[(\gamma_1+\alpha_1)\exp(-\gamma_1|y|) - (\gamma_2+\alpha_2)\exp(-\gamma_2|y|)\right] A_1(\xi) \times$$

$$\times \sin\xi x \; d\xi \tag{16}$$

Next putting

$$A(\xi) = \frac{\alpha_1\gamma_1 - \alpha_2\gamma_2}{\xi} A_1(\xi)$$

the boundary conditions (6) and (7) lead to the following dual integral equations in $A(\xi)$:

$$\int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right) A(\xi)\cos\xi x \; d\xi = -\frac{\pi}{2} v_o \quad , \quad c\le|x|\le 1 \tag{17}$$

and

$$\int_0^\infty A(\xi)\cos\xi x \; d\xi = 0 \quad , \quad |x|<c \; , \; |x|>1 \tag{18}$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (17) and (18) in the form

$$A(\xi) = \int_c^1 t f(t^2)\cos\xi t \; dt \tag{19}$$

where $f(t^2)$ is an unknown function to be determined.

By the choice of $A(\xi)$ given by (19) the relation (18) is satisfied automatically and the equation (17) becomes

$$\int_c^1 tf(t^2)dt\int_0^\infty\left(\frac{\alpha_1}{\alpha_1\gamma_1}-\frac{\alpha_2}{\alpha_2\gamma_2}\right)\cos\xi x\ \cos\xi t\ d\xi = -\frac{\pi}{2}v_o\ ,\quad c\le|x|\le 1 \qquad (20)$$

Using the relation

$$\frac{\sin\xi x\ \sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_o(\xi w)J_o(\xi v)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{d}{dx}\int_c^1 tf(t^2)dt\ \frac{\partial}{\partial t}\int_0^x\int_0^t \frac{vwL_1(v,w)\ dwdv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} = -\frac{\pi}{2}v_o\ ,\qquad c\le|x|\le 1 \qquad (21)$$

where

$$L_1(v,w) = \int_0^\infty\left(\frac{\alpha_1}{\alpha_1\gamma_1}-\frac{\alpha_2}{\alpha_2\gamma_2}\right)J_o(\xi w)J_o(\xi v)\ d\xi \qquad (22)$$

By a contour integration technique, the infinite integral in $L_1(v,w)$ can be converted to the following finite integrals (details have been given in the appendix)

$$L_1(v,w) = -i\left[\int_0^{1/\sqrt{c_{11}}}\frac{c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\ -\right.$$

$$\left.-\int_{1/\sqrt{c_{11}}}^1\frac{c_{11}\eta^2-1+\bar{\gamma}_2'^2}{\bar{\gamma}_2'(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\right],\quad w>v \qquad (23)$$

where
$$\begin{aligned}
\bar{\gamma}_1 &= \left[\frac{1}{2}\left\{X_1-(X_1^2-4X_2)^{1/2}\right\}\right]^{1/2}\\
\bar{\gamma}_2 &= \left[\frac{1}{2}\left\{X_1+(X_1^2-4X_2)^{1/2}\right\}\right]^{1/2}\\
\bar{\gamma}'_1 &= \left[\frac{1}{2}\left\{-X_1+(X_1^2+4X_3)^{1/2}\right\}\right]^{1/2}\\
\bar{\gamma}'_2 &= \left[\frac{1}{2}\left\{X_1+(X_1^2+4X_3)^{1/2}\right\}\right]^{1/2}
\end{aligned} \tag{24}$$

$$X_1 = \frac{1}{c_{22}}\left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\eta^2 + (1+c_{22})\right\}$$

$$X_2 = \frac{c_{11}}{c_{22}}\left(1-\eta^2\right)\left(\frac{1}{c_{11}} - \eta^2\right)$$

$$X_3 = \frac{c_{11}}{c_{22}}\left(1-\eta^2\right)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

The corresponding expression of $L_1(v,w)$ for $w<v$ follows from (23) by interchanging w and v.

Substituting the series expansion of J_0 and $H_0^{(1)}$ in (23) we find after some algebraic manipulation

$$L_1(v,w) = \frac{2}{\pi}\left[\left(\gamma+\log(k_s w/2)-\frac{\pi i}{2}\right)M + N - \frac{(w^2+v^2)}{4}Rk_s^2\log k_s\right] + O(k_s^2) \quad , \; w>v$$

$$= \frac{2}{\pi}\left[\left(\gamma+\log(k_s v/2)-\frac{\pi i}{2}\right)M + N - \frac{(w^2+v^2)}{4}Rk_s^2\log k_s\right] + O(k_s^2) \quad , \; v>w \tag{25}$$

where $\gamma = 0.5772157\ldots\ldots\ldots\ldots$ is Euler's constant,

$$M = \int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\, d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2}{\bar{\gamma}_2'(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\, d\eta \tag{26}$$

$$N = \int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\, \log\eta \; d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2}{\bar{\gamma}_2'(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\, \log\eta \; d\eta \tag{27}$$

and

$$R = \int_0^{1/\sqrt{c_{11}}} \frac{\eta^2(c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2)}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\, d\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\eta^2(c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2)}{\bar{\gamma}_2'(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\, d\eta \tag{28}$$

Now differentiating both sides of the relation (20) with respect to x we obtain

$$\int_c^1 tf(t^2)dt \int_0^\infty \xi \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right) \cos\xi x \; \cos\xi t \; d\xi = 0 \; , \qquad c \le |x| \le 1$$

Following similar procedure as done for deriving equation (21), we obtain

$$\int_c^1 \frac{tf(t^2)}{x^2 - t^2}\, dt = \int_c^1 tf(t^2)dt \; \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{vwL_2(v,w)\; dwdv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \; , \quad c \le |x| \le 1 \tag{29}$$

where

$$L_2(v,w) = \int_0^\infty \left[\xi - \frac{\xi^2}{\theta}\left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right)\right] J_0(\xi w) J_0(\xi v)\; d\xi \tag{30}$$

$$\theta = \frac{c_{11} + N_1 N_2}{N_1 + N_2}$$

$$N_1^2 = \frac{1}{2c_{22}}\left[-(c_{12}^2+2c_{12}-c_{11}c_{22})+\sqrt{(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}}\right]$$

and $$N_2^2 = \frac{1}{2c_{22}}\left[-(c_{12}^2+2c_{12}-c_{11}c_{22})-\sqrt{(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}}\right].$$

We use the contour integration technique mentioned earlier and get from (30)

$$L_2(v,w) = \frac{ik_s^2}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{\eta^2(c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2)}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\, d\eta \right. -$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\eta^2(c_{11}\eta^2-1+\bar{\gamma}_2'^2)}{\bar{\gamma}_2'(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\, d\eta\right],\quad w>v \tag{31}$$

By the process similar to the one which led to the equation (25), (30) for small values of k_s can be written as

$$L_2(v,w) = -\frac{2}{\pi} Pk_s^2\log k_s + O(k_s^2) \tag{32}$$

where

$$P = \frac{1}{\theta} R \quad \text{and } R \text{ is given by (28).}$$

Now, let us consider

$$f(t^2) = f_0(t^2) + k_s^2\log k_s f_1(t^2) + O(k_s^2) \tag{33}$$

Putting the above expansion of $f(t^2)$ and the value of $L_2(v,w)$ given by (32) in the equation (29) and equating the coefficients of like powers of k_s we obtain,

$$\int_c^1 \frac{t f_o(t^2)}{x^2 - t^2}\, dt = 0 \quad , \quad c \le |x| \le 1 \tag{34}$$

and
$$\int_c^1 \frac{t f_1(t^2)}{x^2 - t^2}\, dt = - \frac{2P}{\pi} \int_c^1 t f_o(t^2)\, dt \; , \quad c \le |x| \le 1 \tag{35}$$

Following Srivastava and Lowengrub (1968) the solutions of the above integral equations can be obtained as

$$f_o(t^2) = \frac{D_1}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{36}$$

$$f_1(t^2) = \frac{2}{\pi} PD_1 \left(\frac{t^2-c^2}{1-t^2}\right)^{1/2} + \frac{D_2}{\sqrt{(1-t^2)(t^2-c^2)}} \tag{37}$$

where D_1 and D_2 are constants which can be calculated as follows. We substitute the value of $L_1(v,w)$ from (25) as well as the expansion of $f(t^2)$ obtained from (33), (36) and (37) in the equation (21). When the coefficients of like powers of k_s from both sides of the resulting equation are equated we get the following results :

$$D_1 = - \frac{\pi v_o}{2\left[\left(\gamma + \log(k_s/2) - \frac{\pi i}{2} + \log(1-c^2)^{1/2}\right)M + N\right]} \tag{38}$$

and

$$D_2 = - \frac{2D_1^2}{\pi v_o} \left[\frac{R}{4}(2x^2+c^2+1) - \frac{MP}{2\pi}(1-2x^2+c^2) + \frac{Pv_o(1-c^2)}{2D_1} \right] \tag{39}$$

4. DISPLACEMENT AND STRESS

The vertical displacement $v(x,y)$ on the plane $y=0$ can be obtained from equations (15) and (19) as

$$v(x,0) = -v_0 + \frac{2M}{\pi}\left[D_1 + k_s^2 \log k_s\left\{D_2 + \frac{(1-c^2)PD_1}{\pi}\right\}\right]\sinh^{-1}\sqrt{\frac{(x^2-1)}{(1-c^2)}} + $$

$$+ \frac{2PD_1M}{\pi^2} k_s^2 \log k_s \sqrt{(x^2-1)(x^2-c^2)} + O(k_s^2) \quad, \qquad |x|>1$$

$$= -v_0 \quad, \qquad c \le |x| \le 1$$

$$= -v_0 + \frac{2M}{\pi}\left[D_1 + k_s^2 \log k_s\left\{D_2 + \frac{(1-c^2)PD_1}{\pi}\right\}\right]\sinh^{-1}\sqrt{\frac{(c^2-x^2)}{(1-c^2)}} -$$

$$- \frac{2PD_1M}{\pi^2} k_s^2 \log k_s \sqrt{(1-x^2)(c^2-x^2)} + O(k_s^2) \quad, \qquad |x|<c \tag{40}$$

The normal stress $\tau_{yy}(x,y)$ in the plane $y=0$ can be found from the relation (15) as

$$\tau_{yy}(x,\pm 0) = \mp c_{22}\mu_{12}|x|\left[\frac{D_1}{\sqrt{(1-x^2)(x^2-c^2)}} + k_s^2 \log k_s\left\{\frac{2}{\pi} PD_1\left(\frac{x^2-c^2}{1-x^2}\right)^{1/2} + \right.\right.$$

$$\left.\left. + \frac{D_2}{\sqrt{(1-x^2)(x^2-c^2)}}\right\}\right] + O(k_s^2) \quad, \qquad c \le |x| \le 1$$

$$= 0 \quad, \qquad 0 \le |x| < c, \quad |x|>1 \tag{41}$$

For isotropic medium, putting

$$c_{11} = c_{22} = \frac{\lambda+2\mu}{\mu} , \qquad \mu_{12} = \mu$$

so that $\alpha_1 = \gamma_1$, $\alpha_2 = \xi^2/\gamma_2$, $k_2 = k_s$, $k_1 = k_s/\sqrt{c_{11}}$, $\tau^2 = \frac{1}{c_{11}}$

where $\gamma_i = (\xi^2 - k_i^2)^{1/2}$, $i=1,2,$

the expressions for displacement and stress are found to be

$$v(x,0) = -v_0 - \frac{(1+\tau^2)}{2\tau^2}\left[D_1' + k_2^2 \log k_2 \left\{D_2' - \frac{(3+\tau^4)}{8(1+\tau^2)}(1-c^2)D_1'\right\}\right] \times$$

$$\times \sinh^{-1}\sqrt{\frac{(x^2-1)}{(1-c^2)}} + \frac{(3+\tau^4)}{16\tau^2} D_1' k_2^2 \log k_2 \sqrt{(x^2-1)(x^2-c^2)} + O(k_2^2)$$

$$; \qquad |x|>1$$

$$= -v_0 , \qquad c \le |x| \le 1$$

$$= -v_0 - \frac{(1+\tau^2)}{2\tau^2}\left[D_1' + k_2^2 \log k_2 \left\{D_2' - \frac{(3+\tau^4)}{8(1+\tau^2)}(1-c^2)D_1'\right\}\right] \times$$

$$\times \sinh^{-1}\sqrt{\frac{(c^2-x^2)}{(1-c^2)}} - \frac{(3+\tau^4)}{16\tau^2} D_1' k_2^2 \log k_2 \sqrt{(1-x^2)(c^2-x^2)} + O(k_2^2)$$

$$, \qquad |x|<c \tag{42}$$

$$\tau_{yy}(x,\pm 0) = \mp \frac{\mu}{\tau^2}|x|\left[\frac{D_1'}{\sqrt{(1-x^2)(x^2-c^2)}} + k_2^2 \log k_2 \left\{-\frac{(3+\tau^4)}{4(1+\tau^2)} D_1' \left(\frac{x^2-c^2}{1-x^2}\right)^{1/2}\right.\right.$$

$$\left.\left. + \frac{D_2'}{\sqrt{(1-x^2)(x^2-c^2)}}\right\}\right] + O(k_2^2) , \qquad c \le |x| \le 1$$

$$= 0 , \qquad 0 \le |x| < c , \qquad |x|>1 \tag{43}$$

where

$$D_1' = \frac{2\tau^2 v_o}{\left[q_2+\tau^2 q_1+(1+\tau^2)\log(1-c^2)^{1/2}+\frac{1}{2}(1-\tau^2)\right]} \tag{44}$$

$$D_2' = \frac{D_1'^2(3+\tau^4)}{8v_o(1+\tau^2)}\left[\frac{(1+\tau^2)(1+c^2)}{2\tau^2}+\frac{v_o(1-c^2)}{D_1}\right] \tag{45}$$

$$q_i = \gamma + \log(k_i/4) - \pi i/2 \quad , \quad i=1,2 \tag{46}$$

Now, substituting $v_o=1$, $m_i=k_i$, $i=1,2$,

$$C = \frac{-2}{\pi\left[(q_1\tau^2+q_2)+\frac{1}{2}(1-\tau^2)+(1+\tau^2)\log(1-c^2)^{1/2}\right]}$$

and dropping term involving $k_2^2\log k_2$ the displacement and stress can be written as

$$v(x,0) = -1 + \frac{\pi C}{2}(1+\tau^2)\sinh^{-1}\sqrt{\frac{(c^2-x^2)}{(1-c^2)}} + O(m_2^2) \quad , \quad |x|<c$$

$$= -1 \quad , \quad c\le|x|\le 1$$

$$= -1 + \frac{\pi C}{2}(1+\tau^2)\sinh^{-1}\sqrt{\frac{(x^2-1)}{(1-c^2)}} + O(m_2^2) \quad , \quad |x|>1$$

(47)

$$\tau_{yy}(x,\pm 0) = \pm\frac{\mu\pi C|x|}{\sqrt{(1-x^2)(x^2-c^2)}} + O(m_2^2) \quad , \quad c\le|x|\le 1 \tag{48}$$

which coincide with the results obtained by Jain and Kanwal (1972b).

4. NUMERICAL RESULTS

The vertical displacement field for points near about the rigid strips has been plotted against dimensionless distance for two different types of orthotropic materials whose engineering constants have been listed in table-1. Type I-a, II-a and Type I-b, II-b correspond to the cases of x and y-directional fibre-reinforced composites respectively.

It is interesting to note that in both the cases (c=0.5 and c=0.8) the real part of the displacement viz. $Re(v/v_o)$ increases with the increase in the values of nondimensional frequency k_s [(Fig.2) - (Fig.9)].

The stress intensity factors T_c and T_1 at inner and outer edges of the strips defined by

$$T_c = \left| \underset{x \to c+}{Lt} \; Re\left[\frac{\tau_{yy}(x,0)(x-c)^{1/2}}{c_{22}\mu_{12}} \right] \right|$$

and

$$T_1 = \left| \underset{x \to 1-}{Lt} \; Re\left[\frac{\tau_{yy}(x,0)(1-x)^{1/2}}{c_{22}\mu_{12}} \right] \right|$$

have been plotted against frequency k_s.

It is found from the graphs that for low frequency, stress intensity factors (Fig.10-Fig.13) at both the edges increasegradually, attain maximum values and then go on decreasing. It may be noted further that at the inner edge, stress intensity factor increases with the increase in the values of the

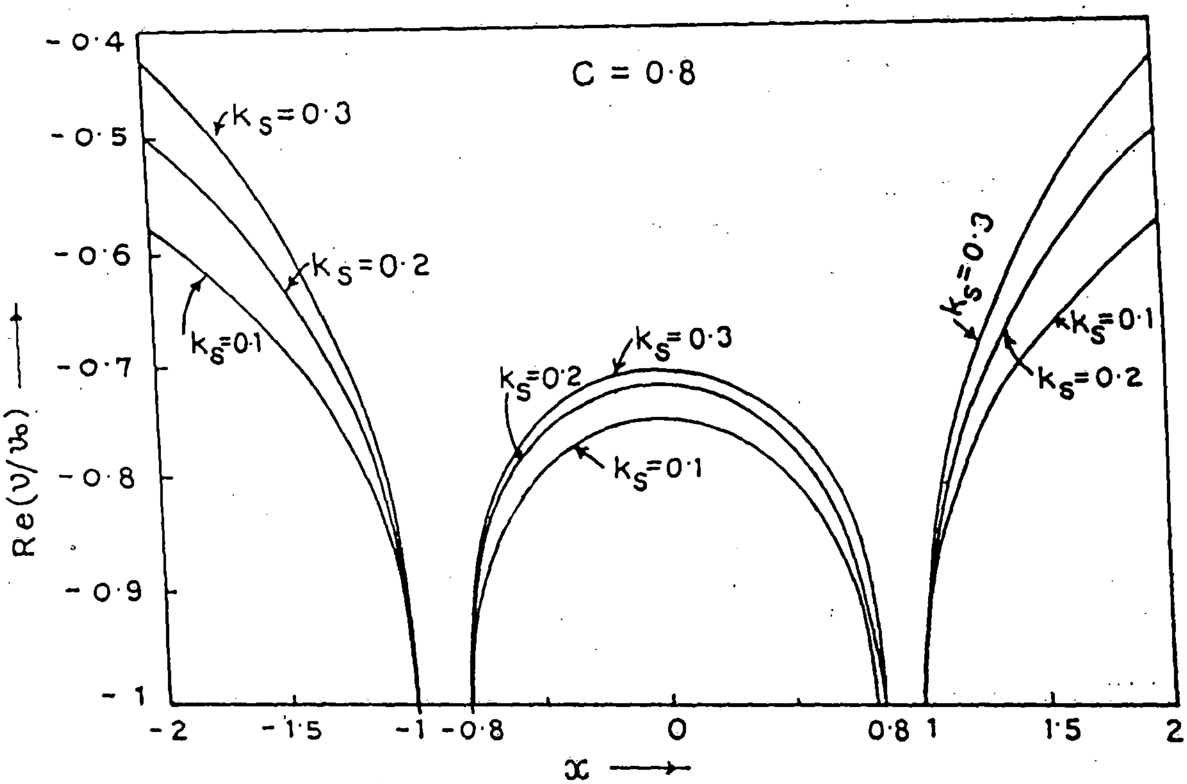

Fig.2. Displacement vs. distance for generalized plane stress (Type Ia, c=0.8).

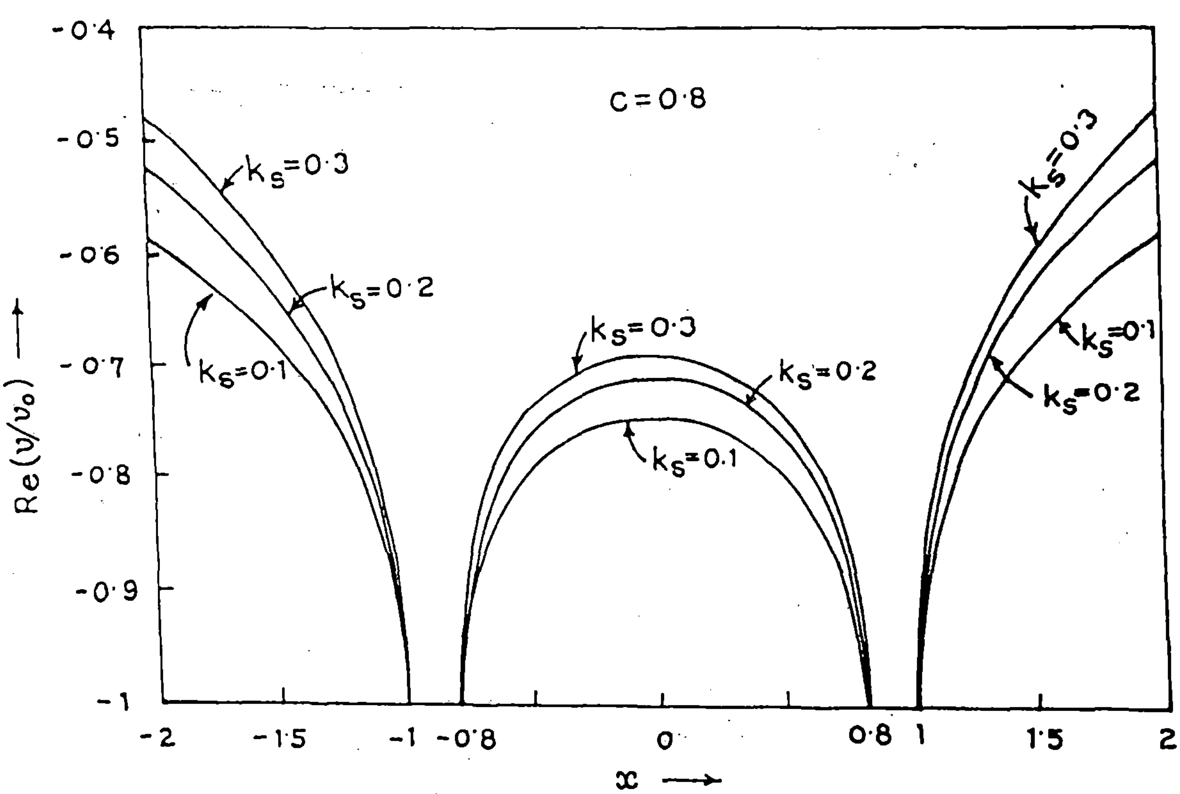

Fig. 3. Displacement vs. distance for generalized plane stress (Type Ib, c=0.8).

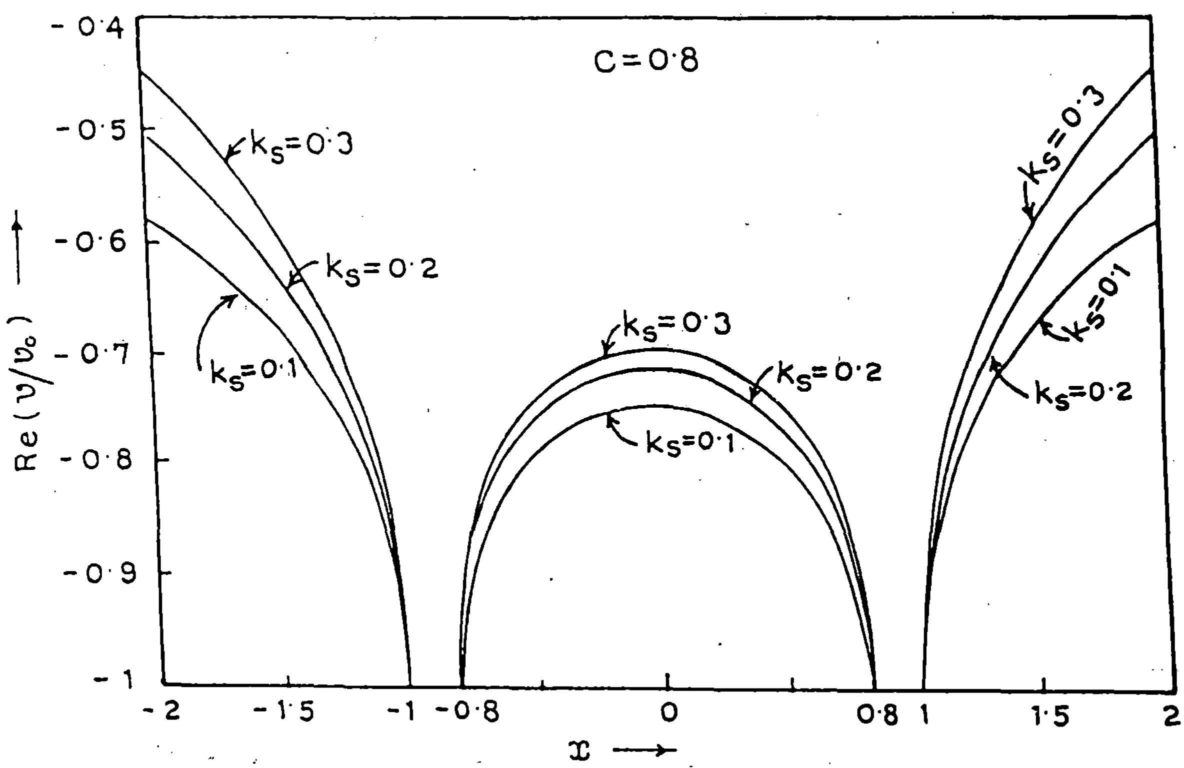

Fig. 4. Displacement vs. distance for generalized plane stress (Type IIa, c=0.8).

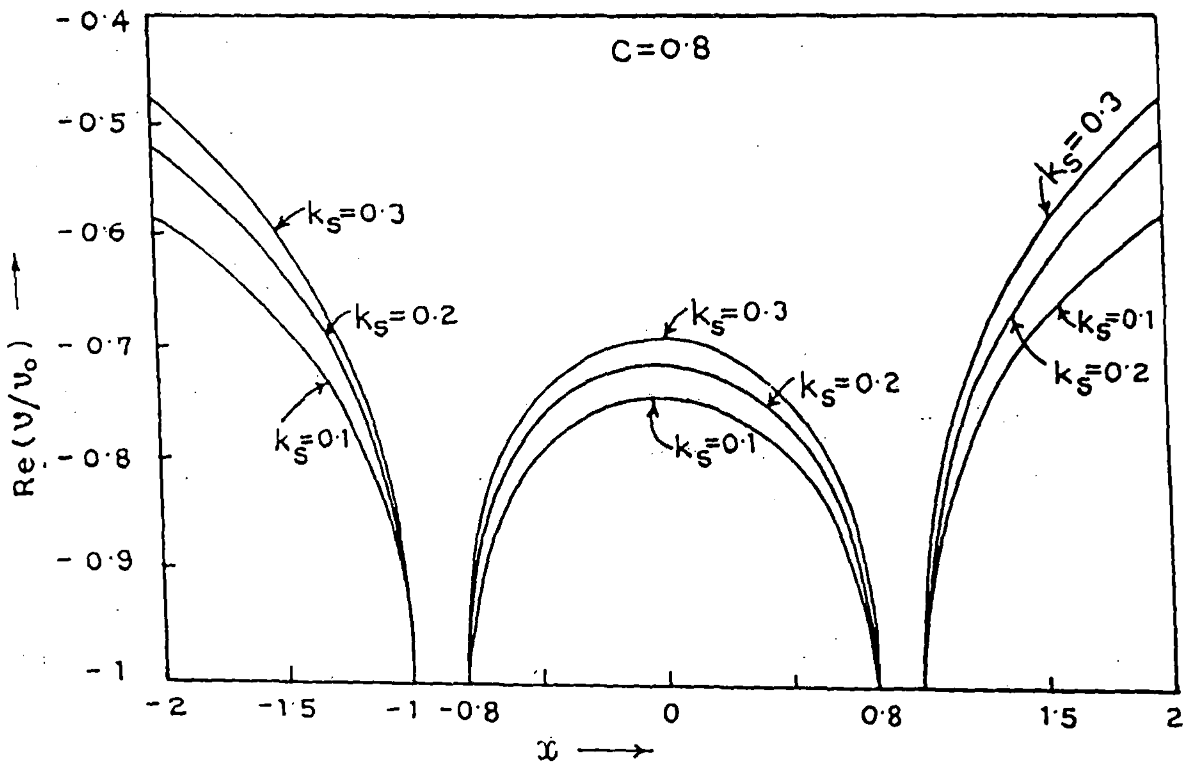

Fig. 5. Displacement vs. distance for generalized plane stress (Type IIb, c=0.8).

