CHAPTER I

SOME ELASTODYNAMIC PROBLEMS ON CRACK PROPAGATION

Paper 1: Two coplanar Griffith cracks moving in a strip under anti-plane shear stress.

Paper 2: Two coplanar Griffith cracks moving along the interface of two dissimilar elastic media.

paper 3: Problem of two coplanar Griffith cracks running steadily under three dimensional loading.

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Paper 5: Three coplanar moving Griffith cracks in an infinite elastic strip.

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TWO COPLANAR GRIFFITH CRACKS MOVING IN A STRIP UNDER ANTI-PLANE SHEAR STRESS

1.Introduction

In fracture mechanics , the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been investigated by several authors. Sih and Chen (1972) investigated the problem of propagation of a crack of finite length in a strip under plane extension. The resulting mixed boundary value problem was reduced to the solution of a Fredholm integral equation of second kind , which was solved numerically . Closed- form solutions for a finite length crack moving in a strip under anti plane shear stress was also obtained by Singh et al. (1981). As regards the dynamic crack problem , research has been restricted mainly to the case of a single crack because of the severe mathematical complexity encountered in finding solutions of two or more cracks . However , using finite Hilbert transform techniques developed by Srivastava and Lowengrub (1968), Lowengrub and Srivastava (1968) solved the statical problem of distribution of stress in an infinitely long elastic strip containing two coplanar Griffith cracks .The scattering of time harmonic normally incident plane waves by two parallel and coplanar Griffith cracks in an infinite elastic medium has been studied by Jain and Kanwal (1972) and more recently by Itou (1980).

In this paper we have considered the problem of propagation of two coplanar Yoffe (1951) cracks moving steadily in an infinitely long finite width strip . Employing Fourier transform and finite Hilbert transform technique closed-form solutions are obtained for two cases of practical interest.Firstly, the case when the rigidly clamped edges are pulled apart in opposite directions are considered. Secondly , we have treated the case when the lateral boundaries are subjected to shearing stresses. Exact expressions for the crack opening displacement and the stress intensity factors have

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been derived in both the cases .Finally numerical results for stress intensity factors are presented graphically to show its variation with crack speed for different values of the lengths of the cracks.

2.Formulation Of The Problem

We consider two cracks of finite length to be placed on the X-axis from -b to -a and from a to b with reference to the rectangular coordinate system (x,y,z) which referred to fixed coordinate system (X,Y,Z) is moving with constant velocity v along X -direction within the strip of elastic material occupying the region $-h' \le Y \le h'$ as shown in Fig.1 .

In dynamic problem of anti plane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2} \tag{2.1}$$

where $c_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity and ρ is the density of the material.The non-vanishing components of stress are

$$\left.\begin{aligned}\sigma_{XZ} &= \mu\frac{\partial W}{\partial X}\\ \sigma_{YZ} &= \mu\frac{\partial W}{\partial Y}\end{aligned}\right\} \tag{2.2}$$

Using Galilean transformation $x' = X - vt$, $y' = Y$, $z' = Z$, $t' = t$, where (x',y',z') is the translating coordinate system shown in Fig.1 and next introducing the dimensionless coordinates x,y,z such that $x' = xb$, $y' = yb$, $z' = zb$, $h' = hb$ equation (2.1) reduces to

$$s^2\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \tag{2.3}$$

with $$s^2 = 1 - v^2/c_2^2 \tag{2.4}$$

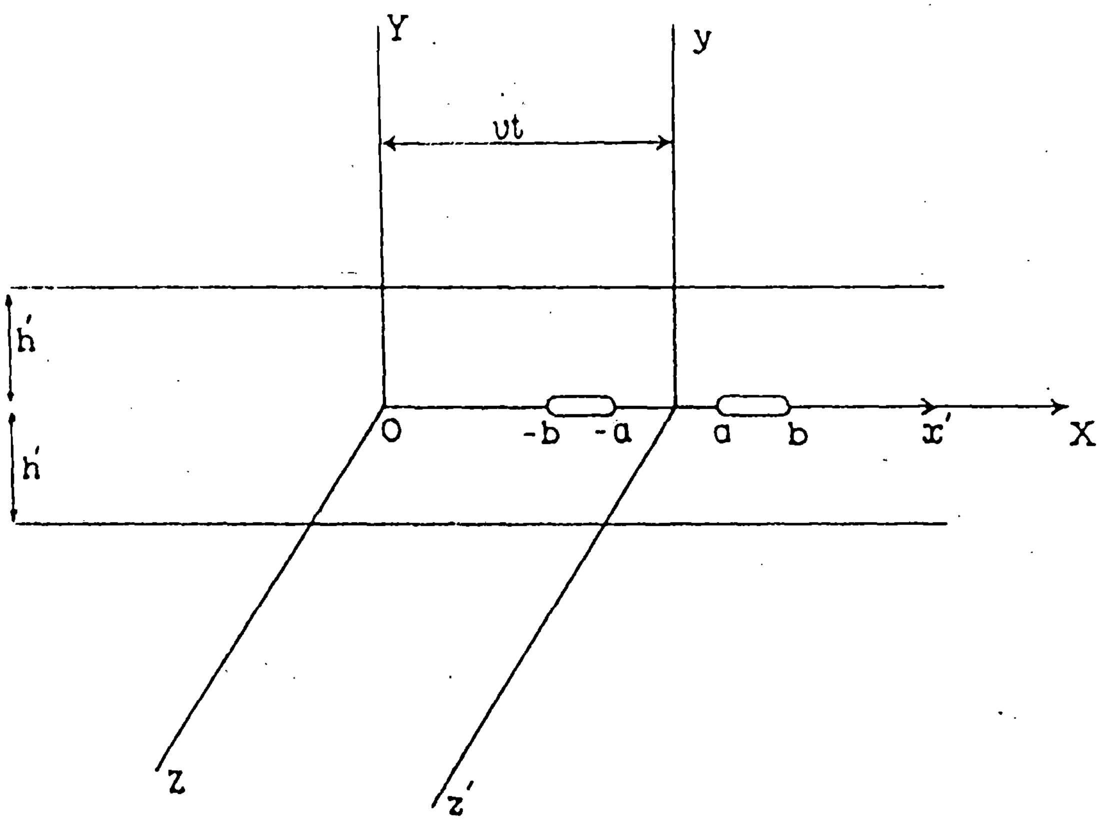

Fig 1. Moving cracks in a strip under antiplane shear.

3. Boundary Conditions

We consider two basic problems of practical interest with different boundary conditions

Problem I. The edges of the strip $y = \pm h$ are assumed to be rigidly clamped and displaced laterally in opposite directions by an equal amount w_o, where w_o is a constant. As a result, anti plane shear motion takes place in z-direction whereas cracks move in the x-direction and the boundary conditions are

$$W(x,\pm h) = \pm w_o , \qquad -\infty < x < \infty \tag{3.1}$$

$$\sigma_{yz}(x,0) = 0 , \qquad d<|x|<1 \tag{3.2}$$

$$W(x,0) = 0 , \qquad 0\leq|x|<d, |x|>1 \tag{3.3}$$

where d=a/b .

In order to apply the integral transform technique it is necessary to solve a different but equivalent problem which can be obtained from the problem of a clamped strip (without any crack) subject to a uniform strain . The equivalent stress condition on the crack are

$$\sigma_{yz}(x,0) = -\frac{\mu w_o}{h} , \qquad d<|x|<1 \tag{3.4}$$

and the displacement must satisfy

$$W(x,0) = 0 , \qquad 0\leq|x|<d, \ |x|>1 \tag{3.5}$$

$$W(x,\pm h) = 0 , \qquad -\infty < x < \infty \tag{3.6}$$

Problem II. In this case uniform shearing stress p_o is applied to the upper and lower boundaries $y = \pm h$ of the strip. The equivalent problem in this case involves the application of the shear stress $-p_o$ to the crack faces at $y = 0$. Accordingly the boundary conditions are

$$\sigma_{yz}(x,\pm h) = 0 , \qquad -\infty < x < \infty \tag{3.7}$$

$$\sigma_{yz}(x,0) = -p_o , \qquad d<|x|<1 \tag{3.8}$$

$$W(x,0) = 0 , \qquad 0\leq|x|<d, |x|>1 \tag{3.9}$$

4. Solutions Of The Problems

Due to symmetry about (x,z)- plane we need consider the region $0<y<h$ only. Employing

$$F_c[A(\xi);\xi \rightarrow x] = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\xi)\cos(\xi x)\, d\xi \tag{4.1}$$

and $$F_s[A(\xi);\xi \rightarrow x] = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\xi)\sin(\xi x)\, d\xi \tag{4.2}$$

we obtain the solution of (2.3) as

$$W(x,y) = F_c[A_1(\xi)\exp(-\xi ys) + A_2(\xi)\exp(\xi ys);\xi \rightarrow x] \tag{4.3}$$

with

$$\sigma_{yz}(x,y) = \mu s F_c\left[\xi\{-A_1(\xi)\exp(-\xi ys) + A_2(\xi)\exp(\xi ys)\};\xi \rightarrow x\right] \tag{4.4}$$

Problem I. Using the expression for $W(x,y)$ given in (4.3) in (3.6) we get

$$A_1(\xi) = \frac{A(\xi)}{1 - \exp(-2\xi hs)}$$

$$A_2(\xi) = \frac{-A(\xi)\exp(-2\xi hs)}{1 - \exp(-2\xi hs)}$$

where $A(\xi)$ is to be determined.

From (3.4) and (3.5) we find that $A(\xi)$ satisfies the set of triple integral equations

$$F_c[\xi A(\xi)\coth(\xi hs)\ ;\ \xi \rightarrow x] = \frac{w_o}{hs} \qquad d<x<1 \tag{4.5}$$

$$F_c[A(\xi);\xi \rightarrow x] = 0 \quad , \qquad 0 \le x < d\ , x>1 \tag{4.6}$$

Let us take

$$A(\xi) = \frac{1}{\xi}\sqrt{\frac{\pi}{2}} \int_d^1 g_1(\tau)\,\mathrm{Sech}^2(c\tau)\sin(\xi\tau)\, d\tau \tag{4.7}$$

It is clear that the above choice of $A(\xi)$ satisfies (4.6) if and only if

$$\int_d^1 g_1(\tau)\ \mathrm{sech}^2(c\tau)\ d\tau = 0 \tag{4.8}$$

Equation (4.5) can be written as

$$\frac{d}{dx} F_s[A(\xi)\ \mathrm{cth}(\xi hs)\ ;\ \xi \longrightarrow x\ ] = \frac{w_o}{hs} \qquad d<x<1 \tag{4.9}$$

Inserting (4.7) in (4.9) and using the result [Gradshteyn and Ryzhik (1965)]

$$\int_0^\infty \frac{\mathrm{cth}(\xi hs)\ \sin(\xi\tau)\ \sin(\xi x)}{\xi}\, d\xi = \frac{1}{2}\log\left|\frac{\mathrm{th}(cx)+\mathrm{th}(c\tau)}{\mathrm{th}(cx)-\mathrm{th}(c\tau)}\right| \tag{4.10}$$

where $c = \pi/2hs$, we obtain

$$\int_d^1 \frac{c g_1(\tau)\mathrm{Sech}^2(c\tau)\ \mathrm{th}(c\tau)}{\mathrm{th}^2(c\tau) - \mathrm{th}^2(cx)}\, d\tau = \frac{w_o}{hs\ \mathrm{Sech}^2(cx)}\ , \quad d<x<1 \tag{4.11}$$

Substituting $\mathrm{th}(c\tau)=T_1$, equation (4.11) is found to reduce to the form

$$\int_{D_1}^{I_1} \frac{T_1 A(T_1^2)}{T_1^2-X_1^2}\, dT_1 = \frac{w_o}{hs(1-X_1^2)} = F(X_1)\ (\text{say}),\ D_1<X_1<I_1 \tag{4.12}$$

where $D_1 = \mathrm{th}(cd)$, $I_1 = \mathrm{th}(c)$, $X_1 = \mathrm{th}(cx)$ and $A(T_1^2) = g_1(\tau)$. Using finite Hilbert transform (1968), the solutions of (4.12) is

$$A(T_1^2) = -\frac{4}{\pi^2}\sqrt{\frac{T_1^2-D_1^2}{I_1^2-T_1^2}}\int_{D_1}^{I_1}\sqrt{\frac{I_1^2-X_1^2}{X_1^2-D_1^2}}\ \frac{X_1 F(X_1)}{(X_1^2-T_1^2)}\, dX_1 + \frac{K_1}{\sqrt{(T_1^2-D_1^2)(I_1^2-T_1^2)}}$$

which can be simplified to

$$g_1(\tau) = \frac{2w_o\ \mathrm{ch}(cd)}{\pi hs(1-T_1^2)\ \mathrm{ch}(c)}\sqrt{\frac{T_1^2-D_1^2}{I_1^2-T_1^2}} + \frac{K_1}{\sqrt{(T_1^2-D_1^2)(I_1^2-T_1^2)}},\quad d<\tau<1 \tag{4.13}$$

Substituting the result (4.13) in (4.8) we obtain

$$K_1\int_{D_1}^{I_1}\frac{dT}{\sqrt{(T^2-D_1^2)(I_1^2-T^2)}} = \frac{4w_o}{hs\pi^2}\int_{D_1}^{I_1}\sqrt{\frac{T^2-D_1^2}{I_1^2-T^2}}\,dT\int_{D_1}^{I_1}\sqrt{\frac{I_1^2-X_1^2}{X_1^2-D_1^2}}\,\frac{X_1\,dX_1}{(X_1^2-T^2)(1-X_1^2)}$$

which is simplified with aid of the results

$$\int_{D_1}^{I_1}\sqrt{\frac{I_1^2-X_1^2}{X_1^2-D_1^2}}\,\frac{X_1\,dX_1}{(X_1^2-T^2)} = -\frac{\pi}{2}$$

and

$$\int_{D_1}^{I_1}\sqrt{\frac{I_1^2-X_1^2}{X_1^2-D_1^2}}\,\frac{X_1\,dX_1}{(1-X_1^2)} = \frac{\pi}{2}\left[1-\sqrt{\frac{1-I_1^2}{1-D_1^2}}\right]$$

to

$$K_1 = \frac{2w_o\,ch(cd)}{\pi hs\ ch(c)}D_1^2\left\{1-\Pi\left(\frac{\pi}{2},\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right)/F\left(\frac{\pi}{2},q\right)\right\} \tag{4.14}$$

where $q=(I_1^2-D_1^2)^{1/2}/I_1$ and $F(\phi,k)$, $\Pi(\phi,n,k)$ are elliptic integrals of first and third kind respectively.

The expressions of displacement and shear stress on the plane of the crack are expressed as

$$W(x,o) = \frac{\pi}{2}\int_x^1 g_1(\tau)\mathrm{Sech}^2(c\tau)\,d\tau \quad , \quad d<x<1 \tag{4.15}$$

and

$$\sigma_{yz}(x,o)=\mu sc\int_d^1\frac{g_1(\tau)\mathrm{Sech}^2(c\tau)th(c\tau)\mathrm{Sech}^2(cx)}{th^2(cx)-th^2(c\tau)}d\tau \quad ,0\le x<d, \quad x>1 \tag{4.16}$$

Now inserting (4.13)in (4.15) and (4.16) we obtain with the aid of the following results

$$\int_{X}^{I_1}\sqrt{\frac{T_1^2-D_1^2}{I_1^2-T_1^2}}\,\frac{dT_1}{1-T_1^2} = -\,I_1^{-1}\left[F(\tfrac{\pi}{2},q) + \frac{1-D_1^2}{1-I_1^2}\,\Pi\!\left(\lambda,\frac{I_1^2-D_1^2}{1-I_1^2},q\right)\right]$$

and
$$\int_{D_1}^{I_1}\sqrt{\frac{T_1^2-D_1^2}{I_1^2-T_1^2}}\,\frac{T_1\,dT_1}{(X^2-T_1^2)} = \frac{\pi}{2}\left[\sqrt{\frac{X^2-D_1^2}{X^2-I_1^2}}-1\right]$$

$$W(x,0) = -\frac{w_o\,ch(cd)}{hsc\;sh(c)}\left[F(\lambda,q)\left(1-D_1^2\left\{1-\Pi\!\left(\frac{\pi}{2},\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right)/F(\tfrac{\pi}{2},q)\right\}\right)+\right.$$

$$\left.+\frac{ch^2(c)}{ch^2(cd)}\,\Pi\left(\lambda,\frac{I_1^2-D_1^2}{1-I_1^2},q\right)\right],\quad d<x<1 \tag{4.17}$$

where $\sin\lambda = \sqrt{\dfrac{I_1^2-X_1^2}{I_1^2-D_1^2}}$

$$\sigma_{yz}(x,o) = \frac{\mu w_o\,ch(cd)}{h\;ch(c)}\left[\sqrt{\frac{th^2(cx)-D_1^2}{th^2(cx)-I_1^2}}-\frac{ch(c)}{ch(cd)}+\right.$$

$$\left.+\left\{1-\Pi\!\left(\frac{\pi}{2},\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right)/F(\tfrac{\pi}{2},q)\right\}\frac{D_1^2\,Sech^2(cx)}{\sqrt{[th^2(cx)-D_1^2][th^2(cx)-I_1^2]}}\right],\quad x>1 \tag{4.18}$$

$$\sigma_{yz}(x,o) = \frac{\mu w_o\,ch(cd)}{h\;ch(c)}\left[\sqrt{\frac{D_1^2-th^2(cx)}{I_1^2-th^2(cx)}}-\frac{ch(c)}{ch(cd)}-\right.$$

$$\left.-\left\{1-\Pi\!\left(\frac{\pi}{2},\frac{I_1^2-D_1^2}{I_1^2(1-D_1^2)},q\right)/F(\tfrac{\pi}{2},q)\right\}\frac{D_1^2\;Sech^2(cx)}{\sqrt{[D_1^2-th^2(cx)][I_1^2-th^2(cx)]}}\right],\quad o<x<d \tag{4.19}$$

where we have used the result

$$\int_d^1 \frac{1}{\sqrt{(t^2-d^2)(1-t^2)}} \frac{t\,dt}{t^2-x^2} = \begin{cases} \dfrac{\pi}{2\sqrt{(d^2-x^2)(1-x^2)}}, & 0<x<d \\ 0, & d<x<1 \\ \dfrac{-\pi}{2\sqrt{(x^2-d^2)(x^2-1)}}, & x>1 \end{cases} \quad (4.20)$$

The stress intensity factor at x = 1 is given by

$$S_{11} = \underset{x\to 1}{Lt} \sqrt{2(x-1)}\, \sigma_{yz}(x,o) = \frac{\mu w_0}{h\mathrm{Sech}(cd)} \left[\sqrt{\frac{I_1^2 - D_1^2}{cI_1}} + \frac{D_1^2(1-I_1^2)}{\sqrt{cI_1(I_1^2-D_1^2)}} \right.$$

$$\left. \times\left\{1- \Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1-D_1^2)}, q\right) / F\left(\frac{\pi}{2}, q\right)\right\}\right] \quad (4.21)$$

and the stress intensity factor at x = d is given by

$$S_{1d} = \underset{x\to d}{Lt} \sqrt{2(d-x)}\, \sigma_{yz}(x,o) = -\frac{\mu w_0 \sqrt{D_1^3(1-I_1^2)}}{h\sqrt{c(I_1^2-D_1^2)}} \times$$

$$\times \left\{1- \Pi\left(\frac{\pi}{2}, \frac{I_1^2 - D_1^2}{I_1^2(1-D_1^2)}, q\right) / F\left(\frac{\pi}{2}, q\right)\right\} \quad (4.22)$$

Letting d = a/b = 0 in the expressions for displacement , stress and stress intensity factors it can be easily shown that the results coincide with the corresponding expressions given by Singh et al. (1981).

Problem II. In this case again we take the general solution of (2.3) as

$$W(x,y) = F_c[\, C_1(\xi)\exp(-\xi ys) + C_2(\xi)\exp(\xi ys) \;;\; \xi \to x] \quad (4.23)$$

and inserting it in (3.7) we find that

$$C_1(\xi) = \frac{D(\xi)}{1 + \exp(-2\xi hs)}$$

$$C_2(\xi) = \frac{D(\xi)\exp(-2\xi hs)}{1 + \exp(-2\xi hs)}$$

From (3.8) and (3.9) it is determined that $D(\xi)$ satisfies the following set of triple integral equation

$$F_c[\xi\ D(\xi)\mathrm{th}(\xi hs)\ ;\xi \longrightarrow x\ ] = \frac{p_o}{\mu s} \quad , \quad d<x<1 \qquad (4.24)$$

$$F_c[D(\xi);\xi \longrightarrow x\ ] = 0 \quad , \quad 0\le x<d\ ,x>1 \qquad (4.25)$$

Proceeding as in problem I , we consider a trial solution

$$D(\xi) = \frac{1}{\xi}\sqrt{\frac{\pi}{2}}\int_d^1 g_2(\tau)\ \mathrm{ch}(c\tau)\sin(\xi\tau)\ d\tau \qquad (4.26)$$

With this choice of $D(\xi)$, equation (4.25) will be satisfied provided the unknown function $g_2(\tau)$ in (4.26) satisfies

$$\int_d^1 g_2(\tau)\ \cosh(c\tau)\ d\tau = 0 \qquad (4.27)$$

Now equation (4.24) can be written as

$$\frac{d}{dx} F_s[D(\xi)\ \mathrm{th}(\xi hs)\ ;\ \xi \longrightarrow x\ ] = \frac{p_o}{\mu s} \qquad d<x<1 \qquad (4.28)$$

Insertion of equation (4.26) in (4.28) and use of the result [Gradshteyn and Ryzhik (1965)]

$$\int_0^\infty \frac{\mathrm{th}(\xi hs)\ \sin(\xi\tau)\ \sin(\xi x)}{\xi}\ d\xi = \frac{1}{2}\ \log\left|\frac{\mathrm{sh}(cx)+\mathrm{sh}(c\tau)}{\mathrm{sh}(cx)-\mathrm{sh}(c\tau)}\right| \qquad (4.29)$$

where $c = \pi\ /\ 2hs$, gives

$$\int_{d}^{1} \frac{c\ g_2(\tau)\ Sh(2c\tau)}{sh^2(c\tau)\ -sh^2(cx)}\, d\tau = \frac{2p_0}{\mu s\ ch(cx)} \quad , \ d<x<1 \tag{4.30}$$

Substituting $T_2 = sh(c\tau)$, $I_2 = sh(c)$, $D_2 = sh(cd)$ and $X_2 = Sh(cx)$ and proceeding as in problem I, we obtain the solution of (4.30) as

$$g_2(\tau) = -\frac{4p_0}{\pi^2 \mu s}\sqrt{\frac{T_2^2 - D_2^2}{I_2^2 - T_2^2}} \times \frac{1}{\sqrt{1+I_2^2}}\left[\Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q''\right) - F\left(\frac{\pi}{2}, q''\right)\right] +$$

$$+ \frac{K_2}{\sqrt{(T_2^2 - D_2^2)(I_2^2 - T_2^2)}} \tag{4.31}$$

where $q' = (I_2^2 - D_2^2)^{1/2}/I_2$, $q'' = q'.th(c)$ and using the result (4.31) in the condition (4.27) the constant K_2 is determined with the aid of the result

$$\int_{D_2}^{I_2} \sqrt{\frac{I_2^2 - X^2}{X^2 - D_2^2}}\ \frac{X\, dX}{(X^2 - T_2^2)\sqrt{1+X^2}} = \frac{1}{\sqrt{1+I_1^2}}\left[\Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q''\right) - F\left(\frac{\pi}{2}, q''\right)\right]$$

as

$$K_2 = \frac{4p_0 th(c)}{\pi^2 \mu s\ F\left(\frac{\pi}{2}, q'\right)} \int_{D_2}^{I_2} \sqrt{\frac{T_2^2 - D_2^2}{I_2^2 - T_2^2}}\left[\Pi\left(\frac{\pi}{2}, \frac{I_2^2 - D_2^2}{I_2^2 - T_2^2}, q''\right) - F\left(\frac{\pi}{2}, q''\right)\right] dT_2 \tag{4.32}$$

The relevant displacement and stress components in the plane of the cracks may be written as

$$W(x,o) = \frac{\pi}{2}\int_{x}^{1} g_2(\tau)\ ch(c\tau)\ d\tau \quad , \quad d<x<1 \tag{4.33}$$

and $$\sigma_{yz}(x,o) = \frac{\mu s c}{2}\int_d^1 \frac{g_2(\tau)\ sh(2c\tau)ch(cx)}{sh^2(cx) - sh^2(c\tau)}\, d\tau \ , \quad o\le x<d, \quad x>1 \qquad (4.34)$$

Now using (4.31) in (4.33) and (4.34) we obtain

$$W(x,o) = -\frac{2p_o}{\pi\mu s\ ch(c)}\left[\int_x^1 \sqrt{\frac{sh^2(c\tau)-sh^2(cd)}{sh^2(c) - sh^2(c\tau)}} \times \left\{\Pi\left(\frac{\pi}{2}, \frac{I_2^2-D_2^2}{I_2^2-T_2^2}, q''\right) - F\left(\frac{\pi}{2}, q''\right)\right\} ch(c\tau)\ d\tau\right] + \frac{K_2\ F(\lambda', q')}{cI_2} \qquad (4.35)$$

where $\sin\lambda' = \sqrt{\frac{I_2^2-X_2^2}{I_2^2-D_2^2}}$

$$\sigma_{yz}(x,o) = -\frac{2p_o ch(cx)}{\pi}\sqrt{\frac{sh^2(cx)-D_2^2}{sh^2(cx)-I_2^2}}\int_{D_2}^{I_2}\sqrt{\frac{I_2^2 - T_2^2}{T_2^2 - D_2^2}} \times \frac{1}{T_2^2 - sh^2(cx)} \times \frac{T_2\ dT_2}{\sqrt{1+T_2^2}} + \frac{\pi\mu s\ ch(cx)K_2}{2.\sqrt{(sh^2(cx)-I_2^2)(sh^2(cx)-D_2^2)}}, \quad \text{for } x>1 \qquad (4.36)$$

$$\sigma_{yz}(x,o) = -\frac{2p_o ch(cx)}{\pi}\sqrt{\frac{D_2^2-sh^2(cx)}{I_2^2-sh^2(cx)}}\int_{D_2}^{I_2}\sqrt{\frac{I_2^2 - T_2^2}{T_2^2 - D_2^2}} \times \frac{1}{T_2^2 - sh^2(cx)}$$

$$\times \frac{T_2 \, dT_2}{\sqrt{1 + T_2^2}} - \frac{\pi\mu s \, ch(cx)K_2}{2.\sqrt{[I_2^2 - sh^2(cx)][D_2^2 - sh^2(cx)]}} , \quad \text{for } o<x<d \tag{4.37}$$

The stress intensity factor at x=1 is given by

$$S_{21} = \underset{x\to 1}{Lt} \sqrt{2(x-1)} \; \sigma_{yz}(x,o) = \frac{2p_o}{\pi} \sqrt{\frac{I_2^2 - D_2^2}{cI_2 ch(c)}} \times F\left(\frac{\pi}{2}, q''\right) + $$

$$+ \frac{\pi\mu s \; K_2}{2\sqrt{c.th(c)[I_2^2 - D_2^2]}} \tag{4.38}$$

and the stress intensity factor at x = d is given by

$$S_{2d} = \underset{x\to d}{Lt} \sqrt{2(d-x)} \quad \sigma_{yz}(x,o) = \frac{-\pi\mu s \; K_2}{2\sqrt{c.th(cd)[I_2^2 - D_2^2]}} \tag{4.39}$$

Again letting d = 0 in the expressions for displacement , stress and stress intensity factors we obtain the corresponding results for a single crack as given by Singh et al. (1981).

5. Numerical results

In this section we present the variation of stress intensity factors with ratio of crack speed v to shear wave speed c_2 for both the problems. The crack length dependance of the stress intensity factors and its variations with v/c_2 have been shown in figures 2 – 5. Figures 2 – 3 depict the fact that in problem I, the stress intensity factors at both the crack tips decrease with the increase in the distance between the cracks.

But for the problem II, as seen from figures 4 – 5, it is found that the behaviour of the stress intensity factors at the crack tips is of different nature as compared to the corresponding nature of problem I. .In the problem I, the stress intensity factors at both the crack edges decrease with the increase in the value of v/c_2 and approaches to zero as $v/c_2 \rightarrow 1$. But in Problem II, the stress intensity factors at both the edges increase gradually with the increase in the value of v/c_2 and approaches infinity as $v/c_2 \rightarrow 1$. In problem II it is also found that the stress intensity factors at both the edges decrease with the increase in the values of the separating distance between the cracks. The dashed line in fig.2 and Fig.4 corresponding to the stress intensity factors at the tip of a single crack as given by Singh et al.(1981) for the case $b/h' = 1$.

