Application

of

" Constant Deflection Countour "

Method

To Problems of Dynamic Response of Structures.

Thesis Submitted

By

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Contents

PREFACE

The present thesis deals mainly with some problems of practical interest in the field of structural and mechanical engineering. The topic covers a wide area in theoretical conception which is beyond the expectation of one's objective to investigate into every aspect of the topic. Hence the present author restricts herself mainly to some problems related to the static and dynamic behaviour of structures, to be more precise, plate and shell structures are considered for illustrative examples.

Considering the importance of vibrational characteristic of structures, the nonlinear analysis of elastic plates and shells received considerable attention in literature since the sixth decade of 20th century. In general Karman type field equations are employed for almost all types of structures. Several methods are available to investigate the static and dynamic response of structures amongst which the "Constant Deflection Contour" method needs special mention. This method has been previously developed by Mazumdar, however the most of his investigations were restricted to linear analysis only. The present thesis aims at extending this method to the nonlinear analysis of plates vibrating at large amplitudes in conjunction with Galerkin procedure. Illustrations of the present studies have been considered to cover those problems which are either new investigations or treated with this new approach to previously investigated problems. The results of the present investigations have always been compared to known available results so far as possible. Starting from structure having regular and common boundaries gradually more and more complicated structures and mixed boundary value problems have been included in the present thesis.

The first chapter concerns with the comprehensive study of the early researches in the area under investigation and with an introduction of the basic need of such problems arising out of the future demand of the twenty first century. A review of the early investigations is cited chronologically so far as possible.

The second chapter contains the basic nonlinear theory of elasticity. This theory constitutes the main frame of operation for the present investigation under consideration.

In the third chapter different methods for solving the non-linear problems have been discussed with their merits and demerits. The third chapter also includes some preliminary remarks about the "Constant Deflection Contour" method.

In all problems considered from Chapter IV the basic governing differential equations have been deduced primarily on the basis of Karman field equations extended to a dynamic case. In chapter IV, a simplified method for solving nonlinear problems using "Constant Deflection Contour" method has been discussed.

The first problem (Chapter IV) aims at investigating the static and dynamic behaviour of elliptic. platesclamped along its boundary. Large vibrations of elliptic plates on elastic foundation have been discussed in the second problem of chapter IV. In problem -3 (Chapter IV), the static and dynamic behaviour of elliptic plates under damping condition is discussed using Berger's hypothesis. Attempts have also been made to justify the use of Karman field equations over other simplified or modified equations. Problem 4, (chapter IV) concerns with effect of varying flexural rigidity on nonlinear vibrations of circular and elliptical plates.

In chapter V, a modified method for solving the nonlinear problems is discussed some typical examples have been discussed with some more complicated problems. These illustrations not only support the proposed theory but also establish the accuracy of numerical results for mixed boundary value problems.

In chapter VI the extension of "Constant Deflection Contour" method to shell structures has been made to break the monotony of plate structures.

In Chapter VII an attempt has been made to extend the present analysis to elastic plastic shells structures.

At last the author has tried to refer all relevant papers published so far. However none can avoid omissions when choosing a few out of thousands of papers, but any omissions is absolutely unintentional.

English being the sencond language of the author, she begs apology beforehand for any linguistic error or absence of any proper expression of the text that may creep-in in the preparation of the work.

The author expresses her heartiest gratefulness to her guide Dr. M.M. Banerjee [Ex Reader, A.C. College, Jalpaiguri, West Bengal, India]. Without his help it would not have been possible for her to complete the thesis.

: Notations :

u	=	In plane displacement along x axis.
v	=	In plane displacement along y axis
w	=	Displacement normal to the middle surface of the structure
h	=	Thickness of the plate
ε_x	=	Strain Component along x axis
ε_y	=	Strain Component along y axis
ε_{xy}	=	Shearing Strain Component in cartesian Co-ordinate.
E	=	Young's Modulus
G	=	Shear Modulus of Elasticity and Pasternak Foundation Parameter
ν	=	Poisson's ratio
U	=	Potential Energy
N_x, N_y	=	Force per Unit length in the x, and y direction
N_{xy}	=	Force per Unit length in x-y direction
M_x	=	Bending moment per unit length of the section of the structure perpendicular to x axis
M_y	=	Bending moment per unit length of the section of the structure perpendicular to x axis.
M_{xy}	=	Twisting moment per unit length
σ_x	=	Normal component of stress parallel to x - axis
σ_y	=	Normal component of stress parallel to y axis
σ_{xy}	=	Normal component of stress along x-y direction
ρ	=	Density of the material of the structure

D	=	$\frac{Eh^3}{12(1-\nu^2)}$ = Flexural rigidity
F	=	Stress Function
K_1	=	Curvature along x- axis
K_2	=	Curvature along y- axis
p	=	Normal load intensity
K	=	Constant related to elastic (Winkler Foundation) parameter
T	=	Time period
e_1	=	First Strain invariant of the middle surface of the plate
e_2'	=	Second Strain invariant of the middle surface of the plate
A	=	Amplitude of vibrations

CHAPTER - 1

BIBLIOGRAPHY

With the progress of modern civilization, the applications of elastic and plastic properties of solids in the field of structural Engineering are gaining momentum day by day. Most of the modern structurs are subjected to severe vibrations and hence it becomes a must to design structural components to withstand high dynamic stresses. Dynamic characteristics of structural system are essential for design and control.

Since Robert Hook (1635-1716) gave a simple relation between stress and strain of elastic substance many research workers become inclined to investigate the elastic behaviour of matter. Since then the linear analysis of plates and shells has long attracted the attention of several investigartors resulting in a wealth of papers published by several authors. An extensive study on this subject has been presented by Gontkevich[1] and later by Leissa [2].

A study on the dynamic response of structures reveals the fact that much has been investigated so far as linear analysis is concerned as determination of the natural frequency plays important role in designing a structure prone of vibration. Some of the works previously made may include the problem of symmetrical bending of circular and rectangular plates of variable thickness investigated by H. Holzer[3], R.G. Olson[4] and H.D. Conway[5].

The linear frequencies of in-plane vibrations of polar orthotropic annular plates with linearly varying thickness have been analyzed by Ganesan and Soamidas[6]. A semi-analytical method of analysis has been used where the radial and tangential displacements are expanded in the circumferential direction as Fourier series and the radial behaviour is solved using finite element method and the frequencies have been studied with respect to various boundary conditions, aspect ratio, thickness ratios, ratio of moduli and two fiber directions. The vibration and stability analysis of polar orthotropic circular plates using the finite element method is discussed by Gerard and C-Pardoen[7].

Free torsional vibrations of conical and cylindrical shells of thickness varying as a power of distance have been studied by Soni, Jain and Prasad[8]. The numerical values of the frequency parameter for the first three modes of vibration are computed for shells of linearly and parabolically varying thickness for different ratio of treminal radii.

Laura et.al. [84] analyzed the vibration and stability of a circular plate elastically restrained against rotation. Forced vibration of a circular plate elastically restrained against rotation has been discussed by Laura el.al.[85].

The resonant response of simply supported thin and thick orthotropic cylindrical shell is determined by Warburton and Soni [9] by using model analysis.

The transverse vibration of free elliptical plates with rectangular orthotropy is analyzed by T. Naritra[10], while the natural frequencies of rectangular and polygonal plates have been obtained by R.B. Bhat [11,12]. Dickinson and Blasio[13] analyzed the use of orthogonal polynomials to study the flexural vibration and buckling of isotropic and orthotropic rectangular plates. Sing and Chakraborty [14,15] studied the transvese vibrations of circular and elliptic plates of variable thickness. Flexural vibration of skew plates was investigated by Singh and Chakraborty [16].

Unfortunately, the linear classical theory is no longer applicable in cases of practical interest and this leads to the non linear analysis of such problem. The analysis of non linear vibration of plates and shells with their symmetrical and unsymmetrical bending character has drawn the attention of many research workers because of their applications in engineering design. It is , however, difficult to solve the vibration problems of plates and shells of practical interest due to their highly non-linear behaviour. A number of research workers tried to solve the necessary differential equation by linearzing those through proper approximation. The results thus obtained do not agree with the experimental results for complicated plate geometry which are actually used in practice. Approximate solutions of such problems can be obtained from Karman [20] field equations. These equations involve the deflection and membrane stress functions as two dependent variables coupled together. Many workers have used Karman equations to solve the vibrational problems of elastic plates, among which Chu and Hermann [21] and Yamaki [22] need special mention.

Several techniques have been used to solve the equations; for example Levy [23] subtitutes a double Fourier series in the equation for rectangular plates. Chi-Teh-Wang [24] wrote the equation for rectangular plates in a finite difference form and solved them by the method of successive approximations. S.Way, [117] solved the circular plate equations by substituting a power series solution into the energy expression determining the coefficient by setting the first variation of the strain energy equal to the first variation of potential energy due to the external loading for any variation of each coefficient. Many investigators used Von-Karman equations to analyze the non-linear vibrational problems of plates of various shape. Rectangular plates were analyzed by Smith, Malme and Gogos [25] Yamaki [22]; Eisely [26]; Murthy and Shebourne [27]; Bayles, Lowery and Boyd [28]; Cawford and Atluri [29]. Circular plates are treated by Crawford and Atluri[30]; Fornsworth and Evan-Iwanowski [31]; Sridhar, Mook, Nayfeh [32].Ring sector plates were treated by Chisyaki and Takashi [33] and elliptic plates were treated by Lobitz, Nayfeh and Mook [34]. Wu and Vinsion [86] investigated the influence of large amplitudes, transverse shear deformation and rotatory inertia on lateral vibrations of transversely isotropic plates.

Vendhan and Das [35] investigated the non-linear vibration of plates by the application of Raylegh Ritz and Galerkin methods to the Von-Karman equation, expressed in terms of the three displacement variables, governing the non-linear dynamic behaviour of thin elastic plate. It was seen that the Rayleigh-Ritz approximation are consistently better than the Galerkin approximation, which however, tend to be equally good after a few terms. It was also noted that the above two aproximations are idential for a linear problems and the difference between them is solely due to non-linearity, even though they tend to ultimately converge to a common value. J. Ramchandran [134] studied the free vibration of rectangular plates carrying concentrated mass .

The large amplititude free flexural vibrations of thin elastic anisotropic skew plates were studied by Prathap and Vardhan [36]. They used Von-Karman field equations in which the governing non-linear dynamic equations are derived in terms of the stress-function and the lateral displacement. Clamped boundary conditions are chosen and in-plane edge conditions considered are either immovable or movable. Solutions are obtained by Galerkin method. The relationship between amplitude and period of frequency was shown to exhibit a hardening type non-linearity, irrespective of the boundary conditions, skew angle, angle of fiber oreientation or aspect ratio.

Non-linear transverse vibrations of elastic orthotropic shells were investigated by Nowinski [37] using Von Karman-Tsien equations, generalized to dynamic and orthotropic case. A sharp decrease of the period of non linear vibrations with an increase in amplitude was corroborated the mode pattern influencing the period more than the degree of anisotropy.

B.R.El. Zaouk and C.L.Dym [38] studied the effect of curvature, meterial orthotropy and internal pressure upon the non-linear vibtarions of shallow shells.

Nath , Mahrenholtz and Varma [39] investigated the non linear response of a doubly curved shallow shell on an elastic foundation. They studied the large dynamic response of a doubly curved shallow spherical shell of rectangular platform, supported on two parameter elastic subgarde and subjected to uniformly distributed step and sinusoidal loading.

Hu-Nan-Chu [40] investigated the influence of large amplitude of fexural vibrations of a thin circular cylindrical shell. Axial body force terms which may be of practical importance are included. Non linear periods are obtained for the free vibration case. The numerical results are compared with a previous study on flat plates. Nonlinear effects are found to be considerably less manifest in cylinders than in corresponding flat plates.

Nonlinear equations of motion for a transversely isotropic plate having initial geometric imprefection are derived by Lin and Chen [41]. The effects of both transverse shear deformation and rotatory intertia are included. Equation of motion for a simply supported imperfect plate is obtained by performing the Galerkin procedure and solved by Runge-Kutta method. It is found that the vibration frequencies are very much dependent on the order of initial amplitude and imperfection.

A number of investigators analyzed the non linear oscillation of anisotropic plates using Von-Karman equations. Yu [42]; Yu and Lai [43] studied the non linear vibrations of sandwich plate. Yu [44] investigaed the nonlinear vibration of layered plates and shells. Hassert and Nowinski [45], Sathyamoorthy and Pandalai [46], Ramachandran [47] treated rectangular plates with special rectangular orthotrophy. Nowinski [48] analyzed orthotropic circular plates, Sathyamoorthy and Pandalai [49] analyzed rectilinear orthotropic skew plates, while Venkateswara Rao, Kanaka Raju and Raju [50] used a finite element method, to orthotropic circular plates. Bert [52] investigated the nonlinear oscillations of an arbitary laminated rectangular plate.

Banerjee, Mazumdar and Chanda [54] investigated the non-linear vibrations of elastic plates and shell applying the Karman field equations, extended to the dynamic case, these equations involved the deflection and membrane stress functions as two dependent variables. As consequence, the solutions for almost all problems require considerable computation. But they found that to study the non linear dynamic behaviour of plates and shells, Karman equations pose difficulties in obtaining the required solution. In such cases other methods may be employed. Berger equation may provide acceptable results when the relative amplitudes assumes value less that 2.0.

Due to very complicated nature of the basic equations governing the motion of a structure exhibiting large deflection it has always been a difficult task for investigator to obtain even an approximate solution. Attempts have also been made to find ways to ease such problems. Berger [55] proposed an alternative method which enabled one to replace the coupled Karman equations by simpler set of uncoupled and quasilinear equations. Bergers assumption was based on the idea that the second strain invariant in the middle plane of the plate can be neglected wihtout including any appreciable error in the solution. However, he did not put forward any physical justification for this assumption.

Following this idea J. Nowinski [57], S.N.Sinha [58] studied the large deflection analysis of plates. Later this technique was extended to the dynamic case by Nash and Mooder [59]. Since then this method has been followed by different authors [60-65] for the analysis and dynamic behaviour of plates exhibiting large deflections.

Nash and Mooder [59] extended the Berger method to a dynamic case. M.M.Banerjee with his co-workers published a large number of papers [65-67] based on Berger's hypothesis. Most of their works are related to the problem concerning the variation of thickness of plates and shells. Neglecting in-plane intertia Nash and Mooder [59] showed that the use of such equation for simply-supported plates yeilds results which are in excellent agreement with those obtained from Karman equations. S.Das and B.Banerjee [68] investigated the damped oscillations of moderatly thick plates of arbitary shapes. They used the concept of "Lines of Equal Deflection".

M.M. Banerjee and S. Chanda [69] investigated the large deflections of thin plats of arbitrary shape placed on elastic foundation and subjected to both uniform and concentrated load at the centre as well. They followed Berger's method in conjunction with the method of "Constant Deflection Contour Lines". Nonlinear free vibrations and thermal buckling of a elastic rectangular plate at elevated temperature has been analyzed by P.Biswas [87]. The analysis was based on Berger approximation. S.Datta [70] analyzed the large deflection of clamped circular plate on elastic foundation under non-uniform but symmetrical loads, following Berger's approximate method. Here the deflections are obtained in the form of an infinite series involving Bessel function. S.Dutta [71] again investigated the large amplitude free vibrations of irregular plates placed on elastic foundation by introducing conformal mapping technique and Galerkin method.

Berger's technique was extensively used till Nowinski and Ohanbe [72] examined Berger equation critically and initiated the criticism on the free hand application of these euqations. They observed that the method may lead to grave inaccuracies and even become meaningless if the edge of the plate is free to move in in-plane directions. Lee,Blotter and Yen [73] found that the errors introduced by applying the Berger's hypothesis to a clamped circular plate, depend on Poisson's ratio and the ratio of the radius to the thickness of the plate. Moreover they found that the error is minimized when the Poisson's ratio increases. Huang and Al-Khattat [74] showed that for radially restrained circular plates, solutions base don Berger's hypothesis are accurate at low amplitude but the accuracy decreases as the amplitude increases. Moreover they found that Berger hypothesis is entirely unsuitable for plates with moveable edges. Banerjee [75] while dealing with the large amplitude vibrations of nonuniform rectangular plates, observed that the value of the relative time period (nonlinear and linear) differ from what has been calculated by Bouer [76] by approximately 2% for unit relative ampitude. In an attempt to explore some limitation on the use of Berger equations for large amplitude vibrations of thin elastic plates Banerjee and Sarker [77] further observed that the acceptability of Berger's hypothesis may be restricted to the cases of clamped square and circular plates with immovable edges, and to some extent to simply supported circular or rectangular plates having smaller aspect ratio. They suggested that Berger's method may be restricted to circular and rectangular plates and to some extent to skew plates with smaller skew angles for clamped immovable edge conditions. Banerjee and Das [78] aimed at finding a few points in support of Berger equation without rejecting them totally. They suggested to be cautious against the freehand application of this method. Mention may also be made regarding the relative exactness of Berger's technique as studied by Vendhan [79], by Prathap and Vardan [80] and by Prathap [81].

Sinharay et.al. [82] proposed some modification of Berger's approximation by expressing the total potential energy expression due to bending and streching of the middle surface of a plate or shell in a different way. They preferred to replace e_2 by a new expression without rejecting it totally. The idea is novel one but the limit of its accuracy is yet to be established. For, like Berger's hypothesis, it lacks in providing with a rigorous physical justification. A simple application of the hypothesis proposed by Sinharay has been made by Banerjee et.al. [83] to test its validity. The method was aplied to the problem of finding the temperature effect on the dynamic response of shallow spherical shells. The findings are not very encouraging. Rather, one of the vital equations in Ref. [82] appeared to be fallacious denying the claim of the accuracy of the new approach, at least on the basis of the very problem treated in Ref. [83]. Moreover, Banerjee et.al. [83] observed that assumption of a certain parameter less than unity appeared to be impractical when the radius of the base circle of the spherical shell is large enough compard to the thickness of the shell whereas the authors of Ref. [82] have assumed values of the parameter less than unity. Taking into consideration the different options expressed by authors working on this method it may be stated that Berger method, simpliest of all the existing ones for the analysis of vibrating structures, cannot be discarded altogether. Its applicability may be restricted to the cases of clamped square and circular plates with immovable edge. There is every scope that the deficiency in Berger's method can be overcome and it will then be applied to all possible cases of structures with various boundary conditions. Further studies which deal with membranes, shells of different shapes, flat plates, spinning disks, spinning membranes have been citied in Ref. [88-105].

Mazumdar [19,137-139] put forward a new method to solve the problems of elastic plates of arbitrary shape. The method as it was termed is "Constant Deflection Contour" method. Mazumdar with his co-workers published a series of papers [19, 137-139] on linear vibration of plates and shells utilizing this technique. The outstanding feature of this method is that it is entirely independent of the shape of the plate. Using this method, Jones, Mazumdar and Fu-Pen-Chiang [106] investigated the vibrations of plates under various boundary and load conditions. As illustrations, the case of circular plate clamped on one part of its boundary and simply - supported on the remainder and the case of clamped elliptical plate under elliptical line loading, have been discussed. A simple method for the analysis of the elastic-plastic bending of plates of arbitrary shape was developed by Jain and Mazumdar [107]. The procedure was based upon the concept of "Constant Deflection Contour" method. Again Mazumdar and Bucco [108] analyzed the transverse vibrations of shells of visco-elastic material under arbitray time - dependent load. Banerjee [109] developed an idea of extending the " Constant Deflection Contour" method to the non-linear analysis of plates vibrating at large amplitudes.

The present thesis is based on this idea and it will be followed in all problems considered in this thesis. The details of the method needs a separate chapter to explain the procedure of deriving the basic equations as well as the method of finding their solutions. [See chapter III]

Chapter - II

NON LINEAR THEORY OF ELASTICITY

When an elastic body undergoes deformation under the action of external forces stresses and strains are developed within the body. The state of stress at a point within the body is specified, at most by nine components of stress.In the linear theory the strains in the middle surface are neglected in which the deflections are small compared with the thickness of the plate. If one wishes to study the exact analysis of the non linear theory of plates he may be refered to the Donnel's work [110]. The relations between the strains and displacement may be put as [110].

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \left[-\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}\right] z$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \left[-\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial w}{\partial x} + \frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2}\right] z$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left[-2\frac{\partial^2 w}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial w}{\partial x}\right.$$

$$+ 2\frac{\partial^2 v}{\partial x \partial y}\cdot\frac{\partial w}{\partial y} + 2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\frac{\partial^2 w}{\partial x \partial y}$$

$$\left. + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\nabla^2 w\right] z \quad \text{............} [2{\cdot}1]$$

in which higher powers and products of the displacements involving z^2 have been omitted. We will confine our attention to cases where both the strains and deflections slopes $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are small compared to unity: and in general, for plates used in mechanics and structures the allowable strains and deflection slopes are very small compared to unity. It is important to note that for the membrane part of the strain terms like $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial x}{\partial y}$ and $\frac{\partial v}{\partial x}$ are important, but the terms involving the squares or product of themselves will be negligible, while the flexural strain terms like $\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right)$ are very small compared to the principal flexural terms like $\frac{\partial^2 w}{\partial x^2}$. Similar arguments may be presented for the ommission of terms like $\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{\partial x}\right)$ and ter-m containing z^2. Moreover the derivation of the strain-displacement relations is based on Love-Kirchhoff hypothesis (i.e., the linear filaments of the plate initially perpendicular to the middle surface remain straight and perpendicular to the deformed middle surface and suffer no extensions).With these assumption and aproximations, the total strains in the layer of the plate parallel to and a distance z from the middle surface, they can be written as

$$\epsilon_x = \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] - z\frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_y = \left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] - z\frac{\partial^2 w}{\partial y^2}$$

$$\epsilon_{xy} = \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] - 2z\frac{\partial^2 w}{\partial x \partial y} \quad \text{-----------} [2\cdot 2]$$

where the terms within third bracket being constants with respect to 'z' are membrane strain and the last terms containing the factor 'z' are flexural strains.

Stress - Strain Relations :

For isotropic materials the elastic stress-strain relations or Hook's Law have been found from experiments to be, in general, as :

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right)$$

$$\epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_z - \nu\sigma_x\right)$$

$$\epsilon_z = \frac{1}{E}\left(\sigma_z - \nu\sigma_x - \nu\sigma_y\right)$$

$$\epsilon_{xy} = \frac{1}{G}\sigma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

$$\epsilon_{yz} = \frac{1}{G}\sigma_{yz} = \frac{2(1+\nu)}{E}\sigma_{yz}$$

$$\epsilon_{xz} = \frac{1}{G}\sigma_{xz} = \frac{2(1+\nu)}{E}\sigma_{xz} \quad \text{------} [2\cdot 3]$$

But for plane stress, i.e., when σ_x, σ_y and σ_{xy} are assumed to be uniform over the thickness and σ_z, σ_{xz}, σ_{yz} are every where zero, the relations given by [2.3] will then reduce to

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \sigma_x = \frac{E}{(1-\nu^2)}\left(\epsilon_x + \nu\epsilon_y\right)$$

$$\epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \sigma_y = \frac{E}{(1-\nu^2)}\left(\epsilon_y + \nu\epsilon_x\right)$$

$$\epsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$$

$$\epsilon_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy} \qquad \sigma_{xy} = \frac{E}{2(1+\nu)}\epsilon_{xy} \qquad [2.4]$$

For the case of plane strain, i.e., when $\epsilon_z = 0$, equation (2.3) will then become

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

$$\epsilon_x = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right)$$

$$\epsilon_y = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right)$$

$$\epsilon_{xy} = \frac{2\left[1+\frac{\nu}{(1-\nu)}\right]}{E/(1-\nu^2)}\sigma_{xy} \qquad [2.5]$$

Principle of Virtual Work :

In obtaining the solutions of elastic problems energy principles and variational methods play an important role. It will be seen later that the governing differential equations are direct consequences of the minimisation of the energy expressions associated with the structure concerned. The method is termed as 'variational method' since it is based on calculus of variation. The basis of 'calculus of variation' is the 'principle of virtual work' enunciated by the great mathematician John Bernouli in 1717, - "If under the action of a certain force a particle undergoes an arbitrary small displacement, called a virtual displacement and if the particle retains its condition of equilibrium then the total workdone by the force is zero."

Since the principle of virtul work is very common in every sphere of mathematics we would better leave it here and carry out the required mathematical operations without giving much emphasis on the theory of the principle.

Let us denote the total workdone against the mutual actions between the particles in an elastic body due to the virtual displacements $\delta u, \delta v, \delta w$, by δV, where $\delta u, \delta v, \delta w$ are the displacements parallel to the axes of a Cartesian system of coordinates wih respect to a certain origin; then the total work done by the mutual actions is $-\delta V$.

If there be forces applied at the boundary of the body and if X, Y, Z be the components of the body forces along the x,y,z directions respectively, and $\bar{X}, \bar{Y}, \bar{Z}$ be the component of the boundary forces per unit area then the work done by them

$$W = \iiint (X\,\delta u + Y\,\delta v + Z\,\delta w)\,dx\,dy\,dz + \iint (\bar{X}\,\delta u + \bar{Y}\,\delta v + \bar{Z}\,\delta w)\,dA \qquad [2{\cdot}6]$$

dA being the elementary area and the integration being taken over the part of the boundary surface of the body, on which displacements are not prescribed. It is important to note here that the part of the surface where forces are prescribed is the same as the part where displacements are not prescribed. We may assume further that the external forces are constants during the virtual displacement, when we put

$\delta(V-W) = 0$ ------------------------ [2.7]

One can interpret the result given by the equation (2.7), in comparing various values of the displacements u,v, and w, the displacements which actually occur in an elastic system under the given external forces are those which lead to zero variation of the total energy (potential energy) of the system for any virtual displacement from the position of equilibrium.

Principle of Minimum Potential Energy and Principle of Complementary Energy :

The expression (V-W), consisting of V, the potential energy of deformation, and -W, the potential energy of the external forces is called the 'Potential Energy' of the system. For stable equilibrium it can be shown that the total potential energy of the system is positive, hence in this case the total potential energy of the system is a minimum. This is the principle of minimum potential energy.

In case of a vibrating plate there is an additional energy, the Kinetic Energy. If we denote it by T then we can form the Lagrangian

$L = T - U$, U is the potential energy ------------------------ [2.8]

Applying Hamilton's principle we can further show that the Hamiltonian

$H = T + U$ = the total energy

If the potential energy is independent of velocities (i.e., independent of u,v,w) remains positive, since by defination, T, the kinetic energy is positive definite.

Instead of varying the displacements from those at equilibrium, one may want to vary the stress components. If $\delta\sigma_x, \delta\sigma_y, \delta\sigma_{xy}$ be the small variations in the stress components $\sigma_x, \sigma_y, \sigma_{xy}$ respectively, then the change in the strain energy per unit thickness of the plate may be written as

$$\delta\Pi = \iint \left[\frac{1}{E}\left(\sigma_x\,\delta\sigma_x + \sigma_y\,\delta\sigma_y - \nu\sigma_x\,\delta\sigma_y - \nu\sigma_y\,\delta\sigma_x\right) + \frac{1}{G}\,\sigma_{xy}\,\delta\sigma_{xy}\right] dA \qquad [2{\cdot}9]$$

The body forces being given external forces, remain unchanged but on that part of the boundary where surface forces are not prescribed, corresponding to the variation of stress components, there will be some variation in the boundary surface forces. In this case also it can be seen that the variation of the total energy 'Π' is zero, i.e.,

$$\delta \Pi = \delta (V^* - W^*) = 0 \qquad [2\cdot10]$$

Where V* isthe strain energy per unit thickness of the plate and W* is the work done by the boundary-surface forces. The expression Π is called the Complementary Energy of the system. We have seen in the case of potential energy, the same deduction may be made and we may conclude that "for all stress satisfying the equilibrium conditions in the interior and on the part of the boundary surface where the surface forces are prescribed, the stresses which satisfy the compatibility equations, are such that the complementary energy assumes a satisfactory value". Note that we have imposed the restriction on the stresses to satisfy the compatibility equations for the deduction of equation (2.10) depends on this restriction.

Deduction of Equation of Equilibrium and Boundary Conditions :

Let us consider a plate of thickness 'h'. The mid-plane of the plate is given by z = 0 and the two surfaces are denoted by S_1 and S_2, defined by $z = \frac{h}{2}$ and $z = -\frac{h}{2}$, respectively.
Hence forth we shall be mainly concerned with plane-stress only. Accordingly, we consider the total strain energy

$$V = \iiint_{-h/2}^{h/2} (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_{xy} \epsilon_{xy}) dz\, dA \qquad [2\cdot11]$$

combining equation (2.9) and (2.4). Using the stress-strain relations given by (2.4) to express the strains in terms of stresses, or the stresses in terms of strains equation (2.11) may be written as

$$V = \frac{1}{2E} \iint_R \int_{-h/2}^{h/2} [\sigma_x^2 - 2\nu\sigma_x\sigma_y + \sigma_y^2 + 2(1+\nu)\sigma_{xy}^2] dz\, dA \qquad [2\cdot12]$$

or,

$$V = \frac{E}{2(1-\nu^2)} \iint_R \int_{-h/2}^{h/2} \left[\epsilon_x^2 + 2\nu\epsilon_x\epsilon_y + \frac{(1-\nu)}{2}\epsilon_{xy}^2 + \epsilon_y^2\right] dz\, dA \qquad [2\cdot13]$$

Replacing the strains in the expression (2.11) and taking the first variation of the strain energy one can write

$$\delta V^{(1)} = \iint_R \int_{-h/2}^{h/2} \left[\sigma_x \left\{ \frac{\partial}{\partial x}\delta u + \frac{\partial w}{\partial x}\frac{\partial}{\partial x}\delta w - z\frac{\partial^2}{\partial x^2}\delta w \right\} \right.$$

$$+ \sigma_{xy} \left\{ \frac{\partial}{\partial y}\delta u + \frac{\partial}{\partial x}\delta v - 2z\frac{\partial^2 \delta w}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial}{\partial y}\delta w + \frac{\partial w}{\partial y}\frac{\partial}{\partial x}\delta w \right\}$$

$$\left. + \sigma_y \left\{ \frac{\partial}{\partial y}\delta v + \frac{\partial w}{\partial y}\frac{\partial}{\partial y}\delta w - z\frac{\partial^2 \delta w}{\partial y^2} \right\} \right] dz\, dA \qquad [2{\cdot}14]$$

We shall now preform the integration with respect to z first and introduce the stress resultants N_x, N_y, N_{xy} and moment intensity M_x, M_y, M_{xy} respectively, defined by

$$N_x = \int_{-h/2}^{h/2} \sigma_x \, dz = h\sigma_{xm} \qquad M_x = \int_{-h/2}^{h/2} \sigma_x \, z \, dz$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \, dz = h\sigma_{ym} \qquad M_y = \int_{-h/2}^{h/2} \sigma_y \, z \, dz$$

$$N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} \, dz = h\sigma_{xym} \qquad M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} \, z \, dz \qquad [2{\cdot}15]$$

Where the symbol N_x represents forces per unit length in the x-direction, N_y in the y-direction and $N_{xy} = N_{yx}$ is a force in the xy-direction. Similarly M_x, M_y and M_{xy} represent the moments per unit length in the respective directions as shown in the **figure 1**

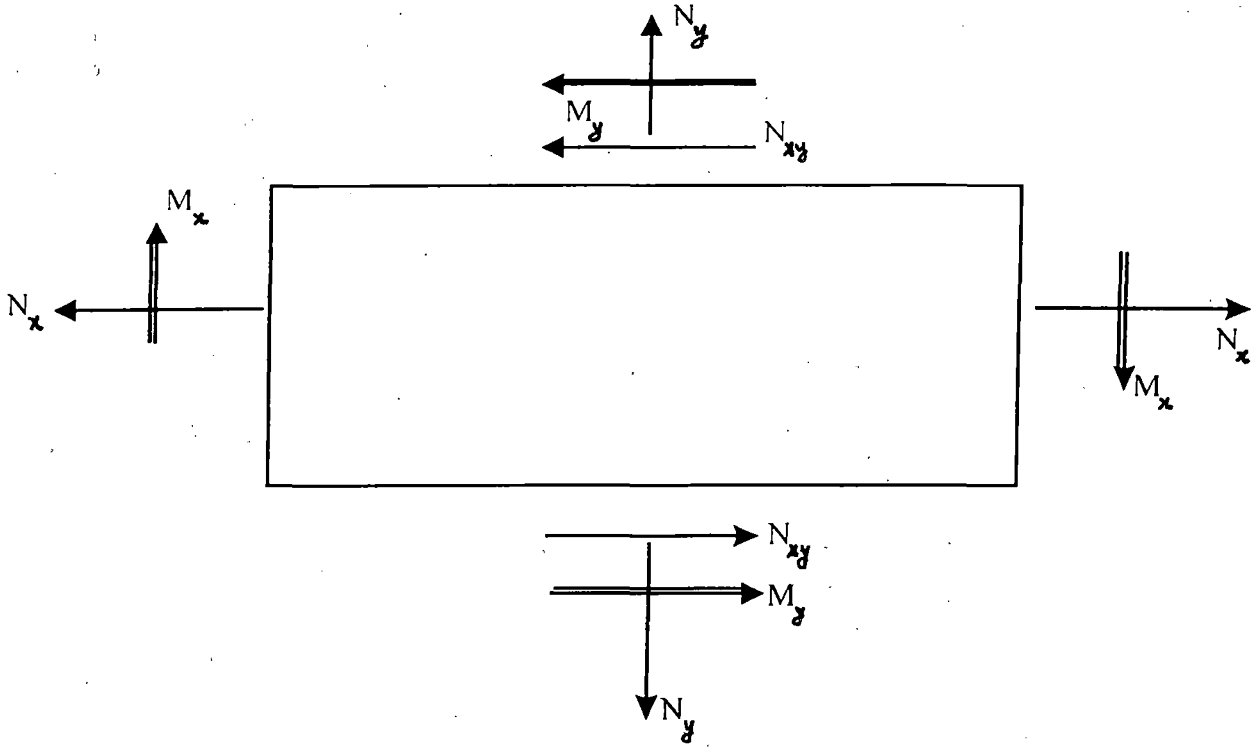

Figure1 : Stress and moment resultants for a rectangular plate.