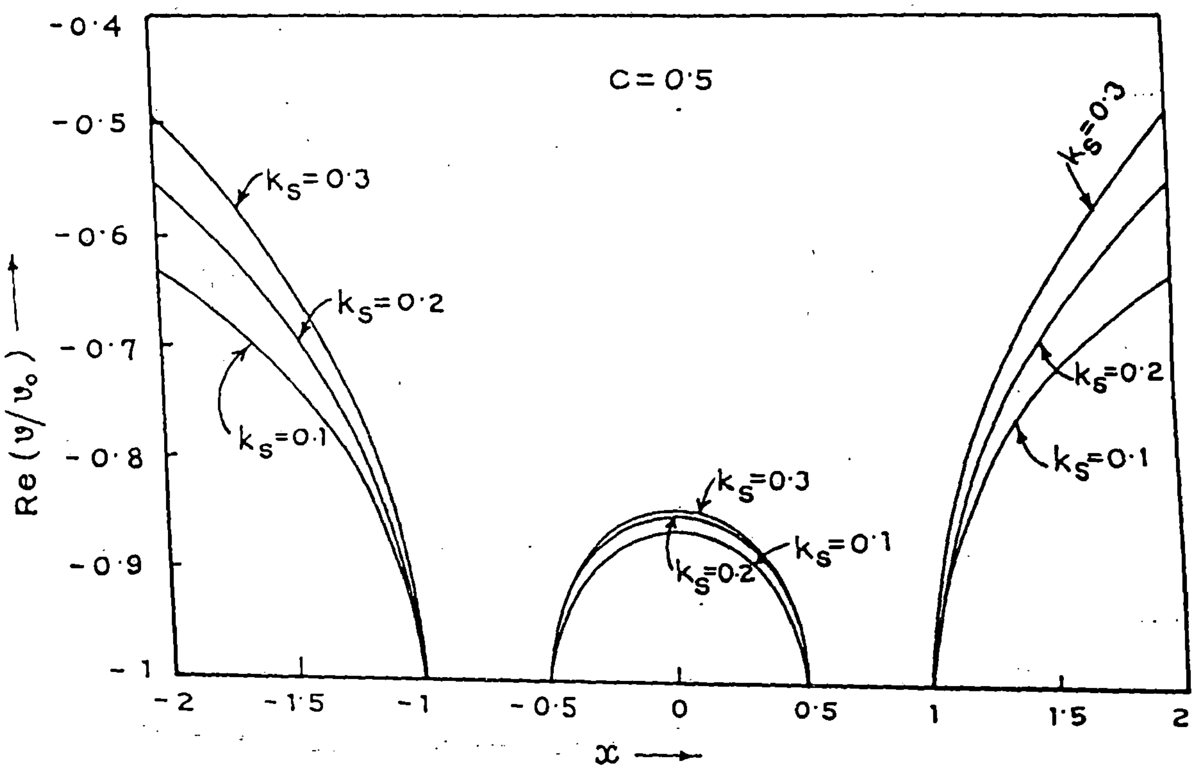

Fig. 6. Displacement vs. distance for generalized plane stress (Type Ia, c=0.5).

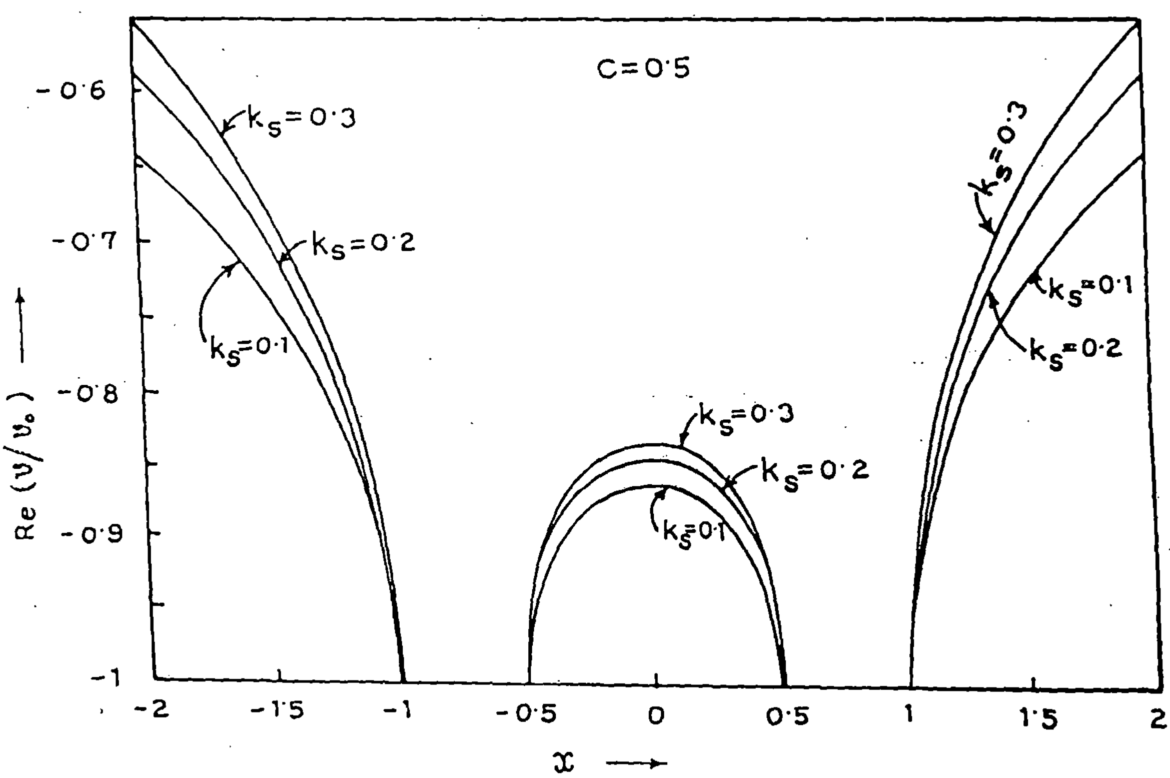

Fig. 7. Displacement vs. distance for generalized plane stress (Type Ib, c=0.5).

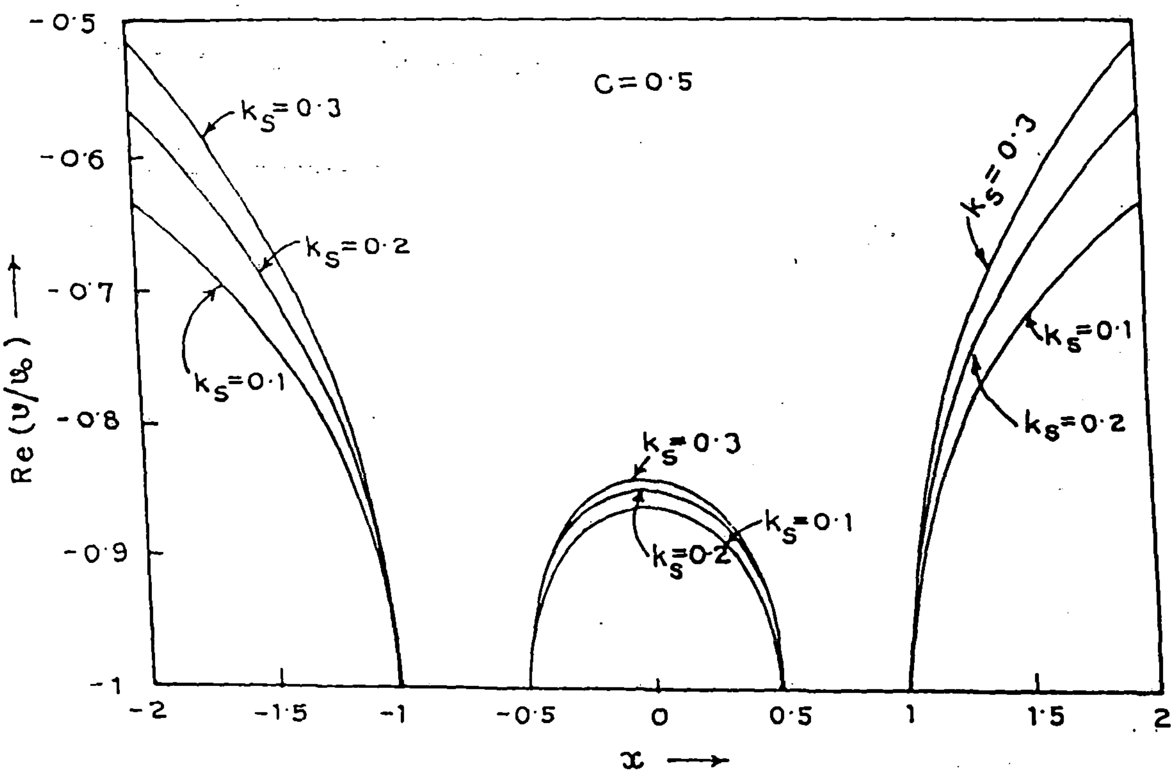

Fig. 8. Displacement vs. distance for generalized plane stress (Type IIa, c=0.5).

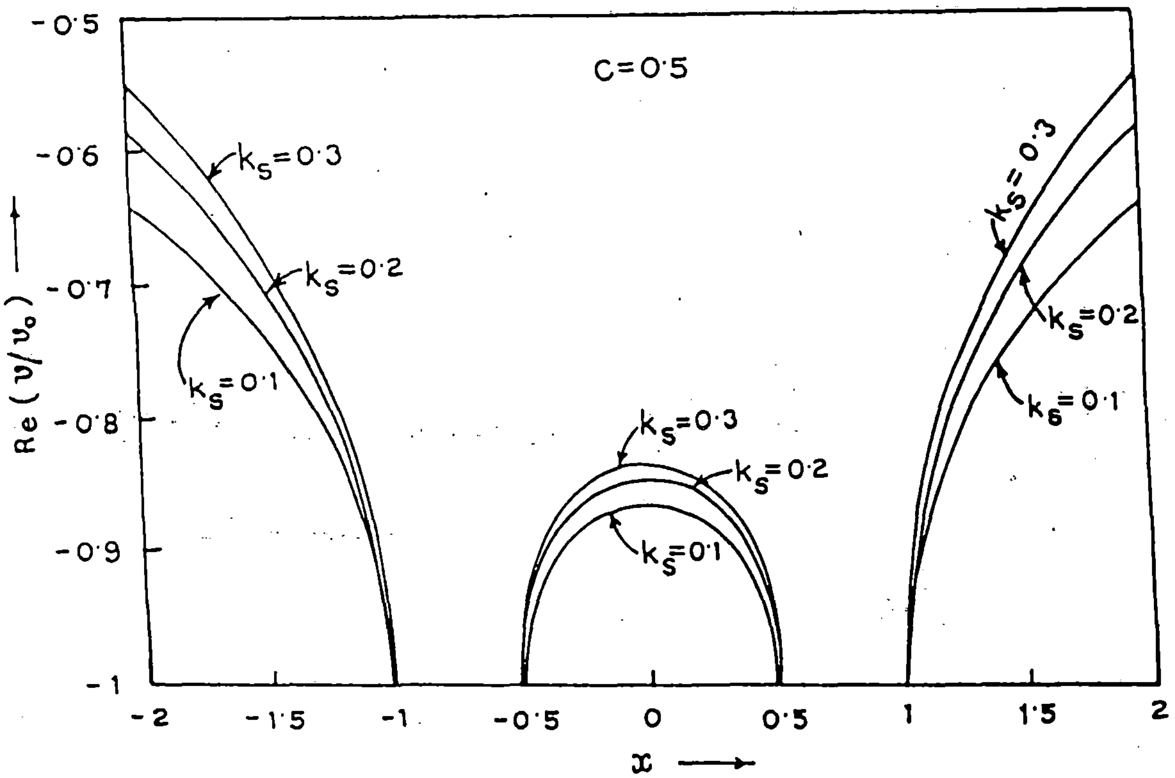

Fig. 9. Displacement vs. distance for generalized plane stress (Type IIb, c=0.5).

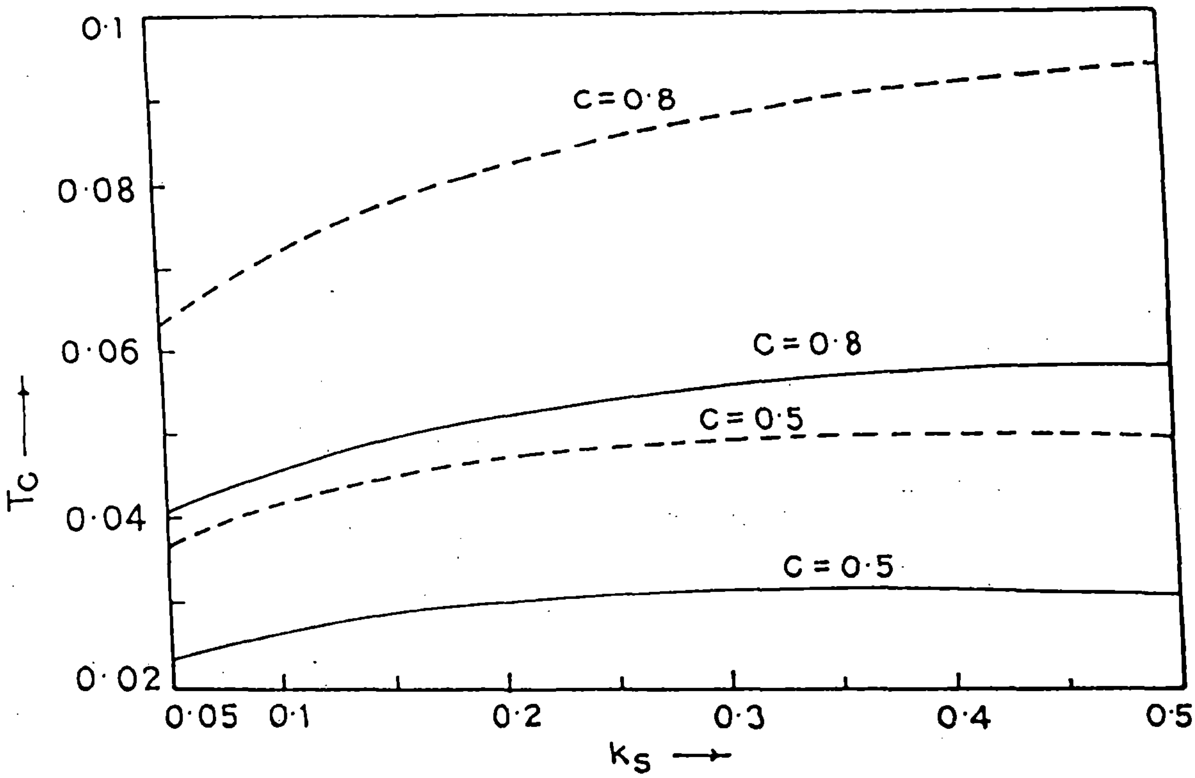

Fig.10. Stress intensity factor T_c vs. frequency k_s for generalized plane stress.
(——— Type Ia, ----- Type IIa).

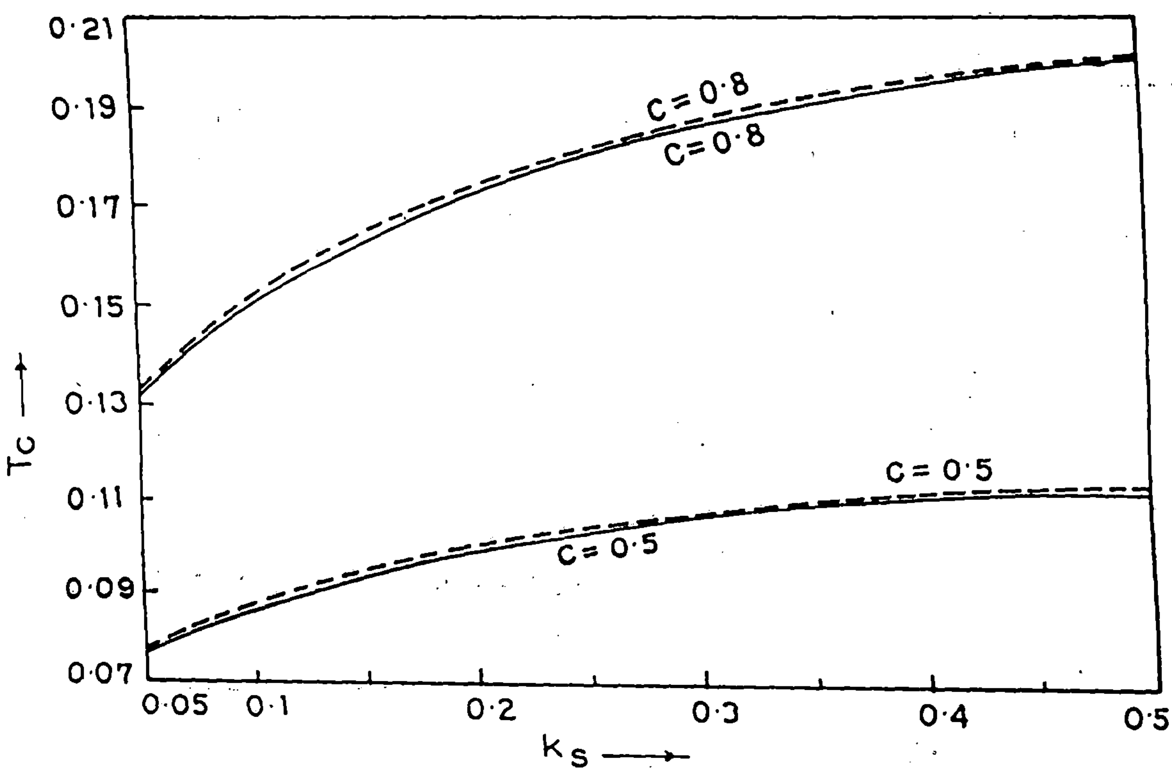

Fig.11. Stress intensity factor T_c vs. frequency k_s for generalized plane stress.
(——— Type Ib, ----- Type IIb).

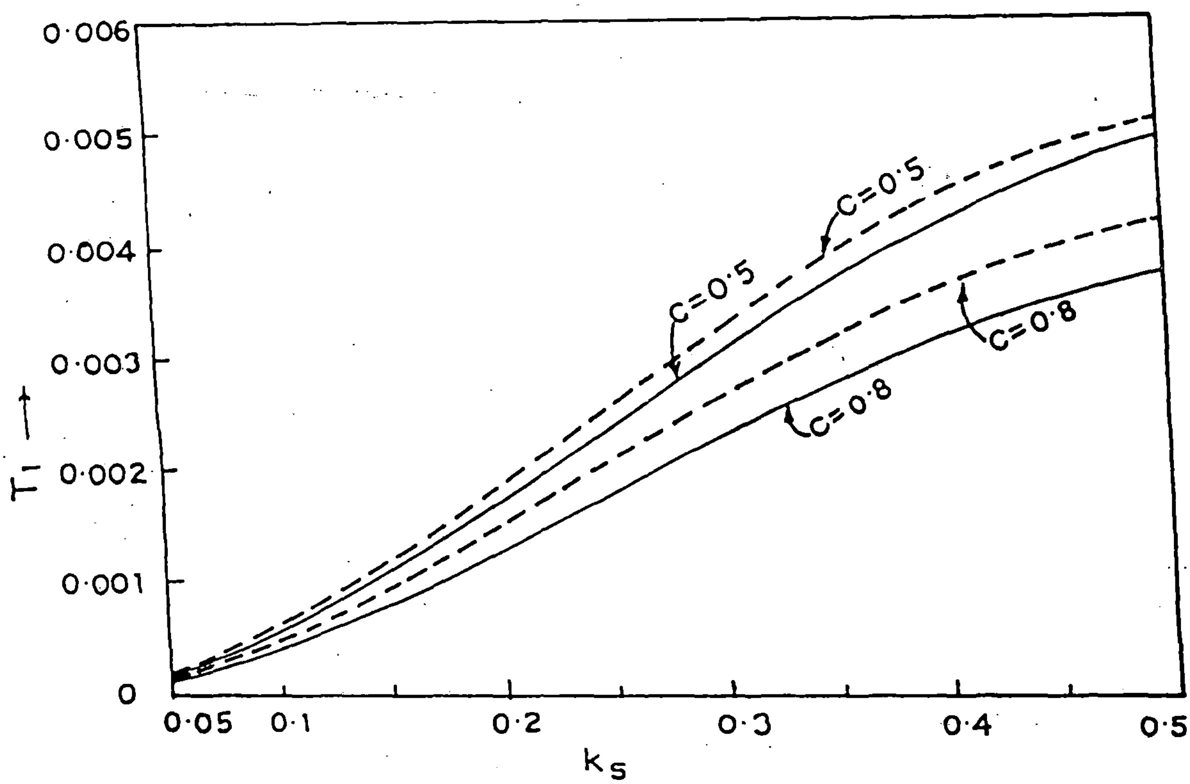

Fig.12. Stress intensity factor T_1 vs. frequency k_s for generalized plane stress. (——— Type Ia, ----- Type IIa).

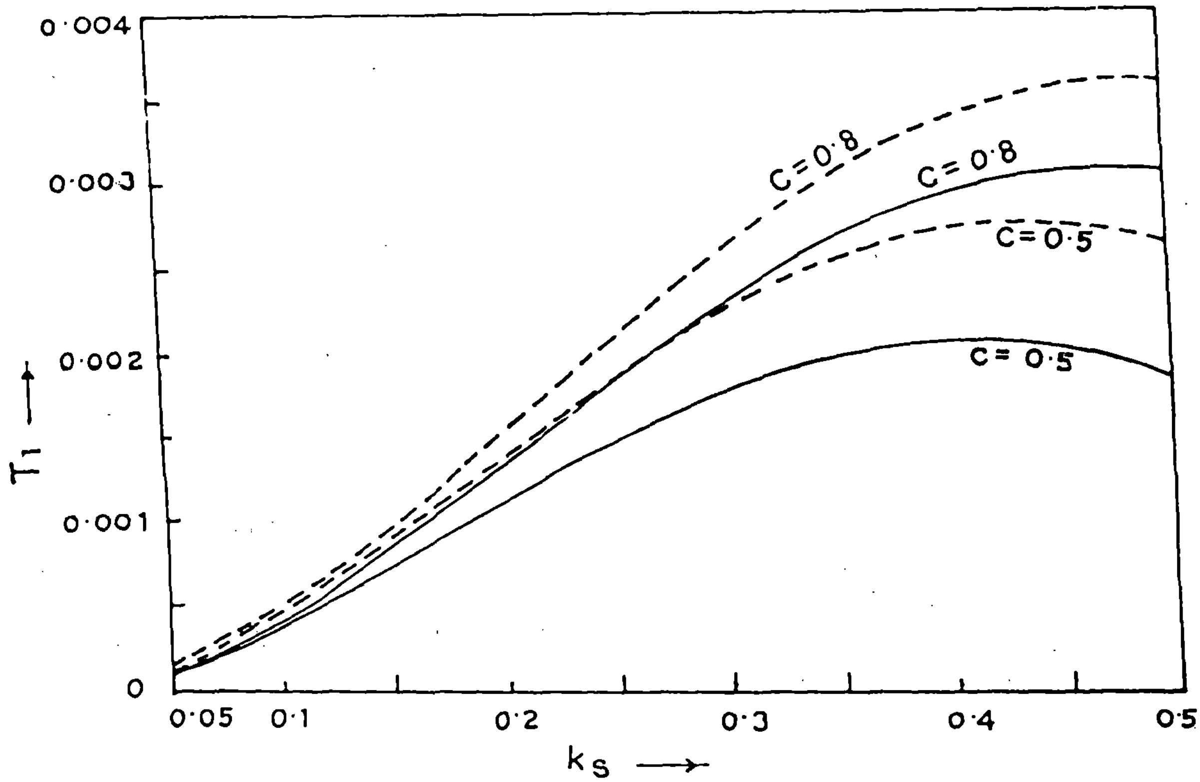

Fig.13. Stress intensity factor T_1 vs. frequency k_s for generalized plane stress. (——— Type Ib, ----- Type IIb).

strip length whereas at the outer edge the stress intensity factor exhibits similar behaviour where the fibres are perpendicular to the strip but in case the fibres are parallel to the strips, the behaviour is just the opposite.

It may also be noted from the graphs that in case the fibres are perpendicular to the strips, the variation of the stress intensity factors at the inner edge do not vary significantly with the material though their variations at the outer edge are prominant.

APPENDIX

EVALUATION OF $L_1(v,w)$:

The integral $L_1(v,w)$ given by (22) is

$$L_1(v,w) = \int_0^\infty K(\xi,\gamma_1,\gamma_2) J_0(\xi w) J_0(\xi v)\, d\xi \tag{A1}$$

where

$$K(\xi,\gamma_1,\gamma_2) = \frac{\alpha_1-\alpha_2}{\alpha_1\gamma_1-\alpha_2\gamma_2} = \frac{c_{11}\xi^2-k_s+\gamma_1\gamma_2}{\gamma_1\gamma_2(\gamma_1+\gamma_2)} \tag{A2}$$

$$\begin{aligned} \gamma_1 &= \left[\frac{1}{2}\left\{-B_1+(B_1^2-4B_2)^{1/2}\right\}\right]^{1/2} \\ \gamma_2 &= \left[\frac{1}{2}\left\{-B_1-(B_1^2-4B_2)^{1/2}\right\}\right]^{1/2} \\ B_1 &= \frac{1}{c_{22}}\left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\right\} \end{aligned} \tag{A3}$$

$$B_2 = \frac{1}{c_{22}} \left(\xi^2 - k_s^2\right)\left(c_{11}\xi^2 - k_s^2\right)$$

To evaluate the integral (A1) we consider two contour integrals :

$$I_1 = \int_{\Gamma_1} K(\xi,\gamma_1,\gamma_2) J_0(\xi v) H_0^{(1)}(\xi w)\, d\xi \quad , \quad w>v$$

$$I_2 = \int_{\Gamma_2} K(\xi,\gamma_1,\gamma_2) J_0(\xi v) H_0^{(2)}(\xi w)\, d\xi \quad , \quad w>v \tag{A4}$$

where Γ_1 and Γ_2 are the closed contours defined in fig.14.

Assuming the relation

$$\left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})(1+c_{22})}{c_{22}^2} + \frac{2(1+c_{11})}{c_{22}}\right\}^2 - \left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})^2}{c_{22}^2} - \frac{4c_{11}}{c_{22}}\right\} \times \left\{\frac{(1+c_{22})^2}{c_{22}^2} + \frac{4}{c_{22}}\right\} < 0 \tag{A5}$$

it is noted the branch points $\xi = \lambda_i (i=1-4)$ corresponding to the roots of the equation $B_1^2 - 4B_2 = 0$ are always complex.