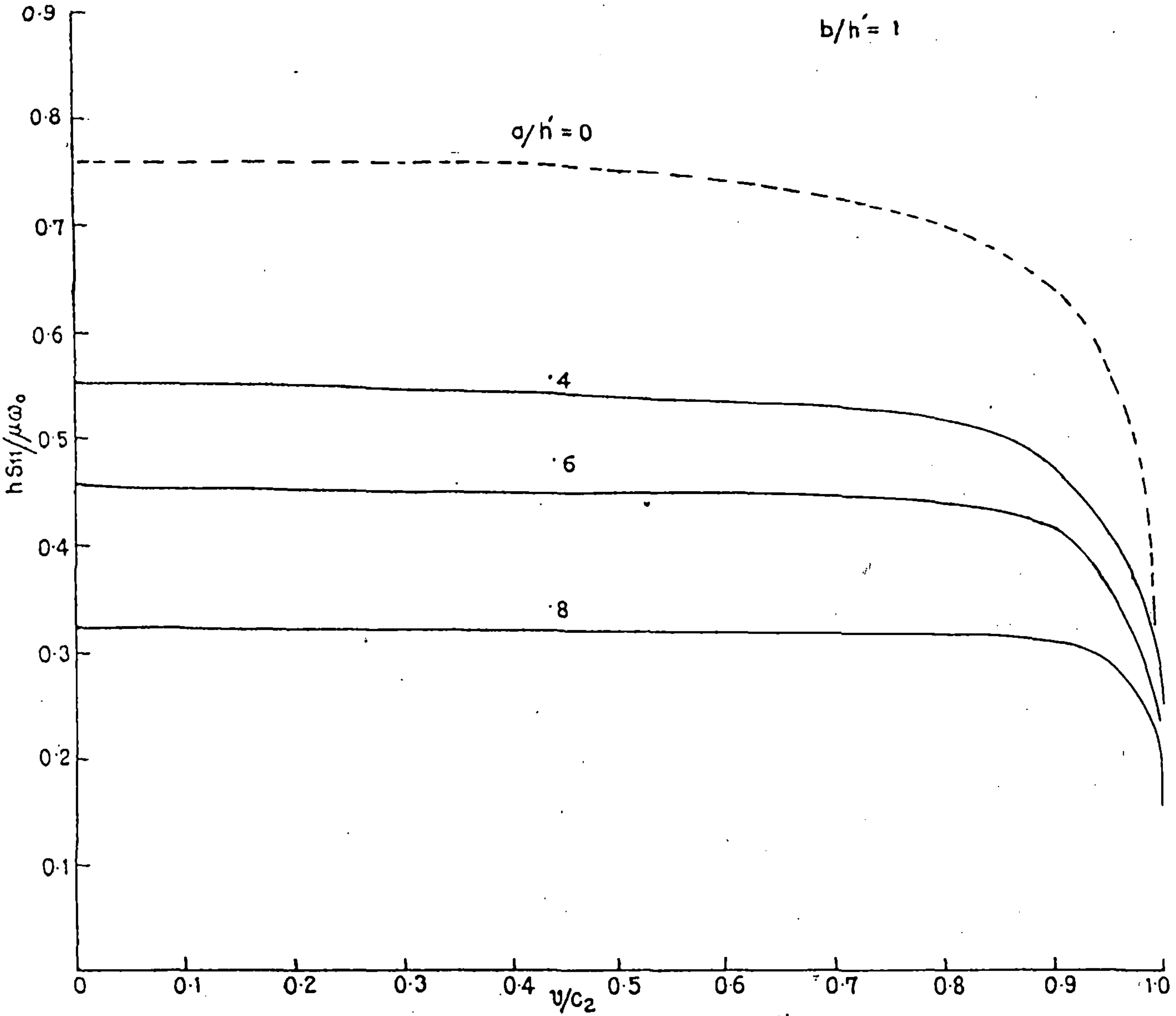

Fig 2: Stress intensity factor at the outer edge vs. υ/c_2, for problem I

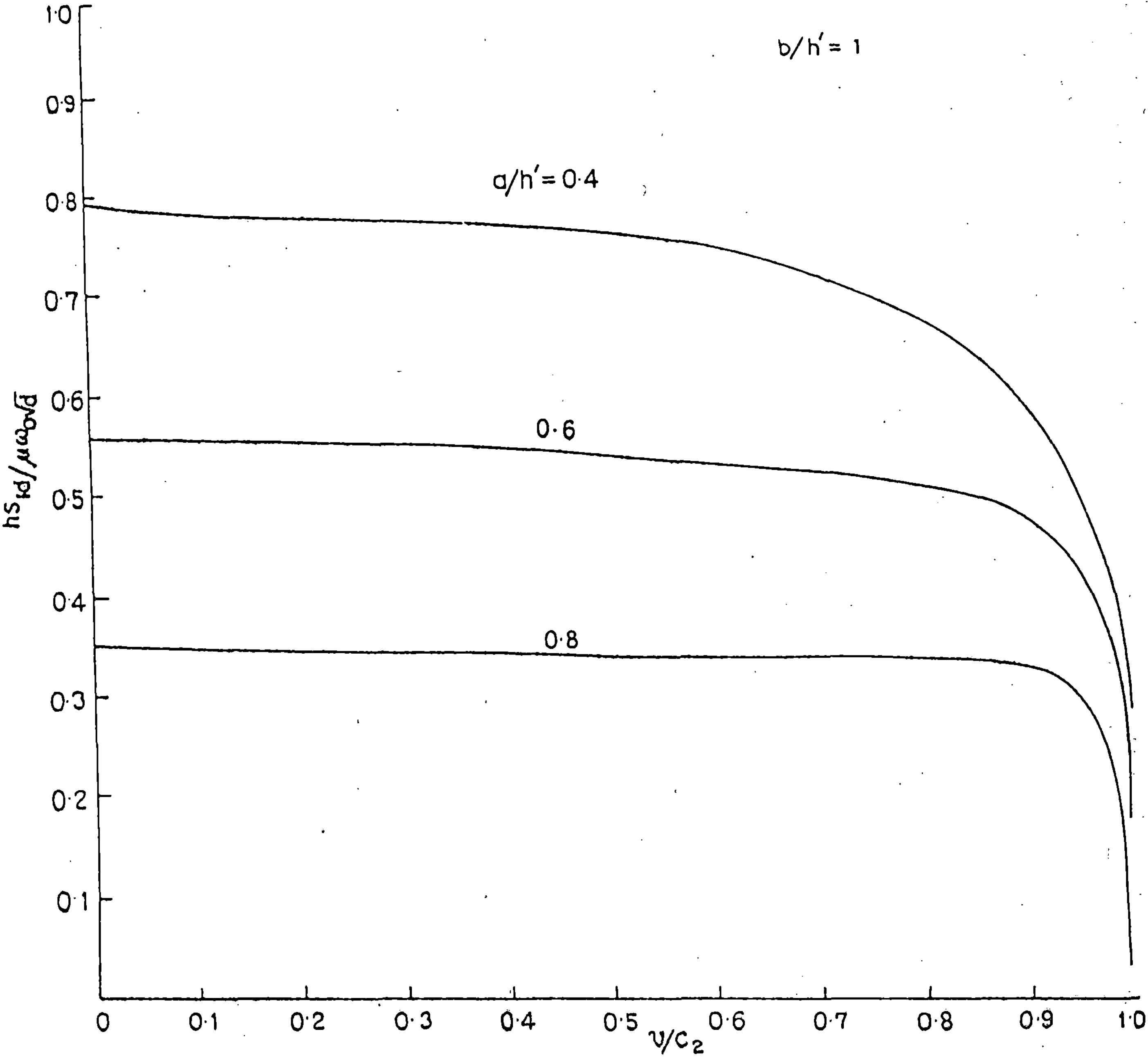

Fig 3. Stress intensity factor at the inner edge vs. v/c_2, for problem I

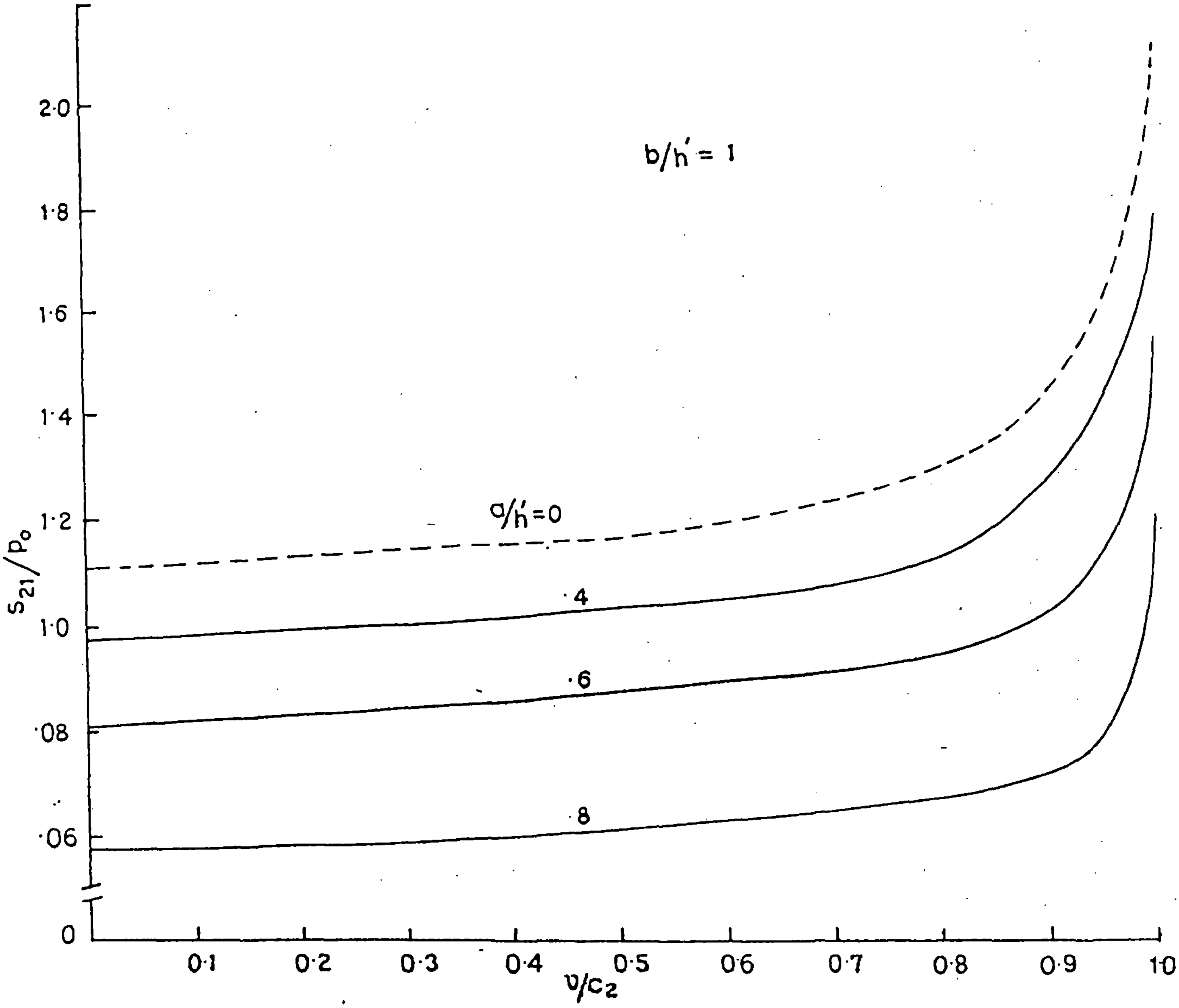

Fig 4. Stress intensity factor at the outer edge vs. v/c_2, for problem II

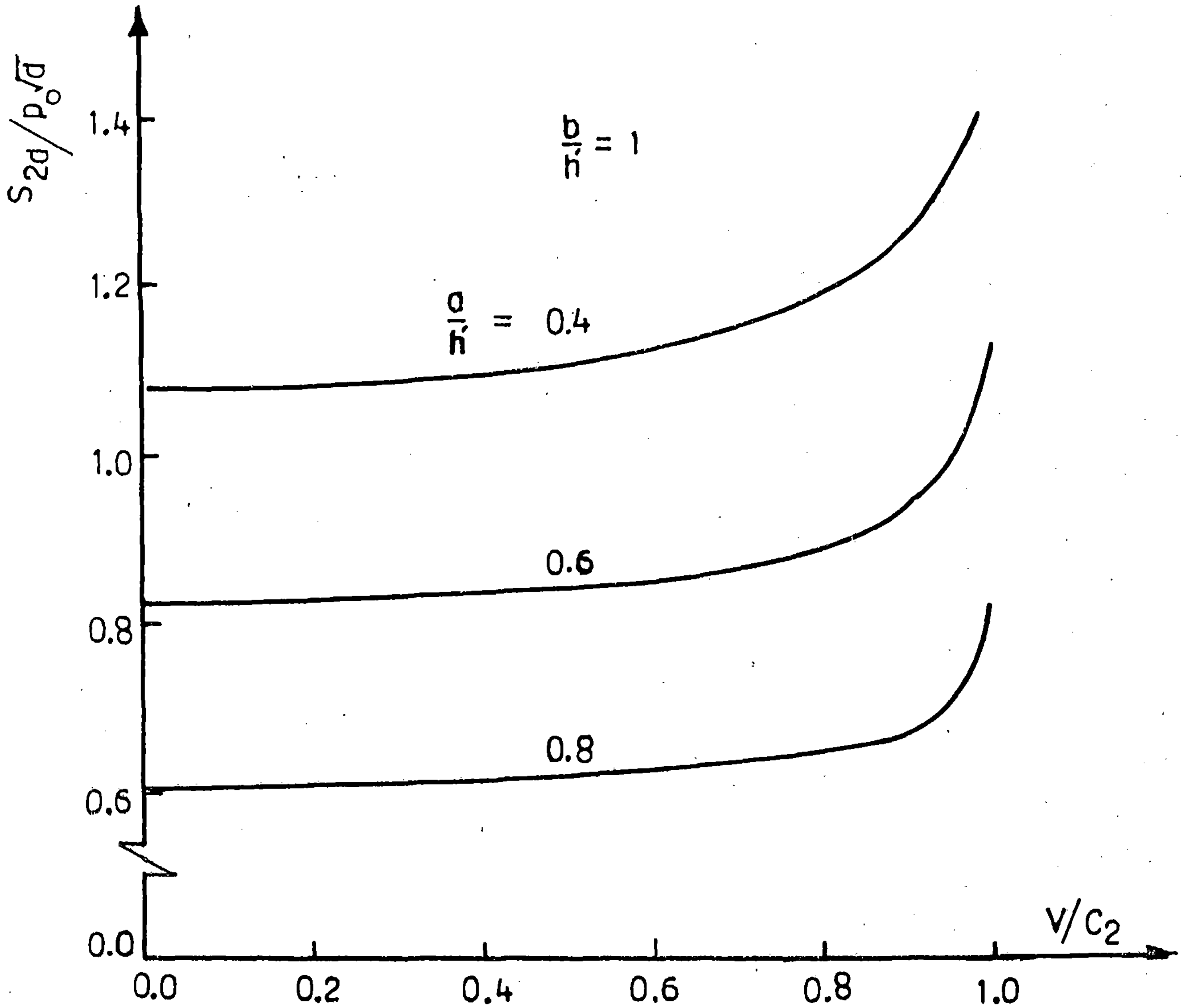

Fig. 5: Stress intensity factor at the inner edge vs v/c_2 for problem II.

TWO COPLANAR GRIFFITH CRACKS MOVING ALONG THE INTERFACE OF TWO DISSIMILAR ELASTIC MEDIA

1. Introduction

Scattering of elastic waves by cracks located in a homogeneous, isotropic medium has important applications in geophysics and seismology. If the cracks are located at the interface of layered media, the study becomes more relevant. Scattering of an elastic wave from an interface crack under anti-plane strain was solved by Bostrom (1987). Srivastava et al. (1980) solved the problem of interaction of an anti-plane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff (1979). As regards the dynamic crack problem, research has been restricted mainly to the cases of a single crack because of the severe mathematical complexity encountered in finding solutions of problems involving two or more cracks. The diffraction of an anti-plane shear wave by two coplanar Griffith cracks in an infinite elastic medium has been treated by Itou (1980). Lowengrub and Srivastava (1968) treated the statical problem of stress distribution in the presence of two coplanar Griffith cracks in an infinite elastic strip. The scattering of time harmonic normally incident plane wave by two coplanar Griffith cracks was also solved by Jain and Kanwal (1972).

To the best knowledge of the authors, diffraction of elastic waves by two cracks moving along the interface of bonded dissimilar elastic media has not been investigated so far. In this paper, we consider the problem of determining the distribution of shear stress in the neighbourhood of the cracks, moving along the interface of the two bonded dissimilar elastic media. Two cases of practical importance have been considered here. Firstly, the case of two coplanar Griffith cracks moving along the interface of two semi-infinite dissimilar elastic media has been treated; secondly, the

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problem of the propagation of two coplanar Griffith cracks along the interface of an elastic layer overlying a semi-infinite medium of different elastic properties has been considered . Employing Fourier transforms the problem has been reduced to solving a set of triple integral equations with cosine kernel and weight functions. These equations are solved using the finite Hilbert transform technique .In the second problem, analytical expressions retain up to the order h^{-4}, where h is the thickness of the upper layer, for deriving the dynamic stress intensity factors and crack opening displacement. Numerical results have also been presented graphically.

2. Formulation Of The Problem

Two cracks of finite width are considered to be placed along the X -axis from -1 to -c and c to 1 with reference to a rectangular coordinate (x,y,z) system which, referred to fixed coordinate system (X,Y,Z), is moving with constant velocity v along X -axis, as shown in Fig.1 .

The coordinates are regarded as dimensionless, referring to the outer edge of the crack. In the dynamic problem of anti- plane shear ,there exists a single non-vanishing component of displacement in the Z -direction $W_i = W_i(X,Y,t)$, i=1,2, where W_1 and W_2 are the displacement component along the Z -direction in media Y>0 and Y<0 respectively . In the absence of body forces the equation of motion is

$$\frac{\partial^2 W_i}{\partial X^2} + \frac{\partial^2 W_i}{\partial Y^2} = \frac{1}{b_i^2} \frac{\partial^2 W_i}{\partial t^2} \tag{1}$$

where $b_i = (\mu_i/\rho_i)^{1/2}$,(i=1,2) are the shear wave speeds and ρ_i are the density of the materials and μ_i are the shear moduli.

Using Galilean transformation x = X - vt, y = Y, z = Z, t'= t, where (x,y,z) represents the translating coordinates system shown in Fig.1 eqn.(1) becomes independent of t and reduces to

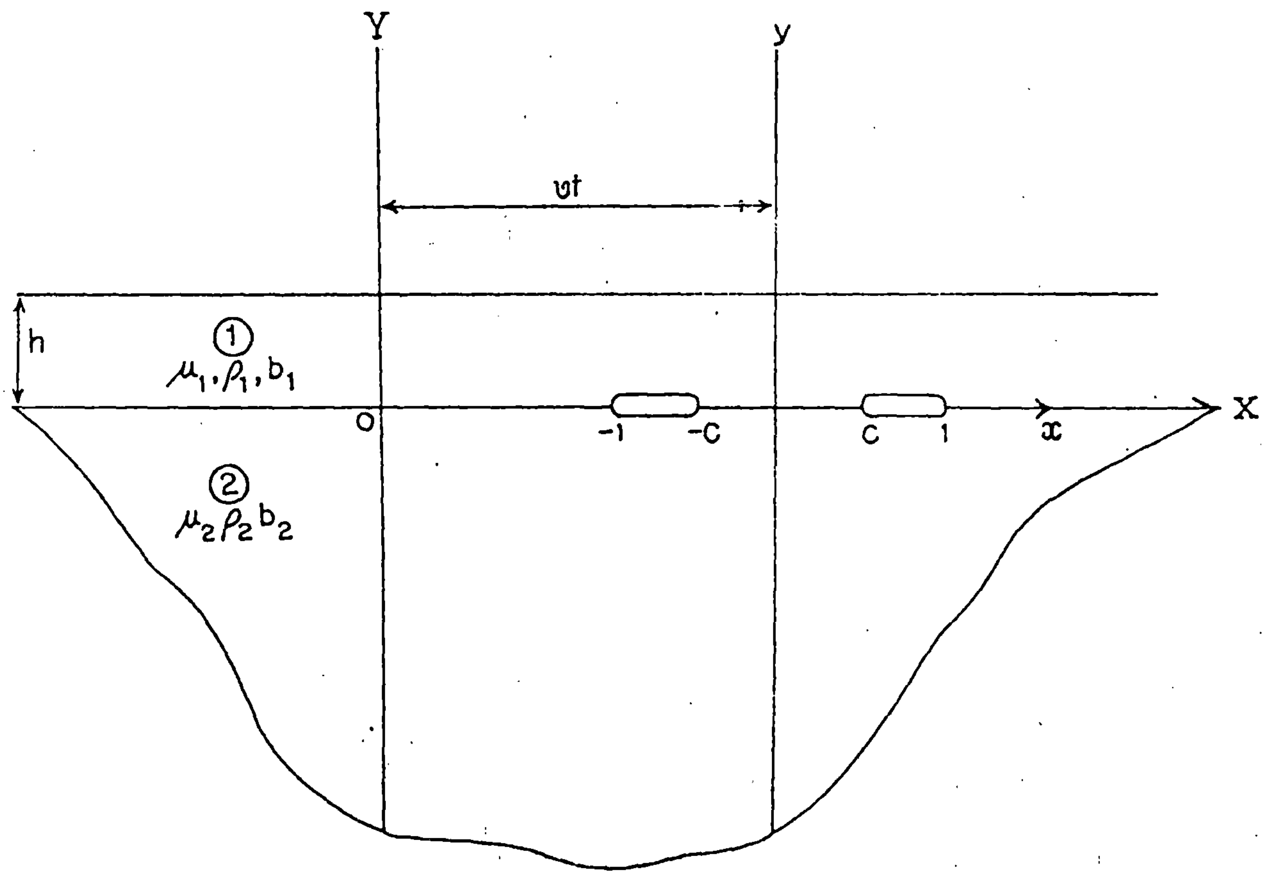

Fig 1. Geometry and coordinate system.

$$s_i^2 \frac{\partial^2 W_i}{\partial x^2} + \frac{\partial^2 W_i}{\partial y^2} = 0 \tag{2}$$

with
$$s_i^2 = 1 - v^2/b_i^2 \tag{3}$$

3.Boundary Conditions

Problem I:

In this case the cracks are placed along the interface of two joined dissimilar elastic half- spaces and are moving along the interface of these media. The cracks are excited by a normally incident anti-plane shear wave. The boundary conditions are

$$\begin{aligned}
&[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 = -p , && c<|x|<1 \\
&[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 , && 0\le|x|<c, |x|>1 \\
&W_1(x,0) = W_2(x,0) , && 0\le|x|<c, |x|>1
\end{aligned} \tag{4}$$

Problem I now consists of solving equation (2) together with the conditions (4).

Problem II:

In this case two coplanar Griffith cracks of finite width are assumed to be moving with uniform velocity under anti-plane shear stress along the interface of an elastic layer kept in welded contact with a semi-infinite medium of different elastic properties. The boundary conditions of this dynamic anti-plane problem are

$$\begin{aligned}
&[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 = -p , && c<|x|<1 \\
&[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 , && 0\le|x|<c, |x|>1 \\
&W_1(x,0) = W_2(x,0) , && 0\le|x|<c, |x|>1 \\
&[\tau_{yz}(x,h)]_1 = 0 && -\infty < x < \infty
\end{aligned} \tag{5}$$

Problem II now consists of solving equation (2) together with the conditions (5).

4.Solution Of The Problem I

Employing Fourier cosine transforms viz.

$$f_c(\xi, y) = \int_0^\infty f(x,y)\cos(\xi x)\,dx, \quad \text{and}$$

$$f(x,y) = \frac{2}{\pi}\int_0^\infty f_c(\xi, y)\cos(\xi x)\,d\xi$$

we obtain the solution of equation (2) as

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty A_1(\xi)\exp(-s_1\xi y)\cos(\xi x)d\xi, \quad \text{for } y>0 \tag{6}$$

$$W_2(x,y) = \frac{2}{\pi}\int_0^\infty A_2(\xi)\exp(s_2\xi y)\cos(\xi x)d\xi, \quad \text{for } y<0 \tag{7}$$

where s_i is the positive root of (3) and $A_i(\xi)$ are unknown functions to be determined.

From (6) and (7) we obtain

$$[\tau_{yz}(x,y)]_1 = -\frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi A_1(\xi)\exp(-s_1\xi y)\cos(\xi x)d\xi, \quad \text{for } y>0 \tag{8}$$

$$[\tau_{yz}(x,y)]_2 = \frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi A_2(\xi)\exp(s_2\xi y)\cos(\xi x)d\xi, \quad \text{for } y<0 \tag{9}$$

Using (4a) and (4b) we get

$$A_2(\xi) = -\frac{\mu_1 s_1}{\mu_2 s_2}A_1(\xi) \tag{10}$$

The crack opening displacement $\Delta w(x)$ is defined as

$$\Delta w(x) = W_1(x,0+) - W_2(x,0-)$$

$$= \frac{2L}{\pi} \int_0^\infty A_1(\xi) \cos(\xi x)\, d\xi \quad , \qquad c<x<1 \qquad (11)$$

$$= 0 \quad , \quad , \qquad 0\le x<c \ , \ x>1 .$$

where

$$L = \frac{\mu_1 s_1 + \mu_2 s_2}{\mu_2 s_2} \qquad (12)$$

From (8) and (4a)

$$\int_0^\infty \xi A_1(\xi) \cos(\xi x)\, d\xi = \frac{p\pi}{2\mu_1 s_1} \ , \quad c<x<1 \qquad (13)$$

Let us take $\quad A_1(\xi) = \dfrac{1}{\xi} \displaystyle\int_c^1 h(t^2) \sin(\xi t)\, dt$, (14)

Substituting (14) in (11) we see that this choice of $A_1(\xi)$ leads to

$$\int_c^1 h(t^2)\, dt = 0 \qquad (15)$$

Inserting (14) in (13) we obtain

$$\int_c^1 \frac{th(t^2)dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1} \ , \quad c<x<1 \qquad (16)$$

Using finite Hilbert transform technique (1968) , the solution of (16) is

$$h(t^2) = \frac{2p}{\pi\mu_1 s_1}\sqrt{\frac{t^2 - c^2}{1 - t^2}} \int_c^1 \sqrt{\frac{1 - x^2}{x^2 - c^2}}\, \frac{x\, dx}{t^2 - x^2} + \frac{K'}{\sqrt{(t^2 - c^2)(1-t^2)}} \qquad (17)$$

where the unknown constant K', determined from (15), is

$$K' = p(c^2 - E/F)/\mu_1 s_1 \qquad (18)$$

where $F = F(\Pi/2, q)$ and $E = E(\Pi/2, q)$ are complete elliptic integrals of the first kind and second kind respectively and $q = \sqrt{1-c^2}$

The relevant expressions for the crack opening displacement and stress component at the interface are

$$\Delta w(x) = L \int_x^1 h(t^2)\, dt , \qquad c \le x \le 1 \tag{19}$$

$$[\tau_{yz}(x,0)]_1 = -\frac{2\mu_1 s_1}{\pi} \int_c^1 \frac{t h(t^2) dt}{t^2 - x^2} , 0 \le x < c,\ x > 1 \tag{20}$$

Substituting the value of $h(t^2)$ from (17) in (19) and (20) we obtain

$$\Delta w(x) = \frac{Lp}{\mu_1 s_1} \left[E(\lambda, q) - \frac{E}{F} F(\lambda, q) \right] \tag{21}$$

where

$$\sin\lambda = \sqrt{(1-x^2)/(1-c^2)} \tag{22}$$

and

$$[\tau_{yz}(x,0)]_1 = p\left[\sqrt{\frac{x^2 - c^2}{x^2 - 1}} - 1 - \frac{E/F - c^2}{\sqrt{(x^2 - c^2)(x^2 - 1)}} \right], \text{ for } x > 1 \tag{23}$$

$$= p\left[\sqrt{\frac{c^2 - x^2}{1 - x^2}} - 1 + \frac{E/F - c^2}{\sqrt{(c^2 - x^2)(1 - x^2)}} \right], \text{ for } x < c \tag{24}$$

where we have used

$$\int_c^1 \frac{t\, dt}{(t^2 - x^2)\sqrt{(t^2 - c^2)(1-t^2)}} = \begin{cases} \dfrac{\pi}{2\sqrt{(c^2-x^2)(1-x^2)}}, & \text{for } 0<x<c \\ 0, & \text{for } c<x<1 \\ \dfrac{-\pi}{2\sqrt{(x^2-c^2)(x^2-1)}}, & \text{for } x>1 \end{cases} \tag{25}$$

The stress intensity factors at the tips of the cracks x=1 and x=c respectively are given by

$$K_1 = \underset{x \to 1}{Lt} \sqrt{2(x-1)}\ [\tau_{yz}(x,0)]_1 = \frac{p(1-E/F)}{\sqrt{1-c^2}} \tag{26}$$

$$K_c = \underset{x \to c}{Lt} \sqrt{2(c-x)}\ [\tau_{yz}(x,0)]_1 = \frac{p(E/F-c^2)}{\sqrt{c(1-c^2)}} \tag{27}$$

5.Solution Of The Problem II

Employing Fourier cosine transform the solutions of is are sought in the form

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty [A_1(\xi)\exp(-s_1\xi y) + A_2(\xi)\exp(s_1\xi y)]\cos(\xi x)d\xi \ , \quad \text{for } 0\le y\le h$$

$$W_2(x,y) = \frac{2}{\pi}\int_0^\infty A_3(\xi)\ \exp(s_2\xi y)\ \cos(\xi x)\ d\xi \ , \qquad \text{for} \quad y<0 \tag{28}$$

Using (28) we obtain the stress components as

$$[\tau_{yz}(x,y)]_1 = \frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi[-A_1(\xi)\exp(-s_1\xi y)+A_2(\xi)\exp(s_1\xi y)]\cos(\xi x)d\xi ,$$

$$\text{for } 0\le y\le h$$

$$[\tau_{yz}(x,y)]_2 = \frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi A_3(\xi)\ \exp(s_2\xi y)\ \cos(\xi x)d\xi , \quad \text{for } y<0 \tag{29}$$

Applying (5a), (5b) and (5c)

$$A_3(\xi) = \frac{\mu_1 s_1}{\mu_2 s_2}[A_2(\xi) - A_1(\xi)] \tag{30}$$

and $$A_2(\xi) = A_1(\xi)\ \exp(-2\xi h s_1) \tag{31}$$

The crack opening displacement $\Delta w(x)$ is defined as

$$\Delta w(x) = W_1(x,0+) - W_2(x,0-)$$

$$= \frac{2L}{\pi} \int_0^\infty f(\xi) \cos(\xi x)\, d\xi \quad , \qquad c<x<1$$

$$= 0 \qquad , \quad 0 \le x < c \ , \ x>1 \tag{32}$$

where

$$f(\xi) = A_1(\xi)\left[1 + \frac{\mu_2 s_2 - \mu_1 s_1}{\mu_2 s_2 + \mu_1 s_1} \exp(-2\xi h s_1) \right] \tag{33}$$

Therefore, by (5c) and (5a) , $f(\xi)$ is found to be the solution of the following triple integral equations

$$\int_0^\infty f(\xi) \cos(\xi x)\, d\xi = 0 \qquad 0 \le x<c,\ x>1 \tag{34}$$

$$\int_0^\infty \xi f(\xi)[1+M(\xi h)] \cos(\xi x)\, d\xi = \frac{p\pi}{2\mu_1 s_1} \ , \quad c<x<1 \tag{35}$$

with

$$M(\xi h) = -\frac{1 - \tanh(\xi h s_1)}{\left[1 + \frac{\mu_1 s_1}{\mu_2 s_2} \tanh(\xi h s_1) \right]} \tag{36}$$

Assuming

$$f(\xi) = \frac{1}{\xi} \int_c^1 h(t^2) \sin(\xi t)\, dt \quad , \tag{37}$$

it is found from (35) and (36)

$$\int_c^1 \frac{t h(t^2) dt}{t^2 - x^2} = \frac{p\pi}{2\mu_1 s_1} - Q(x)$$

where

$$Q(y) = \int_c^1 h(t^2)\, K_1(y,t)\, dt$$

and

$$K_1(y,t) = \int_0^\infty M(\xi h)\ \cos(\xi y)\ \sin(\xi t)\ d\xi . \qquad (38)$$

Now using Hilbert transform technique (1968) we find that $h(x^2)$ is the solution of the following Fredholm integral equation

$$h(x^2) + \int_c^1 h(t^2)\ K(x^2,t)\ dt = F(x^2)\ , \quad c<x<1 \qquad (39)$$

satisfying the condition

$$\int_c^1 h(x^2)\ \ dx = 0 \qquad (40)$$

where

$$K(x^2,t) = -\frac{4}{\pi^2}\sqrt{\frac{x^2-c^2}{1-x^2}}\int_c^1 \sqrt{\frac{1-y^2}{y^2-c^2}} \times \frac{y\ \ K_1(y,t)}{y^2-x^2}\ dy \qquad (41)$$

and

$$F(x^2) = -\frac{2p}{\pi\mu_1 s_1}\sqrt{\frac{x^2-c^2}{1-x^2}}\int_c^1 \sqrt{\frac{1-y^2}{y^2-c^2}}\ \frac{y\ dy}{y^2-x^2} + \frac{K''}{\sqrt{(x^2-c^2)(1-x^2)}}\ , \qquad (42)$$

K'' being an arbitrary constant determined by condition (40). If $h>>1$ is taken, then by substituting $\eta = \xi h$ and expanding $\cos(\eta y/h)$, $\sin(\eta y/h)$, it is possible to write (38) in the form

$$K_1(y,t) = \frac{I_0 t}{h^2} + \frac{I_1 t}{h^4}(t^2 + 3y^2) + O(h^{-6}) \qquad (43)$$

where

$$I_j = \frac{(-1)^j}{(2j+1)!}\int_0^\infty \eta^{2j+1}\ M(\eta)\ d\eta \quad , \ (j=0,1) \qquad (44)$$

and hence

$$K(x^2,t)=\frac{2}{\pi}\sqrt{\frac{x^2-c^2}{1-x^2}}\left[\frac{I_o t}{h^2}+\frac{I_1 t}{h^4}\left(t^2+3x^2-\frac{3}{2}k^2\right)\right]+O(h^{-6}) \tag{45}$$

where $k^2 = 1 - c^2$.

Integrating both sides of (39) with respect to x from c to 1 and using (40)

$$K'' = \frac{p}{\mu_1 s_1}\left[c^2-\frac{E}{F}\right]+\frac{1}{F}\int_c^1 h(t^2)\,K(t)dt \tag{46}$$

with

$$K(t) = \frac{2}{\pi}\left[\frac{I_o t}{h^2}(E-c^2F)+\frac{I_1 t}{h^4}\left\{(t^2-\frac{3}{2}k^2)(E-c^2F)-c^2(E+F)+2E\right\}\right]+O(h^{-6})$$

where E and F defined by $E = E(\pi/2,q)$ and $F = F(\pi/2,q)$ with $q = k$ are known as elliptic integrals of first and second kind respectively. Using the results (46) and (42) in the equation (38) we see that $h(x^2)$ must satisfy the integral equation

$$h(x^2) + \int_c^1 h(t^2)\,M(x^2,t)\,dt = S(x^2) \tag{47}$$

where

$$M(x^2,t)= K(x^2,t)-\frac{K(t)}{F\sqrt{(x^2-c^2)(1-x^2)}} = \frac{2t}{\pi\sqrt{(x^2-c^2)(1-x^2)}}\left[\frac{I_o}{h^2}(x^2-\frac{E}{F})+\frac{I_1}{h^4}\right.$$

$$\left.\times\left\{(t^2+\frac{3}{2}k^2)(x^2-\frac{E}{F})+3x^2(x^2-1)+\frac{E}{F}+c^2-\frac{2c^2E}{F}\right\}\right]+O(h^{-6}) \tag{48}$$

and

$$S(x^2) = \frac{p\left[x^2-\frac{E}{F}\right]}{\mu_1 s_1\sqrt{(x^2-c^2)(1-x^2)}} \tag{49}$$

Since $h \gg 1$, and $|M(x^2,t)| < 1$, the solution of (47) may be written in the form

$$h(x^2) = h_o(x^2) + \frac{1}{h^2} h_1(x^2) + \frac{1}{h^4} h_2(x^2) + O(h^{-6}) \tag{50}$$

where

$$h_o(x^2) = \frac{p[\, x^2 - \frac{E}{F}\,]}{\mu_1 s_1 \sqrt{(x^2-c^2)(1-x^2)}} \tag{51}$$

$$h_1(x^2) = \frac{-\, I_o C_o p[\, x^2 - \frac{E}{F}\,]}{2\mu_1 s_1 \sqrt{(x^2-c^2)(1-x^2)}} \tag{52}$$

$$h_2(x^2) = \frac{p\, C_o}{4\mu_1 s_1 \sqrt{(x^2-c^2)(1-x^2)}} \left[I_o^2 C_o \left\{ x^2 - \frac{E}{F} \right\} - 2I_1 (3x^4 + C_1 x^2 + C_2) \right] \tag{53}$$

with

$$C_o = 1 + c^2 - 2\,\frac{E}{F}$$

$$C_1 = k^4/4C_o - (1 + c^2)$$

$$C_2 = c^2 + \frac{E}{F}\left\{ C_1 - \frac{k^4}{2C_o} \right\}$$

The relevant crack opening displacement and stress component at the interface are

$$\Delta w(x) = L \int_x^1 h(t^2)\, dt\,, \qquad c \le x \le 1 \tag{54}$$

$$[\tau_{yz}(x,0)]_1 = -\frac{2\mu_1 s_1}{\pi} \left[\int_c^1 \frac{t h(t^2) dt}{t^2 - x^2} + \int_c^1 h(t^2) K_1(x,t) dt \right], 0 \le x < c,\ x > 1 \tag{55}$$

where $K_1(x,t)$ is given in (38).

Using (43) and equations (50)– (53)

$$\int_c^1 h(t^2) K_1(x,t) dt = \frac{p\pi}{8\mu_1 s_1} \left[\frac{2I_o C_o}{h^2} - \frac{I_o^2 C_o^2}{h^4} + \frac{2I_1 C_o}{h^4} \left\{ 3x^2 + C_1 + \frac{3}{2}(1+c^2) \right\} \right] + O(h^{-6})$$

(56)

we get for $0<x<c$

$$\int_c^1 \frac{t h(t^2)dt}{t^2-x^2} = \frac{p\pi}{2\mu_1 s_1}\left[\left\{1-\frac{I_o C_o}{2h^2}+\frac{I_o^2 C_o^2}{4h^4}\right\}\times\left\{\frac{x^2-\frac{E}{F}}{X_1}+1\right\}-\frac{I_1 C_o}{2h^4}\times\right.$$

$$\left.\times\left\{\frac{3x^4+C_1x^2+C_2}{X_1}+3\left(\frac{1+c^2}{2}+x^2\right)+C_1\right\}\right]+O(h^{-6}) \quad , \qquad (57)$$

and for $x>1$

$$\int_c^1 \frac{t h(t^2)dt}{t^2-x^2} = \frac{p\pi}{2\mu_1 s_1}\left[\left\{1-\frac{I_o C_o}{2h^2}+\frac{I_o^2 C_o^2}{4h^4}\right\}\times\left\{\frac{\frac{E}{F}-x^2}{X_2}+1\right\}+\frac{I_1 C_o}{2h^4}\times\right.$$

$$\left.\times\left\{\frac{3x^4+C_1x^2+C_2}{X_2}-3\left(\frac{1+c^2}{2}+x^2\right)-C_1\right\}\right]+O(h^{-6}) \quad , \qquad (58)$$

where

$$X_1 = \sqrt{(c^2-x^2)(1-x^2)}$$

$$X_2 = \sqrt{(x^2-c^2)(x^2-1)}$$

Using equations (50) to (53) the crack opening displacement is obtained from (54) after integration as

$$\Delta w(x) = \frac{pL}{\mu_1 s_1}\left[\left\{1-\frac{I_o C_o}{2h^2}+\frac{I_o^2 C_o^2+2I_1 C_o(C_1-k^4/2C_o)}{4h^4}\right\}\times\right.$$

$$\left.\times\left\{E(\lambda,q)-\frac{E}{F}F(\lambda,q)\right\}-\frac{2I_1 C_o}{4h^4}\times\sqrt{(1-x^2)(x^2-c^2)}\right]+O(h^{-6}) \qquad (59)$$

where λ is given by (22).

Substituting the results obtained in (56),(57) and (58) on the right hand side of (55) the stress in the plane of the crack can be derived and from it stress intensiy factors at the crack tips can be determined easily.

The stress intensity factor at x = 1 is given by

$$N_1 = \underset{x \to 1}{Lt} \sqrt{2(x-1)}\,[\tau_{yz}(x,0)]_1 = \frac{-p}{\sqrt{1-c^2}}\left[(E/F-1)\left\{1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right\} + \right.$$

$$\left. + \frac{I_1C_0}{2h^4}(3 + C_1 + C_2)\right] + O(h^{-6}) \qquad (60)$$

and the stress intensity factor at x = c is found to be

$$N_c = \underset{x \to c}{Lt} \sqrt{2(c-x)}\,[\tau_{yz}(x,0)]_1 = \frac{-p}{\sqrt{c(1-c^2)}}\left[(c^2-E/F)\left\{1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right\} - \right.$$

$$\left. - \frac{I_1C_0}{2h^4}(3c^4 + C_1c^2 + C_2)\right] + O(h^{-6}) \qquad (61)$$

5. NUMERICAL RESULTS

In this section numerical results are presented for the stress intensity factors at the crack tips and also the crack opening displacement for different values of the layer thickness and the crack speed and for $b_1/b_2 = 0.6$ .The crack opening displacement is found to increase gradually with increase in the value of the crack speed. Further, for a fixed crack speed, the crack opening displacement increases with decrease in the value of the separating distance between the cracks.

Variation of the stress intensity factors at both the crack tips with the crack speed is depicted in Figures 4 – 7. It is interesting to note that the stress intensity factors at both the crack tips increase very slowly at the onset with increase in the value of v/b_1 but change rapidly and goes to infinity as v/b_1 approaches unity.This fact becomes prominent as the layer thickness becomes large.

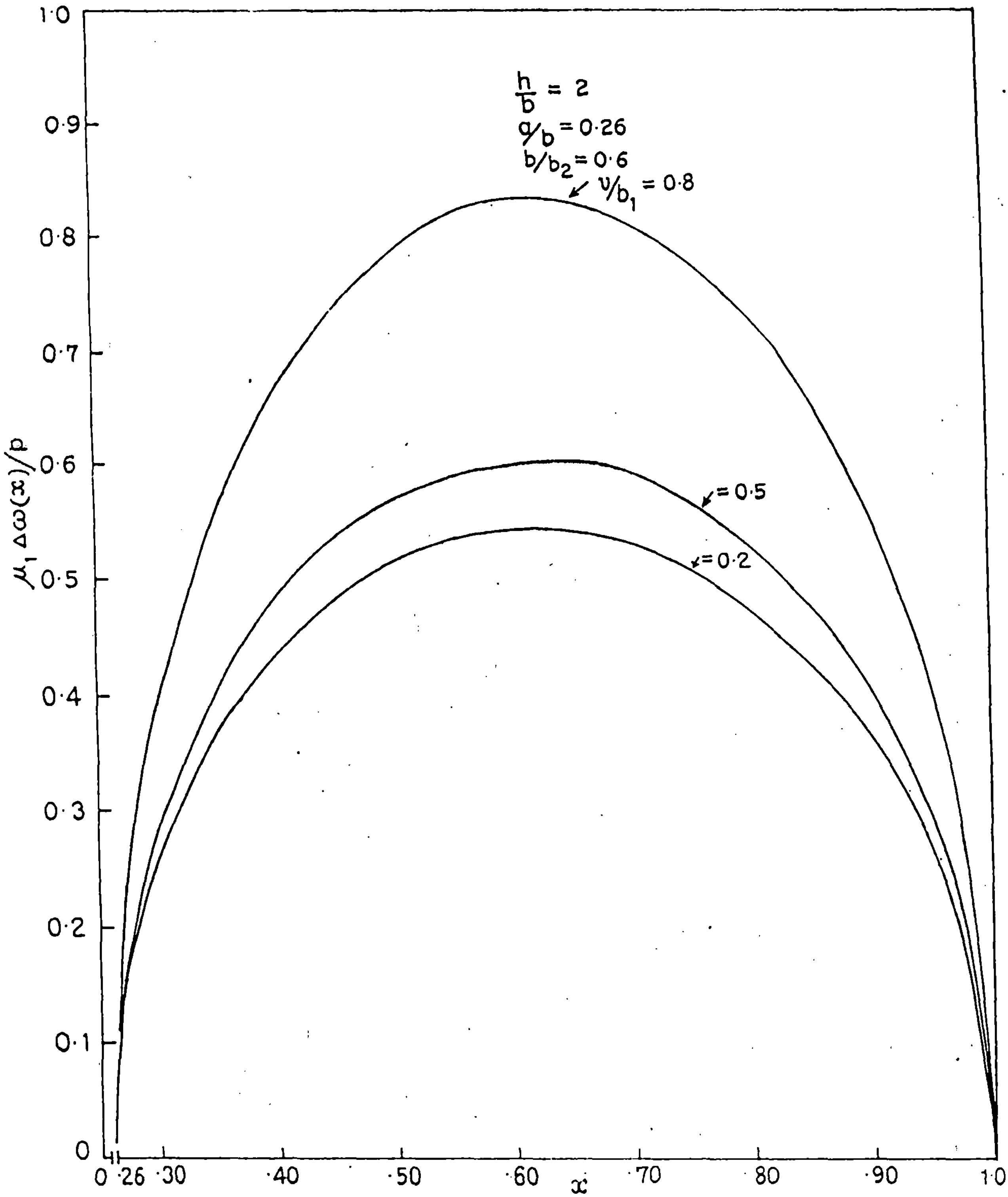

Fig. 2. Variation of crack opening displacement with x for problem II

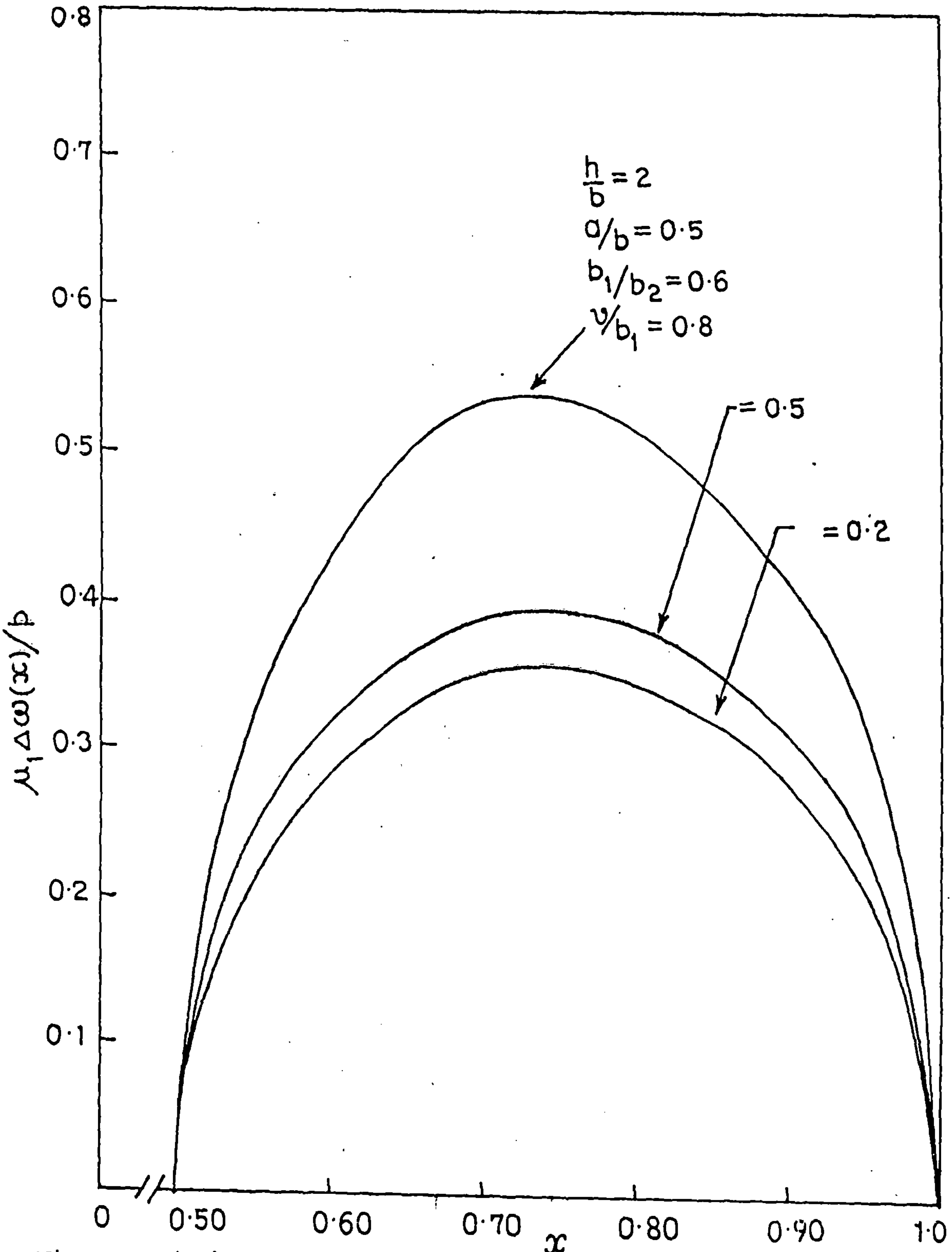

Fig 3. Variation of crack opening displacement with x for problem II .