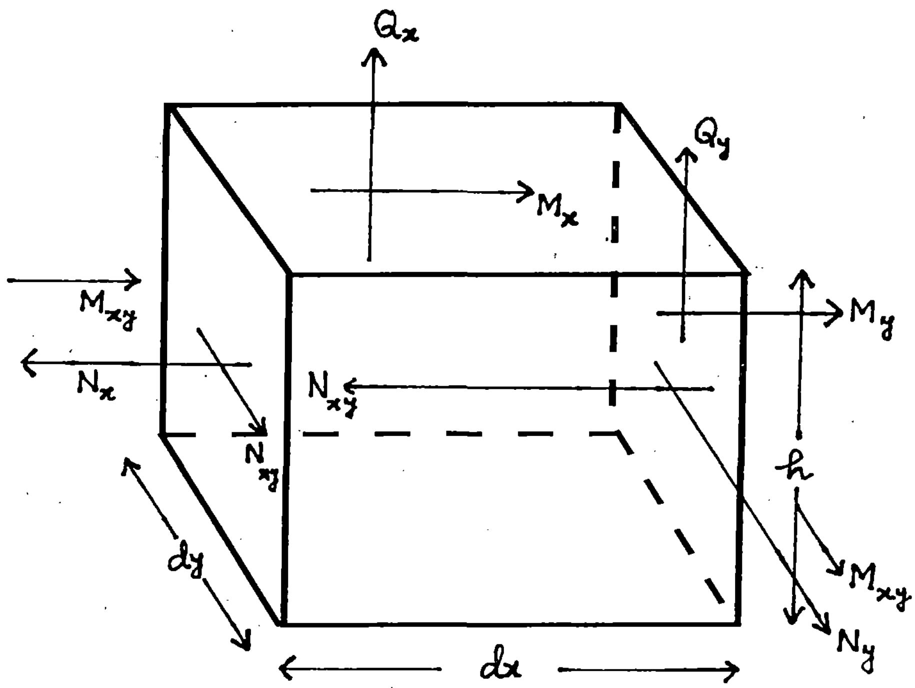

Figure2 : Stress resultants and moments

For practical purpose we may write $N_{xy} = N_{yx}$ and equation (2.9) may be replaced by

$$\delta V^{(1)} = \iint \Big\{ N_x \left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) - M_x \frac{\partial^2 \delta w}{\partial x^2}$$

$$+ N_{xy} \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} \right)$$

$$- 2M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} + N_y \left(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right)$$

$$- M_y \frac{\partial^2 \delta w}{\partial y^2} \Big\} dx\,dy \qquad [2.16]$$

If $\overline{F}_x$, $\overline{F}_y$ and $\overline{F}_z$ be the components of the external forces acting in the directions of x, y and z axes respectively then the virtual workdone is given by

$$\delta^{(1)} W^* = -\iint_{S_1} (\overline{F}_x \delta U + \overline{F}_y \delta V + \overline{F}_z \delta W)\, ds\, dz \qquad [2.17]$$

where S_1 is that part of the side boundary where the external forces are prescribed

$$U = u - z\frac{\partial w}{\partial x}, \quad V = v - z\frac{\partial w}{\partial y}, \quad W = w \qquad [2.18]$$

u,v,w being the displacement components of the middle surface of the plate.

Considering the load intensity 'p' the principle of virtual work for the present problem (for simplicity we are, for the time being, avoiding the expression for the kinetic energy associated with the motion of the plate) may be put in the following form

$$\overset{(1)}{\delta}(\Phi) = \iiint (\sigma_x \delta\epsilon_x + \sigma_y \delta\epsilon_y + \sigma_{xy} \delta\epsilon_{xy})\, dx\, dy\, dz$$

$$- \iint_{S_1} (\bar{F}_x \delta U + \bar{F}_y \delta V + \bar{F}_z \delta W)\, ds\, dz$$

$$- \iint_{S_m} p\, \delta w\, dx\, dy \qquad [2{\cdot}19]$$

where S_m denote the mid-surface region of the plate.
Let us introduce the following integrals.

$$\bar{N}_{x\nu} = \int_{-h/2}^{h/2} \bar{F}_x\, dz \qquad \bar{N}_{y\nu} = \int_{-h/2}^{h/2} F_y\, dz$$

$$\bar{N}_z = \int_{-h/2}^{h/2} \bar{F}_z\, dz$$

$$\bar{M}_{x\nu} = \int_{-h/2}^{h/2} \bar{F}_x z\, dz \qquad \bar{M}_{y\nu} = \int_{-h/2}^{h/2} \bar{F}_y z\, dz \qquad [2{\cdot}20]$$

The first variation of the potential of the applied forces, including the load intensity 'p', corresponding to equation (2.17) may be written as

$$\delta^{(1)} W^{*} = -\int_{S_m} p\, \delta w\, dx\, dy + \iint_{S_1} \bar{N}_{\nu}\, \delta u_{\nu}\, ds + \iint_{S_1} N_{\nu s}\, \delta u_s\, ds \qquad [2.21]$$

where u_ν and u_s are the in-plane displacements of the boundary of the plate in directions normal and tangential respectively to the boundary (Figure 2.3). N_ν is taken as positive in compression as shown in Fig-**3**

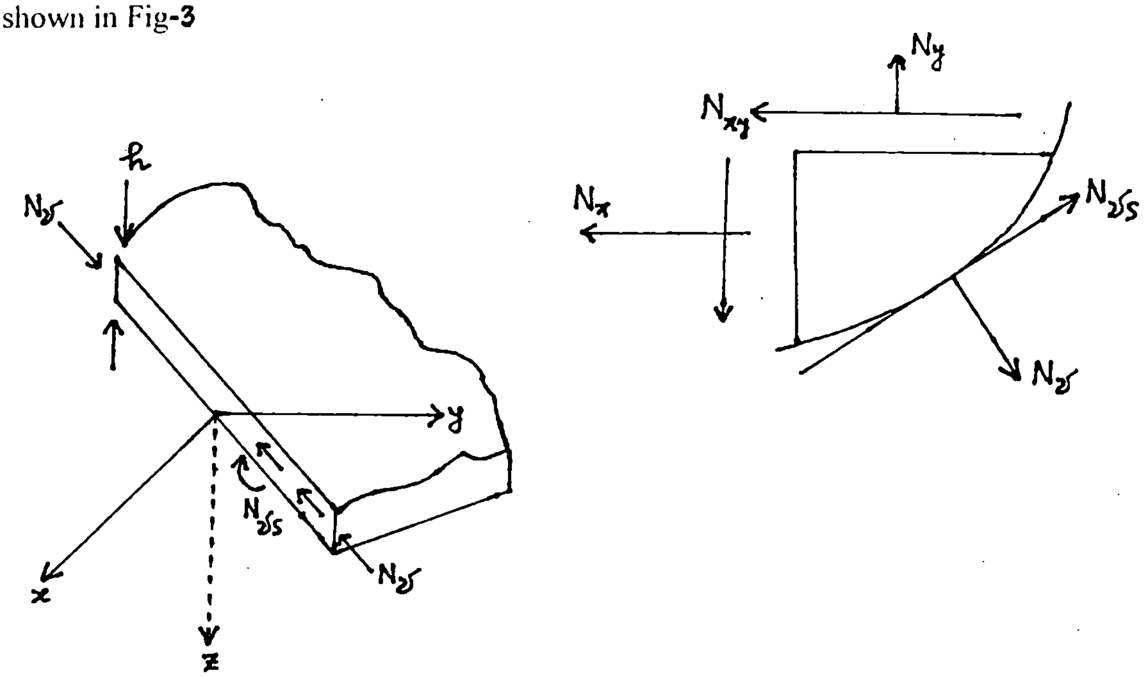

Fig 3 : Boundary loading on the plate segment

If the direction cosines of the normal (drawn outward) be (l,m,o) i.e., $l = \cos(x,\nu)$ and $m = \cos(y,\nu)$, we can write

$$\frac{\partial}{\partial x} = l\frac{\partial}{\partial \nu} - m\frac{\partial}{\partial s} \quad , \quad \frac{\partial}{\partial y} = m\frac{\partial}{\partial \nu} + l\frac{\partial}{\partial s} \qquad [2.22]$$

The displacements along the normal and tangential directions may be put

$u_\nu = lu + mv,\ u_s = -mu + lv$ -------------------- (2.23)

With the above notations we can express

$$N_{\nu} = \ell^2 N_x + 2\ell m N_{xy} + m^2 N_y$$

$$N_{\nu s} = (N_y - N_x)\ell m + N_{xy}(\ell^2 - m^2) \qquad [2{\cdot}24]$$

Also $N_{x\nu} = (\ell N_{\nu} - m N_{\nu s}) = \ell N_x + m N_{xy}$

$$N_{y\nu} = (m N_{\nu} + \ell N_{\nu s})$$

or, $N_{y\nu} = (\ell N_{xy} + N_y m) \qquad [2{\cdot}25]$

$$M_{\nu} = \ell^2 N_x + 2\ell m M_{xy} + M_y m^2$$

$$M_{\nu s} = (M_y - M_x)\ell m + M_{xy}(\ell^2 - m^2) \qquad [2{\cdot}26]$$

$$\bar{M}_{\nu} = \bar{M}_{x\nu}\ell + \bar{M}_{y\nu} m,$$

$$\bar{M}_{\nu s} = -\bar{M}_{x\nu} m + \bar{M}_{y\nu}\ell \qquad [2{\cdot}27]$$

$$M_{x\nu} = \ell M_x + m M_{xy}$$

$$M_{y\nu} = M_{xy}\ell + M_y m$$

Finally we introduce the transverse shear forces of plate theory

$$Q_{\nu} = Q_x \ell + Q_y m$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \qquad [2{\cdot}28]$$

We are now in a position to utilise the extermization process and to apply Green's Theorem so long as the du and dv are involved in the integrals in the following equation obtained from relations given by equations (2.16) - (2.28).

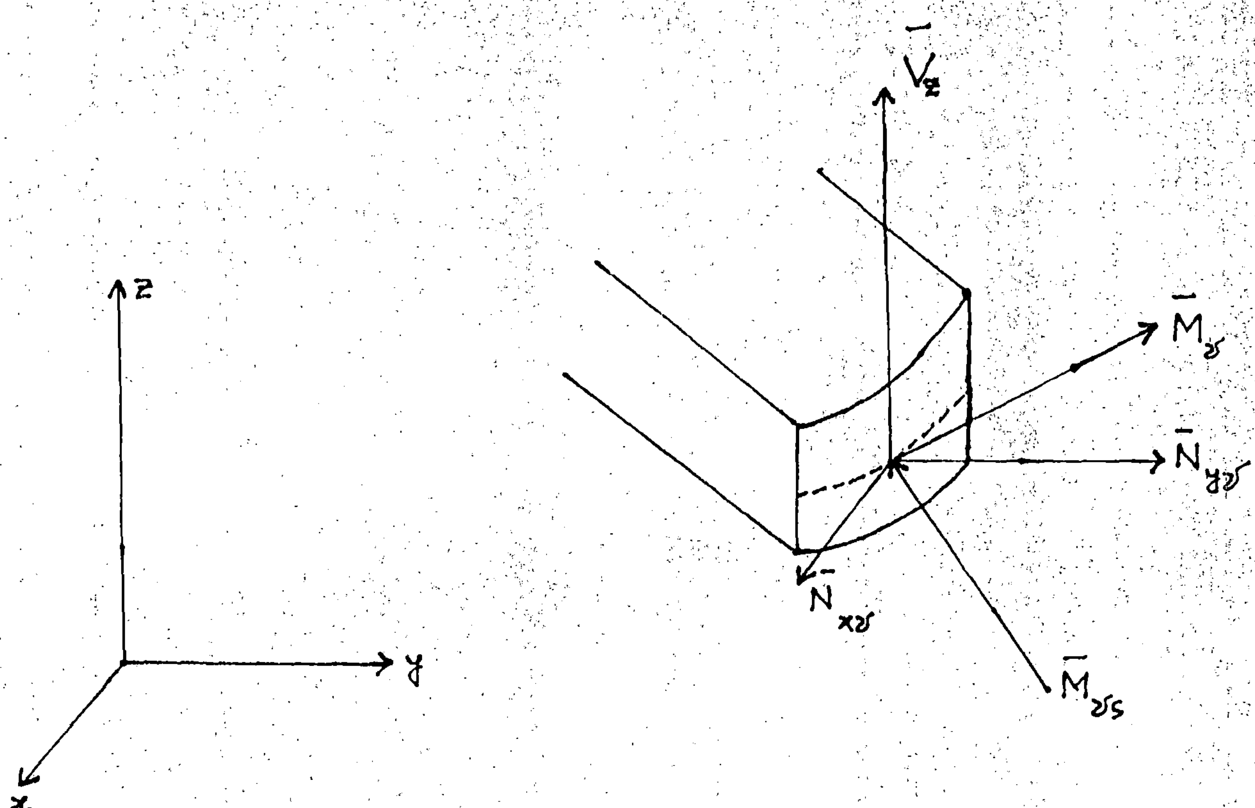

Figure :- 4 Forces on the Plate Filament

Combining equations (2.16) and (2.21) we may rewrite for the total potential energy variation :

$$
\begin{aligned}
\delta\Pi_s^{(1)} = \iint\Big[& N_x\left(\frac{\partial\,\delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial\,\delta w}{\partial x}\right) - M_x\frac{\partial^2\delta w}{\partial x^2} \\
& + N_{xy}\left(\frac{\partial\,\delta u}{\partial y} + \frac{\partial\,\delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial\,\delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial\,\delta w}{\partial x}\right) \\
& - 2M_{xy}\frac{\partial^2\delta w}{\partial x\,\partial y} + N_y\left(\frac{\partial\,\delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial\,\delta w}{\partial y}\right) - M_y\frac{\partial^2}{\partial y^2}\delta w\Big]\,dx\,dy \\
& - \iint p\,\delta w\,dx\,dy + \int \bar{N}_\nu\,\delta u_\nu\,ds + \int N_{\nu s}\,\delta u_s\,ds = 0 \qquad [2.29]
\end{aligned}
$$

We now perform the line integrals considering the equilibrium of the plate element and the notations defined earlier.

$$
\begin{aligned}
\delta\Pi_s^{(1)} = -\iint\Big[& \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\delta u + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)\delta v \\
& + \Big\{\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}\right) \\
& + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_y\frac{\partial w}{\partial y}\right) + p\Big\}\delta w\Big]\,dx\,dy \\
& + \int_\Gamma (N_\nu + \bar{N}_\nu)\,\delta u_\nu\,ds + \int_\Gamma (N_{\nu s} + \bar{N}_{\nu s})\,\delta u_s\,ds \\
& - \int_\Gamma M_\nu\frac{\partial\,\delta w}{\partial \nu}\,ds + \int_\Gamma\left(Q_\nu + \frac{\partial}{\partial s}M_{\nu s} + N_\nu\frac{\partial w}{\partial \nu} + N_{\nu s}\frac{\partial w}{\partial s}\right)\delta w\,ds \\
& - \left(M_{\nu s}\,\delta w\right) = 0 \qquad [2.30]
\end{aligned}
$$

The last expression accounts for corners in the boundary. Considering the condition of equilibrium, one may deduce from equation [2.30]

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad [2.31]$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad [2.32]$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}\right)$$

$$+ \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right) + p = 0 \qquad [2.33]$$

The mechanical boundary conditions are obtained form the remaining line integrals on the boundary

Either $N_\upsilon = - N_\upsilon$ or u is specified --------------------- (2.34)

Either $N_{\upsilon s} = - N_{\upsilon s}$ or u_s is specified ------------------- (2.35)

Either $M_\upsilon = 0$ or $\frac{\partial w}{\partial \upsilon}$ is specified -------------------- (2.36)

Either

$$Q_\upsilon + \frac{\partial M_{\upsilon s}}{\partial s} + N_\upsilon \frac{\partial w}{\partial \upsilon} + N_{\upsilon s}\frac{\partial w}{\partial s} = 0$$

or w is specified ---------------------------- (2.37)

The last term in equation (2.30) which accounts for the corners, indicates

At discontinuties [$M_{\upsilon s} \delta w$] = 0 ---------------------- (2.38)

Equation (2.31 - 2.33) together with the geometrical boundary conditions constitute the problem for a flat plate in large deflection.

With proper transformations, we can get the mechanical boundary conditions otherwise

On the boundary, c_1 :

$$N_{x\nu} = \bar{N}_{x\nu} \quad ; \quad N_{y\nu} = \bar{N}_{y\nu}$$

$$\left(Q_x l + Q_y m + N_{x\nu}\frac{\partial w}{\partial x} + N_{y\nu}\frac{\partial w}{\partial y}\right) + \frac{M_s}{s}\left[= \left(V_z + \frac{\partial M_{\nu s}}{\partial s}\right)\right]$$

$$= \bar{V}_z + \frac{\partial \bar{M}_{\nu s}}{\partial s}$$

$$M_{\nu} = \bar{M}_{\nu} \qquad [2.39]$$

We shall now establish the stress resultant displacement relations from equation (2.15) after performing the necessary integrations and expressing them in terms of partial derivatives of the three displacements u, v, w as :

$$N_x = h\sigma_{xm} = C\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \nu\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}\right] \qquad [2.40]$$

$$N_y = h\sigma_{ym} = C\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \nu\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\}\right] \qquad [2.41]$$

$$N_{xy} = h\sigma_{xym} = \frac{C(1-\nu)}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \qquad [2.42]$$

$$N_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad [2.43]$$

$$N_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \qquad [2.44]$$

$$M_{xy} = -(1-\nu)\, D \frac{\partial^2 w}{\partial x \partial y} \qquad [2.45]$$

where $D = Ch^2 = \frac{Eh^3}{12(1-\nu^2)}$, the flexural rigidity

and $C = \frac{Eh}{12(1-\nu^2)}$, the extensional rigidity

Now introducing the Airy's stress function F defined by

$$N_x = h\sigma_{xm} = h\frac{\partial^2 F}{\partial y^2}$$

$$N_y = h\sigma_{ym} = h\frac{\partial^2 F}{\partial x^2}$$

$$N_{xy} = h\sigma_{xym} = -h\frac{\partial^2 F}{\partial x \partial y} \qquad [2.46]$$

compatible with equations (2.30) and (2.31).

We can now express the membrane strains $[\epsilon_{xm}, \epsilon_{ym}, \epsilon_{xym}]$ in terms of membrane stress (σ_{xm}, σ_{ym}, σ_{xym}) and equate the same in terms of displacements and Airy's stress function in the following form :

$$\epsilon_{xm} = \frac{1}{E}\left(\sigma_{xm} - \nu\sigma_{ym}\right) = \frac{1}{Eh}\left[\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right]$$

$$= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_{ym} = \frac{1}{E}\left(\sigma_{ym} - \nu\sigma_{xm}\right) = \frac{1}{Eh}\left[\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right]$$

$$= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\epsilon_{xym} = \frac{2(1+\nu)}{E}\sigma_{xym} = \frac{-2(1+\nu)}{Eh}\frac{\partial^2 F}{\partial x \partial y}$$

$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad [2.47]$$

Applying the operators $\frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x^2}$; , $\frac{-\partial^2}{\partial x \partial y}$ to the first, second and third of the above equations, respectively and adding them together, in order to eleminate u and v one obtains

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = Eh\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \qquad [2.48]$$

Further assuming that the thickness of the plate is constant, and combining equation (2.33) with equations (2.40) - (2.45) one can write

$$D\nabla^4 w = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + p \qquad [2.49]$$

Equation (2.48) and (2.49) are the well known Von Karman plate equations.
The equation (2.48) is known as the 'compatibility equation' and the equation (2.49) is the equation of equilibrium in the direction of the z axis. These equations henceforth will be termed as Karman equations. These Karman equations may be written in a simplified form by introducing the non linear operator α , defined by

$$\mathcal{L}(a,b) = \frac{\partial^2 a}{\partial x^2}\frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 a}{\partial y^2}\frac{\partial^2 b}{\partial x^2} - 2\frac{\partial^2 a}{\partial x \partial y}\frac{\partial^2 b}{\partial x \partial y} \qquad [2{\cdot}50]$$

$$\nabla^4 F = -\frac{Eh}{2}\mathcal{L}(w,w) \qquad [2{\cdot}51]$$

$$D\nabla^4 w = \mathcal{L}(F,w) + p \qquad [2{\cdot}52]$$

The above two equations are the governing equations for thin plates at large deflections under a static load. However, they may be extended to a dynamic case by changing p by $\left(p - \rho h \frac{\partial^2 w}{\partial t^2}\right)$

While deducing the governing differential equations for a dynamic case, we shall consider the kinetic energy of the plate, given by

$$T_e = \iiint \rho \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dx\,dy\,dz \qquad [2{\cdot}53]$$

In this case we shall have to minimize the integral

$$\delta \Pi^* = \int_{t_1}^{t_2} \left(\Pi_s - T_e\right) dt \qquad [2{\cdot}54]$$

when the integral is obtained by combining equations (2 .29) and (2.53). The equations of equilibrium will thus be transformed to

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho h \frac{\partial^2 u}{\partial t^2} \qquad [2.55]$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = \rho h \frac{\partial^2 v}{\partial t^2} \qquad [2.56]$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right)$$

$$+ \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_{xy} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy} \frac{\partial w}{\partial x}\right)$$

$$+ p = \rho h \frac{\partial^2 w}{\partial t^2} \qquad [2.57]$$

These equations are equivalent to equations (2.31) - (2.33) for the static case differing only by the inertia terms on the right hand side. If we neglect the inertia in the plane of the plate i.e., if we set the right hand side of equations (2.55) and (2.56) to zero, the resulting equations will then be transformed to

$$\nabla^4 F = -\frac{Eh}{2} \mathcal{L}(w, w) \qquad [2.58]$$

$$D \nabla^4 w = \mathcal{L}(F, w) + p - \rho h \frac{\partial^2 w}{\partial t^2} \qquad [2.59]$$

These are the governing differential equations for thin plates at large amplitudes. The deflection function w is dependent on the space coordinates as well as on the time. It is important to note that the load 'p' may be uniform, concentrated at a point or distributed over a segment of the plate: it may be dependent or independent of time, as for example, in case of forced vibration, p becomes function of time.

Chater III

A Genaral Discussion

In the linear theories of motion of bodies, deflections are assumed to be small in comparison with the plate thickness. But in most practical cases this basic assumption is nolonger valid, instead the deflections have the order of the magnitude of the plate thickness. Hence, derivation of governinrg differential equations considering large deflections needs special attention in such analysis. In general Karman type filed equations are employed for almost all types of structures.

The paucity of literature concerning non linear (large amplitude) vibration analysis, probably, due to the fact that the two basic Von Karman field equations extended to the dynamic case, involve the deflection and stress functions in a coupled form. Moreover, these equations are of fourth order, posing analytical problems and necessitating a numerical approach. Several methods are available to investigate such problems and thereby elucidate non linear response for some simpler cases. For one type of method the analytical difficulties are overcome by using modern high speed computers and finite elements or finite differences. Yet, classical approach is still preferable, even for some approximate solution, wher-ever possible .

The basic equations for free flexural vibrations of rectangular elastic plates have been explicitily discussed by G. Hermann (21) These equations are Karman type equations extended to the dynamic case. Chu and Herman studied the influence of large amplitude on free flexural vibration of rectangular plates with hinged immovable edges, by applying Herman's theory. Approximate solution for the nonlinear response of rectangular plate to sinusoidal acoustic pressure were obtained by Kirchman and Greenspon (17) by using the static load-deflection relations previously obtained . Eringen (18) studied vibrations of strings and bars exhibiting large vibrations . During the last thirty years, several problems of practical interest have been investigated by different authors using different approaches. Laminated isotropic and orthotropic plates or sandwiched plates have also gained importance during this period.

Due to the very complicated nature of the basic equations governing the motion of a structure exhibiting large deflections , it has always been a difficult task for an investigator to obtain an approximate solution. Attempts have also been made to find ways to ease such problems .Berger [55] proposed an alternative method which enable one to replace the coupled Karman equations by a simpler set of uncoupled and quasi-linear equations. Following this novel idea authors [56,57,58] studied the large deflection analysis of plates. This technique was extended to the dynamic case by different authors for the analysis of static and dynamic behaviour of plates exhibiting large deflections. Berger's assumption was based on the idea that the second strain invariant in the middle plane of the plate can be neglected without inducing any appreciable error in the solution. However , he didnot give any physical justification for this assumption. This technique was extensively used till Nowinski[72] examined Berger's equations critically and initiated the criticism on the freehand application of these equations . However, this method may be applied to some specific problems with some limitations [78]

Sinharay and Banerjee [82] tried to improve Berger's approach to the solutions of large amplitude vibrations of orthogonal plates with some modification of Berger's hypothesis. But like Berger's one this method lacks in providing with the rigorous physical justification.

Most of the investigations employed Galerkin approximation as a tool for the nonlinear analysis of vibration of plates and shells but it has been observed by Vendhan [53] that the first order Galerkin approximation may involve substantial error in the case of plates with unrestricted boundary conditions. Moreover Bayeles et. al. [140] have pointed out inadequacy of a first order Galerkin approximation to the solution of inplane equilibrium or of the compatibility equation.

The Rayleigh -Ritz method has now become a versatile method for obtaining the approximate solutions of vibrational problems in solid mechanics though the Galerkin method has a wider applicability the one term Rayleigh Ritz approximation yields better results compared to the former one. But it remains wide open to justify the validity of a method merely on the basis of a one term approximation.Yet we have no alternative for the increase of just one term more in the approximation series which will multiply the mathematical and computational labour considerably.Vendhan and Das [35] have made a nice comparative study between the Rayleigh Ritz method and Galerkin method, Investigating on the nonlinear vibrations of triangular and rectangular plates, the authors of Ref[141] have discussed the nature of Rayleigh-Ritz approximation based on the variational for mulation and the Galerkin approximation based on minimization of the error function. Presenting the numerical results for convergence of the relative time periods of the nonlinear and linear vibrations for triangular and rectangular plates of orthotropic materials for various aspect ratios, they have shown that the Rayleigh Ritz approximation is consistently better than the Galerkin approximation but they become equally good after a few terms. It has further been observed that the results obtained from both the methods approach the true value from the opposite sides of it. In fact this is what is expected from the Rayleigh Ritz method for the approximation is associated with a potential energy formutation corresponding to a stiffer structure . On the other hand Galerkin approximation which turns out to be an upper bound to the true value for a given lateral displacement corresponding to a more flexible structure. The bounding property of the Galerkin approximation may be the characteristic of the constrained in palne edge condition and the single mode expression for the deflection function. However, the observation that the modified nonlinear equations gives a less stiff model than the Rayleigh Ritz method may be considered to be of general significance.

The expression for defining the transverse deflections and in plane displacements are often assumed as polynomials in space co-ordinates, or., in terms of trigonometric series. The proper choice of the coordinate functions is very imortant for obtaining good accuracy. For example. Vendhan's [142] investigation into the non linear vibrations of thin plates including inplane inertia effects reveals that sometimes polynomial expressions for the displacement functions may be found to be good enough for obtaining good accuracy . Leissa [111] making on extensive survey on the free vibrations of rectangular plate has shown that the percent difference of the 36- terms solution for eigen-frequency from the single term Rayleigh Ritz solution with respect to the first term solution is negative indicating that the one term solution is more exact than the 36- term solution.

Besides the methods discussed above, several methods based on computer application are being used to analyze the vibration problems. Finite Elements method is now regarded as one of the most powerful method for problems on structural and continuum mechanics[143]

Using Karman type field equations for solution of plate problems having uncommon boundaries is a difficult task and a more complicated one when geometrical non-linearities are involved.

Recently a new idea has been put forward by Banerjee [109] to study the dynamic response of structures of arbitar.y shapes based on "Constant Deflection Contour" method. The method has been previously developed by Mazumdar [137-139]. Further application of this method has been made by Majumder and others [106-108, 115, 116]. However the application of this method has been restricted to linear analysis only.

The present thesis aims at extending this method to the nonlinear analysis of plates vibrating at large amplitudes. It begins with a review of the basic ideas developed by Banerjee [109]. The analysis carried out in this thesis may readily be applied to other geometrical structures and as a by- product, the static deflection is also obtainable. A combination of the ' constant Deflection Contour' method and the Galerkin procedure is employed. The numerical results obtained, are in excellent agreement with those from previous studies. Application of the present analysis to structures with complicated geometry is in progress.

Some preliminary Remarks about the constant Deflection contour Method :

The fundamental concept of the constant deflection contour method may be best explained by considering transverse vibrations of plate, referred to a system of orthogonal coordinates oxyz. for which oz is the transverse direction (positively downward). The horizontal plane oxy coincides with the middle plane of the plate. Such a plate is either statically deflected, vibrating freely or forced to vibrate, all due to normal static or dynamic loads. When the plate vibrates in a normal mode, then at any instant t_o, the intersections between the deflected surface and the parallels z = constant will yield contours which after projection onto z = o surface are a set of level curves, u(x,y) = constant, called the " Lines of Equal Deflection", which are iso-amplitude contours. The boundary of the plate, irrespective of any combinations of support, is also a simple closed curve C_o belonging to the family of lines of Equal Deflections C_u. As defined by Mazumdar [138], the family of non-intersecting curves may be denoted by C_u for $O<u<u^*$, so that C_o (u = O) is the boundary and C_u conicides with the point (s) at which the maximum u = u* is obtained

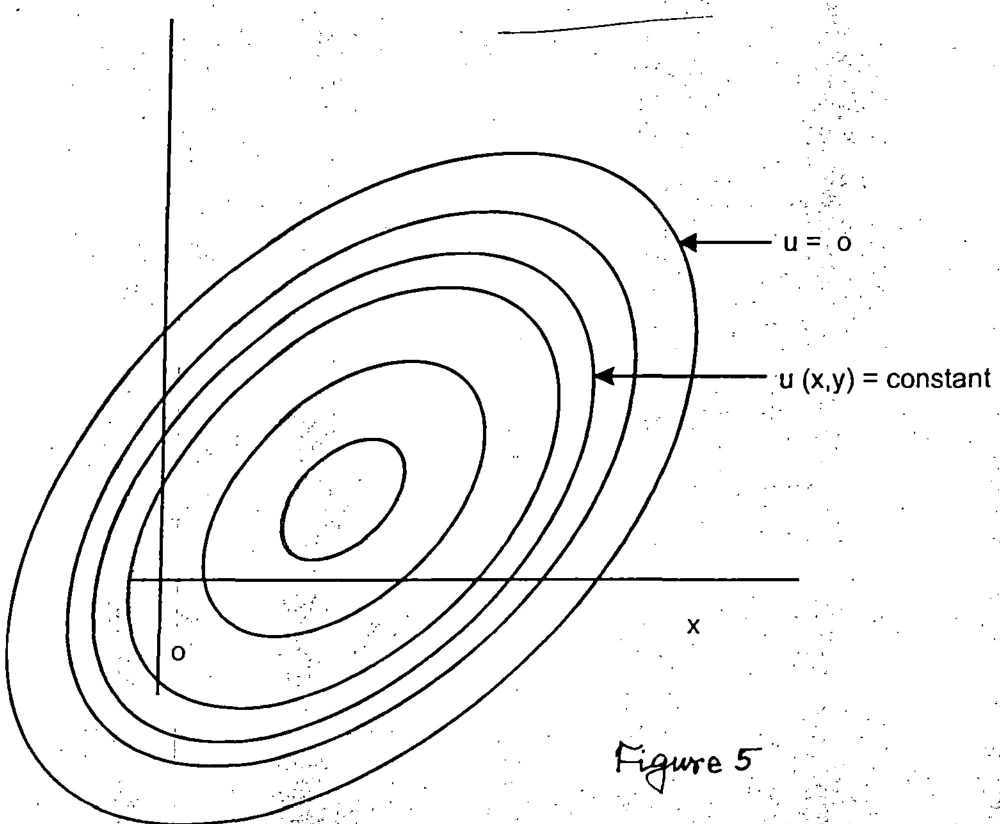

Figure 5

Governing / Basic Equations :

A different apporach will be utilised in establishing the governing equations to that used by Mazumdar [137]. We consider a thin elastic plate which vibrates with a moderately large amplitude in the transverse direction, under the action of an uniform load p. The usual procedure is to consider Karmań type field equations extended to the dynamic case

$$D\nabla^4 w = \mathcal{L}(F, w) + p + \rho h w_{,tt} \qquad [3\cdot1]$$

$$\nabla^4 F = -\frac{Eh}{2}\mathcal{L}(w, w) \qquad [3\cdot2]$$

in which the flexural rigidity D and two dimensional Laplacian operator ∇^2 are defined by

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad [3\cdot3]$$

with h the thickness of plate, E Young's modulus, p the uniform normal load. ρ the material density, w the deflection function and F the Airy stress function. In addition a suffix is taken as an indication of partical differentiqtion with respect to the implied variable and the operator $\mathcal{L}$ is defined by

$$\mathcal{L}(I, J) = I_{,xx} J_{,yy} + J_{,xx} I_{,yy} - 2I_{,xy} J_{,xy} \qquad [3\cdot4]$$

As an alternative to the derivation of the subsequent equations directly from basic equations. as done by Mazumdar [137-139] we will establish them directly from the above equations by introducing the concept of constant deflection contour lines.