Now, the branch points corresponding to the roots of the equations

$$- B_1 + (B_1^2 - 4B_2)^{1/2} = 0 \quad \text{and} \quad - B_1 - (B_1^2 - 4B_2)^{1/2} = 0$$

are $\xi=\pm k_s$ and $\xi=\pm k_s/\sqrt{c_{11}}$ respectively, where it is assumed that

$$c_{11}c_{22} - c_{12}^2 - 2c_{12} > 1 + c_{22}$$

$$\text{and} \qquad c_{12}^2 + 2c_{12} + c_{11} > 0 \tag{A6}$$

Most of the orthotropic materials satisfy the relations (A5) and (A6). Therefore under the above condition, $\xi=\pm k_s/\sqrt{c_{11}}$ and $\xi=\pm k_s$

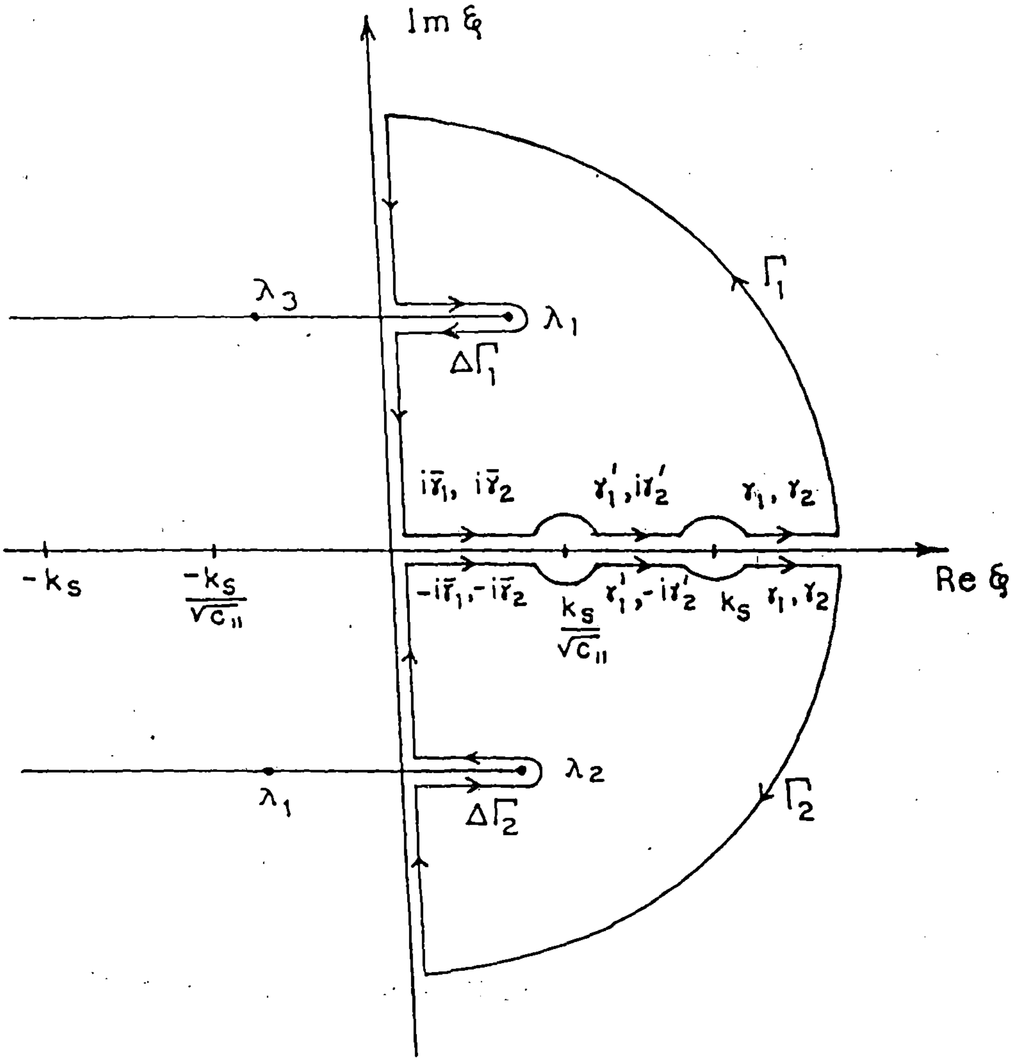

Fig. 14. Contours of integration for integral in equation (A1).

are the branch points of γ_1 and γ_2 respectively.

The integrals in equation (A4) are found to be zero on the contours $\Delta\Gamma_1$ and $\Delta\Gamma_2$ (fig.14) around the branch cuts from λ_1 and λ_2. Thus integrating along the contours Γ_1 and Γ_2 the integral $L_1(v,w)$ for w>v can be finally written as

$$L_1(v,w) = -i\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)} J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\, d\eta \, - \right.$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_2'^{\,2}}{\bar{\gamma}_2'(\bar{\gamma}_1'^{\,2}+\bar{\gamma}_2'^{\,2})} J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\, d\eta \right], \quad w>v \tag{A7}$$

where $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_1'$ and $\bar{\gamma}_2'$ are given by (24).

TABLE - 1. ENGINEERING ELASTIC CONSTANTS

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II Graphite-Epoxy Composite			
a	15.3×10^9	158.0×10^9	5.52×10^9	0.033
b	158.0×10^9	15.3×10^9		0.34
Type II	E-Type Glass-Epoxy Composite			
a	9.79×10^9	42.3×10^9	3.66×10^9	0.063
b	42.3×10^9	9.79×10^9		0.27

INTERACTION OF ELASTIC WAVES WITH TWO COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

1. INTRODUCTION

Dynamic fracture problems involving anisotropic materials weakened by crack-like imperfections have drawn much attention to the investigators because of the increased usage of macroscopically anisotropic construction materials such as fibre reinforced composites. The different possible location of cracks with respect to the planes of material symmetry introduce great modifications in the strain and stress distribution. The problems are also of considerable interest in seismology and exploration geophysics. The problems involving single or two Griffith cracks in isotropic elastic medium have been studied by many authors (Loeber and Sih 1960, Mal 1978, Srivastava et al. 1981, Jain and Kanwal 1972a, Itou 1980b). Mathematical difficulties encountered in solving the governing equations of the anisotropic elasticity theory are responsible for the availability of few results only for special classes of materials. Kassir and Bandyopadhyay (1983) have studied the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the elastodynamic problem of a finite Griffith crack in an orthotropic strip under normal impact was investigated by Shindo et al. (1986).

ENGNG. FRAC. MECH. ,1994 (IN PRESS).
PUBLISHED IN ENGNG. FRAC. MECH., VOL. 49, NO. 3
PP. 411-423 (1994)

Problem involving a moving Griffith crack in an orthotropic strip has also been studied by De and Patra (1990). Recently, Kundu and Bostrom (1991) solved the problem of scattering of elastic waves by a circular crack situated in a transversely isotropic solid.

In our paper, the diffraction of normally incident time harmonic elastic waves by two coplanar Griffith cracks in an infinite orthotropic medium has been investigated. The faces of each of the cracks are assumed to be separated by a small distance so that, during small deformations of the solid, the crack faces do not come into contact. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. Analytical formulae for stress intensity factor and crack opening displacement have been derived. Making the distance between two crack zero the corresponding results for single crack have been presented. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal (1972a). To display the influence of the material orthotropy numerical values of stress intensity factors and crack opening displacement have been plotted for several orthotropic materials.

2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the plane problem of diffraction of normally incident longitudinal wave by two symmetrical coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the region $b \le |X| \le a$, $Y=0$, $|Z| < \infty$. It is convenient to normalize all lengths with respect to 'a' and so setting $X/a=x$, $Y/a=y$, $Z/a=z$, $b/a=c$, the new position of the cracks are defined by $c \le |x| \le 1$, $y=0$, $|z| < \infty$ (Fig.1).

Let a plane time harmonic elastic wave originating at $y=-\infty$ be incident normally on the two cracks is defined by $v_o = \exp[i(ky-\omega t)]$ where $k=a\omega/c_s\sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$ with ρ being the density of the material. In the isotropic solid, c_s, represents the velocity of the shear wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}\, u_{,x} + c_{22}\, v_{,y}$$
$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} , \tag{1}$$

where u, v denote the component of the displacement in the x, y directions respectively and comma denotes partial differentiation with respect to the co-ordinates or time; $c_{ij}(i,j=1,2)$ are nondimensional parameters related to the elastic constants by the relations :

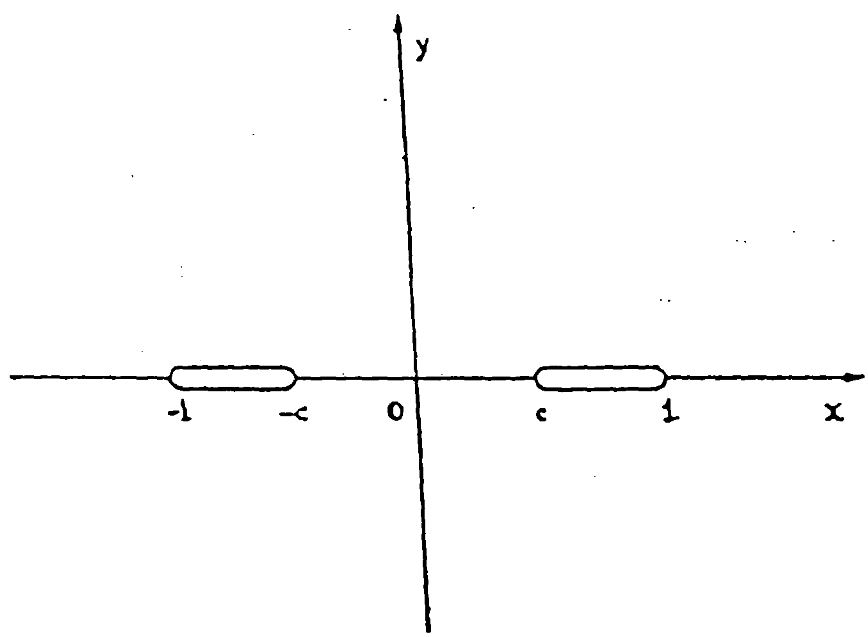

Fig. 1 Geometry of the cracks

$$c_{11}=E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1)$$

$$c_{22}=E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=c_{11}E_2/E_1 \tag{2}$$

$$c_{12}=\nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=\nu_{12}c_{22}=\nu_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11}=(E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32})$$

$$c_{22}=(E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31})$$

$$c_{12}=E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \tag{3}$$

$$=E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32}$$

for plane strain. In the above equations E_i, μ_{ij} and ν_{ij} (i, j=1, 2, 3) denote the engineering ealstic constants of the material where the subscripts 1,2,3 correspond to the x, y, z directions which coincide with the axes of material orthotropy and the constants E_i and ν_{ij} satisfy the Maxwell's relation :

$$\nu_{ij}/E_i = \nu_{ji}/E_j \tag{4}$$

The equations of motion for orthotropic material, in terms of displacements are

$$c_{11}\,u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} = \frac{a^2}{c_s^2}\,u_{,tt}$$

$$c_{22}\,v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} = \frac{a^2}{c_s^2}\,v_{,tt}. \tag{5}$$

Therefore, substituting $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ in equation (5) we obtain

$$c_{11}\,u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} + k_s^2 u = 0$$

and (6)

$$c_{22}\,v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} + k_s^2 v = 0$$

with $k_s^2 = a^2\omega^2/c_s^2$.

The boundary conditions of the problem are

$$\tau_{xy}(x,0) = 0 \quad , \qquad |x|<\infty \tag{7}$$

$$\tau_{yy}(x,0) + \tau_{yy}^{(0)}(x,0) = 0 \quad , \quad c\leq|x|\leq 1 \tag{8}$$

$$v(x,0) = 0 \quad , \qquad |x|<c \ , \ |x|>1. \tag{9}$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of equations (6) can be taken as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)\right]\sin(\xi x)d\xi$$

(10)

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)\right]\cos(\xi x)d\xi$$

$$, \quad y>0 \tag{11}$$

where
$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \quad , \quad i=1,2 \tag{12}$$

and $A_i(\xi)$ $(i=1,2)$ are the unknown function to be determined, γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2-k_s^2)(\xi^2-k_s^2) = 0. \quad (13)$$

From the boundary condition (7), it is found that

$$A_2(\xi) = -\beta A_1(\xi) \quad (14)$$

where

$$\beta = \frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2}. \quad (15)$$

Employing equation (14) the expressions for displacements and stresses reduce to

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi, \quad (16)$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi \quad , \quad y>0 \quad (17)$$

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)\left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi \quad , \quad y>0 \quad (18)$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty\left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi. \quad (19)$$

We further substitute

$$A(\xi) = \frac{\alpha_1-\beta\alpha_2}{\xi}A_1(\xi)$$

so that the boundary conditions (9) and (8) yield the following integral equations in $A(\xi)$:

$$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 \quad , \qquad |x|<c, \; |x|>1 \tag{20}$$

and

$$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)d\xi = -\frac{\pi p_o}{2\mu_{12}} \quad , \qquad c\leq|x|\leq 1 \tag{21}$$

where $p_o = ik\mu_{12}c_{22}$

and

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)} \tag{22}$$

3. METHOD OF SOLUTION

In order to solve the set of integral equations (20) and (21), assume

$$A(\xi) = \frac{1}{\xi}\int_c^1 h(t^2)\sin(\xi t)dt \tag{23}$$

where $h(t^2)$ is an unknown function to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from equation (23) in equation (20) and using the following result (Gradshteyn and Ryzhik, 1965)

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi} d\xi = \begin{cases} \frac{\pi}{2}, & t>x \\ 0, & t<x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 h(t^2)\,dt = 0. \tag{24}$$

Further substitution of $A(\xi)$ from equation (23) in equation (21) leads to

$$\int_c^1 h(t^2)dt\int_0^\infty \sin(\xi t)\cos(\xi x)d\xi$$

$$= q_o - \frac{d}{dx}\int_c^1 h(t^2)dt\int_0^\infty \xi H_1(\xi)\frac{\sin(\xi t)\sin(\xi x)}{\xi^2}\,d\xi \quad , \quad c\le|x|\le 1 \tag{25}$$

where

$$q_o = -\frac{\pi p_o}{2\theta\mu_{12}} \tag{26}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \text{ as } \xi\to\infty \tag{27}$$

$$\theta = \frac{(c_{12}^2+c_{12}-c_{11}c_{22})(c_{12}N_1N_2-c_{11})-c_{22}\left[c_{12}N_1^2N_2^2+c_{11}(N_1^2+N_1N_2+N_2^2)\right]}{c_{11}(1+c_{12})(N_1+N_2)} \tag{28}$$

$$N_1^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}+\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\}$$

$$N_2^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}-\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\} \tag{29}$$

Using the relation

$$\frac{\sin\xi x\ \sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_o(\xi w)J_o(\xi v)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \tag{30}$$

equation (25) can be rewritten in the following form

$$\int_c^1 \frac{th(t^2)}{t^2-x^2}\,dt = q_o - \frac{d}{dx}\int_c^1 h(t^2)dt \int_o^x \int_o^t \frac{vwL(v,w)\;dw\;dv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \quad , \quad c\le|x|\le 1 \tag{31}$$

where

$$L(v,w) = \int_o^\infty \xi H_1(\xi)J_o(\xi w)J_o(\xi v)\;d\xi . \tag{32}$$

Applying a contour integration technique, (Mandal and Ghosh, 1994) the infinite integral in L(v,w) can be converted to the following finite integrals

$$L(v,w) = -ik_s^2\Bigg[\int_o^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2-c_{22}\bar{\alpha}_1\bar{\gamma}_1 - \bar{\beta}(c_{12}\eta^2-c_{22}\bar{\alpha}_2\bar{\gamma}_2)}{\theta(\bar{\alpha}_1-\bar{\beta}\,\bar{\alpha}_2)} \times$$
$$\times\, J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\;d\eta\; -$$
$$- \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2- c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1-\hat{\beta}\,\hat{\alpha}_2)}\, J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\;d\eta\Bigg],\quad w>v \tag{33}$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{R_1-(R_1^2-4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{R_1+(R_1^2-4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\left\{-R_1+(R_1^2+4R_2')^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\left\{R_1+(R_1^2+4R_2')^{1/2}\right\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}} \left\{ (c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22}) \right\}$$

$$\overline{R}_2 = \frac{c_{11}}{c_{22}} \left[1 - \eta^2\right]\left[\frac{1}{c_{11}} - \eta^2\right]$$

$$R_2' = \frac{c_{11}}{c_{22}} \left[1 - \eta^2\right]\left[\eta^2 - \frac{1}{c_{11}}\right]$$

$$\overline{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \overline{\gamma}_i^{\,2}}{(1 + c_{12})\overline{\gamma}_i} \qquad (i = 1, 2)$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i \hat{\gamma}_i^{\,2}}{(1 + c_{12})\hat{\gamma}_i} \qquad (i = 1, 2)$$

$$\overline{\beta} = \frac{\overline{\alpha}_1 - \overline{\gamma}_1}{\overline{\alpha}_2 - \overline{\gamma}_2} \quad \text{and} \quad \hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2} \tag{34}$$

The corresponding expression of $L(v,w)$ for $w<v$ follows from (33) by interchanging w and v.

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in equation (33), it is found that

$$L(v,w) = \frac{2}{\pi} P \, k_s^2 \log k_s + O(k_s^2) \tag{35}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - c_{22}\overline{\alpha}_1\overline{\gamma}_1 - \overline{\beta}(c_{12}\eta^2 - c_{22}\overline{\alpha}_2\overline{\gamma}_2)}{(\overline{\alpha}_1 - \overline{\beta}\,\overline{\alpha}_2)} \, d\eta \; - \right.$$

$$- \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\,\hat{\alpha}_2)}\, d\eta \Bigg].$$

Now, let us expand $h(t^2)$ in the form

$$h(t^2) = h_o(t^2) + k_s^2 \log k_s \, h_1(t^2) + O(k_s^2). \tag{36}$$

Inserting the above expansion of $h(t^2)$ and the value of $L(v,w)$ given by equation (35) into equation (31) and equating the coefficients of like powers of k_s, we obain the equations

$$\int_c^1 \frac{t h_o(t^2)}{t^2 - x^2}\, dt = q_o \quad , \quad c \le |x| \le 1 \tag{37}$$

and

$$\int_c^1 \frac{t h_1(t^2)}{t^2 - x^2}\, dt = - \frac{2P}{\pi} \int_c^1 t h_o(t^2) dt \ , \quad c \le |x| \le 1. \tag{38}$$

Using the finite Hilbert transform technique (Srivastava and Lowengrub, 1968), the solutions of the above integral equations can be obtained as

$$h_o(t^2) = \frac{2}{\pi} q_o \sqrt{\frac{t^2 - c^2}{1 - t^2}} + \frac{D_1}{\sqrt{(t^2 - c^2)(1 - t^2)}} \tag{39}$$

$$h_1(t^2) = - \frac{2}{\pi} P \left[\frac{q_o(1 - c^2)}{\pi} + D_1 \right] \sqrt{\frac{t^2 - c^2}{1 - t^2}} + \frac{D_2}{\sqrt{(t^2 - c^2)(1 - t^2)}} \tag{40}$$

where D_1 and D_2 are consants to be determined using the condition given by equation (24) so that

$$\int_c^1 h_o(t^2)\,dt = 0 \quad \text{and} \quad \int_c^1 h_1(t^2)\,dt = 0. \tag{41}$$

Substitution of the values of $h_o(t^2)$ and $h_1(t^2)$ given by equations (39) and (40) in (41), yields

$$D_1 = \frac{2}{\pi} q_o \left[c^2 - \frac{E}{F}\right] \tag{42}$$

$$D_2 = \frac{2}{\pi^2} q_o \left[1 + c^2 - \frac{2E}{F}\right]\left[\frac{E}{F} - c^2\right], \tag{43}$$

where

$$F = F\left[\frac{\pi}{2}, \sqrt{1-c^2}\right] \quad \text{and} \quad E = E\left[\frac{\pi}{2}, \sqrt{1-c^2}\right]$$

are the elliptic integrals of first and second kind, respecively. Substitituting the value of D_1 and D_2 given by equations (42) and (43) into equations (39-40), we obtain

$$h_o(t^2) = -\frac{p_o}{\mu_{12}\theta} \frac{\left[t^2 - \frac{E}{F}\right]}{\sqrt{(t^2-c^2)(1-t^2)}} \tag{44}$$

$$h_1(t^2) = -\frac{P\, p_o}{\pi\mu_{12}\theta} \frac{\left[t^2 - \frac{E}{F}\right]\left[1+c^2 - \frac{2E}{F}\right]}{\sqrt{(t^2-c^2)(1-t^2)}} \tag{45}$$

4. CRACK OPENING DISPLACEMENT AND STRESS INTENSITY FACTORS

The crack opening displacement and the normal stress component in the plane of the crack can be written as

$$\Delta v(x,0) = v(x,0+) - v(x,0-) = 2\int_x^1 h(t^2)\,dt \quad , \quad c \le x \le 1 \tag{46}$$

and

$$\tau_{yy}(x,0) = \frac{2\mu_{12}\theta}{\pi} \int_c^1 \frac{th(t^2)}{t^2 - x^2}\, dt \quad , \quad 0<x<c \tag{47}$$

$$= -\frac{2\mu_{12}\theta}{\pi} \int_c^1 \frac{th(t^2)}{x^2 - t^2}\, dt \quad , \quad x>1 \tag{48}$$

Expressions (47) and (48) with the aid of the equations (36), (44) and (45) yield

$$\tau_{yy}(x,0) = -p_o\left[1+\frac{\left[x^2-\frac{E}{F}\right]}{\sqrt{(c^2-x^2)(1-x^2)}}\right]\left[1-\frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\}k_s^2\log k_s\right] + O(k_s^2)$$

$$, \quad 0<x<c \tag{49}$$

$$\tau_{yy}(x,0) = -p_o\left[1-\frac{\left[x^2-\frac{E}{F}\right]}{\sqrt{(x^2-c^2)(x^2-1)}}\right]\left[1-\frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\}k_s^2\log k_s\right] + O(k_s^2)$$

$$, \quad x>1 \tag{50}$$

The stress intensity factors are defined as (in physical units)

$$K_c = \lim_{x\to c-}\left[\frac{\sqrt{(c-x)}\ \tau_{yy}(x,0)}{p_o}\right]_{0<x<c} \tag{51}$$

$$K_1 = \lim_{x\to 1+}\left[\frac{\sqrt{(x-1)}\ \tau_{yy}(x,0)}{p_o}\right]_{x>1} \tag{52}$$

Substituting equations (49-50) into equations (51-52) it can be shown that

$$K_c = -\frac{\left[c^2 - \frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1+c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s\right] + O(k_s^2) \qquad (53)$$

$$K_1 = \frac{\left[1 - \frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1+c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s\right] + O(k_s^2) \qquad (54)$$

Further substituting equations (36), (44-45) in the expression given by equation (46), the crack opening displacement is obtained as

$$\Delta v(x,0) = \frac{2p_o}{\mu_{12}\theta}\left[1 - \frac{P}{\pi}\left\{1+c^2 - \frac{2E}{F}\right\} k_s^2 \log k_s\right]\left[\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} F(\lambda,q) - E(\lambda,q)\right] +$$

$$+ O(k_s^2) \quad , \quad c \le x \le 1 \qquad (55)$$

where

$$\sin\lambda = \sqrt{\frac{1-x^2}{1-c^2}} \quad \text{and} \quad q = \sqrt{1-c^2} \; .$$

Letting $c \to 0$ in the expression for stress intensity factor and crack opening displacement, the results for a single crack occupying the region $|x| \le 1$, $y=0$, $|z| < \infty$ are found to be

$$K_1 = \frac{1}{\sqrt{2}}\left[1 - \frac{P}{\pi} k_s^2 \log k_s\right] + O(k_s^2) \qquad (56)$$

$$\Delta v(x,0) = -\frac{2p_o}{\mu_{12}\theta}\sqrt{1-x^2}\left[1 - \frac{P}{\pi} k_s^2 \log k_s\right] + O(k_s^2) \quad , \quad 0 \le x \le 1 \qquad (57)$$

For isotropic medium, putting

$$c_{11}=c_{22}=\frac{\lambda+2\mu}{\mu}, \quad \mu_{12}=\mu, \quad c_{12}=c_{11}-2=\frac{\lambda}{\mu}$$

so that $\alpha_1=\gamma_1$, $\alpha_2=\xi^2/\gamma_2$, $k_s=m_2$, $k_s/\sqrt{c_{11}}=m_1$, $\tau=\frac{1}{c_{11}}$

$N_1=1=N_2$, $\theta=-2(1-\tau^2)$ and $P=\frac{\pi}{2}c_1$,

where

$$c_1=\frac{3\tau^4-4\tau^2-3}{4(1-\tau^2)}, \quad \gamma_i=(\xi^2-m_i^2)^{1/2} \quad \text{and} \quad m_i=\frac{a\omega}{c_i} \quad (i=1,2)$$

the expressions for displacemet and stress are found to be

$$\Delta v(x,\pm 0) = \mp\frac{p_o}{2\mu(1-\tau^2)}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]\times$$

$$\times\left[\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}F(\lambda,q)-E(\lambda,q)\right]+O(m_2^2), \quad c\le x\le 1$$

$$=0, \quad |x|<c, \quad |x|>1$$

and

$$\tau_{yy}(x,0) = -p_o\left[1+\frac{\left[x^2-\frac{E}{F}\right]}{\sqrt{(c^2-x^2)(1-x^2)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2), \quad 0<x<c$$

$$=-p_o, \quad c\le|x|\le 1$$

$$=-p_o\left[1-\frac{\left[x^2-\frac{E}{F}\right]}{\sqrt{(x^2-c^2)(x^2-1)}}\right]\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2), \quad |x|>1.$$

Now, the crack opening displacement and stress intensity factors are found to be

$$\Delta v(x,0) = -\frac{p_0}{\mu(1-\tau^2)}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]\times$$

$$\times\left[\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}F(\lambda,q)-E(\lambda,q)\right]+O(m_2^2)\quad,\quad c\le x\le 1$$

and

$$K_c = -\frac{\left[c^2-\frac{E}{F}\right]}{\sqrt{2c(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)$$

$$K_1 = \frac{\left[1-\frac{E}{F}\right]}{\sqrt{2(1-c^2)}}\left[1-\frac{c_1}{2}\left\{1+c^2-\frac{2E}{F}\right\}m_2^2\log m_2\right]+O(m_2^2)$$

which coincide with the results obtained by Jain and Kanwal (1972a) up to the order of $m_2^2\log m_2$ in the isotropic case.

When $c\to 0$, we recover the stress intensity factor and crack opening displacement for a single crack

$$K_1 = \frac{1}{\sqrt{2}}\left[1-\frac{c_1}{2}m_2^2\log m_2\right]+O(m_2^2)$$

$$\Delta v(x,0) = \frac{p_0}{\mu(1-\tau^2)}\sqrt{1-x^2}\left[1-\frac{c_1}{2}m_2^2\log m_2\right]+O(m_2^2)\quad,\quad 0\le x\le 1$$

which agrees with the result of Mal (1978)

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_c and K_1 given by (53) and (54) at the inner and outer tips of the cracks and crack opening displacements (COD) given by (55) have been plotted against

TABLE - 1. ENGINEERING ELASTIC CONSTANTS.

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II Graphite-Epoxy Composite :			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type Glass-Epoxy Composite :			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless Steel-Aluminium Composite :			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

dimensionless frequency k_s and distance, respectively for three different types of orthotropic materials whose engineering constants have been listed in table 1.

From Fig.2 it is found that SIF K_c at the inner tip of the crack increases at a slow rate with the increase in the value of frequency $k_s (0.1 \le k_s \le 0.6)$. On the other hand the rate of increase of the SIF K_1 (Fig.3) with frequency k_s at the outer tip of the crack is found to be higher than that of K_c.