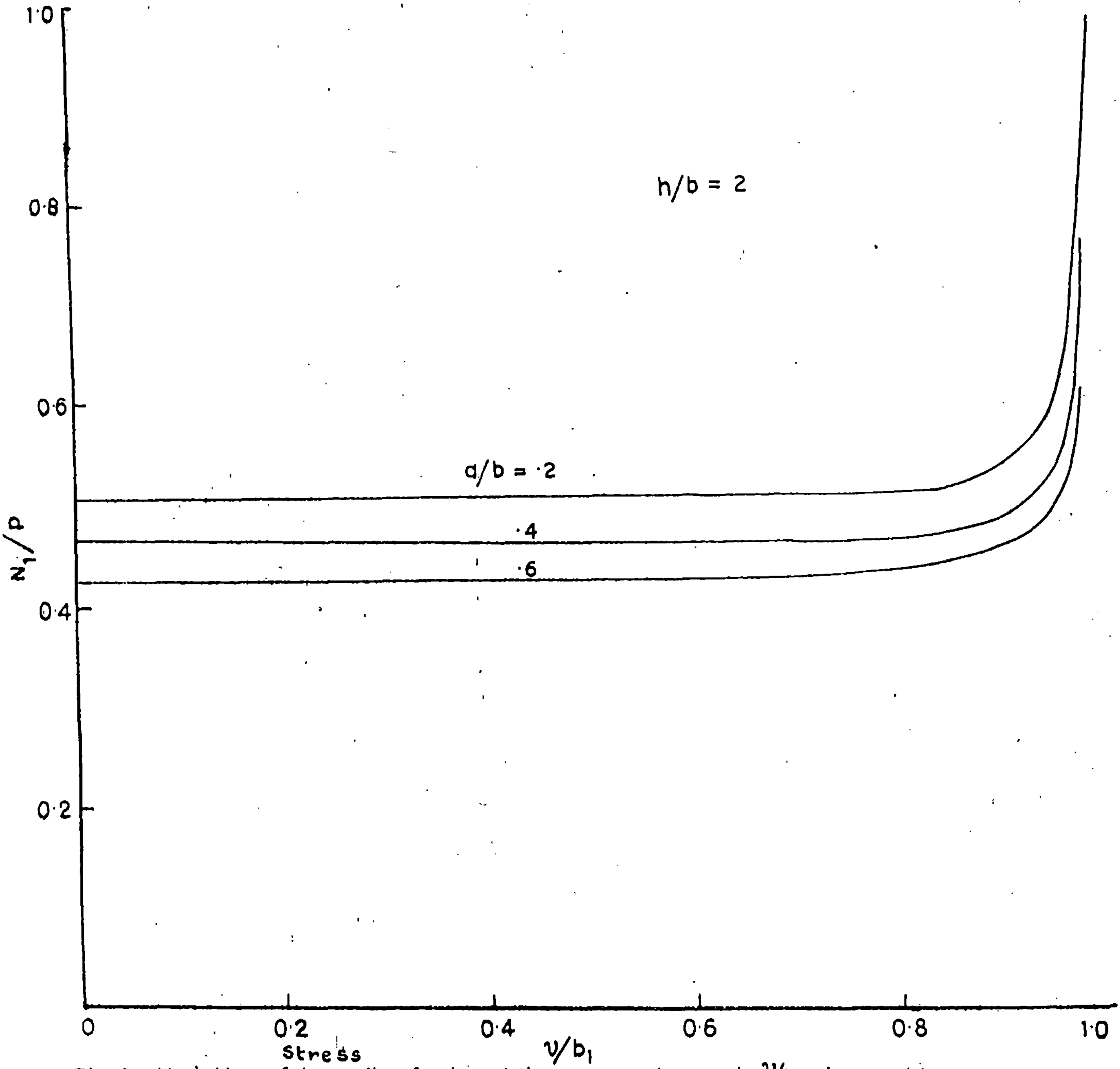

Fig 4. Variation of stress intensity factor at the outer edge with ν/b_1 for problem II

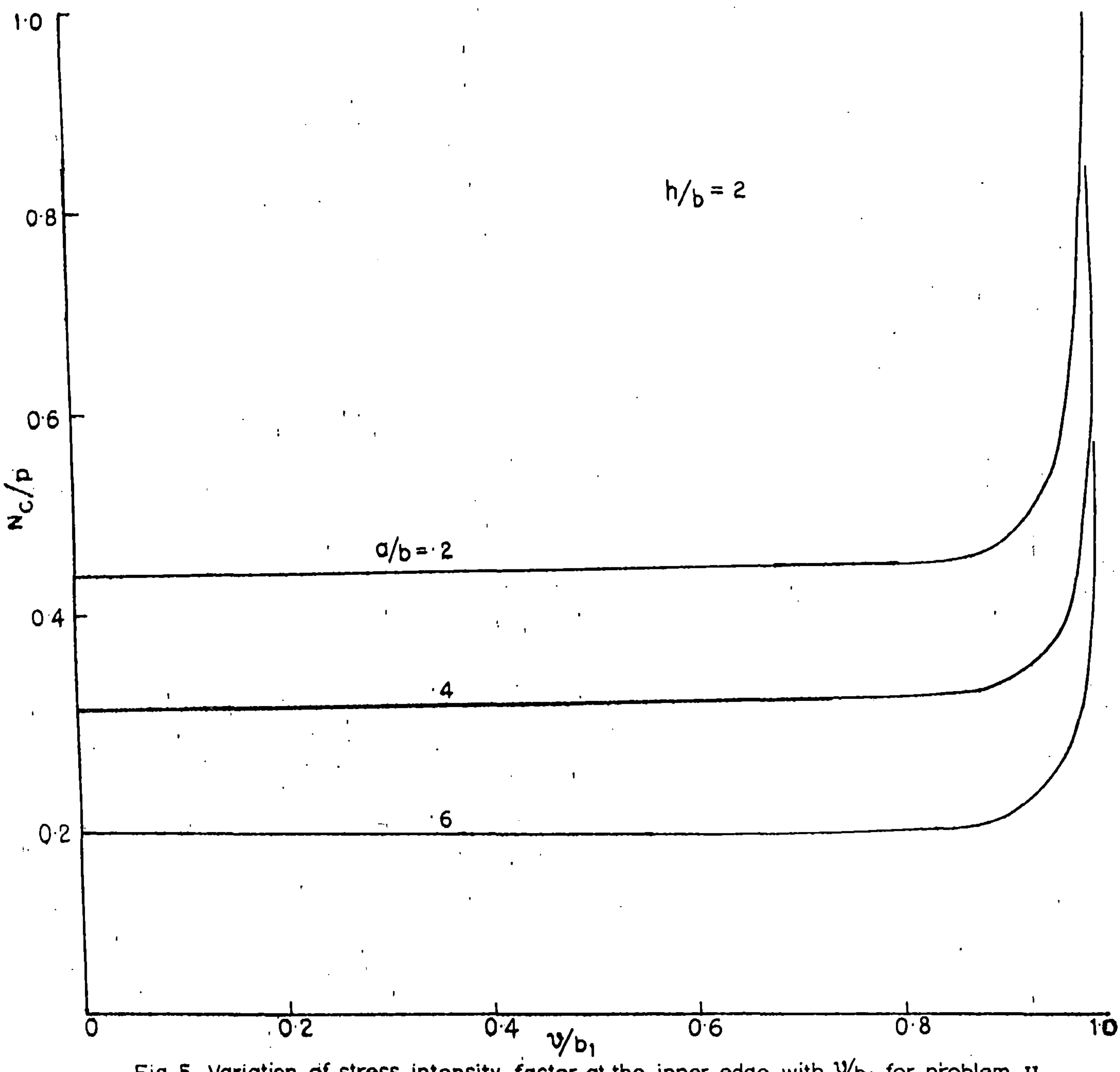

Fig 5. Variation of stress intensity factor at the inner edge with υ/b_1 for problem II.

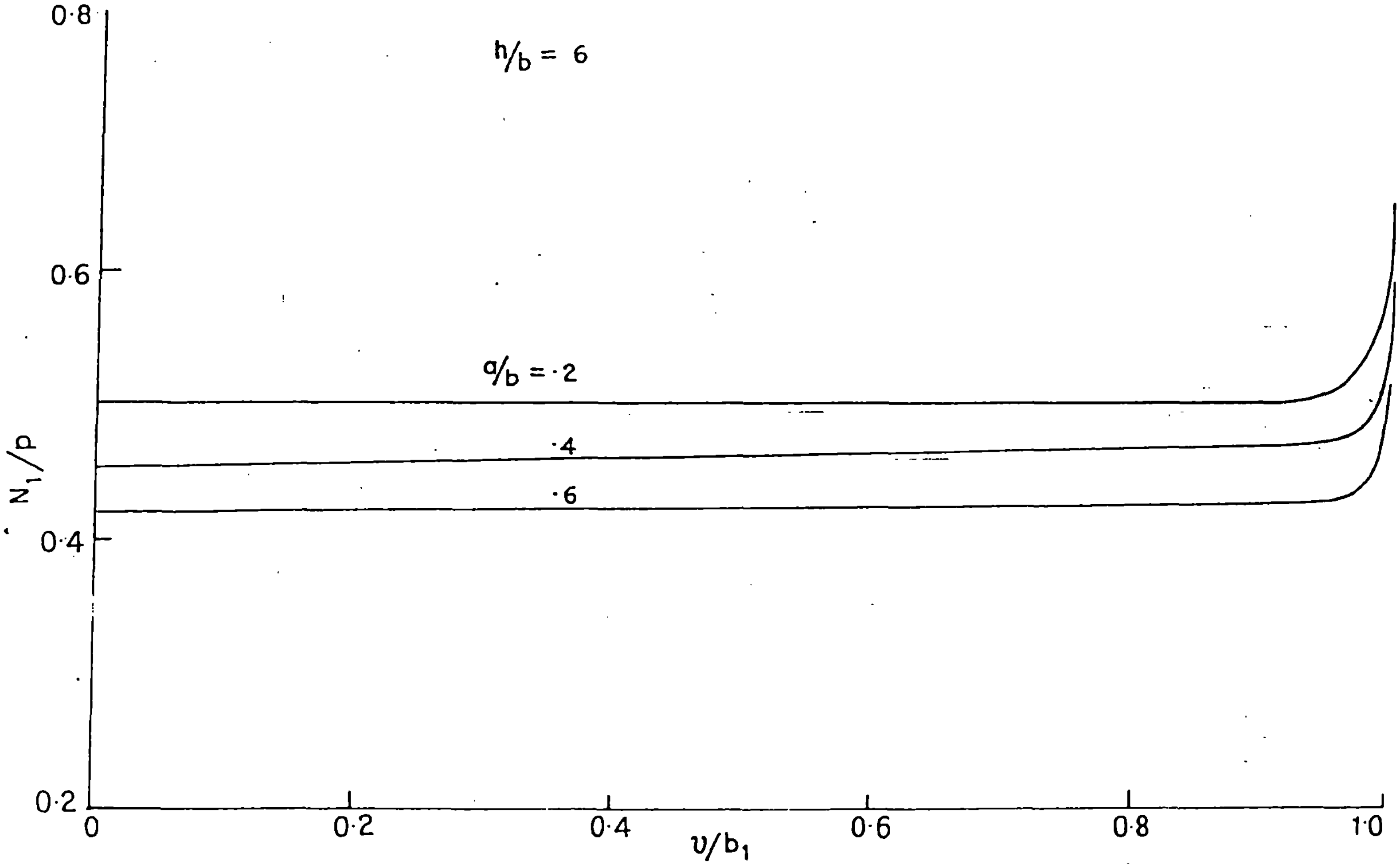

Fig 6. Variation of stress intensity factor at the outer edge with v/b_1 for problem II .

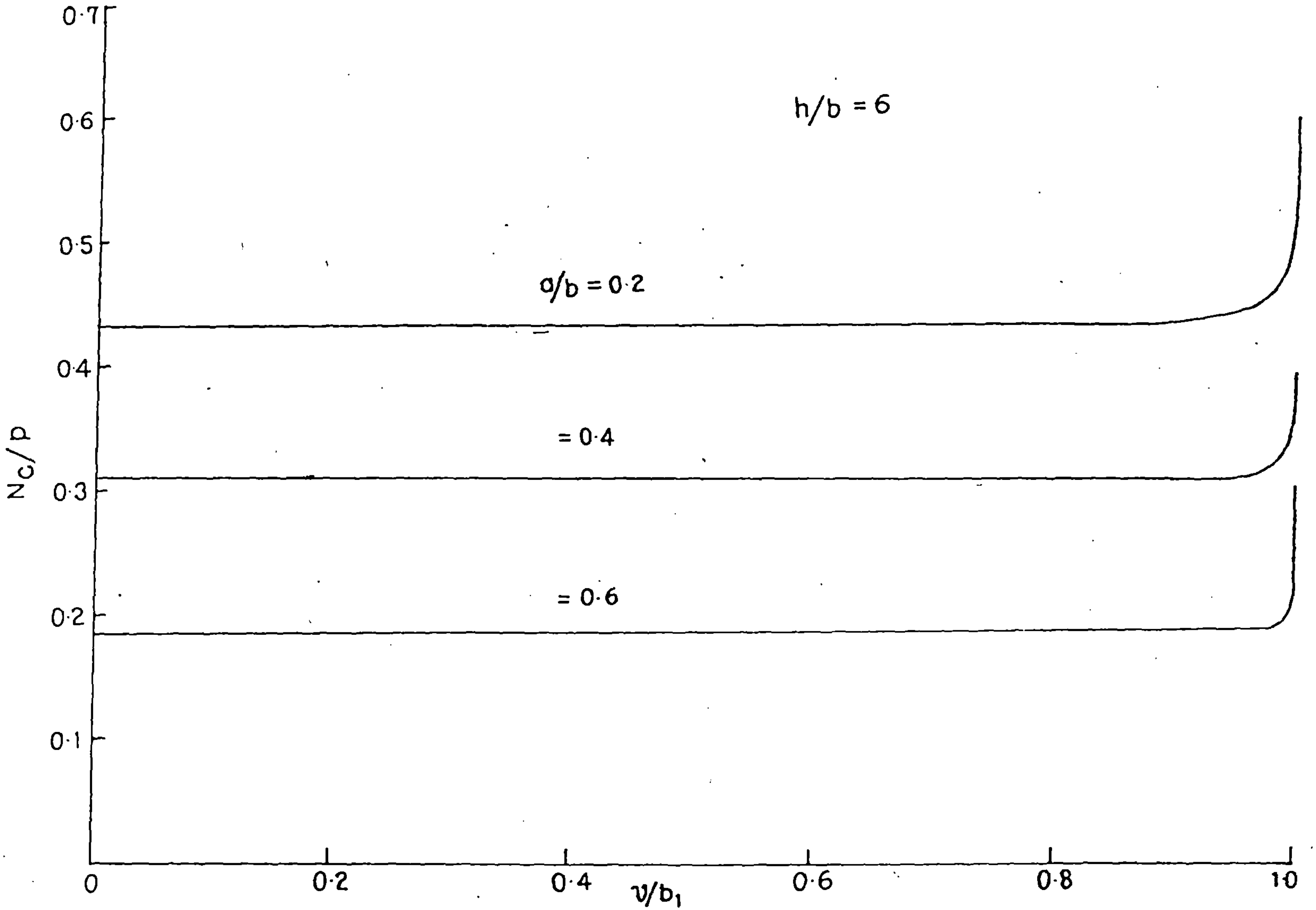

Fig 7. Variation of stress intensity factor at the inner edge with ν/b_1 for problem II.

PROBLEM OF TWO COPLANAR GRIFFITH CRACKS RUNNING STEADILY UNDER THREE- DIMENSIONAL LOADING

1. Introduction

Yoffe (1951) considered the problem of propagation of a crack of fixed length at a constant speed through a stretched isotropic elastic solid of infinite extent. In recent years, Yoffe's investigation was extended to include different types of materials and different material geometries. Sih and Chen (1972) considered the problem of uniformly propagating finite crack in a strip of isotropic elastic material. Recently, Kassir and Tse (1983) solved the plane stress problem of a moving crack in an infinite orthotropic stressed medium by using integral transform technique and the same technique has been employed by De and Patra (1990) to solve Yoffe's problem in an stressed orthotropic strip of finite thickness.

However all the problems mentioned above have been solved using dynamic equation of elasticity in two dimension. But in most instance, cracks are subjected to a state of stress that is triaxial in nature. Cracks problems involving three dimensional loading have generally not been attempted so far.

Recently, Angel and Achenbach (1985) derived the elastodynamic stress intensity factor for three dimensional loading of a cracked half- space. Freund (1971) also solved the three dimensional problem of the oblique reflection of a Rayleigh wave from the edge of a semi- infinite crack employing a Wiener- Hopf technique. The problem of a uniformly propagating finite crack in an elastic medium has been solved by Itou (1979) using dynamic equations of elasticity in three dimension.

Regarding the dynamic crack problem, research has

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been restricted mainly to a single crack because of severe mathematical complexity encountered in finding the solution of two or more cracks. Recently Jain and Kanwal (1972) presented the low frequency solution of diffraction of normally incident longitudinal waves by two coplanar Griffith cracks in an isotropic elastic medium. They used the finite Hilbert transform technique developed by Srivastava and Lowengrub (1968) to solve the mixed boundary value problem. Using a completely different technique Itou (1980) solved the diffraction problem of elastic waves by two coplanar Griffith cracks in an infinite elastic medium.

In this paper, we have considered the problem of propagation of two coplanar Griffith cracks propagating steadily with uniform velocity under three dimensional loading. The application of two dimensional Fourier transform reduced this problem to that of solving triple integral equations in which the double Fourier transform of the crack opening displacement appear as the unknown. In an attempt to solve the problem the transformed surface displacement has been expanded in a series of a function which is automatically zero outside the cracks. Finally, Schmidt method (1978) has been employed to solve the integral equations. The dynamic stress intensity factors and the crack opening displacement have been evaluated numerically for various values of crack speed and distance between the cracks.

2. Formulation of the problem.

Let (X,Y,Z) be a fixed rectangular coordinate system. Two coplanar Griffith cracks of infinite length but of finite width located in the XZ- plane, the Z- axis being in the direction of the length of the cracks, are, assumed to be moving steadily with velocity U in the direction of X- axis. It is convenient to introduce Galilean transformation $x = X-UT$, $y = Y$, $z = Z$, $t = T$ where (x,y,z) represents the translating coordinate system shown in Fig.1. Referred to this moving system of the coordinate the cracks are assumed to occupy the positions $b<|x|<a, y=0, |z|<\infty$.

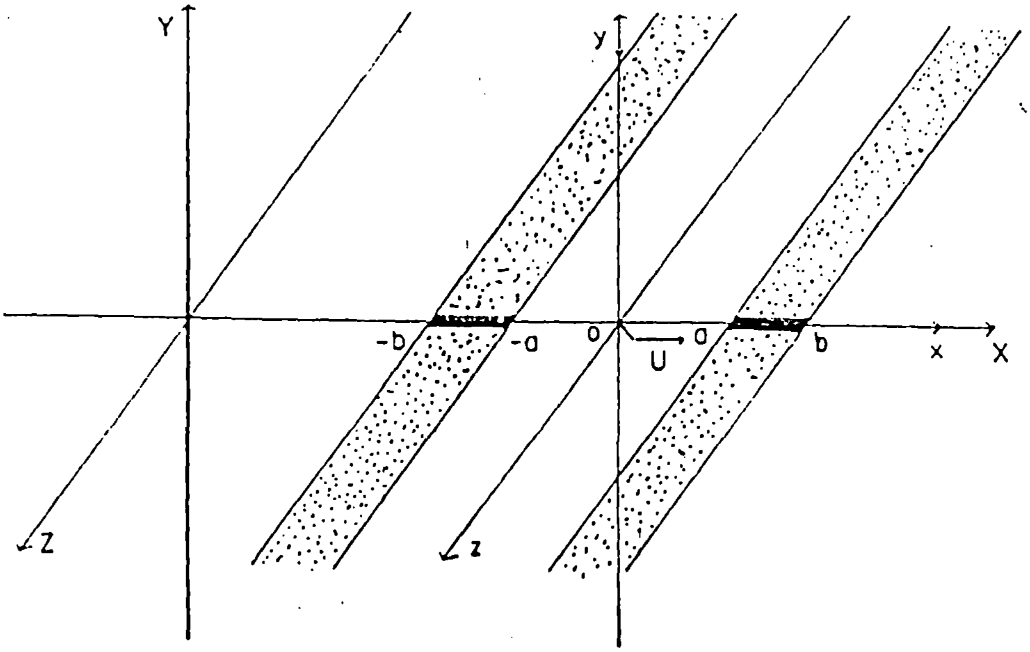

Fig. 1: Geometry and coordinate system.

The equations on motion in the absence of body force are

$$(\lambda+\mu)\frac{\partial}{\partial X}\left[\frac{\partial u^*}{\partial X}+\frac{\partial v^*}{\partial Y}+\frac{\partial w^*}{\partial Z}\right]+\mu\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)u^*=\rho\frac{\partial^2 u^*}{\partial T^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial Y}\left[\frac{\partial u^*}{\partial X}+\frac{\partial v^*}{\partial Y}+\frac{\partial w^*}{\partial Z}\right]+\mu\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)v^*=\rho\frac{\partial^2 v^*}{\partial T^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial Z}\left[\frac{\partial u^*}{\partial X}+\frac{\partial v^*}{\partial Y}+\frac{\partial w^*}{\partial Z}\right]+\mu\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right)w^*=\rho\frac{\partial^2 w^*}{\partial T^2} \qquad (2.1)$$

where u^*, v^*, w^* are the displacement components, λ and μ are Lame's constants and ρ is the material density. Using Galilean transformation

$$x = X - UT, \quad y = Y, \quad z = Z, \quad t = T$$

(2.1) reduces to

$$(\lambda+\mu)\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u=\rho U^2\frac{\partial^2 u}{\partial x^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)v=\rho U^2\frac{\partial^2 v}{\partial x^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)w=\rho U^2\frac{\partial^2 w}{\partial x^2} \qquad (2.2)$$

where u, v, w are the displacement components in the moving coordinate system so that

$$u^*(X,Y,Z,T) = u(x,y,z)$$

$$v^*(X,Y,Z,T) = v(x,y,z)$$

$$w^*(X,Y,Z,T) = w(x,y,z)$$

The stress components for the three dimensional problem are

$$\sigma_x = (\lambda+2\mu)\frac{\partial u}{\partial x} + \lambda\left[\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right], \quad \tau_{xy} = \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

$$\sigma_y = (\lambda+2\mu)\frac{\partial v}{\partial y} + \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right], \quad \tau_{yz} = \left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

$$\sigma_z = (\lambda+2\mu)\frac{\partial w}{\partial z} + \lambda\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right], \quad \tau_{xz} = \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \qquad (2.3)$$

The boundary conditions are

$$\sigma_y/2\mu = -p(x,z), \quad \text{for } y=0, \; a\le|x|\le b, \; |z|<\infty,$$

$$v=0, \quad \text{for } y=0, \; |x|>b, |x|<a, |z|<\infty,$$

$$\tau_{xy} = 0 = \tau_{yz}, \quad \text{for } y=0, \; |x|<\infty, |z|<\infty. \qquad (2.4)$$

3. Solution of the problem

Using Fourier transformations viz.

$$\bar{\bar{g}}(\xi,y,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y,z)\, e^{i(\xi x+\zeta z)}\, dx\, dz,$$

and

$$g(x,y,z) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\bar{g}}(\xi,y,\zeta)\, e^{-i(\xi x+\zeta z)}\, d\xi\, d\zeta, \qquad (3.1)$$

(2.2) reduces to

$$\left\{\frac{d^2}{dy^2} - (\alpha^2-M^2)\xi^2 - \zeta^2\right\}\bar{\bar{u}} - i(\alpha^2-1)\xi\frac{d\bar{\bar{v}}}{dy} - (\alpha^2-1)\xi\zeta\bar{\bar{w}} = 0,$$

$$-i(\alpha^2-1)\xi\frac{d\bar{\bar{u}}}{dy} + \left\{\alpha^2\frac{d^2}{dy^2} - (1-M^2)\xi^2 - \zeta^2\right\}\bar{\bar{v}} - i(\alpha^2-1)\xi\frac{d\bar{\bar{w}}}{dy} = 0,$$

$$-(\alpha^2-1)\xi\zeta\bar{\bar{u}} - i(\alpha^2-1)\zeta\frac{d\bar{\bar{v}}}{dy} + \left\{\frac{d^2}{dy^2} - (1-M^2)\xi^2 - \zeta^2\alpha^2\right\}\bar{\bar{w}} = 0, \qquad (3.2)$$

with $\alpha^2 = (\lambda+2\mu)/\mu$, $\beta^2=\rho/\mu$ and $M^2 = \beta^2U^2$.

Due to symmetry given in (2.4), we need to consider the region $y\geq0$ only. The solutions of (3.2) in the region $y\geq0$ can easily be found to be of the form

$$\bar{\bar{u}} = A_1\, e^{-s_1 y} + B_1\, e^{-s_2 y}$$

$$\bar{\bar{v}} = A_2\, e^{-s_1 y} + B_2\, e^{-s_2 y}$$

$$\bar{\bar{w}} = A_3\, e^{-s_1 y} + B_3\, e^{-s_2 y} \tag{3.3}$$

where

$$s_1 = \sqrt{(1-M^2/\alpha^2)\xi^2+\zeta^2}$$

$$s_2 = \sqrt{(1-M^2)\xi^2+\zeta^2} \tag{3.4}$$

and

$$A_1 = i\xi A_2/s_1, \quad A_3 = i\zeta A_2/s_1, \quad B_2 = -i(\xi B_1+\zeta B_3)/s_2 \tag{3.5}$$

The transformed stress components $\bar{\bar{\sigma}}_x$, $\bar{\bar{\sigma}}_y$, $\bar{\bar{\tau}}_{xy}$, $\bar{\bar{\tau}}_{yz}$ obtained from (3.3),(3.5) and (2.3) are

$$\bar{\bar{\sigma}}_x/2\mu = [\, \xi^2M^2(1-2/\alpha^2) + 2\xi^2 \,]\, A_2 e^{-s_1 y}/2s_1 - i\xi B_1 e^{-s_2 y},$$

$$\bar{\bar{\sigma}}_y/2\mu = [\, \xi^2M^2 - 2(\xi^2+\zeta^2)]\, A_2 e^{-s_1 y}/2s_1 + i(\xi B_1+\zeta B_3) e^{-s_2 y},$$

$$\bar{\bar{\tau}}_{xy}/2\mu = -i\xi A_2\, e^{-s_1 y} - [\, \xi\zeta B_3 + (s_2^2+\xi^2)B_1]\, e^{-s_2 y}/2s_2$$

$$\bar{\bar{\tau}}_{yz}/2\mu = -i\zeta A_2\, e^{-s_1 y} - [\, \xi\zeta B_1 + (s_2^2+\zeta^2)B_3]\, e^{-s_2 y}/2s_2 \tag{3.6}$$

Using the conditions (2.4.3) B_1 and B_3 can be expressed in terms of

A_2 as

$$B_1 = \frac{-2i\xi s_2 A_2}{(2-M^2)\xi^2+2\zeta^2}$$

$$B_3 = \frac{-2i\zeta s_2 A_2}{(2-M^2)\xi^2+2\zeta^2} \tag{3.7}$$

Hence we find that all the components of stress and displacement can be expressed in terms of the unknown function $A_2(\xi,\zeta)$. Now insertion of (3.5) and (3.7) in $\bar{\bar{v}}$ given in (3.3) yields

$$A_2 = -\frac{(2-M^2)\xi^2+2\zeta^2}{M^2\xi^2}\,\bar{\bar{v}}_0 \tag{3.8}$$

where $\bar{\bar{v}}_0$ is the transformed displacement on y=0.
Using (3.7) and (3.8) we obtain from (3.6)

$$\bar{\bar{\sigma}}_x/2\mu = -\bar{\bar{v}}_0\left[\left\{(2-M^2)\xi^2+2\zeta^2\right\}\left\{2+M^2(1-2/\alpha^2)\right\}\frac{e^{-s_1y}}{2M^2s_1} - 2s_2\frac{e^{-s_2y}}{M^2}\right],$$

$$\bar{\bar{\sigma}}_y/2\mu = \bar{\bar{v}}_0\left[\left\{(2-M^2)\xi^2+2\zeta^2\right\}^2\frac{e^{-s_1y}}{2M^2s_1\xi^2} - 2(\xi^2+\zeta^2)s_2\frac{e^{-s_2y}}{M^2\xi^2}\right],$$

$$\bar{\bar{\tau}}_{xy}/2\mu = i\xi\bar{\bar{v}}_0\left\{(2-M^2)\xi^2+2\zeta^2\right\}\frac{e^{-s_1y}-e^{-s_2y}}{M^2\xi^2},$$

$$\bar{\bar{\tau}}_{yz}/2\mu = i\zeta\bar{\bar{v}}_0\left\{(2-M^2)\xi^2+2\zeta^2\right\}\frac{e^{-s_1y}-e^{-s_2y}}{M^2\xi^2}. \tag{3.9}$$

Using the conditions (2.4.1) and (2.4.2) we obtain the following triple integral equations

$$\bar{\sigma}_y/2\mu = (2\pi)^{-1}\int_{-\infty}^{\infty}\bar{v}_0\,G(\xi,\zeta)\,e^{-i\xi x}\,d\xi = -\bar{p}(x,\zeta), \quad \text{for } a<|x|<b$$

and

$$\bar{v}_o = (2\pi)^{-1} \int_{-\infty}^{\infty} \bar{\bar{v}}_o \quad e^{-i\xi x} \, d\xi = 0, \text{ for } |x|>b, \ |x|<a. \tag{3.10}$$

with

$$G(\xi,\zeta) = \frac{1}{2M^2\xi^2 s_1} \left[\{(2-M^2)\xi^2+2\zeta^2)^2 - 4(\xi^2+\zeta^2)s_1 s_2 \right] \tag{3.11}$$

Taking p(x,z) as the even function of x, the solution may be assumed as

$$\bar{v}_o(x,\zeta) = \sum_{n=1}^{\infty} c_n(\zeta) \frac{(-1)^{n+1}}{n} \sin\left[n \cos^{-1}\left\{ \frac{a+b-2|x|}{b-a}\right\}\right], \text{ for } a\le|x|\le b$$

$$= 0, \text{ for } 0\le|x|<a, \ |x|>b, \tag{3.12}$$

where $c_n(\zeta)$ are the unknown functions to be determined.

Applying Fourier tranformation on (3.12) and using the result

$$\int_a^b \sin\left[n \cos^{-1}\left\{\frac{a+b-2x}{b-a}\right\}\right]\cos(\xi x)dx = (-1)^{n+1} \frac{n\pi}{\xi} \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)$$

we obtain

$$\bar{\bar{v}}_o(\xi,\zeta) = 2\pi\xi^{-1} \sum_{n=1}^{\infty} c_n(\zeta) \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right), \tag{3.13}$$

where $J_n()$ are Bessels functions.

Insertion of the expression (3.13) in the first equation of (3.10) yields

$$2\sum_{n=1}^{\infty} c_n(\zeta)\int_0^{\infty} \frac{G(\xi,\zeta)}{\xi} \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x)d\xi = -\bar{p}(x,\zeta),$$

$$\text{for } a<x<b. \tag{3.14}$$

Using the following results [Gradshteyn and Ryzhik (1965)]

$$\int_0^\infty \cos(a_1\xi)\, J_n(a_2\xi)d\xi = \frac{\cos(n\varepsilon)}{\sqrt{a_2^2 - a_1^2}}, \qquad \text{for } a_2>a_1>0$$

$$= \frac{a_2^n \sin(n\pi/2)}{\sqrt{a_1^2 - a_2^2}\,\left[a_1 + \sqrt{a_1^2 - a_2^2}\,\right]^n}, \quad \text{for } a_1>a_2>0,$$

and

$$\int_0^\infty \sin(a_1\xi)\, J_n(a_2\xi)d\xi = \frac{\sin(n\varepsilon)}{\sqrt{a_2^2 - a_1^2}}, \qquad \text{for } a_2>a_1>0$$

$$= \frac{a_2^n \cos(n\pi/2)}{\sqrt{a_1^2 - a_2^2}\,\left[a_1 + \sqrt{a_1^2 - a_2^2}\,\right]^n}, \quad \text{for } a_1>a_2>0,$$

where $\varepsilon = \sin^{-1}(a_1/a_2)$

in (3.14) we obtain,

$$\sum_{n=1}^{\infty} c_n(\zeta)\Bigg[\Bigg[\int_0^\infty\left\{\frac{G(\xi,\zeta)}{\xi} - \frac{G(\delta,\zeta)}{\delta}\right\}\Bigg[\cos(n\pi/2)\left\{\sin\left(\frac{a+b+2x}{2}\,\xi\right)+\sin\left(\frac{a+b-2x}{2}\,\xi\right)\right\}$$

$$- \sin(n\pi/2)\left\{\cos\left(\frac{a+b+2x}{2}\,\xi\right)+\cos\left(\frac{a+b-2x}{2}\,\xi\right)\right\}\Bigg]J_n\left(\frac{b-a}{2}\,\xi\right)d\xi + \frac{G(\delta,\zeta)}{\delta}\times$$

$$\times\left[\left(\frac{b-a}{2}\right)^n / \left[\sqrt{\left(\frac{a+b+2x}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2}\left\{\frac{a+b+2x}{2} + \sqrt{\left(\frac{a+b+2x}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2}\right\}^n\right] +\right.$$

$$+ \sin\left\{n\sin^{-1}\left\{\frac{a+b-2x}{b-a}\right\} - \frac{n\pi}{2}\right\} / \sqrt{\left(\frac{b-a}{2}\right)^2 - \left(\frac{a+b-2x}{2}\right)^2}\;\Bigg]\Bigg] = -\overline{p}(x,\zeta),$$

(3.16)

where

$$\underset{\xi \to \infty}{\mathrm{Lt}} \frac{G(\xi,\zeta)}{\xi} = \frac{G(\delta,\zeta)}{\delta} = \left[(2-M^2)^2 - 4\sqrt{1-M^2}\sqrt{1-M^2/\alpha^2}\right]/2M^2\sqrt{1-M^2/\alpha^2} \qquad (3.17)$$

since the function $\frac{G(\xi,\zeta)}{\xi} - \frac{G(\delta,\zeta)}{\delta}$ behaves as ξ^{-2} for large ξ, the semi-infinite integral on the left hand side of (3.16) can easily be evaluated by Filon's method.