It should be noted here that the use of the stress function is equivalent to disregard of intertia terms in the equations of inplane motions of the particles of the plate. This assumption is legitimate when the oscillations primarily take place in the transverse direction, perpendicular to the middle plane of the plate. We choose the deflection function and the stress function in the separable form

$$w(x, y, t) = W(x, y)\, h\, \psi(t)$$
$$F(x, y, t) = F'(x, y)\, h\, \psi^2(t) \qquad [3\cdot5]$$

where ψ (t) is an unknown function of time to be determined. Let us make the following transformations

$$\frac{\partial W}{\partial x} = W_{,x} = u_{,x}\frac{dW}{du}$$

$$W_{,xx} = u_{,x}^2\frac{d^2W}{du^2} + u_{,xx}\frac{dW}{du}$$

$$W_{,y} = u_{,y}\frac{dW}{du}$$

$$W_{,yy} = u_{,y}^2\frac{d^2W}{du^2} + u_{,yy}\frac{dW}{du}$$

$$W_{,xy} = u_{,xy}\frac{dW}{du} + u_{,x}u_{,y}\frac{d^2W}{du^2} \quad ; \text{ etc } \left(u_{,x} = \frac{\partial u}{\partial x}\right) \qquad [3.6]$$

with transformations examplified by those shown in equations (3.5) and (3.6), equations (3.1) and (3.2) are transformed to

$$D\left[A_1\frac{d^4W}{du^4} + A_2\frac{d^3W}{du^3} + A_3\frac{d^2W}{du^2} + A_4\frac{dW}{du}\right]\psi(t)$$

$$= \left[\left\{A_5\frac{dW}{du}\frac{dF'}{du}\right\} + \left\{A_6\frac{d}{du}\left(\frac{dW}{du}\frac{dF'}{du}\right)\right\}\right]\psi^3(t) + \left[p - \rho h W\psi_{,tt}\right] \qquad [3.7]$$

$$\left[A_1\frac{d^4F'}{du^4} + A_2\frac{d^3F'}{du^3} + A_3\frac{d^2F'}{du^2} + A_4\frac{dF'}{du}\right]$$

$$= -\frac{Eh}{2}\left[A_5\left(\frac{dW}{du}\right)^2 + A_6\frac{d}{du}\left(\frac{dW}{du}\right)^2\right] \qquad [3.8]$$

where

$$A_1 = \left(u_{,x}^2 + u_{,y}^2\right)^2$$

$$A_2 = 6\left(u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy}\right) + 8u_{,x}u_{,y}u_{,xy} + 2\left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx}\right)$$

$$A_3 = 3\left(u_{,xx}^2 + u_{,yy}^2\right) + 4u_{,xy}^2 + 4\left(u_{,x}u_{,xxx} + u_{,y}u_{,yyy}\right) + 2u_{,xx}u_{,yy} + 4\left(u_{,x}u_{,xyy} + u_{,y}u_{,xxy}\right)$$

$$A_4 = u_{,xxxx} + u_{,yyyy} + 2u_{,xxyy}$$

$$A_5 = 2\left(u_{,xx}u_{,yy} - u_{,xy}^2\right)$$

$$A_6 = u_{,xx}u_{,y}^2 + u_{,yy}u_{,x}^2 - 2u_{,x}u_{,y}u_{,xy}$$

Since equations (3.7) and (3.8) are valid for all points of the whole domain, it is clear that

$$\iint_{\Omega_u} D\left[A_1\frac{d^4W}{du^4} + A_2\frac{d^3W}{du^3} + A_3\frac{d^2W}{du^2} + A_4\frac{dW}{du}\right]\psi(t)\,d\Omega$$
$$= \iint_{\Omega_u}\left[A_5\frac{dW}{du}\frac{dF'}{du} + A_6\frac{d}{du}\left\{\frac{dW}{du}\frac{dF'}{du}\right\}\right]\psi^3(t)\,d\Omega + \iint_{\Omega_u}\left[p - \rho h W\psi_{,tt}\right]d\Omega \qquad [3.9a]$$

$$\iint_{\Omega_u}\left[A_1\frac{d^4F'}{du^4} + A_2\frac{d^3F'}{du^3} + A_3\frac{d^2F'}{du^2} + A_4\frac{dF'}{du}\right]d\Omega + \frac{Eh}{2}\iint_{\Omega}\left[A_5\left(\frac{dW}{du}\right)^2 + A_6\frac{d}{du}\left(\frac{dW}{du}\right)^2\right]d\Omega = 0 \qquad [3.9b]$$

Where the region of integration is taken over the region enclosed by some contour C_u. To integrate equations (3 .8) and (3.9) previous authors have usually employed Green's theorem. However we pursue a different approach and change the variables using the general formula

$$\iint_{\Omega_u} f\left(u, u_{,x}, u_{,xx}, u_{,y}, u_{,yy}, u_{,xy} \text{---------} \frac{dW}{du}, \frac{d^2W}{du^2}, \text{--------} \frac{d^nW}{du^n}\right) d\Omega$$

$$= -\int_{u^*}^{u} \phi_1(u) \left\{ \oint_{C_u} \phi_2(x,y) \frac{ds}{\sqrt{A_1}} \right\} du \qquad [3\cdot 10]$$

Which is a generalisation of a formula adopted in Ref[115]. Often it has been encountered that the contour integral in (3.10) turns out to be dependent of u and hence care should be taken to evaluate first the contour integral to avoid any confusion that may arise from equation (25) of Ref. [180].

On evaluation of integrals (3.9a) and (3.9b), they may be further reduced to the forms

$$\Lambda_1\left[W, \psi(t), \psi_{,tt}(t)\right] \equiv 0$$

or,
$$D\left[f_{11}\frac{d^3W}{du^3} + f_{12}\frac{d^2W}{du^2} + f_{13}\frac{dW}{du} + f_{14}W\right]\psi(t) + f_{15}\left(\frac{dW}{du}\frac{d\psi}{du}\right)\psi^3(t)$$

$$+ f_{16}\, p + \rho h\, \psi_{,tt}(t) \int_{u^*}^{u} W\, du_0 = 0 \qquad [3\cdot 11]$$

$$\Lambda_2\left[F'(t), \frac{dW}{du}\right] \equiv 0$$

$$\left[g_1\frac{d^3F'}{du^3} + g_2\frac{d^2F'}{du^2} + g_3\frac{dF'}{du} + g_4F'\right]$$

$$+ \frac{Eh}{2} g_5\left(\frac{dW}{du}\right)^2 = 0 \qquad [3\cdot 12]$$

To avoid the integral appearing in equation (3.11) Mazumdar and others [108] have taken the derivative of it making the equation of order four again, viz

$$D\left[f_{21}\frac{d^4W}{du^4} + f_{22}\frac{d^3W}{du^3} + f_{23}\frac{d^2W}{du^2} + f_{24}\frac{dW}{du}\right]\psi(t)$$

$$+\left[f_{25}\frac{dW}{du}\frac{dF'}{du} + f'_{25}\frac{d}{du}\left(\frac{dW}{du}\frac{dF'}{du}\right)\right]\psi^3(t) + f_{26}\,p + \rho h\,\psi_{,tt}(t) W = 0 \qquad [3\cdot 13]$$

Equations (3.11) and (3.12) or equations (3.12) and (3.13) form the basic equations governing the vibrations of any structure. These equations have been derived without specifying the geometry of the structure and they may therefore be specialized to deal with any type of geometry. Moreover (3.11 - 3.13) form a system of ordinary differential equations which may be solved for a variety of structures and subjected to several forms of boundary conditions.

Method of Solution :

The method of solution may be considered in a two fold way.

Considering (3.11) and (3.12) as the basic equations with appropriate boundary conditions, it starts with finding the exact or approximate solution for ψ (t) from equation (3.12). However the exact solution for $\psi(t)$ may only be feasible for linear analysis like what has been followed by Mazumder[137]For non-linear analysis one may has to seek for approximate solution for which the form of deflection function must be first assumed compatible with the boundary condition, next to solve for ψ (t) from equation (3.12) in conjunction with a Galerkin procedure. With this expression of ψ (t), equation (3.11)or equation (3.13) will again yield an ordinary time differential equation in combination with the Galerkin procedure. Mathematically, the above steps may be explained in the following way.

Let u(x,y) = u be the representative of one of the family of the iso-deflection curves. Then for any prescribed boundary conditions the deflection function w (u,t) can be assumed to take the form

$$w = \sum_{i=1}^{n} A_i W(u)\psi(t) \qquad [3{\cdot}14]$$

$$F = F'(u)\psi^2(t) \qquad [3{\cdot}15]$$

Equation (3.12) in combination with (3.14) and (3.15) will yield

F = F (u) .... (3.16)

Substituting this value of F with u was in (3.14) equation (3.11) will yield the error function

$$\epsilon = \Lambda\left[u, \psi(t), \psi_{,tt}(t)\right] \qquad [3{\cdot}17]$$

rather an approximation. The associated error function may be minimised using Galerkin method. The appropriate orthogonality condition applied to equation (3.17) will yield the following " Time Differential Equation"

$$Z\,\psi_{,tt}(t) + C_1\,\psi(t) + C_2\,\psi^3(t) = C_3\,p \qquad [3\cdot18]$$

Equation (3.18) is a Dufling type equation and its solution is well known. Equation (3.18) will enable one to find the frequency response for various case in respect of linear and non-linear response and free or forced vibrations. Additionally, it may also be used to determine static deflection to a uniform load. This equation and the method just detailed may be applied to any structure provided the contour - lines are known. Before we proceed to give any specific illustration, we must have the expressions for the boundary conditions to be satisfied by the deflection function as well as by the stress function.

Boundary conditions and their transformations for u- variables :

The boundary conditions imposed on structures play an important role both in obtaining the exact or approximate solution and also on the class of possible response. We now therefore look at several commonly used boundary conditions and their implications in respect of the concept of Constant Deflection Contour method. In two dimensional problems the following boundary conditions are usually imposed.

3·1 Supporting Conditions and their Transformations :

Case - I Clamped edges

a) For rectangular plates

$$w = \frac{dw}{dx}\left(= w_{,x}\right) = 0,$$ on the edge perpendicular to x-direction

$$w = \frac{dw}{dy}\left(= w_{,y}\right) = 0,$$ on the edge perpendicular to y-direction

or, $$w = \frac{dw}{dn}\left(= w_{,n}\right) = 0,$$ when n is the outward unit-normal to the edge [3·1·1]

The appropriate form of these conditions in terms of the variable u are given by

$$w = \frac{dw}{du} = 0, \text{ on the boundary } (\text{i.e., } u = 0) \qquad [3.1.2]$$

Case - II Simply-Supported edges

In the usual notation the conditions for a simply supported edge may be expressed in the form

$$\begin{aligned} w_{,xx} + \nu w_{,yy} &= 0, \text{ on the edge normal to } x\text{-direction} \\ w_{,yy} + \nu w_{,xx} &= 0, \text{ on the edge normal to } y\text{-direction} \end{aligned} \qquad [3.1.3]$$

which when transformed in terms of u variable become

$$\begin{aligned} \frac{d^2w}{du^2}\left(u_{,x}^2 + \nu u_{,y}^2\right) + \frac{dw}{du}\left(u_{,xx} + \nu u_{,yy}\right) &= 0 \\ \frac{d^2w}{du^2}\left(u_{,y}^2 + \nu u_{,x}^2\right) + \frac{dw}{du}\left(u_{,yy} + \nu u_{,xx}\right) &= 0 \end{aligned} \qquad [3.1.4]$$

on the edge normal to x and y directions respectively.

Case - III Free edges :

For free edge conditions, on an edge normal to the x - direction the usual boundary conditions are

$$\begin{aligned} w_{,xxx} + (2-\nu)\, w_{,xyy} &= 0 \\ w_{,xx} + \nu w_{,yy} &= 0 \end{aligned} \qquad [3.1.5]$$

Together with two similar conditions for the edge normal to the y - direction, if it is also free obtained by interchanging x and y.

The above boundary conditions may be recast in terms of u to yield

$$\left(u_{,y}^2 + \nu u_{,x}^2\right)\frac{d^2w}{du^2} + \left(u_{,xx} + \nu u_{,yy}\right)\frac{dw}{du} = 0$$

$$\left[u_{,x}^3 + (2-\nu)u_{,x}u_{,y}^2\right]\frac{d^3w}{du^3} + \left[3u_{,x}u_{,xx} + (2-\nu)\left(2u_{,x}u_{,xy}\right.\right. \qquad [3.1.6]$$

$$\left.\left. + u_{,x}u_{,yy}\right)\right]\frac{d^2w}{du^2} + \left[u_{,xxx} + (2-\nu)u_{,xyy}\right]\frac{dw}{du} = 0$$

for free edge normal to x-direction.

and a similar expressions for a free edge normal to y-direction changing x to y and y to x

3.2 - Stress conditions and their transformation to u - variables :

a) Stress free edge (normal to x - direction)

$F_{yy} = F_{xy} = 0$ .... (3.2.1)

which on transformation to u variable turns

$$u_{,y}^2\frac{d^2F}{du^2} + u_{,yy}\frac{dF}{du} = 0 \qquad [3.2.2]$$

$$u_{,x}u_{,y}\frac{d^2F}{du^2} + u_{,xy}\frac{dF}{du} = 0$$

Two similar expression if needed for a free edge normal to y-direction are obtained by interchanging x and y .

b) *Immovable* edge (normal to x - direction)

$$F_{,xy} = 0$$

$$u_{,x}\, u_{,y} \frac{d^2F}{du^2} + u_{,xy} \frac{dF}{du} = 0$$

$$U = \int_0^x \left[\frac{1}{E}\left(F_{,yy} - \nu F_{,xx}\right) - \frac{1}{2} w_{,x}^2 \right] dx = 0$$

$$U = \int_0^x \left[\frac{1}{E} \left\{ \left(u_{,y}^2 - \nu u_{,x}^2\right) \frac{d^2F}{du^2} + \left(u_{,yy} - \nu u_{,xx}\right) \frac{dF}{du} \right\} - \frac{1}{2} u_{,x}^2 \left(\frac{dw}{du}\right)^2 \right] dx = 0 \qquad [3.2.3]$$

Similarly for an edge normal to y - direction may be directly put with variable u as

$$u_x u_{,y} \frac{d^2F}{du^2} + u_{,xy} \frac{dF}{du} = 0$$

$$V = \int_0^y \left[\frac{1}{E} \left\{ \left(u_{,x}^2 - \nu u_{,y}^2\right) \frac{d^2F}{du^2} + \left(u_{,xx} - \nu u_{,yy}\right) \frac{dF}{du} \right\} - \frac{1}{2} \left(\frac{dw}{du}\right)^2 u_{,y}^2 \right] dy = 0 \qquad [3.2.4]$$

c) Movable edge :

The stress condition for edge normal to x -direction may be directly written here as

$$u_{,x}\, u_{,y} \frac{d^2F}{du^2} + u_{,xy} \frac{dF}{du} = 0$$

$$U = \text{Constant}$$

and for the edge normal to y direction

$$u_{,x}\, u_{,y} \frac{d^2F}{du^2} + u_{,xy} \frac{dF}{du} = 0$$

$$V = \text{Constant} \qquad [3.2.5]$$

3·3 Since in most practical cases polar co ordiantes having circular symmetry are of importance. We rewrite here the transformed boundary conditions for deflection and stress functions below in polar co ordinates (r, θ)

Case - I : Clamped

$$w = w_r = 0 \Rightarrow w\Big|_{u=0} = \sqrt{1-u}\, \frac{dw}{du}\Big|_{u=0} = 0 \qquad [3.3.1]$$

Case - II simply supported

$$w = w_{,rr} + \frac{\nu}{r} w_{,r} = 0 \Longrightarrow w\Big/_{u=0} = \left\{2(1-u)\frac{d^2 w}{du^2} - (1+\nu)\frac{dw}{du}\right\}\Big/_{u=0} = 0$$

[3.3.2]

For stress conditions, in brief we may state

a) Stress free edge

$$\sqrt{1-u}\;\frac{dF}{du}\Big/_{u=0} = 0 \qquad [3.3.3]$$

b) Immovable edge

$$\left\{2(1-u)\frac{d^2F}{du^2} - (1-\nu)\frac{dF}{du}\right\}\Big/_{u=0} = 0 \qquad [3.3.4]$$

Perhaps it would be proper to mention here that though the above boundary conditions are in fact the transformed expressions from cartesian or polar co ordinates to u - variables in the light of the isodeflection contour method; sometimes in general "it is impossible in the simply supported case to find the exact functions u and w such that they satisfy mechanical boundary conditions" as observed by Mazumdar et.al [106] . In such cases some conditions should be imposed on demand without violating the normal conditions [106]. This will be further discussed in the foregoing illustrations whenever such cases arise.

CHAPTER - IV

" A Simplified Method for Solving Non-linear Problems using "Constant Deflection Contour" Method.

As stated earlier the method of " Lines of Equal Deflection" is one of the existing methods applied in studying the non-linear behaviour of structures subject to moderately large vibrations. With reference to the idea expressed by Banerjee and Rogerson [122] equations (3.11) and (3.12) or equations (3.12) and (3.14) may be applied to study the vibration analysis of structures. However the present investigator has some reservations in accepting the free hand use of any one and the present investigator has to add that the first choice of using equations (3.11) and (3.12), though simplifies the mathematical computations, in the sense that it involves third order ordinary differential equations, may not yield the diserable and accurate result in comparison with the second set of equations (3.12) and (3.13). In the foregoing chapters both the set of governing equations will be used for the analysis and a comparative study will be made thereafter.

In the present chapter all the problems considered here will involve the first set of governing equations viz. equations (3.11) and (3.12).

PROBLEM 4.1

Non - linear Vibrations of Rigid Elliptic Plate With Uniform Thickness

Let us consider a problem in establishing the applicability of this method to one of the useful structures, such as " an elliptic plate with uniform thickness vibrating at large amplitudes."

As usual the dynamic Von-Karman equations for a plate subjected to a normal uniform load may be put in the following form [vide equation (3.1) and (3.2)]

$$D\nabla^4 w = h\mathcal{L}(F,w) + p - \rho h w_{,tt}$$

$$\nabla^4 F = -\frac{E}{2}\mathcal{L}(w,w) \qquad \text{where } D = \frac{Eh^3}{12(1-\nu^2)}$$

With in-plane inertial effect ignored, where w is the deflection function, F is the stress function, p is the load, h is the plate thickness, ρ is the mass density. D is the flexural rigidity, E is the plate modulus of elasticity. υ is the Poison's ratio.

For an elliptic plate clamped along the edges, the family of contour lines of the deflected surface may be represented by

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad [4.1.1]$$

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Where u = 0 defines the boundary and u = 1 defines the centre of plate, the boundary conditions imposed are

w = 0 at u = 0

$$\text{and } \frac{dw}{du} = 0, \text{ at } u = 0, 1 \qquad [4.1.2]$$

Then performing the integrati ons of equations(3·9a) and (3·9b), using equation (4.11) one may arrive at the following equations after a lengthy calculations (for brevity the trivial but lengthy calculations being ommited)

$$\frac{2D(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)^2\frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2}\right]$$

$$+\frac{8h}{a^2b^2}(1-u)\frac{dF}{du}\frac{dw}{du} + p(1-u) + \rho h\int_1^u w_{,tt}\,du = 0 \qquad [4.1.3]$$

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)^2\frac{d^3F}{du^3} - 2(1-u)\frac{d^2F}{du^2}\right]$$

$$= \frac{2E}{a^2b^2}(1-u)\left(\frac{dw}{du}\right)^2 \qquad [4.1.4]$$

Nowinski [121] has shown that when a plate vibrates principally in the transverse directions and in-plane movements are restricted then without only loss of generality the spacial part of the deflection as well as of the stress function may be considered as the same

$$w = \sum_{i=2}^{\alpha} A_i u^i \psi(t) \cong Au^2\psi(t)$$

$$F = Au^2\phi(t) \qquad [4.1.5]$$

A is a constant, $\Phi(t)$ and $\psi(t)$ are unknown functions of time.

Since equation (4.1.5) does not represent the exact solution, Galerkin procedure may be applied to minimize the error. Substitution of equation (4.1.5) in equation (4.1.4) and performing the required integration a relation between $\Phi(t)$ and $\psi(t)$ is first established

$$\Phi(t) = -\frac{6}{5}\,\frac{a^2b^2AE}{(3a^4+3b^4+2a^2b^2)}\,\psi^2(t) \qquad [4.1.6]$$

while equation (4.1.3) will then reduce to

$$\frac{2}{3}D\,\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\,A\psi(t) + 1\cdot 28\,\frac{Eh}{(3a^4+3b^4+2a^2b^2)}\,A^3\psi^3(t)$$

$$+\frac{1}{18}\rho hA\psi_{,tt}(t) = \frac{p}{12} \qquad [4.1.7]$$

Equation (4.1.7) may be put in a simplified form

$$\psi_{,tt}(t) + B_1\psi(t) + B_3\psi^3(t) = Cp \qquad [4.1.8]$$

$$\text{where } B_1 = \left(\frac{12D}{\rho h}\right)\left(\frac{3a^4+3b^4+2a^2b^2}{a^4b^4}\right)$$

$$B_3 = 23\cdot 04\,\frac{EA^2}{\rho(3a^4+3b^4+2a^2b^2)}$$

$$C = \frac{3}{2\rho hA}$$

4.1. a) ***Free Linear Vibration :***

For free vibration p = o and equation (4.8) becomes.

$$\psi_{,tt}(t) + B_1 \psi(t) + B_3 \psi^3(t) = 0 \qquad [4.1.9]$$

For free linear vibration (with $B_3 = 0$) equation (4.1.9) reduces to

$$\psi_{,tt}(t) + B_1 \psi(t) = 0$$

The linear frequency parameter is given by

$$B_1^{1/2} = \left[\left(\frac{12D}{\rho h}\right)\left(\frac{3a^4+3b^4+2a^2b^2}{a^4 b^4}\right)\right]^{1/2}$$

$$\text{or,} \quad B_1^{1/2} = \left[\left(\frac{12D}{\rho h a^4}\right)\left(3m^4+2m^2+3\right)\right]^{1/2}$$

where $m = \frac{a}{b}$

Putting m = 1 we get linear frequency for a circular plate as $9.797\sqrt{\frac{D}{\rho h a^4}}$

4.1.b) ***Free Non-Linear Vibration :***

If T and T* be the corresponding time periods of linear and non-linear oscillations then, the ratio as

$$\frac{T^*}{T} = \left[1 + \frac{3}{4}\frac{B_3}{B_1}\right]^{-1/2} \qquad [4.1.10]$$

$$\text{where } \frac{B_3}{B_1} = \frac{23.04\,(1-\nu^2)\, m^4}{(3m^4+2m^2+3)^2}\left(\frac{A_0}{h}\right)^2 \qquad [4.1.11]$$

$\xi = \frac{A_o}{\lambda}$ represents the relative amplitude

Numerical results have been computed and shown in table [1&2]

4.1.c) ***Static Case :***

Neglecting the inertial terms in equation (4.1.8) one gets, for analyzing the large deflection behaviour

$$B_1 \psi(t) + B_3 \psi^3(t) = C_p \qquad [4.1.12]$$

On further simplification one gets the relation between the non-dimensional central defelction $\left(\frac{W_0}{h}\right)$ and the load parameter $\left[\frac{pa^4}{Eh^4}\right]$ as

$$\frac{2}{3} \frac{(3m^4+2m^2+3)}{(1-\nu^2)} \left(\frac{W_0}{h}\right) + 15{\cdot}36 \frac{m^4}{(3m^4+2m^2+3)} \left(\frac{W_0}{h}\right)^3$$

$$= \frac{pa^4}{Eh^4} \qquad [4.1.13]$$

Numerical results have been shown in tables [4&5]

For circular plate (m=1) and for $\upsilon = 0.3$ equation (4.13) reduces to

$$5{\cdot}8608\left(\frac{W_0}{h}\right) + 1{\cdot}92\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4} \qquad [4.1.14]$$

while $5{\cdot}848\left(\frac{W_0}{h}\right) + 2{\cdot}754\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4}$ [22]

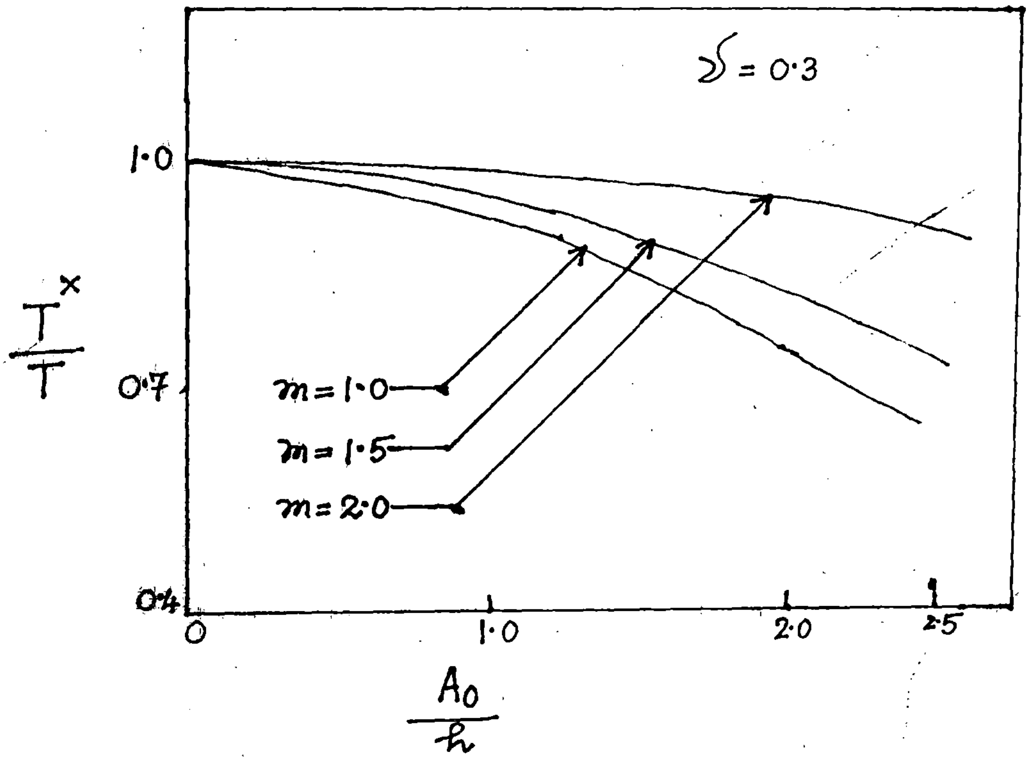

Figure I : Time Period Ratio against Relative Amplitude for Elliptic Plates for Different Values of 'm'.

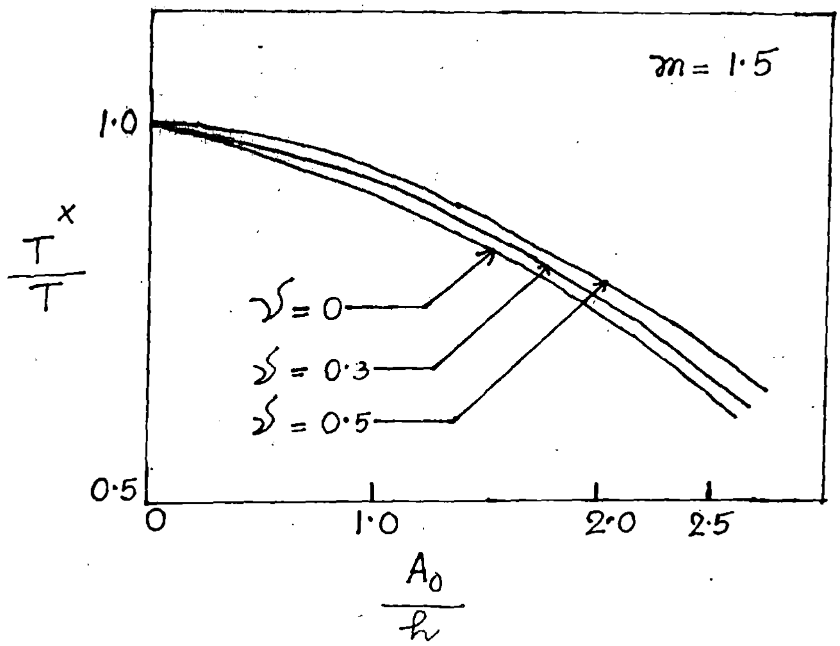

Figure II : Loa Time Period Ratio against Relative Amplitude $\frac{A_0}{h}$ for an Elliptic Plate for Different Values of Poisson's Ratio.

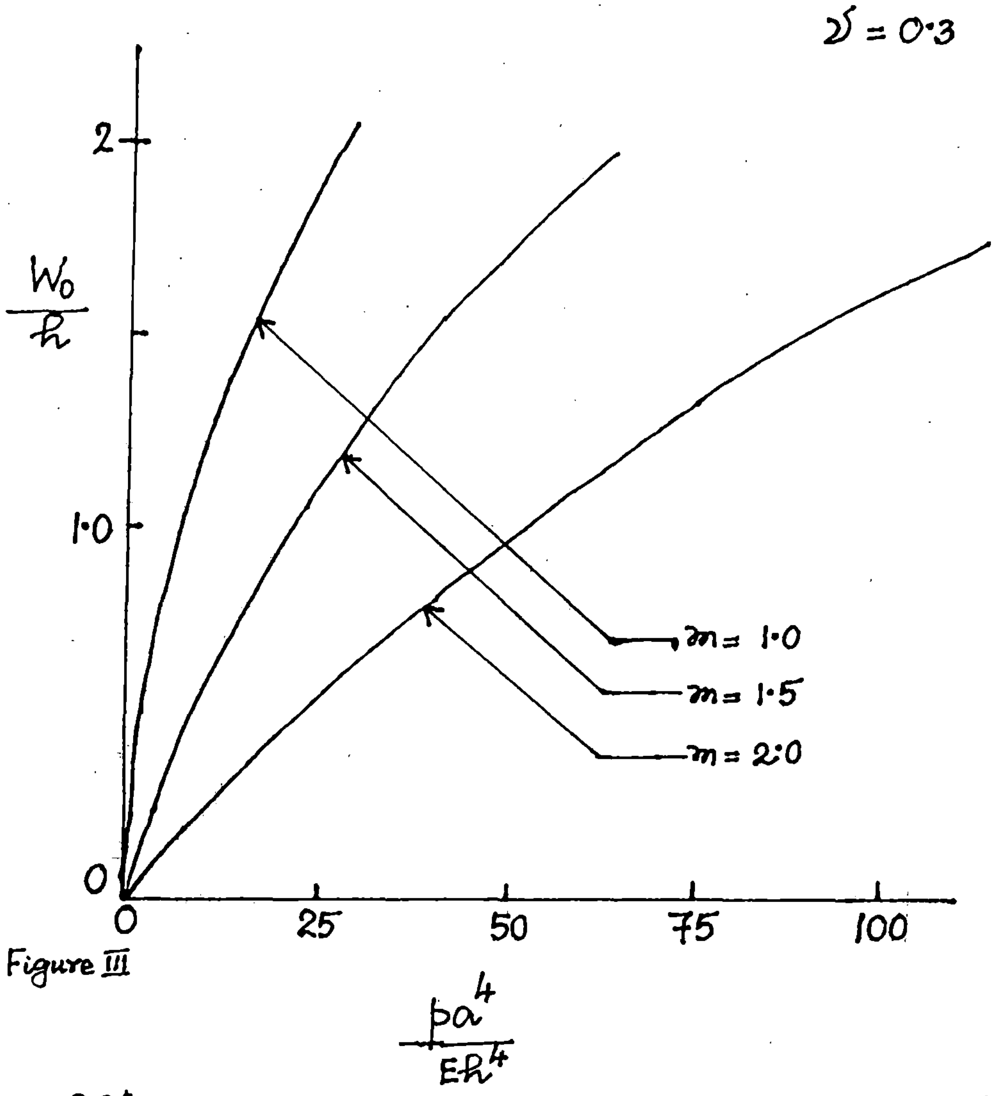

Figure III

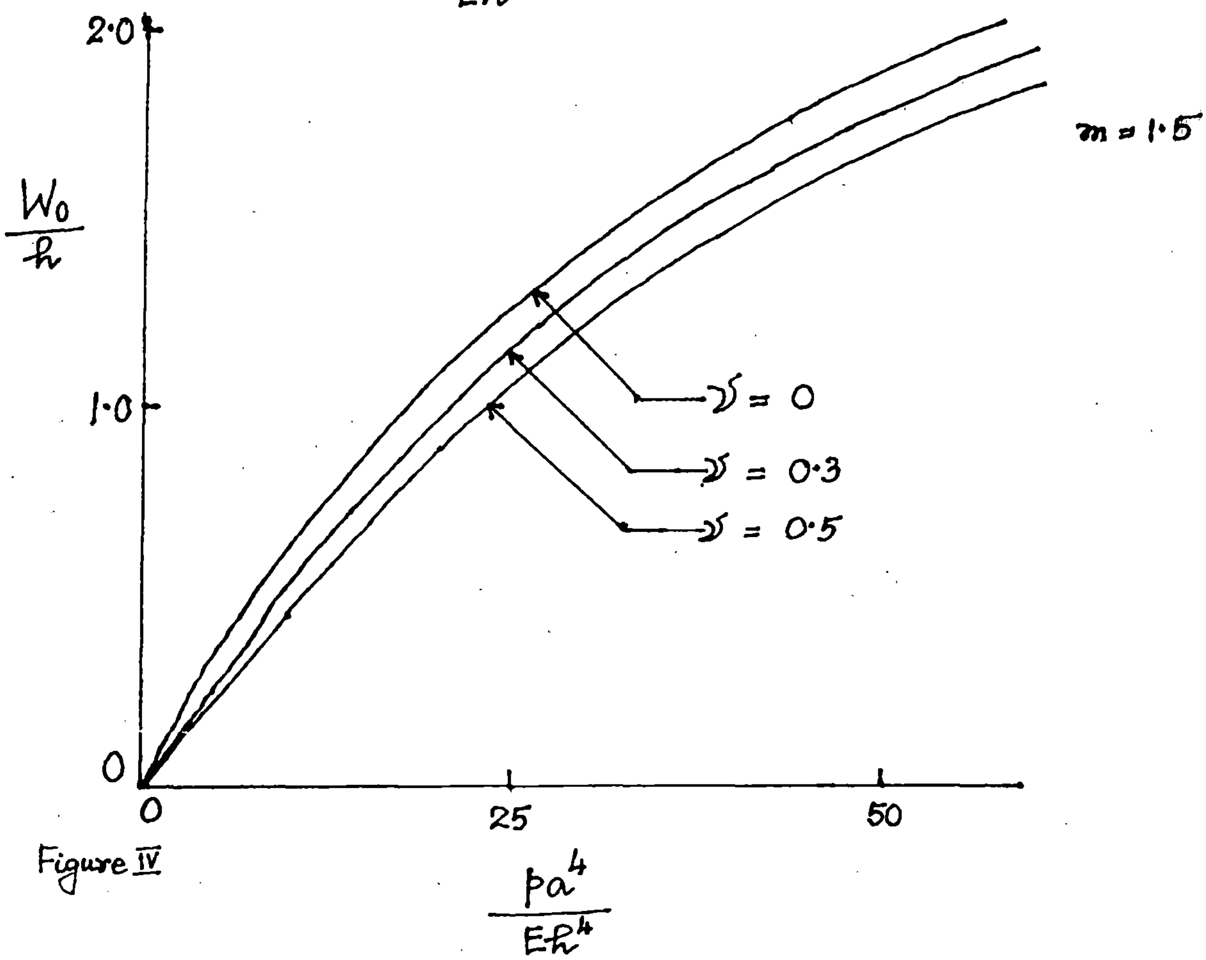

Figure IV

Figure III & IV : Non-linear Static Behaviour of Elliptic Plates.