In both the cases the value of SIF is higher for small values of c, i.e., for greater crack length SIF is higher. But it is interesting to note that for different materials the variation of SIFs in both the cases are not significant. In the case of single crack (c=0) the variation of SIF with material properties has been shown in Fig.4.

The COD has been plotted for different crack length. In each case COD increases gradually from zero, attains maximum value and then decreases to zero. It is found that with the increase in the values of c (i.e., for small crack length) the values of COD decreases (Figs.5-6). For a fixed material the variation of COD with frequency is found to be insignificant, but it is noticed that for smaller values of c (Fig.5, Fig.7) the variation of COD with frequency is palpable. c=0 (Fig.7) correspond to the case of single crack.

In all the cases where different values of c has been considered the variation of COD is found to be prominent for different orthotropic materials.

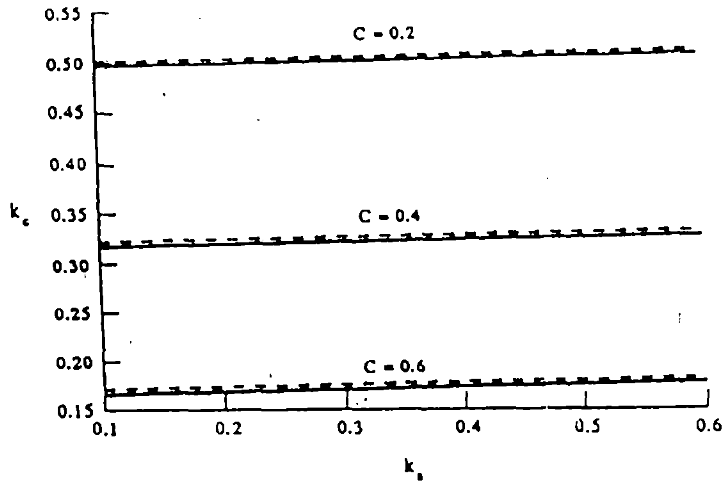

Fig. 2. Stress intensity factor K_c vs frequency k_s for generalized plane stress. (—, Type I; - - - -, Type II).

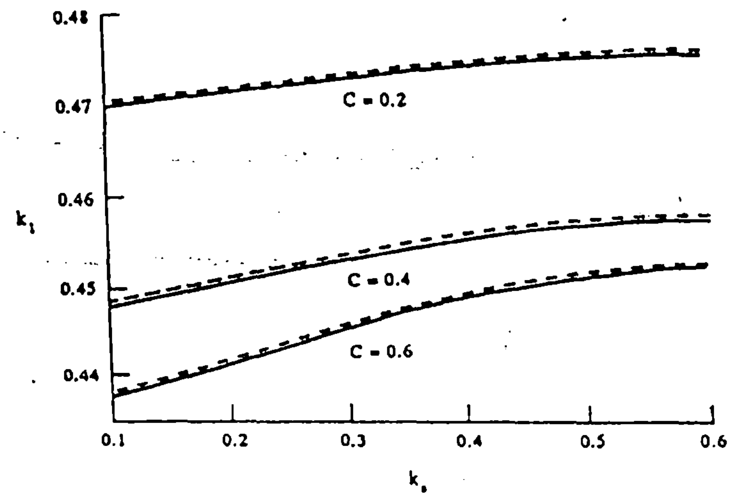

Fig. 3. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (—, Type I; - - - -, Type II).

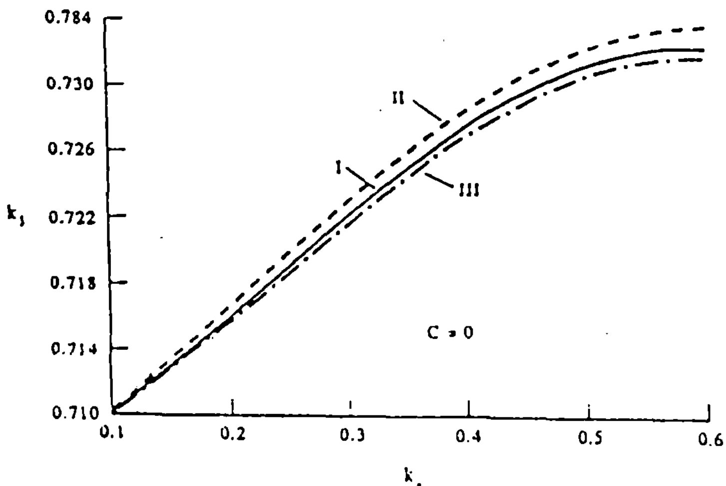

Fig. 4. Stress intensity factor K_1 vs frequency k_s for generalized plane stress. (Single crack, $c = 0$).

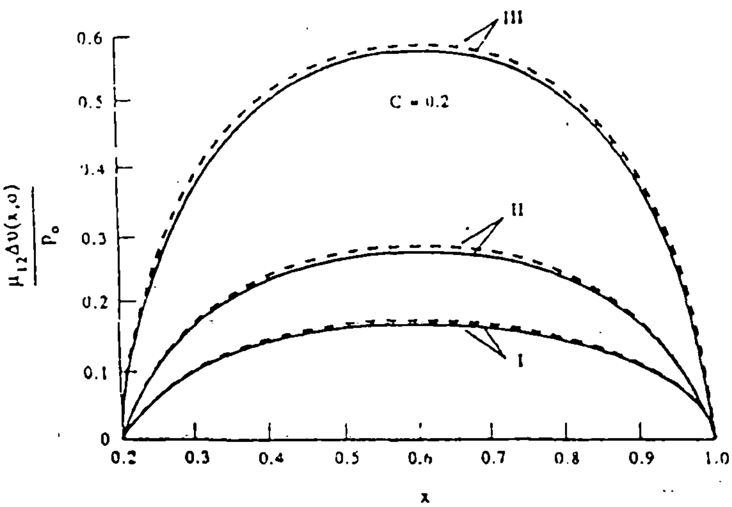

Fig. 5. Crack opening displacement (COD) vs distance ($c = 0.2$) for generalized plane stress. (—, $k_s = 0.2$; - - - -, $k_s = 0.6$).

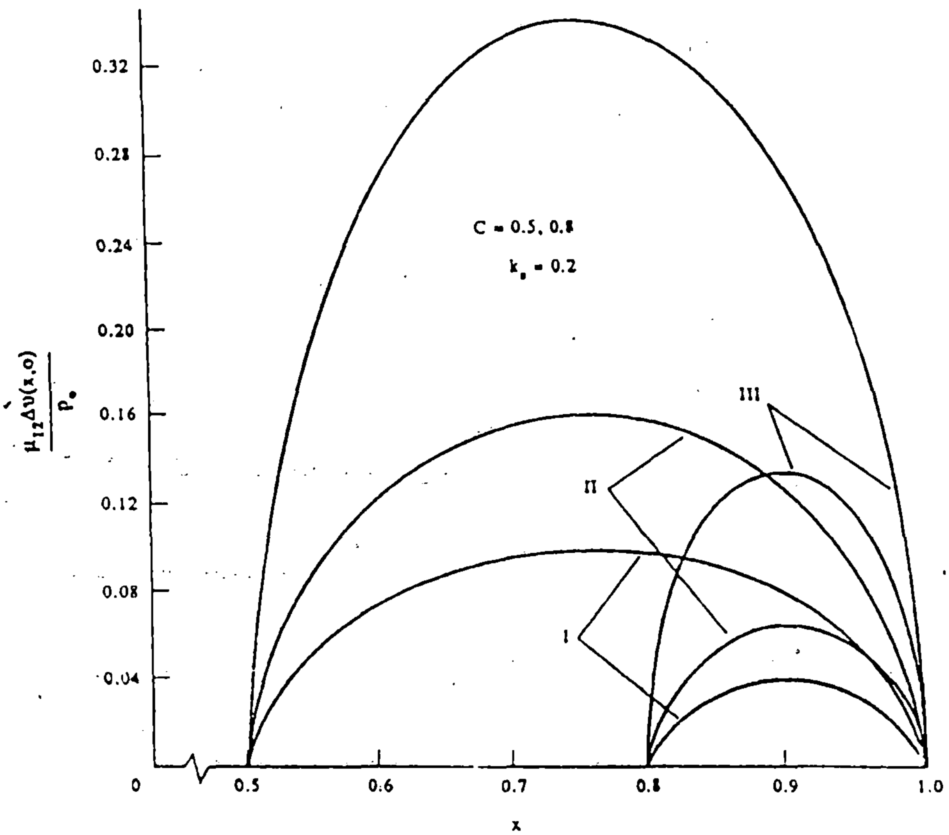

Fig. 6. Crack opening displacement (COD) vs distance ($c = 0.5$ and $c = 0.8$) for generalized plane stress.

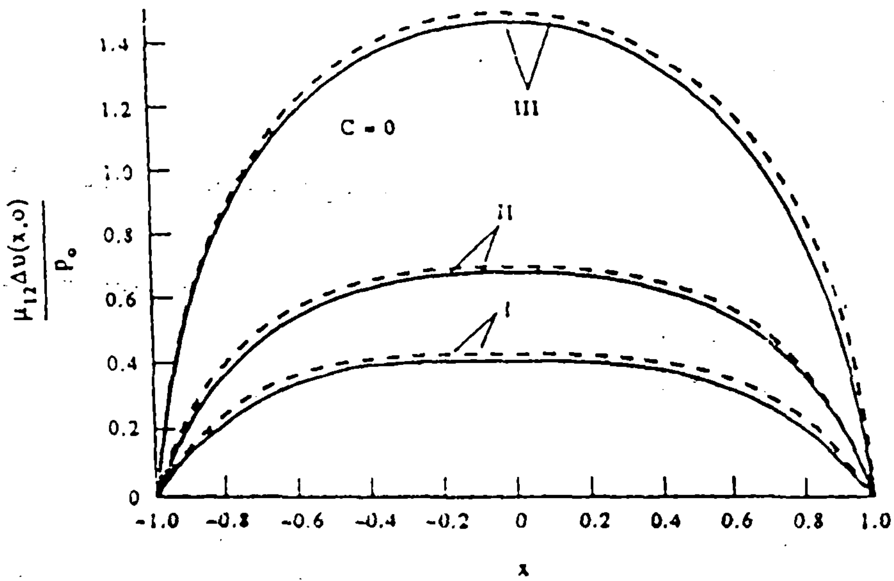

Fig. 7. Crack opening displacement (COD) vs distance (single crack, $c = 0$) for generalized plane stress. (—, $k_r = 0.2$; - - - -, $k_r = 0.6$).

DIFFRACTION OF ELASTIC WAVES BY THREE COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

1. INTRODUCTION

Recently, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced materials, the study of diffraction of elastic waves with cracks or inclusions has attracted the attention of scientists. The different possible location of cracks with respect to the planes of material symmetry is of great interest in Seismology and Exploration Geophysics. The problem of scattering of elastic waves by cracks of finite dimension in isotropic medium has been investigated by several investigators. Many investigators (Mal 1970b, Lowengrub et al. 1968a, Itou 1980b, Jain and Kanwal 1972a, Srivastava et al. 1981, Das and Ghosh 1992a) have solved the diffraction problem involving single or two cracks in isotropic medium. Dhawan and Dhaliwal (1978) solved the statical problem involving three coplanar cracks in an infinite transversely isotropic medium. The dynamic problem of singular stresses around cracks in orthotropic medium are few in number. Kassir and Bandyopadhyay (1983) solved the problem of elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading. The problem of normal impact response of a finite Griffith crack in an orthotropic strip has been solved by Shindo (1986). De and Patra (1990) have also solved the problem involving a moving Griffith crack in an orthotropic strip. Recently Kundu and Bostrom (1991) treated the diffraction problem of a circular crack in orthotropic medium.

To the best knowledge of the authors, the problem of diffraction of elastic waves by three coplanar Griffith cracks in an orthotropic material has not been considered. In our paper, the interaction of normally incident time harmonic elastic waves with three coplanar Griffith cracks in an orthotropic medium has been investigated. It is assumed that the faces of each of the cracks do not come into contact during small deformation of the solid. The resulting mixed boundary value problem is reduced to the solution of a set of four integral equations which has been reduced to the solution of an integro-differential equation. Iteration method has been used to obtain the low frequency solution of the problem. This enables us to obtain approximate value of the crack opening displacements and stress intensity factors. Making the length of the central crack tend to zero, the corresponding results for two Griffith cracks have been obtained. Numerical results for stress intensity factors and crack opening displacements have been plotted against dimensionless frequency and distance respectively for different orthotropic materials which have been shown graphically.

2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the interaction of normally incident longitudinal wave with three coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The cracks are assumed to occupy the position $|X|\le d_1$, $d_2\le|X|\le d$, $Y=0$, $|Z|<\infty$. Let E_i, μ_{ij} and ν_{ij} $(i,j= 1,2,3)$ denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the X, Y, Z directions chosen to coincide with the axes of material orthotropy. Normalizing all the lengths with respect to 'd' and setting $X/d=x$, $Y/d=y$, $Z/d=z$, $d_1/d=b$, $d_2/d=c$, the cracks are defined by $|x|\le b$, $c\le|x|\le 1$, $y=0$, $|z|<\infty$ (fig.1).

Displacement components are also made dimensionless with respect to 'd' so that dimensionless components of displacement in x, y directions are assumed to be u, v respectively, where

$$u = u(x,y,t) \text{ and } v = v(x,y,t).$$

Let a time harmonic plane elastic wave originating at $y=-\infty$ and incident normally on the three cracks be given by $v=v_o\exp\left[i\left(ky-\omega t\right)\right]/d$ where $k=d\omega/c_s\sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$, v_o is a constant, ω and v_o/d are the frequency and dimensionless amplitude of the incident wave respectively, ρ being the density of the material. In the isotropic solid, c_s represents the velocity of the shear-wave.

The non-zero stress components τ_{yy} and τ_{xy} are given by

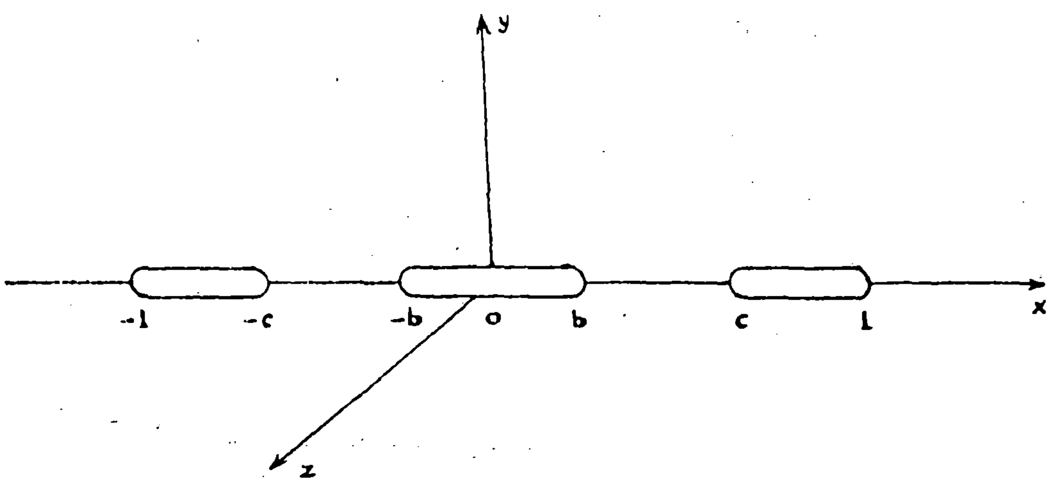

Fig. 1. Geometry of the cracks.

$$\tau_{yy}/\mu_{12} = c_{12}\, u_{,x} + c_{22}\, v_{,y}$$
$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} \qquad (2.1)$$

where u, v denote the component of the displacement in the x, y directions respectively and comma denotes partial differentiation with respect to the co-ordinates or time ; $c_{ij}(i,j=1,2)$ are nondimensional parameters related to the elastic constant by the relations :

$$c_{11}=E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1)$$
$$c_{22}=E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=c_{11}E_2/E_1 \qquad (2.2)$$
$$c_{12}=\nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=\nu_{12}c_{22}=\nu_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11}=(E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32})$$
$$c_{22}=(E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31})$$
$$c_{12}=E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \qquad (2.3)$$
$$=E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32}$$

for plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation :

$$\nu_{ij}/E_i = \nu_{ji}/E_j\,. \qquad (2.4)$$

The displacement equations of motion for orthotropic material are

$$c_{11}\, u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} = \frac{d^2}{c_s^2}\, u_{,tt}$$

$$c_{22}\, v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} = \frac{d^2}{c_s^2}\, v_{,tt} \qquad (2.5)$$

Substitution of $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ in equations (2.5) reduces them to

$$c_{11}\, u_{,xx} + u_{,yy} + (1+c_{12})v_{,xy} + k_s^2 u = 0$$

$$c_{22}\, v_{,yy} + v_{,xx} + (1+c_{12})u_{,xy} + k_s^2 v = 0 \qquad (2.6)$$

with $k_s^2 = d^2\omega^2/c_s^2$, which are to be solved subject to the boundary conditions

$$v(x,0) = 0 \quad , \quad b\le|x|\le c \;,\; |x|\ge 1 \qquad (2.7)$$

$$\tau_{xy}(x,0) = 0 \quad , \quad |x|<\infty \qquad (2.8)$$

$$\tau_{yy}(x,0) + \tau_{yy}^{(0)}(x,0) = 0 \quad , \quad |x|<b,\; c<|x|<1 \qquad (2.9)$$

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

Using the condition (2.8), the solutions of equations (2.6) may be written as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\exp(-\gamma_1|y|) - \beta\exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi \tag{2.10}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi \quad , \quad y>0 \tag{2.11}$$

and the stress components are given by

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)\left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi \quad , \quad y>0 \tag{2.12}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty\left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x) \tag{2.13}$$

where
$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \quad , \quad i=1,2 \tag{2.14}$$

$$\beta = \frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2} \tag{2.15}$$

$A_1(\xi)$ is the unknown function to be determined, and γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2-k_s^2)(\xi^2-k_s^2) = 0 \tag{2.16}$$

With the aid of the boundary conditions, (2.7) and (2.9) $A(\xi)$ is found to satisfy the integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 \quad , \qquad x \in I_2 \ , \ I_4 \tag{2.17}$$

and
$$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)d\xi = -\frac{\pi p_o}{2\mu_{12}} \quad , \qquad x \in I_1 \ , \ I_3 \tag{2.18a,b}$$

where $I_1 = (0,b)$, $I_2 = (b,c)$, $I_3 = (c,1)$, $I_4 = (1,\infty)$

and

$$p_o = ik\mu_{12}c_{22}v_o/d \tag{2.19}$$

$$A(\xi) = \frac{\alpha_1-\beta\alpha_2}{\xi} A_1(\xi) \tag{2.20}$$

$$H(\xi) = \frac{c_{12}\xi^2-c_{22}\alpha_1\gamma_1-\beta(c_{12}\xi^2-c_{22}\alpha_2\gamma_2)}{(\alpha_1-\beta\alpha_2)} \tag{2.21}$$

3. METHOD OF SOLUTION

The solution of the integral equations (2.17) and (2.18) is taken in the form

$$A(\xi) = \frac{1}{\xi}\int_0^b h(t)\sin(\xi t)dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin(\xi u)du \tag{3.1}$$

where $h(t)$ and $g(u^2)$ are the unknown functions to be determined. Substituting the value of $A(\xi)$ from (3.1) in (2.17) and using the following result (Gradshteyn et al, 1965)

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\, d\xi = \begin{cases} \frac{\pi}{2}, & t>x \\ 0, & t<x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equation

$$\int_c^1 g(u^2)\, du = 0. \tag{3.2}$$

Further substituting $A(\xi)$ from (3.1) in (2.18a) and using the result (Srivastava et al., 1968)

$$\int_0^\infty \xi^{-1}\sin(\xi u)\sin(\xi x)\, d\xi = \frac{1}{2}\log\left|\frac{u+x}{u-x}\right|$$

we obtain

$$\frac{d}{dx}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right| dt + \frac{d}{dx}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right| du$$

$$= 2\Big[\, q_o - \frac{d}{dx}\int_0^b h(t)dt\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi t)\sin(\xi x)d\xi -$$

$$- \frac{d}{dx}\int_c^1 g(u^2)du\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi u)\sin(\xi x)d\xi\Big], \quad x\in I_1 \tag{3.3}$$

where

$$q_o = -\frac{\pi p_o}{2\theta\mu_{12}} \tag{3.4}$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \text{ as } \xi\to\infty \tag{3.5}$$

$$\theta = \frac{(c_{12}^2+c_{12}-c_{11}c_{22})(c_{12}N_1N_2-c_{11})-c_{22}\left[c_{12}N_1^2N_2^2+c_{11}(N_1^2+N_1N_2+N_2^2)\right]}{c_{11}(1+c_{12})(N_1+N_2)} \tag{3.6}$$

$$N_1^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}+\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\}$$

$$N_2^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}-\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\} \tag{3.7}$$

Using the relation

$$\frac{\sin\xi x \,\sin\xi t}{\xi^2} = \int_0^x \int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

equation (3.3) can now be rewritten in the form

$$\frac{d}{dx}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right| dt + \frac{d}{dx}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right| du$$

$$= 2\left[q_0 - \frac{d}{dx}\int_0^b h(t)dt\int_0^x\int_0^t \frac{vwL(v,w)\,dwdv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} -\right.$$

$$\left. - \frac{d}{dx}\int_c^1 g(u^2)du\int_0^x\int_0^u \frac{vwL(v,w)\,dwdv}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}}\right], \quad x\in I_1 \tag{3.8}$$

where

$$L(v,w) = \int_0^\infty \xi H_1(\xi)J_0(\xi w)J_0(\xi v)\,d\xi \tag{3.9}$$

and $J_0(\ )$ is the Bessel function of order zero.

Applying a contour integration technique (Mandal and Ghosh, 1994) the infinite integral in L(v,w) can be converted to the following finite integrals.

$$L(v,w) = -ik_s^2\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2-\bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2-\bar{\alpha}_2\bar{\gamma}_2 c_{22})}{\theta(\bar{\alpha}_1 - \bar{\beta}\,\bar{\alpha}_2)} \times \right.$$

$$\times\ J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\ d\eta\ -$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\,\hat{\alpha}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\ d\eta\right],\quad w>v \tag{3.10}$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\left\{-R_1 + (R_1^2 + 4R_2')^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\left\{R_1 + (R_1^2 + 4R_2')^{1/2}\right\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\left\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1+c_{22})\right\}$$

$$\bar{R}_2 = \frac{c_{11}}{c_{22}}\left(1-\eta^2\right)\left(\frac{1}{c_{11}} - \eta^2\right) \tag{3.11}$$

$$R_2' = \frac{c_{11}}{c_{22}} \left(1-\eta^2\right)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

$$\bar{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_i^2}{(1+c_{12})\bar{\gamma}_i} \quad , \quad i=1,2$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i \hat{\gamma}_i^2}{(1+c_{12})\hat{\gamma}_i} \quad , \quad i=1,2$$

$$\bar{\beta} = \frac{\bar{\alpha}_1 - \bar{\gamma}_1}{\bar{\alpha}_2 - \bar{\gamma}_2}$$

$$\hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}$$

The corresponding expression of $L(v,w)$ for $w<v$ is obtained by interchanging v and w in (3.10).

Employing the series expansions for the Bessel function J_o and the Hankel function $H_o^{(1)}$ in equation (3.10), it is found that

$$L(v,w) = \frac{2}{\pi} P \, k_s^2 \log k_s + O(k_s^2) \tag{3.12}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2 - \bar{\alpha}_2\bar{\gamma}_2 c_{22})}{(\bar{\alpha}_1 - \bar{\beta}\,\bar{\alpha}_2)} d\eta - \right.$$

$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\,\hat{\alpha}_2)} d\eta \right]. \tag{3.13}$$

Let us now expand h(t) and $g(u^2)$ in the form

$$h(t) = h_o(t) + k_s^2 \log k_s\, h_1(t) + O(k_s^2)$$

(3.14)

and $$g(u^2) = g_o(u^2) + k_s^2 \log k_s\, g_1(u^2) + O(k_s^2).$$

Substituting the above equations (3.14) and the value of L(v,w) given by (3.10) in equations (3.8) and (3.2) and equating the coefficients of like powers of k_s, the following equations are derived.

$$\frac{d}{dx}\int_o^b h_o(t)\,\log\left|\frac{t+x}{t-x}\right|\,dt + 2\int_c^1 \frac{ug_o(u^2)}{u^2-x^2}\,du = 2q_o \quad , \quad x \in I_1 \;,\; I_3$$

(3.15a,b)

$$\frac{d}{dx}\int_o^b h_1(t)\,\log\left|\frac{t+x}{t-x}\right|\,dt + 2\int_c^1 \frac{ug_1(u^2)}{u^2-x^2}\,du =$$

$$= -\frac{4P}{\pi}\left[\int_o^b th_o(t)dt + \int_c^1 ug_o(u^2)du\right] \;,\quad x \in I_1 \;,\; I_3 \qquad (3.16a,b)$$

and $$\int_c^1 g_i(u^2)\,du = 0 \qquad (i=0,1) \qquad (3.17a,b)$$

Rewriting equation (3.15a) as

$$\int_o^b h_o(t)\,\log\left|\frac{t+x}{t-x}\right|\,dt = \pi F_1(x) \quad , \quad x \in I_1 \qquad (3.18)$$

where

$$F_1(x) = -\int_0^x \left[\frac{p_o}{\mu_{12}\theta} + \frac{2}{\pi}\int_c^1 \frac{u g_o(u^2)}{u^2-y^2}\, du \right] dy.$$

The solution of the integral equation (3.18) with the help of Cook's result (1970) is found to be

$$h_o(t) = -\frac{p_o}{\mu_{12}\theta}\frac{t}{(b^2-t^2)^{1/2}} - \frac{2}{\pi}\frac{t}{(b^2-t^2)^{1/2}}\int_c^1 \frac{\sqrt{u^2-b^2}\, g_o(u^2)}{u^2-t^2}\, du. \tag{3.19}$$

Substitution of the value of $h_o(t)$ from (3.19) in (3.15b) with the aid of the result

$$\int_0^b \frac{1}{(b^2-t^2)^{1/2}}\frac{t^2\, dt}{(x^2-t^2)(u^2-t^2)} = \frac{\pi}{2}\left[\frac{x}{(x^2-b^2)^{1/2}} - \frac{u}{(u^2-b^2)^{1/2}} \right], \quad x \in I_3$$

yields the singular integral equation

$$\int_c^1 \frac{\sqrt{u^2-b^2}\, g_o(u^2)}{u^2-x^2}\, du = -\frac{\pi}{2}\frac{p_o}{\mu_{12}\theta} \quad , \quad x \in I_3 \tag{3.20}$$

Next using the finite Hilbert transform technique (Srivastava et al, 1968) the solution of the integral equation is found to be

$$g_o(u^2) = -\frac{p_o}{\mu_{12}\theta}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_1}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}} \tag{3.21}$$

where D_1 is unknown constant to be determined from equation (3.17a).

Now substituting the value of $g_o(u^2)$ from (3.21) in (3.19) and performing the integrations, $h_o(t)$ is obtained in the following form

$$h_o(t) = -\frac{p_o}{\mu_{12}\theta}\left[\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} + \frac{tD_1}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}\right]. \qquad (3.22)$$

By the procedure similar to one which led to the derivations of the solutions of (3.15) as given by (3.21) and (3.22), the solutions of equation (3.16a,b) can also be obtained and they are found to be

$$h_1(t) = -\frac{4PR}{\pi^2}\left[\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} - \frac{tD_2}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}\right] \qquad (3.23)$$

$$g_1(u^2) = -\frac{4PR}{\pi^2}\left[\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uD_2}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}}\right] \qquad (3.24)$$

where

$$R = -\frac{p_o}{\mu_{12}\theta}\left[I_o^b + I_c^1\right] - D_1\left[J_o^b - J_c^1\right]$$

$$I_m^n = \int_m^n \frac{t^2\sqrt{(c^2-t^2)}}{\sqrt{(b^2-t^2)(1-t^2)}}\,dt \qquad (3.25)$$

$$J_m^n = \int_m^n \frac{t^2\, dt}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} .$$

The constant D_2 is to be determined from equation (3.17b).