To solve (3.16) for unknown coefficients $c_n(\zeta)$ we adopt the Schmidt method (1958) and write (3.16) as

$$\sum_{n=1}^{\infty} c_n(\zeta)\, F_n(\zeta,x) = -f(\zeta,x), \qquad \text{for } a<x<b, \qquad (3.18)$$

where $F_n(\zeta,x)$ and $f(\zeta,x) = \bar{p}(\zeta,x)$ are known functions. Let $H_n(\zeta,x)$'s be a set of orthogonal functions which satisfy

$$\int_a^b H_n(\zeta,x) H_m(\zeta,x)\, dx = N_n \delta_{nm} ,$$

where $$N_n = \int_a^b H_n^2(\zeta,x)\, dx \qquad (3.19)$$

Then $H_n(\zeta,x)$'s can be constructed from the functions $F_n(\zeta,x)$ in the following way

$$H_n(\zeta,x) = \sum_{i=1}^{\infty} \frac{C_{in}}{C_{nn}} F_i(\zeta,x) \qquad (3.20)$$

with C_{in} as the cofactor of the element e_{in} of D_n which is defined as

$$D_n = \begin{vmatrix} e_{11}, e_{12} \cdots\cdots\cdots e_{1n} \\ e_{21}, e_{22} \cdots\cdots\cdots e_{2n} \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ e_{n1}, e_{n2} \cdots\cdots\cdots e_{nn} \end{vmatrix}, \quad e_{in} = \int_a^b F_n(\zeta, x) F_i(\zeta, x)\, dx. \tag{3.21}$$

Now in terms of the set of orthogonal functions $H_n(\zeta, x)$, the function $-f(\zeta, x)$ can be expressed as

$$-f(\zeta, x) = \sum_{i=1}^{\infty} h_i H_i(\zeta, x) \tag{3.22}$$

Substituting values of $H_n(\zeta, x)$ from (3.20) into (3.22), we obtain from (3.18) after some rearrangement

$$\sum_{n=1}^{\infty} c_n(\zeta)\, F_n(\zeta, x) = \sum_{n=1}^{\infty} F_n(\zeta, x) \sum_{i=n}^{\infty} h_i \frac{C_{ni}}{C_{ii}} \tag{3.23}$$

Comparing the coefficients of $F_n(\zeta, x)$ from both sides of (3.23) we find

$$c_n = \sum_{i=n}^{\infty} h_i \frac{C_{ni}}{C_{ii}} \tag{3.24}$$

where

$$h_i = -\frac{1}{N_i} \int_a^b f(\zeta, x)\, H_i(\zeta, x)\, dx \tag{3.25}$$

4. Stress intensity factors and crack opening displacement

To evaluate the stress intensity factors at the vicinity of the crack ends we put $x=b+r\cos\theta$. $y=r\sin\theta$ for the stress intensity factor at the outer edge and $x=a-r\cos\theta$, $y=r\sin\theta$ for the stress intensity factor at the inner edge.

The required stress σ_θ given by

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta \cos\theta \tag{4.1}$$

is to be evaluated for small values of r.

Using asymptotic values of $J_n\left(\frac{b-a}{2}\xi\right)$ for large value of ξ, we obtain

$$\sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x) = \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4\pi\xi(b-a)}}\left[\cos\frac{n\pi}{2}\left\{\sin(b-x)\xi - \sin(x-a)\xi\right.\right.$$

$$\left.+\left[\cos(x-a)\xi - \cos(b-x)\xi\right]\tan\left(\frac{2n+1}{\pi}\right)\right\} - \sin\frac{n\pi}{2}\left\{\cos(b-x)\xi + \cos(x-a)\xi +\right.$$

$$\left.\left.+\left[\sin(x-a)\xi + \sin(b-x)\xi\right]\tan\left(\frac{2n+1}{\pi}\right)\right\}\right].$$

Further using the following results [Gradhsteyn and Ryzhik (1965)]

$$\int_0^\infty x^{\mu-1} e^{-\beta x} \sin(\delta x)\, dx = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\mu/2}} \sin\left[\mu \tan^{-1}\left(\frac{\delta}{\beta}\right)\right], \quad \mu>-1,\ \beta>0$$

$$\int_0^\infty x^{\mu-1} e^{-\beta x} \cos(\delta x)\, dx = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\mu/2}} \cos\left[\mu \tan^{-1}\left(\frac{\delta}{\beta}\right)\right], \quad \mu>0,\ \beta>0$$

it is found that for small values of r

$$\int_0^\infty e^{-\sqrt{1-q^2}\,\xi y} \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x)\,d\xi = -\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}} \times$$

$$\left[\cos(n\pi/2)\sqrt{\frac{(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}} + \sin(n\pi/2)\sqrt{\frac{-(-1)^n\cos\theta+\sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]$$

$$+ O(r^0), \quad \text{for } x>b \tag{4.2}$$

$$= \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}} \left[\cos(n\pi/2)\sqrt{\frac{(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}} - \right.$$

$$\left. - \sin(n\pi/2)\sqrt{\frac{-(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right] + O(r^o), \text{ for } x<a \tag{4.3}$$

and

$$\int_0^\infty e^{-\sqrt{1-q^2}\,\xi y} \sin\left[\frac{a+b}{2}\xi - \frac{n\pi}{2}\right] J_n\left(\frac{b-a}{2}\xi\right)\sin(\xi x)d\xi = -\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}} \times$$

$$\left[\cos(n\pi/2)\sqrt{\frac{-(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}} - \sin(n\pi/2)\sqrt{\frac{(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right]$$

$$+ O(r^o), \text{ for } x>b \tag{4.4}$$

$$= \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{4r(b-a)}} \left[\cos(n\pi/2)\sqrt{\frac{-(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}} + \right.$$

$$\left. + \sin(n\pi/2)\sqrt{\frac{(-1)^n\cos\theta + \sqrt{1-q^2\sin^2\theta}}{1-q^2\sin^2\theta}}\right] + O(r^o), \text{ for } x<a \tag{4.5}$$

Inserting (3.13) into (3.9) and taking inverse Fourier transform of (3.9) we obtain the stress intensity factor at x=b with the aid of (4.2)-(4.5) as

$$K_b = \frac{\sigma_\theta}{2\mu}\sqrt{r}\,\Big|_{r\to 0} = \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{b-a}} \left[\frac{2-M^2}{2M^2\sqrt{1-M^2/\alpha^2}}\, Q_1^+\left\{(2+M^2(1-2/\alpha^2))\sin^2\theta - \right.\right.$$

$$- (2 - M^2)\cos^2\theta\Big\} + \frac{2\sqrt{1-M^2}\,\cos 2\theta}{M^2}\,Q_2^+ - \frac{2-M^2}{M^2}\,(P_1^- - P_2^-)\,\sin 2\theta\Big]\times$$

$$\times\,\frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)e^{-i\zeta z}\,d\zeta \qquad (4.6)$$

and also the stress intensity factor at x=a is found to be

$$K_a = \frac{\sigma_\theta}{2\mu}\sqrt{r}\,\Big|_{r\to 0} = \sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n+1}{4}\right)\pi}{\sqrt{b-a}}\Big[\frac{-(2-M^2)}{2M^2\sqrt{1-M^2/\alpha^2}}\,Q_1^-\Big\{(2+M^2(1-2/\alpha^2))\sin^2\theta -$$

$$- (2 - M^2)\cos^2\theta\Big\} - \frac{2\sqrt{1-M^2}\,\cos 2\theta}{M^2}\,Q_2^- - \frac{2-M^2}{M^2}\,(P_1^+ - P_2^+)\,\sin 2\theta\Big]\times$$

$$\times\,\frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)e^{-i\zeta z}\,d\zeta \qquad (4.7)$$

where

$$\left.\begin{aligned} Q_i^{\pm} &= \Big[\cos(n\pi/2)\sqrt{q_i+(-1)^n\cos\theta} \pm \sin(n\pi/2)\sqrt{q_i-(-1)^n\cos\theta}\Big]/q_i \\ P_i^{\pm} &= \Big[\cos(n\pi/2)\sqrt{q_i-(-1)^n\cos\theta} \pm \sin(n\pi/2)\sqrt{q_i+(-1)^n\cos\theta}\Big]/q_i \end{aligned}\right\}\quad i=1,2$$

and

$$q_1 = \sqrt{1-M^2\alpha^{-2}\sin^2\theta}$$

$$q_2 = \sqrt{1-M^2\sin^2\theta}$$

It is to be noted that in (4.6) $\theta = \tan^{-1} y/(x-b)$ whereas in (4.7) it is given by $\theta = \tan^{-1} y/(a-x)$.

Taking Fourier inversion of (3.12) we obtain the crack surface displacement as

$$v_0(x,z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left[n \cos^{-1}\left\{ \frac{a+b-2|x|}{b-a}\right\}\right]\frac{1}{2\pi}\int_{-\infty}^{\infty} c_n(\zeta)e^{-i\zeta z}\,d\zeta$$

$$\text{for } a \le |x| \le b \qquad (4.8)$$

5. Numerical discussions

In order to evaluate the stress intensity factors and crack surface displacement we take the function $p(x,z)$ as

$$p(x,z) = \frac{P}{1+d^2z^2},$$

where d governs the distribution of the applied force and P is a constant. Numerical calculations have been done taking $\lambda=\mu$ and $d=1$. The semi infinite integral in (3.16) is evaluated by Filon's method as the integral converges rapidly because of the rapid decay of the function

$$\left\{\frac{G(\xi,\zeta)}{\xi} - \frac{G(\delta,\zeta)}{\delta}\right\}$$

with the increase in ξ. Adopting the first seven terms of the infinite series given in the left hand side of (3.18) we used the Schmidt method to determine the coefficients $c_n(\zeta)$. For the check of accuracy the value of $\sum_{n=1}^{7} c_n(\zeta)F_n(\zeta,x)/Pb$ and $-f(\zeta,x)/Pb$ are given in Table 1 for ζb = 0.0, 0.2, M=0.4 and for a/b = 0.3, 0.4.

Table 1.

ζb	a/b	x/b	$\sum_{n=1}^{7} c_n(\zeta)F_n(\zeta,x)/Pb$	$-f(\zeta,x)/Pb$
0.0	0.3	0.3	-3.140993	-3.140994
		0.4	-3.140995	
		0.5	-3.140993	
		0.6	-3.140996	
		0.7	-3.140991	
		0.8	-3.140994	
		0.9	-3.140993	
		1.0	-3.140992	
	0.4	0.4	-3.140995	
		0.5	-3.140994	
		0.6	-3.140994	
		0.7	-3.140994	
		0.8	-3.140994	
		0.9	-3.140995	
		1.0	-3.140994	
0.2	0.3	0.3	-2.572111	-2.572113
		0.4	-2.572113	
		0.5	-2.572111	
		0.6	-2.572116	
		0.7	-2.572110	
		0.8	-2.572113	
		0.9	-2.572108	
		1.0	-2.572106	
	0.4	0.4	-2.572114	
		0.5	-2.572114	
		0.6	-2.572114	
		0.7	-2.572113	
		0.8	-2.572113	
		0.9	-2.572113	
		1.0	-2.572113	

Table 2.

ζb	$c_1(\zeta)$	$c_2(\zeta)$	....	$c_7(\zeta)$
0.0	-0.165871×10^{1}	-0.923569×10^{-4}	....	-0.759039×10^{-8}
0.2	-0.135194×10^{1}	-0.734980×10^{-4}	....	0.105638×10^{-6}
0.4	-0.109342×10^{1}	-0.556495×10^{-4}	....	0.357814×10^{-6}
....	....	....	....	....
....	....	....	....	....
3.0	-0.578184×10^{-3}	-0.601254×10^{-7}	....	0.114694×10^{-5}
4.0	-0.182994×10^{-3}	0.883491×10^{-7}	....	0.659423×10^{-6}
5.0	-0.573139×10^{-4}	0.489839×10^{-7}	....	0.342023×10^{-6}
....	....	....	....	....
....	....	....	....	....
9.6	0.366305×10^{-5}	-0.816894×10^{-8}		-0.907244×10^{-7}
9.8	0.362848×10^{-5}	-0.829789×10^{-8}		-0.938769×10^{-7}
10.0	0.358409×10^{-5}	-0.843117×10^{-8}		-0.967438×10^{-7}

From Table 1 it is clear that the Schmidt method is carried out satisfactorily. The values of $c_n(\zeta)$ are given in Table 2 for M=0.4, a/b =0.4.

The variation of stress intensity factor at the outer edge and at the inner edge with M is shown in Fig.2. and Fig.3. respectively for $\theta = 0^\circ, 18^\circ, 36^\circ$ and a/b = 0.2, 0.3, 0.4. Fig.2 depicts the fact that the value of stress intensity factor at the outer edge decreases with the increase in the values of a/b, whereas from Fig.3 it is evident that the stress intensity factor at the inner edge is of an opposite character. It increases with the increases in the values of a/b.

The variations of stress intensity factors both at the inner edge and outer edge with z have been presented in Figs. 4-7 for different values of a/b, M and θ. The values of the stress intensity factor in all the cases are found to decrease gradually with the increase in the values of z, which is expected from physical stand point.

The variation of stress intensity factor corresponding to the circumferential stress σ_θ given by (4.1) with θ at both the crack tips has been shown in in Figs. 8-12 for different values of a/b and M.

It is known that there are several factors which contribute to crack curving and branching. One factor, of course, is based upon the criterion that a crack may propagate in a direction normal to the maximum tensile stress and it is interesting to note from Fig.8 and Fig.10, there is the possibility of curving and branching of the cracks at the outer edge at very low velocities of the cracks whereas from Fig.9, Fig.11 and Fig.12 it is clear that for a/b = 0.3, the crack tends to become curved at the inner edge for values of M about 0.65.

Finally the crack opening displacement in the plane z=0 has been shown by means of graphs in Figs. 15-16 for different values of a/b and M. The variation of crack opening displacement with z for some fixed values of M and a/b has been depicted in Figs.13-14.

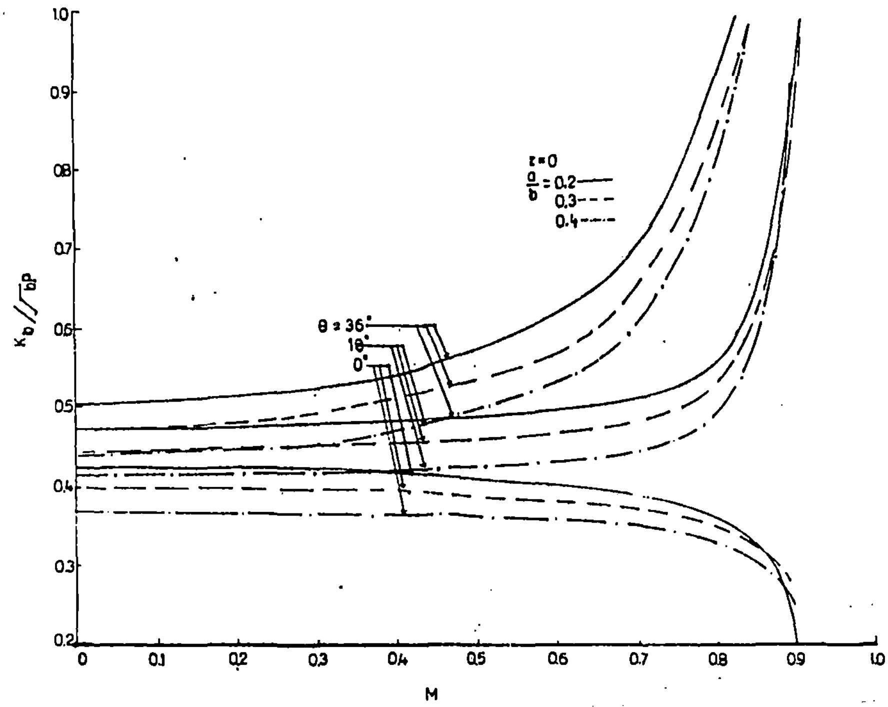

Fig. 2: Variation of stress intensity factor at the outer edge with M.

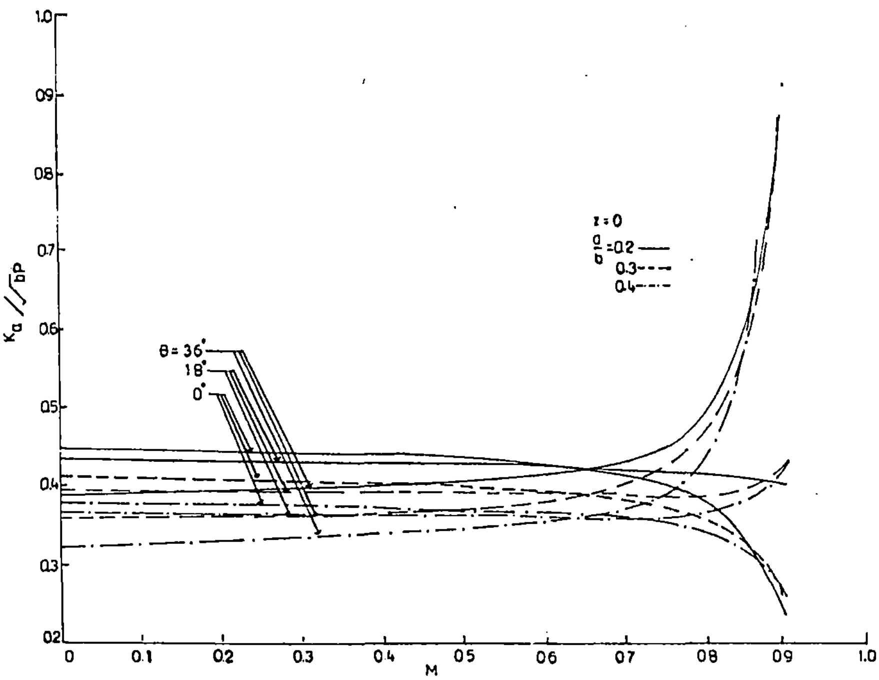

Fig. 3: Variation of stress intensity factor at the inner edge with M.

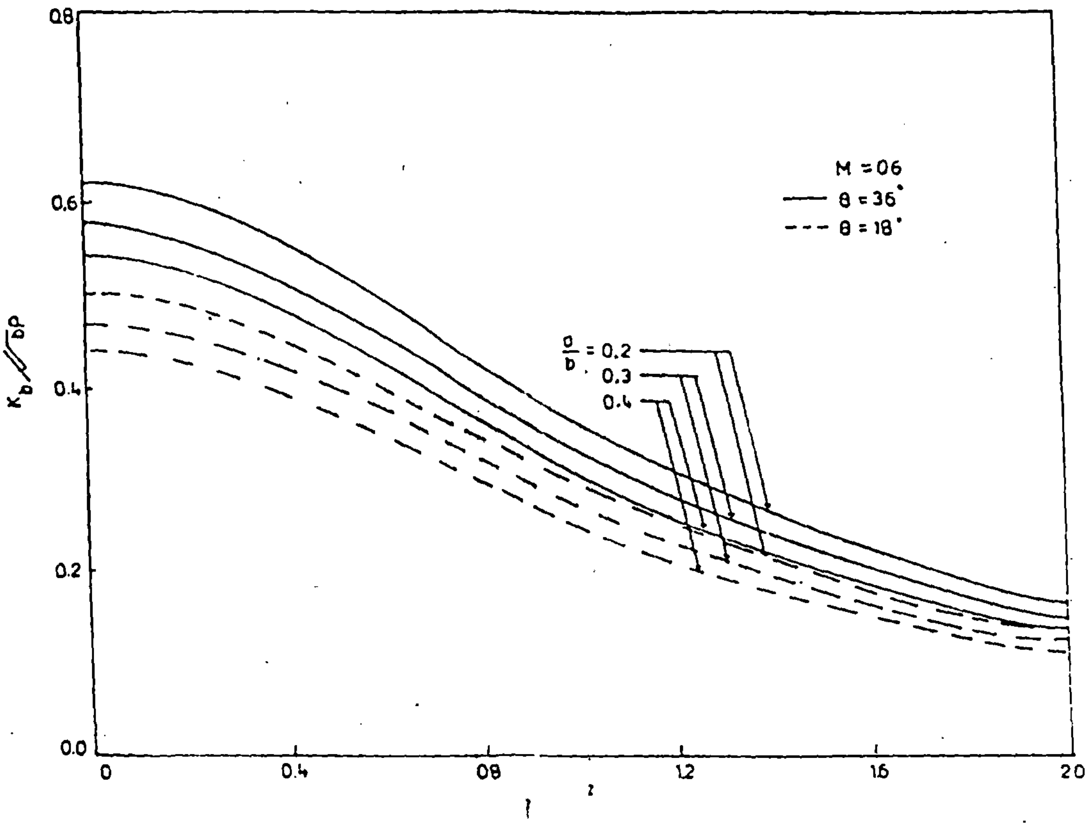

Fig. 4: Stress intensity factor at the outer edge vs z.

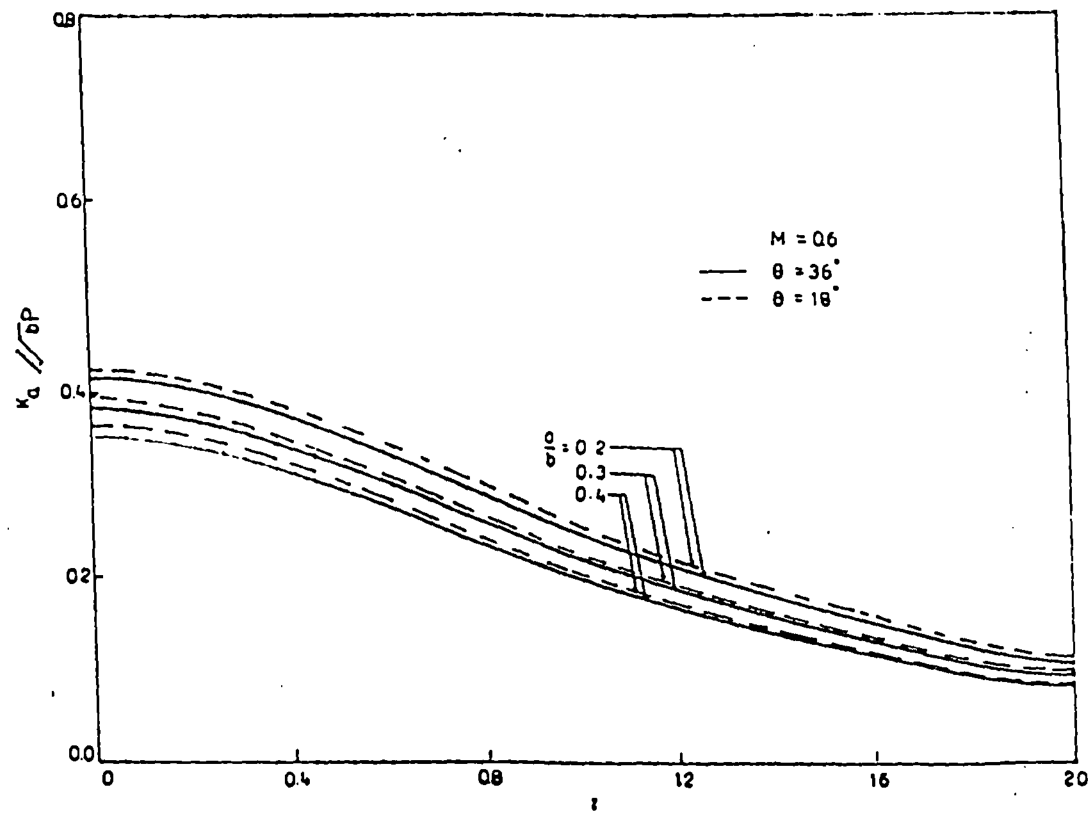

Fig.5: Stress intensity factor at the inner edge vs z.

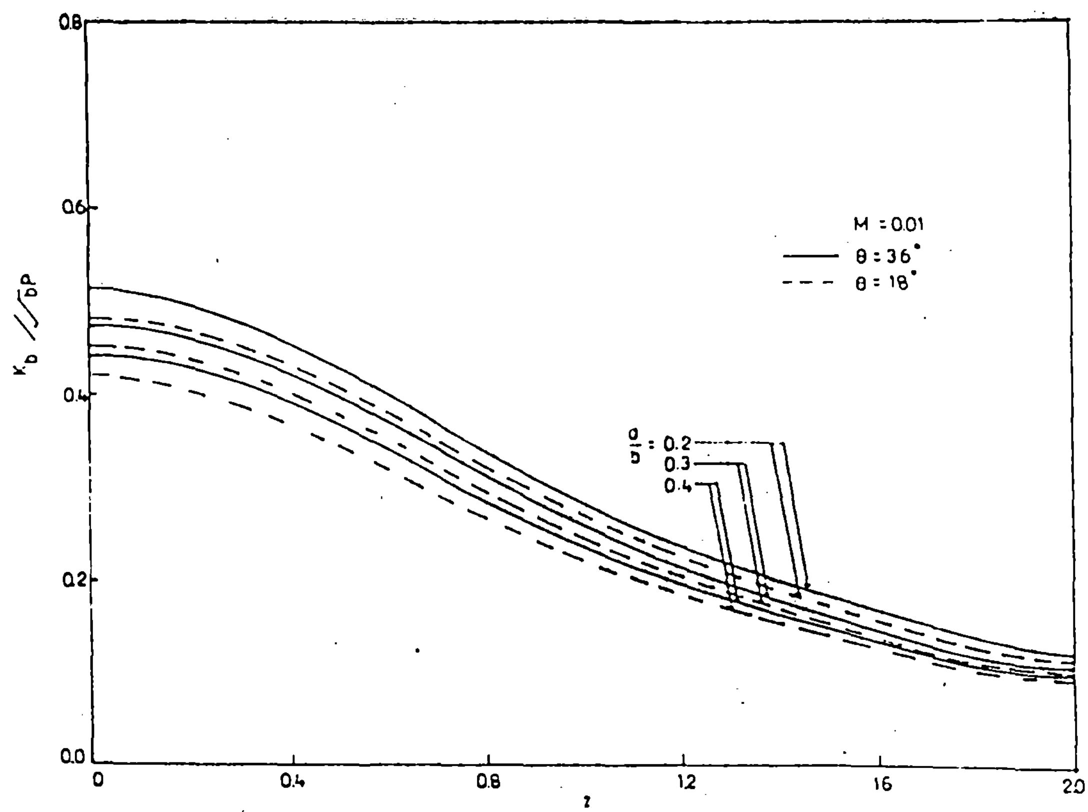

Fig. 6: Stress intensity factor at the outer edge vs z.

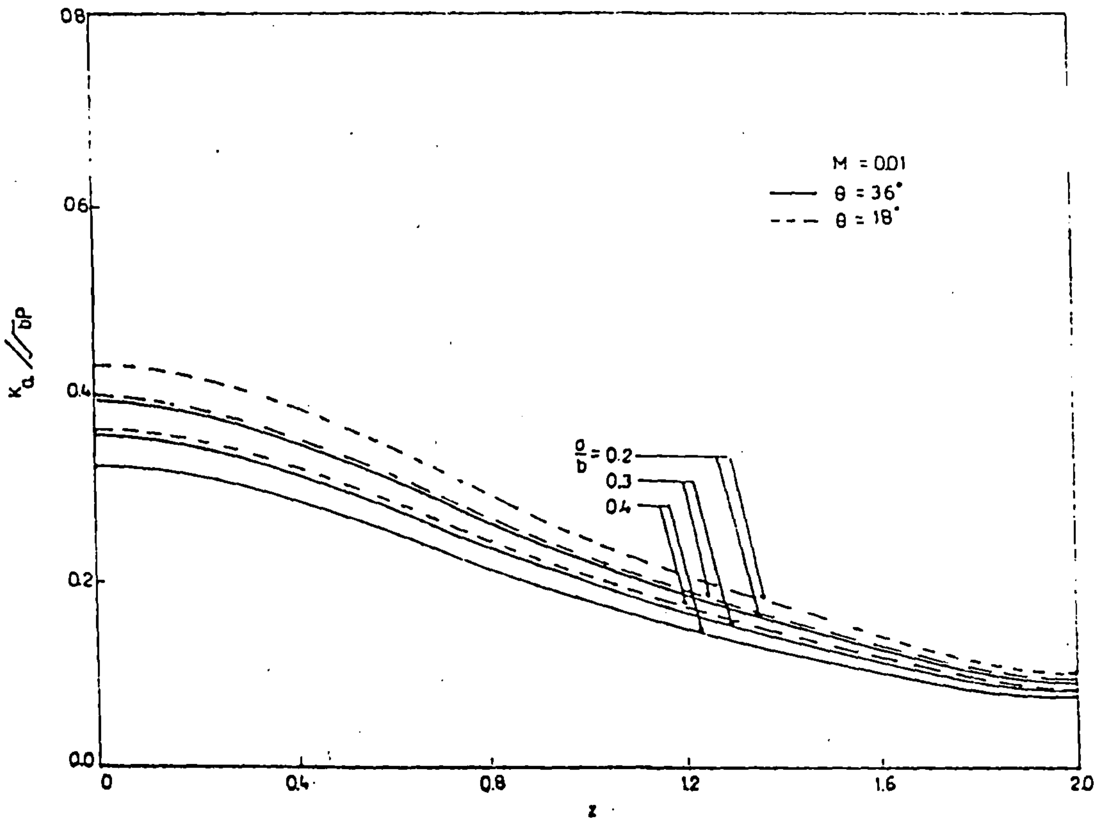

Fig. 7: Stress intensity factor at the inner edge vs z.

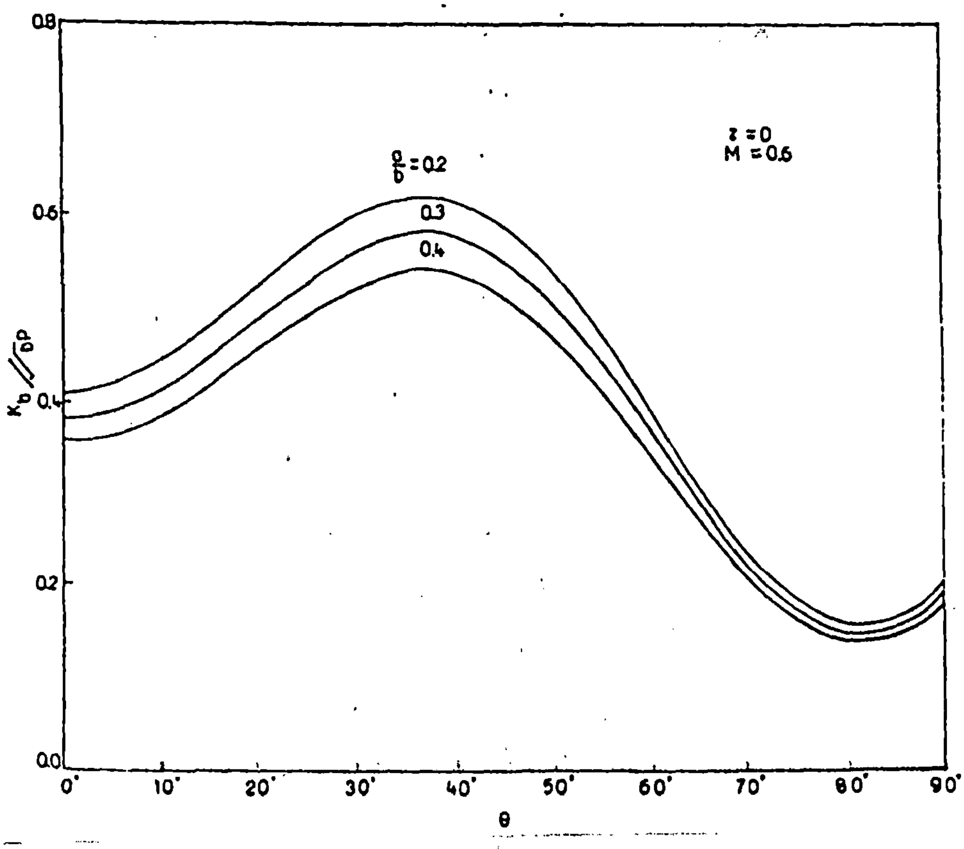

Fig. 8: Variation of stress intensity factor at the outer edge with θ.

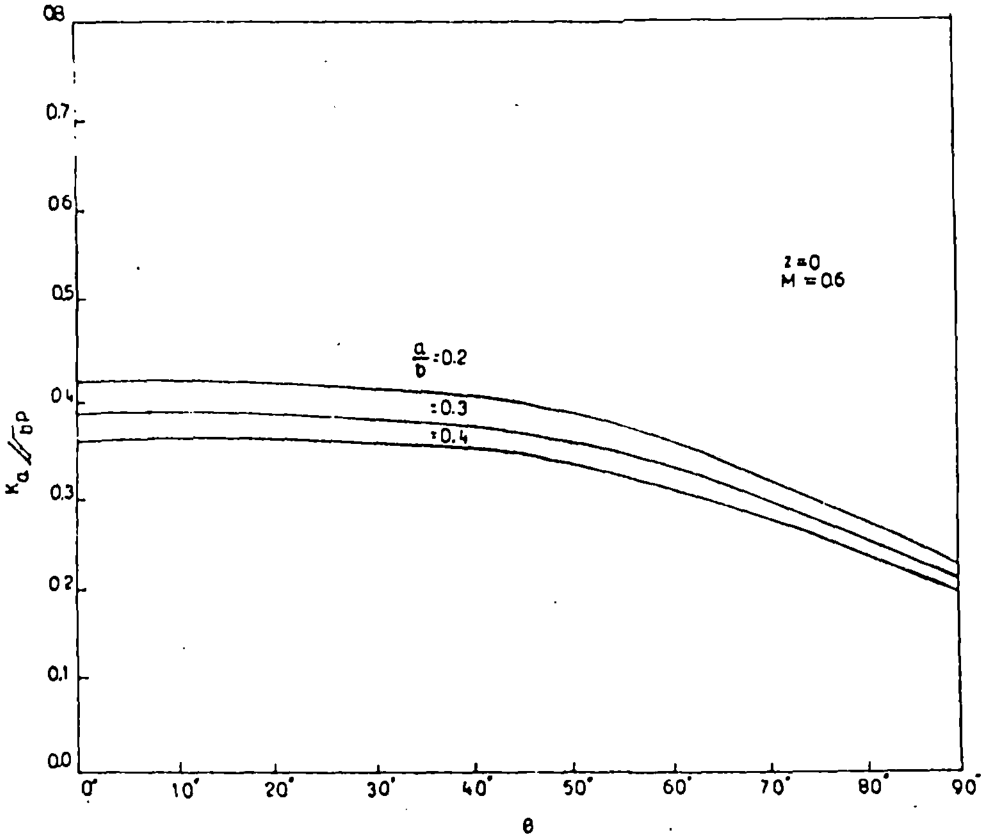

Fig. 9: Variation of stress intensity factor at the outer edge with θ.

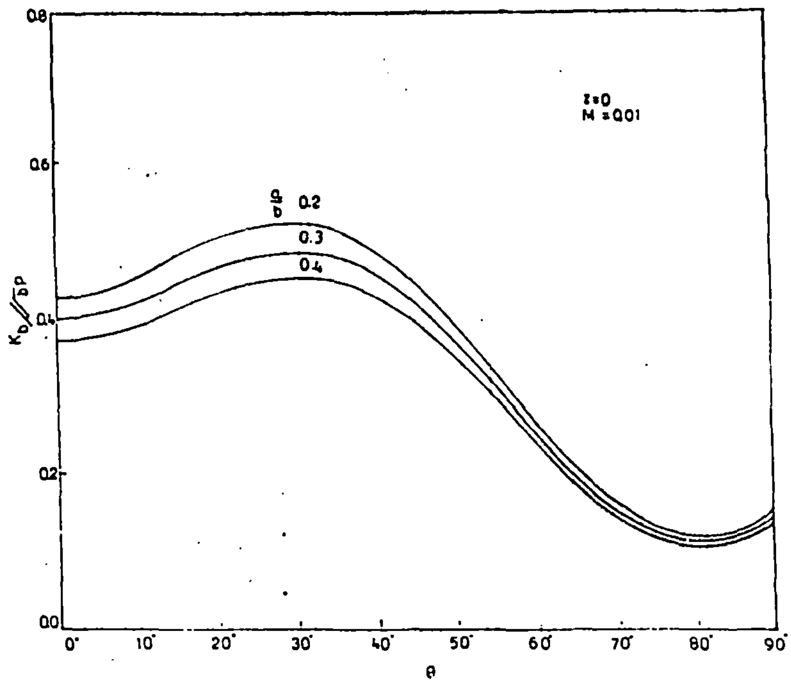

Fig.10: Variation of stress intensity factor at the outer edge with θ.

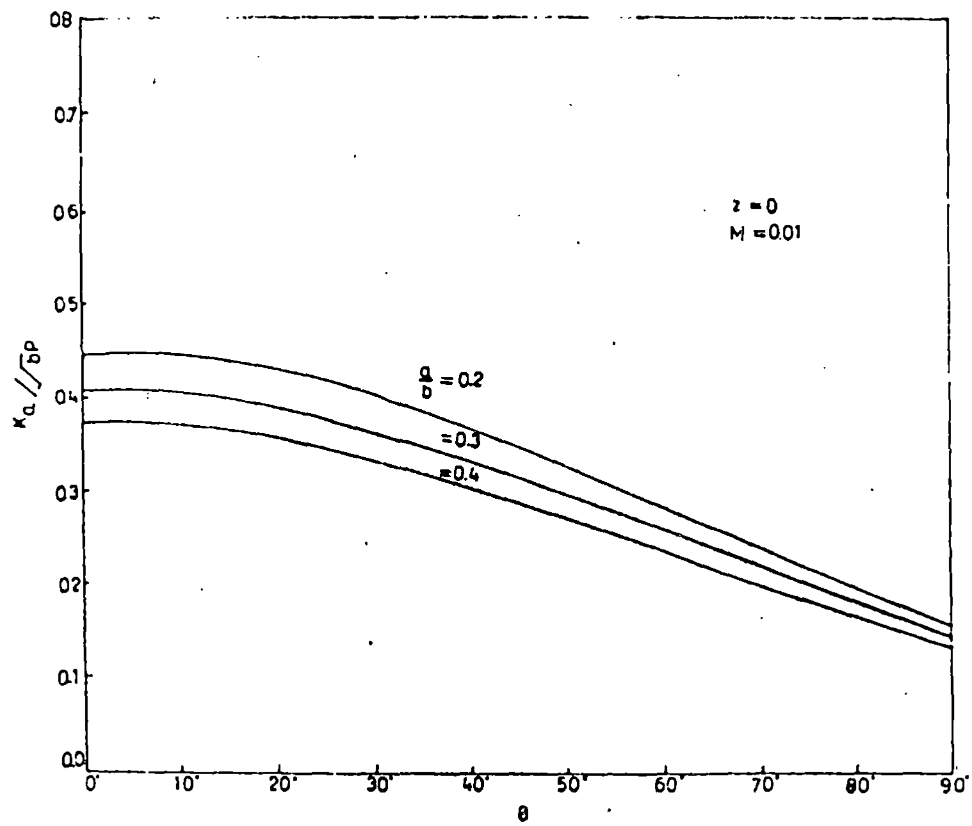

Fig.11: Variation of stress intensity factor at the outer edge with θ.