Numerical Results :

Table [1]: Dependence of the relative time period of nonlinear and linear vibrations ($T^*/_T$) on relative amplitude ($\frac{A_0}{h}$) for different values of m = ($a/_b$), $\upsilon = 0.3$

$\frac{A_0}{h}$	$T^*/_T$			
	m = 1	m = 1.5	m = 2	m = 2.5
0	1.000	1.000	1.000	1.000
0.5	0.9705	0.9811	0.9911	0.9956
1.0	0.8946	0.9300	0.9656	0.9829
1.5	0.7989	0.8593	0.9296	0.9627
2.0	0.7045	0.7822	0.8795	0.9362
2.5	0.6209	0.7076	0.8276	0.9050

Table [2]: Dependence of relative time period (T*/T) on $\left(\frac{A_0}{h}\right)$ for different values of ν, m = 2.0

$\frac{A_0}{h}$	$T^*/_T$			
	υ= 0.2	υ = 0.3	υ = 0.4	υ = 0.5
0	1.000	1.000	1.000	1.000
0.5	0.9905	0.9911	0.9917	0.9926
1.0	0.9636	0.9656	0.9680	0.9713
1.5	0.9230	0.9269	0.9318	0.9385
2.0	0.8734	0.8795	0.8870	0.8973
2.5	0.8197	0.8276	0.8375	0.8513

Table [3]: Comparative study of relative time periods of non-linear and linear vibration [$T^*/_T$] versus [$\frac{A_0}{\sqrt{\mu_1}}$] where μ_1 = 12 (3m^4 + 2m^2 + 3) for a circular plate as obtained in the present study and the results give by Sinharay and Banerjee [124], υ = 0.3, m = 1

$\frac{A_0}{\sqrt{\mu_1}}$	$T^*/_T$	
	Present Study	[124]
0	1.00	1.000
0.02	0.9937	0.992
0.04	0.9757	0.970
0.06	0.9477	0.936
0.08	0.912	0.894
0.10	0.8722	0.848
0.12	0.8296	0.7997
0.14	0.7865	0.752
0.16	0.7443	0.707

Table [4] : Dependence of Central deflection $\frac{W_0}{h}$ on load parameter $\frac{pa^4}{Eh^4}$ for different values of m and $\upsilon = 0.3$

$\frac{W_0}{h}$	pa^4/Eh^4			
	m = 1	m = 1.5	m = 2	m = 2.5
0	0	0	0	0
0.2	1.187	3.3513	8.6772	19.476
0.4	2.446	6.866	17.5542	39.168
0.6	3.93	10.7107	26.8305	58.608
0.8	5.67	15.047	36.7059	80.064
1.0	7.78	20.040	47.385	101.722
1.4	13.47	32.652	71.923	148.416
1.6	17.24	40.6003	86.1913	173.952
2.0	27.08	60.600	119.7200	230.400
2.4	40.606	87.166	161.2358	295.488

Table [5] : Dependence of central deflection W_0/h on load parameter pa^4/Eh^4 for different values of υ, m = 1.5

$\frac{W_0}{h}$	pa^4/Eh^4		
	$\upsilon = 0.2$	$\upsilon = 0.3$	$\upsilon = 0.5$
0	0	0	0
0.2	3.1784	3.3513	4.0598
0.4	6.3240	6.866	8.2867
0.6	10.1934	10.7107	12.8476
0.8	14.3589	15.047	17.9097
1.0	19.1826	20.04	23.6400
1.2	24.8287	25.8537	30.2054
1.4	31.4619	32.652	37.7731
1.6	39.2469	40.6003	46.5100
2.0	58.2926	60.600	67.8400
2.4	85.1927	87.1660	96.4915

Discussions :

Tables (1,2,3) and figures (I,II) show the dependence of the ratio of non-linear to linear time periods T^*/T on the relative amplitude A_0/h for elliptic plates. It may be observed in table (1) and figure (I) that there is hardly any effect of non linearity so far as the free oscillations of elliptic plates having higher eccentricity is concerned. However, in case of circular plate the non-linear effect is notable. . In table (3) it may also be noted that the results of the present study for circular plates are in excellent agreement with those obtained by Sinharay and Banerjee [124]

Tables [4& 5] and figures (III & IV) show the variation of central deflection (W_0/h) for different values of m and υ. Table (4) and figure (III) shows that for a particular value of central deflection, the value of load parameter (pa^4/Eh^4) increases with the increase of the vlaues of m. If implies that to obtain a particular central deflection, more load is needed for an elliptic plate than for a circular plate. Again table (5) shows that for a particular value of load parameter, the central deflection for the plate having higher Possion ratio are smaller than that for the plate meterial having lower Poisson ratio.

From equation [4.1.4], the results for a rigid circular plate do not tally with those of Yamaki[22]. The reason for this may be due to the procedural difference. Also the assumption of retaining the same spacial part for the deflection function and stress function may not be valid for the present case . Also as indicated in the beginning of this chpater the use of equation (3.11) and (3.12) for simplification appears to be unjustified. The use of equation (3.12) and (3.13) will be made later on to justify the above agreement and the results become more accurate.

Problem - 4.2

Non-linear Vibrations of Plates on Elastic Foundation

With the increasing demands for improved efficiency in material usage in structures and the emphasis on high strength/weight ratio, the geometric non-linear behaviour of plates has become more significant. The majority of analysis into the non-linear deflection of plates on elastic foundation has been restricted to the determination of static deflection for an elastic foundation with a Winkler deflection characteristic at the plate/foundation interface. Non-linear static or dynamic analysis of plates under viscous damping and placed on an elastic foundation of Pasternak model is presented.

The 'Constant Deflection Contour' method will be used here in support of its application to a little more complicated problem for which the governing differential equation are of Von Karman type extended to a dynamic case including the effect of elastic foundation and vicous damping. As it has already been stated that this method can easily be applied to investigate large amplitude behaviour of vibrating plates having uncommon or complicated boundary.

The dynamic Von Karman equations for plate placed in an elastic foundation of the Pasternak model and subjected to a normal uniform load may be put in following form [125].

$$D\nabla^4 w - h\mathcal{L}(F,w) - p + \rho h w_{,tt} + \rho h K_v w_{,t} + Kw - G\nabla^2 w = 0 \qquad [4.2.1]$$

$$\nabla^4 F = -\frac{E}{2}\mathcal{L}(w,w) \qquad [4.2.2]$$

with in-plane inertia effect ignored; where, D is the flexural rigidity $= \frac{Eh^3}{12(1-\nu^2)}$

p is the applied load, E is the plate modulus of elasticity, $(Kw - G\nabla^2 w)$ is the linear foundation reaction for Pasternak model, K and G are foundation constants, w is the vertical deflection, ρ is the density of the plate material, ν is the Poisson's ratio, K_v is the viscous damping constant.

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It is assumed that the set of Equations (4.2.1) and (4.2.2) satisfied in every region bounded by the contour line C_u. Using the transformations in Chapter III and integrating over the region equations (4.2.1) and (4.2.2) will reduce to

$$\iint_{\Omega}\left[A_1\left(\frac{d^4w}{du^4}\right)+A_2\left(\frac{d^3w}{du^3}\right)+A_3\left(\frac{d^2w}{du^2}\right)+A_4\left(\frac{dw}{du}\right)\right]d\Omega$$

$$=\iint_{\Omega}\left[A_5\left(\frac{dw}{du}\right)\left(\frac{dF}{du}\right)+A_6\frac{d}{du}\left(\frac{dw}{du}\,\frac{dF}{du}\right)+\left(p-\rho h w_{,tt}-\rho h K_v w_{,t}\right)\right.$$

$$\left.-Kw+A_7\left(\frac{dw}{du}\right)+A_8\left(\frac{d^2w}{du^2}\right)\right]d\Omega \qquad [4.2.3]$$

$$\iint_{\Omega}\left[A_1\left(\frac{d^4F}{du^4}\right)+A_2\left(\frac{d^3F}{du^3}\right)+A_3\left(\frac{d^2F}{du^2}\right)+A_4\left(\frac{dF}{du}\right)\right]d\Omega$$

$$=-\frac{E}{2}\iint_{\Omega}\left[A_9\left(\frac{dw}{du}\right)^2+A_{10}\frac{d}{du}\left(\frac{dw}{du}\right)^2\right]d\Omega \qquad [4.2.4]$$

where $A_1 = \left(u_{,x}^2+u_{,y}^2\right)^2$

$$A_2 = 6\left(u_{,x}^2 u_{,xx}+u_{,y}^2 u_{,yy}\right)+2\left(u_{,x}^2 u_{,yy}+u_{,y}^2 u_{,xx}\right)$$
$$+8u_{,x}u_{,y}u_{,xy}$$

$$A_3 = 4\left(u_{,x}u_{,xxx}+u_{,y}u_{,yyy}\right)+3\left(u_{,xx}^2+u_{,yy}^2\right)+2u_{,xx}u_{,yy}$$
$$+u_{,xy}^2+4\left(u_{,x}u_{,xyy}+u_{,y}u_{,xxy}\right)$$

$$A_4 = u_{,xxxx}+u_{,yyyy}+2u_{,xxyy}$$

$$A_5 = 2\left(u_{,xx}u_{,yy}-u_{,xy}^2\right)$$

$$A_6 = u_{,xx} u_{,y}^2 + u_{,yy} u_{,x}^2 - 2u_{,x} u_{,y} u_{,xy}$$

$$A_7 = G\left(u_{,xx} + u_{,yy}\right)$$

$$A_8 = G\left(u_{,x}^2 + u_{,y}^2\right)$$

$$A_5 = A_9$$

$$A_6 = A_{10} \qquad [4.2.5]$$

Equations (4.2.3), (4.2.4) can be further reduced to simpler form on application of Green's theorem wherever possible.

On transformation to line integrals equations (5.3) and (5.4) will then become.

$$f_1(u)\frac{d^3w}{du^3} + f_2(u)\frac{d^2w}{du^2} + f_3(u)\frac{dw}{du}\frac{dF}{du} + f_4(u)p$$
$$+ f_5(u)\frac{dw}{du} + \rho h\int_1^u w_{,tt}\,du + \rho h K_v \int_1^u w_{,t}\,du$$
$$+ K\int_1^u w\,du = 0 \qquad [4.2.6]$$

$$g_1(u)\frac{d^3F}{du^3} + g_2(u)\frac{d^2F}{du^2} + g_3(u)\frac{dF}{du} + g_4(u)\left(\frac{dw}{du}\right)^2 = 0 \qquad [4.2.7]$$

where $f_i(u)$ and $g_i(u)$ are functions of u only

Equations (4.2.6) and (4.2.7) are the two basic equations to study the dynamic response of structures of arbitrary shape. Hereforth, unless the contour lines are defined one cannot proceed further. The following illustration may be cited for studying the dynamic response of a given shape.

Damped Oscillation of elliptic Plates on An Elastic Foundation :

Considered here an elliptic plate clamped along the edges. The family contour lines of deflected surface may be represented as usual, by

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

where $u = 0$ defines the boundary. The boundary conditions imposed are

$$w = 0 \quad \text{at } u = 0$$

$$\frac{dw}{du} = 0 \quad \text{at } u = 0, 1 \qquad [4.2.8]$$

Then performing the integratins of equations (4.2.3 , 4.2.4) one may arrive at the following equations after a lengthy calculations

$$2D\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)^2\frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2}\right]$$

$$+\frac{8h}{a^2b^2}(1-u)\frac{dw}{du}\frac{dF}{du} + p(1-u) + \rho h\int_1^u\left[w_{,tt} + K_v\, w_{,t}\right]du$$

$$+ K\int_1^u w\,du - 2G\left(\frac{1}{a^2}+\frac{1}{b^2}\right)(1-u)\frac{dw}{du} = 0 \qquad [4.2.9]$$

$$\frac{3a^4+3b^4+2a^2b^2}{a^4b^4}\left[(1-u)^2\frac{d^3F}{du^3} - 2(1-u)\frac{d^2F}{du^2}\right]$$

$$= \frac{2E}{a^2b^2}(1-u)\left(\frac{dw}{du}\right)^2 \qquad [4.2.10]$$

Considering that the plate vibrates primarily in the transverse direction and the plate is restrained from in-plane movements, one can assume without any loss of generality [121]

$$w = Au^2\Psi(t)$$
$$F = Au^2\Phi(t) \qquad [4.2.11]$$

Since equation (4.2.11) does not represent the exact solution, Galerkin procedure may be applied to minimize the error. Substituting equation (4.2.11) in equation (4.2.10) and (4.2.9) and performing the required integrations a relation between $\Phi(t)$ and $\psi(t)$ is first established

$$\Phi(t) = -\frac{6}{5}\frac{a^2 b^2 AE}{(3a^4+3b^4+2a^2b^2)}\psi^2(t) \qquad [4.2.12]$$

and equation (4.2.9) will then reduce to

$$\left[\frac{2D}{3}\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}+\frac{K}{18}+\frac{G}{5}\frac{(a^2+b^2)}{a^2b^2}\right]\psi(t)$$

$$+1\cdot 28\frac{hE}{(3a^4+3b^4+2a^2b^2)}A^2\psi^3(t)-\frac{p}{12}+\frac{\rho h}{18}\left[\psi_{,tt}(t)+K_v\psi_{,t}(t)\right]$$

$$=0 \qquad [4.2.13]$$

Equation (4.2.13) can be put in a simplified form

$$\psi_{,tt}(t)+\mu_0\psi_{,t}(t)+\mu_1^2\psi(t)+\mu_3A^2\psi^3(t)=0 \qquad [4.2.14]$$

where p has been set to zero for free vibration and

$$\mu_0 = K_v$$

$$\mu_1^2 = \frac{12D}{\rho h}\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}+\frac{K}{\rho h}+\frac{18G}{5\rho h}\frac{(a^2+b^2)}{a^2b^2}$$

$$\mu_3 = 23\cdot 04\frac{EA^2}{\rho(3a^4+3b^4+2a^2b^2)}$$

The solution of equation (4.2.14) may be taken as [68]

$$\psi(t) = a_0 e^{-\frac{\mu_0 t}{2}} \sin\left[\mu_1 t\left(1+\frac{3}{8} a_0 A^2 \frac{\mu_3}{\mu_1^2} e^{-\mu_0 t}\right) + \psi_0\right] \qquad [4.2.15]$$

If T and T* be the corresponding time periods of linear and non linear oscillations then

$$\frac{T^*}{T} = \frac{1}{1+\frac{3}{8} a_0 A^2 \frac{\mu_3}{\mu_1^2} e^{-\mu_0 t}}$$

The Dependence of T*/T on the relative amplitude has been presented in Tables [11 and 12]

Static Case

Neglecting the inertial term in equation (4.2.13) the static deflection is given by

$$\left[\frac{2}{3}\frac{(3m^4+2m^2+3)}{(1-\nu^2)} + \frac{0.0555}{(1-\nu^2)} K^* + \frac{1}{5}\frac{(1+m^2)}{(1-\nu^2)} G^*\right]\frac{W_0}{h}$$

$$+ \frac{15.38\, m^4}{(3m^4+2m^2+3)}\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4} \qquad [4.2.16]$$

where $m = \frac{a}{b}$, $K^* = \frac{Ka^4}{D}$, $G^* = \frac{Ga^2}{D}$

$\frac{W_0}{h}$ presents the central deflection and K* and G* are dimensionless parameters

Numerical Results :

Table 6 : Dependence of Central Deflection ($\frac{W_0}{h}$) on Load Parameter ($\frac{pa^4}{Eh^4}$) for different values of K*. υ 0.3, G* =0 , m = 1

$\frac{W_0}{h}$	pa^4/Eh^4					
	K*=0	K*=40	K*=80	K*=120	K*=160	K*=200
0.2	1.74	2.16	2.65	3.13	3.62	4.25
0.4	2.47	3.44	4.42	5.39	6.37	7.34
0.6	3.23	5.39	6.85	8.32	9.78	10.95
0.8	5.67	7.62	9.57	11.52	13.47	15.43
1.0	7.78	10.22	12.66	15.10	17.54	19.98
	9.0[125]	11.8[125]	14.7 [125]	17.0 [125]	20.0 [125]	22.7[125]
1.2	10.34	13.27	16.20	19.13	22.06	24.98
1.4	13.47	16.88	20.30	23.72	27.13	30.55
1.6	17.24	21.14	25.04	28.95	32.86	36.76
1.8	21.75	26.13	30.53	34.92	39.31	43.71

Table7 : Dependence of Central Deflection on Load Parameter for diferent values of K*, G* = 20, υ = 0.3, m = 1.5

$\frac{W_0}{h}$	pa^4/Eh^4		
	K* = 50	K* = 100	K* = 150
0.2	6.82	7.43	8.04
0.4	13.80	15.02	16.24
0.6	21.11	22.94	24.77
0.8	28.91	31.36	33.79
1.0	37.38	40.43	43.47
1.2	46.65	50.32	53.97
1.4	56.65	61.20	65.46
1.6	68.33	73.22	79.01
2.0	95.27	101.37	167.46

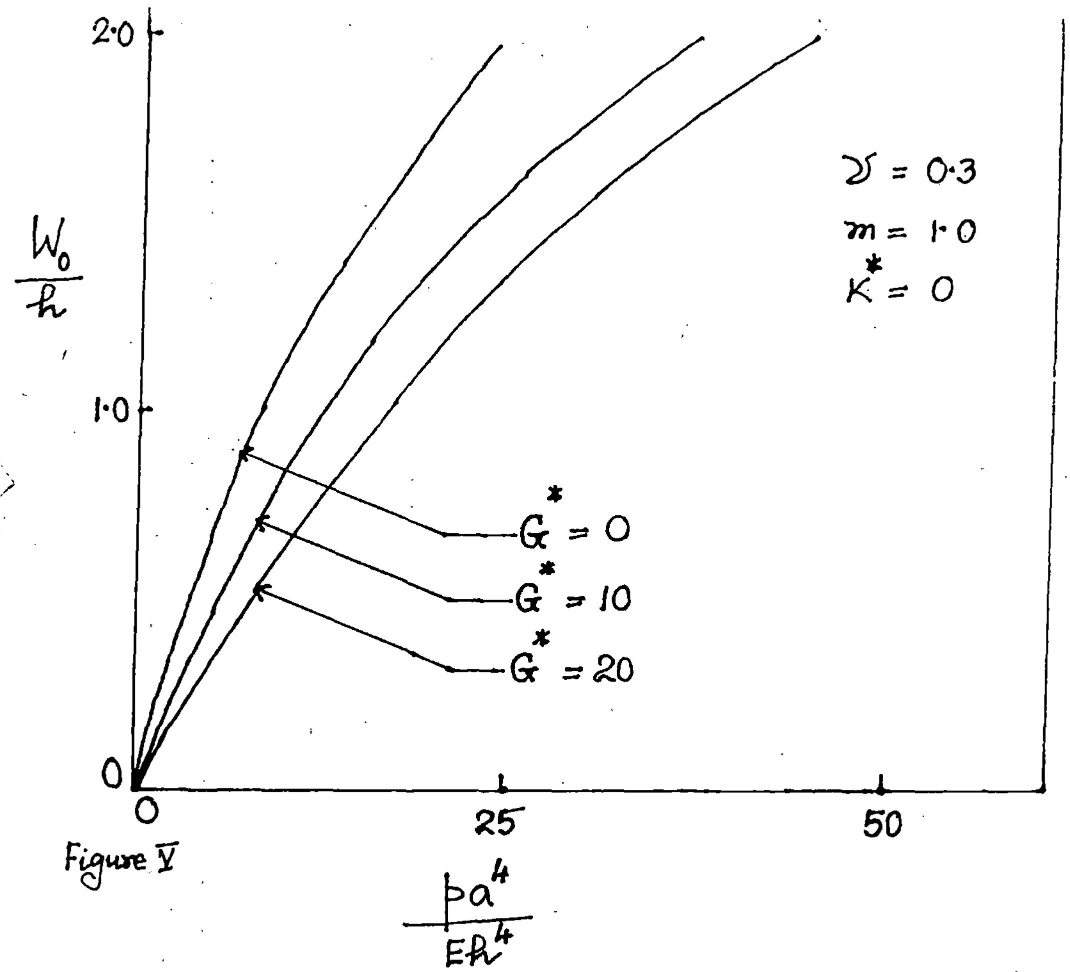


Figure V

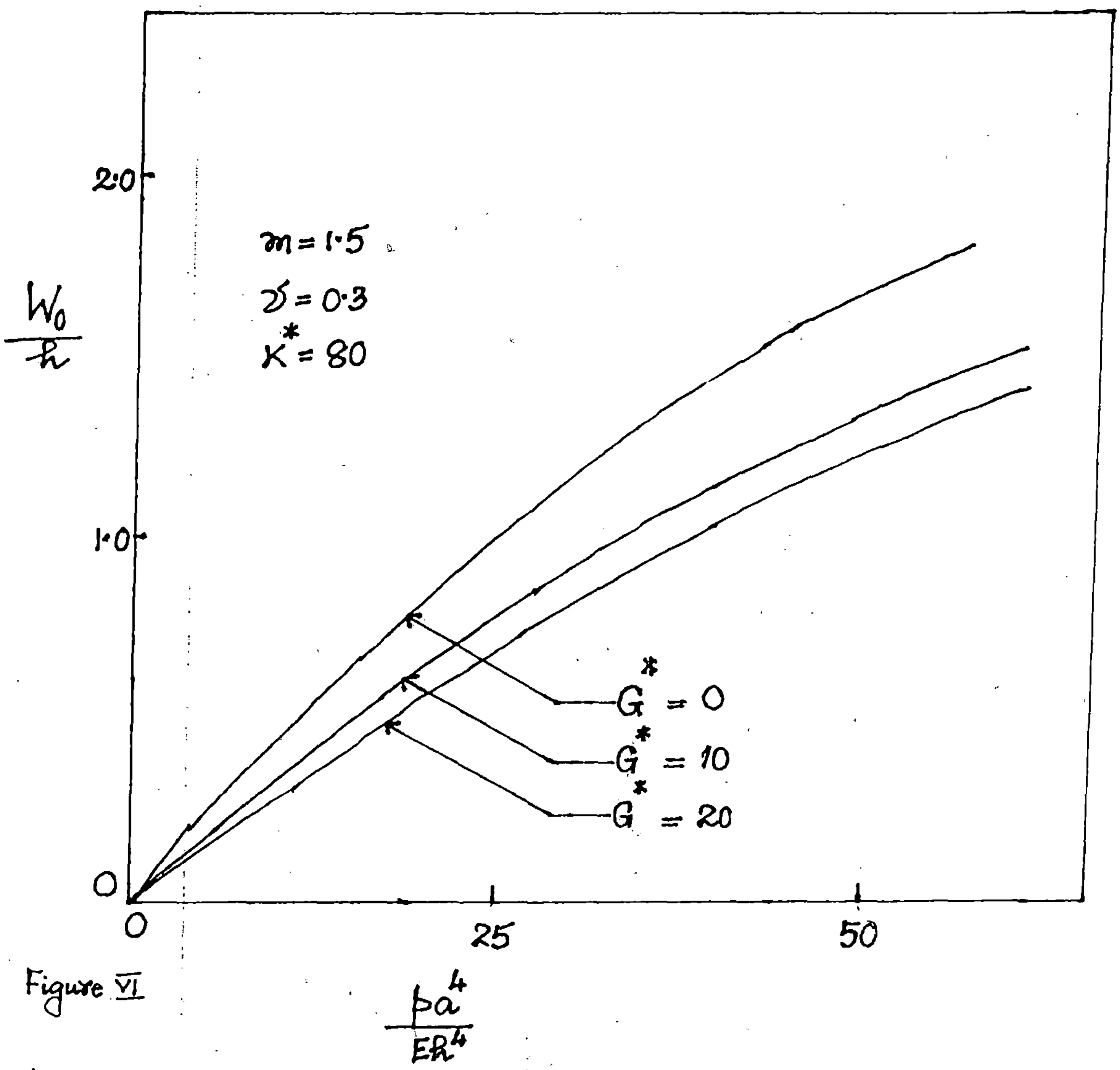


Figure VI

Figure V & VI : Non-linear Central Deflection vs Uniform Lateral Load (Clamped) for Different values of Pasternak Foundation Parameter G^*

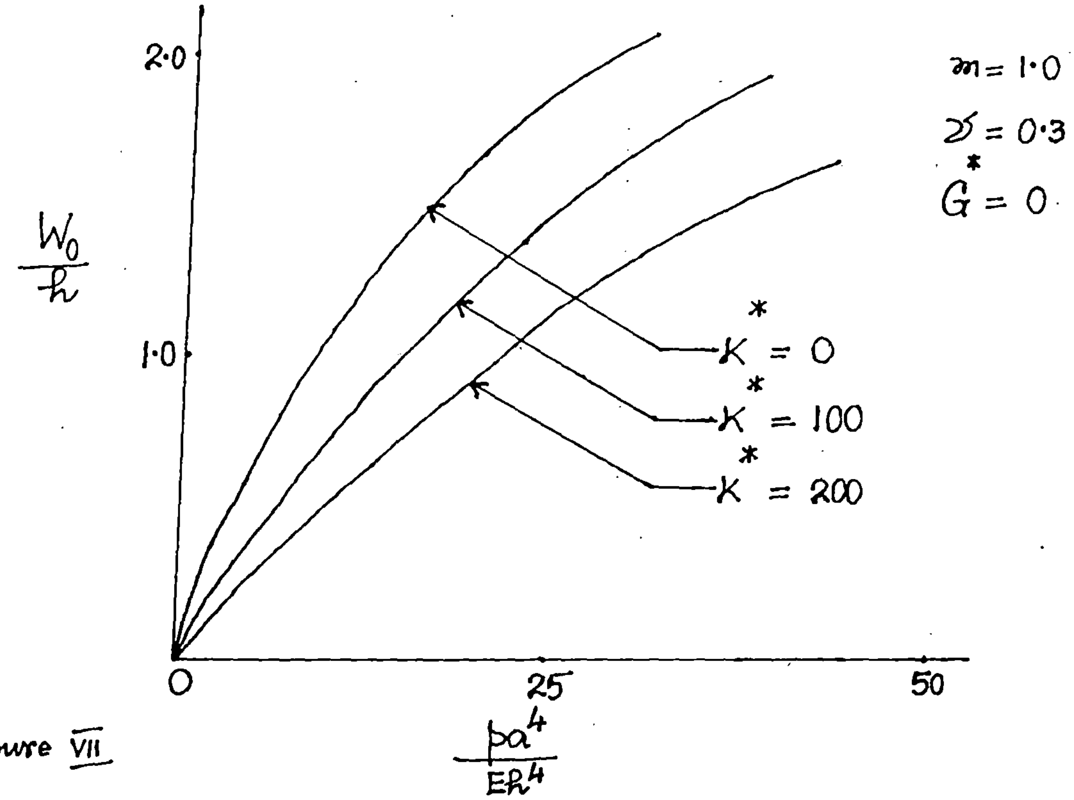


Figure VII

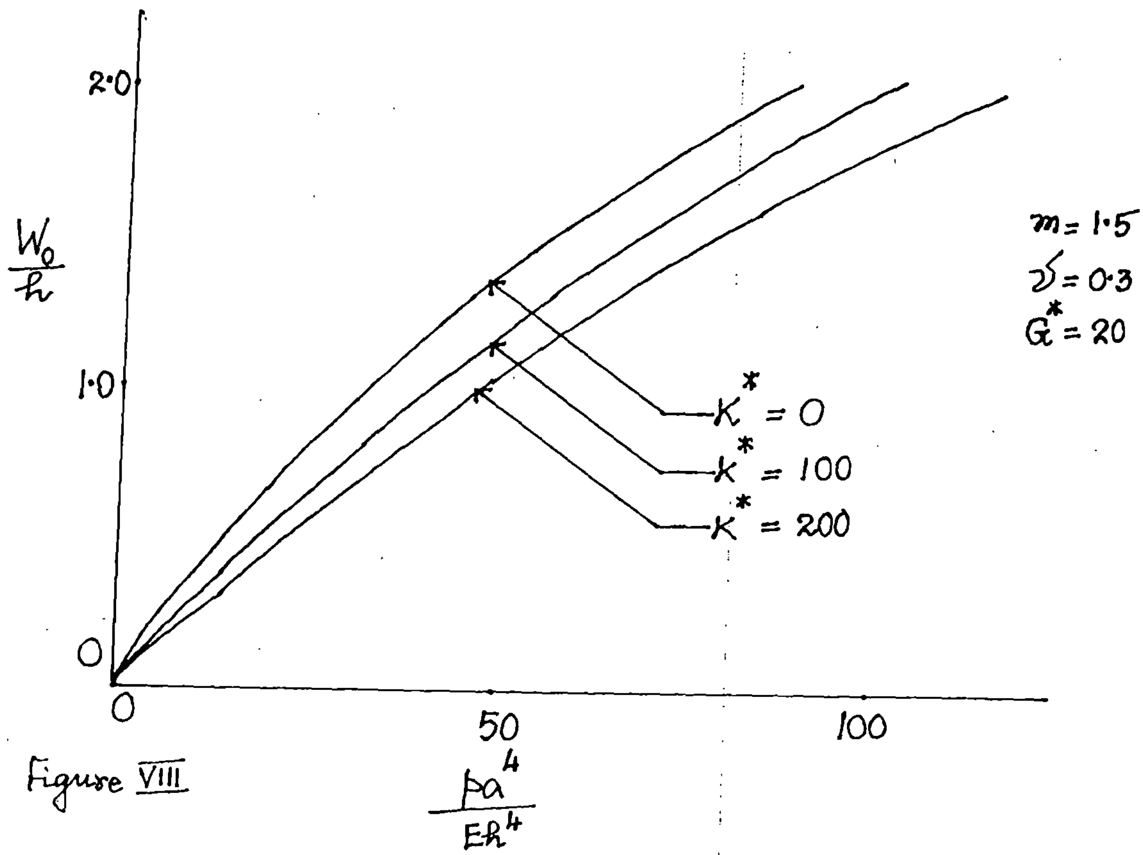


Figure VIII

Figure VII & VIII : Non-linear Central Deflection vs Uniform Lateral Load for a Clamped Elastic Plate for Different Values of Winkler Foundation Parameter K^*

Table (8) : StaticDeflection of an elliptic plate on Elastic Foundation. $\upsilon = 0.3$, m= 1.5, G* = 10

$\frac{W_0}{h}$ pa^4/Eh^4

	K* 0	K*=40	K*=80	K*=120	K*=160	K*=200
0.2	4.78	5.27	5.75	6.24	6.73	7.22
0.4	9.72	10.69	11.67	12.65	13.62	14.60
0.6	15.00	16.45	17.92	19.38	20.84	22.31
0.8	20.75	22.76	24.66	26.61	28.56	30.51
1.0	27.17	29.61	32.05	34.49	36.93	39.37
1.2	34.41	37.34	40.27	43.20	46.12	49.05
1.4	42.63	46.05	49.47	52.88	56.30	59.71
1.6	52.01	55.91	59.82	36.72	67.62	71.53
2.0	74.86	79.74	84.62	89.50	94.38	99.26

Table 9 : values of load parameter for various values of m and υ for K* = 80 and G*= 10

pa^4/Eh^4

$\frac{W_0}{h}$	J = 0.3			m = 1.5		
	m=1	m=1.5	m=2	ν =0.2	ν =0.3	ν =0.5
0.2	3.04	5.75	11.85	5.0189	5.75	6.98
0.4	6.18	11.67	23.91	10.0703	11.67	14.12
0.6	9.50	17.92	36.36	15.1869	17.92	21.60
0.8	13.10	24.66	49.41	20.4014	24.66	29.55
1.0	17.06	32.05	63.27	25.7454	32.05	38.17
1.2	21.49	40.27	78.11	31.2517	40.27	47.61
1.4	26.46	49.47	94.16	36.9545	49.47	58.04
1.6	32.09	59.82	111.60	42.884	59.82	69.61
2.0	45.64	84.62	151.48	55.5582	84.62	96.86

Table 10 : The nonlinear central deflection for a rigidly clamped plate on a Pasternak foundation subject to a static load p = 20, $\upsilon = 0.3$, m = 1

$\frac{W_0}{h}$	G*		
	K* = 50	K* = 100	K* = 150
0.2	207.08	200.32	193.23
0.4	92.15	86.00	79.00
0.6	54.63	43.20	40.14
0.8	33.81	27.00	19.96
1.0	20.86	14.06	7.00
	20.0 [125]	14.0 [125]	5.2 [125]

Table 11 : Dependence on foundation parameter of the relative time period of nonlinear and linear vibration T*/T for elliptic plate for various values of relative amplitude $\frac{A_0}{h}$ for p=0 and m = 2 , υ = 0.3

[T*/T]

$\frac{A_0}{h}$	G*=0			K*=0		
	K*=40	K*=120	K*=200	G*=50	G*=100	G*=200
0.5	0.9994	0.9995	0.99957	0.9997	0.9998	0.9999
1	0.9979	0.9981	0.9983	0.9990	0.9994	0.9996
1.5	0.9952	0.9957	0.9961	0.9978	0.9986	0.9992
2.0	0.9916	0.9924	0.9932	0.9960	0.9975	0.9985
2.5	0.9870	0.9882	0.9894	0.9940	0.9961	0.9977

Table 12 : Comparison of variation of T*/T with relative amplitude $\frac{A_0}{h}$ between cases for circular (m=1) and an elliptic (m=2.0) plate ; υ =0.3 = 0 K* = 40 G* = 100, p= 0

A_0/h	T*/T	
	m=1	m=2
0.0	1.000	1.000
0.5	0.9965	0.9998
101.0	0.9864	0.9994
1.5	0.9700	0.9986
2.0	0.9211	0.9962

Discussions:

Tables (6-9) and figures (V,VI.VII,VIII) show the static behaviour of an elliptic plate for various values of the parameters υ , m and the foundation parameters G* and K*. The results show a very good agreements with those of Smaill [125] in the limiting case when a=b

Table (9) shows that the static deflections are as expcted dependent on the ratio $\frac{a}{b}$ appreciably.

Table (9) also depicts the dependence of the central deflection on poisson's ratio. The choice of material having higher poisson ratio increases the load bearing capacity.

Tables (6,7,8) show the static behaviour of elliptic plate for different values of Winkler foundation parameter K* when Pasternak foundation parameter G* is kept fixed.

Table (10) identifies the characteristics of Pasternak foundation G* for different values of Winkler foundation parameter K*. Though a single term approximation has been made the results are inquite good agreement with those of Smaill [125] for circular plate.

Table (6) when compared with Smaill's [125]. results show that the values of non-dimensional central deflections are little higher than those given by Smaill [125] for all values of K*.

All the tables, presented here, are for undamped cases only with $K_v = 0$. This is the reson for which the results given in tables [6] differ Smaill's results (Fig-5 of Ref [125]). It appears that the central deflections are higher for undamped cases than those for a damped oscillatory motion for a fixed load.

In table(11) and (12) the dependence of $^{T*}/_{T}$ on relative amplitude has been presented. It may be observed that there hardly any effect of nonlinearity so far as undamped free oscillation of elliptic plates are concerned, irrespective of variation in the values of foundation parameter. However, in case of circular plate the nonlinear effect may observed. [Table (12)]

Thus the concept of "Constant Deflection Contour" method may safely be applied for the study of static and dynamic behaviour of plates on elastic foundations.

Problem - 4.3

The non-linear damped vibration of moderately thick plates has been studied by using the method of "Constant Deflection Contour lines" and well-known Berger [55] method . Berger offered a simplified approach to study the non-linear behaviour of thin plates.

Some points on Berger's Method

Combined the potential energy due to the bending and stretching of the middle surface of a plate/shell may be represented by

$$V = \iint \frac{D}{2}\left\{\nabla^2 w\right\}^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu)\left[\frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\Bigg\} dx\,dy \qquad [4.3.1a]$$

in terms of the displacement w, $e_1 = e_{xx} + e_{yy}$, $e_2 = e_{xx}\, e_{yy} - \frac{1}{4} e_{xy}^2$, e_1 and e_2 being the first and second strain invariants. And E, ν, h are Young modulus, Poisson ratio and thickness respectively and with usual notations the in-plane strain components are given by

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - wK_1$$

$$e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - wK_2$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad [4.3.2a]$$

For the shallow shells K_1 and K_2 denote the principal curvatures at a point of the middle surface. For plate problem they are put to zero.