In order to determine the values of the unknown constants D_1 and D_2, $g_0(u^2)$ and $g_1(u^2)$ as given by (3.21) and (3.24) respectively are substituted in (3.17a,b) and it is found that

$$D_j = A_j\left[(1-b^2)\frac{E}{F} - (c^2-b^2)\right] \quad , \quad (j=1,2) \tag{3.26}$$

and

$$A_1 = \frac{p_0}{\mu_{12}\theta} \quad , \quad A_2 = \frac{4PR}{\pi^2} \tag{3.27}$$

where $F = F(\frac{\pi}{2}, q)$ and $E = E(\frac{\pi}{2}, q)$ are the elliptic integrals of first and second kind respectively and $q = \sqrt{\dfrac{1-c^2}{1-b^2}}$.

Substitution of the values of $D_j (j=1,2)$ given by equations (3.26) in equations (3.21) - (3.24) yields

$$h_{j-1}(t) = -A_j\left[(1-b^2)\frac{E}{F} + (b^2-t^2)\right] \frac{t}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}$$

$$(j=1,2) \tag{3.28}$$

$$g_{j-1}(u^2) = -A_j\left[(1-b^2)\frac{E}{F} - (u^2-b^2)\right] \frac{u}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}}$$

$$(j=1,2) \tag{3.29}$$

4. STRESS INTENSITY FACTORS AND CRACK OPENING DISPLACEMENTS

The stress intensity factors are defined as (in physical units)

$$N_b = \lim_{x\to b+} \left[\frac{\sqrt{(x-b)}\ \tau_{yy}(x,0)}{p_o} \right]_{b<x<c} \quad (4.1)$$

$$N_c = \lim_{x\to c-} \left[\frac{\sqrt{(c-x)}\ \tau_{yy}(x,0)}{p_o} \right]_{b<x<c} \quad (4.2)$$

$$N_1 = \lim_{x\to 1+} \left[\frac{\sqrt{(x-1)}\ \tau_{yy}(x,0)}{p_o} \right]_{x>1} \quad (4.3)$$

and the crack opening displacement can now be shown to be given by

$$\Delta v(x,0) = v(x,0+)-v(x,0-) = 2\int_x^b h(t)\,dt \quad , \quad 0\le x\le b \quad (4.4)$$

$$= 2\int_x^1 g(u^2)\,du \quad , \quad c\le x\le 1. \quad (4.5)$$

Substituting the values of the function h(t) and $g(u^2)$, the stress component τ_{yy} can be evaluated from the expressions (2.13), (2.21) and (3.1). After evaluation of the value of τ_{yy} and putting it in relations (4.1) - (4.3) it is found that

$$N_b = \sqrt{\frac{b(1-b^2)}{2(c^2-b^2)}}\ \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} \left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2) \quad (4.6)$$

$$N_c = \sqrt{\frac{c}{2(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - (c^2-b^2)\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \quad (4.7)$$

$$N_1 = \sqrt{\frac{(1-b^2)}{2(1-c^2)}}\left[1 - \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \quad (4.8)$$

where

$$M_2 = \left[I_0^b + I_c^1 + \left\{(1-b^2)\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - (c^2-b^2)\right\}\left\{ J_0^b - J_c^1 \right\}\right].$$

Expressions (4.4) - (4.5) with the aid of the equations (3.28) - (3.29) yield

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\, F(\beta,q)\left\{\frac{E(\beta,q)}{F(\beta,q)} - \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\right\} - \right.$$

$$\left. - \sqrt{\frac{(1-x^2)(b^2-x^2)}{(c^2-x^2)}}\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \quad , \quad 0 \le x \le b \quad (4.9)$$

and

$$\Delta v(x,0) = \frac{2p_0}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\, F(\lambda,q)\left\{\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - \frac{E(\lambda,q)}{F(\lambda,q)}\right\}\right]\times$$

$$\times\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \quad , \quad c \le x \le 1 \quad (4.10)$$

where

$$\sin\beta = \sqrt{\frac{b^2-x^2}{c^2-x^2}} \quad \text{and} \quad \sin\lambda = \sqrt{\frac{1-x^2}{1-b^2}} .$$

When $b \to 0$, we recover the stress intensity factor and the crack opening displacement for two Griffith cracks occupying the region $c \leq |x| \leq 1$, $y=0$, $|z| < \infty$:

$$N_c = -\frac{[c^2 - \frac{E}{F}]}{\sqrt{2c(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\} k_s^2 \log k_s\right] + O(k_s^2)$$

$$N_1 = -\frac{[1 - \frac{E}{F}]}{\sqrt{2(1-c^2)}}\left[1 - \frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\} k_s^2 \log k_s\right] + O(k_s^2) \tag{4.11}$$

and

$$\Delta v(x,0) = \frac{2p_o}{\mu_{12}\theta}\left[1 - \frac{P}{\pi}\left\{1+c^2-\frac{2E}{F}\right\} k_s^2 \log k_s\right]\left[\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} F(\lambda,q) - E(\lambda,q)\right] + O(k_s^2) , \quad c \leq x \leq 1 \tag{4.12}$$

where $M_2 = \frac{\pi}{4}(1+c^2-2E/F)$ has been used.

It is noted that if further $c \to 0$, the cracks merge into a single crack of width two units. In this case $F \to \infty$ and $M_2 \to \pi/4$; so the results for stress intensity factor and crack opening displacements corresponding to the single crack are found to be

$$N_1 = \frac{1}{\sqrt{2}}\left[1 - \frac{P}{\pi} k_s^2 \log k_s\right] + O(k_s^2) \tag{4.13}$$

and

$$\Delta v(x,0) = -\frac{2p_o}{\mu_{12}\theta}\sqrt{(1-x^2)}\left[1 - \frac{P}{\pi}k_s^2 \log k_s\right] + O(k_s^2) \quad , \quad 0\le x\le 1. \quad (4.14)$$

The results given by (4.11) - (4.14) are found to be in agreement with the results of Sarkar, Mandal and Ghosh (1994a).

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) N_b, N_c and N_1 given by (4.6), (4.7) and (4.8) at the tips of the cracks and crack opening displacements (COD) given by (4.9) and (4.10) have been plotted against dimensionless frequency k_s and distance respectively for three different types of orthotropic materials whose engineering constants have been listed in table 1.

Keeping the length of the central crack fixed (b=0.2) SIFs at the tips of the central and outer cracks have been plotted against frequency k_s ($0.1\le k_s\le 0.6$) for different lengths (c=0.5, 0.6, 0.7) of the outer crack (fig.2-fig.4). It is noted from the graphs (fig.2-fig.4) that with the decrease in the value of outer crack length, i.e. with the increase in the value of the distance between inner and outer cracks the rate of increase in the SIF is higher with the increase in the value of the frequency k_s.

The same nature of SIFs are seen (fig.5-fig.7) in the case when the length of the outer cracks are fixed (c=0.7) and the length of

TABLE - 1. ENGINEERING ELASTIC CONSTANTS.

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II Graphite-Epoxy Composite :			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type Glass-Epoxy Composite :			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless Steel-Aluminium Composite :			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

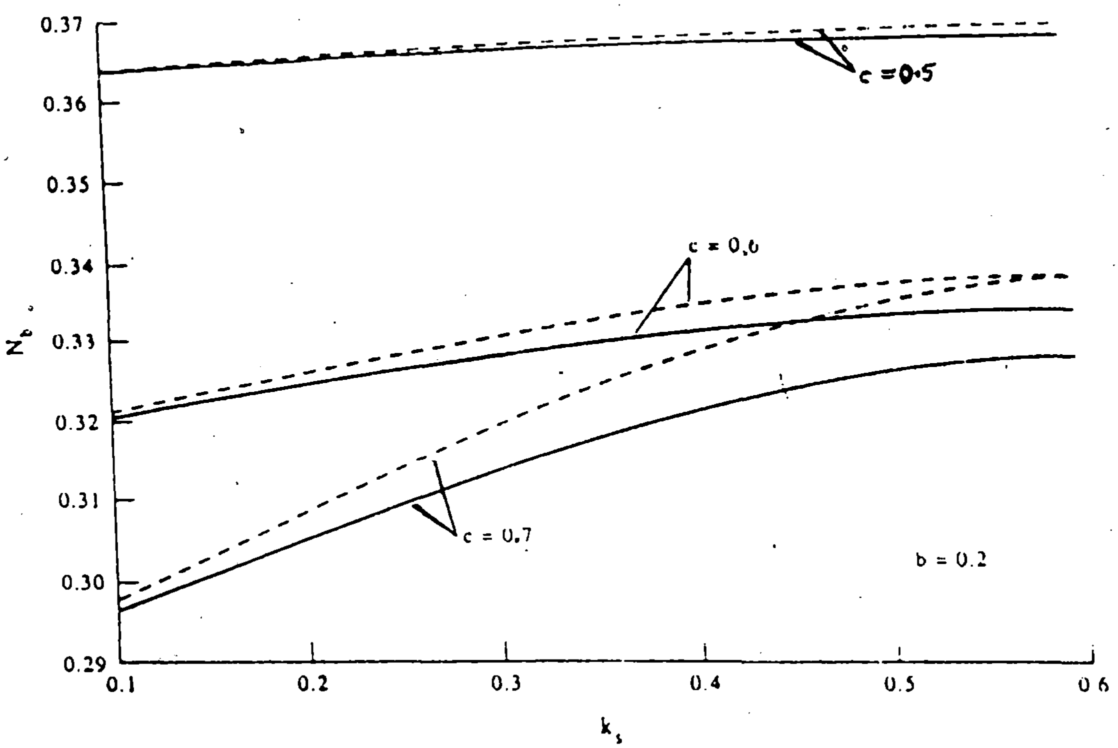

Fig. 2. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

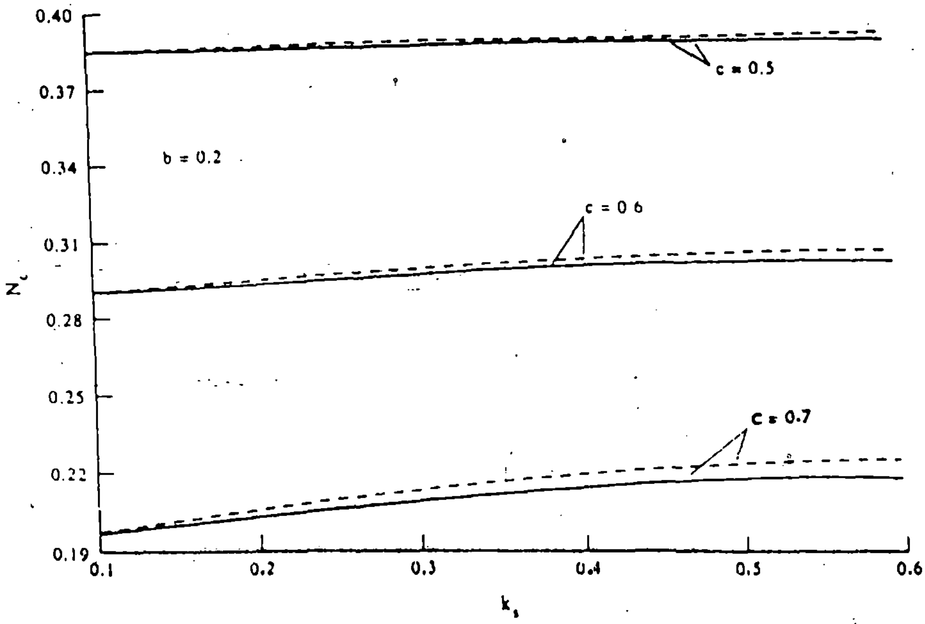

Fig. 3. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I; (- - - -) type III.

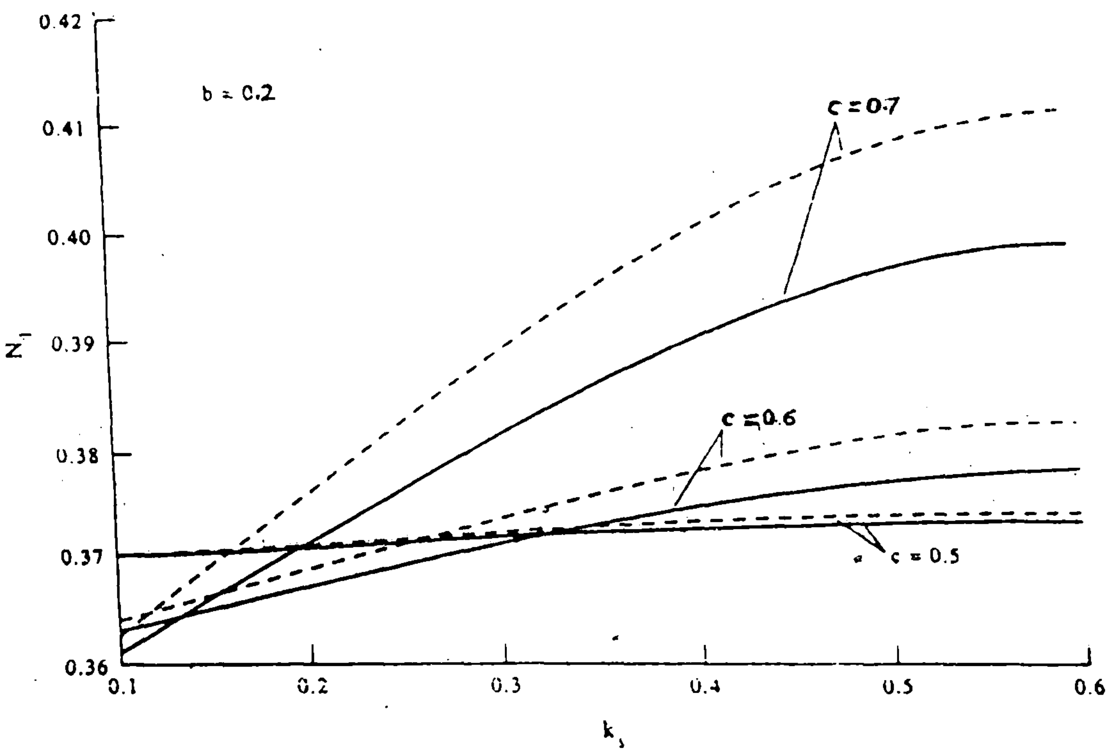

Fig. 4. Stress intensity factor N_1 vs frequency k_s for generalized plane stress. (——) type I; (- - - -) type III.

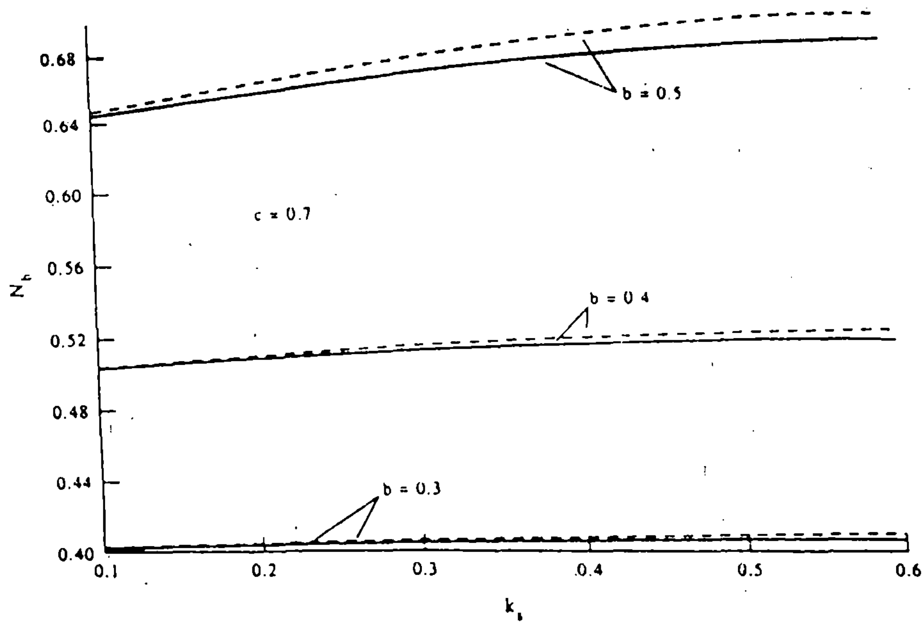

Fig. 5. Stress intensity factor N_b vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

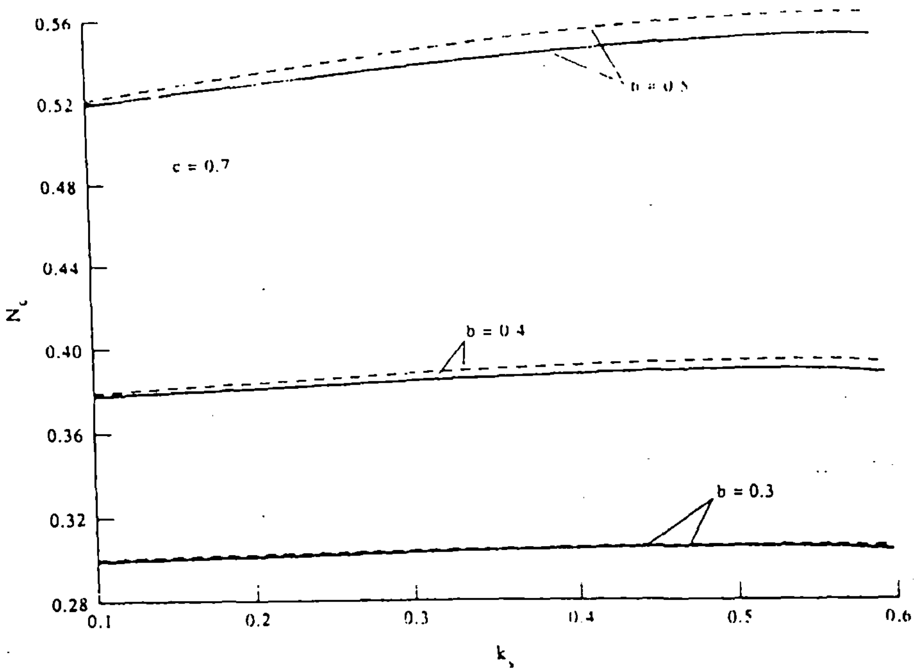

Fig. 6. Stress intensity factor N_c vs frequency k_s for generalized plane stress. (——) type I, (-----) type III.

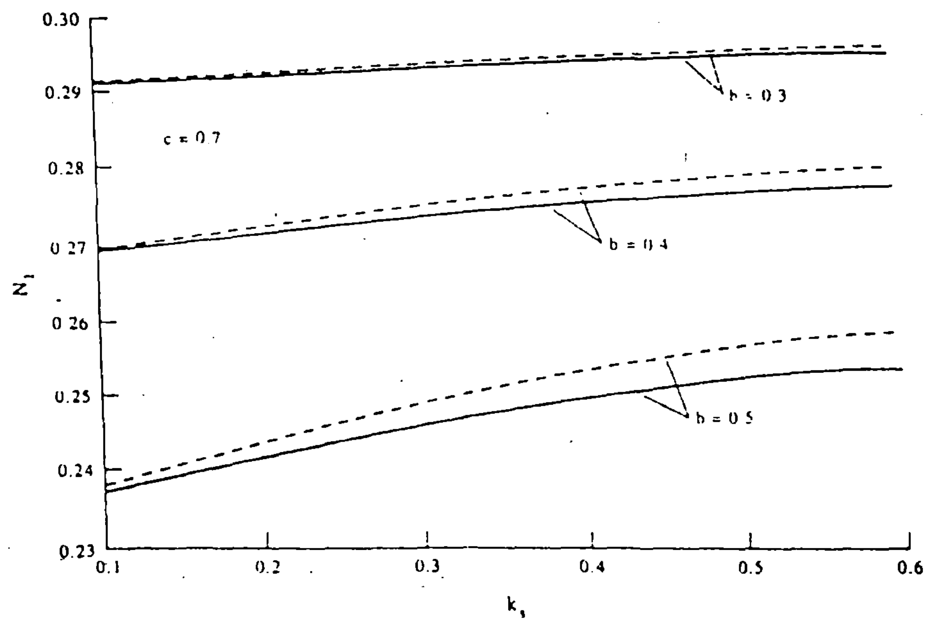

Fig. 7. Stress intensity factor N_1 vs frequency k_s for generalized plane stress. (——) type I; (-----) type III.

the central crack increases (b=0.3, 0.4, 0.5). It is interesting to note that for fixed c(=0.7) the SIFs N_b and N_c increase with the increase in the value of b, but the effect is just reverse in case of N_1.

The COD $\mu_{12}\Delta v(x,0)/p_0$ has been plotted for different crack lengths. It is found from fig.8 and fig.9 that with the increase in the value of crack length the value of COD increases. For a fixed material the variation of COD with frequency is found to be insignificant.

In all the cases where different values of crack length have been considered the variation of COD is found to be prominant for different orthotropic materials.

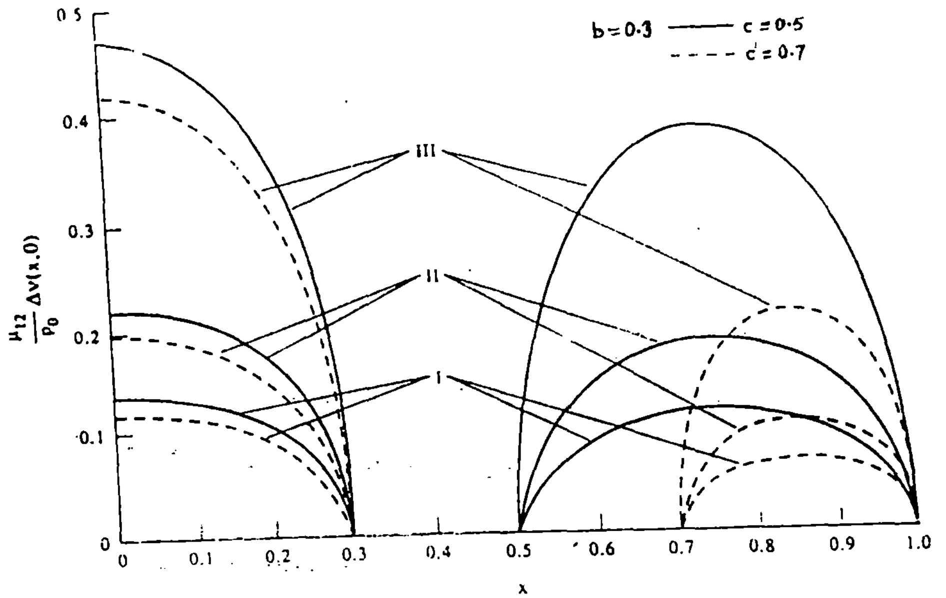

Fig. 8. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, b = 0.3, c = 0.5, 0.7).

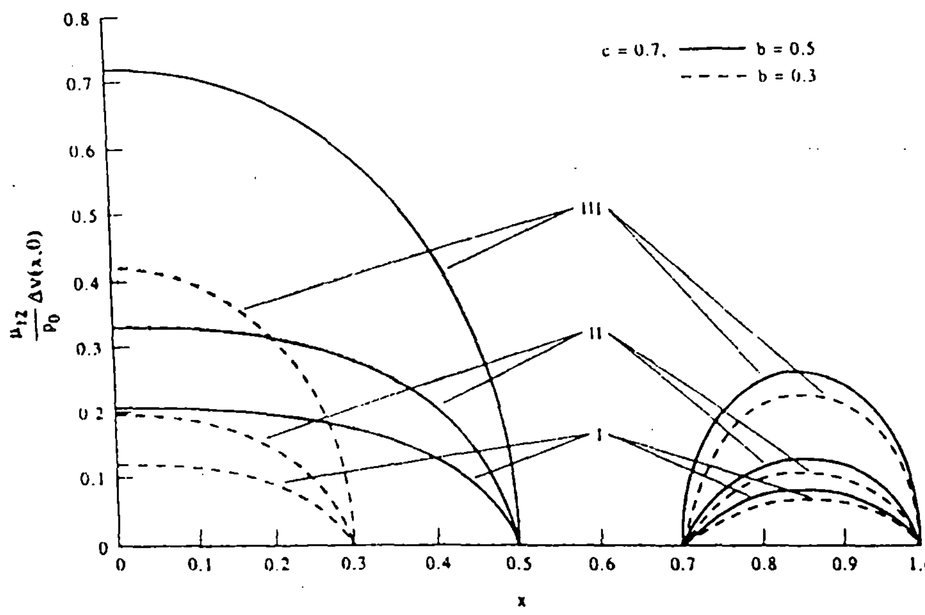

Fig. 9. Crack opening displacement vs distance for generalized plane stress ($k_s = 0.5$, b = 0.3, 0.5, c = 0.7) 0.7).

ELASTIC WAVE SCATTERING FROM FOUR COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC MEDIUM

1. INTRODUCTION

With the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced composite, the study of an anisotropic material with crack-like imperfections has become a matter of great importance in fracture analysis of composites (G.C.Sih et al. 1975). The different possible location of cracks with respect to the plane of symmetry is of great importance in seismology and exploration Geophysics. The problems involving the diffraction of elastic waves by cracks in an isotropic medium have been investigated by several investigators (Mal 1970b, Lowengrub et al. 1968a, Itou 1980b, Jain and Kanwal 1972a, Srivastava et al. 1981, Das and Ghosh 1992a, Dhawan et al. 1978), but perhaps, due to mathematical complexity, elastodynamic problems involving two or more Griffith cracks in an anisotropic medium for low frequency have not been treated earlier. Kassir and Tse (1983) have studied the plane stress problem of a moving Griffith crack in an infinite orthotropic stresses medium by using integral transform technique and the same technique has also been employed by De and Patra (1990) to solve the Yoffe's problem in a prestressed orthotropic strip of finite thickness. Kassir and Bandyopadhyay (1983) solved the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the problem of normal impact response of an orthotropic strip with a central crack have also been studied by Shindo et al. (1986).

In the present paper, we investigate the problem of diffraction of normally incident time harmonic elastic waves by four coplanar Griffith cracks in an infinite orthotropic medium. The faces of each of the cracks are assumed to be separated by a small distance so that during small deformation of the solid, the crack faces do not come into contact. The resulting mixed boundary value problem has been reduced to solving a set of five integral equations. Iterative method has been used to obtain the low frequency solution of the problem. Employing finite Hilbert transform technique (Srivastava and Lowengrub 1968) the integral equations have been solved to derive crack opening displacement and stress intensity factors. Finally, making the distance between two inner cracks tend to zero, the corresponding results for three cracks have been derived. To display the influence of the material orthotropy, numerical results of stress intensity factors and crack opening displacements have been plotted graphically against the dimensionless frequency and distance respectively for several orthotropic materials.

2. STATEMENT AND FORMULATION OF THE PROBLEM

Consider the problem of diffraction of normally incident elastic waves by four coplanar Griffith cracks situated in an infinite orthotropic elastic medium. The position of the cracks referred to a set of cartesian co-ordinate system (X,Y,Z) are assumed to be $d_1 \le |X| \le d_2$, $d_3 \le |X| \le d$, $Y=0$, $|Z|<\infty$. Let E_i, μ_{ij} and ν_{ij} (i,j= 1,2,3) denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the X, Y, Z directions chosen to coincide with the axes of material orthotropy. Normalizing all the lengths with respect to 'd' and setting X/d=x, Y/d=y, Z/d=z, $d_1/d=a$, $d_2/d=b$, $d_3/d=c$ the cracks are defined by $a \le |x| \le b$, $c \le |x| \le 1$, $y=0$, $|z|<\infty$ (Fig.1).