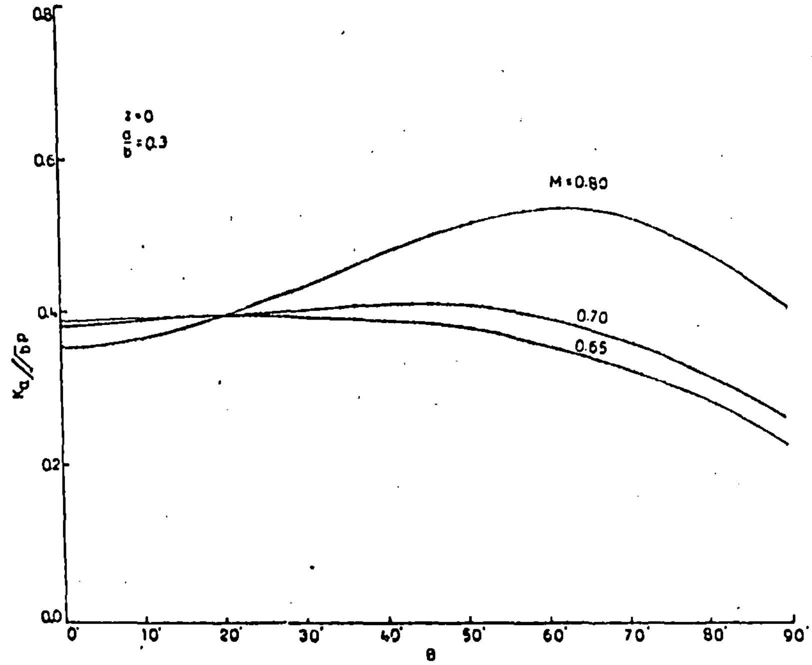

Fig.12: Variation of stress intensity factor at the outer edge with θ.

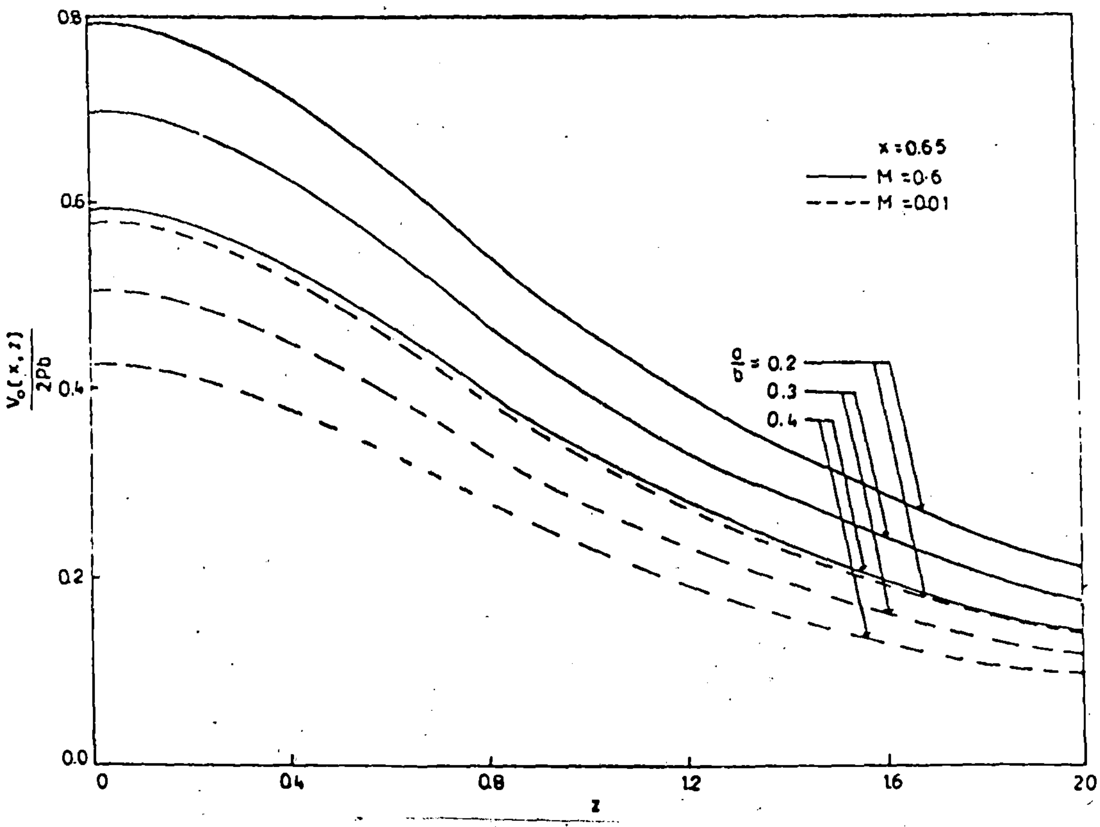

Fig.13: Crack opening displacement vs z.

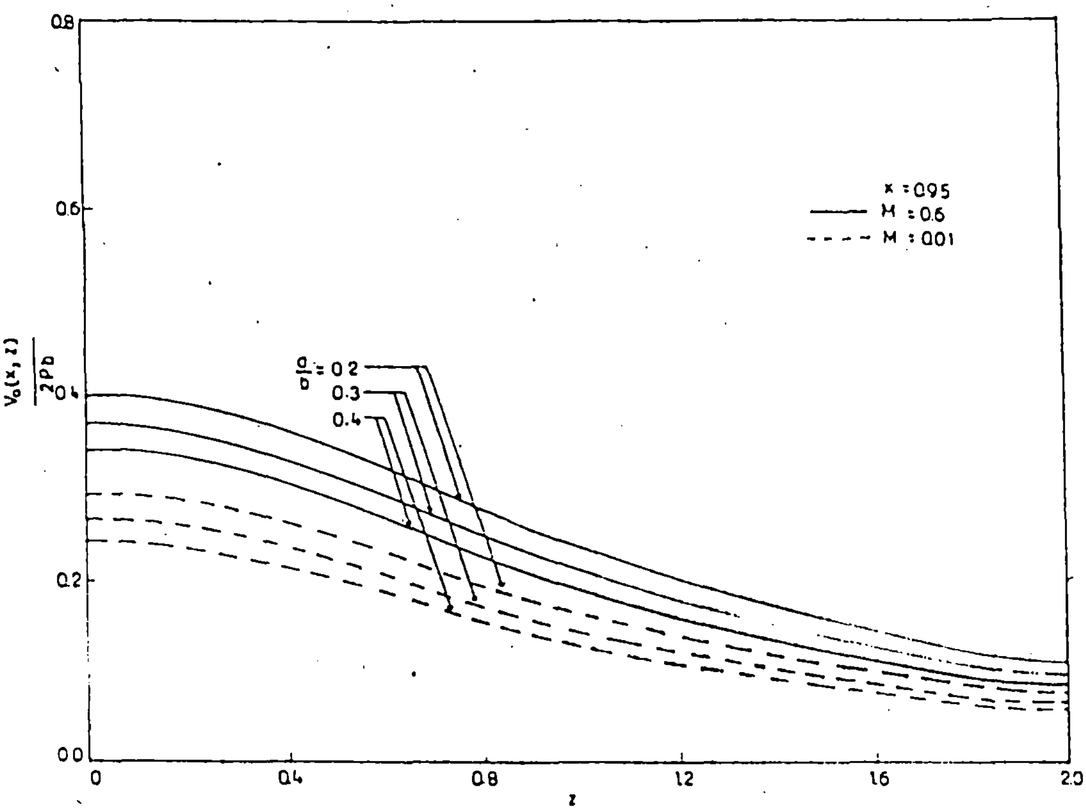

Fig.14: Crack opening displacement vs z.

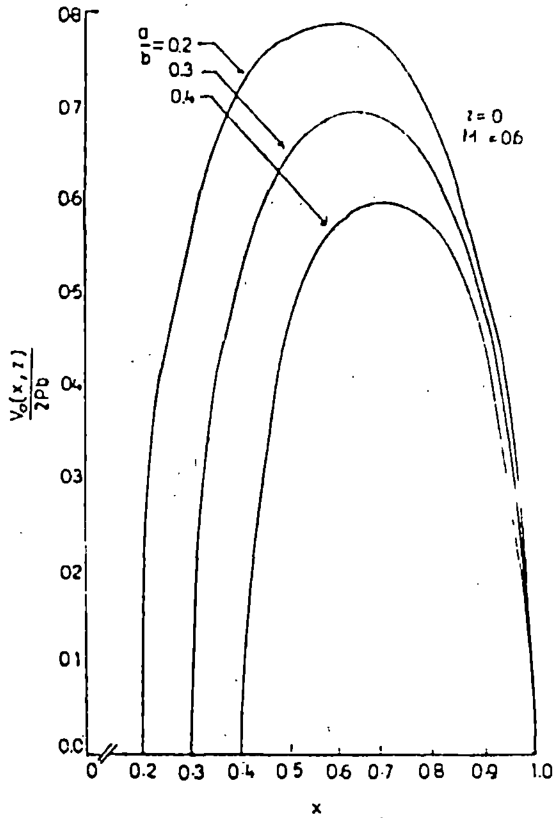

Fig.15: Variation of crack opening displacement with x.

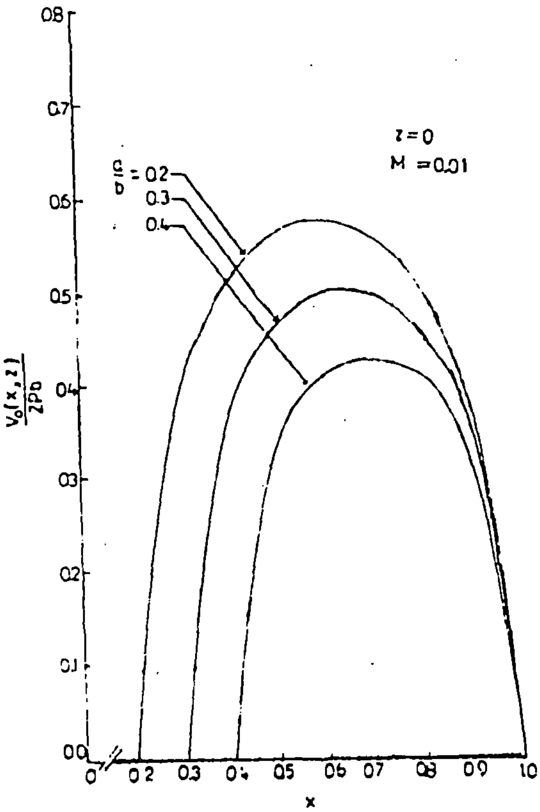

Fig.16: Variation of crack opening displacement with x.

THREE CO-PLANAR MOVING GRIFFITH CRACKS IN AN INFINITE ELASTIC MEDIUM

1. Introduction

In fracture mechanics, scattering of elastic waves by cracks of finite dimension in an infinite elastic medium has been investigated by several investigators.The problem of scattering of elastic wave from an interface crack was solved by Bostrom (1987). Srivastava et. al. (1980) solved the problem of interaction of anti-plane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff (1979). Itou (1980) solved problem of diffraction of anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. The scattering of time harmonic normally incident plane wave by two co-planar Griffith cracks was solved by Jain and Kanwal (1972). Itou (1978) also solved the problem of stress concentration around two co-planar Griffith cracks in an infinite elastic medium. Yoffe (1951) considered the problem of propagation of a crack of fixed length at a constant speed through a stretched isotropic elastic solid of infinite extent. The problem of diffraction of horizontal shear waves by a moving interface crack has been solved by Nishida et. al. (1984). Recently Kassir and Tse (1983) have solved the plane stress problem of a moving Griffith crack in an infinite orthotropic stressed medium by using integral transform technique and the same technique has been employed by De and Patra (1990) to solve Yoffe's problem in a stressed orthotropic strip of finite thickness.

As regards the crack problem, research has been restricted mainly to the case of single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal (1978)solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar cracks

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To the best knowledge of the author, the problem of stress distribution around three co-planar moving Griffith cracks in an infinite isotropic elastic medium has not been investigated so far. In this paper, two cases regarding stress distribution around three co-planar Griffith cracks in an infinite homogeneous, isotropic medium have been investigated. In the first case, cracks are assumed to be moving steadily along a fixed direction with constant velocity V. In the second case, the statical problem of determining the stress and displacement in an infinite homogeneous, isotropic medium weakened by three co-planar Griffith cracks has been considered. Using Fourier integral transform both the problems have been reduced to solving a set of four integral equations. Employing finite Hilbert transform technique (1968) and Cook's result (1970) the integral equations have been solved to derive crack opening displacement and stress intensity factors which are presented in the form of graphs.

2. Statement Of Problem I And Its Formulation

Consider an infinite homogeneous isotropic material weakened by three co-planar Griffith cracks, moving steadily at a constant velocity V in the X- direction referred to a fixed coordinate system (X,Y,Z) as shown in the Fig 1. In absence of body force equations of motion in terms of displacement are

$$(\lambda+2\mu)\left[u_{,XX} + v_{,XY} \right] + \mu\left[u_{,YY} - v_{,XY} \right] = \rho\, u_{,TT} \tag{2.1}$$

$$(\lambda+2\mu)\left[u_{,XY} + v_{,YY} \right] + \mu\left[v_{,XX} - u_{,XY} \right] = \rho\, v_{,TT} \tag{2.2}$$

where u,v denote the displacement components in X and Y directions and λ,μ are Lame's constants and $u_{,X}$ represents partial derivatives of u with respect to X.

For cracks moving with constant velocity V in the X- direction it is convenient to introduce the Galilean transformation

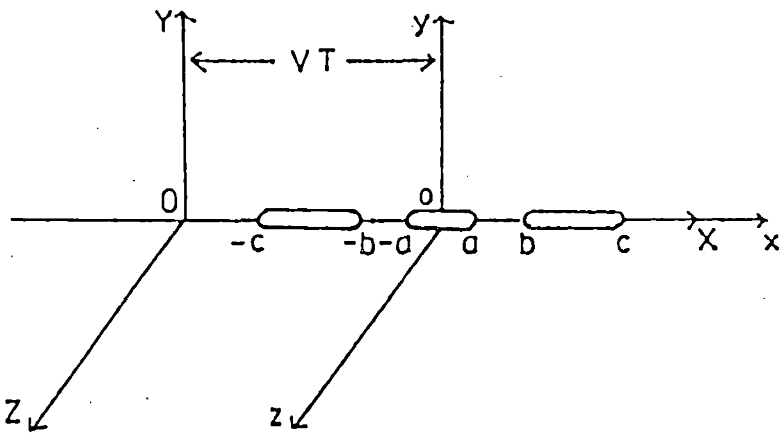

Fig. 1. Geometry and coordinate system.

$$x = X - VT,\ y = Y,\ z = Z,\ t = T \tag{2.3}$$

where (x,y,z) represents the translating coordinate system as shown in Fig 1.

Let the positions of the co-planar Griffith cracks referred to translating coordinate (x,y,z) be $-a<x<a$, $-c<x<-b$, $b<x<c$ on $y=0$.

In the moving coordinates, The equations of motion (2.1) and (2.2)become independent of time and take the form

$$(\lambda+2\mu-\rho V^2)u_{,xx} + (\lambda+\mu)v_{,xy} + \mu u_{,yy} = 0$$

$$(\lambda+2\mu)v_{,yy} + (\mu-\rho V^2)v_{,xx} + (\lambda+\mu)u_{,xy} = 0 \tag{2.4}$$

The cracks are assumed to be moving steadily in an infinite medium subjected to a homogeneous stress such that the state of stress at infinity is given by $\sigma_{yy}^{\infty} = p$, $\sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0$.

For symmetry about the x- axis, only a half plane need be considered.

The state conditions at $y=\infty$ can all be made zero by superposing the simple static problem $\sigma_{yy}^{\infty} = -p$, $\sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0$.

The boundary conditions of the resulting dynamic problem are in terms of moving coordinates.

$$\begin{aligned} v &= 0, && y=0,\ a\le|x|\le b,\ |x|\ge c \\ \sigma_{xy} &= 0, && |x|<\infty \\ \sigma_{yy} &= -p, && |x|<a,\ b<|x|<c \end{aligned} \tag{2.5}$$

In view of the symmetry of the proposed problem with respect to y-axis, we introduce

$$\bar{u}_s(\xi,y) = \int_0^{\infty} u(x,y)\sin(\xi x)\,dx$$

$$\overline{v}_c(\xi,y) = \int_0^\infty v(x,y)\cos(\xi x)\,dx$$

and
$$u(x,y) = \frac{2}{\pi}\int_0^\infty \overline{u}_s(\xi,y)\sin(\xi x)\,d\xi$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \overline{v}_c(\xi,y)\cos(\xi x)\,d\xi$$

in equation (2.4) so that equations given by (2.4) reduce to

$$\mu\overline{u}_{s,yy} - \xi(\lambda+\mu)\overline{v}_{c,y} - \xi^2(\lambda+2\mu-\rho V^2)\overline{u}_s = 0$$

$$(\lambda+2\mu)\overline{v}_{c,yy} + \xi(\lambda+\mu)\overline{u}_{s,y} - \xi^2(\mu-\rho V^2)\overline{v}_c = 0 \qquad (2.6)$$

Elimination of $\overline{u}_s$ from (2.6) yields the following ordinary differential equation

$$\left[\left\{\frac{d^2}{dy^2} - (1-M^2k^2)\xi^2\right\}\left\{\frac{d^2}{dy^2} - (1-M^2)\xi^2\right\}\right]\overline{v}_c = 0 \qquad (2.7)$$

where $M = V/c_2$, $k = c_2/c_1$.

The solution of the differential equation given by (2.7), for $y \geq 0$, is

$$\overline{v}_c(\xi,y) = A(\xi)\,e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\,e^{-\xi y\sqrt{1-M^2}} \qquad (2.8)$$

where the unknown functions $A(\xi)$ and $B(\xi)$ are to be determined using the boundary conditions of the proposed problem.

Employing (2.8) in equations (2.6) it can be shown that

$$\overline{u}_s(\xi,y) = \frac{A(\xi)}{\sqrt{1-M^2k^2}}\,e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\,B(\xi)\,e^{-\xi y\sqrt{1-M^2}}, \quad y\geq 0 \qquad (2.9)$$

Therefore, the stress components given by

$$\sigma_{yy} = \lambda(u_{,x} + v_{,y}) + 2\mu v_{,y}$$

$$\sigma_{xy} = \mu(u_{,y} + v_{,x}) \tag{2.10}$$

become

$$\sigma_{yy}(x,y) = -\frac{2\mu}{\pi}\int_0^\infty \xi\left[\frac{2-M^2}{\sqrt{1-M^2k^2}}A(\xi)e^{-\xi y\sqrt{1-M^2k^2}} + 2\sqrt{1-M^2}\;B(\xi)e^{-\xi y\sqrt{1-M^2}}\right].$$
$$.\cos(\xi x)\;d\xi$$

$$\sigma_{xy}(x,y) = -\frac{2\mu}{\pi}\int_0^\infty \xi\left[2A(\xi)e^{-\xi y\sqrt{1-M^2k^2}} + (2-M^2)B(\xi)e^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)d\xi$$

$$\tag{2.11}$$

with

$$u(x,y) = \frac{2}{\pi}\int_0^\infty\left[\frac{A(\xi)}{\sqrt{1-M^2k^2}}e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\;B(\xi)e^{-\xi y\sqrt{1-M^2}}\right]\sin(\xi x)d\xi$$

and

$$v(x,y) = \frac{2}{\pi}\int_0^\infty\left[A(\xi)\;e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)e^{-\xi y\sqrt{1-M^2}}\right]\cos(\xi x)\;d\xi \tag{2.12}$$

On account of symmetry with respect to y-axis the boundary conditions (2.5) can be rewritten as

$$v(x,0) = 0\;, \qquad x \in I_2, I_4 \tag{2.13}$$

$$\sigma_{xy}(x,0) = 0, \qquad 0<x<\infty \tag{2.14}$$

$$\sigma_{yy}(x,0) = -p, \quad x \in I_1, I_3 \tag{2.15}$$

where $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,\infty)$

Using the condition (2.14) in (2.11.2) it is found that $A(\xi)$, $B(\xi)$ are related by

$$B(\xi) = -\frac{2}{2-M^2}A(\xi) \tag{2.16}$$

With the help of the boundary condition (2.13), equation (2.12.2) reduces to

$$\int_0^\infty A(\xi)\cos(\xi x)\, d\xi = 0, \quad x \in I_2, I_4 \tag{2.17}$$

Substitution of (2.11.1) in (2.15) yields with the aid of (2.16)

$$\int_0^\infty \xi\, A(\xi)\cos(\xi x)\, d\xi = \frac{P\pi}{2\mu}, \quad x \in I_1, I_3 \tag{2.18}$$

where $\quad P = \frac{p}{K}, \quad K = \frac{(2- M^2)^2 - 4\sqrt{(1-M^2k^2)(1-M^2)}}{(2- M^2)\sqrt{1- M^2k^2}}$

3.Method Of Solution

In order to solve the set of four integral equations given in equations (2.17) and (2.18) let us take

$$A(\xi) = \frac{1}{\xi}\int_0^a h(s)\, \sin(\xi s)\, ds + \frac{1}{\xi}\int_b^c g(t^2)\, \sin(\xi t)\, dt \tag{3.1}$$

where h(s) and $g(t^2)$ are unknown functions to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from equation (3.1) in equation (2.17) and using the following result [Gradshteyn and Ryzhik (1965)]

$$\int_0^\infty \frac{\sin(\xi x)\cos(\xi y)}{\xi}\, d\xi = \begin{cases} \pi/2, & x>y>0 \\ \pi/4, & x=y>0 \\ 0, & y>x>0 \end{cases}$$

it is found that this choice of $A(\xi)$ leads to the equation

$$\int_b^c g(t^2)\, dt = 0 \tag{3.2}$$

Further substitution of $A(\xi)$ from equation (3.1) in (2.18.1) and use of the result [Gradshteyn and Ryzhik (1965)]

$$\int_0^\infty \frac{\sin(\xi x)\sin(\xi u)}{\xi}\, d\xi = \frac{1}{2}\log\left|\frac{u+x}{u-x}\right|$$

yields

$$\frac{d}{dx}\int_0^a h(s)\log\left|\frac{s+x}{s-x}\right| ds + \frac{d}{dx}\int_b^c g(t^2)\log\left|\frac{t+x}{t-x}\right| dt = \frac{\pi P}{\mu}, \quad x \in I_1$$

Rewriting this equation as

$$\int_0^a h(s)\log\left|\frac{s+x}{s-x}\right| ds = \pi\, F(x), \quad x \in I_1$$

where $\qquad F(x) = \int_0^x \left[\frac{P}{\mu} - \frac{2}{\pi}\int_c^d \frac{t g(t^2)}{t^2 - x'^2} dt\right] dx'$

and using Cook's result (1970) it is found that

$$h(s) = \frac{P}{\mu}\frac{s}{\sqrt{a^2 - s^2}} - \frac{2}{\pi}\frac{s}{\sqrt{a^2 - s^2}}\int_b^c \frac{\sqrt{t^2 - a^2}\; g(t^2)}{t^2 - s^2} dt \qquad (3.3)$$

where the result

$$\int_0^a \sqrt{a^2 - x^2}\frac{dx}{(s^2 - x^2)(t^2 - x^2)} = \frac{\pi}{2}\frac{\sqrt{t^2 - a^2}}{t}\frac{1}{t^2 - s^2}$$

has been used.

Substituting the value of h(s) from (3.3) in (3.1) and using the resulting value of $A(\xi)$ in the boundary condition (2.18.2) and using the results

$$\int_0^a \frac{1}{\sqrt{a^2 - s^2}}\frac{s^2\, ds}{(s^2 - x^2)(t^2 - s^2)} = \frac{\pi}{2}\left[\frac{t}{\sqrt{t^2 - a^2}} - \frac{x}{\sqrt{x^2 - a^2}}\right]\frac{1}{t^2 - x^2},$$

and $\qquad \int_0^a \frac{1}{\sqrt{a^2 - s^2}}\frac{s^2\, ds}{(s^2 - x^2)} = \frac{\pi}{2}\left[1 - \frac{x}{\sqrt{x^2 - a^2}}\right], \qquad$ for $x \in I_3$

it can be shown that $g(t^2)$ is solution of the singular integral equation

$$\int_b^c \sqrt{\frac{t^2 - a^2}{t^2 - x^2}}\, g(t^2)\, dt = \frac{\pi P}{2\mu}, \quad x \in I_3$$

Using finite Hilbert transform technique (1968) the solution of this integral equation is obtained with the aid of the result

$$\int_b^c \sqrt{\frac{c^2 - x^2}{x^2 - b^2}}\, \frac{x\, dx}{(x^2 - v^2)} = -\frac{\pi}{2}, \qquad \text{for } x \in I_3$$

as

$$g(t^2) = \frac{P}{\mu}\sqrt{\frac{t^2(t^2 - b^2)}{(t^2 - a^2)(c^2 - t^2)}} + \frac{t\, C_1}{\sqrt{(t^2 - a^2)(t^2 - b^2)(c^2 - t^2)}} \tag{3.4}$$

the constant C_1 is to be determined using the condition given by equation (3.2).

Next substituting the value of $g(t^2)$ from (3.4) in equation (3.3) and finally using the following results

$$\int_b^c \sqrt{\frac{t^2 - b^2}{c^2 - t^2}}\, \frac{t\, dt}{(t^2 - s^2)} = \frac{\pi}{2}\left[1 - \sqrt{\frac{b^2 - s^2}{c^2 - s^2}}\right]$$

$$\int_b^c \frac{t\, dt}{(t^2 - s^2)\sqrt{(t^2 - b^2)(c^2 - t^2)}} = \frac{\pi}{2\sqrt{(c^2 - s^2)(b^2 - s^2)}} \quad \text{for } s \in I_1$$

h(s) is derived in the form

$$h(s) = \frac{P}{\mu}\sqrt{\frac{s^2(b^2 - s^2)}{(a^2 - s^2)(c^2 - s^2)}} - \frac{s\, C_1}{\sqrt{(a^2 - s^2)(b^2 - s^2)(c^2 - s^2)}} \tag{3.5}$$

Now insertion of (3.4) in condition (3.2) yields

$$C_1 = -\frac{P}{\mu}\left[(c^2 - a^2)\frac{E(\pi/2, l)}{F(\pi/2, l)} - (b^2 - a^2)\right] \tag{3.6}$$

where $F(\phi, l)$ and $E(\phi, l)$ are elliptic integrals of first kind and second kind respectively and $l = \sqrt{\frac{c^2 - b^2}{c^2 - a^2}}$

The relevant displacement and stress components in the plane of crack can now be shown to be given by

$$v(x,0) = \int_x^a h(s)\, ds\,, \qquad 0 \le x \le a$$

$$= \int_x^c g(t^2)\, dt\,, \qquad b \le x \le c \tag{3.7}$$

and

$$[\sigma_{yy}(x,o)]_{a<x<b} = \frac{2\mu K}{\pi}\left[\int_0^a \frac{sh(s)}{x^2 - s^2}\, ds - \int_b^c \frac{tg(t^2)}{t^2 - x^2}\, dt\right]$$

$$[\sigma_{yy}(x,o)]_{x>c} = \frac{2\mu K}{\pi}\left[\int_0^a \frac{sh(s)}{x^2 - s^2}\, ds + \int_b^c \frac{tg(t^2)}{x^2 - t^2}\, dt\right] \tag{3.8}$$

Insertion of the values of $h(s)$ and $g(t^2)$ as given by the equations (3.5) and (3.4) in the expressions (3.8) yields after some algebraic manipulation,

$$[\sigma_{yy}(x,o)]_{a<x<b} = \frac{2\mu K}{\pi}\left[F_1(x) - F_2(x) + F_3(x) - F_5(x) - F_6(x)\right]$$

$$[\sigma_{yy}(x,o)]_{x>c} = \frac{2\mu K}{\pi}\left[F_1(x) - F_2(x) + F_4(x) - F_5(x) + F_6(x)\right] \tag{3.9}$$

where

$$F_1(x) = \left[\frac{P}{\mu}(b^2-a^2) - C_1 \right]\left[\sqrt{\frac{x^2}{x^2-a^2}} - 1 \right] \frac{\pi}{2\sqrt{(c^2-a^2)(b^2-a^2)}}$$

$$F_2(x) = \int_0^a \left[\frac{P}{\mu}(c^2-b^2) - C_1 \frac{2u^2-b^2-c^2}{b^2-u^2} \right] \frac{g_1(u,x)}{c^2-u^2}\, du$$

$$F_{3,4}(x) = \left\{ \frac{P}{\mu}\left[\sqrt{\frac{b^2-x^2}{c^2-x^2}} - 1 \right] \mp \frac{C_1}{\sqrt{(c^2-x^2)(b^2-x^2)}} \right\} \frac{\pi c}{2\sqrt{c^2-a^2}}$$

$$F_5(x) = \frac{P}{\mu}a^2 \int_b^c \left[\tan^{-1}\sqrt{\frac{v^2-b^2}{c^2-v^2}} - \sqrt{\frac{b^2-x^2}{c^2-x^2}} \tan^{-1}\sqrt{\frac{(c^2-x^2)(v^2-b^2)}{(b^2-x^2)(c^2-v^2)}} \right] \frac{dv}{\sqrt{(v^2-a^2)^3}}$$

$$F_6(x) = \frac{a^2 C_1}{\sqrt{(c^2-x^2)(b^2-x^2)}} \int_b^c \frac{\tan^{-1}\sqrt{\frac{(u^2-b^2)(x^2-c^2)}{(c^2-u^2)(x^2-b^2)}}}{\sqrt{(u^2-a^2)^3}}\, du$$

$$g_1(u,x) = \frac{u}{\sqrt{(b^2-u^2)(c^2-u^2)}} \left[\sin^{-1}\left(\frac{u}{a}\right) - \frac{x}{\sqrt{x^2-a^2}} \tan^{-1}\sqrt{\frac{(x^2-a^2)u^2}{(a^2-u^2)x^2}} \right] \quad (3.10)$$

The dynamic stress intensity factors are given by

$$N_a = \underset{x \to a+}{\mathrm{Lt}} \sqrt{2(x-a)} \left[\sigma_{yy}(x,0) \right]_{a<x<b}$$

$$N_b = \underset{x \to b-}{\mathrm{Lt}} \sqrt{2(b-x)} \left[\sigma_{yy}(x,0) \right]_{a<x<b}$$

$$N_c = \underset{x \to c+}{\mathrm{Lt}} \sqrt{2(x-c)} \left[\sigma_{yy}(x,0) \right]_{x>c} \quad (3.11)$$

Employing (3.9) in (3.11) it can be shown that

$$N_a = p\sqrt{a}\sqrt{\frac{c^2-a^2}{b^2-a^2}}\,\frac{E(\pi/2,l)}{F(\pi/2,l)}$$

$$N_b = \frac{p\sqrt{b}}{\sqrt{(c^2-b^2)(b^2-a^2)}}\left[(c^2-a^2)\frac{E(\pi/2,l)}{F(\pi/2,l)} - (b^2-a^2)\right]$$

$$N_c = p\sqrt{c}\sqrt{\frac{c^2-a^2}{c^2-b^2}}\left[1 - \frac{E(\pi/2,l)}{F(\pi/2,l)}\right]$$

Now using the values of h(s) and $g(t^2)$ from (3.5) and (3.4) in the expressions given by equations (3.7) displacement on the cracks are obtained as

$$[v(x,0)]_{0\le x\le a} = \frac{p}{\mu}\sqrt{c^2-a^2}\,F(\beta,l)\left[\frac{E(\pi/2,l)}{F(\pi/2,l)} - \frac{E(\beta,l)}{F(\beta,l)}\right] + \frac{p}{\mu}\,\frac{\sqrt{(c^2-x^2)(a^2-x^2)}}{\sqrt{b^2-x^2}}$$

$$[v(x,0)]_{b\le x\le c} = \frac{p}{\mu}\sqrt{c^2-a^2}\,F(\lambda,l)\left[\frac{E(\lambda,l)}{F(\lambda,l)} - \frac{E(\pi/2,l)}{F(\pi/2,l)}\right]$$

where $\sin\lambda = \sqrt{\dfrac{c^2-x^2}{c^2-b^2}}$ and $\sin\beta = \sqrt{\dfrac{a^2-x^2}{b^2-x^2}}$.

It is interesting to note that the crack opening displacements depend on the crack velocity V but in the plane of the cracks the stresses and stress intensity factors are independent of the velocity of the moving cracks in an infinite elastic medium.

4. Statement Of Problem II And Its Formulation

In this case, consider an infinite homogeneous isotropic material with three coplanar Griffith cracks, located at $Y=0, -a\le X\le a$, $b\le |X|\le c$ and subjected to uniform internal pressure q. In absence of body force equation of equilibrium in terms of displacement are

$$(\lambda+2\mu)[\, u_{,XX} + v_{,XY}\,] + \mu[\, u_{,YY} - v_{,XY}\,] = 0$$

and $$(\lambda+2\mu)[\, u_{,XY} + v_{,YY}\,] + \mu[\, v_{,XX} - u_{,XY}\,] = 0 \qquad (4.1)$$

Since the problem exhibits a state of symmetry about $Y = 0$, attention can be given to a single half-space occupying the region $Y\ge 0$.

The equations (4.1) are to be solved subject to the boundary conditions

$$v(X,0) = 0, \qquad a\le |X|\le b, |X|\ge c \qquad (4.2)$$

$$\sigma_{XY}(X,0) = 0, \qquad -\infty < X < \infty \qquad (4.3)$$

$$\sigma_{YY}(X,0) = -q, \quad |X|\le a, \; b\le |X|\le c \qquad (4.4)$$

In view of the boundary conditions, appropriate integral solutions of equation (4.1) are

$$u(X,Y) = \frac{2}{\pi}\int_0^\infty \left[C(\xi) + D(\xi)\left\{ Y - \frac{1}{\xi}\,\frac{\lambda+3\mu}{\lambda+\mu} \right\} \right] e^{-\xi Y} \sin(\xi X)\, d\xi$$

and $$v(X,Y) = \frac{2}{\pi}\int_0^\infty \left[C(\xi) + Y\, D(\xi) \right] e^{-\xi Y} \cos(\xi X)\, d\xi \qquad (4.5)$$

Therefore,

$$\sigma_{YY}(X,Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{ Y\xi - \frac{\mu}{\lambda+\mu} \right\} D(\xi) \right] e^{-\xi Y} \cos(\xi X)\, d\xi$$

$$\sigma_{XY}(X,Y) = -\frac{4\mu}{\pi}\int_0^\infty\left[\xi C(\xi) + \left\{Y\xi - \frac{\lambda+2\mu}{\lambda+\mu}\right\}D(\xi)\right]e^{-\xi Y}\sin(\xi X)\; d\xi \qquad (4.6)$$

It may be noted that the displacement and stress components given by (4.5) and (4.6) can not be derived from the corresponding expressions of the dynamic problem given in (2.11) and (2.12) on setting $M = 0$.

The functions $C(\xi)$ and $D(\xi)$ are to be determined from the boundary conditions (4.2)-(4.4), which yield

$$C(\xi) = \frac{1}{\xi}\,\frac{\lambda+2\mu}{\lambda+\mu}\,D(\xi) \qquad (4.7)$$

and the following set of four integral equations

$$\int_0^\infty C(\xi)\cos(\xi X)\; d\xi = 0, \qquad X \in I_2, I_4 \qquad (4.8)$$

$$\int_0^\infty \xi\; C(\xi)\cos(\xi X)\; d\xi = \frac{Q\pi}{2\mu}, \qquad X \in I_1, I_3 \qquad (4.9)$$

where

$Q = \dfrac{(\lambda+2\mu)}{2(\lambda+\mu)}\, q$ and I_j $(j=1,2,3,4)$ are the intervals defined earlier in problem I.

5. Method Of Solution And Quantities Of Physical Interest

Integral equations given by (4.8) and (4.9) are found to be the same as given by equations (2.17) and (2.18) with the exception that P is replaced by Q. Therefore, the same technique as that used in problem I can be employed to obtain

$$\left[v(X,0)\right]_{0\le X\le a} = \frac{Q}{\mu}\sqrt{c^2 - a^2}\; F(\beta',1)\left[\frac{E(\pi/2,1)}{F(\pi/2,1)} - \frac{E(\beta',1)}{F(\beta',1)}\right]$$

$$+ \frac{Q}{\mu}\,\frac{\sqrt{(c^2 - X^2)(a^2 - X^2)}}{\sqrt{b^2 - X^2}}$$

$$[v(X,0)]_{b\le X\le c} = \frac{Q}{\mu}\sqrt{c^2-a^2}\;F(\lambda',1)\left[\frac{E(\lambda',1)}{F(\lambda',1)} - \frac{E(\pi/2,1)}{F(\pi/2,1)}\right] \qquad (5.1)$$

where $\sin\lambda' = \sqrt{\dfrac{c^2-X^2}{c^2-b^2}}$ and $\sin\beta' = \sqrt{\dfrac{a^2-X^2}{b^2-X^2}}$.

Stresses in the regions $a<X<b$, $X>c$ are found to be the same as that given in (3.9), the only change being that P is to replaced by Q.

6. Numerical Results and Discussions

Numerical results for the stress intensity factors and crack opening displacement, defined as $\Delta v(x,0) = v(x,0^+) - v(x,0^-)$, for different values of the parameters and $\lambda = \mu$ are presented in this section. Numerical calculations have been carried out for both the dynamic and static problems. As the crack velocity is less than Rayleigh wave velocity, it is reasonable to take the value of M less than 0.9194 .