In 1955 Berger [55] proposed that the so called strain invariant of the membrane strain to the strain energy of the plate may be neglected without appreciably impairing the accuracy of the results. On neglection of e_2, equation (4.3 1a) will reduce to

$$V = \frac{1}{2}\iint D\left[(\nabla^2 w)^2 + \frac{12 e_1^2}{h^2} - 2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right] dx\,dy - \iint q w \, dx\,dy \qquad [4.3.3a]$$

with $e_1 = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) + \frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}$

The governing equation will now reduce to

$$\nabla^2(D\nabla^2 w) - C f(t)\nabla^2 w - q + \rho h w_{,tt} = 0 \qquad [4.3.4a]$$

$$\text{with} \quad e_1 = C(1-\nu^2) f(t) / Eh$$

where c is a normalized constant of integration and function of time, f(t) to be determined. Here the present author aims to verify the applicability of the "Constant Deflection Contour method" to Berger equations with regard to specific problem.

Non-Linear Damped Oscillations of Moderately Thick Plate of Arbitrary Shape

Many workers utilize Berger's equation in their respective field of investigations and obtained satisfactory results. In most cases the effects of transverse shear deformation and rotatory inertia has not been taken into account. Sathya moorthy and Chia [133] show that the effect of transverse shear and rotatory inertia play an important role in the large amplitude vibrations of moderately thick plates of different shape. Banerjee and Bhattacharya [132] investigated the effect of transverse shear and rotatory inertia on large amplitude vibration of thick plates.

The works so far carried out on the theory of non-linear vibrations of thick plates are restricted to the plates of regular shapes only. The present investigation concerns with the study of the non-linear static and dynamic behaviour of moderately thick plates of arbitrary shape by using the idea of "Lines of Equal Deflection". To study the dynamic behaviour a damping factor has been introduced. Numerical results for elliptic and circular plates have been computed and compared with the other available known results.

The set of decoupled differential equations governing the vibrations of plates are given by R. Bhattacharya and B. Banerjee [132].

$$\nabla^4 w + \frac{6K}{5(1-\nu^2)}\frac{E}{G_c}\frac{\bar{\alpha}^2 h^2}{12}\tau(t)\nabla^4 w - \frac{6\rho}{5G_c}\frac{\partial^2}{\partial t^2}(\nabla^2 w)$$

$$-\bar{\alpha}^2\tau(t)\nabla^2 w + \frac{12}{h^2 C_p^2}\frac{\partial^2 w}{\partial t^2} = 0 \qquad [4.3.1]$$

where $$\frac{\bar{\alpha}^2 h^2}{12}\tau(t) = \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \qquad [4.3.2]$$

w is the vertical deflection, k is tracing constant characterising the effects of transverse shear deformation, υ is Poisson's ratio, E is Young's modulus, G_c is shear modulus. $\bar{\alpha}$ is coupling parameter, h is the thickness of the plate, $\tau(t)$ is non-linear time dependent function, ρ is the density of the material.

$C_p = \left[\frac{E}{\rho(1-\upsilon^2)}\right]^{1/2}$, speed of the wave propagation along the surface of the plate. The deflections are of the same order of magnitude as the plate thickness.

∇^2 is the two dimensional Laplacian operator.

Putting

$$\left.\begin{aligned} A' &= 1 + \frac{6K}{5(1-\upsilon^2)}\,\frac{E}{G_c}\,\frac{\bar{\alpha}^2 h^2}{12}\,\tau(t) \\ B' &= \frac{6\rho}{5G_c} \\ C' &= \bar{\alpha}^2\,\tau(t) \\ D' &= \frac{12}{h^2 C_p^2} \end{aligned}\right\} \qquad [4.3.3]$$

Equation (3.1) becomes

$$A'\nabla^4 w - B'\frac{\partial^2}{\partial t^2}\left(\nabla^2 w\right) - C'\nabla^2 w + D'\frac{\partial^2 w}{\partial t^2} = 0$$

For damping we introduce another term in the above equation

$$A'\nabla^4 w - B'\frac{\partial^2}{\partial t^2}\left(\nabla^2 w\right) - C'\nabla^2 w + D'\frac{\partial^2 w}{\partial t^2} + K_v\frac{\partial w}{\partial t} = 0 \qquad [4.3.4]$$

where K_v is the damping constant

It is assumed that the equation (4.3.2) and (4.3.4) are satisfied in every region bounded by the contour line C_u. Using the transformations done in Chapter I and integrating over the region, eqations (4.3.4) and (4.3.2) will respectively reduce to

$$\iint_{\Omega} A' \left[A_1 \frac{d^4 w}{du^4} + A_2 \frac{d^3 w}{du^3} + A_3 \frac{d^2 w}{du^2} + A_4 \frac{dw}{du} \right] d\Omega$$

$$- \iint_{\Omega} B' \frac{\partial^2}{\partial t^2} \left[A_5 \frac{d^2 w}{du^2} + A_6 \frac{dw}{du} \right] d\Omega$$

$$- \iint_{\Omega} C' \left[A_5 \frac{d^2 w}{du^2} + A_6 \frac{dw}{du} \right] d\Omega + \iint_{\Omega} D' \frac{\partial^2 w}{\partial t^2} \, d\Omega$$

$$+ \iint_{\Omega} K_v \frac{\partial w}{\partial t} \, d\Omega = 0 \qquad [4.3.5]$$

$$\iint_{\Omega} \frac{\bar{\alpha}^2 h^2}{12} \tau(t) \, d\Omega = \iint \frac{1}{2} A_5 \left(\frac{dw}{du} \right)^2 d\Omega \qquad [4.3.6]$$

where

$$A_1 = \left(u_{,x}^2 + u_{,y}^2 \right)^2$$

$$A_2 = 6 \left(u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy} \right) + 2 \left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx} \right) + 8 u_{,x} u_{,y} u_{,xy}$$

$$A_3 = 4 \left(u_{,x} u_{,xxx} + u_{,y} u_{,yyy} \right) + 3 \left(u_{,xx}^2 + u_{,yy}^2 \right) + 4 \left(u_{,x} u_{,xyy} + u_{,y} u_{,xxy} \right)$$
$$- 2 u_{,xx} u_{,yy} + 4 u_{,xy}^2$$

$$A_4 = u_{,xxxx} + u_{,yyyy} + 2 u_{,xxyy}$$

$$A_5 = u_{,x}^2 + u_{,y}^2$$

$$A_6 = u_{,xx} + u_{,yy}$$

On transformation to line intergals, utilising Green's theorem, equation (4.3.5) and (4.3.6) becomes

$$A'\left[f_1(u)\frac{d^3w}{du^3}+f_2(u)\frac{d^2w}{du^2}+f_3(u)\frac{dw}{du}\right]-B'\frac{\partial^2}{\partial t^2}\left[g_5\frac{dw}{du}\right]$$

$$-C'g_5\frac{dw}{du}+D'\int_1^u w_{,tt}\,du+K_v\int_1^u w_{,t}\,du=0 \qquad [4.3.8]$$

and

$$\frac{\bar{\alpha}^2 h^2}{12}\tau(t)=E'\int_1^u j(u)\left(\frac{dw}{du}\right)^2 du \qquad [4.3.9]$$

where E' is a constant and $f_i(u)$, $g_i(u)$ and $j(u)$ are functions of u only.

Illustration :
Damped Oscillation of Elliptic Plate

Considering an elliptic plate clamped along the boundary, the family of contour lines of the deflected surface may be presented by

$$u=1-\frac{x^2}{a^2}-\frac{y^2}{b^2} \qquad [4.3.10]$$

where u = 0 defines the boundary .
The boundary conditions for a clamped plate are

$$\left.\begin{array}{l} w=0 \quad \text{at} \quad u=0 \\ \text{and } \dfrac{dw}{du}=0 \quad \text{at} \quad u=0,1 \end{array}\right\} \qquad [4.3.11]$$

Performing the integrations, of equations (4.3.5) and (4.3.6) with the boundary conditions represented by (4.3.11). One may arrive at the following equations for elliptic plate after a lengthy calculations

$$A'\left[(1-u)^2\frac{d^3w}{du^3}-2(1-u)\frac{d^2w}{du^2}\right]-M^2(1-u)\frac{dw}{du}-N^2(1-u)\frac{dw}{du}$$

$$+P^2\int_1^u w_{,tt}\,du+Q^2\int_1^u w_{,t}\,du=0 \qquad [4.3.12]$$

$$\frac{\bar{\alpha}^2 h^2}{12}\tau(t)=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_1^u (1-u)\left(\frac{dw}{du}\right)^2 du \qquad [4.3.13]$$

where

$$\left.\begin{aligned}
M^2 &= \frac{C' a^2 b^2 (a^2+b^2)}{(3a^4+3b^4+2a^2b^2)} \\
N^2 &= \frac{B' a^2 b^2 (a^2+b^2)}{(3a^4+3b^4+2a^2b^2)} \\
p'^2 &= \frac{D' a^4 b^4}{2(3a^4+3b^4+2a^2b^2)} \\
Q^2 &= \frac{K_d a^4 b^4}{2(3a^4+3b^4+2a^2b^2)}
\end{aligned}\right\} \qquad [4.3.14]$$

Considering that the plate vibrates primarily in the transverse direction and the plate is restrained from in-plane movements one can assume without any loss of generality

$$w \sum_{i=2}^{\alpha} W_i u^i \psi(t) \simeq W_0 u^2 \psi(t) \qquad [4.3.15]$$

Method of Solution

Since equation (4.3.15) does not represent the exact solution, Galerkin procedure may be applied to minimize the error. Substituting equation (4.3.15) in equation (4.3.12) and (4.3.13) and after performing the integration satisfying necessary condition as required in Galerkin procedure one gets

$$\frac{1}{3} A' \psi(t) + \frac{1}{10} M^2 \psi(t) + \left[\frac{1}{10} N^2 + \frac{1}{18} p'^2\right] \psi_{,tt}(t) + \frac{1}{18} Q^2 \psi_{,t}(t) = 0 \qquad [4.3.16]$$

$$\tau(t) = \frac{4 W_0}{\alpha^2 h^2} \frac{a^2+b^2}{a^2 b^2} \psi^2(t) \qquad [4.3.17]$$

Substituting $\tau(t)$ given by equation (4.3.17) into the first two terms on the left hand side of equation (4.3.16) it reduces to

$$\psi_{,tt}(t) + \mu \psi_{,t}(t) + \mu_1^2 \psi(t) + \mu_2 W_0 \psi^3(t) = 0 \qquad [4.3.18]$$

where

$$\left.\begin{aligned}
\mu &= \frac{K_d}{\dfrac{12}{h^2 C_p^2} + \dfrac{108}{25}\dfrac{\rho}{G_c}\left(\dfrac{a^2+b^2}{a^2 b^2}\right)} \\[2ex]
\mu_1^2 &= \frac{3a^4+3b^4+2a^2b^2}{\dfrac{a^4b^4}{h^2C_p^2} + \dfrac{9}{25}\dfrac{\rho}{G_c}a^2b^2(a^2+b^2)} \\[2ex]
\mu_2 &= \frac{2}{5}\,\frac{3\left(\dfrac{a^2+b^2}{h}\right)^2 + \dfrac{KE}{G_c(1-\nu^2)}\left(\dfrac{a^2+b^2}{a^2b^2}\right)(3a^4+3b^4+2a^2b^2)}{\dfrac{a^4b^4}{h^2C_p^2} + \dfrac{9}{25}\dfrac{\rho}{G_c}a^2b^2(a^2+b^2)}
\end{aligned}\right\} \quad [4.3.19]$$

The solution of equation (3.18) may be taken as

$$\psi(t) = a_0 e^{-\mu t/2} \operatorname{Sin}\left[\mu_1 t\left(1+\frac{3}{8}a_0^2 W_0^2 \frac{\mu_2}{\mu_1^2} e^{-\mu t}\right) + \psi_0\right] \qquad [4.3.20]$$

The time period of non-linear oscillation

$$T^* = \frac{2\pi}{\mu_1\left(1+\frac{3}{8}a_0^2 W_0^2 \frac{\mu_2}{\mu_1^2} e^{-\mu t}\right)}$$

The corresponding time period of linear oscillation is

$$T = \frac{2\pi}{\mu_1} \quad (\text{for linear oscillation } \mu_2 = 0)$$

$$\text{Thus } \frac{T^*}{T} = \frac{1}{1+\frac{3}{8}a_0^2 W_0^2 \frac{\mu_2}{\mu_1^2} e^{-\mu t}} \qquad [4.3.21]$$

where

$$\frac{\mu_2}{\mu_1^2} = \frac{\frac{6}{5}\left(\frac{a^2+b^2}{h}\right)^2}{(3a^4+3b^4+2a^2b^2)} + \frac{2}{5}\frac{KE}{G_c(1-\nu^2)}\frac{(a^2+b^2)}{a^2b^2}$$

[4.3.22]

Static Case

For mechanical loading the inertial terms in equation (4.3.1) are neglected to consider the static case and the required differential equation for the static deflection of a thick elastic plate is

$$\nabla^4 w + \frac{6K}{5(1-\nu^2)}\frac{E}{G_c}\frac{\bar{\alpha}^2 h^2}{12}\tau(t)\nabla^4 w - \bar{\alpha}^2\tau(t)\nabla^2 w - \frac{p}{D} = 0 \quad [4.3.23]$$

where p is the uniform load and the coupling parameter $\frac{\bar{\alpha}^2 h^2}{12}\tau(t)$ is given by equation (4.3.2) (.) and

$D = \frac{Eh^3}{12(1-\nu^2)}$ is flexural rigidity.

Equation (4.3.23) can be written as

$$A'\nabla^4 w - C'\nabla^2 w - \frac{p}{D} = 0 \qquad [4.3.24]$$

where A' and C' are given by equation (4.3.3)

Now introducing the idea of constant deflection contour lines as it has been done in the case of dynamic loading and proceeding as before for elliptic plate where the contour lines are represented by $u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$

One gets the equation

$$A'\left[(1-u)^2\frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2}\right] - M^2(1-u)\frac{dw}{du} + q_1(1-u) = 0 \qquad [4.3.25]$$

where M^2 is given by equation (4.3.14)

$$\text{and} \quad q_1 = \frac{a^4b^4p}{2D(3a^4+3b^4+2a^2b^2)} \qquad [4.3.26]$$

For the solution of equation (3.25) is assumed

$$w = W_0 u^2 \qquad [4.3.27]$$

Substituting equation (3.27) in equation (4.3.25) and applying Galerkin procedure to minimize the error one gets

$$\left(\frac{W_0}{h}\right) + \beta \left(\frac{W_0}{h}\right)^3 = \gamma \left(\frac{p a^4}{E h^4}\right) \qquad [4.3.28]$$

where

$$\beta = \frac{6}{5} \frac{(a^2+b^2)^2}{(3a^4+3b^4+2a^2b^2)} + \frac{2}{5} \frac{KE}{G_c} \frac{h^2}{(1-\nu^2)} \frac{(a^2+b^2)}{a^2 b^2}$$

$$\gamma = \frac{3}{2} \frac{(1-\nu^2)\, b^4}{(3a^4+3b^4+2a^2b^2)}$$

Table 13 : Free vibrations of clamped elliptical plate

$$\frac{KE}{G_c} = 0, \quad \nu = 0.3, \quad \mu = 0, \quad m = 1.5$$

$\frac{W_0 a_0}{h}$	$T^*/_T$		
	Present Study	Das and Banerjee (68)	Sathyamoorthy (135)
0	1.000	1.000	1.000
0.5	0.9502	0.9502	0.9615
1	0.8268	0.8268	0.8700
1.5	0.6797	0.6797	0.7654
2	0.5441	0.5441	0.6500

Table 14 : Free vibrations of clamped Elliptic plate

$$\frac{KE}{G_e} = 0, \quad \nu = 0.3, \quad \mu = 0, \quad m = 2$$

$\frac{W_0 a_0}{h}$		$T^*/_T$	
	Present Study	Das and Banerjee	Sathyamoonthy
0	1.000	1.000	1.000
0.5	0.9545	0.9545	0.9615
1.0	0.8399	0.8400	0.8750
1.5	0.6998	0.6999	0.7538
2.0	0.5674	0.5674	0.6500

Table 15 : Free vibration of clamped elliptic plate

$$\frac{KE}{G_c} = 1, \quad \nu = 0\cdot3, \quad \mu = 0, \quad \frac{a}{b} = 1\cdot5$$

$\frac{W_0 a_0}{h}$	T*/T								
	h/a=0.2			h/a=0.1			h/a=0.066		
	Present Study	Das & Banerj	Sathya moorthy	Present Study	Das & Banerjee	Sathya moorthy	Present Study	Das & Banerjee	Satya moorthy
0	1.000	1.000	1.111	1.000	1.000	1.025	1.000	1.000	1.0077
0.5	0.9454	0. 9454	1.0538	0.9490	0.9490	0.9846	0.9497	0.9497	0.9730
1	0.8124	0.8124	0.9154	0.8232	0.8238	0.8808	0.8252	0.8252	0.8700
1.5	0.6581	0.658	0.7800	0.6742	0.6742	0.7712	0.6773	0.6773	0.7700
2.0	0.5199	0.5198	0.6600	0.5379	0.5379	0.6550	0.5411	0.5411	0.6500

Table 16 : Free vibration of clamped elliptic plate

$$\frac{KE}{G_c} = 1, \quad \nu = 0\cdot3, \quad \mu = 0, \quad \frac{a}{b} = 2$$

	T*/T								
	h/a=0.2			h/a=0.1			h/a=0.066		
	Present Study	Das & Banerj	Sathya moorthy	Present Study	Das & Banerjee	Sathya moorthy	Present Study	Das & Banerjee	Satya moorthy
0	1.000	1.000	1.1807	1.000	1.000	1.0423	1.000	1.000	1.0135
0.5	0.9470	0. 9471	1.0942	0.9526	0.9526	0.9846	0.9537	0.9537	0.9731
1	0.8172	0.8172	0.9270	0.8341	0.8341	0.8865	0.8373	0.8373	0.8570
1.5	0.6653	0.6653	0.800	0.6909	0.6908	0.7827	0.6960	0.6959	0.7769
2.0	0.5279	0.5279	0.6731	0.5570	0.5569	0.6600	0.5629	0.5628	0.6500

Table 17 : Free vibration of clamped circular plate

$m = 1, \; \nu = 0.3, \; \mu = 0$

$\frac{W_0 a_0}{h}$	$T^*/_T$								
	h/a=0.2 $\frac{KE}{G_c}=8.197$			h/a=0.1, $\frac{KE}{G_c}=8.8133$			h/a=0.066, $\frac{KE}{G_c}=19.3165$		
	Present Study	Das & Banerjee	K.K.Raju [136]	Present Study	Das & Banerjee	K.K.Raju (136)	Present Study	Das & Banerjee	K.K.Raju (136)
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.2	0.9868	0. 9869	1.9921	0.9885	0.9885	0.9924	0.9904	0.9910	0.9927
0.4	0.9494	0.9494	0.9699	0.9556	09556	0.9710	0.9628	0.9629	0.9718
0.6	0.8929	0.89303	0.9366	0.9053	0.9054	0.9388	0.9210	0.9202	0.9402
0.8	0.8242	0.8244	0.8965	0.8433	0.8433	0.8995	0.8664	0.8664	0.9015
1.0	0.7501	0.7503	0.8533	0.7749	0.7751	0.8568	0.8058	0.8058	0.8591

Table18 : Damped oscillations of clamped elliptical plate

$K_v = 0.5, \quad \frac{1}{h^2 C_p^2} = 1, \quad \frac{\rho}{b^2 G_c} = 0.5, \quad m = 1.5, \; \frac{h}{a} = 0.2$

$\frac{W_0 a_0}{h}$	$T^*/_T$					
	KE/Gc=2.5		KE/Gc=10		KE/Gc=20	
	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68)
0	1.000	1.000	1.000	1.000	1.000	1.000
0.25	0.9862	0.9892	0.9780	0.9873	0.9673	0.9874
0.5	0.9472	0.9582	0.9177	0.9510	0.8809	0.9416
0.75	0.8885	0.9105	0.8321	0.8661	0.7790	0.8775
1	0.8177	0.8513	0.7359	0.8291	0.6492	0.8012

Table 19 : Damped oscillations of clamped elliptical plate

$$K_v = 0\cdot5, \frac{1}{h^2 C_p^2} = 1, \frac{\rho}{b^2 G_c} = 0\cdot5, t = 5 \text{ secs}, m = 2, \frac{h}{a} = 0\cdot2$$

$\frac{W_0 a_0}{h}$	$T^*/_T$					
	KE/Gc=2.5		KE/Gc=10		KE/Gc=20	
	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68)
0	1.000	1.000	1.000	1.000	1.000	1.000
0.25	0.9858	0.9892	0.9733	0.9867	0.9571	0.9828
0.5	0.9455	0.9582	0.9011	0.9489	0.8479	0.9345
0.75	0.8853	0.9105	0.8020	0.8920	0.7125	0.8638
1	0.8128	0.8513	0.6949	0.8228	0.5829	0.7811

Table 20 : Damped Oscillations of clamped circular plate

$$K_v = 0\cdot5, \quad m = 1, \quad \frac{1}{h^2 C_p^2} = 1, \quad \frac{\rho}{b^2 G_c} = 0\cdot5, \quad t = 5 \text{ secs} \quad \frac{h}{a} = 0\cdot2$$

$\frac{W_0 a_0}{h}$	$T^*/_T$					
	KE/Gc=2.5		KE/Gc=10		KE/Gc=20	
	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68)	Present Study	Das & Banerjee (68)
0	1.000	1.000	1.000	1.000	1.000	1.000
0.25	0.9863	0.9889	0.9877	0.98122	0.9744	0.9862
0.5	0.9475	0.9570	0.9527	0.9289	0.9051	0.9470
0.75	0.8892	0.9983	0.8995	0.8531	0.8091	0.8881
1	0.8187	0.8478	0.8343	0.7655	0.7045	0.8171

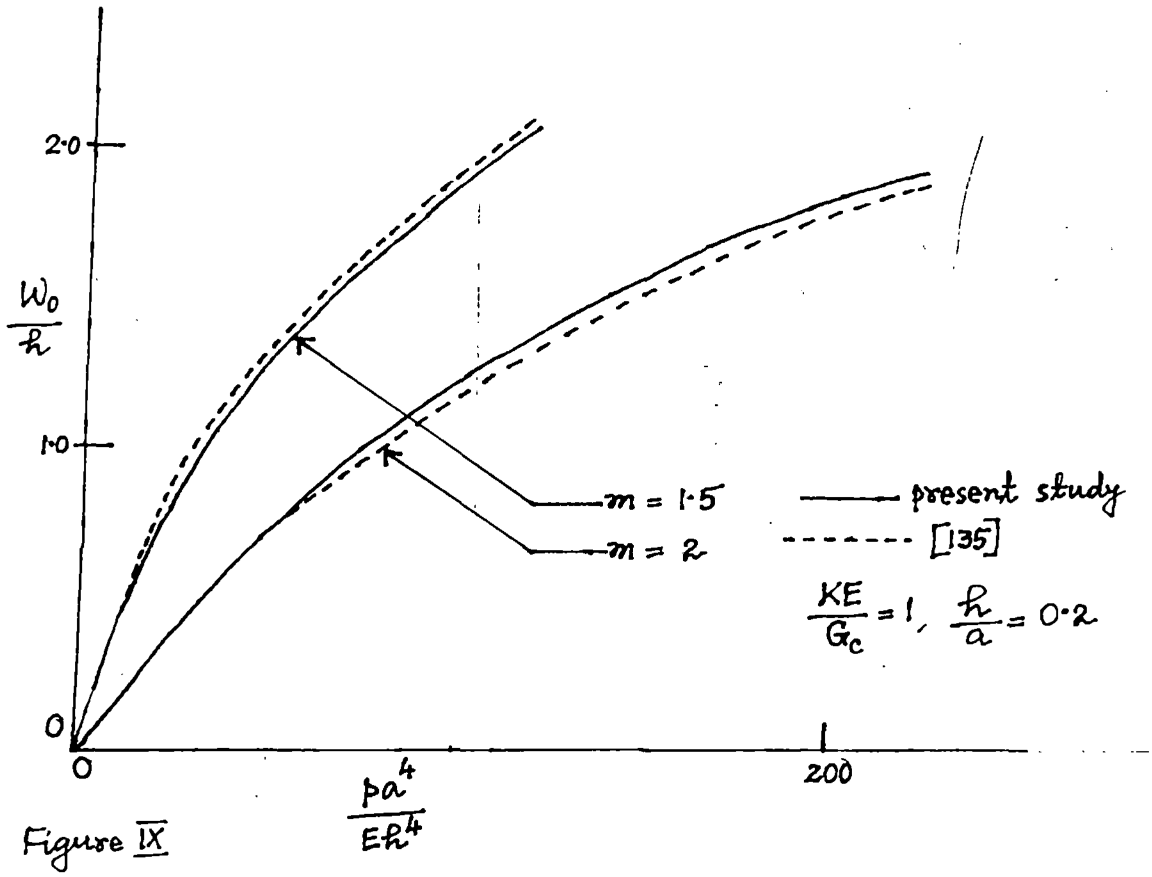


Figure IX

:- Non-linear Static Behaviour of Damped Elastic Plates

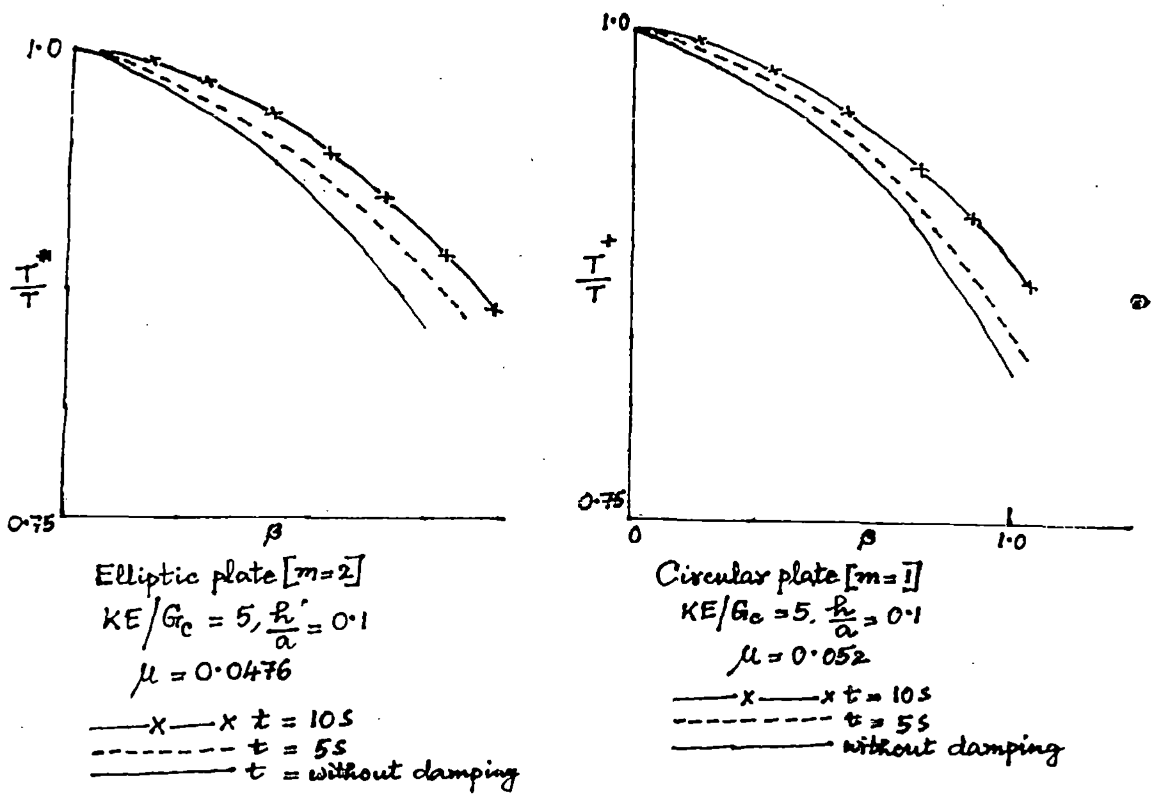


Figure X :- Time-Period Ratio vs Relative Amplitude for Elliptic & Circular Plates [Damped]

Table 21 : Static Deflection for Thick Clamped Circular Plate.

$\frac{KE}{G_c} = 1, \frac{h}{a} = 0.2, \nu = 0.3, m = 1$

W_0/h	Present Study	Sathyamoorthy [135]
0.5	3.3961	3.2756
1.0	9.5844	8.6227
1.5	21.356	18.1126
2.0	41.505	33.8169

Discussion

Tables [13-17] represent the dependence of $\frac{T^*}{T}$ on central deflection $\frac{W_0 a_0}{h}$ for elliptical and circular plates for free oscillations and the results are compared with those of Das and Banerjee [68] and Sathya-moorthy [135]. The results show a very good agreement with those of Das and Banerjee [68].

Tables [18-20] show the dependence of $\frac{T^*}{T}$ on $\frac{W_0 a_0}{h}$ for damped oscillations. ($K_v = 0.5$) for elliptic and circular plates and the results are compared with those of Das and Banerjee [68].

Table 21 shows the static behaviour of plates. The non-dimensional deflection parameters $\frac{W_0}{h}$ are obtained for different values of the load parameter pa^4/Eh^4 . The results show a very good agreement with those of Sathyamoorthy [135] for small values of p.

It is observed that the numerical results of the present study showing the role of rotatory inertia are in good agreement with those obtained by other methods. The discrepancies in some cases between the present results and those of K.Kanakaraju and G. Venkateswara Rao [136], and M. Sathyamoothy [135] are due to the fact that K. KankaRaju and G. Venkateswara Rao use Fine Element Method whereas classical VonKarman equation has been solved by Sathyamoorthy, but present study uses- Berger's approximation.

The present investigation while checking the work of Das and Banerjee has observed some salient points which are unfortunatehy not in favour of the authors of Ref [68].

For example Equation (5) of referance [68] though appears to be true in the concept of "constant Deflection contour Method" but it becomes totally erroneous when equation (7) is simultaneously considered. The reason is obvious as the expressions R,G.F. in reference [68] can never be identically equal to those A_1, A_2, A_3, A_4, A_5, A_6 of equation(3.7))obtained in the present study.Probably the authors of referenee [68] have not checked the deductions.

The main purpose of the present problem is to establish the applicability of the concept of "Constant Contour Deflection Method" for the study of static and dynamic behaviour of moderately thick plate of arbitrary shape. The advantage of this proposal is that the basic equations (4.3.8) and (4.3.9) established here are ordinary differential equations of third order while equation (4.3.1) and (4.3.2) are partial differential equations of fourth order. Moreover, modified equations will describe the nature ofnonlinear oscillations of plates of arbitrary shape provided the equation of its deflection contour u,(x,y) = constant, is known. As for example if

$x(x, y) = y\,[a/2\,(2/a - y) - x^2]$, we get the results of the uniformly loaded parabolic plate with a clamped edge.

Problem - 4.4

Non liner vibrations of elastic Plates with varying Flexural Rigidity

Non-homogeneous materials, with varying flexural rigidity have received a considerable attention.The governing differential equations are of Karman type, extended to a dynamic case, including the effect of varying flexural rigidity. Assuming the Young's Modulus to be inhomogeneous, the govering differential equations are solved with the boundary conditions for clamped edge and by Galerk-in method. The "constant deflection contour" method is employed here.

The dynamic Von-Karman equations for a non-homogenous plate having varying flexural rigidity and subjected to a normal uniform load may be put in the form

$$\nabla^2\left(D\nabla^2 w\right) - (1-\nu)\mathcal{L}(D, w) = p - \rho h w_{,tt} + \mathcal{L}(F, w) \qquad [4.4.1]$$

$$\nabla^2\left(\mu\nabla^2 F\right) - (1+\nu)\mathcal{L}(\mu, F) = -\frac{E}{2}\mathcal{L}(w, w) \qquad [4.4.2]$$

with in plane inertia effect ignored

and $\mu = \frac{1}{h}$

considering E as function of x and y, and h and υ as constants equations (4.4.1) and (4.4.2) become

$$\frac{h^3}{12(1-\nu^2)}\Big[E\nabla^4 w + \left(E_{,xx} + \nu E_{,yy}\right) w_{,xx} + \left(E_{,yy} + \nu E_{,xx}\right) w_{,yy}$$

$$+ 2(1-\nu) E_{,xy}\, w_{,xy} + 2\left\{E_{,x}\frac{\partial}{\partial x}(\nabla^2 w) + E_{,y}\frac{\partial}{\partial y}(\nabla^2 w)\right\}\Big]$$

$$= p - \rho h w_{,tt} + \alpha(F, w) \qquad [4.4.3]$$

$$\nabla^4 F = hE\left(w_{,xy} - w_{,xx}\, w_{,yy}\right) \qquad [4.4.4]$$

It is assumed that the set of equations (4.4.3) and (4.4.4) is satisfied in every region bounded by the contour line C_u As a necess ity for the application of "Constant Deflection Countour" method we integrate equqtions (4.4.3) and (4.4.4)over the region.

$$\frac{h^3}{12(1-\nu^2)}\Big[\iint E\left\{A_1\frac{d^4 w}{du^4} + A_2\frac{d^3 w}{du^3} + A_3\frac{d^2 w}{du^2} + A_4\frac{dw}{du}\right\} d\Omega$$

$$+ \iint\Big\{B_1\frac{dE}{du}\frac{dw}{du} + B_2\frac{d^2E}{du^2}\frac{dw}{du} + B_3\frac{dE}{du}\frac{d^2 w}{du^2}$$

$$+ B_4\frac{d^2 w}{du^2}\frac{d^2 E}{du^2} + B_5\frac{dE}{du}\frac{d^3 w}{du^3}\Big\} d\Omega\Big]$$

$$= \iint\left(p - \rho h w_{,tt}\right) d\Omega + \iint A_5\frac{dw}{du}\frac{dF}{du}\, d\Omega$$

$$+ \iint A_6\frac{d}{du}\left\{\left(\frac{dw}{du}\right)\left(\frac{dF}{du}\right)\right\} d\Omega \qquad [4.4.5]$$

$$\iint \left[A_1 \frac{d^4F}{du^4} + A_2 \frac{d^3F}{du^3} + A_3 \frac{d^2F}{du^2} + A_4 \frac{dF}{du} \right] d\Omega$$

$$= -\iint \frac{Eh}{2} \left\{ A_5 \left(\frac{dw}{du}\right)^2 + A_6 \frac{d}{du}\left(\frac{dw}{du}\right)^2 \right\} d\Omega \qquad [4.4.6]$$

where

$$A_1 = \left(u_{,x}^2 + u_{,y}^2\right)^2$$

$$A_2 = 6\left(u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy}\right) + 2\left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx}\right) + 8 u_{,x} u_{,y} u_{,xy}$$

$$A_3 = 4\left(u_{,x} u_{,xxx} + u_{,y} u_{,yyy}\right) + 3\left(u_{,xx}^2 + u_{,yy}^2\right)$$

$$+ 4\left(u_{,x} u_{,xyy} + u_{,y} u_{,xxy}\right) + 2 u_{,xx} u_{,yy} + 4 u_{,xy}^2$$

$$A_4 = u_{,xxxx} + u_{,yyyy} + 2u_{,xxyy}, \qquad A_5 = 2\left(u_{,xx} u_{,yy} - u_{,xy}^2\right)$$

$$A_6 = u_{,xx} u_{,y}^2 + u_{,yy} u_{,x}^2 - 2u_{,x} u_{,y} u_{,xy}$$

$$B_1 = u_{,xx}^2 + u_{,yy}^2 + 2\nu u_{,xx} u_{,yy} + 2(1-\nu) u_{,xy}^2 + 2u_{,x} u_{,xxx}$$

$$+ 2u_{,y} u_{,yyy} + 2u_{,x} u_{,xyy} + 2u_{,y} u_{,xxy}$$

$$B_2 = u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy} + 2\nu u_{,y}^2 u_{,xx} + 2\nu u_{,x}^2 u_{,yy}$$

$$+ 2(1-\nu) u_{,x} u_{,y} u_{,xy}$$

$$B_3 = 7\left(u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy}\right) + (\nu+2)\left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx}\right)$$

$$+ 2(5-\nu) u_{,x} u_{,y} u_{,xy}$$

$$B_4 = \left(u_{,x}^2 + u_{,y}^2\right)^2$$

$$B_5 = 2B_4 = 2\left(u_{,x}^2 + u_{,y}^2\right)^2 \qquad [4.4.7]$$

Equations (4.4.5) and (4.4.6) can be further reduced to a simpler form on application of Green's theorem wherever possible. On transformation to line integrals equations (4.4.5) and (4.4.6) will become

$$f_1(u)\frac{d^3w}{du^3} + f_2(u)\frac{d^2w}{du^2} + f_3(u)\frac{dw}{du} + f_4(u)p$$

$$+ \rho h\int_1^u w_{,tt}\,du + g_5(u)\frac{dF}{du}\frac{dw}{du} = 0 \qquad [4.4.8]$$

$$g_1(u)\frac{d^3F}{du^3} + g_2(u)\frac{d^2F}{du^2} = g_3(u)\left(\frac{dw}{du}\right)^2 + \int_1^u g_4(u)\left(\frac{dw}{du}\right)^2 du$$

$$[4.4.9]$$

Equations (4.4.8) and (4.4.9) are the two basic equations to study the dynamic response of structures of arbitrary shapes.