Components of the displacement are also made dimensionless with respect to 'd' so that dimensionless components of displacement in x, y directions are assumed to be u, v respectively, where

$$u = u(x,y,t) \quad \text{and} \quad v = v(x,y,t).$$

Let a time harmonic plane elastic wave given by u=0 and $v=v_o \exp[i(ky-\omega t)]/d$ where $k=\omega d/c_s\sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$, ρ the density of the material and v_o a constant, travelling in the direction of positive y-axis be incident normally on the four cracks.

The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}\, u_{,x} + c_{22}\, v_{,y}$$
$$\tau_{xy}/\mu_{12} = u_{,y} + v_{,x} \qquad (2.1a,b)$$

in which a comma denotes partial differentiation with respect to the co-ordinates or the time and c_{ij}(i,j=1,2) are non-dimensional

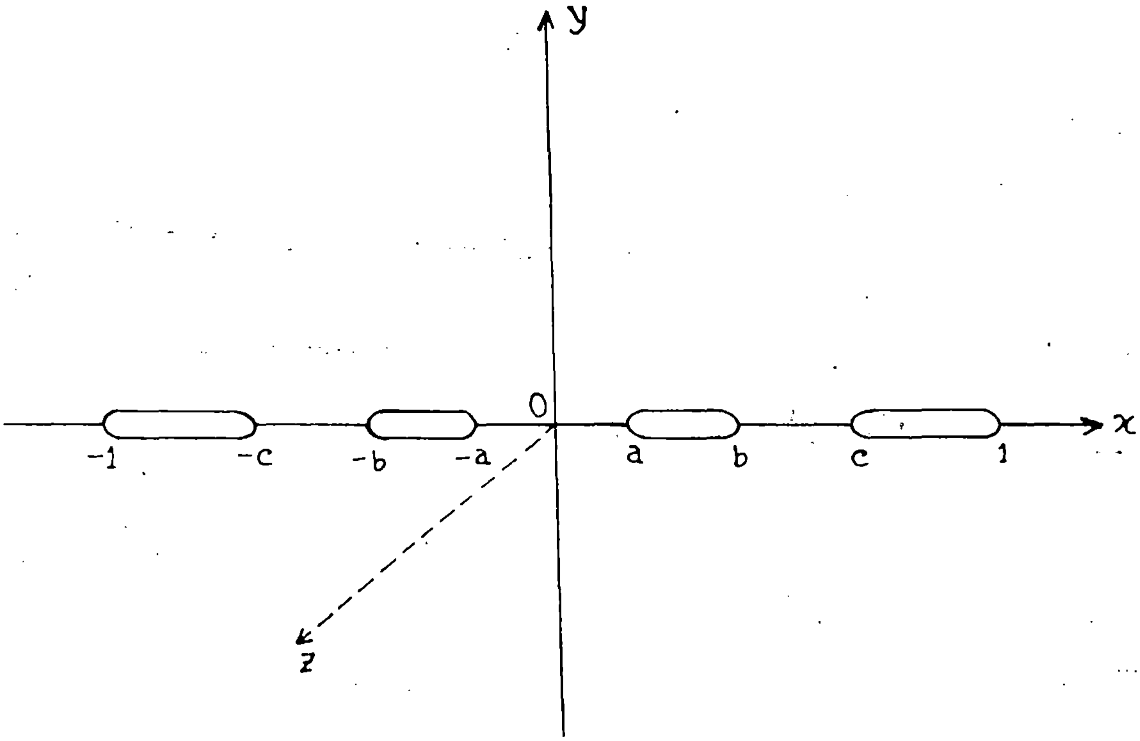

Fig.1. Geometry of the cracks.

parameters related to the elastic constants by the relations :

$$c_{11}=E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1)$$

$$c_{22}=E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=c_{11}E_2/E_1 \tag{2.2}$$

$$c_{12}=\nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1)=\nu_{12}c_{22}=\nu_{21}c_{11}$$

for generalized plane stress, and by

$$c_{11}=(E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32})$$

$$c_{22}=(E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31})$$

$$c_{12}=E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \tag{2.3}$$

$$=E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12}$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32}$$

for plane strain. The constants E_i and ν_{ij} satisfy Maxwell's relation :

$$\nu_{ij}/E_i = \nu_{ji}/E_j \tag{2.4}$$

The displacement equations of motion for orthotropic material are

$$c_{11}\,u_{,xx}+u_{,yy}+(1+c_{12})v_{,xy}=\frac{d^2}{c_s^2}\,u_{,tt}$$

$$\tag{2.5}$$

$$c_{22}\,v_{,yy}+v_{,xx}+(1+c_{12})u_{,xy}=\frac{d^2}{c_s^2}\,v_{,tt}$$

Substitution of $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ in equations (2.5) reduces them to

$$c_{11}\, u_{,xx} + u_{,yy} + (1+c_{12}) v_{,xy} + k_s^2 u = 0$$

and (2.6)

$$c_{22}\, v_{,yy} + v_{,xx} + (1+c_{12}) u_{,xy} + k_s^2 v = 0$$

with $k_s^2 = d^2\omega^2/c_s^2$.

The boundary conditions of the problem on account of the symmetry with respect to the y-axis are

$$\tau_{xy}(x,0) = 0 \quad , \qquad |x|<\infty \tag{2.7}$$

$$\tau_{yy}(x,0) + \tau_{yy}^{(0)}(x,0) = 0 \quad , \qquad x \in I_2,\ I_4 \tag{2.8}$$

$$v(x,0) = 0 \quad , \qquad x \in I_1,\ I_3,\ I_5 \tag{2.9}$$

where $I_1=(0,a)$, $I_2=(a,b)$, $I_3=(b,c)$, $I_4=(c,1)$, $I_5=(1,\infty)$.

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables is suppressed throughout the analysis.

The solution of equations (2.6) are taken as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \Big[A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)\Big]\sin(\xi x)d\xi ,$$

(2.10)

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\Big[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)\Big]\cos(\xi x)d\xi$$

$$, \quad y>0 \tag{2.11}$$

where $$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i} \quad , \quad i=1,2 \tag{2.12}$$

$A_i(\xi)$ $(i=1,2)$ are the unknown functions to be determined and γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_a^2\right\}\gamma^2 + (c_{11}\xi^2-k_a^2)(\xi^2-k_a^2) = 0 \tag{2.13}$$

Using the condition (2.7), it is found that

$$A_2(\xi) = -\frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2} A_1(\xi) \tag{2.14}$$

By the help of the relation (2.14), the expressions for displacements and stresses reduce to

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\exp(-\gamma_1|y|) - \beta \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi, \tag{2.15}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1\exp(-\gamma_1|y|) - \beta\alpha_2 \exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi \quad , \quad y>0 \tag{2.16}$$

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty (\gamma_1+\alpha_1)\left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin(\xi x)d\xi \quad , \quad y>0 \tag{2.17}$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty\left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \right.$$

$$\left. - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos(\xi x)d\xi . \tag{2.18}$$

where $\beta = \dfrac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2}$.

Finally, with the aid of the boundary conditions (2.9) and (2.8) the following set of five integral equations

$$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 \quad , \qquad x \in I_1 , I_3 , I_5 \qquad (2.19a\text{-}c)$$

and $$\int_0^\infty H(\xi)A(\xi)\cos(\xi x)d\xi = -\frac{\pi p_0}{2\mu_{12}} \quad , \qquad x \in I_2 , I_4 \qquad (2.20a,b)$$

are obtained for the determination of the unknown function $A(\xi)$

where $$p_0 = ik\mu_{12}c_{22}v_0/d$$

$$A(\xi) = \frac{\alpha_1-\beta\alpha_2}{\xi} A_1(\xi) \qquad (2.21)$$

$$H(\xi) = \frac{c_{12}\xi^2-c_{22}\alpha_1\gamma_1-\beta(c_{12}\xi^2-c_{22}\alpha_2\gamma_2)}{(\alpha_1-\beta\alpha_2)} .$$

3. METHOD OF SOLUTION

The solution of the integral equations (2.19) and (2.20) is taken in the form

$$A(\xi) = \frac{1}{\xi}\int_a^b h(t^2)\sin(\xi t)dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin(\xi u)du \qquad (3.1)$$

where $h(t^2)$ and $g(u^2)$ are the unknown functions to be determined. Substituting the value of $A(\xi)$ from (3.1) in (2.19) and using the following result (Gradshteyn and Ryzhik, 1965)

$$\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi} d\xi = \begin{cases} \frac{\pi}{2} , & t>x \\ 0 , & t<x \end{cases}$$

it is found that the choice of $A(\xi)$ leads to the equations

$$\int_a^b h(t^2)\,dt = 0 \qquad \text{and} \qquad \int_c^1 g(u^2)\,du = 0 \qquad (3.2a,b)$$

Further substitution of $A(\xi)$ from (3.1) in (2.20) leads to

$$\int_a^b \frac{th(t^2)}{t^2-x^2}\,dt + \int_c^1 \frac{ug(u^2)}{u^2-x^2}\,du =$$

$$= q_o - \frac{d}{dx}\int_a^b h(t^2)dt\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi t)\sin(\xi x)d\xi -$$

$$- \frac{d}{dx}\int_c^1 g(u^2)du\int_0^\infty H_1(\xi)\xi^{-1}\sin(\xi u)\sin(\xi x)d\xi \quad , \; x\in l_2,\; I_4 \qquad (3.3)$$

where

$$q_o = -\frac{\pi p_o}{2\theta\mu_{12}} \qquad (3.4)$$

$$H_1(\xi) = \frac{H(\xi)}{\xi\theta} - 1 \to 0 \text{ as } \xi\to\infty \qquad (3.5)$$

$$\theta = \frac{(c_{12}^2+c_{12}-c_{11}c_{22})(c_{12}N_1N_2-c_{11})-c_{22}\left[c_{12}N_1^2N_2^2+c_{11}(N_1^2+N_1N_2+N_2^2)\right]}{c_{11}(1+c_{12})(N_1+N_2)} \qquad (3.6)$$

$$N_1^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}+\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\}$$

$$N_2^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}-\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\} \qquad (3.7)$$

Using the relation

$$\frac{\sin\xi x\ \sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_o(\xi w)J_o(\xi v)dvdw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

equation (3.3) can now be rewritten in the form

$$\int_a^b \frac{th(t^2)}{t^2-x^2}\,dt + \int_c^1 \frac{ug(u^2)}{u^2-x^2}\,du =$$

$$= q_o - \frac{d}{dx}\int_a^b h(t^2)dt \int_o^x \int_o^t \frac{vwL(v,w)\ dwdv}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} -$$

$$- \frac{d}{dx}\int_c^1 g(u^2)du \int_o^x \int_o^u \frac{vwL(v,w)\ dwdv}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}} \quad , \ x \in I_2 \ , \ I_4$$

(3.8)

where

$$L(v,w) = \int_o^\infty \xi H_1(\xi) J_o(\xi w) J_o(\xi v)\ d\xi \tag{3.9}$$

and $J_o(\ )$ is the Bessel function of order zero.

Applying a contour integration technique (Mandal and Ghosh, 1994) the infinite integral in L(v,w) can be converted to the following finite integrals

$$L(v,w) = -ik_s^2\Bigg[\int_o^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2-\bar{\alpha}_1\bar{\gamma}_1 c_{22} - \bar{\beta}(c_{12}\eta^2-\bar{\alpha}_2\bar{\gamma}_2 c_{22})}{\theta(\bar{\alpha}_1 - \bar{\beta}\ \bar{\alpha}_2)} \times$$

$$\times\ J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\ -$$

$$- \int_{1/\sqrt{c_{11}}}^1 \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{\theta(\hat{\alpha}_1 - \hat{\beta}\ \hat{\alpha}_2)}\ J_o(k_s\eta v)H_o^{(1)}(k_s\eta w)\ d\eta\Bigg],\quad w>v$$

(3.10)

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{R_1 - (R_1^2 - 4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{R_1 + (R_1^2 - 4\bar{R}_2)^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_1 = \left[\frac{1}{2}\left\{-R_1 + (R_1^2 + 4R_2')^{1/2}\right\}\right]^{1/2}$$

$$\hat{\gamma}_2 = \left[\frac{1}{2}\left\{R_1 + (R_1^2 + 4R_2')^{1/2}\right\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\left\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\eta^2 + (1 + c_{22})\right\}$$

$$\overline{R}_2 = \frac{c_{11}}{c_{22}}\left(1-\eta^2\right)\left(\frac{1}{c_{11}} - \eta^2\right)$$

$$R_2' = \frac{c_{11}}{c_{22}}\left(1-\eta^2\right)\left(\eta^2 - \frac{1}{c_{11}}\right)$$

$$\overline{\alpha}_i = \frac{c_{11}\eta^2 - 1 + \overline{\gamma}_i^{\,2}}{(1+c_{12})\overline{\gamma}_i}, \quad i=1,2$$

$$\hat{\alpha}_i = \frac{c_{11}\eta^2 - 1 + (-1)^i \hat{\gamma}_i^{\,2}}{(1+c_{12})\hat{\gamma}_i}, \quad i=1,2$$

$$\overline{\beta} = \frac{\overline{\alpha}_1 - \overline{\gamma}_1}{\overline{\alpha}_2 - \overline{\gamma}_2} \quad \text{and} \quad \hat{\beta} = \frac{\hat{\alpha}_1 + \hat{\gamma}_1}{\hat{\alpha}_2 - \hat{\gamma}_2}. \tag{3.11}$$

The corresponding expression of $L(v,w)$ for $w<v$ is obtained by interchanging v and w in (3.10).

Employing the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$ in equation (3.10), it is found that

$$L(v,w) = \frac{2}{\pi} P\, k_s^2 \log k_s + O(k_s^2) \tag{3.12}$$

where

$$P = \frac{1}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{c_{12}\eta^2 - \overline{\alpha}_1\overline{\gamma}_1 c_{22} - \overline{\beta}(c_{12}\eta^2 - \overline{\alpha}_2\overline{\gamma}_2 c_{22})}{(\overline{\alpha}_1 - \overline{\beta}\,\overline{\alpha}_2)}\, d\eta \;-\right.$$

$$- \int_{1/\sqrt{c_{11}}}^{1} \frac{\hat{\beta}(c_{12}\eta^2 - c_{22}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1 - \hat{\beta}\,\hat{\alpha}_2)} d\eta \Bigg] \qquad (3.13)$$

Next expanding $h(t^2)$ and $g(u^2)$ in the form

$$h(t^2) = h_o(t^2) + k_s^2 \log k_s \; h_1(t^2) + O(k_s^2)$$

(3.14)

and

$$g(u^2) = g_o(u^2) + k_s^2 \log k_s \; g_1(u^2) + O(k_s^2)$$

and substituting this expansion as well as the result (3.12) in equation (3.8) and finally equating the coefficients of like powers of k_s , the following equations are derived.

$$\int_a^b \frac{th_o(t^2)}{t^2-x^2} dt + \int_c^1 \frac{ug_o(u^2)}{u^2-x^2} du = q_o \quad , \qquad x \in I_2 \; , \; I_4$$

(3.15a,b)

$$\int_a^b \frac{th_1(t^2)}{t^2-x^2} dt + \int_c^1 \frac{ug_1(u^2)}{u^2-x^2} du =$$

$$= - \frac{4P}{\pi}\Big[\int_a^b th_o(t^2)dt + \int_c^1 ug_o(u^2)du \Big] \; , \qquad x \in I_2 \; , \; I_4$$

(3.16a,b)

and also equation (3.2) with the aid of equation (3.14) yields

$$\int_a^b h_i(t^2) \; dt = 0 \qquad (i=0,1)$$

(3.17a-d)

$$\int_c^1 g_i(u^2) \; du = 0 \qquad (i=0,1)$$

Rewriting equation (3.15a) as

$$\int_a^b \frac{t h_o(t^2)}{t^2-x^2}\, dt = \frac{\pi}{2} F_1(x) \quad , \quad x \in I_2 \tag{3.18}$$

where $$F_1(x) = -\left[\frac{p_o}{\mu_{12}\theta} + \frac{2}{\pi}\int_c^1 \frac{u g_o(u^2)}{u^2-x^2}\, du \right]$$

Applying finite Hilbert transform technique (Srivastava and Lowengrub, 1968), the solution of the integral equation (3.18) is found to be

$$h_o(t^2) = -\frac{p_o}{\mu_{12}\theta}\sqrt{\frac{t^2-a^2}{b^2-t^2}} - \frac{2}{\pi}\sqrt{\frac{t^2-a^2}{b^2-t^2}} \int_c^1 \sqrt{\frac{u^2-b^2}{u^2-a^2}}\, \frac{u\, g_o(u^2)}{u^2-t^2}\, du + \frac{C_1}{\sqrt{(t^2-a^2)(b^2-t^2)}} \tag{3.19}$$

where C_1 is the unknown constant to be determined from equation (3.17a).

Substitution of the value of $h_o(t^2)$ from (3.19) in (3.15b) with the aid of the results

$$\int_a^b \sqrt{\frac{t^2-a^2}{b^2-t^2}}\, \frac{t\, dt}{(x^2-t^2)(u^2-t^2)} = \frac{\pi}{2(u^2-x^2)}\left[\sqrt{\frac{x^2-a^2}{x^2-b^2}} - \sqrt{\frac{u^2-a^2}{u^2-b^2}}\right]$$

$$\int_a^b \frac{t\, dt}{(x^2-t^2)\sqrt{(t^2-a^2)(b^2-t^2)}} = \frac{\pi}{2\sqrt{(t^2-a^2)(b^2-t^2)}} \quad \text{for} \quad x \in I_4$$

yields the singular integral equation

$$\int_c^1 \sqrt{\frac{u^2-b^2}{u^2-a^2}}\, \frac{u\, g_o(u^2)}{u^2-x^2}\, du = \frac{\pi}{2} F_2(x) \quad , \quad x \in I_4 \tag{3.20}$$

where

$$F_2(x) = -\frac{p_o}{\mu_{12}\theta} + \frac{C_1}{x^2-a^2}.$$

Next using the finite Hilbert transform technique (Srivastava and Lowengrub, 1968) the solution of the integral equation (3.20) is found to be

$$g_o(u^2) = -\frac{p_o}{\mu_{12}\theta}\sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \sqrt{\frac{1-a^2}{c^2-a^2}}\,\frac{C_1\sqrt{(u^2-c^2)}}{\sqrt{(u^2-a^2)(u^2-b^2)(1-u^2)}} +$$

$$+\frac{C_2\sqrt{(u^2-a^2)}}{\sqrt{(u^2-b^2)(u^2-a^2)(1-u^2)}} \qquad (3.21)$$

where C_2 is unknown constant to be determined from equation (3.17c).

Further substituting the value of $g_o(u^2)$ from (3.21) in (3.19) and performing the resulting integrations, $h_o(t^2)$ is obtained in the following form

$$h_o(t^2) = -\frac{p_o}{\mu_{12}\theta}\sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}} + \sqrt{\frac{1-a^2}{c^2-a^2}}\,\frac{C_1\sqrt{(c^2-t^2)}}{\sqrt{(t^2-a^2)(b^2-t^2)(1-t^2)}} -$$

$$-\frac{C_2\sqrt{(t^2-a^2)}}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \qquad (3.22)$$

By the procedure similar to one which led to the derivations of the solutions of (3.15) as given by (3.21) and (3.22), the solutions of equation (3.16a,b) can also be obtained and they are found to be

$$h_1(t^2) = -\frac{4PR}{\pi^2}\sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}} + \sqrt{\frac{1-a^2}{c^2-a^2}}\,\frac{D_1\sqrt{(c^2-t^2)}}{\sqrt{(t^2-a^2)(b^2-t^2)(1-t^2)}} - $$

$$- \frac{D_2\sqrt{(t^2-a^2)}}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}} \qquad (3.23)$$

$$g_1(u^2) = -\frac{4PR}{\pi^2}\sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \sqrt{\frac{1-a^2}{c^2-a^2}}\,\frac{D_1\sqrt{(u^2-c^2)}}{\sqrt{(u^2-a^2)(u^2-b^2)(1-u^2)}} + $$

$$+ \frac{D_2\sqrt{(u^2-a^2)}}{\sqrt{(u^2-b^2)(u^2-a^2)(1-u^2)}} \qquad (3.24)$$

where

$$R = -\frac{p_o}{\mu_{12}\theta}\left[R_a^b + R_c^1 \right] + \left[\sqrt{\frac{1-a^2}{c^2-a^2}}\, C_1 + C_2 \right] J_2$$

$$R_m^n = \int_m^n \sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}}\, dt \qquad (3.25)$$

$$J_2 = \frac{(c^2-b^2)}{\sqrt{(c^2-a^2)(1-b^2)}}\left[\Pi\left(\frac{\pi}{2}, \frac{b^2-a^2}{c^2-a^2}, r\right) + \Pi\left(\frac{\pi}{2}, \frac{1-c^2}{1-b^2}, r\right) - F\left(\frac{\pi}{2}, r\right)\right]$$

$$r = \sqrt{\frac{(1-c^2)(b^2-a^2)}{(1-b^2)(c^2-a^2)}} \; .$$

The constants D_1 and D_2 are to be determined from (3.17b) and (3.17d). In equations (3.25), F() is the elliptic integral of the first kind and $\Pi()$ is the elliptic integral of the third kind. Substitution of the values of $h(t^2)$ and $g(u^2)$ given by equations (3.21-3.24) in equations (3.17a-d)yield

$$C_i = \frac{p_o}{\mu_{12}\theta} Q_i \qquad (i=1,2) \tag{3.26}$$

$$D_i = \frac{4PR}{\pi^2} Q_i \qquad (i=1,2) \tag{3.27}$$

where

$$Q_1 = \left[\frac{K_a^b \; I_c^1 + K_c^1 \; I_a^b}{K_a^b \; J_c^1 + K_c^1 \; J_a^b} \right] \sqrt{\frac{(c^2-a^2)}{(1-a^2)}}$$

$$Q_2 = \left[\frac{J_a^b \; I_c^1 - J_c^1 \; I_a^b}{K_a^b \; J_c^1 + K_c^1 \; J_a^b} \right]$$

$$I_m^n = \int_m^n \sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}}\, du \tag{3.28}$$

$$J_m^n = \int_m^n \frac{\sqrt{(u^2-c^2)}}{\sqrt{(u^2-a^2)(u^2-b^2)(1-u^2)}}\, du$$

$$K_m^n = \int_m^n \frac{\sqrt{(u^2-a^2)}}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}}\, du$$

Substitution of the values of C_i and D_i given by equations (3.26) and (3.27) in equations (3.21-3.24) yields

$$h_{i-1}(t^2) = -A_i\left[1 - \frac{Q_1}{t^2-a^2}\sqrt{\frac{(1-a^2)}{(c^2-a^2)}} + \frac{Q_2}{c^2-t^2}\right]\sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}} \quad (i=1,2)$$

$$g_{i-1}(u^2) = -A_i\left[1 - \frac{Q_1}{u^2-a^2}\sqrt{\frac{(1-a^2)}{(c^2-a^2)}} - \frac{Q_2}{u^2-c^2}\right]\sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}} \quad (i=1,2) \qquad (3.29)$$

where

$$A_1 = \frac{p_o}{\mu_{12}\theta} \quad , \quad A_2 = \frac{4PR}{\pi^2} .$$

4. STRESS INTENSITY FACTORS AND CRACK OPENING DISPLACEMENTS

The stress intensity factors are defined as (in physical units)

$$N_a = \lim_{x\to a-}\left[\frac{\sqrt{(a-x)}\,\tau_{yy}(x,0)}{p_o}\right]_{0<x<a}$$

$$N_b = \lim_{x\to b+}\left[\frac{\sqrt{(x-b)}\,\tau_{yy}(x,0)}{p_o}\right]_{b<x<c}$$

$$N_c = \lim_{x\to c-}\left[\frac{\sqrt{(c-x)}\,\tau_{yy}(x,0)}{p_o}\right]_{b<x<c}$$

$$N_1 = \lim_{x\to 1+}\left[\frac{\sqrt{(x-1)}\,\tau_{yy}(x,0)}{p_o}\right]_{x>1} \qquad (4.1a\text{-}d)$$

and the crack opening displacement can now be shown to be given by

$$\Delta v(x,0) = v(x,0+)-v(x,0-) = 2\int_x^b h(t^2)\,dt \quad , \qquad a\le x\le b$$

$$= 2\int_x^1 g(u^2)\,du \quad , \qquad c\le x\le 1 \qquad (4.2a\text{-}b)$$

The stress component τ_{yy} can be evaluated from the equations (2.18), (2.21) and (3.1) when the values of the functions $h(t^2)$ and $g(u^2)$ as obtained above from (3.29) are substituted. Next substitution of the value of τ_{yy} in the relations (4.1a-d) yields finally,

$$N_a = \sqrt{\frac{1}{2a(b^2-a^2)}}\,Q_1\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2)$$

$$N_b = \left[\sqrt{\frac{(b^2-a^2)(c^2-b^2)}{2b(1-b^2)}} - Q_1\sqrt{\frac{(c^2-b^2)(1-a^2)}{2b(b^2-a^2)(1-b^2)(c^2-a^2)}} + \right.$$

$$\left. + Q_2\sqrt{\frac{(b^2-a^2)}{2b(c^2-b^2)(1-b^2)}}\right]\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2)$$

$$N_c = \sqrt{\frac{(c^2-a^2)}{2c(c^2-b^2)(1-c^2)}}\,Q_2\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2)$$

$$N_1 = \left[\sqrt{\frac{(1-a^2)(1-c^2)}{2(1-b^2)}} - Q_1\sqrt{\frac{(1-c^2)}{2(1-b^2)(c^2-a^2)}} - Q_2\sqrt{\frac{(1-a^2)}{2(1-b^2)(1-c^2)}}\right]\times$$

$$\times\left[1 - \frac{4P}{\pi^2}M_2k_s^2\log k_s\right] + O(k_s^2) \qquad (4.3a\text{-}d)$$

where $\quad M_2 = [\,R_a^b + R_c^1\,] - \left[\sqrt{\frac{1-a^2}{c^2-a^2}}\,Q_1 + Q_2\right]J_2\,.$

Expressions (4.2a-b) with the aid of the equations (3.14) and

(3.29) yield

$$\Delta v(x,0) = -2\left[A_1 + A_2\, k_s^2 \log k_s \right]\int_x^b \sqrt{\frac{(t^2-a^2)(c^2-t^2)}{(b^2-t^2)(1-t^2)}} \times$$

$$\times \left[1 - \frac{Q_1}{t^2-a^2}\sqrt{\frac{(1-a^2)}{(c^2-a^2)}} + \frac{Q_2}{c^2-t^2}\right]dt + O(k_s^2)\ , \qquad a\le x\le b$$

$$= -2\left[A_1 + A_2\, k_s^2 \log k_s \right]\int_x^1 \sqrt{\frac{(u^2-a^2)(u^2-c^2)}{(u^2-b^2)(1-u^2)}} \times$$

$$\times \left[1 - \frac{Q_1}{u^2-a^2}\sqrt{\frac{(1-a^2)}{(c^2-a^2)}} - \frac{Q_2}{u^2-c^2}\right]du + O(k_s^2)\ , \qquad c\le x\le 1$$

(4.4a-b)

When $a=d_1/d \to 0$, the stress intensity factor and the crack opening displacement for three Griffith cracks occupying the region $|x|\le b$, $c\le|x|\le 1$, $y=0$, $|z|<\infty$ are recovered (Sarkar et al. , 1994b)

$$N_b = \sqrt{\frac{b(1-b^2)}{2(c^2-b^2)}}\ \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$N_c = \sqrt{\frac{c}{2(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - (c^2-b^2)\right]\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$N_1 = \sqrt{\frac{(1-b^2)}{2(1-c^2)}}\left[1 - \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\right]\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s \right] + O(k_s^2)$$

$$\Delta v(x,0) = \frac{2p_o}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\; F(\beta,q)\left\{\frac{E(\beta,q)}{F(\beta,q)} - \frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)}\right\} - \right.$$

$$\left. - \sqrt{\frac{(1-x^2)(b^2-x^2)}{(c^2-x^2)}}\right]\left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s\right] + O(k_s^2) \;, \qquad 0 \le x \le b$$

and

$$\Delta v(x,0) = \frac{2p_o}{\mu_{12}\theta}\left[\sqrt{(1-b^2)}\; F(\lambda,q)\left\{\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - \frac{E(\lambda,q)}{F(\lambda,q)}\right\}\right] \times$$

$$\times \left[1 - \frac{4P}{\pi^2} M_2 k_s^2 \log k_s\right] + O(k_s^2) \;, \qquad c \le x \le 1$$

where

$$M_2 = \left[R_o^b + R_c^1 + \left\{(1-b^2)\frac{E(\frac{\pi}{2},q)}{F(\frac{\pi}{2},q)} - (c^2-b^2)\right\}\left(L_o^b - L_c^1\right)\right]$$

$$L_m^n = \int_m^n \frac{t^2\, dt}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}$$

$$\sin\beta = \sqrt{\frac{b^2-x^2}{c^2-x^2}} \qquad \text{and} \qquad \sin\lambda = \sqrt{\frac{1-x^2}{1-b^2}} \;.$$

and $E(\frac{\pi}{2},q)$ is the elliptic integral of the second kind with

$$q = \sqrt{\frac{1-c^2}{1-b^2}} \;.$$

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) N_a, N_b, N_c and N_1 given by (4.3a-d) at the tips of the cracks and crack opening displacements (COD) given by (4.4a-b) have been plotted against dimensionless frequency k_s and distance respectively for three different types of orthotropic materials whose engineering constants have been listed in table 1.