Problem I: Variations of crack opening displacement for different values of crack speed, crack lengths and the separating distance between the cracks have been plotted in Figures 2-4. It is interesting to note from the Fig.2 that crack opening displacement on both the cracks decreases with the increase in the value of M at the onset and takes its minimum value at M=0.7415 ,after which it increases with the increase in the value of M. It has also been depicted in figures 3-4 that on each of the cracks,crack opening displacement decreases as the crack length decreases.

It has been mentioned earlier that the stress intensity factors at the crack tips are independent of crack speed and are found to depend on the crack lengths and the separating distance between the cracks. Variation of stress intensity factors with a/b for different values of c/b, and that with b/a for different values c/a are plotted in Fig.5 and Fig.6 respectively.

It has been found form these graphs that when the separating distance between the inner crack and outer pair of cracks decreases the variations of stress intensity factors at the tips x=a and x=b become more prominent than that at the edge x=c. Fig.7 shows that the stress intensity factors at the edges of the inner crack and outer pair of cracks increases as the length of the outer pair of cracks increases keeping the separating distance between the inner crack and outer pair of cracks fixed.

Problem II: Fig.8 shows the variations of crack opening displacement for different values of the parameters a/b, c/b, They exhibit that crack opening displacement on a crack of fixed length increases with the increase in the length of the other crack as expected from physical stand point.

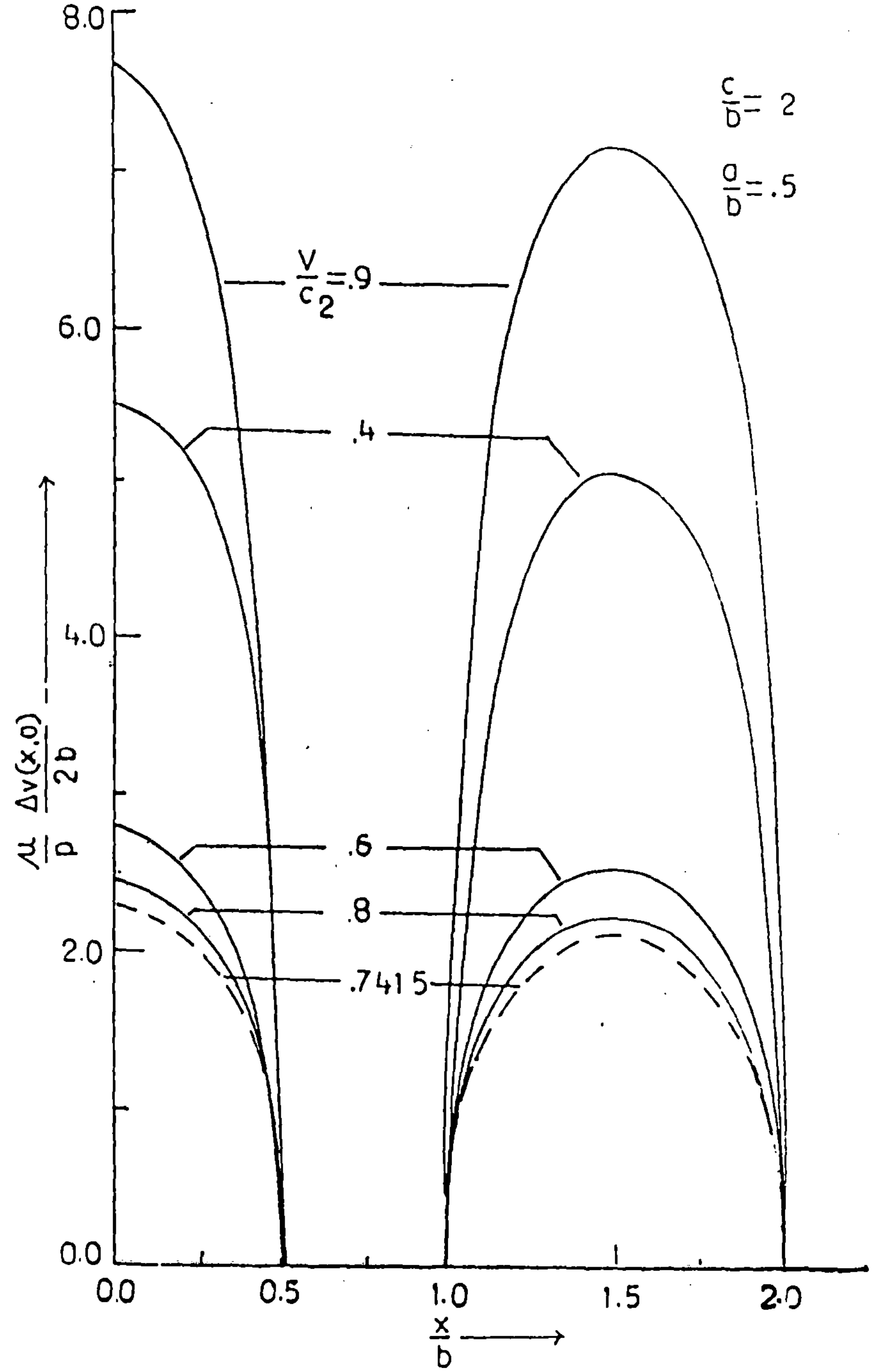

Fig. 2. Variation of crack opening displacement with x/b on both the cracks for the problem I.

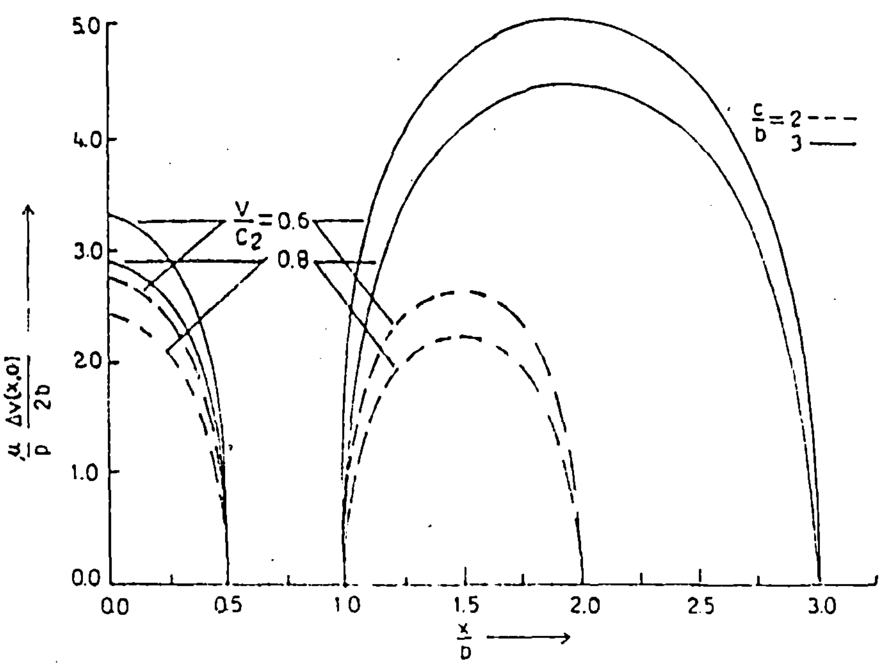

Fig. 3. Variation of crack opening displacement with x/b on both the cracks for the problem I.

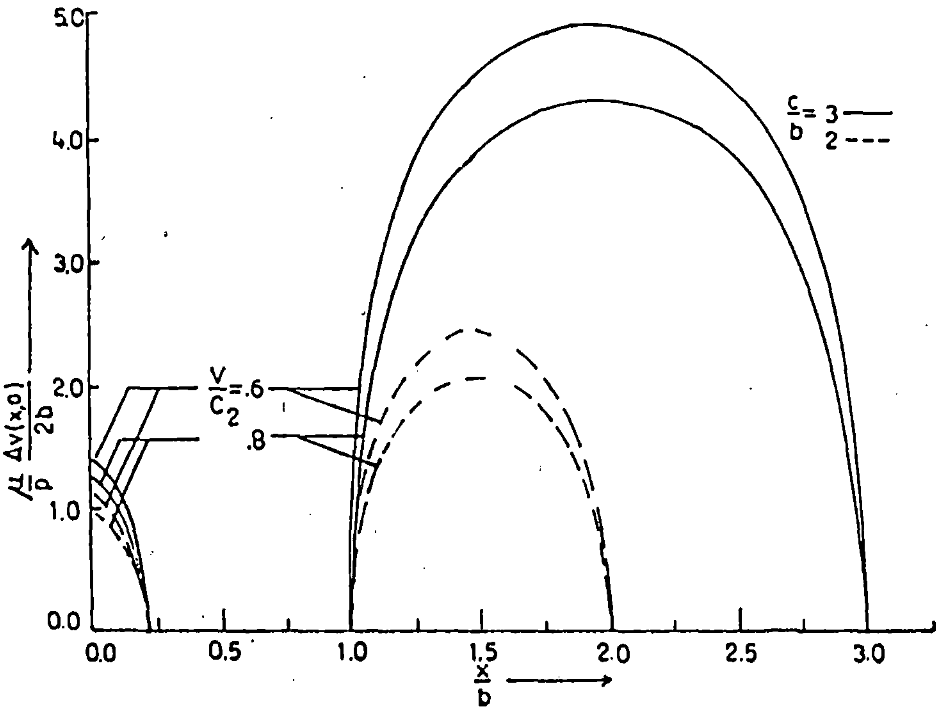

Fig. 4. Variation of crack opening displacement with x/b on both the cracks for the problem I.

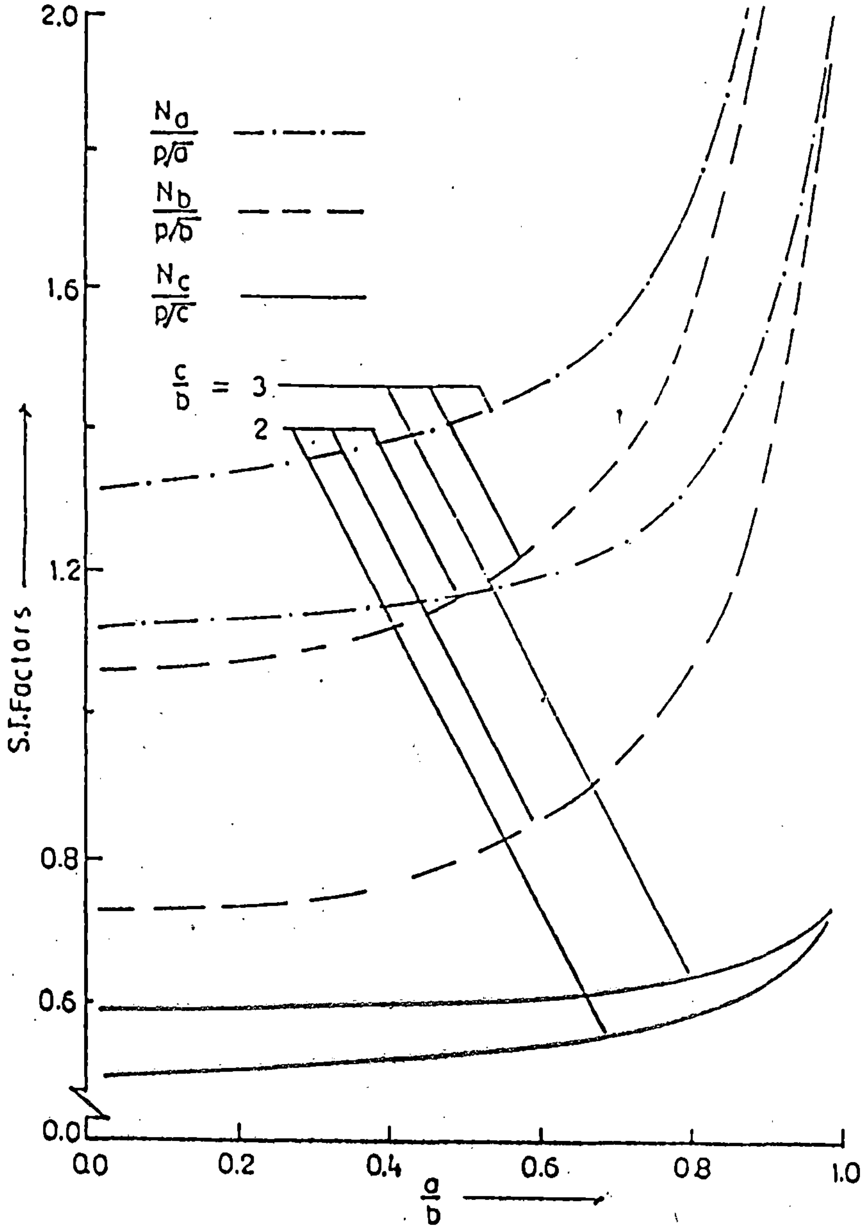

Fig. 5. Stress intensity factors Vs. a/b.

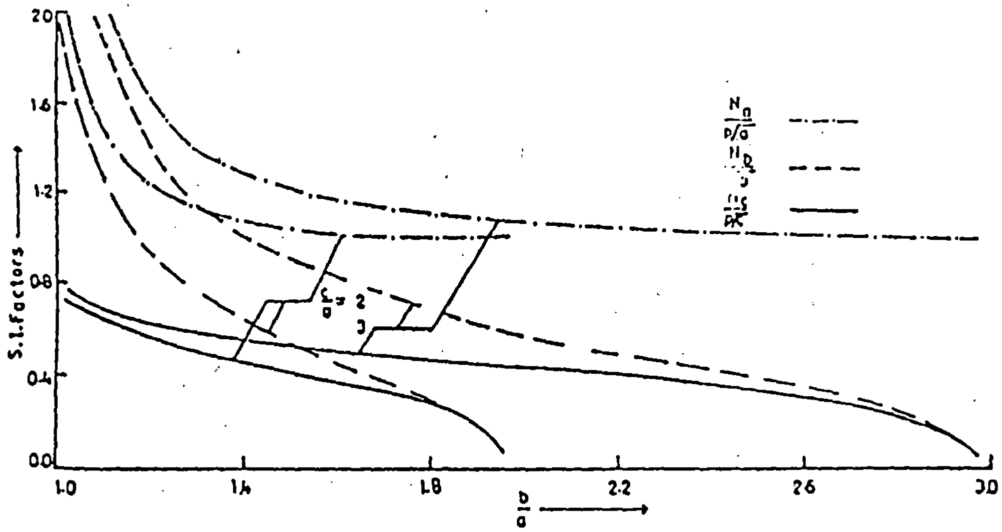

Fig. 6. Stress intensity factors Vs. *b a*.

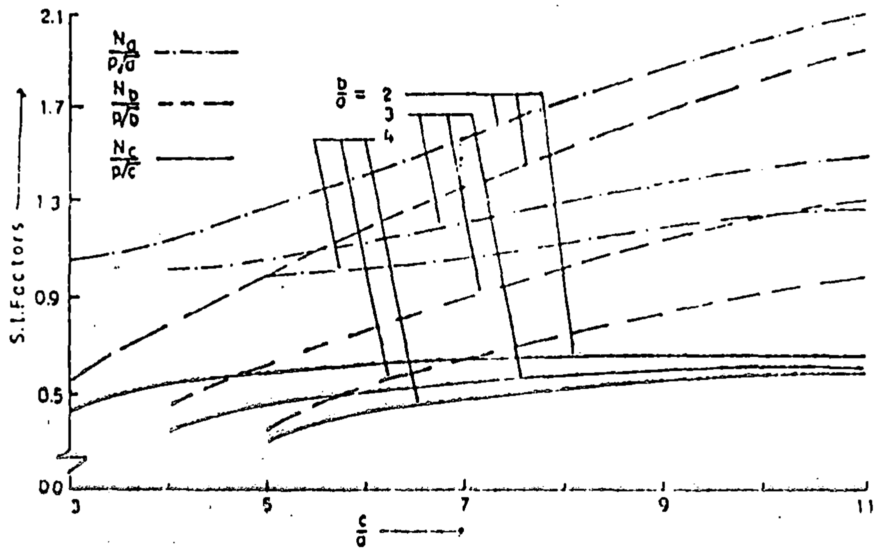

Fig. 7. Stress intensity factors Vs. c/a.

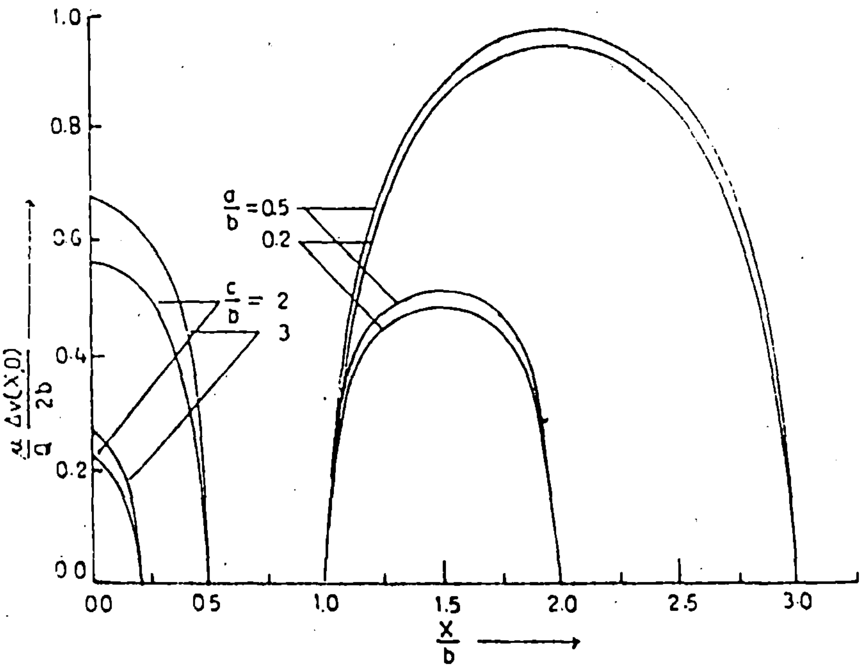

Fig. 8 Variation of crack opening displacement with X/b on both the cracks for the problem II.

THREE CO-PLANAR MOVING GRIFFITH CRACKS IN AN INFINITE ELASTIC STRIP

1. Introduction

In fracture mechanics, the problem of diffraction of elastic waves by cracks of finite dimension in a strip of elastic material has been investigated by several investigators. Sih and Chen (1972) investigated the problem of propagation of a crack of finite length in a strip under plane extension. Closed-from solutions for a finite length crack moving in a strip under anti-plane shear stress was obtained by Singh et. al (1981). Using finite Hilbert transform technique developed by Srivastava and Lowengrub (1968), Lowengrub and Srivastava (1968) solved the statical problem of distribution of stress and displacement in an infinitely long elastic strip containing two co-planar Griffith cracks.

As regards the crack problem, research has been restricted mainly to the case of a single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal (1978) solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar Griffith cracks.

To the best knowledge of the author, the problem of stress distribution around three co-planar moving Griffith cracks in an infinite elastic strip has not been investigated so far. In this paper, the problem of propagation of three co-planar Griffith cracks in a fixed direction with constant velocity V in an infinitely long but of finite width elastic strip has been considered. Employing Fourier integral transform the problem, when the lateral boundaries are subjected to shearing stress, has been reduced to solving a set of four integral equations which are solved using finite Hilbert transform technique and Cook's result (1970) to derive the exact form of stress intensity factors and crack opening displacement. Numerical results for stress intensity factors are

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presented graphically to show its variations with crack speed, crack lengths and the separating distance between the cracks.

2. Statement Of The Problem

Consider an infinitely long elastic strip occupying the region $-h \le Y \le h$, weakened by three co-planar Griffith cracks moving steadily at a constant velocity V in the X-direction referred to a fixed coordinate system (X,Y,Z) as shown in the Fig.1.

In dynamic problem of anti-plane shear, the non-vanishing component of displacement W directed in the Z-direction satisfies the equation of motion

$$W_{,XX} + W_{,YY} = \frac{1}{C_2^2} W_{,TT} \qquad (2.1)$$

where $C_2 = (\mu/\rho)^{1/2}$ is the shear wave velocity, ρ is the material density and $W_{,X}$ represents partial derivatives of W with respect to X.

For cracks moving with constant velocity V in the X-direction it is convenient to introduce the Galilean transformation

$$x = X - VT,\ y = Y,\ z = Z,\ t = T \qquad (2.2)$$

where (x,y,z) represents the translating coordinate system as shown in the Fig.1.

Let the positions of the co-planar Griffith cracks referred to coordinate (x,y,z) be $-c<x<-b$, $-a<x<a$ and $b<x<c$ on $y=0$, and let the uniform shearing stress p be apllied to the lataral boundaries $y = \pm h$ of the strip. The equivalent problems involves the application of shear stress $-p$ to the crack faces at $y=0$. Accordingly, the boundary conditions of the proposed problem are

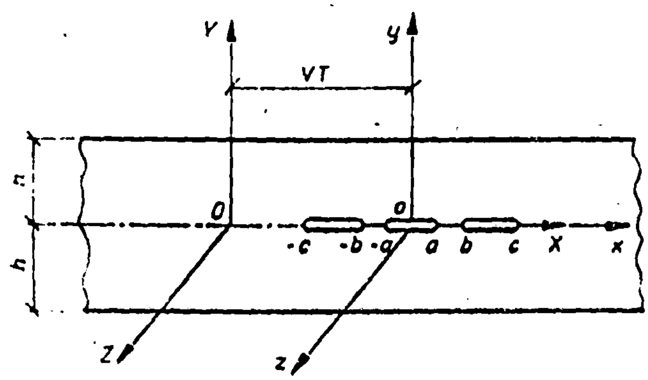

FIG. 1. Geometry and coordinate system.

$$\sigma_{yz}(x,0) = -p, \quad |x|<a, \; b<|x|<c \tag{2.3}$$

$$\sigma_{yz}(x,\pm h) = 0, \qquad -\infty<x<\infty \tag{2.4}$$

$$W(x,0) = 0 , \quad a<|x|<b, \; |x|>c \tag{2.5}$$

In the moving coordinate system , the equation of motion becomes independent of time and takes the form

$$s^2 W_{,xx} + W_{,yy} = 0 \tag{2.6}$$

with

$$s = \sqrt{1 - V^2/C_2^2} \tag{2.7}$$

Due to the symmetry about x-z plane we need to consider the region $0<y\leq h$ only. Introducing the Fourier transform

$$\overline{W}_c(\xi,y)=\int_0^\infty W(x,y)\cos(\xi x)dx$$

$$W(x,y)= \frac{2}{\pi}\int_0^\infty \overline{W}_c(\xi,y)\cos(\xi x)d\xi \tag{2.8}$$

in equation (2.6), the solution of equation (2.6) is obtained as

$$W(x,y)= \frac{2}{\pi}\int_0^\infty\left[C_1(\xi)e^{-\xi ys}+C_2(\xi)e^{\xi ys}\right]\cos(\xi x)d\xi \tag{2.9}$$

with

$$\sigma_{yz}(x,y) = -\frac{2\mu s}{\pi}\int_0^\infty\xi\left[C_1(\xi)e^{-\xi ys}-C_2(\xi)e^{\xi ys}\right]\cos(\xi x)d\xi \tag{2.10}$$

Using the expression for $\sigma_{yz}(x,y)$ given in (2.10) in equation (2.4) it has been found that

$$C_1(\xi) = \frac{C(\xi)}{1 + e^{-2\xi hs}}$$

$$C_3(\xi) = \frac{C(\xi)e^{-2\xi hs}}{1 + e^{-2\xi hs}}$$

where the unknown function $C(\xi)$ is to be determined.
From conditions (2.3) and (2.5) it is determined that $C(\xi)$ satisfies the following quadruple integral equations

$$\int_0^\infty \xi C(\xi)\,\mathrm{th}(\xi hs)\cos(\xi x)d\xi = \frac{\pi p}{2\mu s}, \quad x \in I_1, I_3 \qquad (2.11)$$

and

$$\int_0^\infty C(\xi)\cos(\xi x)d\xi = 0, \qquad x \in I_2, I_4 \qquad (2.12)$$

where $I_1=(0,a)$, $I_2=(a,b)$, $I_3=(b,c)$, $I_4=(c,\infty)$

3. Method Of Solution

In order to solve the quadruple integral equations given by equations (2.11) and (2.12), let us take

$$C(\xi) = \frac{1}{\xi}\int_0^a h(u)\sin(\xi u)du + \frac{1}{\xi}\int_b^c g(v^2)\,\mathrm{ch}(ev)\sin(\xi v)dv \qquad (3.1)$$

where $h(u)$ and $g(v^2)$ are the unknown functions to be determined from the boundary conditions of the proposed problem.

Substituting the value of $C(\xi)$ given by(3.1) in (2.12) and using the well known result

$$\int_0^\infty \frac{\sin(\xi x)\cos(\xi y)}{\xi}d\xi = \begin{cases} \pi/2, & x>y>0 \\ \pi/4, & x=y>0 \\ 0, & y>x>0 \end{cases}$$

it is found that this choice of $C(\xi)$ leads to the condition

$$\int_b^c g(v^2)\,ch(ev)\,dv = 0 \tag{3.2}$$

Rewriting equation (2.11.1) as

$$\frac{d}{dx}\int_0^\infty C(\xi)\,th(\xi hs)\sin(\xi x)\,d\xi = \frac{\pi p}{2\mu s}, \quad x \in I_1 \tag{3.3}$$

and inserting the value of $C(\xi)$ from equation (3.1) in (3.3) it is found that $h(u)$ is the solution of the following singular integral equation

$$\int_0^a h(u)\log\left|\frac{sh(ex)+sh(eu)}{sh(ex)-sh(eu)}\right|du = \pi f(x), \qquad x \in I_1 \tag{3.4}$$

with $$f(x) = \int_0^x \left[\frac{p}{\mu s} - \frac{1}{\pi}\int_b^c \frac{e g(v^2)\,ch(ex')\,sh(2ev)}{sh^2(ev) - sh^2(ex')}dv\right]dx'$$

where the following result [Gradshteyn, I.S. and Ryzhik, I.M. (1965)] has been used

$$\int_0^\infty th(\xi hs)\frac{\sin(\xi x)\sin(\xi u)}{\xi}d\xi = \frac{1}{2}\log\left|\frac{sh(ex)+sh(eu)}{sh(ex)-sh(eu)}\right|, \qquad e = \frac{\pi}{2hs} \tag{3.5}$$

Now using the Cook's result (1970), the solution of (3.4) has been obtained with the aid of following result

$$\int_0^a \frac{\sqrt{sh^2(ea)-sh^2(ex)}\; e\, ch(ex)\, dx}{[sh^2(ex)-sh^2(eu)][sh^2(ev)-sh^2(ex)]} = -\frac{\pi}{2sh(ev)}\frac{\sqrt{sh^2(ev)-sh^2(ea)}}{sh^2(ev)-sh^2(eu)}$$

for $u \in I_1$ and $v \in I_3$,

$$h(u)=\frac{-e\ sh(2eu)}{\pi\sqrt{sh^2(ea)-sh^2(eu)}}\left[\frac{p}{\mu s}\int_0^a\frac{\sqrt{sh^2(ea)-sh^2(ex)}}{sh^2(ex)-sh^2(eu)}dx+\int_b^c\frac{\sqrt{sh^2(ev)-sh^2(ea)}}{sh^2(ev)-sh^2(eu)}\times g(v^2)ch(ev)dv\right] \qquad (3.6)$$

Substituting the resulting value of $C(\xi)$, obtained using equation (3.6) in equation (3.1), in condition (2.11.2) and making use of the following results

$$\int_0^a\frac{e\ sh^2(eu)ch(eu)\ du}{[sh^2(eu)-sh^2(ex)][sh^2(ev)-sh^2(eu)]\sqrt{sh^2(ea)-sh^2(eu)}}=$$

$$=\frac{\pi}{2[sh^2(ev)-sh^2(ex)]}\left[\frac{sh(ev)}{\sqrt{sh^2(ev)-sh^2(ea)}}-\frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}}\right],$$

$$\int_0^a\frac{e\ sh^2(eu)ch(eu)\ du}{[sh^2(eu)-sh^2(ex)][sh^2(ey')-sh^2(eu)]\sqrt{sh^2(ea)-sh^2(eu)}}=$$

$$=\frac{\pi}{2[sh^2(ex)-sh^2(ey')]}\frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}},\ \text{for } x,v\in l_3 \text{ and } y'\in l_1$$

it can be shown that $g(v^2)$ is the solution of the following singular integral equation

$$\int_b^c\frac{\sqrt{sh^2(ev)-sh^2(ea)}}{sh^2(ev)-sh^2(ex)}eg(v^2)ch(ev)\ dv=\frac{\pi p}{\mu s}\left[\frac{\sqrt{sh^2(ex)-sh^2(ea)}}{sh(2ex)}+\right.$$

$$\left.+\frac{1}{\pi}\int_0^a\frac{\sqrt{sh^2(ea)-sh^2(ey')}}{sh^2(ex)-sh^2(ey')}dy'\right],\ \text{for } x\in l_3. \qquad (3.7)$$

Using finite Hilbert transform technique (1968) and the following result

$$\int_b^c \sqrt{\frac{sh^2(ec)-sh^2(ex)}{sh^2(ex)-sh^2(eb)}}\ \frac{sh(2ex)\ \ dx}{[sh^2(ex)-sh^2(ey')][sh^2(ex)-sh^2(ev)]} =$$

$$= -\ \frac{\pi}{e[sh^2(ev)-sh^2(ey')]}\sqrt{\frac{sh^2(ec)-sh^2(ey')}{sh^2(eb)-sh^2(ey')}}\ ,$$

the solution of equation (3.7) is found as

$$g(v^2) = -\ \frac{2ep}{\mu\pi s}\ \frac{sh(ev)\sqrt{sh^2(ev)-sh^2(eb)}}{\sqrt{[sh^2(ev)-sh^2(ea)][sh^2(ec)-sh^2(ev)]}}\left[\ \int_b^c \sqrt{\frac{sh^2(ec)-sh^2(ex)}{sh^2(ex)-sh^2(eb)}}\ \times\right.$$

$$\left.\times\ \sqrt{\frac{sh^2(ex)-sh^2(ea)}{sh^2(ex)-sh^2(ev)}}\ dx\ -\ \int_0^a \sqrt{\frac{sh^2(ec)-sh^2(ey')}{sh^2(eb)-sh^2(ey')}}\ \sqrt{\frac{sh^2(ea)-sh^2(ey')}{sh^2(ev)-sh^2(ey')}}\ dy'\ \right] +$$

$$+\ \frac{C_1\ \ sh(ev)}{\sqrt{[sh^2(ev)-sh^2(ea)][sh^2(ev)-sh^2(eb)][sh^2(ec)-sh^2(ev)]}} \qquad (3.8)$$

Next substituting the value of $g(v^2)$ from equation (3.8) in equation (3.6) and finally using the following result

$$\int_b^c \sqrt{\frac{sh^2(ev)-sh^2(eb)}{sh^2(ec)-sh^2(ev)}}\ \frac{sh(2ev)\ \ dv}{[sh^2(ev)-sh^2(eu)][sh^2(ex')-sh^2(ev)]} =$$

$$= \frac{\pi}{e[sh^2(eu)-sh^2(ex')]}\left[\ \sqrt{\frac{sh^2(eb)-sh^2(eu)}{sh^2(ec)-sh^2(eu)}}\ -\ \sqrt{\frac{sh^2(eb)-sh^2(ex')}{sh^2(ec)-sh^2(ex')}}\ \right],$$

for $u,\ x' \in l_1$

h(u) is derived in the form

$$h(u) = -\frac{2ep}{\mu\pi s}\frac{ch(eu)sh(eu)\sqrt{sh^2(eb)-sh^2(eu)}}{\sqrt{[sh^2(ea)-sh^2(eu)][sh^2(ec)-sh^2(eu)]}}\Bigg[\int_0^a\sqrt{\frac{sh^2(ea)-sh^2(ey')}{sh^2(eb)-sh^2(ey')}}\times$$

$$\times\sqrt{\frac{sh^2(ec)-sh^2(ey')}{sh^2(ey')-sh^2(eu)}}\,dy' - \int_b^c\sqrt{\frac{sh^2(ec)-sh^2(ex)}{sh^2(ex)-sh^2(eb)}}\sqrt{\frac{sh^2(ex)-sh^2(ea)}{sh^2(ex)-sh^2(eu)}}\,dx\Bigg] -$$

$$-\frac{C_1\ sh(eu)ch(eu)}{\sqrt{[sh^2(ea)-sh^2(eu)][sh^2(eb)-sh^2(eu)][sh^2(ec)-sh^2(eu)]}} \qquad (3.9)$$

Substitution of the value of $g(v^2)$ from equation (3.8) in the condition (3.2) yields

$$C_1 = -\frac{2ep}{\pi\mu s}\Bigg[\int_b^c\sqrt{\frac{sh^2(ec)-sh^2(ex)}{sh^2(ex)-sh^2(eb)}}\sqrt{sh^2(ex)-sh^2(ea)}\Bigg\{\frac{sh^2(ex)-sh^2(eb)}{sh^2(ec)-sh^2(ex)}\times$$

$$\Pi\Big\{\frac{\pi}{2},\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(ex)};q\Big\}/F(\frac{\pi}{2},q)+1\Bigg\}dx+\int_0^a\sqrt{\frac{sh^2(ec)-sh^2(es)}{sh^2(eb)-sh^2(es)}}\sqrt{sh^2(ea)-sh^2(es)}$$

$$\Bigg\{1-\frac{sh^2(eb)-sh^2(es)}{sh^2(ec)-sh^2(es)}\Pi\Big\{\frac{\pi}{2},\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(es)},q\Big\}/F(\frac{\pi}{2},q)\Bigg\}\,ds \qquad (3.10)$$

where $F(\phi,q)$ and $\Pi(\phi,n,q)$ are elliptic integrals of first and third kind respectively and $q = \sqrt{\dfrac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(ea)}}$

The relevant displacement and stress components in the plane of the crack can now be shown to be given by

$$W(x,0) = \int_x^a h(u)du \,, \qquad 0 \le x \le a$$

$$= \int_x^c g(v^2) ch(ev) dv \,, \qquad b \le x \le c \qquad (3.11)$$

and

$$\left[\sigma_{yz}(x,0)\right]_{a<x<b} = \frac{2\mu s}{\pi}\left[\int_0^a \frac{eh(u)sh(eu)\,du}{sh^2(ex)-sh^2(eu)} - \int_b^c \frac{eg(v^2)sh(ev)ch(ev)}{sh^2(ev) - sh^2(ex)} dv\right] ch(ex)$$

$$\left[\sigma_{yz}(x,0)\right]_{x>c} = \frac{2\mu s}{\pi}\left[\int_0^a \frac{eh(u)sh(eu)\,du}{sh^2(ex)-sh^2(eu)} + \int_b^c \frac{eg(v^2)sh(ev)ch(ev)}{sh^2(ex) - sh^2(ev)} dv\right] ch(ex)$$

(3.12)