An elliptic plate clamped along the edges is considered. The family of contour lines of the deflected surface are as usual represented by

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad [4.4.10]$$

where u = 0 defines the boundary. The boundary conditions for clamped edges are equations (3.1.1) and (3.1.2)

Suppose the non-linearity is governed by the equation

$$E = E_0\left[1 + \beta\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right]$$

$$\text{i.e. } E = E_0\left[1 + \beta(1-u)\right] \qquad [4.4.11]$$

Where β and E_0 are constants.

Inserting the expressions for E, u, u_x etc into equations (4.4.5) and (4.4.6) integrating over the region bounded by C_u, subject to the boundary conditions for clamped edges, one gets respectively

$$\frac{h^3}{12(1-\nu^2)}\left[2E_0P\left\{(1-u)^2+\beta(1-u)^3\right\}\frac{d^3w}{du^3}-2E_0P\left\{2(1-u)+3\beta(1-u)^2\right\}\frac{d^2w}{du^2}+4\beta E_0P'(1-u)\frac{dw}{du}\right]+p(1-u)$$
$$+\rho h\int_1^u w_{,tt}\,du+\frac{8}{a^2b^2}(1-u)\frac{dF}{du}\frac{dw}{du}=0 \qquad [4.4.12]$$

$$(1-u)^2\frac{d^3F}{du^3}-2(1-u)\frac{d^2F}{du^2}=\frac{2E_0h}{a^2b^2P}\left[\left\{(1-u)+\beta(1-u)^2\right\}\left(\frac{dw}{du}\right)^2\right.$$
$$\left.+\beta\int_1^u(1-u)\left(\frac{dw}{du}\right)^2du\right] \qquad [4.4.13]$$

$$\left.\begin{aligned} P &= \frac{3a^4+3b^4+2a^2b^2}{a^4b^4} \\ P' &= \frac{a^4+b^4+2\nu a^2b^2}{a^4+b^4} \end{aligned}\right\} \qquad [4.4.14]$$

Considering that the plate vibrates primarily in the transverse direction and the plate is restained from in-plane movements, one can assume without any loss of generality [121]

$$\begin{aligned} w &= Au^2\psi(t) \\ F &= Au^2\Phi(t) \end{aligned} \qquad [4.4.15]$$

Since equation (4.4.15) does not represent the exact solution, Galerkin procedure is applied to minimise the error. Substituting equation (4.4.15) into equation (4.4.13) a relation between $\Phi(t)$ and $\psi(t)$: is first established

$$\Phi(t) = -\frac{6}{5}\frac{E_0 h}{P a^2 b^2}\left(1+\frac{7\beta}{144}\right)\psi^2(t) \qquad [4.4.16]$$

while equation (4.4.12) will reduce to

$$\frac{E_0 h^3}{6(1-\nu^2)}\left[P\left(\frac{1}{3}+\frac{\beta}{5}\right)-\beta\frac{P'}{5}\right]A\psi(t)$$

$$+\,1{\cdot}28\,\frac{E_0 h}{P a^4 b^4}\left(1+\frac{7\beta}{144}\right)A^3\psi^3(t)+\frac{1}{18}\rho h A\psi_{,tt}(t)$$

$$=\frac{p}{12} \qquad [4.4.17]$$

Equation (4.4.17) may be put in a simple form

$$\psi_{,tt}+C_1\psi(t)+C_3\psi^3(t)=C\,p \qquad [4.4.18]$$

$$C_1=\frac{3E_0h^2}{\rho(1-\nu^2)}\left[P\left(\frac{1}{3}+\frac{\beta}{5}\right)-\frac{\beta}{5}P'\right]$$

$$C_3=\frac{23{\cdot}04\,EA^2}{\rho P a^4 b^4}\left(1+\frac{7\beta}{144}\right)$$

$$C=\frac{3p}{2\rho h A} \qquad [4.4.19]$$

a) Free Linear Vibration

For free vibration p = 0 equation (4.4.18) will become

$$\psi_{,tt}(t) + C_1 \psi(t) + C_3 \psi^3(t) = 0 \qquad [4.4.20]$$

The linear frequency parameter is given by

$$\omega = B_1^{1/2} = \left[\frac{3 E_0 h^2}{\rho (1-\nu^2)} \left\{ P\left(\frac{1}{3} + \frac{\beta}{5}\right) - \frac{\beta P'}{5} \right\} \right]^{1/2} \qquad [4.4.21]$$

b) Non-Linear Vibration

If T and T* be the corresponding time periods of linear and non-linear free oscillations then the ratio

$$\frac{T^*}{T} = \left[1 + \frac{3}{4} \frac{C_3}{C_1} \right]^{-1/2} \qquad [4.4.22]$$

where

$$\frac{C_3}{C_1} = \frac{7\cdot 68 \left(1 + \frac{7\beta}{144}\right)(1-\nu^2)\, m^4 \left(\frac{A_0}{h}\right)^2}{\left(3m^4 + 2m^2 + 3\right)\left[\left(3m^4 + 2m^2 + 3\right)\left(\frac{1}{3} + \beta/5\right) - \beta/5 \left(m^4 + 2\nu m^2 + 1\right)\right]} \qquad [4.4.23]$$

Where $m = a/b$, A_0/h = represents the relative amplitude. Numerical results have been computed and shown in tables (22—31)

c) Static case

Negleting the inertial term in equation (4.4.18) one gets for analyzing the large defection behaviour

$$C_1 \psi(t) + C_3 \psi^3(t) = Cp \qquad [4.4.24]$$

On further simplification one gets the relation between the non dimesional central deflection (W_0/h) and the load parameter (pa^4/Eh^4)

$$\frac{2}{(1-\nu^2)}\left[(3m^4+2m^2+3)\left(\frac{1}{3}+\beta/5\right)-\frac{\beta}{5}\left(m^4+2\nu m^2+1\right)\right]\frac{W_0}{h}$$

$$+\ \frac{15\cdot 36\left(1+\frac{7\beta}{144}\right)m^4}{\left(3m^4+2m^2+3\right)}\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4} \qquad [4.4.25]$$

Numerical results are shown in tables(32 — 35)

Table - [22]: Dependence of relative time period of non - linear and linear vibrations[$T^*/_T$] on relative amplitudes [A_0/h] for circular plate for different values of β, $\upsilon = 0.3$, m=1

A_0/h	$T^*/_T$				
	$\beta = -2$	$\beta = -1$	$\beta = 0$	$\beta = 1$	$\beta = 2$
0	1.00	1.000	1.000	1.000	1.000
0.5	0.8782	0.9539	0.9705	0.9777	0.9818
1.0	0.6730	0.8449	0.8946	0.9184	0.9323
1.5	0.5171	0.7233	0.7989	0.8388	0.8635
2.0	0.4121	0.6165	0.7045	0.7549	0.7878
2.5	0.3399	0.5300	0.6209	0.6764	0.7142

Table - [23] Dependence of relative time period of non - linear and linear vibrations $[T^*/_T]$ on relative amplitudes $[A_0/h]$ for circular plate for different values of β, υ=0.3, m=1.5

A_0/h	$T/_{T^*}$				
	β= -2	β= -1	β = 0	β = 1	β = 2
0	1.000	1.000	1.000	1.000	1.000
0.5	0.9190	0.9703	0.9811	0.9858	0.9904
1	0.75643	0.8942	0.9300	0.9464	0.9633
1.5	0.60873	0.7982	0.8593	0.8894	0.9222
2.0	0.4978	0.7036	0.7822	0.8240	0.8722
2.5	0.4167	0.6200	0.7076	0.7574	0.8181

Table - [24] Dependence of $[T^*/_T]$ on relative amplitudes $[A_0/h]$ for circular plate for different values of β, υ = 0.3, m = 2

A_0/h	$T^*/_T$				
	β = -2	β = -1	β = 0	β = 1	β = 2
0	1.000	1.000	1.000	1.000	1.000
0.5	0.9569	0.9858	0.9911	0.9933	0.9945
1.0	0.8553	0.9465	0.9656	0.9739	0.9786
1.5	0.7354	0.8896	0.9269	0.9438	0.9536
2.0	0.6300	0.8244	0.8795	0.9057	0.9215
2.5	0.5435	0.7579	0.8276	0.8626	0.8843

Table - [25] Dependence of relative time period of non - linear and linear vibrations $[T^*/_T]$ on relative amplitudes $[A_0/h]$ for circular plate for different values of m, υ = 0.3, β = 1

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.9777	0.9858	0.9933
1.0	0.9184	0.9464	0.9739
1.5	0.8388	0.8894	0.9438
2.0	0.7549	0.8240	0.9057
2.5	0.6764	0.7574	0.8626

Table - [26] : Dependence of $[T/_T]$ on relative amplitudes $[A_0/h]$ for circular plate for different values of m, υ =0.3, β = 2

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.9818	0.9904	0.9945
1.0	0.9323	0.9633	0.9786
1.5	0.8635	0.9222	0.9536
2.0	0.7878	0.8722	0.9215
2.5	0.7142	0.8181	0.8843

Table - [27] : Dependence of relative time period of non - linear and linear vibrations$[T^*/_T]$ on relative amplitudes $[A_0/h]$ for circular plate for different values of m, υ =0.3, β = -1

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.9539	0.9703	0.9858
1.0	0.8449	0.8942	0.9465
1.5	0.7233	0.7982	0.8896
2.0	0.6165	0.7036	0.8244
2.5	0.5300	0.6200	0.7579

Table - [28] : Dependence of relative time period on relative amplitudes $[A_0/h]$ for elliptic plate for different values of m, υ = 0.3, β = -2

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.8782	0.9190	0.9569
1.0	0.6730	0.7564	0.8553
1.5	0.5171	0.6087	0.7354
2.0	0.4121	0.4978	0.6300
2.5	0.3399	0.4167	0.5435

Table - [29] : Dependence of $T^*/_T$ on relative amplitudes $[A_0/h]$ for circular plate for different values of m, $\upsilon = 0.3$, $\beta = -0.7$

A_0/h	$T^*/_T$		
	m = 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.5	0.9608	0.9784	0.9880
1.0	0.8647	0.9087	0.9543
1.5	0.7523	0.8221	0.9046
2.0	0.6492	0.7335	0.8460
2.5	0.5630	0.6525	0.7847

Table - [30] : Dependence of relative time period of non - linear and linear vibrations $T^*/_T$ on relative amplitudes $[A_0/h]$ for circular plate for different values of υ, m = 1, $\beta = 2$

A_0/h	$T^*/_T$		
	$\upsilon = 0.2$	$\upsilon = 0.3$	$\upsilon = 0.5$
0	1.000	1.000	1.000
0.5	0.9808	0.9812	0.9844
1.0	0.9289	0.9323	0.9415
1.5	0.8574	0.8635	0.88032
2.0	0.7796	0.7878	0.81116
2.5	0.7047	0.7142	0.7418

Table - [31] : Dependence of relative time period of non - linear and linear vibrations $T^*/_T$ on relative amplitudes $[A_0/h]$ for circular plate for different values of υ, $\beta = -1$, m =2

A_0/h	$T^*/_T$		
	$\upsilon = 0.2$	$\upsilon = 0.3$	$\upsilon = 0.5$
0	1.000	1.000	1.000.
.5	0.9849	0.9858	0.9886
1.0	0.9432	0.9465	0.9565
1.5	0.8835	0.8896	0.9089
2.0	0.8156	0.8244	0.85239
2.5	0.7472	0.7579	0.79267

Table [32] : Dependence of central deflection (W_0/h) on load parameter (pa^4/Eh^4) for different values of β, $\upsilon = 0.3$, m = 1

W_0/h	pa^4/Eh^4				
	$\beta = -2$	$\beta = -1$	$\beta = 0$	$\beta = 1$	$\beta = 2$
0	0	0	0	0	0
0.2	0.2365	0.3319	1.187	1.6629	2.1384
0.4	0.5563	0.7515	2.466	3.4106	4.3780
0.6	1.0425	1.3465	3.93	5.3754	6.8198
0.8	1.7782	2.2045	5.67	7.6183	9.5649
1.0	2.8468	3.4133	7.78	10.2477	12.7146
1.2	4.3313	5.0602	10.34	13.3601	16.3698
1.4	6.31507	7.2336	13.47	17.0525	20.6317
1.6	8.8814	10.0206	17.24	21.4214	25.6014
2.0	16.0935	17.7861	27.08	32.5752	38.0688

Table [33] : Dependence of central deflection (W_0/h) on load parameter (pa^4/Eh^4) for different values of β, m = 2, $\upsilon = 0.3$

W_0/h	pa^4/Eh^4				
	$\beta = -2$	$\beta = -1$	$\beta = 0$	$\beta = 1$	$\beta = 2$
0	0	0	0	0	0
0.2	1.712	5.1949	8.6772	12.1609	15.6437
0.4	3.6046	10.58	17.5542	24.1609	31.5068
0.6	5.8581	16.3456	26.5542	37.3214	47.8087
0.8	8.6531	22.6818	36.7059	50.74047	64.7686
1.0	12.17	29.779	47.385	64.998	82.606
1.2	16.5892	37.8272	5.9.0524	80.3039	101.5401
1.4	22.0884	47.016	71.923	96.8677	121.7904
1.6	28.8568	57.538	86.1913	114.8993	143.5763
2.0	46.9	83.336	119.72	156.204	192.632

Table [34] : Dependence of central deflection (W_0/h) on load parameter (pa^4/Eh^4) for different values of m, $\upsilon = 0.3$, $\beta = 2$

W_0/h	pa^4/Eh^4		
	m = 1	m = 1.5	m = 2
0	0	0	0
0.2	2.1384	6.04	15.6437
0.4	4.378	12.2606	31.5068
0.6	6.8198	18.8421	47.8087
0.8	9.5649	25.9651	64.7686
1.0	12.7146	33.81	82.606
1.2	16.3698	42.5572	101.5401
1.4	20.6317	52.3874	121.7904
1.6	25.6014	63.464	143.5763
2.0	38.0688	90.18	192.632

Table [35] : Dependence of central deflection (W_0/h) on load parameter (pa^4/Eh^4) for different values of υ, m = 2, $\beta = 1$

W_0/h	pa^4/Eh^4		
	$\upsilon = 0.2$	$\upsilon = 0.3$	$\upsilon = 0.5$
0	0	0	0
0.2	11.5949	12.1509	14.5771
0.4	23.3995	24.5315	29.3639
0.6	35.6234	37.3214	44.5694
0.8	48.4764	50.7404	60.4052
1.0	62.168	64.998	77.079
1.2	76.9079	80.3039	94.8011
1.4	92.9	96.8677	113.7811
1.6	110.3713	114.8993	134.2289
2.0	150.544	156.204	180.366

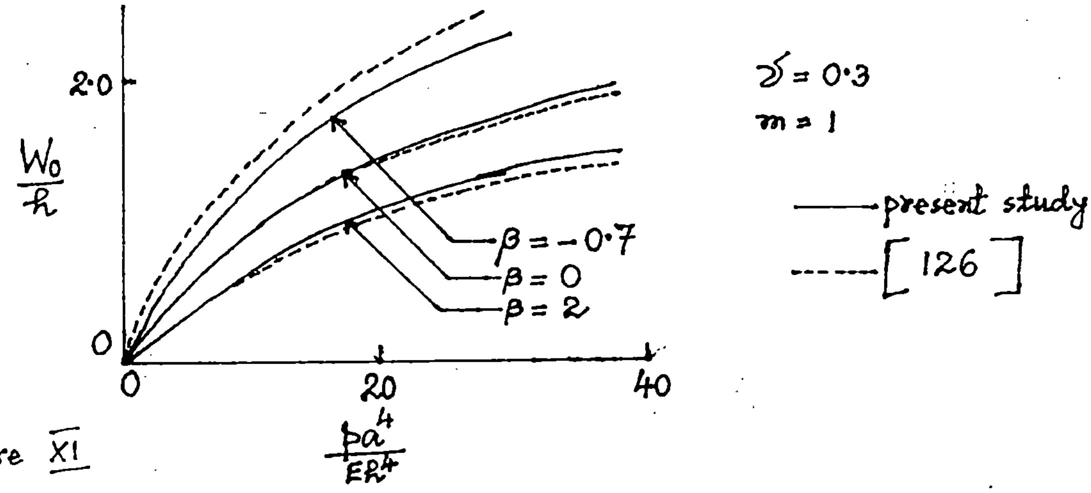


Figure XI :- Non-linear Static Behaviour of a Clamped Circular Plate with varying Flexural Rigidity

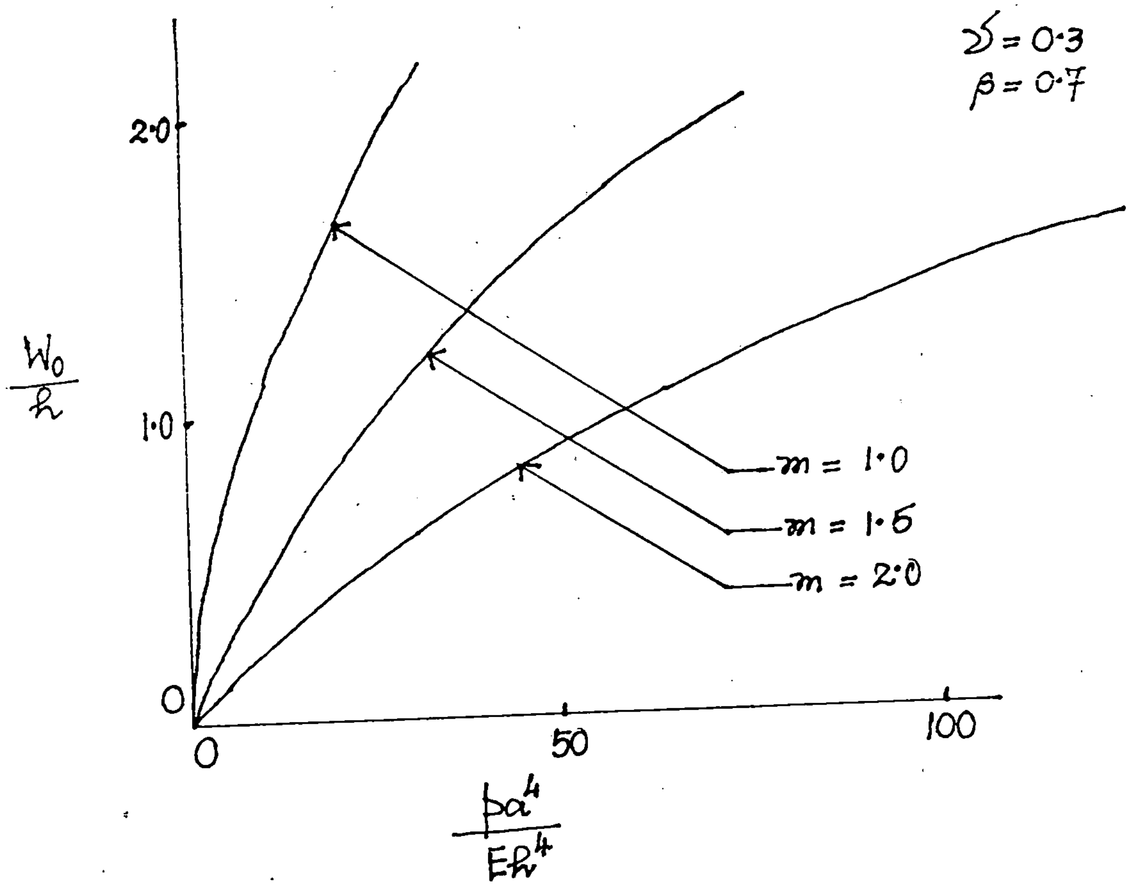


Figure XII :- Non-linear Static Behaviour of Clamped Elastic Plates with varying Flexural Rigidity for Different Values of Aspect Ratios

Discussion :

From tables(23 & 24), it is observed that for a particular value of m the effect of non-linearity increases with the decreasing value of β whereas for a particular value of β the effect of nonlinearity decreases with increasing value of m from tables(25 to 29).

Normally the variation of Poisson's ratio is overlooked since the effect in large vibration is marginal. However tables (30 & 31) show that for moderately large vibration the non-linear effect may be taken into account in the sense that the effect of non linearity appears to be appreciable when value of Poisson's ratio for the corresponding material deereases.

The numerical results for static deflection for the present study have been compared with those of [126] and what has been observed is that the present results are not in exact agreement with those of [126], particularly when $\beta < 0$, but for $\beta > 0$ the results agree well with those of [126]. The first reason for slight difference may be caused due to what has been explained in the very beginning of this chapter (page - 39)

The second reason is the procedural difference, Ohanabe el.al [126] has used Berger equations whereas in the present analysis Karman equations have been employed. Since Karman equations are reliable than those of Berger, hence the present results may be more acceptable than those presented in Ref. [126].

Chapter - V A Modified Method for Solving Non-linear Problems using "Constant Deflection Contour" Method.

During the process of investigation while using two sets of governing equations i.e. equations (3.11)and (3.12) or equations (3.12) and (3.13) both have been utilized. Since the first set (3.11) and (3.12) though simplifies the computational hazards yields not very satisfactory results. This prompts the present investigator to have a little more careful examination and application of the second set of equations.

In the present chapter the investigator used the second set of equations (3.12) and (3.13) for all illustration. In order to make a comparative study some of the problems treated in the previous chapter have been reinvestigated and some of new problems have also been treated in the present chapter. The second set of a equations appears to be more effective in the vibrational analysis. Obvious reason is that it involves a fourth order differential equations whereas the first one involves a third order differential equation. To verify the application of the " Constant deflection Contour " method with equations (3.12) and (3.13), four specific cases will be considered.

1) Circular plate with built in edge (considered immovable)
2) Annular plate with outer boundary clamped and inner boundary free
3) Annular plate with outer boundary simply supported and inner boundary free.
4) Elliptic plate with clamped edge

5.1

Problem - 1

Non-Linear Vibration of a Clamped Rigid Circular Plate :

A rigid circular plate which is clamped along its boundary is considered. The family of isodeflection curves are concentric circles represented by

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{a^2} \qquad [5.1.1]$$

Clearly u = 0 defines the boundary and u = 1 is the centre of the plate where the deflection is maximum under an uniform load p .
The deflection and stress functions are assumed to be

$$w = W(u)\Psi(t)h$$

$$F = F'(u)\Psi^2(t)h \qquad [5.1.2]$$

For Such variable u equations(3.12) and (3.13) as deduced by Banerjee and Rogerson [122]
in chapter III will reduce to

$$D\left[(1-u)^2\frac{d^4W}{du^4} - 4(1-u)\frac{d^3W}{du^3} + 2\frac{d^2W}{du^2}\right]h\psi(t)$$

$$+\frac{h^3}{2}\left[(1-u)\frac{d}{du}\left(\frac{dW}{du}\frac{dF'}{du}\right) - \left(\frac{dW}{du}\frac{dF'}{du}\right)\right]\psi^3(t)$$

$$= \frac{pa^4}{16} + \rho\frac{h^2a^4}{16}W\psi_{,tt}(t) \qquad [5.1.3]$$

$$(1-u)^2\frac{d^3F'}{du^3} - 2(1-u)\frac{d^2F'}{du^2} = \frac{E}{4}(1-u)\left(\frac{dW}{du}\right)^2 \qquad [5.1.4]$$

The first step of the method of solution as explained in Chapter- III, W can be assumed as $W = \sum_i A_i u^{2i}$
compatible with the boundary conditions for a clamped boundary, where A_i 's are constants which may be evaluated while applying Galerkin procedure due to orthogonal property of the error function. However, since we are concerned more with the applicability rather then exact solution, we may try with a rough approximation by considering a first term only $W \cong u^2$

The first integral of equation (5.1.4) yields

$$(1-u)\frac{d^2F'}{du^2} - \frac{dF'}{du} = \frac{Eu^3}{3} + B_1 \qquad [5.1.5]$$

While the second integral becomes

$$(1-u)\frac{dF'}{du} = \frac{Eu^2}{12} + B_1u + B_2 \qquad [5.1.6]$$

B_1 and B_2 are constants subject to immovable condition

$$\left[2(1-u)\frac{d^2F'}{du^2} - (1-\nu)\frac{dF'}{du}\right]_{u=0} = 0 \qquad [5.1.7]$$

Further equations (5.1.5) and (5.1.6) are valid for the whole domain bounded by C_u, then for u = 0

$$\left.\frac{d^2F'}{du^2}\right|_{u=0} - \left.\frac{dF'}{du}\right|_{u=0} = B_1, \quad \left.\frac{dF'}{du}\right|_{u=0} = B_2 \qquad [5.1.8]$$

Also when u = 1, equation (5.1.6) reduces to

$$B_1 + B_2 + \frac{E}{12} = 0 \quad \ldots\ldots\ldots\ldots\ldots (5.1.9)$$

Solving equations(5.1.7), (5.1.8) and (5.1.9) one gets B_1 and B_2 and equation (.5.1.6) reduces to

$$(1-u)\frac{dF'}{du} = \frac{Eh}{12(1-\nu)}\left[-2+(1+\nu)u+(1-\nu)u^4\right] \qquad [5.1.10]$$

Combining equations (5.1.3), (5.1.6) and (5.1.10) one can get the error function

$$\epsilon_1 = 4h\psi(t) + \frac{Eh^4}{12D(1-\nu)} + \left[-2+2(1+\nu)u+5(1-\nu)u^4\right]\psi^3(t) + \rho\frac{ha^4}{16D}\psi_{,tt}(t)\,u^2 - \frac{pa^4}{16D} \qquad [5.1.11]$$

Minimizing the error function by application of Galerkin procedure

$$\iint_{\Omega_u} \epsilon_1 u^2 du = 0$$

which on evaluation yields the following time differential equations as

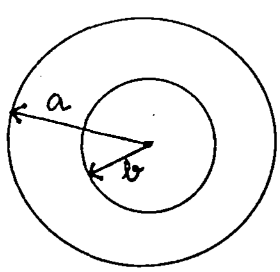
$$\rho h\psi_{,tt}(t) + \frac{80}{9}\frac{Eh^4\psi(t)}{(1-\nu^2)a^4} + \frac{Eh^4}{a^4}\frac{(230-90\nu)}{63(1-\nu)}\psi^3(t) = \frac{5}{3}p \qquad [5.1.12]$$

The above equation is in exact agreement with the result of Yamaki [22] for vibration of a circular plate with clamped immovable edge.

5.2 Non-Linear Vibration of Annular Plates

An Annular plate vibrating at moderately large amplitude is considered. The geometry of the structure has been shown in the adjoining figure.

The region occupied by the plate lies between two concentric circles of radii a and b respectively (a>b).



The isodeflection curves are given by

$$u = 1 - \frac{x^2+y^2}{a^2} \qquad [5.2.1]$$

Where the contour C_0 defines the outer boundary for u = 0 and C_u for u = u_1 (say), for the inner boundary, i.e.

$$u_1 = 1 - \frac{x^2+y^2}{b^2} \qquad [5.2.2]$$

In the present case evaluating the integrals (3.9a) and (3.9b) the limits of u will be u_1 to u instead of ' 1 to u. The nature of the basic equations remains the same except for some constant term dependent on u_1. For example the first term of $\iint_\Omega \nabla^4 W d\Omega$

may be evaluated as

$$\int_{u_1}^{u} (1-u)^2 \frac{d^4W}{du^4}\, du \oint \frac{ds}{\sqrt{u_{,x}^2+u_{,y}^2}}$$

$$= (1-u)^2 \frac{d^3W}{du^3} - (1-u_1)^2\left(\frac{d^3W}{du^3}\right)_{u=u_1} +$$

$$+2\int_{u_1}^{u} (1-u)\frac{d^3W}{du^3}\, du \oint \frac{ds}{\sqrt{u_{,x}^2+u_{,y}^2}} \qquad [5.2.3]$$

In the case for a rigid circular plate u_1 = 1, making the second term to vanish, so, the representative of equation (3.11) for the present case of annular circular plate reduces to

$$D\left[(1-u)^2\frac{d^3W}{du^3} - 2(1-u)\frac{d^2W}{du^2}\right] h\psi(t) + \frac{h^3}{2}(1-u)\frac{dF'}{du}\frac{dW}{du}\psi^3(t)$$

$$+ \rho\frac{h^2a^4}{16}\psi_{,tt}(t)\int_{u_1}^{u} W(u)\,du + \frac{pa^4}{16}(1-u)$$

$$+ MN(t) = 0 \qquad [5.2.4]$$

Where M is a constant term involving parameters u_1, $\left(\frac{dW}{du}\right)_{u_1}$ and N (t) is dependent on t only. But when equation (5.2.4) differentiated with respect to u it reduces to

$$D\left[(1-u)^2\frac{d^4W}{du^4} - 4(1-u)\frac{d^3W}{du^3} + 2\frac{d^2W}{du^2}\right]k\psi(t) + \frac{k^3}{2}\frac{d}{du}\left[(1-u)\frac{dF'}{du}\frac{dW}{du}\right]\psi^3(t)$$
$$+ \rho\frac{k^2a^4}{16}W\psi_{,tt}(t) = \frac{pa^4}{16} \qquad [5.2.5]$$

However the transformed equation equivalent to equation (5.1.4) differs from the present one by a constant

$$(1-u)^2\frac{d^3F'}{du^3} - 2(1-u)\frac{d^2F'}{du^2} = \frac{E}{4}(1-u)\left(\frac{dW}{du}\right)^2$$
$$+ B_1 \qquad [5.2.6]$$

Hence the two governing equations for the present problem are equations (5.2.5) and (5.2.6) Two specific cases will be considered.
Circular plate with mixed boundary condition namely annular plate

a) Outer boundary clamped and inner boundary free.

b) Outer boundary simply supported and inner boundary free.

5.2 a) Problem - 2 a

<u>Non- Linear vibration of a annular plate with immovable outer boundary and free inner boundary :</u>

It can be verified that the expression like

$$W(u) \cong u^2 + 16{\cdot}888u^3 \cong u^2 + 0{\cdot}8042u^4$$
$$\cong u^2 - 1{\cdot}37766u^3 + 0{\cdot}8698u^4 \qquad [5.2a.1]$$

all satisfy the given condition [as deduced from (3.1.2.) and (3.1.6) mathematically]

$$W\Big|_{u=0} = \frac{dW}{du}\Big|_{u=0} = 0$$

$$\left[2(1-u)\frac{d^3W}{du^3} - (5-2\nu)\frac{d^2W}{du^2}\right]_{u=u_1} = 0$$

$$\left[2(1-u)\frac{d^2W}{du^2} - (1+2\nu)\frac{dW}{du}\right]_{u=u_1} = 0 \qquad [5.2a.2]$$

depending on assumption

$W = Au^2 + Bu^3$, $= Au^2 + Bu^4$ or $W = Au^2 + Bu^3 + Cu^4$

But it will be proper to assume

$W = u^2 - 1.3776u^3 + 0.8698\, u^4$ ....(5.2a.3)

With the above value of W, the third integral of equation (5.2.6) will yield

$$(1-u)\frac{dF'}{du} = E\left[0.0833u^4 - 0.2066u^5 + 0.2583u^6 - 0.1712u^7 + 0.063u^8\right] - B_1\left[(1-u)\log(1-u)\right] + B_2 u + B_3 \qquad [5.2a.4]$$

The stress conditions, for the inner boundary being free and outer boundary being clamped immovable are not sufficient to evaluate the constants B_1, B_2, B_3. So one must therefore impose certain legitimate condition further. In general $\frac{d^2F'}{du^2}$ or $\frac{dF'}{du}$ may or may not be zero on the simply supported boundary [106]. However , here it has been observed that vanishing of any one of them implies vanishing of the other. So, we are to choose either of the following conditions

a) if $\frac{d^2F'}{du^2}$ or $\frac{dF'}{du} = 0$ on $u = 0$, then $B_2 = B_3 = 0$

b) if $\frac{d^3F'}{du^3} = 0$, then none of B_i's are zero.