Keeping the length of the outer cracks and distance between inner and outer cracks fixed (b=0.6, c=0.8) SIFs at the tips of the cracks have been plotted against frequency k_s ($0.1 \le k_s \le 0.6$) for different lengths of the inner cracks (a=0.2,0.3,0.4). It is noted from the graphs (Fig.2-Fig.5) that with the decrease in the value of inner crack length i.e. with the increase in the value of the distance between inner cracks the rate of increase in the SIF is higher with the increase in the value of the frequency k_s.

It is also found that the value of SIF is higher for lower value of a. When lengths of the outer cracks and the distance between inner cracks are kept fixed (a=0.2, c=0.8) it is noted from the graphs (Fig.6-Fig.9) that with the increase in the value of b(0.4, 0.5,0.6) i.e. with the decrease in the value of the distance between inner and outer cracks the rate of decrease of SIFs are higher. It is interesting to note that the value of SIF N_a is lower for higher values of b but in case of the SIFs N_b, N_c and N_1 the effect is just reverse.

Next, keeping the lengths of the inner cracks fixed (a=0.2, b=0.4)

TABLE - 1. ENGINEERING ELASTIC CONSTANTS.

	E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I	Modulite II Graphite-Epoxy Composite :			
	15.3×10^9	158.0×10^9	5.52×10^9	0.033
Type II	E-Type Glass-Epoxy Composite :			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless Steel-Aluminium Composite :			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

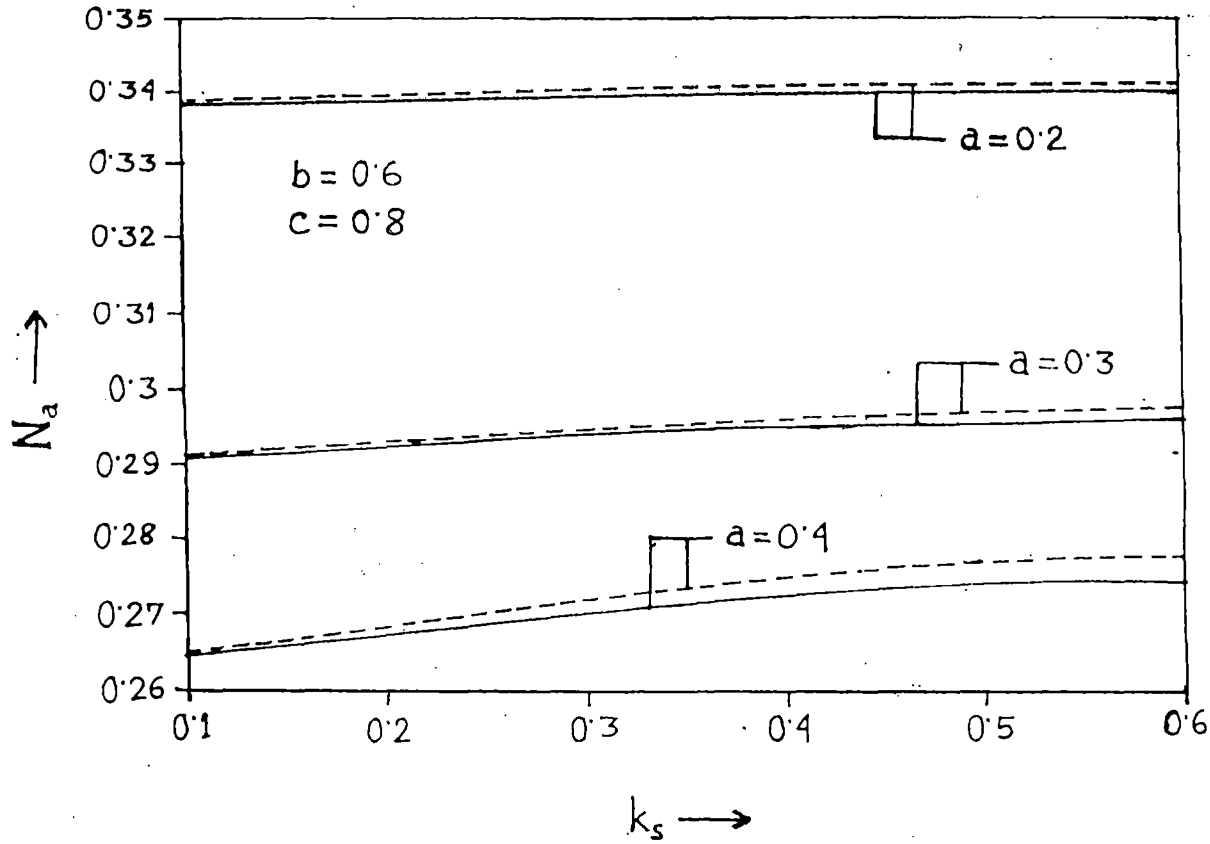

Fig. 2. Stress intensity factor N_a vs. frequency k_s for generalized plane stress.
(——— Type I, - - - - - - Type III).

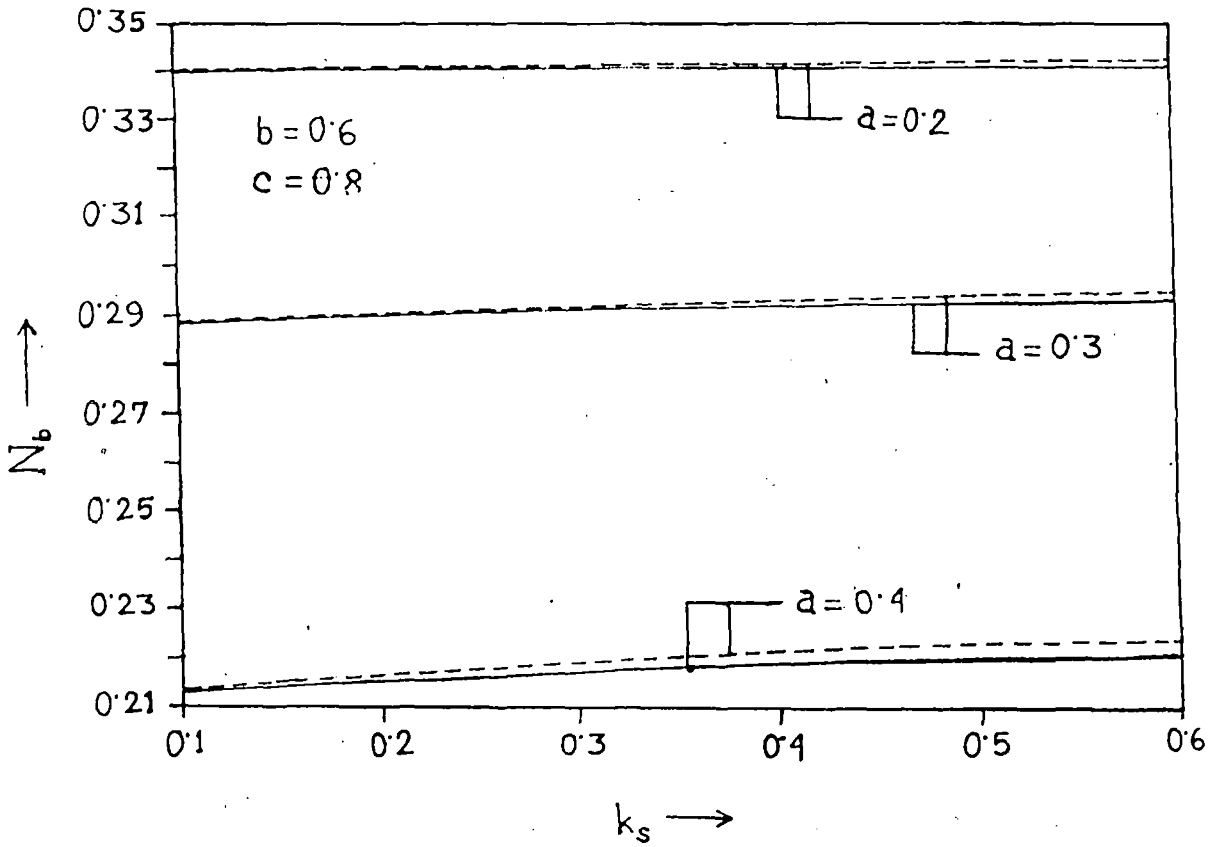

Fig. 3. Stress intensity factor N_b vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

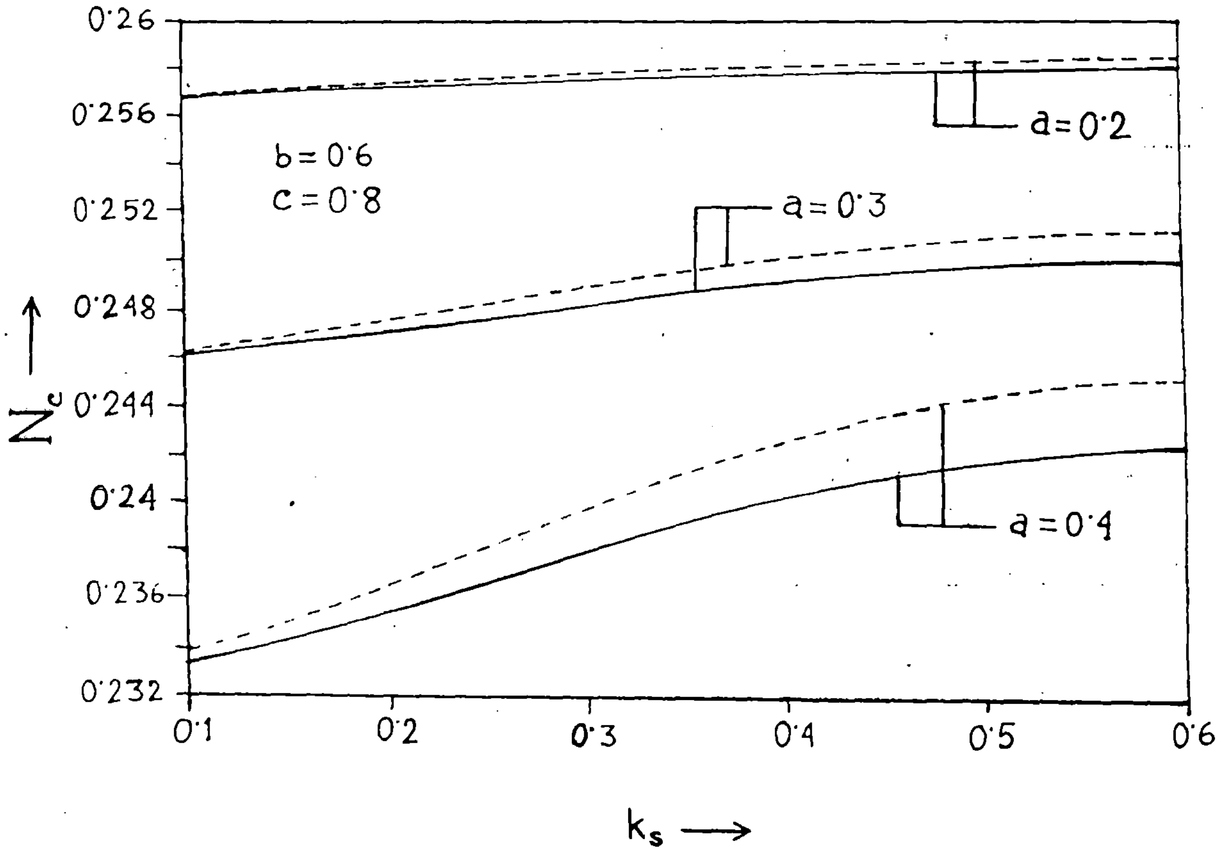

Fig. 4. Stress intensity factor N_c vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

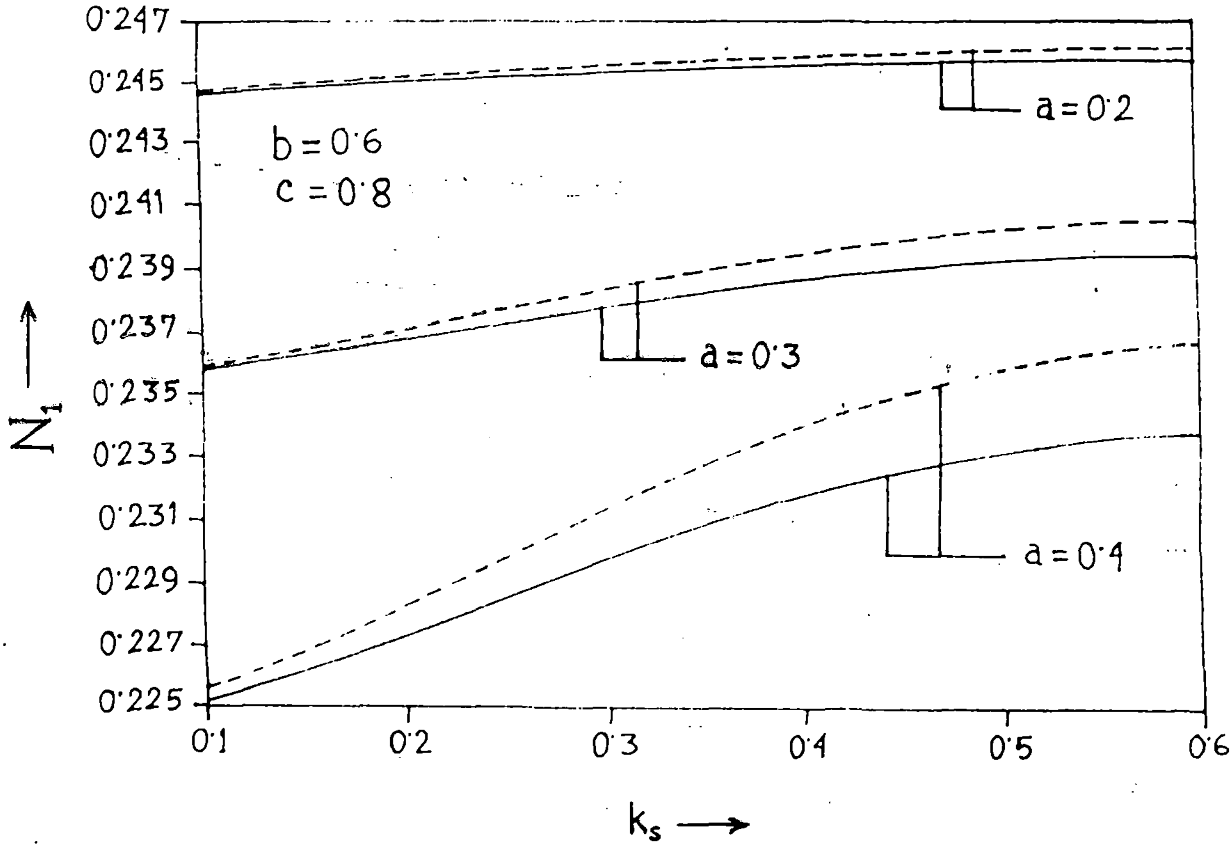

Fig. 5. Stress intensity factor N_1 vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

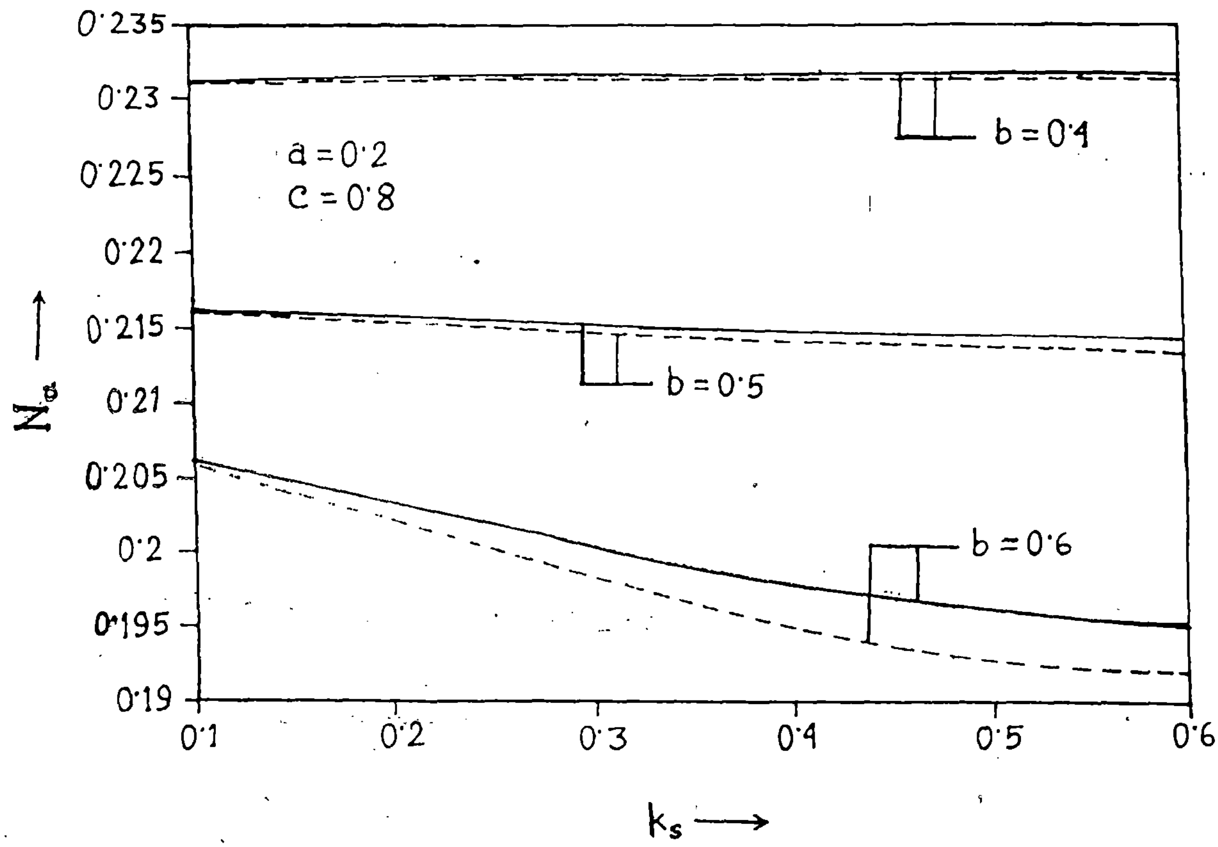

Fig.6. Stress intensity factor N_a vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

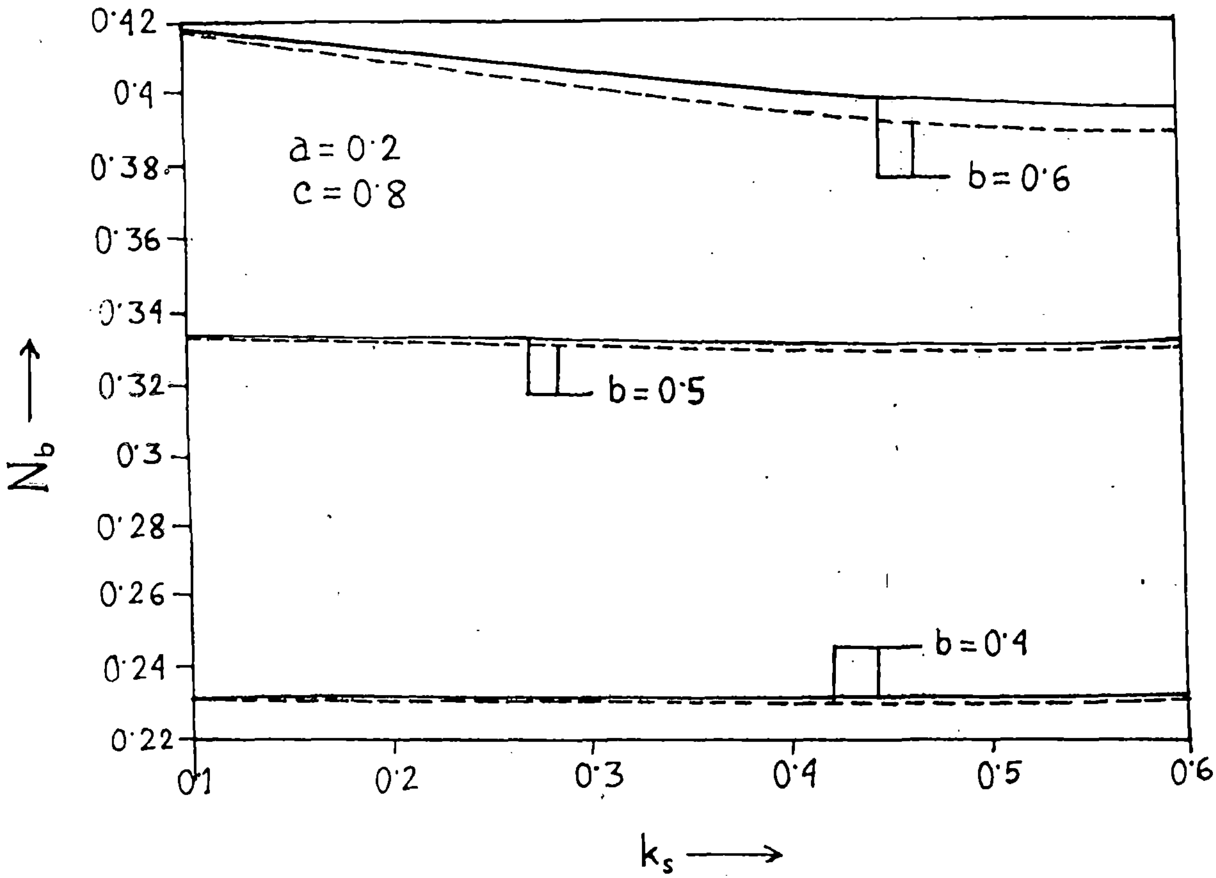

Fig. 7. Stress intensity factor N_b vs. frequency k_s for generalized plane stress. (——— Type I, ------ Type III).

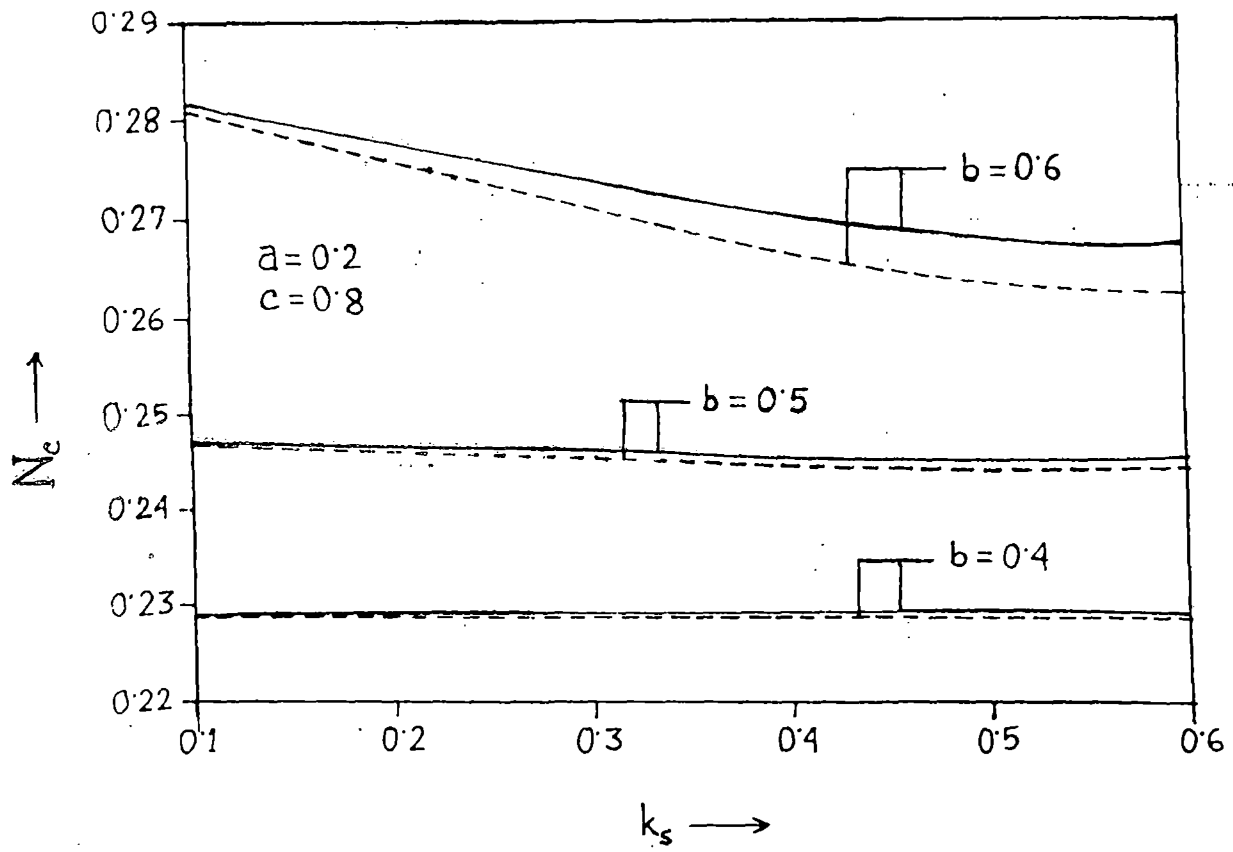

Fig.8. Stress intensity factor N_c vs. frequency k_s for generalized plane stress. (——— Type I, ------ Type III).