Now insertion of the values of $h(u)$ and $g(v^2)$ as given by equations (3.9) and (3.8), in the expressions (3.12) yields after some algebraic manipulations

$$\left[\sigma_{yz}(x,0)\right]_{a<x<b} = \frac{2pe}{\pi}\left[-\sqrt{\frac{sh^2(eb)-sh^2(ea)}{sh^2(ec)-sh^2(ea)}} \frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}}\left\{\int_0^a F_2(u,x)du \; + \right.\right.$$

$$\left. + \int_b^c F_2(v,x)dv \right\} - \frac{2e[sh^2(ec)-sh^2(eb)]}{\pi} \left\{ \int_0^a F_2(u',x)du' \int_0^a F_4(c,u) \times \right.$$

$$\left. \times F_3(0,x,u)\; du + \int_b^c F_2(v,x)dv \int_0^a F_4(c,u)F_3(v,x,u)\; du \right\} + \frac{\mu s}{ep}\, C_1\left\{ \frac{\pi}{2} \times \right.$$

$$\left. \times \frac{1 - sh(ex)/\sqrt{sh^2(ex)-sh^2(ea)}}{\sqrt{[sh^2(eb)-sh^2(ea)][sh^2(ec)-sh^2(ea)]}} + e\int_0^a F_4(c,u)F_5(u,x)\; du \right\} +$$

$$+\ \frac{e[sh^2(eb)-sh^2(ea)]}{\pi}\left\{\int_b^c F_2(v',x)dv'\int_b^c F_4(a,v)F_6(v',x,v)dv+\int_0^a F_2(u,x)du\ \times\right.$$

$$\left.\int_b^c F_4(a,v)F_6(u,x,v)dv-\frac{sh^2(ec)-sh^2(eb)}{sh^2(eb)-sh^2(ea)}\int_0^a F_1(u,x)du\int_0^a F_4(c,u')F_9(u,u')du'\right\}$$

$$-\ \frac{\mu s}{pe}\ \frac{C_1}{X_1}\left\{\frac{\pi}{2}\ \frac{sh(ec)}{\sqrt{sh^2(ec)-sh^2(ea)}}\ +\ e\ sh^2(ea)\int_b^c F_7(x,v)\ dv\right\}\Bigg]ch(ex)$$

and

$$\left[\sigma_{yz}(x,0)\right]_{x>c}\ =\ \frac{2pe}{\pi}\Bigg[-\sqrt{\frac{sh^2(eb)-sh^2(ea)}{sh^2(ec)-sh^2(ea)}}\ \frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}}\ \left\{\ \int_0^a F_2(u,x)du\ +\right.$$

$$\left.+\ \int_b^c F_2(v,x)dv\ \right\}\ -\ \frac{2e[sh^2(ec)-sh^2(eb)]}{\pi}\ \left\{\ \int_0^a F_2(u',x)du'\ \int_0^a F_4(c,u)\times\right.$$

$$\left.\times\ F_8(0,x,u)\ du\ +\ \int_b^c F_2(v,x)dv\ \int_0^a F_4(c,u)F_8(v,x,u)\ du\ \right\}\ +\ \frac{\mu s}{ep}\ C_1\left\{\ \frac{\pi}{2}\ \times\right.$$

$$\left.\times\ \frac{1\ -\ sh(ex)/\sqrt{sh^2(ex)-sh^2(ea)}}{\sqrt{[sh^2(ec)-sh^2(ea)][sh^2(eb)-sh^2(ea)]}}\ +\ e\int_0^a F_4(c,u)F_5(u,x)du\ \right\}\ -$$

$$-\ \frac{e[sh^2(eb)-sh^2(ea)]}{\pi}\left\{\int_b^c F_2(v',x)dv'\int_b^c F_4(a,v)F_8(v',v,x)dv\ +\int_0^a F_2(u,x)du\ \times\right.$$

$$\left.\int_b^c F_4(a,v)F_8(u,v,x)dv+\ \frac{sh^2(ec)-sh^2(eb)}{sh^2(eb)-sh^2(ea)}\int_0^a F_1(u,x)du\int_0^a F_4(c,u')F_9(u,u')du'\right\}$$

$$+\frac{\mu s C_1}{p e X_1}\left\{\frac{\pi}{2}\frac{sh(ec)}{\sqrt{sh^2(ec)-sh^2(ea)}}+e\; sh^2(ea)\int_b^c F_7(x,v)\;dv\right\}-\sqrt{\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(ea)}}\times$$

$$\times\frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ec)}}\left\{\int_0^a F_2(u,x)du+\int_b^c F_2(v,x)dv\right\}\Bigg]ch(ex) \qquad (3.13)$$

where

$$F_1(u,x) = \sqrt{\frac{sh^2(ec)-sh^2(eu)}{sh^2(eb)-sh^2(eu)}}\;\frac{sh(eu)}{sh^2(ex)-sh^2(eu)}$$

$$F_2(v,x) = \sqrt{\frac{sh^2(ec)-sh^2(ev)}{sh^2(ev)-sh^2(eb)}}\;\frac{\sqrt{sh^2(ev)-sh^2(ea)}}{sh^2(ev)-sh^2(ex)}$$

$$F_3(v,x,u) = \frac{sh(ex)}{\sqrt{sh^2(ex)-sh^2(ea)}}\tan^{-1}\left\{\frac{sh(eu)}{sh(ex)}\sqrt{\frac{sh^2(ex)-sh^2(ea)}{sh^2(ea)-sh^2(eu)}}\right\}-$$

$$-\frac{sh(ev)}{\sqrt{sh^2(ev)-sh^2(ea)}}\tan^{-1}\left\{\frac{sh(eu)}{sh(ev)}\sqrt{\frac{sh^2(ev)-sh^2(ea)}{sh^2(ea)-sh^2(eu)}}\right\}$$

$$F_4(\omega,u) = \frac{ch(eu)\;sh(eu)}{\sqrt{[sh^2(e\omega)-sh^2(eu)]^3[sh^2(eb)-sh^2(eu)]}}$$

$$F_5(u,x) = [\;2sh^2(eu)-sh^2(ec)-sh^2(eb)]\left\{\sin^{-1}\left(\frac{sh(eu)}{sh(ea)}\right)-F_3(0,x,u)\right\}$$

$$F_6(u,x,v) = \frac{sh(ex)}{\sqrt{sh^2(ec)-sh^2(ex)}} \times$$

$$\times \log\left|\frac{sh(ex)\sqrt{sh^2(ec)-sh^2(ev)}+sh(ev)\sqrt{sh^2(ec)-sh^2(ex)}}{sh(ex)\sqrt{sh^2(ec)-sh^2(ev)}-sh(ev)\sqrt{sh^2(ec)-sh^2(ex)}}\right| - \frac{sh(eu)}{\sqrt{sh^2(ec)-sh^2(eu)}}$$

$$\times \log\left|\frac{sh(eu)\sqrt{sh^2(ec)-sh^2(ev)}+sh(ev)\sqrt{sh^2(ec)-sh^2(eu)}}{sh(eu)\sqrt{sh^2(ec)-sh^2(ev)}-sh(ev)\sqrt{sh^2(ec)-sh^2(eu)}}\right|$$

$$F_7(x,v) = \tan^{-1}\left(\frac{\sqrt{sh^2(ec)-sh^2(ex)}}{\sqrt{sh^2(ec)-sh^2(ev)}}\,\frac{\sqrt{sh^2(ev)-sh^2(eb)}}{\sqrt{sh^2(eb)-sh^2(ex)}}\right)\frac{ch(eu)}{\sqrt{[sh^2(ev)-sh^2(ea)]^3}}$$

$$F_8(u,v,x) = -\frac{2sh(ex)}{\sqrt{sh^2(ex)-sh^2(ec)}}\tan^{-1}\left\{\frac{sh(ev)}{sh(ex)}\sqrt{\frac{sh^2(ex)-sh^2(ec)}{sh^2(ec)-sh^2(ev)}}\right\} +$$

$$+ \frac{sh(eu)}{\sqrt{sh^2(ec)-sh^2(eu)}}\log\left|\frac{sh(eu)\sqrt{sh^2(ec)-sh^2(ev)}+sh(ev)\sqrt{sh^2(ec)-sh^2(eu)}}{sh(eu)\sqrt{sh^2(ec)-sh^2(ev)}-sh(ev)\sqrt{sh^2(ec)-sh^2(eu)}}\right|$$

$$F_9(u,u') = \log\left|\frac{sh(eu)\sqrt{sh^2(ea)-sh^2(eu')}+sh(eu')\sqrt{sh^2(ea)-sh^2(eu)}}{sh(eu)\sqrt{sh^2(ea)-sh^2(eu')}-sh(eu')\sqrt{sh^2(ea)-sh^2(eu)}}\right|$$

and $$X_1 = \sqrt{[sh^2(eb)-sh^2(ex)][sh^2(ec)-sh^2(ex)]} \qquad (3.14)$$

The dynamic stress intensity factors are defined by

$$N_a = \underset{x \to a+}{\text{Lt}} \sqrt{2(x-a)} \left[\sigma_{yz}(x,0) \right]_{a<x<b}$$

$$N_b = \underset{x \to b-}{\text{Lt}} \sqrt{2(b-x)} \left[\sigma_{yz}(x,0) \right]_{a<x<b}$$

$$N_c = \underset{x \to c+}{\text{Lt}} \sqrt{2(x-c)} \left[\sigma_{yz}(x,0) \right]_{x>c} \tag{3.15}$$

Substitution of the results given by equations (3.13) in expressions (3.15) yields

$$N_a = \sqrt{\frac{sh(2ea)}{e}} \left[-\sqrt{\frac{sh^2(eb)-sh^2(ea)}{sh^2(ec)-sh^2(ea)}} \frac{2pe}{\pi} \left\{ \int_0^a F_2(u,a)du + \int_b^c F_2(v,a)dv \right\} - \right.$$

$$\left. - \frac{\mu s\, C_1}{\sqrt{[sh^2(eb)-sh^2(ea)][sh^2(ec)-sh^2(ea)]}} \right]$$

$$N_b = - \frac{\mu s\, C_1}{\sqrt{[sh^2(eb)-sh^2(ea)][sh^2(ec)-sh^2(eb)]}} \sqrt{\frac{sh(2eb)}{e}}$$

$$N_c = \sqrt{\frac{sh(2ec)}{e}} \left[-\sqrt{\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(ea)}} \frac{2pe}{\pi} \left\{ \int_0^a F_2(u,c)du + \int_b^c F_2(v,c)dv \right\} + \right.$$

$$\left. + \frac{\mu s\, C_1}{\sqrt{[sh^2(ec)-sh^2(ea)][sh^2(ec)-sh^2(eb)]}} \right] \tag{3.16}$$

Again insertion of the values of h(u) and $g(v^2)$, given by equations (3.8) and (3.9), in the expressions for displacements given by equations (3.11) yields

$$\Big[W(x,0)\Big]_{0\le x\le a} = -\frac{p}{\mu\pi s}\Bigg[\frac{2[sh^2(eb)-sh^2(ea)]}{\sqrt{sh^2(ec)-sh^2(ea)}}\Bigg\{\int_b^c \Pi\Big\{\lambda,\frac{sh^2(ev)-sh^2(eb)}{sh^2(ev)-sh^2(ea)},q\Big\}\times$$

$$\times\sqrt{\frac{sh^2(ec)-sh^2(ev)}{sh^2(ev)-sh^2(eb)}}\;\frac{dv}{\sqrt{sh^2(ev)-sh^2(ea)}} - \int_0^a \Pi\Big\{\lambda,\frac{sh^2(eb)-sh^2(eu)}{sh^2(ea)-sh^2(eu)},q\Big\}\times$$

$$\times\sqrt{\frac{sh^2(ec)-sh^2(eu)}{sh^2(eb)-sh^2(eu)}}\;\frac{du}{\sqrt{sh^2(ea)-sh^2(eu)}}\Bigg\}\Bigg] - \frac{C_1 F(\lambda,q)}{e\sqrt{sh^2(ec)-sh^2(ea)}}$$

and

$$\Big[W(x,0)\Big]_{b\le x\le c} = \Bigg[\frac{2p}{\mu\pi s}\Bigg(\int_b^c\sqrt{\frac{sh^2(ec)-sh^2(ev)}{sh^2(ev)-sh^2(eb)}}\;\sqrt{sh^2(ev)-sh^2(ea)}\Bigg\{F(\lambda',q)\; +$$

$$+\frac{sh^2(ev)-sh^2(eb)}{sh^2(ec)-sh^2(ev)}\,\Pi\Big\{\lambda',\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(ev)},q\Big\}\Bigg\}\,dv + \int_0^a\sqrt{\frac{sh^2(ec)-sh^2(eu)}{sh^2(eb)-sh^2(eu)}}\times$$

$$\sqrt{sh^2(ea)-sh^2(eu)}\Bigg\{F(\lambda',q) - \frac{sh^2(eb)-sh^2(eu)}{sh^2(ec)-sh^2(eu)}\Pi\Big\{\lambda',\frac{sh^2(ec)-sh^2(eb)}{sh^2(ec)-sh^2(eu)},q\Big\}\Bigg\}du\Bigg] +$$

$$+\frac{C_1}{e}F(\lambda',q)\Bigg]\frac{1}{\sqrt{sh^2(ec)-sh^2(ea)}} \qquad (3.17)$$

where $\sin\lambda = \sqrt{\dfrac{sh^2(ea)-sh^2(ex)}{sh^2(eb)-sh^2(ex)}}$, $\sin\lambda' = \sqrt{\dfrac{sh^2(ec)-sh^2(ex)}{sh^2(ec)-sh^2(eb)}}$ and $F(\phi,q)$, $\Pi(\phi,n,q)$ and q have been defined earlier.

On putting b=c and simplifying, it may be noted that, the results (3.16.1) and (3.17.1) become those given by equation (4.18) and (4.19) of Singh et. al (1981) and for a=0 the results given by (3.16.2),(3.16.3) and (3.17.2) coincide with those given by equation (4.38), (4.39) and (4.35) of Das and Ghosh (1991).

4. Numerical Results and Discussions.

Numerical results for stress intensity factors at the tips of the cracks for different values of crack speed, crack lengths and the separating distance between the cracks have been presented in this section. The crack length dependence of the stress intensity factors and its variations with V/C_2 have been shown in Figures 2-5. It has been depicted in Figures 2-3 that stress intensity factors at the edges of the cracks have a prominent variation when $V/C_2 \rightarrow 1$ and variations of stress intensity factors at the edge x=a become more prominent than that at the tips x=b and x=c when the length of the inner crack increases.

Variations of stress intensity factors at the edges of the cracks with a/b for different values of c/b and that with b/a for different values of c/a are plotted in Figures 4-5 respectively. It has been found that when the separating distance between the inner crack and outer pair of cracks decreases the stress intensity factors at the tips x=a and x=b become greater than that at the edge x=c.

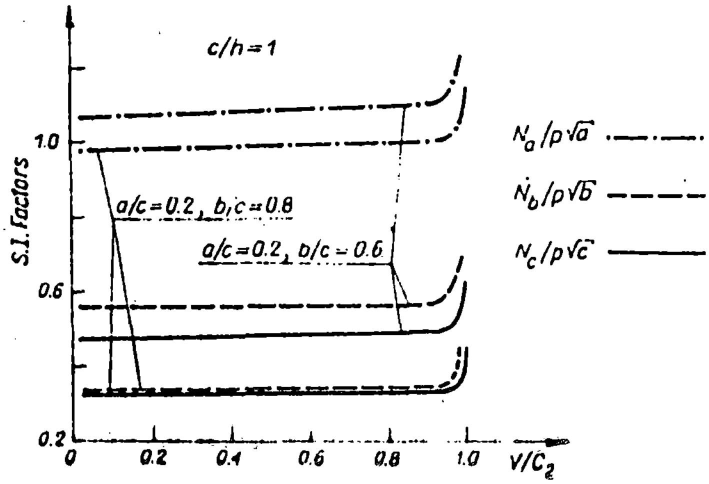

FIG. 2. Variations of stress intensity factors with V/C_2.

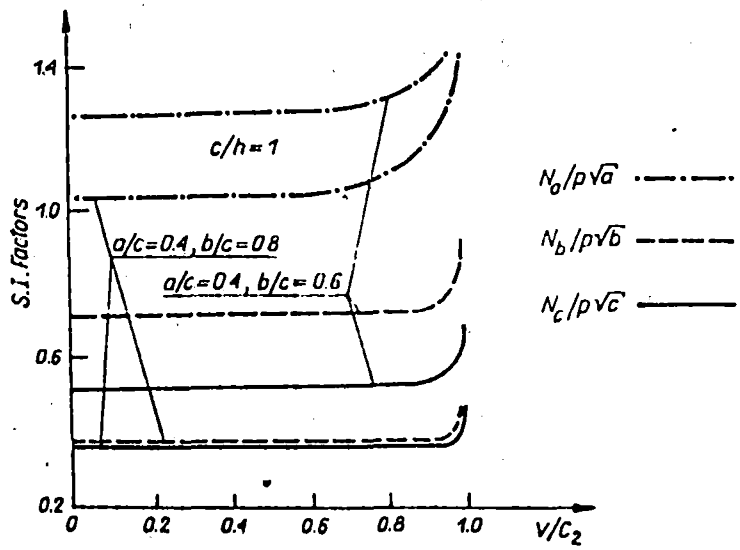

FIG. 3. Variations of stress intensity factors with V/C_2.

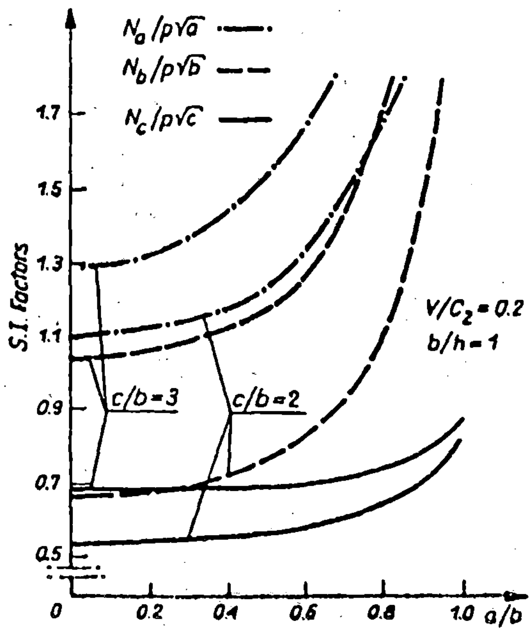

FIG. 4. Stress intensity factors Vs. a/b.

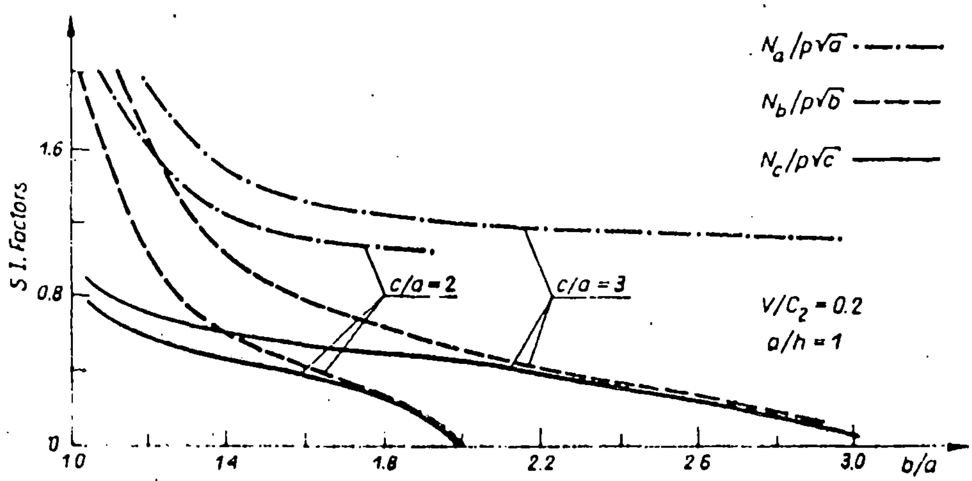

FIG. 5. Stress intensity factors Vs. b/a.

FOUR CO-PLANAR GRIFFITH CRACKS IN AN INFINITE ELASTIC MEDIUM

1.Introduction

In fracture mechanics, scattering of elastic waves by cracks of finite dimension in an infinite elastic medium has been investigated by several investigators.The problem of scattering of elastic wave from an interface crack was solved by Bostrom (1987).Srivastava et. al. (1980) solved the problem of interaction of anti-plane shear wave by an interface crack. The problem of diffraction of Love waves by a crack of finite width in the plane interface of a layered composite has been solved by Neerhoff (1979). Itou (1980) solved problem of diffraction of anti-plane shear wave by two co-planar Griffith cracks in an infinite elastic medium. The scattering of time harmonic normally incident plane wave by two co-planar Griffith cracks was solved by Jain and Kanwal (1972). Itou (1978) also solved the problem of stress concentration around two co-planar Griffith cracks in an infinite elastic medium.

As regards the crack problem, research has been restricted mainly to the case of single crack or a pair of cracks because of severe mathematical complexity encountered in solving the problems of three or more cracks. Recently, Dhawan and Dhaliwal (1978) solved the statical problem of determining the stress distribution in an infinite transversely isotropic medium containing three co-planar cracks.

To the best knowledge of the authors, the problem of stress distribution around four co-planar Griffith cracks has not been investigated so far. In this paper, we consider two cases regarding stress distribution around four co-planar Griffith cracks in an infinite homogeneous, isotropic medium. In the first case, cracks are assumed to be moving steadily along a fixed direction with constant velocity V. In the second case, the statical problem of determining the stress and displacement in an infinite homogeneous, isotropic medium weakened by four co-planar Griffith

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cracks has been considered. Using Fourier integral transform both the problems have been reduced to solving a set of five integral equations. Employing finite Hilbert transform technique (1968) the integral equations have been solved to derive crack opening displacement and stress intensity factors which are presented in the form of graphs.

2. Statement Of Problem I And Its Formulation

Consider an infinite homogeneous isotropic material weakened by four co-planar Griffith cracks, moving steadily at a constant velocity V in the X- direction referred to a fixed coordinate system (X,Y,Z) as shown in the Fig 1. In absence of body force equations of motion in terms of displacement are

$$(\lambda+2\mu)\left[u_{,XX} + v_{,XY} \right] + \mu\left[u_{,YY} - v_{,XY} \right] = \rho\, u_{,TT}$$

and $$(\lambda+2\mu)\left[u_{,XY} + v_{,YY} \right] + \mu\left[v_{,XX} - u_{,XY} \right] = \rho\, v_{,TT} \qquad (1a,b)$$

where u,v denote the displacement components in X and Y directions and λ,μ are Lame's constants and $u_{,X}$ represents partial derivatives of u with respect to X.

For cracks moving with constant velocity V in the X- direction it is convenient to introduce the Galilean transformation

$$x = X - VT,\ y = Y,\ z = Z,\ t = T \qquad (2)$$

where (x,y,z) represents the translating coordinate system as shown in Fig 1.

In the moving coordinates, The equations of motion (1) become independent of time and take the form

$$(\lambda+2\mu-\rho V^2)u_{,xx} + (\lambda+\mu)v_{,xy} + \mu u_{,yy} = 0$$

$$(\lambda+2\mu)v_{,yy} + (\mu - \rho V^2)v_{,xx} + (\lambda+\mu)u_{,xy} = 0 \qquad (3a,b)$$

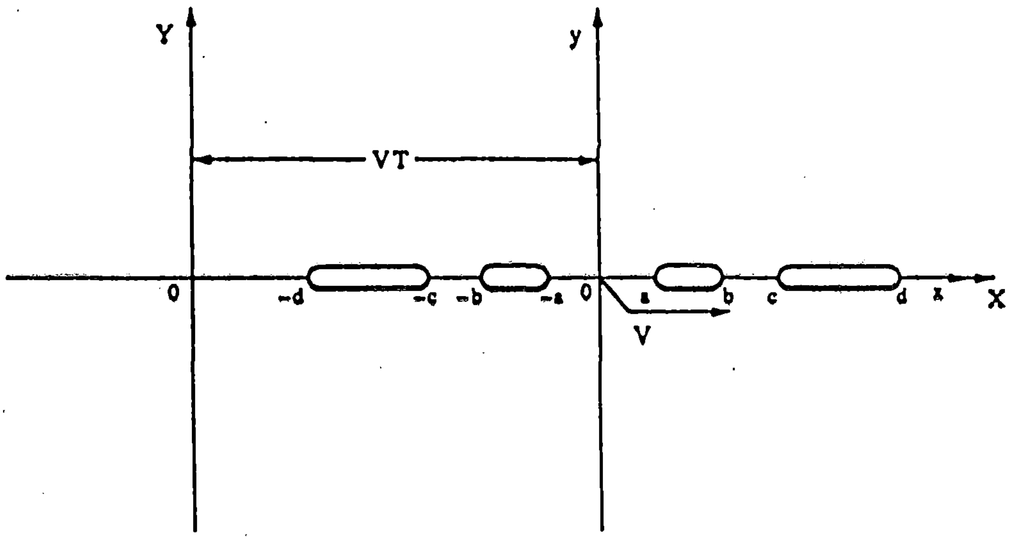

Fig. 1. Geometry and coordinate system.

Introducing $$\bar{u}_s(\xi,y) = \int_0^\infty u(x,y)\ \sin(\xi x)\ dx$$

$$\bar{v}_c(\xi,y) = \int_0^\infty v(x,y)\ \cos(\xi x)\ dx \qquad (4a,b)$$

and $$u(x,y) = \frac{2}{\pi}\int_0^\infty \bar{u}_s(\xi,y)\ \sin(\xi x)\ d\xi$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \bar{v}_c(\xi,y)\ \cos(\xi x)\ d\xi \qquad (5a,b)$$

in equation (3) we obtain

$$\mu\bar{u}_{s,yy} - \xi(\lambda+\mu)\bar{v}_{c,y} - \xi^2(\lambda+2\mu-\rho V^2)\bar{u}_s = 0$$

$$(\lambda+2\mu)\bar{v}_{c,yy} + \xi(\lambda+\mu)\bar{u}_{s,y} - \xi^2(\mu-\rho V^2)\bar{v}_c = 0 \qquad (6a,b)$$

Elimination of $\bar{u}_s$ from (6a,b) yields the following ordinary differential equation

$$\left[\left\{\frac{d^2}{dy^2} - (1-M^2k^2)\xi^2\right\}\left\{\frac{d^2}{dy^2} - (1-M^2)\xi^2\right\}\right]\bar{v}_c = 0 \qquad (7)$$

where $M = V/c_2$, $k = c_2/c_1$.

The solution of the differential equation given by (7), for $y \geq 0$, is

$$\bar{v}_c(\xi,y) = A(\xi)\ e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\ e^{-\xi y\sqrt{1-M^2}} \qquad (8)$$

where the unknown functions $A(\xi)$ and $B(\xi)$ are to be determined using the boundary conditions of the proposed problem.

Employing (8) in equations (6a,b), we obtain

$$\bar{u}_s(\xi, y) = \frac{A(\xi)}{\sqrt{1-M^2k^2}} e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\, B(\xi) e^{-\xi y\sqrt{1-M^2}}, \quad y \geq 0 \tag{9}$$

Therefore, the stress components given by

$$\sigma_{yy} = \lambda(u_{,x} + v_{,y}) + 2\mu v_{,y}$$

$$\sigma_{xy} = \mu(u_{,y} + v_{,x}) \tag{10a,b}$$

become

$$\sigma_{yy}(x,y) = -\frac{2\mu}{\pi}\int_0^\infty \xi \left[\frac{2-M^2}{\sqrt{1-M^2k^2}} A(\xi) e^{-\xi y\sqrt{1-M^2k^2}} + 2\sqrt{1-M^2}\, B(\xi) e^{-\xi y\sqrt{1-M^2}}\right]. \cos(\xi x)\, d\xi$$

$$\sigma_{xy}(x,y) = -\frac{2\mu}{\pi}\int_0^\infty \xi \left[2A(\xi) e^{-\xi y\sqrt{1-M^2k^2}} + (2-M^2) B(\xi) e^{-\xi y\sqrt{1-M^2}}\right] \sin(\xi x)\, d\xi \tag{11a,b}$$

with

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\frac{A(\xi)}{\sqrt{1-M^2k^2}} e^{-\xi y\sqrt{1-M^2k^2}} + \sqrt{1-M^2}\, B(\xi) e^{-\xi y\sqrt{1-M^2}}\right] \sin(\xi x)\, d\xi$$

and

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \left[A(\xi)\, e^{-\xi y\sqrt{1-M^2k^2}} + B(\xi)\, e^{-\xi y\sqrt{1-M^2}}\right] \cos(\xi x)\, d\xi \tag{12a,b}$$

Let four co-planar Griffith cracks of finite length located along X-axis be moving steadily with velocity V in the direction of X axis so that their position referred to translating coordinate (x,y,z) are $a \leq |x| \leq b$, $c \leq |x| \leq d$ on $y=0$.

The boundary conditions of the proposed problem on account of symmetry with respect to y-axis are

$$v(x,0) = 0\ , \qquad x \in I_1, I_3, I_5 \qquad (13a\text{-}c)$$

$$\sigma_{xy}(x,0) = 0, \qquad 0<x<\infty \qquad (14)$$

$$\sigma_{yy}(x,0) = -p, \quad x \in I_2, I_4 \qquad (15a,b)$$

where $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,d)$, $I_5 = (d,\infty)$

Using the condition (14) in (11b) we find that $A(\xi), B(\xi)$ are related by

$$B(\xi) = -\frac{2}{2-M^2} A(\xi) \qquad (16)$$

With the help of the boundary condition (13), we obtain from (12b)

$$\int_0^\infty A(\xi)\cos(\xi x)\ d\xi = 0, \qquad x \in I_1, I_3, I_5 \qquad (17a\text{-}c)$$

Substitution of (11a) in (15) yields with the aid of (16)

$$\int_0^\infty \xi\ A(\xi)\cos(\xi x)\ d\xi = \frac{P\pi}{2\mu}\ , \quad x \in I_2, I_4 \qquad (18a,b)$$

where $\qquad P = \frac{p}{K}\ ,\quad K = \dfrac{(2-M^2)^2 - 4\sqrt{(1-M^2k^2)(1-M^2)}}{(2-M^2)\sqrt{1-M^2k^2}}$

3.Method Of Solution

In order to solve the set of five integral equations given in equations (17) and (18) we assume

$$A(\xi) = \frac{1}{\xi}\int_a^b h(s^2)\ \sin(\xi s)\ ds + \frac{1}{\xi}\int_c^d g(t^2)\ \sin(\xi t)\ dt \qquad (19)$$

where $h(s^2)$ and $g(t^2)$ are unknown functions to be determined from the boundary conditions.

Inserting the value of $A(\xi)$ from equation (19) in equation (17) it is found that this choice of $A(\xi)$ leads to the equations

$$\int_a^b h(s^2)\,ds = 0 \text{ and } \int_c^d g(t^2)\,dt = 0 \qquad (20a,b)$$

Further substituting $A(\xi)$ from equation (19) in (18a), we obtain

$$\int_a^b \frac{sh(s^2)}{s^2-x^2}\,ds + \int_c^d \frac{tg(t^2)}{t^2-x^2}\,dt = \frac{\pi P}{2\mu}, \quad x \in I_2$$

Rewriting this equation as

$$\int_a^b \frac{sh(s^2)}{s^2-x^2}\,ds = \frac{\pi}{2}F(x), \quad x \in I_2$$

where

$$F(x) = \frac{P}{\mu} - \frac{2}{\pi}\int_c^d \frac{tg(t^2)}{t^2-x^2}\,dt$$

and using finite Hilbert transform technique (1968), we obtain

$$h(s^2) = \frac{P}{\mu}\sqrt{\frac{s^2-a^2}{b^2-s^2}} - \frac{2}{\pi}\sqrt{\frac{s^2-a^2}{b^2-s^2}}\int_c^d\sqrt{\frac{t^2-b^2}{t^2-a^2}}\,\frac{tg(t^2)}{t^2-s^2}\,dt + \frac{C_1}{\sqrt{(s^2-a^2)(b^2-s^2)}}$$

where we have used (21)

$$\int_a^b \sqrt{\frac{b^2-x^2}{x^2-a^2}}\,\frac{x\,dx}{(s^2-x^2)(t^2-x^2)} = \frac{\pi}{2}\sqrt{\frac{t^2-b^2}{t^2-a^2}}\,\frac{1}{t^2-s^2}$$

The constant C_1 is to be determined from equation (20).

Substituting the value of $h(s^2)$ from (21) in (19) and using the resulting value of $A(\xi)$ in the boundary condition (18b) we obtain, using the results

$$\int_a^b \sqrt{\frac{s^2-a^2}{b^2-s^2}}\ \frac{s\ ds}{(s^2-x^2)(t^2-s^2)} = \frac{\pi}{2}\left[\sqrt{\frac{t^2-a^2}{t^2-b^2}} - \sqrt{\frac{x^2-a^2}{x^2-b^2}}\right]\frac{1}{t^2-x^2}$$

and $\displaystyle\int_a^b \frac{s\ ds}{(s^2-x^2)\sqrt{(s^2-a^2)(b^2-s^2)}} = -\frac{\pi}{2\sqrt{(x^2-a^2)(x^2-b^2)}}$ for $x \in I_4$

the singular integral equation

$$\int_c^d \sqrt{\frac{t^2-b^2}{t^2-a^2}}\ \frac{tg(t^2)}{t^2-x^2}\ dt = \frac{\pi}{2}\left[\frac{P}{\mu} + \frac{C_1}{x^2-a^2}\right],\quad x \in I_4$$

Again using finite Hilbert transform technique [9], we obtain

$$g(t^2) = \frac{P}{\mu}\sqrt{\frac{(t^2-a^2)(t^2-c^2)}{(t^2-b^2)(d^2-t^2)}} + \sqrt{\frac{d^2-a^2}{c^2-a^2}}\ \frac{C_1\sqrt{t^2-c^2}}{\sqrt{(t^2-a^2)(t^2-b^2)(d^2-t^2)}} +$$

$$+ \frac{C_2\sqrt{t^2-a^2}}{\sqrt{(t^2-b^2)(t^2-c^2)(d^2-t^2)}} \qquad (22)$$

where we have used

$$\int_c^d \sqrt{\frac{d^2-x^2}{x^2-c^2}}\ \frac{x\ dx}{(x^2-a^2)(x^2-t^2)} = -\frac{\pi}{2}\sqrt{\frac{d^2-a^2}{c^2-a^2}}\ \frac{1}{t^2-a^2}$$

and the constant C_2 is to be determined using the condition given by equation (20).

Next substituting the value of $g(t^2)$ from (22) in equation (21) and finally using the following results

$$\int_c^d \sqrt{\frac{t^2-c^2}{d^2-t^2}}\ \frac{t\ dt}{(t^2-a^2)(t^2-s^2)} = \frac{\pi}{2}\left[\sqrt{\frac{c^2-a^2}{d^2-a^2}} - \sqrt{\frac{c^2-s^2}{d^2-s^2}}\right]\frac{1}{s^2-a^2}$$

$$\int_c^d \frac{t\,dt}{(t^2-s^2)\sqrt{(t^2-c^2)(d^2-t^2)}} = \frac{\pi}{2\sqrt{(c^2-s^2)(d^2-s^2)}} \quad \text{for } s \in I_2$$

$h(s^2)$ is derived in the form

$$h(s^2) = \frac{P}{\mu}\sqrt{\frac{(s^2-a^2)(c^2-s^2)}{(b^2-s^2)(d^2-s^2)}} + \sqrt{\frac{d^2-a^2}{c^2-a^2}}\;\frac{C_1\sqrt{c^2-s^2}}{\sqrt{(s^2-a^2)(b^2-s^2)(d^2-s^2)}} - \frac{C_2\sqrt{s^2-a^2}}{\sqrt{(b^2-s^2)(c^2-s^2)(d^2-s^2)}} \qquad (23)$$

To determine the values of the unknown constants C_1 and C_2 we substitute $g(t^2)$ and $h(s^2)$ given by (22) and (23) in (20) and obtain

$$C_1 = \frac{K^{c,d}_{a,b}\, I^{a,b}_{c,d} + K^{a,b}_{c,d}\, J^{c,d}_{a,b}}{I^{c,d}_{a,b}\, I^{a,b}_{c,d} + J^{c,d}_{a,b}\, J^{a,b}_{c,d}}\;\frac{P}{\mu}\sqrt{\frac{c^2-a^2}{d^2-a^2}}$$

$$C_2 = \frac{K^{a,b}_{c,d}\, I^{c,d}_{a,b} - K^{c,d}_{a,b}\, J^{a,b}_{c,d}}{I^{c,d}_{a,b}\, I^{a,b}_{c,d} + J^{c,d}_{a,b}\, J^{a,b}_{c,d}}\;\frac{P}{\mu}$$

where

$$I^{r,s}_{p,q} = \int_p^q \frac{\sqrt{x^2-r^2}\;dx}{\sqrt{(x^2-p^2)(x^2-q^2)(s^2-x^2)}}$$

$$J^{r,s}_{p,q} = \int_p^q \frac{\sqrt{x^2-p^2}\;dx}{\sqrt{(x^2-q^2)(x^2-r^2)(s^2-x^2)}}$$

$$K_{p,q}^{r,s} = - \int_p^q \frac{\sqrt{(x^2 - p^2)(x^2 - r^2)}}{\sqrt{(x^2 - q^2)(s^2 - x^2)}}\, dx$$

The relevant displacement and stress components in the plane of crack can now be shown to be given by

$$v(x,0) = \int_x^b h(s^2)\, ds\ , \qquad a \le x \le b$$

$$= \int_x^d g(t^2)\, dt\ , \qquad c \le x \le d \qquad (24a,b)$$

and

$$[\sigma_{yy}(x,o)]_{0<x<a} = -\frac{2\mu K}{\pi}\left[\int_a^b \frac{s h(s^2)}{s^2 - x^2}\, ds + \int_c^d \frac{t g(t^2)}{t^2 - x^2}\, dt\right]$$

$$[\sigma_{yy}(x,o)]_{b<x<c} = \frac{2\mu K}{\pi}\left[\int_a^b \frac{s h(s^2)}{x^2 - s^2}\, ds - \int_c^d \frac{t g(t^2)}{t^2 - x^2}\, dt\right]$$

$$[\sigma_{yy}(x,o)]_{x>d} = \frac{2\mu K}{\pi}\left[\int_a^b \frac{s h(s^2)}{x^2 - s^2}\, ds + \int_c^d \frac{t g(t^2)}{x^2 - t^2}\, dt\right] \qquad (25a\text{-}c)$$

Insertion of the values of $h(s^2)$ and $g(t^2)$ as given by the equations (22) and (23) in the expressions (25) yields after some algebraic manipulation,

$$[\sigma_{yy}(x,o)]_{0<x<a} = -\frac{2\mu K}{\pi}\left[F_1(x)+F_2(x)+F_3(x)+F_4(x)+F_6(x)+F_7(x)\right]$$

$$[\sigma_{yy}(x,o)]_{b<x<c} = -\frac{2\mu K}{\pi}\left[F_1(x)+F_2(x)+F_3(x)+F_4(x)-F_5(x)-F_8(x)\right]$$

$$[\sigma_{yy}(x,o)]_{x>d} = -\frac{2\mu K}{\pi}\left[F_1(x)+F_2(x)+F_3(x)+F_4(x)-F_6(x)-F_7(x)\right]$$

(26a-c)

where

$$F_1(x) = \left[\frac{P}{\mu}(c^2-a^2) - C_2 \right]\left[1 - \sqrt{\frac{a^2-x^2}{b^2-x^2}} \right] \frac{\pi}{2\sqrt{(c^2-a^2)(d^2-a^2)}}$$

$$F_2(x) = \int_a^b \left[\frac{P}{\mu}(d^2-c^2) - C_2 \frac{2u^2-d^2-c^2}{c^2-u^2} \right] \frac{g_1(u,x)}{d^2-u^2}\, du$$

$$F_3(x) = \left[\frac{P}{\mu}(c^2-a^2) + C_1 \sqrt{\frac{d^2-a^2}{c^2-a^2}} \right]\left[1 - \sqrt{\frac{c^2-x^2}{d^2-x^2}} \right] \frac{\pi}{2\sqrt{(c^2-a^2)(c^2-b^2)}}$$

$$F_4(x) = \int_c^d \left[\frac{P}{\mu}(b^2-a^2) + C_1 \sqrt{\frac{d^2-a^2}{c^2-a^2}}\ \frac{2u^2-a^2-b^2}{u^2-a^2} \right] \frac{g_2(u,x)}{u^2-b^2}\, du$$

$$F_{5,6}(x) = \frac{\pi}{2}\sqrt{\frac{d^2-a^2}{d^2-b^2}} \left[\frac{C_1}{X_1}\sqrt{\frac{c^2-b^2}{c^2-a^2}} \mp \frac{C_2}{X_2} \right]$$

$$F_{7,8}(x) = \frac{C_1}{X_1}\sqrt{\frac{d^2-a^2}{c^2-a^2}}\ L^{a,b}_{c,d}(x) \mp \frac{C_2}{X_2}\ L^{c,d}_{a,b}(x)$$

$$g_1(u,x) = \frac{u}{\sqrt{(d^2-u^2)(c^2-u^2)}} \left[\sqrt{\frac{a^2-x^2}{b^2-x^2}}\ \tan^{-1}\sqrt{\frac{(a^2-x^2)(b^2-u^2)}{(b^2-x^2)(u^2-a^2)}} - \tan^{-1}\sqrt{\frac{b^2-u^2}{u^2-a^2}} \right]$$

$$g_2(u,x) = \frac{u}{\sqrt{(u^2-b^2)(u^2-a^2)}} \left[\sqrt{\frac{c^2-x^2}{d^2-x^2}}\ \tan^{-1}\sqrt{\frac{(c^2-x^2)(d^2-u^2)}{(d^2-x^2)(u^2-c^2)}} - \tan^{-1}\sqrt{\frac{d^2-u^2}{u^2-c^2}} \right]$$

$$X_1 = \sqrt{(x^2 - a^2)(x^2 - b^2)}$$

$$X_2 = \sqrt{(x^2 - c^2)(x^2 - d^2)}$$

$$L^{p,q}_{r,s}(x) = \int_p^q \frac{(s^2 - r^2)u \tan^{-1}\sqrt{\dfrac{(u^2 - p^2)(x^2 - q^2)}{(q^2 - u^2)(x^2 - p^2)}}}{\sqrt{(s^2 - u^2)^3(r^2 - u^2)}}\, du$$

(27a-k)

4.Stress Intensity Factors:

The dynamic stress intensity factors are given by

$$N_a = \operatorname{Lt}_{x \to a-} \sqrt{2(a-x)}\left[\sigma_{yy}(x,0)\right]_{0<x<a}$$

$$N_b = \operatorname{Lt}_{x \to b+} \sqrt{2(x-b)}\left[\sigma_{yy}(x,0)\right]_{b<x<c}$$

$$N_c = \operatorname{Lt}_{x \to c-} \sqrt{2(c-x)}\left[\sigma_{yy}(x,0)\right]_{b<x<c}$$

$$N_d = \operatorname{Lt}_{x \to d+} \sqrt{2(x-d)}\left[\sigma_{yy}(x,0)\right]_{x>d} \qquad (28a\text{-}d)$$

Employing (26) in (28) we obtain

$$N_a = -\frac{\mu K C_1}{\sqrt{a(b^2 - a^2)}}$$

$$N_b = \mu K \left[\frac{P}{\mu}\sqrt{\frac{(b^2 - a^2)(c^2 - b^2)}{b(d^2 - b^2)}} + C_1\sqrt{\frac{(d^2 - a^2)(c^2 - b^2)}{b(b^2 - a^2)(d^2 - b^2)(c^2 - a^2)}} - \right.$$

$$\left. - C_2\sqrt{\frac{(b^2 - a^2)}{b(c^2 - b^2)(d^2 - b^2)}} \right]$$

$$N_c = -\frac{\mu K C_2}{\sqrt{c(d^2-c^2)}}\sqrt{\frac{c^2-a^2}{c^2-b^2}}$$

$$N_d = \mu K\left[\frac{P}{\mu}\sqrt{\frac{(d^2-a^2)(d^2-c^2)}{d(d^2-b^2)}} + C_1\sqrt{\frac{(d^2-c^2)}{d(c^2-a^2)(d^2-b^2)}} + C_2\sqrt{\frac{(d^2-a^2)}{d(d^2-c^2)(d^2-b^2)}}\right] \tag{29}$$

It is interesting to note that the crack opening displacements depend on the crack velocity V but in the plane of the cracks the stresses and stress intensity factors are independent of the velocity of the moving cracks in an infinite elastic medium.