Hence we assume that F' is such a function of u which makes its third derivative to vanish from the outer boundary. We then reject the condition(a) and accept (b) as it is more probable; in which case equation (.2a.4) becomes

$$(1-u)\frac{dF'}{du} = E\left[0.8033u^4 - 0.2066u^5 + 0.2583u^6 - 0.1712u^7 + 0.063u^8\right] + 0.017842\left[(1-u)\log(1-u) + u\right] - 0.01657u + 0.029373 \qquad [5.2a.5]$$

The time differential equation likewise as derived in sec (5.1), for the present problem becomes

$$\rho h^2 \psi_{,tt}(t) + \frac{Eh^4}{a^4}\left[28.6676\psi(t) + 4.7355\psi^3(t)\right] = 6.3302\, p \qquad [5.2a.6]$$

5.2 b Problem - 2 b

Non-Linear Vibration of Annular Plate with Simply Supported Outer Boundary and Free Inner Boundary :

It can be verified that that the expression like

$W = u + 0.325\,u^2 - 0.1632\,u^3$ ....(5.2 b.1)
satisfies the boundary conditions [as deduced from (3.1.3), (3.1.4), (3.1.5) and (3.1.6) mathematically] for b/a = ½

$$2(1-u)\frac{d^2W}{du^2} = (1+2\nu)\frac{dW}{du} \quad \text{for } u=0$$

$$\left.\begin{aligned} (1-u)\frac{d^2W}{du^2} &= 2(1+2\nu)\frac{dW}{du} \\ (1-u)\frac{d^3W}{du^3} &= 2(5-2\nu)\frac{d^2W}{du^2} \end{aligned}\right\} \text{for } u=u_1$$

depending on assumption $W = Au + Bu^2 + Cu^3$ with above value of W, the third integral of equation (5.2.6) is given by

$$(1-u)\frac{dF'}{du} = E\left[0.125u^2 + 0.05416u^3 - 0.01223u^4 - 0.0082u^5 + 0.0021u^6\right] + B_1\left[(1-u)\log(1-u) + u\right] + B_2u + B_1 \qquad [5.2b.2]$$

Stress conditions are given by

$$\frac{d^2F'}{du^2} = \frac{dF'}{du} = 0 \quad \text{for } u=0$$

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and $\frac{dF}{du} = 0$ for $u = u_1$

which leads to

$B_2 = B_3 = 0$, and $B_1 = -\ 0.1463E$

Then equation (5.2b.2) becomes

$$(1-u)\frac{dF'}{du} = E\left[0{\cdot}125u^2 + 0{\cdot}05416u^3 - 0{\cdot}01223u^4 - 0{\cdot}0082u^5 + 0{\cdot}0021u^6\right]$$

$$- 0{\cdot}1463\left[(1-u)\log(1-u) + u\right] \qquad [5.2b.3]$$

The time differential equation likewise as derived in sec. (5.1)

$$\rho h^2 \psi_{,tt}(t) + \frac{Eh^4}{a^4}\left[6{\cdot}9715\psi^3(t) + 2{\cdot}4471\psi(t)\right]$$

$$= 1{\cdot}3028\, p \qquad [5.2b.4]$$

5. 2 C

Numerical Results and Discussion :

A close look into the final equations for non linear vibrations of plates considered here [viz. equation (5.1.12),(5.2a.6), (5. 2b.4)] they may be represented in general by

$$\rho h^2 \psi_{,tt}(t) + C_1\psi(t) + C_3\psi^3(t) = Q$$

Where C_1, C_3 are the coefficients of $\psi(t)$ and $\psi^3(t)$ equations

(5. 1.12), (5.2a.6), (5.2b.4) as the case may be and Q is the load parameter. We will now discuss different aspect of the analysis.

5.2c.1] Free - Linear Vibrations :

If the non-linear term and the load parameter be set to zero, the liner frequency parameter may be easily obtained

$$\Omega^* = \omega a^2\sqrt{\frac{\rho h}{D}}$$

= 10.3279 (circular plate)

= 17.693 (annular plate with clamped outer boundary and free inner boundary)

= 5.169 (annular plate with simply supported outer boundary and free inner boundary)

being the linear frequency, with $\nu = 0{\cdot}3$

and $\frac{b}{a} = \frac{1}{2}$

$\frac{b}{a}=\frac{1}{2}, \nu=0.3$

$$\Omega^* = \omega a^2\sqrt{\frac{\rho h}{D}}$$

Table : 36

Rigid Circular Plate	10.3279 [present]
Annular plate clamped outer boundary and free inner boundary	17.693 [present], 17.70 [111], 17.85 FEM[120], 17.747 Finite strip[116]
Annular plate simply supported outer boundary free inner boundary	5.169 [present], 5.138 [116],

5. 2 C. 2 .

Non-Linear free vibration of Annular Plates :

For non-linear free vibration one sets p = 0 to get

$$\rho h \psi_{,tt}(t) + C_1 \psi(t) + C_3 \psi^3(t) = 0$$

Which is the Duffing type equation and its solution is well known. If T * and T be the time periods of non-linear and linear vibrations, respectively, then the relative time period may be expressed as [22]

$$\frac{T^*}{T} = \left[1+\frac{3}{4}\xi^2\right]^{-1/2} \qquad [5.2c.1]$$

ξ is the non- dimensional relative amplitude of vibration representing W_{max}/h in [22]

Equation (4.2 c.1) may be recast for the three problems with ν = 0.3

$$\frac{T^*}{T} = \left[1+\frac{3}{4}\times 0.47125\,\xi^2\right]^{-1/2}$$ For rigid circular plate

$$= \left[1+\frac{3}{4}\times 2.51055\,\xi^2\right]^{-1/2}$$ For annular plate with outer boundary clamped and inner boundary free

$$= \left[1+\frac{3}{4}\times 3.8228\,\xi^2\right]^{-1/2}$$ For annular plate with outer boundary simply supported and inner boundary free

The numerical results showing the variation of $\frac{T^*}{T}$ with relative amplitude $\left(\frac{W_{max}}{h}\right)$ for rigid circular plate, annular plate with outer boundary clamped and inner boundary free and annular plate with simply -supported outer boundary and free inner boundary have been depicted in table : $\nu = 0.3$

Table 37: T*/T

Relative Amplitude ξ	Rigid Circular Plate and [22]	Annular Plate: Clamped outer edge and free inner edge (present)	= 0.5: Simply Supported outer edge and free inner edge (present)
0	1.000	1.000	1.000
0.25	0.9891	0.9459	0.92086
0.5	0.9585	0.8246	0.7632
0.75	0.9133	0.6969	0.5392
1.0	0.8596	0.5890	0.5085
1.25	0.8026	0.5037	0.4276
1.5	0.7463	0.4370	0.4276
1.75	0.6930	0.3844	0.3667
2.0	0.6437	0.3424	0.2832

5. 2 C. 3]

Static deflection for the same problems may be evaluated from equations (5.1.11), (5.2a.6), (5.2b.4) after rejecting the inertial, terms

$$\frac{pa^4}{Eh^4} = \left.\begin{array}{l} 5\cdot861\left(\frac{W_m}{h}\right)+2\cdot762\left(\frac{W_m}{h}\right)^3 \text{ [present]} \\ = 5\cdot848\left(\frac{W_m}{h}\right)+2\cdot754\left(\frac{W_m}{h}\right)^3 \text{ Ref [22]} \end{array}\right\} \text{For rigid circular plate}$$

$$= 17\cdot655\left(\frac{W_m}{h}\right)+44\cdot3237\left(\frac{W_m}{h}\right)^3$$

for outer boundary clamped and inner boundary free

$$= 2\cdot1715\left(\frac{W_m}{h}\right)+8\cdot3010\left(\frac{W_m}{h}\right)^3$$

For outer boundary Simply supported and inner boundary free

Annular plate

Table 38: The Numerical results showing the static deflection shown in Table : pa^4/Eh^4

$\frac{Wm}{h}$	Rigid Circular Plate	Annular Plate with outer boundary clamped inner boundary free	Annular Plate with outer boundary simply supported and inner boundary free
0	0	0	0
0.25	1.5084	5.1063	0.6724
0.5	3.2758	14.3680	2.12275
0.75	5.5610	31.9400	5.128
1	8.6230 8.6020[22] 9.000[117]	61.9787	10.47
1.25	12.7208	106.6385	18.924
1.50	18.1133	176.0750	31.2672
1.75	25.0593	268.4430	48.2881
2.00	33.818	389.9000	68.751

The values of the load parameter for rigid circular plate as deduced in the present analysis are more close to the values of Way [117] even slightly better than those of Yamaki [22]. . Also the result are in excellent agreement with those of Ref [118].

Discussion and Conclusion :

(for the Problem 5.1, 5.2 a and 5.2 b)

From the results given here for static and dynamic analysis of plates vibrating at large amplitude, it appears that the application of the present method is justified.
The present results are in exact agreement with those obtained by Banerjee and Rogerson [122].
For the linear analysis the results obtained by this method are more close to exact results in comparison with those obtained by other method. For static deflection of annular plates the results cannot be compared for non-availability of such studies.

In conclusion it may be accepted that the application of " Constant Deflection Contour " method is justified and appears to be easier than the other existing methods. The most important point is that the method can be applied to study static and dynamic behaviour of structures having uncommon or complex boundaries for which other methods may fail to analyse. The application of polynomial expressions for the deflection function and stress function in conjunction with Galerkin procedure appears to produce excellent results. In case of free non-linear vibration for plates considered here, the results are compared very well with those previously obtained [22, 72]. Additionally the load deflection relation for rigid circular plate coincides very close with that of Way [117]

5.4 Problem - 4
Non-Linear Vibrations of Elliptic Plate Clamped along its Boundary :

During the process of investigation it has already been observed in chapter IV (problem -1) that the result for vibration of elliptic plates using the equations (3.11) and (3.12) though simplifies the computational hazards, yields not very satisfactory results.

This prompts the present investigator to reinvestigate the same problem using the second set of equations (3.12) and (3.13) for a comparative study .

For an elliptic plate, clamped along its boundary, the family of isodeflection curves are represented by

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad [5.4.1]$$

a and b are the semi-major and semi-minor axes. u = o defines the boundary. u = 1 is the centre of the plate where the deflection is maximum under an uniform load p

Deflection and stress functions are given by

$$w = W(u)\psi(t)$$
$$F = F'(u)\psi^2(t) \qquad [5.4.2]$$

For such variable u equation (3.12) and (3.13) in Chapter III will reduce to

$$\frac{3a^4+3b^4+2a^2b^2}{a^4b^4}\left[(1-u)^2\frac{d^3F'}{du^3} - 2(1-u)\frac{d^2F'}{du^2}\right] = 2E(1-u)\left(\frac{dW}{du}\right)^2 \qquad [5.4.3]$$

$$2D\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)^2\frac{d^4W}{du^4} - 4(1-u)\frac{d^3W}{du^3} + 2\frac{d^2W}{du^2}\right]\psi(t)$$

$$+\frac{8h}{a^2b^2}\frac{d}{du}\left[(1-u)\frac{dF'}{du}\frac{dW}{du}\right]\psi^3(t) - p + \rho h W\psi_{,tt}(t) = 0 \qquad [5.4.4]$$

$$\text{Let } W(u) = \sum_{i=1}^{n} A_i u^{2i} \qquad [5.4.5]$$

Compatible with the boundary conditions for a clamped boundary A_i's are evaluated. Since we are concerned more with the applicability rather than exact solution we may try with a rough approximation by considering the first term only i.e.

$W = u^2$...... (5.4.6)
Since equation (5.4.6) does not represent the exact solution, Galerkin procedure may be applied to minimize the error

We first solve the equation (5.4.3) with $W=u^2$,

The first integral of equation (5.4.3.) yields

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)\frac{d^2F'}{du^2}-\frac{dF'}{du}\right]=\frac{8E}{3a^2b^2}u^3+B_1 \qquad [5.4.7]$$

While the second integral becomes

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left[(1-u)\frac{dF'}{du}\right]=\frac{2}{3}\frac{E}{a^2b^2}+B_1u+B_2 \qquad [5.4.8]$$

B_1 and B_2 are constants subject to immovable condition

$$\left[2(1-u)\frac{d^2F'}{du^2}-(1-2\nu)\frac{dF'}{du}\right]_{u=0}=0 \qquad [5.4.9]$$

Further (5.4.7) and (5.4.8) are valid for the whole domain bounded by C_u then for $u=0$

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}\left.\frac{dF'}{du}\right|_{u=0}=B_2 \qquad [5.4.10]$$

$$\frac{3a^4+3b^4+2a^2b^2}{a^4b^4}\left[\frac{d^2F'}{du^2}-\frac{dF'}{du}\right]_{u=0}=B_1 \qquad [5.4.11]$$

When u = 1 equation (5.4.8) reduces to

$$B_1+B_2=-\frac{2}{3}\frac{E}{a^2b^2} \qquad [5.4.12]$$

Solving equations (5.4.10), (5.4.11) and (5.4.12)

We get

$$B_1=\frac{2E}{3a^2b^2}\frac{(1+2\nu)}{(1-2\nu)}$$

$$B_2=\frac{-4E}{3a^2b^2(1-2\nu)} \qquad [5.4.13]$$

Then equation (5.4.8) reduces to

$$\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}(1-u)\frac{dF'}{du} = \frac{2E\left[(1-\nu)u^4+(1+\nu)u-2\right]}{3a^2b^2(1-\nu)} \qquad [5.4.14]$$

Combing equations (5.4.4), (5.4.6) and (5.4.14) we get error function.

$$\epsilon_1 = -\frac{8D(3a^4+3b^4+2a^2b^2)}{a^4b^4}(1-u)A\psi(t) + \frac{16Eh\left[(1-\nu)u^4+(1+\nu)u-2\right]}{3(1-\nu)(3a^4+3b^4+2a^2b^2)}A^3\psi^3(t) + p(1-u)+\rho hA\psi_{,tt}(t)\left[\frac{u^3}{3}-\frac{1}{3}\right]$$

Minimizing the error function by application of Galerkin procedure
Which on evaluation yields the following time differential equation for $\nu = 0.3$

$$\frac{2}{3}D\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}A\psi(t) + \frac{1.84\,Eh}{(3a^4+3b^4+2a^2b^2)}A^3\psi^3(t) + \frac{1}{18}\rho hA\psi_{,tt}(t) = \frac{p}{12} \qquad [5.4.15]$$

Equn (5.4.15) may be put in a simpler form

$$\psi_{,tt}(t) + C_1\psi(t) + C_3\psi^3(t) = Cp \qquad [5.4.16]$$

where
$$C_1 = \frac{12D}{\rho h}\frac{(3a^4+3b^4+2a^2b^2)}{a^4b^4}$$

$$C_3 = 33.12\frac{E}{\rho}\frac{A^2}{(3a^4+3b^4+2a^2b^2)}$$

$$C = \frac{3}{2A\rho h}$$

a) For free liner vibration $p = 0$, $C_3 = 0$

$$\psi_{,tt}(t) + C_1\psi(t) = 0$$

The linear frequency parameter

$$C_1^{1/2} = \left[\frac{12D}{\rho h}\,\frac{(3a^4+3b^4+2a^2b^2)}{a^4 b^4}\right]^{1/2}$$

For circular plate b = a

Linear frequency for a circular plate = $9{\cdot}797\sqrt{\dfrac{D}{\rho h a^4}}$ where a is the radius of the plate

b) Non linear free vibration

For Non-linear free vibration one puts p = 0 in the equation (5.4.16)

$$\psi_{,tt}(t) + C_1\psi(t) + C_3\psi^3(t) = 0$$

If one designates T* and T as the time periods of non- linear and linear vibrations respectively then the relative period is given by

$$\frac{T^*}{T} = \left[1 + \frac{3}{4}\frac{C_3}{C_1}\xi^2\right]^{-1/2}$$

ξ is the non-dimensional relative amplitude

$$\frac{T^*}{T} = \left[1 + \frac{3}{4}\times\frac{33{\cdot}12\,(1-\nu^2)\,m^4}{(3m^4+2m^2+3)^2}\left(\frac{A_0}{h}\right)^2\right]^{-1/2}$$

Where m = a/b.

For circular plate m = 1 and for ν = $0{\cdot}3$ one gets

$$\frac{T^*}{T} = \left[1 + \frac{3}{4}\times 0{\cdot}4709\,\xi^2\right]^{-1/2}$$

The numerical results showing the variation of T*/T with relative amplitude ξ ($= \frac{A_0}{h}$) for rigid elliptical plate have been depicted in the Table - (39)

C) Static deflection for the same problem may be evaluated from equation (5.4.16) after rejecting the inertial term

$$\frac{pa^4}{Eh^4} = \frac{2}{3}\frac{(3m^4+2m^2+3)}{(1-\nu^2)}\left(\frac{W_0}{h}\right) + \frac{22.08\,m^4}{(3m^4+2m^2+3)}\left(\frac{W_0}{h}\right)^3$$

Which for circular plate and for $\nu = 0.3$ reduces to

$$\frac{pa^4}{Eh^4} = \left\{5.8608\left(\frac{W_0}{h}\right) + 2.76\left(\frac{W_0}{h}\right)^3\right\} \quad \text{present study}$$

$$= \left\{5.848\left(\frac{W_0}{h}\right) + 2.754\left(\frac{W_0}{h}\right)^3\right\} \quad \text{Ref (22)}$$

Numerical results showing the dependence of central deflection $\frac{W_0}{h}$ on load parameter $\frac{pa^4}{Eh^4}$ are shown in table (40)

Table [3 9]
Dependence of relative time period T*/T of non-linear and linear vibrations on relative amplitudes for different values of aspect ratio (m), $\nu = 0.3$

$\frac{W_0}{h}$	T*/T		
	m= 1	m = 1.5	m = 2
0	1.000	1.000	1.000
0.2	0.9930	0.9955	0.9979
0.4	0.9728	0.9826	0.9917
0.6	0.9419	0.9622	0.9818
0.8	0.9031	0.9356	0.9683
1.0	0.8596	0.9054	0.9517
1.2	0.8416	0.8703	0.9386
1.4	0.7708	0.8345	0.9114
1.6	0.7246	0.7983	0.8887
1.8	0.6828	0.7624	0.8649
2.0	0.6437	0.7275	0.8404
2.2	0.6075	0.69405	0.8157
2.5	0.5752	0.6469	0.7786

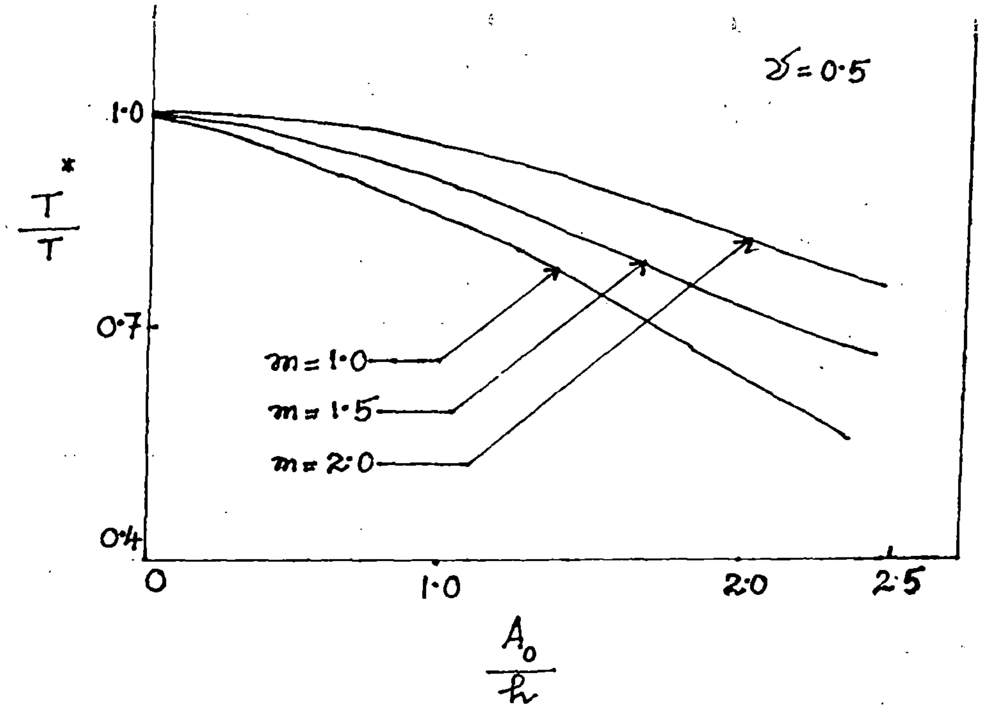


Figure $\overline{XIII}$: Time Period Ratio $\frac{T^*}{T}$ against Relative Amplitude for Elliptic Plates

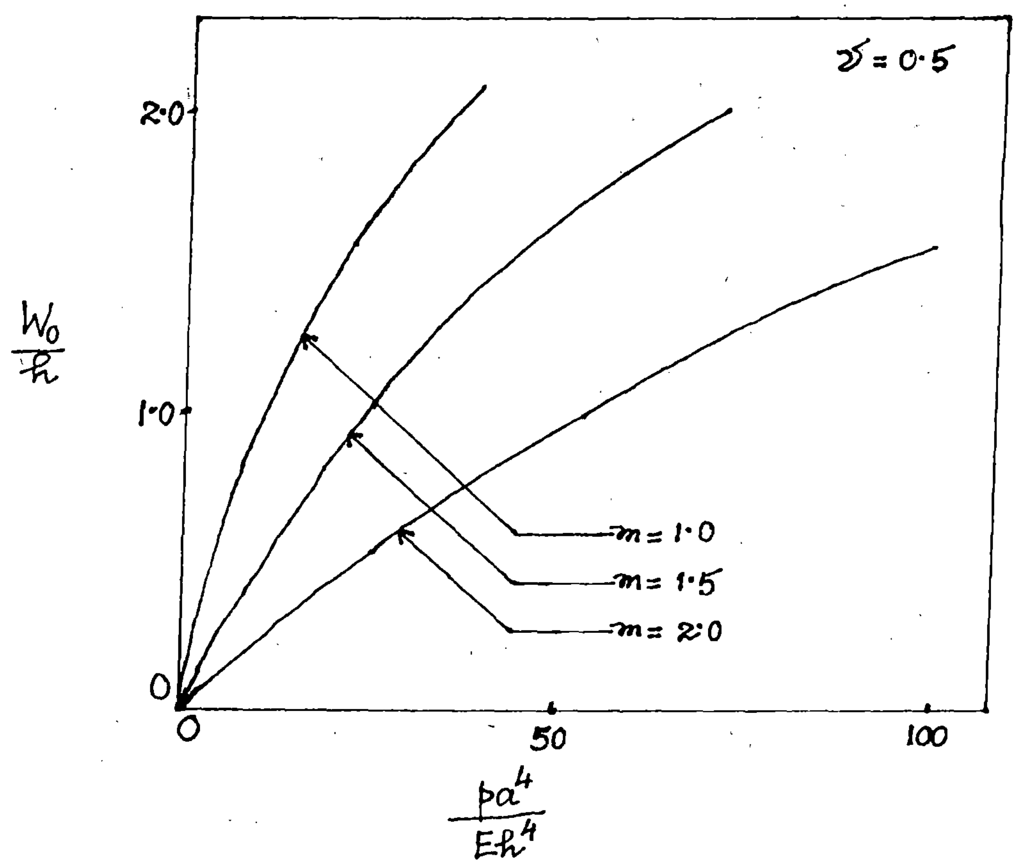


Figure $\overline{XIV}$: Load Deflection Curve for Elliptic Plates

Table [40]

Dependence of Central Deflection $\frac{W_0}{h}$ on Load Parameter $\frac{pa^4}{Eh^4}$ for elliptical plate $\nu = 0.3$, for different aspect ratio , m

W_0/h	pa^4/Eh^4		
	m = 1	m = 1.5	m= 2.0
0	0	0	0
0.2	1.1742	3.3599	8.6918
0.4	2.4619	6.9636	17.6711
0.6	4.1725	11.03668	27.2251
0.8	6.1017	15.8.92	37.6413
1.0	8.6206	21.5478	49.207
1.2	11.802	28.4587	60.4852
1.6	20.6818	46.7742	93.6747
1.8	26.64508	58.6516	112.712
2.0	33.818	72.6576	134.427

It has already been studied that the result for vibration of elliptic plates using the first set (3.11) and (3.12) is not satisfactory. But in this chapter utility of the second set of equations (3.12) and (3.13) has been observed. Hence in conclusion it may be said that the second set of equations should preferably be used to investigate large amplitude, vibration problems. However for rough approximation the first set may be used in cases where the structures have complicated boundaries or the governing differential equations become too much complicated in nature.

Chapter - VI

The Non-liner Vibration Analysis of Elastic Shells.

The main purpose of the thesis is to make an extensive study on the application of "Constant Deflection Contour Method". In previous Chapters it has been shown that the application of this method, on isotropic elastic plate problems is effective and the analysis appears to be easier than the other existing methods.

To make further investigation the application of this method to shell structures will be made in this chapter. However the projection of the curve of intersection of the plane $z = 0$ with the structure must be known in all cases, how much complicated the structure may be.

Though several studies [8, 9,38,39,40,54,59,109,115] have been made on vibration analysis of shell structures in which most of the problems deal with linear analysis. The study of the non-liner analysis is very restrictive in nature and it appears that more detailed study on large amplitude vibration of shell structures should be made.

Regarding the application of " Constant deflection contour Method " on shell structures Mazumdar and Jones[115] had made some useful studies on shell structures. However, all of them concerned with the linear approach. Recently Banerjee [109] has made an attempt to extend the "Constant Deflection Contour" method to shell problems. But no specific illustration has been made in support of the theoretical study. This prompts the present investigator to make an attempt to do some work in this sphere in support of the applicability of " Constant Deflection Contour" method to shell problems involving statical or dynamical cases.

Let an elastic isotropic shallow shell be considered. The equation of its middle surface referred to a system of orthogonal Co-ordinates (x, y, z) represented by

$$Z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2} + \frac{xy}{R_{12}}$$

For a shallow shell $r = (x^2+y^2)^{1/2}$ is small compared to the radii of curvature R_1, R_2, aid R_{12} everywhere in the region. R_1, R_2, aid R_{12} may be taken to be constant.

The well-known Von Karman equation, extended to a dynamic case for shallow shell may be written as [109]

$$D\nabla^4 w = h\mathcal{L}(F, w) + p + \rho h w_{,tt} - K_1 h F_{,yy} - K_2 h F_{,xx} \qquad [6\cdot1]$$

$$\nabla^4 F = -\frac{E}{2}\mathcal{L}(w, w) + EK_1 w_{,yy} + EK_2 w_{,xx} \qquad [6\cdot2]$$

K_1, and K_2 are principal curvatures.

If $u = u(x,y)$ defines the lines of equal deflection equations (6.1) and (6.2) becomes

$$D\left[A_1\frac{d^4w}{du^4} + A_2\frac{d^3w}{du^3} + A_3\frac{d^2w}{du^2} + A_4\frac{dw}{du}\right] = h\left[A_5\frac{dw}{du}\frac{dF}{du} + A_6\frac{d}{du}\left(\frac{dw}{du}\frac{dF}{du}\right)\right] - h\left[A_7\frac{dF}{du} + A_8\frac{d^2F}{du^2}\right] + p - \rho h w_{,tt} \qquad [6\cdot 3]$$

$$A_1\frac{d^4F}{du^4} + A_2\frac{d^3F}{du^3} + A_3\frac{d^2F}{du^2} + A_4\frac{dF}{du} = -\frac{E}{2}\left[A_5\left(\frac{dw}{du}\right)^2 + A_6\frac{d}{du}\left(\frac{dw}{du}\right)^2\right] + EA_7\frac{dw}{du} + EA_8\frac{d^2w}{du^2} \qquad [6\cdot 4]$$

Where

$$A_1 = \left(u_{,x}^2 + u_{,y}^2\right)^2$$

$$A_2 = 6\left(u_{,x}^2 u_{,xx} + u_{,y}^2 u_{,yy}\right) + 2\left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx}\right) + 8u_{,x}u_{,y}u_{,xy}$$

$$A_3 = 4\left(u_{,x}u_{,xxx} + u_{,y}u_{,yyy}\right) + 4\left(u_{,x}u_{,xyy} + u_{,y}u_{,xxy}\right) + 2u_{,xx}u_{,yy} + 4u_{,xy}^2 + 3\left(u_{,xx}^2 + u_{,yy}^2\right)$$

$$A_4 = u_{,xxxx} + u_{,yyyy} + 2u_{,xxyy}$$

$$A_5 = 2u_{,xx}u_{,yy} - 2u_{,xy}^2$$

$$A_6 = \left(u_{,x}^2 u_{,yy} + u_{,y}^2 u_{,xx} - 2u_{,x}u_{,y}u_{,xy}\right)$$

$$A_7 = K_1 u_{,yy} + K_2 u_{,xx}$$

$$A_8 = K_1 u_{,y}^2 + K_2 u_{,x}^2$$

Integrating equation (6.3) and (6.4) over area

$$\iint_{\Omega} D\left[A_1\frac{d^4w}{du^4} + A_2\frac{d^3w}{du^3} + A_3\frac{d^2w}{du^2} + A_4\frac{dw}{du}\right]d\Omega$$

$$= h\iint_{\Omega}\left[A_5\frac{dw}{du}\frac{dF}{du} + A_6\frac{d}{du}\left(\frac{dw}{du}\frac{dF}{du}\right)\right]d\Omega$$

$$- h\iint_{\Omega}\left[A_7\frac{dF}{du} + A_8\frac{d^2F}{du^2}\right]d\Omega - \iint_{\Omega}\rho h w_{,tt}\, d\Omega$$

$$+ \iint_{\Omega} p\, d\Omega \qquad [6\cdot 5]$$

$$\iint_{\Omega}\left[A_1\frac{d^4F}{du^4} + A_2\frac{d^3F}{du^3} + A_3\frac{d^2F}{du^2} + A_4\frac{dF}{du}\right]d\Omega$$

$$= -\frac{E}{2}\iint_{\Omega}\left[A_5\left(\frac{dw}{du}\right)^2 + A_6\frac{d}{du}\left(\frac{dw}{du}\right)^2\right]d\Omega$$

$$+ E\iint_{\Omega}\left[A_7\frac{dw}{du} + A_8\frac{d^2w}{du^2}\right]d\Omega \qquad [6\cdot 6]$$

For an elliptical shell the lines of equal deflection are represented by

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Where u=o defines the boundary and u=1 is the centre of concentric ellipses, where the deflection is maximum under an uniform load p

Deflection and stress functions are assumed to be

$$w = AW(u)\Psi(t)$$
$$F = AF'(u)\Psi^2(t) \qquad [6\cdot 7]$$

For such a variable u, equations (6.5) and (6.6), on integration reduce to

$$2DP\left[(1-u)^2\frac{d^3W}{du^3} - 2(1-u)\frac{d^2W}{du^2}\right]A\Psi(t)$$
$$+\frac{8h}{a^2b^2}(1-u)\frac{dF'}{du}\frac{dW}{du}A^2\Psi^3(t)$$
$$+2h\left(\frac{K_1}{b^2}+\frac{K_2}{a^2}\right)(1-u)\frac{dF'}{du}A\Psi^2(t)$$
$$+\rho hA\Psi_{,tt}(t)\int_1^u W du + p(1-u) = 0 \qquad [6\cdot 8]$$

$$P\left[(1-u)^2\frac{d^3F'}{du^3} - 2(1-u)\frac{d^2F'}{du^2}\right]A\Psi^2(t)$$
$$= \frac{2E}{a^2b^2}(1-u)\left(\frac{dW}{du}\right)^2A^2\Psi^2(t)$$
$$+E\left(\frac{K_1}{b^2}+\frac{K_2}{a^2}\right)(1-u)\frac{dW}{du}A\Psi(t) \qquad [6\cdot 9]$$

$$\text{where } P = \frac{3a^4+3b^4+2a^2b^2}{a^4b^4}$$

$$\text{Let } W(u) = \sum_{i=2}^{\alpha} A_i u^i$$

Compatible with the boundary conditions for a clamped boundary, A_i's are evaluated. Let a rough approximation be considered with the first term only.