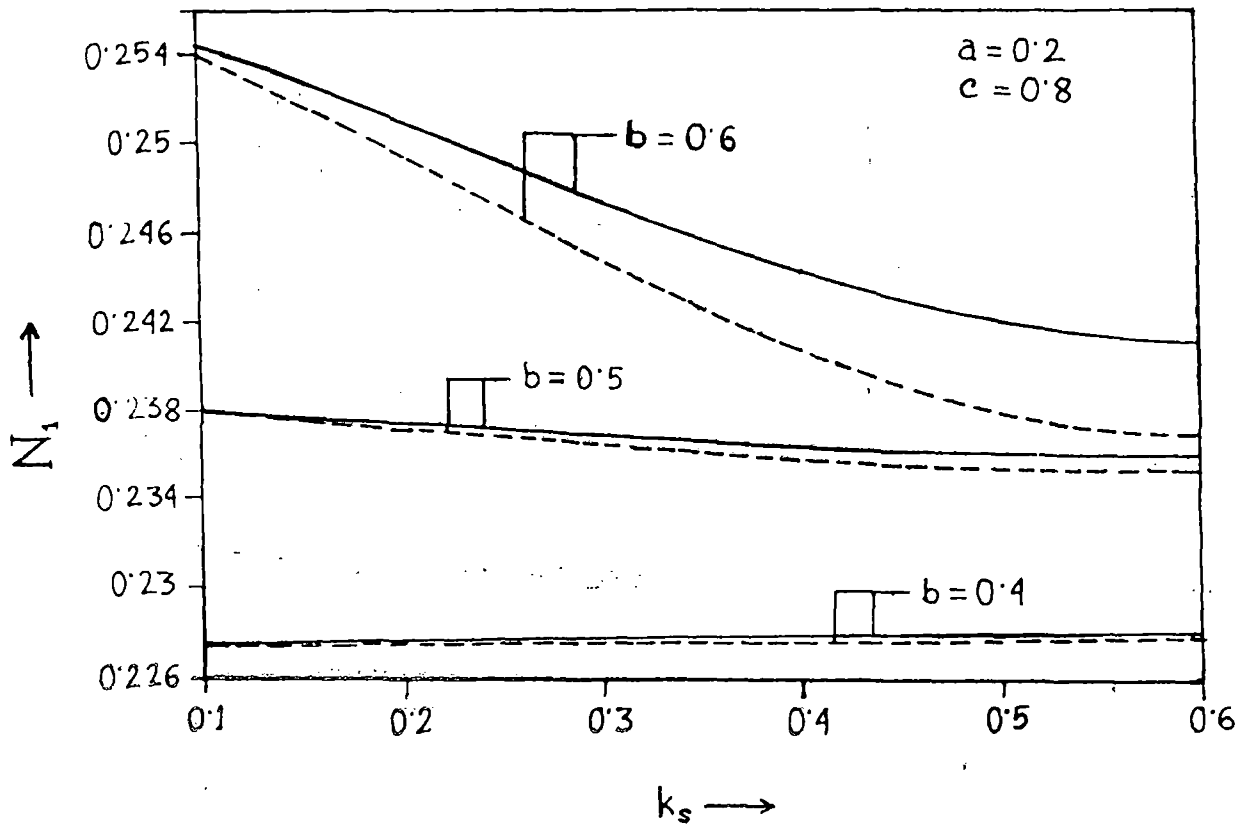

Fig. 9. Stress intensity factor N_1 vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

it is seen from the graphs (Fig.10-Fig.13) that SIFs increase with the increase in the value of k_s for lower values of c(0.6,0.7) but decrease for higher values of c(0.8). The value of SIF N_a is higher for higher values of c. But the nature is opposite in case of N_b, N_c and N_1.

The COD $\mu_{12}\Delta v(x,0)/p_0$ has been plotted for different crack lengths. It is found from Fig.14-Fig.16 that with the increase in the value of crack length the value of COD increases. For a fixed material the variation of COD with frequency is found to be insignificant.

In all the cases where different values of crack length have been considered the variation of COD shows marked difference for different orthotropic materials.

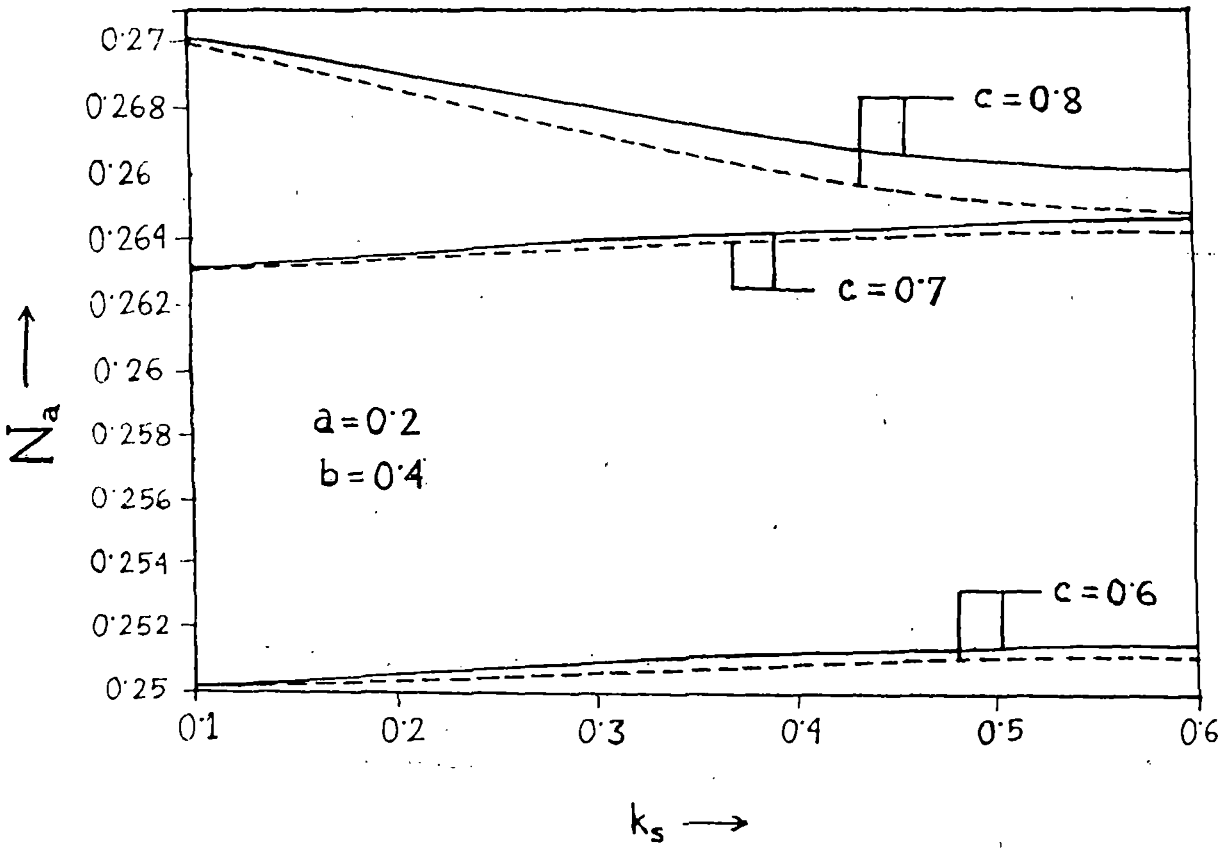

Fig.10. Stress intensity factor N_a vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

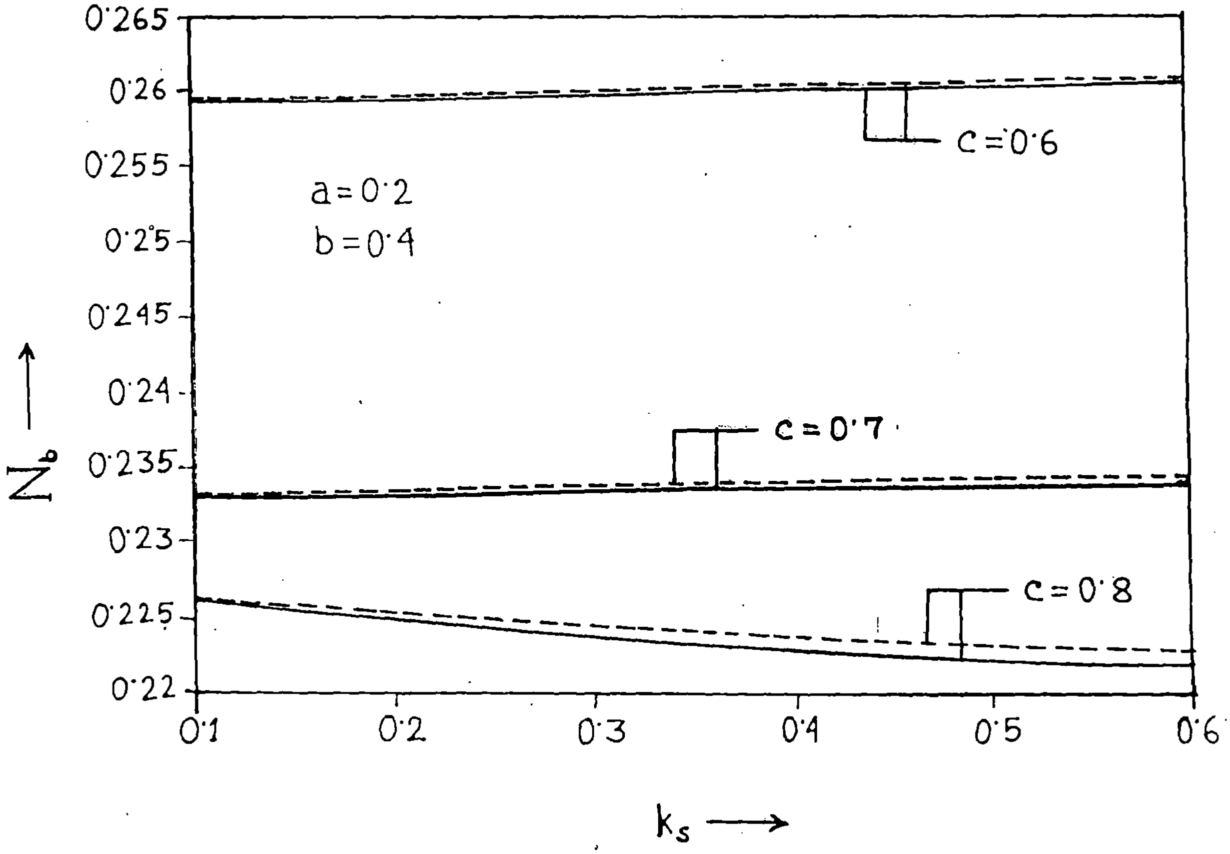

Fig.11. Stress intensity factor N_b vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

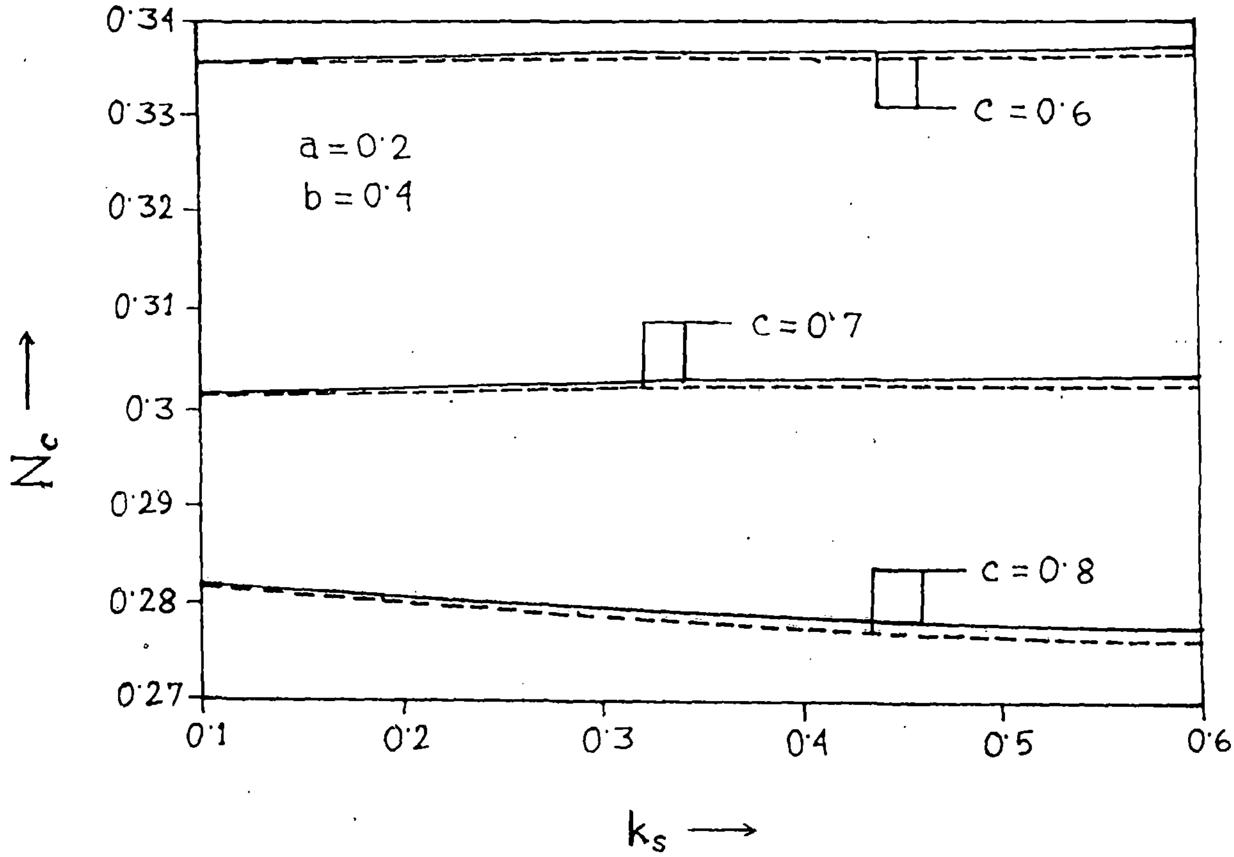

Fig.12. Stress intensity factor N_c vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

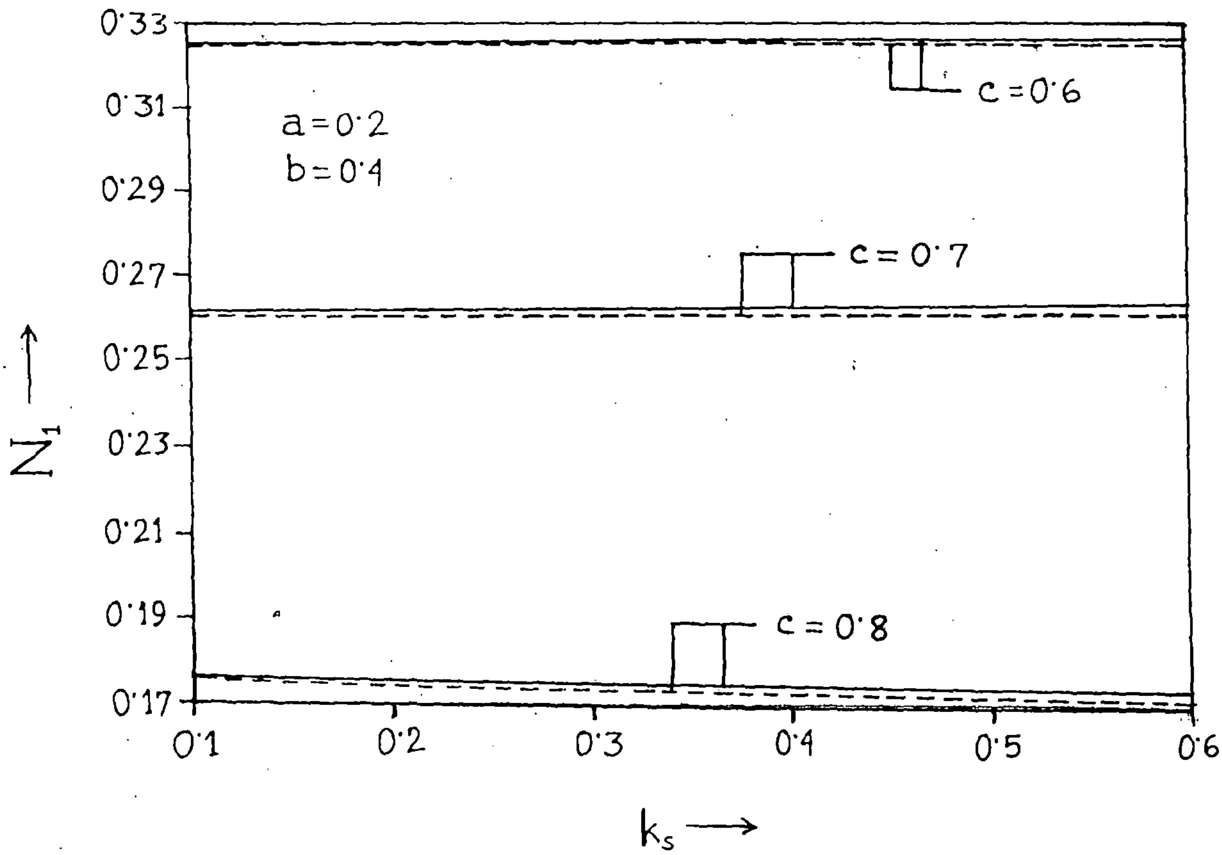

Fig. 13. Stress intensity factor N_1 vs. frequency k_s for generalized plane stress.
(——— Type I, ------ Type III).

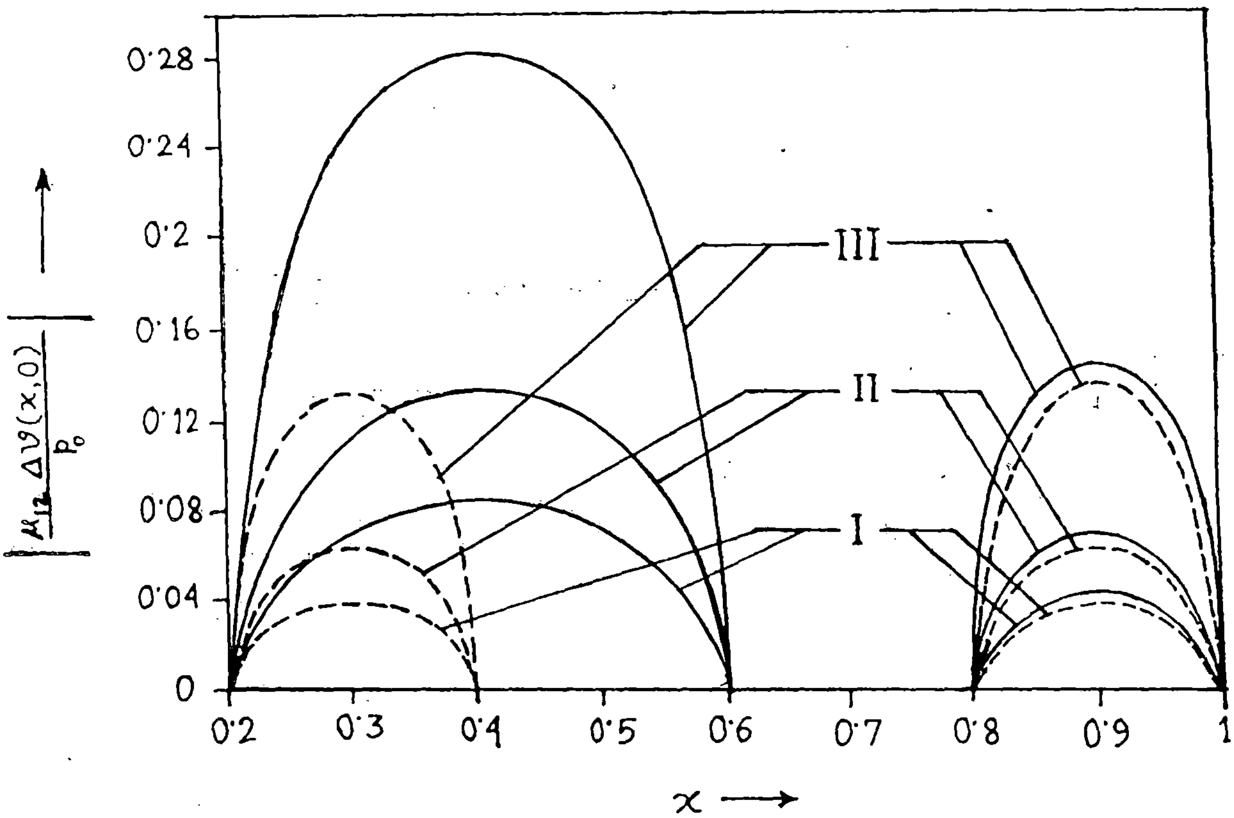

Fig.14. Crack opening displacement vs. distance for generalized plane stress.
(k_s=0.5, a=0.2, b=0.4,0.6, c=0.8).

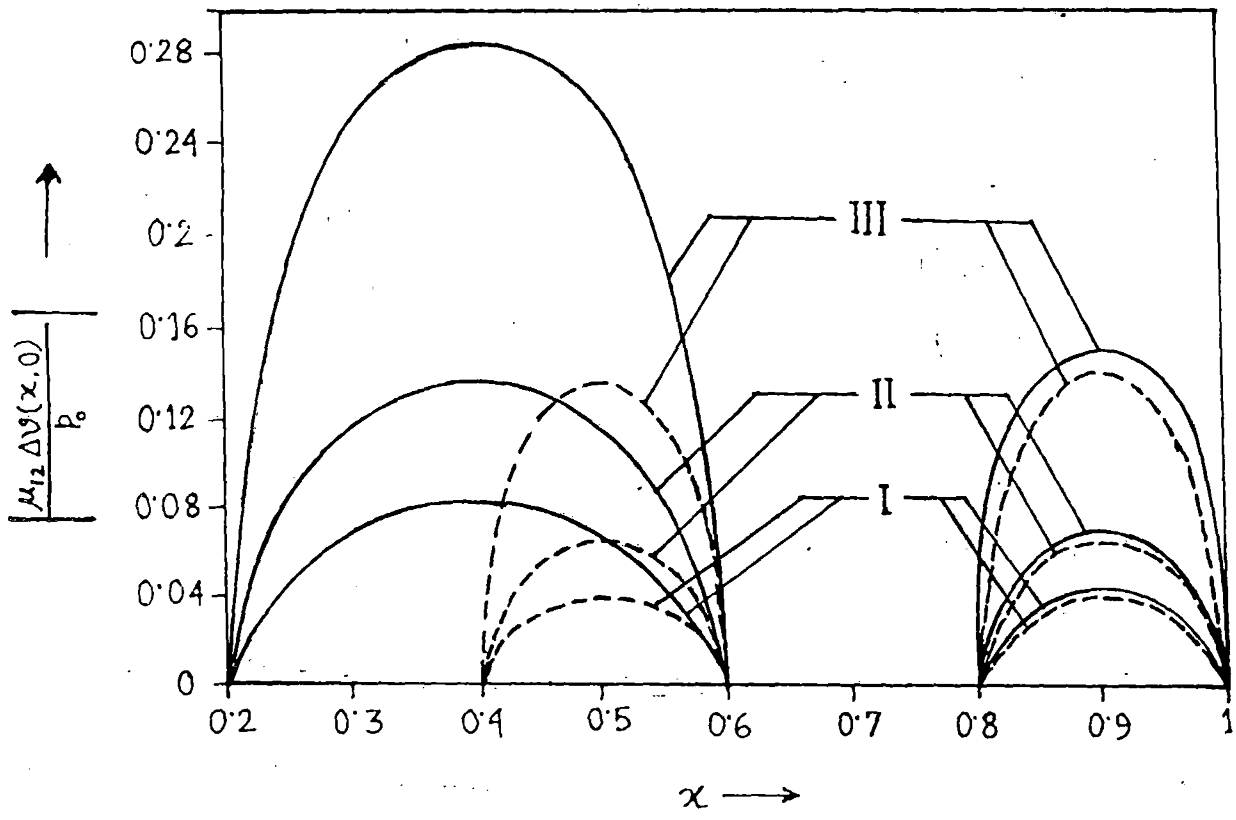

Fig.15. Crack opening displacement vs. distance for generalized plane stress. $(k_s=0.5,\ a=0.2, 0.4,\ b=0.6,\ c=0.8)$.

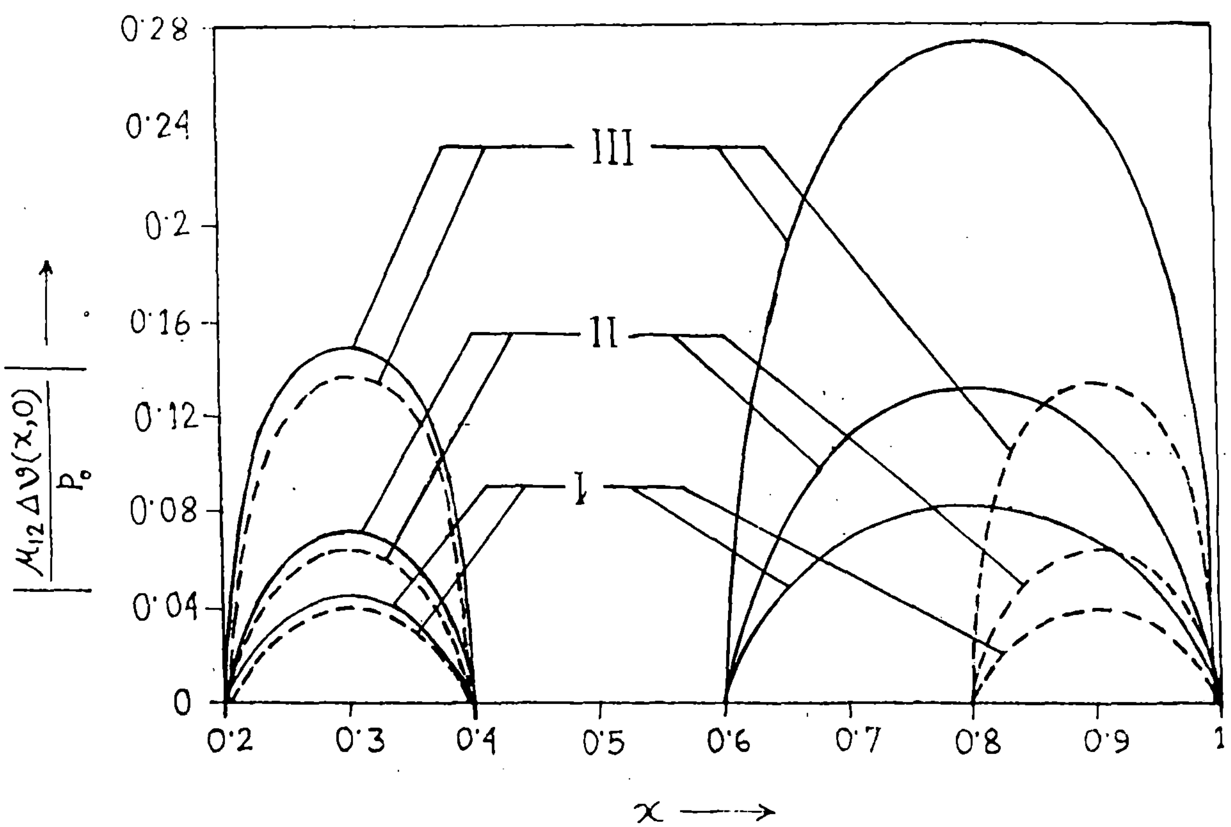

Fig.16. Crack opening displacement vs. distance for generalized plane stress.
(k_s=0.5, a=0.2, b=0.4, c=0.6, 0.8).