5. Statement Of Problem II And Its Formulation

In this case, we consider an infinite homogeneous isotropic material with four coplanar Griffith cracks located at $Y = 0$, $a \le |X| \le b$, $c \le |X| \le d$ and subjected to uniform internal pressure q. In absence of body force, the equations of equilibrium in terms of displacement are

$$(\lambda+2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = 0$$

and

$$(\lambda+2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = 0 \tag{30a,b}$$

Since the problem exhibits a state of symmetry about $Y = 0$, we can restrict our attention to a single half-space occupying the region $Y \ge 0$.

The equations (30) are to be solved subject to the boundary conditions

$$v(X,0) = 0, \quad |X| \le a, \quad b \le |X| \le c, \quad |X| \ge d \qquad (31a\text{-}c)$$

$$\sigma_{XY}(X,0) = 0, \quad -\infty < X < \infty \qquad (32)$$

$$\sigma_{YY}(X,0) = -q, \quad a \le |X| \le b, \quad c \le |X| \le d \qquad (33a,b)$$

In view of the boundary conditions, appropriate integral solutions of equation (30) are

$$u(X,Y) = \frac{2}{\pi}\int_0^\infty \left[C(\xi) + D(\xi)\left\{ Y - \frac{1}{\xi}\frac{\lambda+3\mu}{\lambda+\mu} \right\} \right] e^{-\xi Y} \sin(\xi X)\, d\xi$$

and $$v(X,Y) = \frac{2}{\pi}\int_0^\infty \left[C(\xi) + Y\, D(\xi) \right] e^{-\xi Y} \cos(\xi X)\, d\xi \qquad (34a,b)$$

Therefore,

$$\sigma_{YY}(X,Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{ Y\xi - \frac{\mu}{\lambda+\mu} \right\} D(\xi) \right] e^{-\xi Y} \cos(\xi X)\, d\xi$$

$$\sigma_{XY}(X,Y) = -\frac{4\mu}{\pi}\int_0^\infty \left[\xi C(\xi) + \left\{ Y\xi - \frac{\lambda+2\mu}{\lambda+\mu} \right\} D(\xi) \right] e^{-\xi Y} \sin(\xi X)\, d\xi \qquad (35a,b)$$

It may be noted that the displacement and stress components given by (34) and (35) can not be derived from the corresponding expressions of the dynamic problem given in (11) and (12) on setting M = 0.

The functions $C(\xi)$ and $D(\xi)$ are to be determined from the boundary conditions (31)-(33), which yield

$$C(\xi) = \frac{1}{\xi}\frac{\lambda+2\mu}{\lambda+\mu} D(\xi) \qquad (36)$$

and the following set of five integral equations

$$\int_0^\infty C(\xi)\cos(\xi X)\, d\xi = 0, \quad X \in I_1, I_3, I_5 \qquad (37a\text{-}c)$$

$$\int_0^{\infty}\xi\ C(\xi)\cos(\xi X)\ d\xi = \frac{Q\pi}{2\mu}, \quad X \in I_2, I_4 \qquad (38a,b)$$

where

$Q = \dfrac{(\lambda+2\mu)}{2(\lambda+\mu)}\, q$ and I_j $(j=1,2,..,5)$ are the intervals defined earlier in problem I.

6.Method Of Solution And Quantities Of Physical Interest

Integral equations given by (37) and (38) are found to be the same as given by equations (17) and (18) with the exception that P is replaced by Q. Therefore, the same technique as that used in problem I can be employed to obtain

$$v(X,0)=\int_X^b\left[\frac{Q}{\mu}\sqrt{\frac{(s^2-a^2)(c^2-s^2)}{(b^2-s^2)(d^2-s^2)}}+\sqrt{\frac{d^2-a^2}{c^2-a^2}}\ \frac{C_1\sqrt{c^2-s^2}}{\sqrt{(s^2-a^2)(b^2-s^2)(d^2-s^2)}}-\right.$$

$$\left.-\frac{C_2\sqrt{s^2-a^2}}{\sqrt{(b^2-s^2)(c^2-s^2)(d^2-s^2)}}\right]ds, \qquad a\leq X\leq b$$

$$=\int_X^d\left[\frac{Q}{\mu}\sqrt{\frac{(t^2-a^2)(t^2-c^2)}{(t^2-b^2)(d^2-t^2)}}+\sqrt{\frac{d^2-a^2}{c^2-a^2}}\ \frac{C_1\sqrt{t^2-c^2}}{\sqrt{(t^2-a^2)(t^2-b^2)(d^2-t^2)}}+\right.$$

$$\left.+\frac{C_2\sqrt{t^2-a^2}}{\sqrt{(t^2-b^2)(t^2-c^2)(d^2-t^2)}}\right]dt, \qquad c\leq X\leq d$$

(39a,b)

Stresses in the regions $0<X<a$, $b<X<c$, $X>d$ are found to be the same as that given in (26), the only change being that P is to replaced by Q.

Amount of energy in opening the cracks $a \le |X| \le b$, $c \le |X| \le d$ are given by $E = 2E_1 + 2E_2$, where

$$E_1 = 2\left| \int_a^b [\sigma_{YY}(X,0)\ v(X,0)]\ dX \right|$$

$$E_2 = 2\left| \int_c^d [\sigma_{YY}(X,0)\ v(X,0)]\ dX \right| \qquad (40a,b)$$

Equations (40) can be simplified, with the aid of (33) and (39), to

$$E_1 = -2q\left[\frac{Q}{\mu} M_{a,b}^{c,d} + (c^2-b^2)L_1\, \Pi\left\{\frac{\pi}{2}, \frac{b^2-a^2}{c^2-a^2}, r\right\} + \frac{(c^2-a^2)C_2 - c^4\frac{Q}{\mu}}{\sqrt{(d^2-b^2)(c^2-a^2)}} F(\frac{\pi}{2}, r)\right]$$

$$E_2 = 2q\left[\frac{Q}{\mu} M_{c,d}^{a,b} - (d^2-a^2)L_2\, \Pi\left\{\frac{\pi}{2}, \frac{c^2-d^2}{c^2-a^2}, r\right\} - \frac{\sqrt{(c^2-a^2)(d^2-a^2)}\ C_1 + a^4\frac{Q}{\mu}}{\sqrt{(d^2-c^2)(c^2-a^2)}} F(\frac{\pi}{2}, r)\right]$$

where

$$L_{1,2} = \frac{\left[(a^2+c^2)\frac{Q}{\mu} \mp C_1\sqrt{\frac{d^2-a^2}{c^2-a^2}} \mp C_2\right]}{\sqrt{(d^2-b^2)(c^2-a^2)}},$$

$$r = \sqrt{\frac{(d^2-c^2)(b^2-a^2)}{(d^2-b^2)(c^2-a^2)}}, \quad 2M_{p,q}^{r,s} = \int_{p^2}^{q^2} \frac{z^2\, dz}{\sqrt{(z-p^2)(z-q^2)(z-r^2)(s^2-z)}}$$

and $F(\phi, r)$, $\Pi(\phi, n, r)$ are the elliptic integrals of first and third kinds respectively.

7. Numerical Results and Discussions

Numerical results for the stress intensity factors and crack opening displacement, defined as $\Delta v(x,0) = v(x,0^+) - v(x,0^-)$, for different values of the parameters are presented in this section. Numerical calculations have been carried out for both the dynamic and static problems. As the crack velocity is less than Rayleigh wave velocity, it is reasonable to take the value of M less than 0.9194 .

Problem I: Variations of crack opening displacement for different values of crack speed, crack lengths and the separating distance between the cracks have been plotted in figures 2-4. It is interesting to note from these graphs that crack opening displacement on both the cracks decreases with the increase in the value of M at the onset and takes its minimum value at M=0.7415 ,after which it increases with the increase in the value of M. It has also been depicted in figures 3-4 that on each of the cracks,crack opening displacement decreases as the crack length decreases.

It has been mentioned earlier that the stress intensity factors at the crack tips are independent of crack speed and are found to depend on the crack lengths and the separating distance between the cracks. Variation of stress intensity factors with a/b for different values of c/b, d/b and that with c/b for different values a/b, d/b are plotted in figures 5-8 and figures 9-12 respectively.

It has been found that the effect of variation of the length of either the inner or the outer pair of cracks is more prominent on the stress intensity factors at the edges of the cracks whose lengths are varying compared to its effect on the stress intensity factors at the tips of the cracks whose lengths are kept fixed.

Problem II: Figures 13-15 show the variations of crack opening displacement for different values of the parameters a/b, c/b, d/b. They exhibit that crack opening displacement on a crack of fixed length increases with the increase in the length of the other crack as expected from physical stand point.

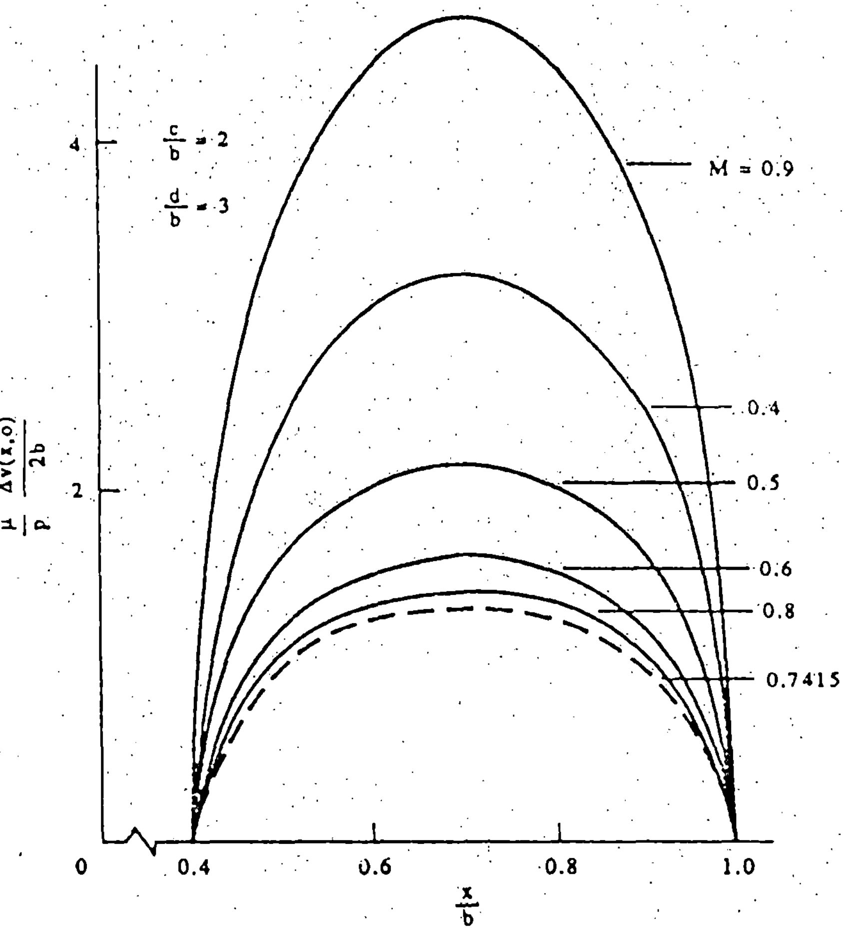

Fig. 2. Variation of crack opening displacement with x/b on the crack of the outer pair for problem I.

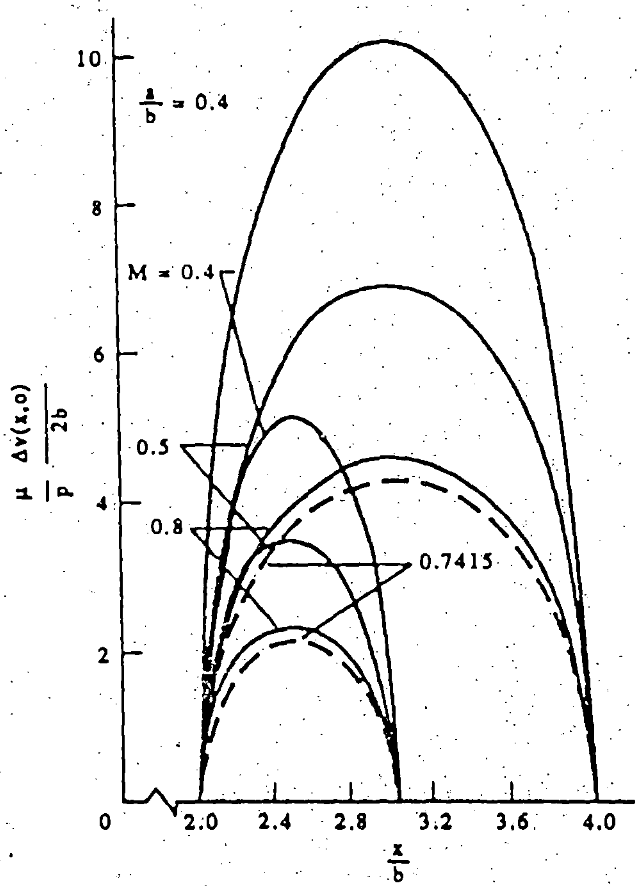

Fig. 3. Variation of crack opening displacement with x/b on the crack of the inner pair for problem I.

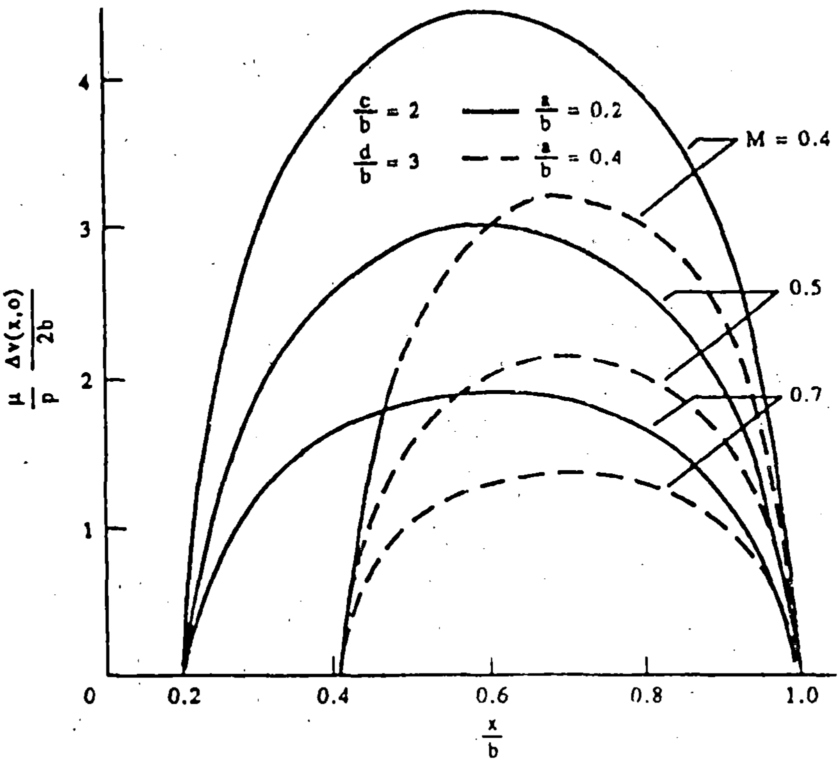

Fig. 4. Variation of crack opening displacement with x/b on the crack of the inner pair for problem I.

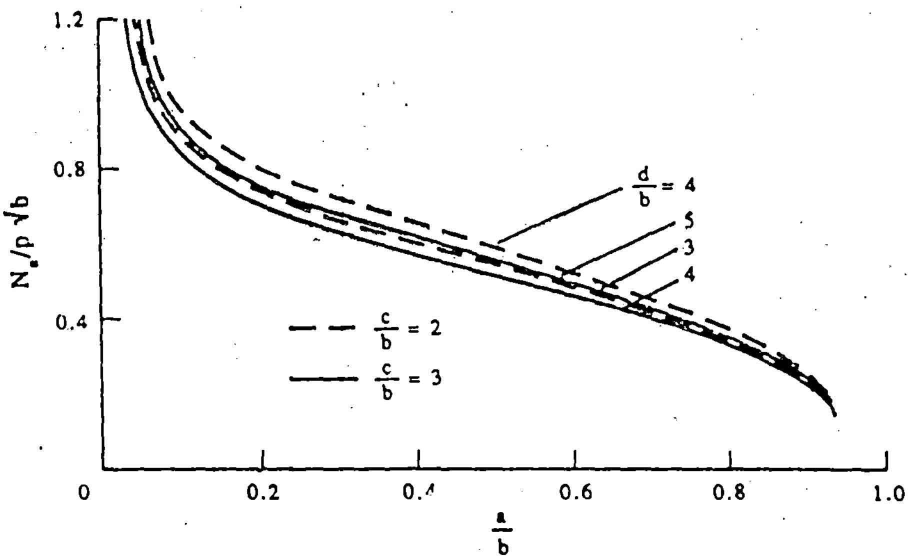

Fig. 5. Stress intensity factor vs a/b at the edge $x = a$.

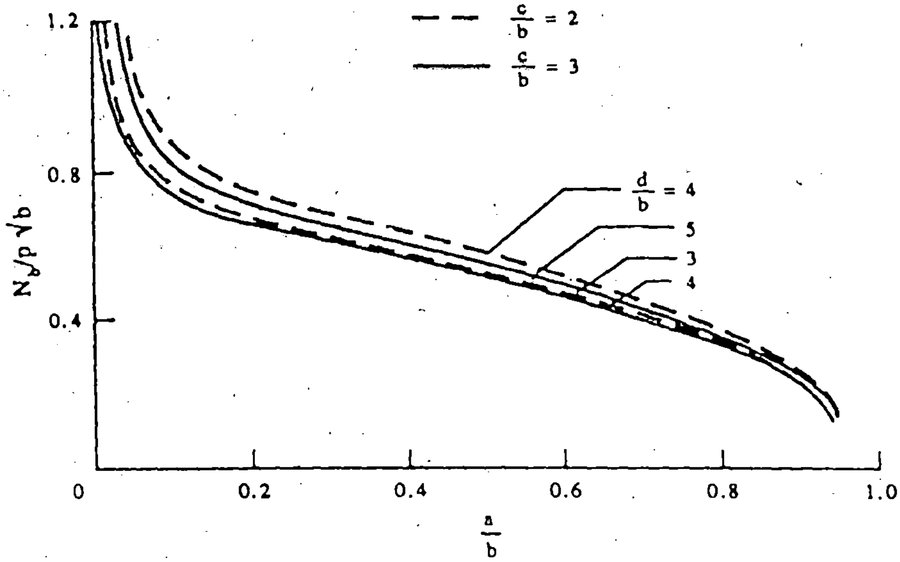

Fig. 6. Stress intensity factor vs a/b at the edge $x = b$.

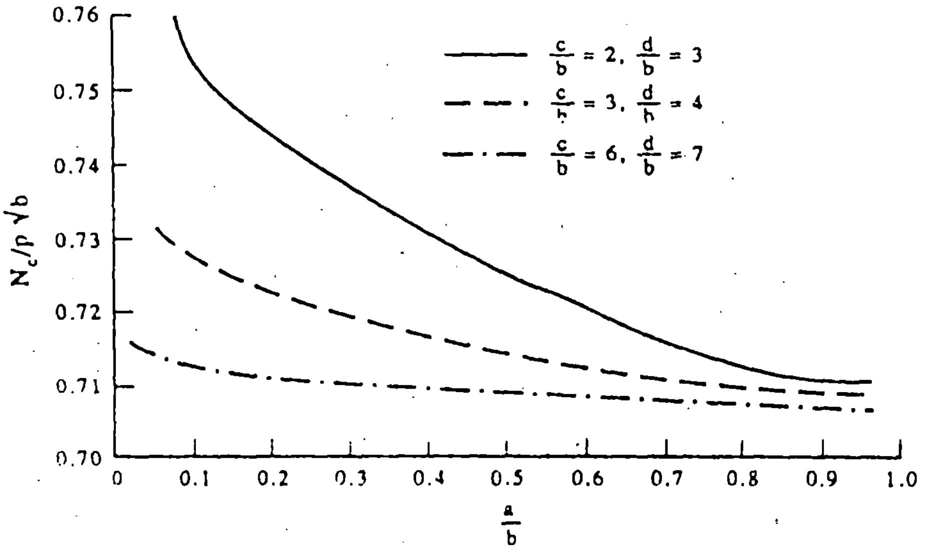

Fig. 7. Stress intensity factor vs a/b at the edge $x = c$

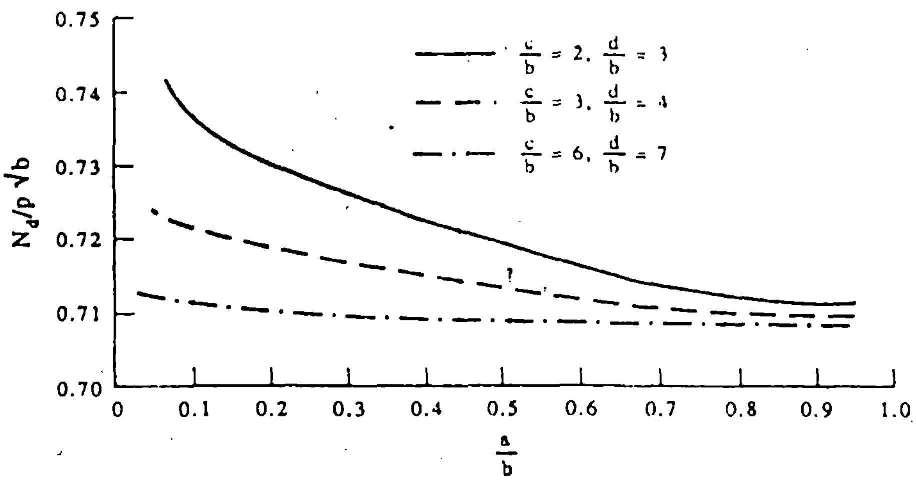

Fig. 8. Stress intensity factor vs a/b at the edge $x = d$.

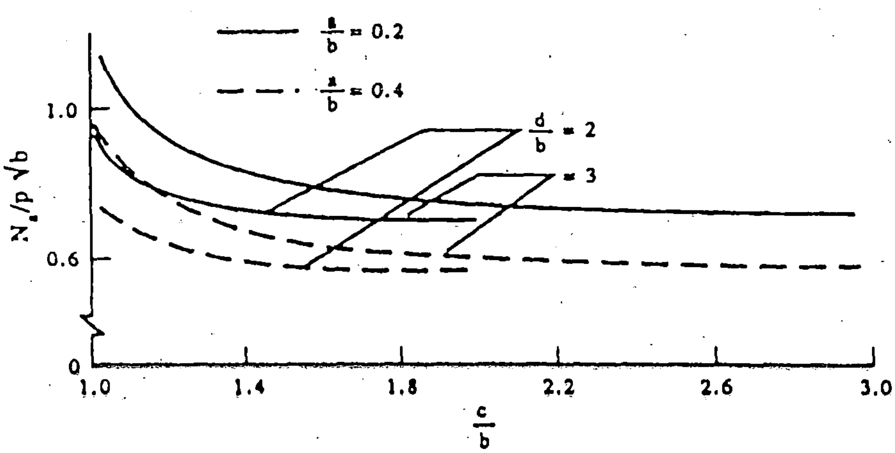

Fig. 9. Stress intensity factor vs c/b at the edge $x = a$.

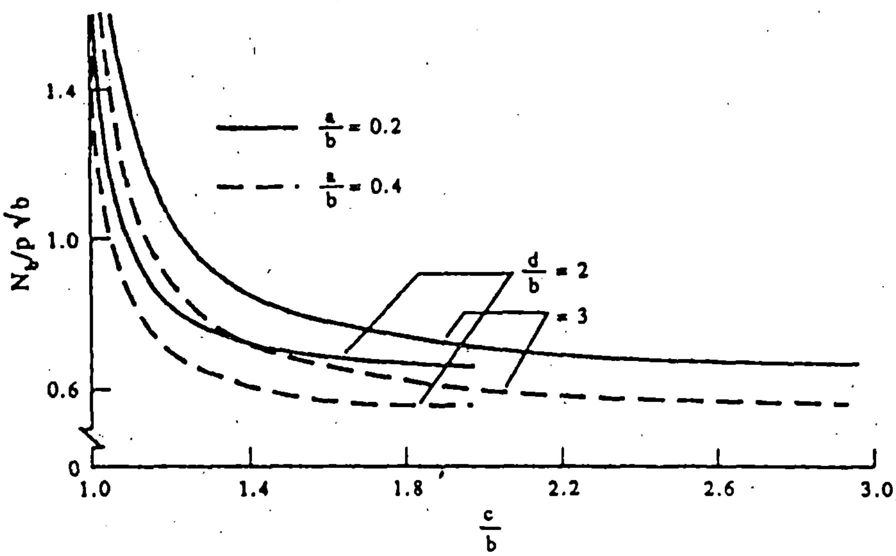

Fig. 10. Stress intensity factor vs c/b at the edge $x = b$.

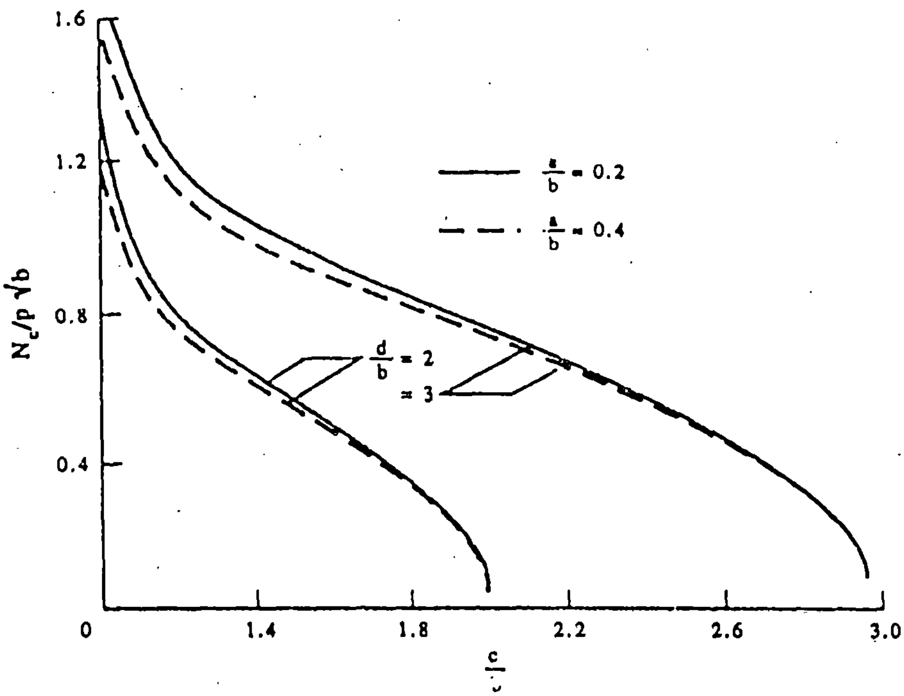

Fig. 11. Stress intensity factor vs c/b at the edge $x = c$.

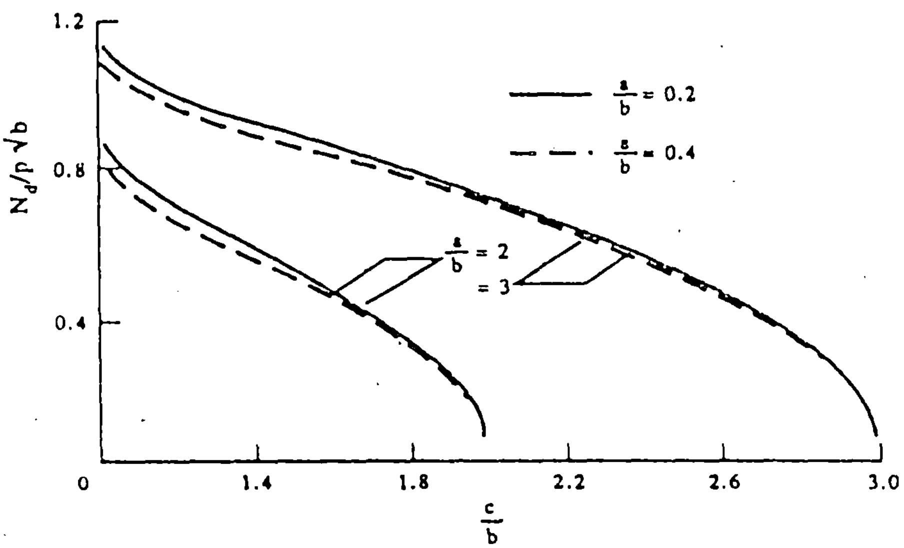

Fig. 12. Stress intensity factor vs c/b at the edge $x = d$.

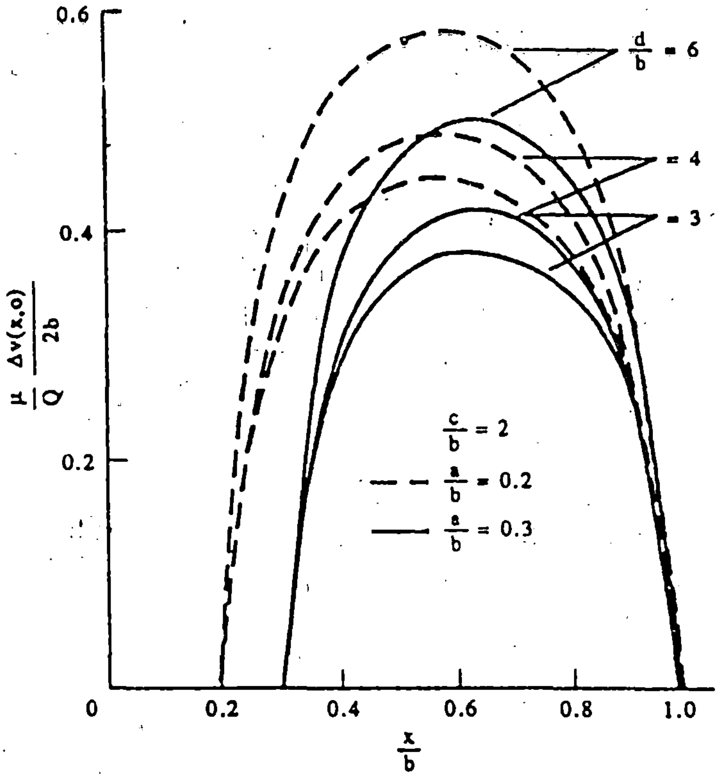

Fig. 13. Variation of crack opening displacement with X/b on the crack of the inner pair for problem II.

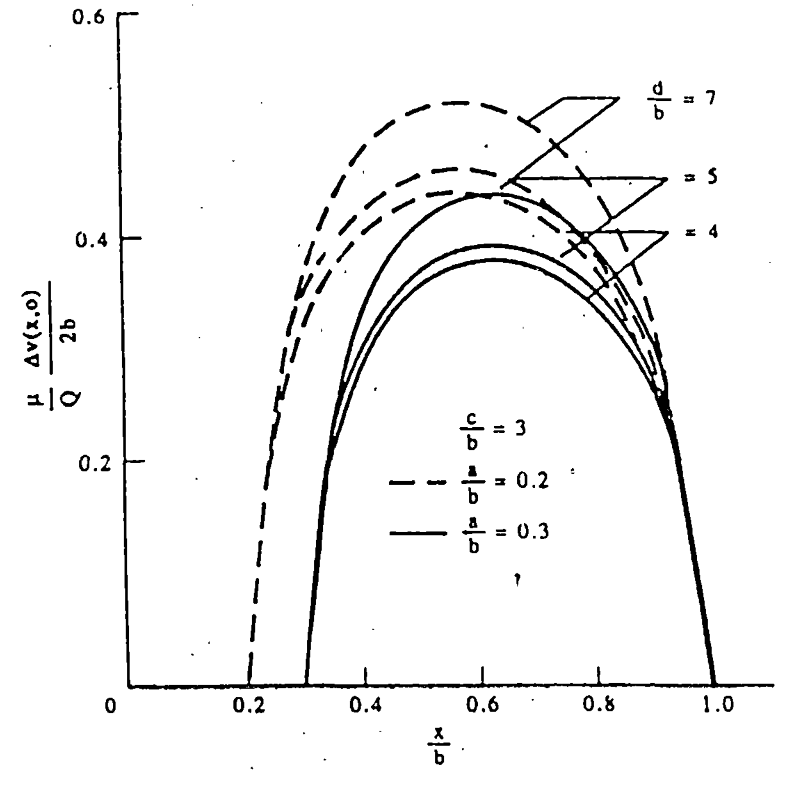

Fig. 14. Variation of crack opening displacement with X/b on the crack of the inner pair for problem II.

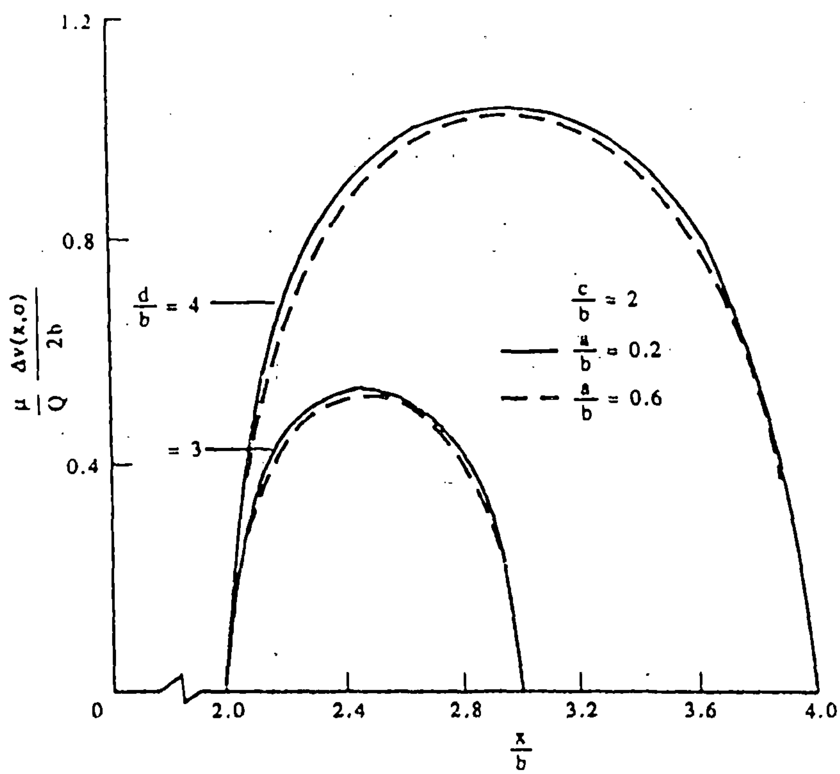

Fig. 15. Variation of crack opening displacement with X/b on the crack of the outer pair for problem II.