$W = Au^2$

With $W = Au^2$, the first integral of equation (6.9)

$$\left[(1-u)\frac{d^2F'}{du^2} - \frac{dF'}{du}\right]A\Psi^2(t) = \frac{8E}{3a^2b^2P}u^3A^2\Psi^2(t)$$
$$+E\Upsilon u^2A\Psi(t) + B_1 \qquad [6.10]$$

$$\text{where} \quad \Upsilon = \frac{1}{P}\left[\frac{K_1}{b^2} + \frac{K_2}{a^2}\right]$$

While the second integral becomes

$$(1-u)\frac{dF'}{du} A\psi^2(t) = \frac{2E}{3a^2b^2P} u^4 A^2\psi^2(t) + \frac{1}{3}\Upsilon E u^3 A\psi(t) + B_1 u + B_2 \qquad [6.11]$$

B_1 and B_2 are constant subject to immovable condition

$$\left[2(1-u)\frac{d^2F'}{du^2} - (1-\nu)\frac{dF'}{du}\right]_{u=0} = 0 \qquad [6.12]$$

Further equations (6.10) and (6.11) are valid for whole domain bounded by C_u then for u = 0

$$A\psi^2(t)\left[\frac{d^2F'}{du^2}\bigg|_{u=0} - \frac{dF'}{du}\bigg|_{u=0}\right] = B_1, \qquad [6\cdot13]$$

$$\frac{dF'}{du}\bigg|_{u=0} A\psi^2(t) = B_2 \qquad [6\cdot14]$$

For u = 1 , equation (6.11) becomes

$$\frac{2E}{3a^2b^2P} A^2\psi^2(t) + \frac{1}{3}\Upsilon E A\psi(t) + B_1 + B_2 = 0 \qquad [6\cdot15]$$

Solving equations (6.12) , (6.13), 6.14 and (6.15) one gets

$$B_1 = \frac{2E(1+\nu)}{3a^2b^2P(1-\nu)} A^2\psi^2(t) + \frac{1}{3}\Upsilon E\frac{(1+\nu)}{(1-\nu)} A\psi(t)$$

$$B_2 = -\frac{4}{3}\frac{E A^2\psi^2(t)}{a^2b^2P(1-\nu)} - \frac{2}{3}\frac{\Upsilon E}{(1-\nu)} A\psi(t)$$

and equation (6.11) reduces to

$$(1-u)\frac{dF'}{du} A\psi^2(t) = \frac{2E}{3a^2b^2P(1-\nu)}\left[(1-\nu)u^4 + (1+\nu)u - 2\right]A^2\psi^2(t) + \frac{1}{3}\frac{\Upsilon E}{(1-\nu)}\left[(1-\nu)u^3 + (1+\nu)u - 2\right]A\psi(t) \qquad [6.16]$$

Taking the first derivative of equation (3.8) in agreement with the proposed theory explained in chapter - V

$$2DP\left[(1-u)^2\frac{d^4W}{du^4}-4(1-u)\frac{d^3W}{du^3}+2\frac{d^2W}{du^2}\right]A\psi(t)$$
$$+\frac{8h}{a^2b^2}\frac{d}{du}\left[(1-u)\frac{dF'}{du}\frac{dW}{du}\right]A^2\psi^3(t)+2\Upsilon hP\frac{d}{du}\left[(1-u)\frac{dF'}{du}\right]A\psi^2(t)$$
$$+\rho hAW\psi_{,tt}(t)-p=0 \qquad [6\cdot17]$$

Substituting equation (6.16) into equation (6.17) and using Galerkin procedure to minimize the error one gets.

$$\left[\frac{8}{3}DP+0\cdot8162\Upsilon^2 EhP\right]A\psi(t)+5\cdot142\frac{\Upsilon Eh}{a^2b^2}A^2\psi^2(t)$$
$$+7\cdot3651\frac{Eh}{a^4b^4P}A^3\psi^3(t)+\frac{1}{5}\rho hA\psi_{,tt}(t)=\frac{p}{3} \qquad [6\cdot18]$$

Equation (6.18) can be put in a simpler form

$$\psi_{,tt}(t)+C_1\psi(t)+C_2\psi^2(t)+C_3\psi^3(t)=C\cdot p \qquad [6\cdot19]$$

$$C_1=\frac{5P}{\rho h}\left[\frac{8}{3}D+0\cdot8126\Upsilon^2 Eh\right]$$

$$C_2=25\cdot71\frac{\Upsilon E}{\rho a^2b^2}A$$

$$C_3=36\cdot8255\frac{EA^2}{\rho a^4b^4P}$$

$$C=\frac{5}{3}\frac{p}{\rho hA}$$

Free linear vibration ;
For free vibration putting p = 0 in equation (6.19)

$$\psi_{,tt}(t) + C_1\psi(t) + C_2\psi^2(t) + C_3\psi^3(t) = 0$$

The linear frequency parameter is given by

$$\omega = C_1^{1/2} = \left[\frac{5P}{\rho h}\left(\frac{8}{3}p + 0.8126\, r^2 E h\right)\right]^{1/2}$$

$$\omega a^2\sqrt{\rho/Eh^2} = (3m^4+2m^2+3)^{1/2}\left[\frac{1.111}{(1-\nu^2)} + 1.0155\left(\frac{2r}{h}\right)^2\right]^{1/2}$$

The variation of $\omega a^2\sqrt{\rho/Eh^2}$ for elliptic circular shell are shown in Table (41)

b) Non-linear free vibration
For non-linear free vibration $p = 0$

$$\psi_{,tt}(t) + C_1\psi(t) + C_2\psi^2(t) + C_3\psi^3(t) = 0$$

If we designate ω^* and ω as the frequencies of non-linear and linear vibration respectively

$$\frac{\omega^*}{\omega} = \left[1 + \frac{3}{4}\frac{C_3}{C_1} - \frac{5}{6}\left(\frac{C_2}{C_1}\right)^2\right]^{1/2}$$

ξ is the non-dimensional relative amplitude

$$\frac{\omega^*}{\omega} = \left[1 + \frac{5.5238\, m^4\xi^2}{(3m^4+2m^2+3)^2\left[0.2442+0.2031\left(\frac{2r}{h}\right)^2\right]} - \frac{5.508\left(\frac{2r}{h}\right)^2 m^4\xi^2}{(3m^4+2m^2+3)^2\left\{0.2442+0.2031\left(\frac{2r}{h}\right)^2\right\}^2}\right]^{1/2}$$

for $\nu = 0.3$

Numerical results showing the variation of $\frac{\omega^*}{\omega}$ with relative amplitude $\xi = \frac{A_0}{h}$ for elliptical shell have been shown in the table (42–46)

C The static deflection of the same problem may be evaluated from equation (6.18) neglecting the inertial term as

$$\left[0.7326 + 0.6093\left(\frac{2r}{h}\right)^2\right](3m^4+2m^2+3)\left(\frac{W_0}{h}\right) + 7.713\left(\frac{2r}{h}\right)m^2\left(\frac{W_0}{h}\right)^2 + \frac{22.09\, m^4}{(3m^4+2m^2+3)}\left(\frac{W_0}{h}\right)^3 = \frac{pa^4}{Eh^4}$$

The numerical results showing the dependence of central deflection on load parameter are shown in tables [47 - 51]

Table : 41 Variation of linear frequency in units of corresponding flat-plate frequency for a complete spectrum of aspect ratio and ($\frac{2r}{h}$)

	m=1		m=1.5		m=2	
	Present Study	[115]	Present Study	[115]	Present Study	[115]
0	1.	1.05	1.68	1.77	2.71	2.85
0.5	1.099	1.15	1.88	1.93	2.98	3.12
1.0	1.35	1.40	2.279	2.36	3.67	3.80
1.5	1.694	1.75	2.85	2.95	4.60	4.75
2.0	2.08	2.16	3.50	3.64	5.64	5.87
3.0	2.91	2.99	4.90	5.73	7.91	8.12
5.0	4.66	4.51	7.68	7.59	12.69	12.25
10	9.17	7.62	15.44	12.83	24.91	20.69
20	18.26	14.1	30.21	23.74	49.61	38.29

Table: 42. Variation of $\frac{\omega^*}{\omega}$ on relative amplitudes $\frac{A_0}{h}$ for different values of m.

$\nu = 0.3$, $\frac{2r}{h} = 0$

	ω^*/ω		
A_0/h	$m=1$	$m=1.5$	$m=2$
0	1.000	1.000	1.000
0.2	1.007	1.0044	1.0020
0.4	1.0272	1.0176	1.0082
0.6	1.0617	1.0392	1.0185
0.8	1.1073	1.0688	1.0327
1.0	1.1633	1.1056	1.0506
1.2	1.2283	1.1490	1.0722
1.4	1.3010	1.1982	1.0971
1.6	1.3801	1.2527	1.1251
1.8	1.4645	1.3117	1.1561
2.0	1.5535	1.3746	1.1897

Table : 43 Dependence of $\frac{\omega^*}{\omega}$ on $\frac{A_0}{h}$

$\nu = 0.3$, $\frac{2r}{h} = 1$

A_0/h	ω^*/ω		
	m=1	m=1.5	m=2.0
0	1.000	1.000	1.000
	0.9925	0.9953	0.9978
0.5	0.9699	0.9811	0.991
1.0	0.8734	0.9224	0.9648
1.5	0.6831	0.8153	0.9191
2.0	0.228	0.6356	0.8508
2.5		0.2622	0.7541

Table : (44 .) Dependence of $\frac{\omega^*}{\omega}$ on $\frac{A_0}{h}$

$\nu = 0.3$, $\frac{2r}{h} = 5$

A_0/h	ω^*/ω		
	m=1	m=1.5	m=2
0	1.000	1.000	1.000
.5	0.9925	0.995	0.9977
1.0	0.9700	0.9813	0.9914
1.5	0.8876	0.9562	0.9806
2.0	0.8740	0.9230	0.9654
2.5	0.7945	0.87678	0.9453

Table : 45 Dependence of $\frac{\omega^*}{\omega}$ on $\frac{A_0}{h}$

$\nu = 0.3$, $\frac{2r}{h} = 10$

A_0/h	ω^*/ω		
	m=1	m=1.5	m=2
0	1.000	1.000	1.000
.5	0.9979	0.9983	0.9994
1.0	0.9919	0.9935	0.9977
1.5	0.9818	0.9854	0.9946
2.0	0.9674	0.9740	0.9905
2.5	0.9486	0.9591	0.9852

Table : 46 Dependence of $\frac{\omega^*}{\omega}$ on $\frac{A_0}{h}$

$\nu = 0.3$, $\frac{2r}{h} = 20$

A_0/h	ω^*/ω		
	m=1	m=1.5	m=2
0	1.000	1.000	1.000
.5	0.9994	0.9996	0.9998
1.0	0.9979	0.9986	0.9993
1.5	0.9953	0.9970	0.9986
2.0	0.9917	0.9947	0.9975
2.5	0.9871	0.9981	0.9962

Table : 47 Dependence of Central Deflection $\frac{W_0}{h}$ on Load Parameter for different values of m

$\nu = 0.3$, $\frac{2r}{h} = 0$

	pa^4/Eh^4		
W_0/h	m=1	$m=1.5$	$m=2$
0	0	0	0
0.2	1.1742	3.3599	8.6918
0.4	2.4619	6.9636	17.6711
0.6	4.1725	11.03668	27.2251
0.8	6.1017	15.8192	37.6413
1.0	8.6206	21.5478	49.207
1.2	11.802	28.4587	60.4852
1.6	20.6818	46.7742	93.6747
2.0	33.818	72.6576	134.427

Table : 48 Dependence of Central Deflection on Load Parameter

$\nu = 0.3$ $\frac{2r}{h} = 1$

W_0/h	pa^4/Eh^4		
	m=1	m=1.5	m=2
0	0	0	0
0.2	2.4765	6.5565	16.4539
0.4	5.5716	14.3336	37.0076
0.6	9.804	23.478	59.905
0.8	14.922	34.248	86.147
1.0	21.203	42.974	116.0125
1.2	28.751	68.681	149.774
1.6	48.18	98.468	230.182
1.8	60.40	120.974	277.396
2.0	74.392	146.54	329.668

Table - 49 Dependence of Central Deflection on Load Parameter

$\nu = 0.3 \quad \frac{2r}{h} = 5$

W_0/h	pa^4/Eh^4		
	m=1	m=1.5	m=2
0	0	0	0
0.2	27.106	86.585	194.597
0.4	54.43	180.347	401.823
0.6	91.108	281.529	605.763
0.8	128.256	390.354	833.706
1.0	169.04	507.08	1102.15
1.2	213.563	613.906	1362.76
1.6	314.372	906.928	1926.47
2.0	431.78	1217.2	2494.76

Table - 50 Dependence of Central deflection on load parameter pa^4/Eh^4 for different values of m

$\nu = 0.3 \quad \frac{2r}{h} = 10$

W_0/h	pa^4/Eh^4		
	m=1	m=1.5	m=2
0	0	0	0
0.2	101.762	286.679	739.987
0.4	209.836	587.273	1504.943
0.6	324.336	902.61	2295.15
0.8	445.413	1232.30	3110.91
1.0	573.19	1576.9	3952.51
1.2	707.789	1935.11	4820.21
1.6	998.03	2495.9	6635.14
2.0	1317.2	3530.2	8557.92

Table - 51

Dependence of Central deflection $\left(\frac{W_0}{h}\right)$ on load parameter for different values m

$\nu = 0\cdot 3 \quad \frac{2r}{h} = 20$

W_0/h	pa^4/Eh^4		
	m=1	$m=1\cdot 5$	$m=2$
0	0	0	0
0.2	397.19	569.91	2909.12
0.4	806.856	1167.83	5867.89
0.6	1229.126	1793.97	8876.59
0.8	1664.133	2448.5	11935.98
1.0	2112.02	3131.9	15045.6
1.2	2572.86	3844.147	18205.07
1.6	3534.2	5356.37	24666.8666
2.0	4549.12	6986.4	31360.6

Table (41) shows that the results tally well with those of Jones & Mazumdar [115] for lower values of measure of shallowness ($2r/h$). The present values of the linear frequency are little greater than these obtained in Reference [115] and the difference increases for higher values of ($2r/h$). This may be due to the rough approximation made for deflection function.

Table - (42) shows the values of relative frequency for different values of m (= aspect ratio) for ν = 0.3 and ($2r/h$) = 0 i.e. when the structure assumes that of a plate . From table (42) the frequency ratio (non-linear : linear) increases with the increase of the value of $\left(\frac{A_0}{h}\right)$ (relative amplitude) for all values of m. However the variation of (ω^*/ω) is not so significant for higher values of m which is a obvious expectation as m increases the structure behaves like a beam whereas the results of tables (43 – 46) in which the variation of the measure of shallowness has been considered and the effect is just the reverse what has been observed in the previous case (Table - 42). In these tables the value of relative frequency increases with the increase of $\frac{A_0}{h}$ and the effect is most significant for m = 1 and the effect of non-linearity decreases with the increase of $2r/h$ and m.

As it has always been treated to find the static behaviour of the structure in all the previous problems as a by-product of the ongoing analysis, tables (47- 51) show the dependence of central deflection W_0/h. on the load parameter pa^4/Eh^4 . For $2r/h = 0$ the results for m = 1 are in excellent agreement with those of Yamaki [22]. The effect of the dependence become more significant with the increasing values of the aspect ratio m. However as the measure of shallowness ($2r/h$) increases the deflections are more significant for smaller values of m and it is a maximum for m = 1 when a particular value of the load parameter is concerned.

CHAPTER - VII

Elastic - Plastic Analysis of Shallow Shells of Arbitrary Shape

In the previous chapters it has been observed that the "Constant Deflection Contour" method can be effectively applied to study the vibrations of elastic plates and shells and the analysis appears to be easier than the other existing methods.

To make further investigation, this method is applied to study the vibration of elastic plastic shallow shells in this chapter.

Though several studies have been made . . on elastic-plastic analysis of plates and shells [107, 129, 130], most of them deal with the linear analysis . This initiates the present investigator to make an attempt to apply the " the Constant Deflection Contour " method to study the non-linear vibration of elastic plastic shallow shells. Regarding the application of " Constant Deflection Contour " method, on elastic plastic analysis of plates, Mazumdar et - al [107] made some useful studies on this sphere.

Considering an elastic plastic shallow shell of thickness h , the equation of the middle surface of the shell is given by

$$z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2} + \frac{xy}{R_{12}}$$

The shell is called shallow if $r = (x^2+y^2)^{1/2}$ is small compared to the least of the radii of curvature R_1, R_2, R_{12}

The basic equations to study the elastic plastic analysis of shell may be written as :

$$\frac{d^3w}{du^3}\oint(1-\Omega)R\,ds + \frac{d^2w}{du^2}\oint(1-\Omega)J\,ds + \frac{dw}{du}\oint(1-\Omega)G\,ds$$

$$+\frac{d^2w}{du^2}\oint D\left(\frac{\partial\Omega}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial\Omega}{\partial y}\frac{\partial u}{\partial y}\right)t^{1/2}\,ds$$

$$+\frac{dw}{du}\oint\frac{D}{t^{1/2}}\left(\frac{\partial\Omega}{\partial x}K + \frac{\partial\Omega}{\partial y}L\right)ds + \iint\left[\rho h w_{,tt} + \frac{1}{R_1}F_{,yy} + \frac{1}{R_2}F_{,xx}\right.$$

$$\left. - \frac{2}{R_{12}}F_{,xy} - p\right]dx\,dy = 0 \qquad [7.1]$$

$$\frac{d^3F}{du^3}\oint R\,ds + \frac{d^2F}{du^2}\oint J\,ds + \frac{dF}{du}\oint G\,ds - \frac{12D^2(1-\nu^2)(1-\Omega)}{h^2}\frac{dw}{du}\oint\left[\frac{u_{,x}^2}{R_2} + \frac{u_{,y}^2}{R_1}\right]t\;ds$$

$$= 0 \qquad [7.2]$$

where $\Omega = 0$, when $e \leqslant 1$ and the region is elastic

$$\Omega = \lambda\left[1 - \frac{3}{2e} + \frac{1}{2e^3}\right] \text{ when } e \geqslant 1 \text{ and the region is plastic} \quad [7.3a]$$

where $$e = \frac{h}{\sqrt{3}\, e_s}\left(w_{,xx}^2 + w_{,yy}^2 + w_{,xx}\, w_{,yy} + w_{,xy}^2\right)^{1/2} \quad [7.3b]$$

$$t = u_{,x}^2 + u_{,y}^2$$

$$R = -D t^{3/2}$$

$$J = \frac{-D}{t^{1/2}}\left[3 u_{,xx}\, u_{,x}^2 + 3 u_{,yy}\, u_{,y}^2 + u_{,xx}\, u_{,y}^2 + u_{,yy}\, u_{,x}^2 + 4 u_{,xy}\, u_{,x}\, u_{,y}\right]$$

$$G = \frac{-D}{t^{3/2}}\Big[u_{,xxx}\, u_{,x}^3 + u_{,yyy}\, u_{,y}^3 + (2-\nu)\big(u_{,xxx}\, u_{,x}\, u_{,y}^2$$

$$+ u_{,yyy}\, u_{,y}\, u_{,x}^2\big) + u_{,xyy}\, u_{,x}^3 + u_{,xxy}\, u_{,y}^3 + (2\nu-1)\big(u_{,xyy}\, u_{,x}\, u_{,y}^2$$

$$+ u_{,xxy}\, u_{,x}^2\, u_{,y}\big) - 2(1-\nu)\, u_{,xy}\big(u_{,x}\, u_{,y}\, u_{,xx} - u_{,y}^2\, u_{,xy}$$

$$- u_{,x}^2\, u_{,xy} + u_{,x}\, u_{,y}\, u_{,yy}\big) + (1-\nu)\big(u_{,xx} - u_{,yy}\big)$$

$$\big(u_{,xx}\, u_{,y}^2 - u_{,yy}\, u_{,x}^2\big) + \frac{2D(1-\nu)}{t^{5/2}}\Big[u_{,xy}\big(u_{,x}^2 - u_{,y}^2\big)$$

$$- u_{,x}\, u_{,y}\big(u_{,xx} - u_{,yy}\big)\Big]^2$$

$$K = u_{,x}\left(u_{,xx} + \nu u_{,yy}\right) + u_{,y}\left[(1-\nu)\, u_{,xy} - H/D\right]$$

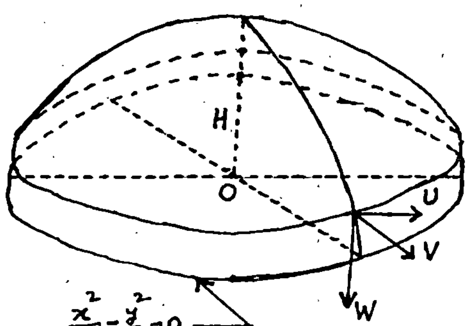
$$L = u_{,y}\left(u_{,yy} + \nu u_{,xx}\right) + u_{,x}\left[(1-\nu)u_{,xy} + \frac{H}{D}\right]$$

$$H = \frac{D(1-\nu)}{t}\left[u_{,xy}\left(u_{,x}^2 - u_{,y}^2\right) - u_{,x}u_{,y}\left(u_{,xx} - u_{,yy}\right)\right]$$

[7.4]

Equations (7.1) and (7.2) are the basic equations to study the vibration of elastic plastic shallow shell . One cannot proceed further unless the geometry of the shell is known.

A clamped dome of non-zero curvature upon an elliptic base is considered (Fig - 6).



The first approximation for the lines of constant deflection for this case due to symmetry consideration may be taken as

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad [7.5]$$

R_{12} has been assumed to be zero in accordance with the shallow shell theory with the form of u given by equation (7.5) and carrying out the necessary but lengthy calculation the differential equations (7.1) and (7.2) takes the form

$$(1-\Omega)(1-u)^2\frac{d^3w}{du^3} - 2(1-\Omega)(1-u)\frac{d^2w}{du^2}$$

$$-\frac{d\Omega}{du}\left[(1-u)^2\frac{d^2w}{du^2} - 2\frac{P'}{P}(1-u)\frac{dw}{du}\right] - \frac{\gamma}{D}(1-u)\frac{dF}{du}$$

$$= \frac{-p(1-u)}{2DP} - \frac{\rho h}{2DP}\int_1^u w_{,tt}\,du \qquad [7.6]$$

$$(1-u)\frac{d^3F}{du^3} - 2\frac{d^2F}{du^2} + \gamma(1-\Omega)Eh\frac{dw}{du} = 0 \qquad [7.7]$$

where $P_1 = \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{2\nu}{a^2b^2}\right)$

$$\gamma = \frac{1}{P}\left[\frac{1}{R_1 b^2} + \frac{1}{R_2 a^2}\right] \qquad [7.8]$$

Using the following non-dimensional parameters

$$w^* = \frac{wh}{e_s a^2} \qquad F^* = \frac{F}{Ee_s a^2} \qquad p^* = \frac{pa^2h}{2De_s} \qquad [7.9]$$

Equation (7.6) and (7.7) takes the form

$$(1-\Omega)(1-u)^2\frac{d^3w^*}{du^3} - 2(1-\Omega)(1-u)\frac{d^2w^*}{du^2}$$

$$-\frac{d\Omega}{du}\left[(1-u)^2\frac{d^2w^*}{du^2} - 2\frac{P_1}{P}(1-u)\frac{dw^*}{du}\right]$$

$$-\frac{Eh\gamma}{D}(1-u)\frac{dF^*}{du} + \frac{\rho h}{2DP}\int_1^u w^*_{,tt}\,du = -\frac{p^*(1-u)}{Pa^4} \qquad [7.10]$$

$$(1-u)\frac{d^3F^*}{du^3} - 2\frac{d^2F^*}{du^2} + \gamma(1-\Omega)\frac{dw^*}{du} = 0 \qquad [7.11]$$

Considering the shell completely clamped along the boundary, the boundary conditions can be expressed in terms of the deflection function and its derivatives with respect to u.

$w'=0$ at $u=0$ and $F=0$ at $u=0$ and

$$\frac{dw}{du} = 0 \text{ at } u=0 \qquad \frac{dF}{du} = 0 \text{ at } u=0 \qquad [7.12]$$

With these conditions expressed in equation (7.12), equations (7.10) and (7.11) are to be solved for W* and F*.

Let $w = \sum_{i=2}^{\alpha} A_i u^i \psi(t)$ compatible with the boundary condition expressed by equation (7.12), A_i's are to be evaluated. Let a rough approximation be considered with the first term only.

$$w = A u^2 \psi(t)$$

$$F = A u^2 \phi(t) \qquad [7.13]$$

Where $\psi(t)$ and $\phi(t)$ are functions of time only. Since equation (7.13) does not represent the exact solution, Galerkin procedure is applied to minimize the error.

Substituting equation (7.13) in to equation (7.11) a relation between $\phi(t)$ and $\psi(t)$ is first established

$$\phi(t) = \frac{3}{8}\gamma(1-\Omega)\psi(t) \qquad [7.14]$$

Making use of equations (7.30) and (7. 3b) $\frac{d\Omega}{du}$ may be evaluated in the following way.

$$e = \frac{h}{\sqrt{3}\, e_s}\left(w_{,xx}^2 + w_{,yy}^2 + w_{,xx} w_{,yy} + w_{,xy}^2\right)^{1/2}$$

$$\text{i.e.,} \quad e = \frac{h}{\sqrt{3}\, e_s}\left[M\left(\frac{dw}{du}\right)^2 + N\frac{dw}{du}\frac{d^2w}{du^2} + t^2\left(\frac{d^2w}{du^2}\right)^2\right]^{1/2} \qquad [7.15]$$

$$\text{where } M = u_{,xx}^2 + u_{,yy}^2 + u_{,xx} u_{,yy} + u_{,xy}^2$$

$$N = 2u_{,xx} u_{,x}^2 + 2u_{,yy} u_{,y}^2 + u_{,xx} u_{,y}^2 + u_{,yy} u_{,x}^2 + 2u_{,xy} u_{,x} u_{,y}$$

$$t = u_{,x}^2 + u_{,y}^2 \qquad [7.16]$$

$$\Omega = \lambda\left[1 - \frac{3}{2e} + \frac{1}{2e^3}\right]$$

$$\frac{d\Omega}{du} = \frac{d\Omega}{de}\,\frac{de}{du} \qquad [7.17]$$

$$\frac{de}{du} = \frac{h^2}{6e_s^2 e}\left[\frac{d}{du}\left\{M\left(\frac{dw}{d}\right)^2 + N\frac{dw}{du}\frac{d^2w}{du^2} + t^2\left(\frac{d^2w}{du^2}\right)^2\right\}\right]$$

[7.18]

Substituting the values of M, N, t from equation (7.16), equation (7.18) takes the form

$$\frac{d\Omega}{du} = 8Q_2\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 + 2\left(\frac{\cos^2\theta}{a^4}+\frac{\sin^2\theta}{b^4}\right)\right]\frac{dw^*}{du}\,\frac{d^2w^*}{du^2}$$

$$-\,8(1-u)\left[\left(2\frac{\cos^2\theta}{a^4}+2\frac{\sin^2\theta}{b^4}+\frac{1}{a^2b^2}\right)\right.$$

$$\left.-\,4\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^2\right]\left(\frac{d^2w^*}{du^2}\right)^2$$

$$-\,8(1-u)\left[2\left\{\frac{\cos^2\theta}{a^4}+\frac{\sin^2\theta}{b^4}\right\}+\frac{1}{a^2b^2}\right]\frac{dw^*}{du}\,\frac{d^3w^*}{du^3}$$

$$+\,32(1-u)^2\left[\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right]^2\frac{d^2w^*}{du^2}\,\frac{d^3w^*}{du^3} \qquad [7.19]$$

where $Q_2 = \dfrac{\lambda a^4}{4}\,\dfrac{e^2-1}{e^5}$

θ = eccentric angle

substituting equations (7. 13) and (7. 19) into equation (7.10) and applying Galerkin procedure to minimize the error one can arrive at the final equation

$$(1-\Omega)\left[\frac{1}{3}+\frac{3}{80}\left(\frac{2r}{h}\right)^{2}(1-\nu^{2})\right]A\psi(t)$$

$$+\frac{32}{5}Q_2\left[\frac{1}{6}P_2+\frac{1}{12}P-\frac{1}{6}P_1-\frac{P_1P_2}{P}\right]A^3\psi^3(t)$$

$$+\frac{1}{36}\rho\frac{hA\psi_{,tt}(t)}{DP} = \frac{p^*}{12Pa^4} \qquad [7.20]$$

$$\text{where } P_2 = \frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{a^2b^2}$$

$$\text{and } Q_2 = \frac{\lambda a^4}{4}\,\frac{e^2-1}{e^5}$$

and e should be replaced by the average value of e i.e.

$$\bar{e}^2 = \frac{8}{9}A^2(3m^4+2m^2+3) \quad \text{where } m=\frac{a}{b}$$

Equation (7. 20) can easily be utilized to study the static and dynamic behaviours of an elastic-plastic shallow shell. No numerical results have been presented in the sense that one of the co-research workers is engaged in such studies on the basis of the equation(7.20).

It can be concluded that the application of " Constant Deflection Contour" method for elastic plastic bending analysis of plates and shell is quite straight forward and efficient. Although the method is illustrated to study the elastic plastic analysis of a shell upon an elliptic base, its application to other plate or shell geometries is quite simple. This method highly relies on the accuracy of the choice of the isodeflection contour lines u (x, y), however it is very difficult to find out the exact form of contour lines for a plate or shell of arbitrary shape. In the present study, the contour line function is assumed to be that for the corresponding fully elastic case. The present investigator wishes to continue further studies in this sphere in near future .

CONCLUDING CHAPTER :

The main objective of the present thesis is to investigate the feasibility of the application of " Constant Deflection Contour " method in extending it to the non-linear analysis, so far as the static and dynamic behaviour of plates and shells under various geometrical as well as boundary conditions. It has already been accepted that the brilliant works have been made by Mazumdar. J, of Department of Applied Mathematics, University of Adelaide, Australia and some of his co-research workers in the field of static and dynamic response of structures. However their works were mainly restricted to linear cases only. The novelty of this method was, perhaps, missed the attention of the research workers on an international basis during the seventies of the last century . An attempt to extend this method to a quasi-linear problem was made by S. Das and B. Banerjee [68] . Unfortunately the essence of the " Constant Deflection Contour " method was lost in dealing with the illustrative example. It appears that , perhaps, the authors in reference [68] have not deduced the required equations following the method offered by Mazumdar [19, 137 - 139]. The present investigator has deduced the same equations which are not in exact agreement with those obtained by [68] .

In 1997 Banerjee [109], probably first initiated the extension and application of ' Constant Deflection Contour ' method to non-liner analysis of structures vibrating at large amplitude. Mention may be made the work of Chanda [131] dealing with some linear and quasi - linear problems using the ' Constant Deflection Contour ' method. The present investigator started her work under the guidance of M. M. Banerjee to study the application of 'Constant Deflection Contour' method to strictly non-linear problems in 1997. Later on Banerjee and Rogerson [122] put forward a general study on the application of ' Constant Deflection Contour " method . Hence it may humbly be stated that the present work may be considered as the first concrete attempt to extend the " Constant Deflection Contour " method to problems associated with large vibrations.

In the present thesis some problems dealing with structures having a little complicated boundary have been considered to pave way for considering problems with more complicated and complex boundaries. In the fourth Chapter an elliptic plate clamped along the edges has been considered to start with. And the results have been compared with known available results as far as possible. The use of " Constant Deflection Contour " method has been found to be effective for such problems.

In all problems considered from Chap - IV onwards the basic governing differential equations have been deduced primarily on the basis of Karman field equations extended to a dynamic case. In some problems such equations have been used straight forward and in some cases where external applied forces or geometrical non-linearity or inhomogeneity has been considered, the basic equations have been deduced on the basis of the present theory. The results so deduced have been found satisfactory enough and support the applicability of this method. For example the problem dealing with large vibration of elliptic plates on elastic foundation have been deduced. The comparison of results so remarkably well with available results obtained using by a different approaches. Differences if there be any arise out of a different approach, method of solution and the approximating functions (limited number of terms out of a polynomial expressions). Of course the variation of results is very insignificant .

In problem-3, Chapter IV, an attempt has been made to justify the use of Karmań field equations over other simplified or modified equations. While considering the problem of static and dynamic behaviour of elliptic plates under damping condition, the problem has been rechecked using " Constant Deflection Contour ", method and the same set of equation as used in [68]. The present investigator states with much hesitance that the results cited in ref [68] are not those obtained from the expressions deduced by the author. Moreover the use of "Constant Deflection Contour" method is not at per with the basic idea of "Constant Deflection Contour" method; the detailed criticism has been given at the end of problem-3, Chap -IV.

In chapter VI, problem 3 and 4concern with the effect of rotatory inertia, damping and varying flexural rigidity. In all cases the present analysis appears to be in conformity with the proposed theory establishing the objective and applicability of "Constant Deflection Contour" method. Moreover two problems referred to above, establish the accuracy of Karmań equations over the other, as stated in the concluding remark made at the end of problem - 4, Chapter IV.

During the process of investigation, for a simplified approach equations (3.11) and (3.12) were utilized, later on Banerjee and Rogerson [122] proposed the use of fourth order equations (3.12) and (3.13) and they have been used throughout the whole Chapter-V and VI. Some typical examples have been dělt with in Chapter V and some more complicated problems have been considered besides the illustrative examples put forward by Banerjee and Rogerson [1.22]. These illustrations not only support the proposed theory but also establish the accuracy of numerical results for mixed boundary value problems. The concluding remark made at the end of Chapter-V includes that even in some cases the present approach provides results more accurate than those obtained by other author with different approaches. Some authors have utilized some simpler form of Karmań equations at the cost of accuracy and even inviting some absurities. For example equation (5.4.20) and equation (4.3.28) are compared with the relevant equation of reference [22], for static case, $\nu=0.3$, m=1 (wherever admissible)

For static case,

$$\frac{pa^4}{Eh^4} = 5\cdot8608\left(\frac{W_0}{h}\right) + 2\cdot76\left(\frac{W_0}{h}\right)^3 \qquad \text{from Karman Equations}$$

$$= 5\cdot8608\left(\frac{W_0}{h}\right) + 3\cdot516\left(\frac{W_0}{h}\right)^3 \qquad \text{from Berger's Equation}$$

[Putting $KE/G_c = 0$ in Equation (4.3.28) in problem 3, chapter IV]

$$= 5\cdot848\left(\frac{W_0}{h}\right) + 2\cdot754\left(\frac{W_0}{h}\right)^3 \qquad \text{[Yamaki, Ref(22)]}$$

For dynamic case,

$$\frac{T^*}{T} = \left[1 + 0\cdot3531\left(A_0/h\right)^2\right]^{-1/2} \qquad \text{from Karman Equation}$$

$$= \left[1 + 0\cdot4499\left(A_0/h\right)^2\right]^{-1/2} \qquad \text{from Berger's Equation}$$

$$\left\{\begin{array}{l}\text{Putting } KE/G_c = 0,\ \mu = 0 \\ \text{in Equation (4.3.21), problem 3} \\ \text{chapter IV}\end{array}\right\}$$

$$= \left[1 + 0\cdot3531\left(A_0/h\right)^2\right]^{-1/2} \qquad \left\{\text{Yamaki, Ref (22)}\right\}$$

Clearly the variation in coefficients of the non-linear term signifies the validity of Banerjee's approach. The same question had already been raised by Nowinski [72] and later on by Banerjee et al [77].

The latest problem utilizing Berger equations has been treated by Mondal and Biswas [128] years later cautioned by Banerjee et al [77]. Hence the present investigation also support the criticism first initiated by Nowinski [72] and later on by others. In Chapter IV problem - 3 Berger's equations have been utilized not to encourage to avail of the mathematical simplicity at the cost of accuracy.

In Chapter VI the extension of " Constant Deflection Contour " method to shell structures, has been made to break the monotony of plate structures.

An attempt has also been made to extend the present analysis to elastic plastic shell structures. The Governing differential equations have been deduced and they provide the primary tools for investigation of elastic plastic structures. The present thesis does not consider any numerical results in the sense that one of the co-research worker is engaged in such studies.

Concluding Remark

The following remarks may be made considering all aspects and investigations made during the research period.

1. For establishing the governing differential equations a new approach has been made different from what had been proposed by previous users of " Constant Deflection Contour " method.

2. The previous investigations based on " Constant Deflection Contour " method concern only with linearised problems whereas the present study is based on a non-linear approach.

3. Starting from structures having regular and common boundaries, gradually more and more complicated structures and mixed boundary value problems have been included in the present thesis.

4. Numerical results presented for various illustrative examples have been compared with all results available to present investigator. Failure in this respect, if there be any, may be due to the non-availability of required information and lack of information sources.

5. A comparative study has always been made in dealing with a specific problem with regard to different approaches made to investigate the problem.

6. The present investigator also humbly suggests that for simplicity the very essence of non-linear analysis should not be lost and basic equations like Berger or modified Berger equations should be avoided in the interest of future study.

7. True research activities never stop. The present investigation has to be cut off so as to finish present project within the specified period. However lot of works still remain to be done in different sphere not considered in the present study. The present investigator wishes to continue further studies in the related topic in near future.

8. Some of the new spheres which the present investigator thinks should attract the co-research workers and contemporary researchers interested in this field are :

i> Dynamic and static response of structures like triangular, annular and polygonal shaped plate structures.

ii> Shell structures, other than spherical and cylindrical shells.

iii> Extension of the " Constant Deflection Contour " method to elastic plastic behaviour of structures.

In conclusion, the following remarks may be added with reference to different illustrations on " Constant Deflection Contour " method :

i> The method is based on solid mathematical foundation.

ii> Unlike Karman equations it ultimately reduces to a problem of solving two ordinary differential equations.

iii> The method may be also applied to structures of arbitrary shape.

iv) The only disadvantage of this method is that the equation of the lines of equal deflection should be known.

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