CHAPTER - 4

RADIATIVE TRANSFER PROBLEM

IN

SPHERICAL ATMOSPHERES

4. Solution of a radiative transfer problem in a spherical atmosphere with Isotropic scattering using a modified form of spherical harmonic method.

4.1 Introduction.

The single interval spherical harmonic method has been widely utilized for solving radiative transfer, neutron transport and heat transfer problems in spherical medium. Marshak [1948] proposed a simple model of neutron transport problem and solved it by single interval method. This method was also used by Chandrasekhar [1943], Sen [1949] and Davison and Sykes [1958] for solving problems of radiative transfer ane neutron transfer in both plane and spherical geometry. Poisy [1961] used a scheme for making iterative corrections for curvature. All the single interval spherical harmonic methods suffer from the limitations that the exact boundary conditions cannot be used at the free surface.

Wilson and Sen [1965 a,b,c] used the double interval spherical harmonic method and demonstrated this for a neutron transport problem in spherical geometry. They considered the same model as has been used by Marshak in case of single interval SHM. They [1964 b] used this method to the classical problem of diffusion of radiation through a homogeneous sphere.

We propose to solve the equation of radiative transfer in case of spherical geometry using the double interval spherical method where the phase function is isotropic. We consider the same forms of intensity as has been done in Sec. (2) and (3). (Equations (3a) & (3b))

4.2 The equation of transfer and the boundary conditions.

The equation of radiative transfer for the problem of diffusion of radiation in a homogeneous sphere is given by

$$\mu\frac{\partial I(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I(r,\mu)}{\partial\mu} + I(r,\mu) = \frac{1}{2}\int_{-1}^{1} I(r,\mu)d\mu \tag{4.2.1}$$

where we have assumed that phase function is isotropic. Further r is the distance measured outward from the center of the sphere and μ is the cosine of the angle measured from the positive direction of the radius vector, $I(r,\mu)$ is the specific intensity of radiation at a distance r in the direction of θ which is given by

$$\theta = cos^{-1}(\mu).$$

The equation of transfer (4.2.1) is to be solved subject to the boundary condition,

$$I(R,\mu) \equiv 0 \text{ for } -1 \le \mu \le 0 \tag{4.2.2}$$

We consider the two forms of intensity as

$$I^+(r,\mu) = I(0,0)\left[\phi(r) + \psi(\mu) + \sum_{l=0}^{L}(2l+1)\mu I_l^+(r)P_l(2\mu-1)\right], \quad 0 \le \mu \le 1 \tag{4.2.3a}$$

$$I^-(r,\mu) = I(0,0)\left[\phi(r) + \psi(\mu) + \sum_{l=0}^{L}(2l+1)\mu I_l^-(r)P_l(2\mu+1)\right], \quad -1 \le \mu \le 0 \tag{4.2.3b}$$

Here ϕ *(r)* is a function of *r* only and we define as usual

$$\psi(\mu) = \begin{cases} 1 & \text{if } 0 \le \mu \le 1 \\ 0 & \text{if } -1 \le \mu \le 0 \end{cases}$$

With these two forms of intensity the equation of transfer becomes

$$\mu\frac{\partial I^{+}(r,\mu)}{\partial r} + \frac{1-\mu^{2}}{r}\frac{\partial I^{+}(r,\mu)}{\partial \mu} + I^{+}(r,\mu) =$$

$$= \frac{1}{2}\left[\int_{-1}^{0} I^{-}(r,\mu)d\mu + \int_{0}^{1} I^{+}(r,\mu)d\mu\right] \tag{4.2.4}$$

and

$$\mu\frac{\partial I^{-}(r,\mu)}{\partial r} + \frac{1-\mu^{2}}{r}\frac{\partial I^{-}(r,\mu)}{\partial \mu} + I^{-}(r,\mu) =$$

$$= \frac{1}{2}\left[\int_{-1}^{0} I^{-}(r,\mu)d\mu + \int_{0}^{1} I^{+}(r,\mu)d\mu\right] \tag{4.2.5}$$

If we use the forms (4.2.3a) and (4.2.3b) in the last two equations we get respectively,

$$\mu\frac{\partial I^{+}(r,\mu)}{\partial r} + \frac{1-\mu^{2}}{r}\frac{\partial I^{+}(r,\mu)}{\partial \mu} + I^{+}(r,\mu) =$$

$$= I(0,0)\left[\phi(r) + \frac{1}{2} + \frac{1}{4}\left(I_{0}^{+} + I_{1}^{+} - I_{0}^{-} + I_{1}^{-}\right)\right] \tag{4.2.6a}$$

and

$$\mu \frac{\partial I^-(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I^-(r,\mu)}{\partial \mu} + I^-(r,\mu) =$$

$$= I(0,0)\left[\phi(r) + \frac{1}{2} + \frac{1}{4}\left(I_0^+ + I_1^+ - I_0^- + I_1^-\right)\right] \qquad (4.2.6b)$$

We know that the well-known recurrence formula for Legendre polynomials is given by

$$\mu P_l(2\mu \pm 1) = \frac{1}{2(2l+1)}\left[(l+1)P_{l+1}(2\mu \pm 1) \mp (2l+1)P_l(2l \pm 1) + lP_{l-1}(2\mu \pm 1)\right] \quad (4.2.7)$$

If we use this in relation (4.2.5) we find

$$\mu \frac{\partial I^+(r,\mu)}{\partial r} = I(0,0)\left[\mu \phi'(r) + \sum_{l=0}^{L} \frac{I_l^{+'}(r)}{4}\left(\frac{l^2+3l+2}{2l+3}P_{l+2}(2\mu - 1) + \right.\right.$$

$$+ 2(l+1)P_{l+1}(2\mu - 1) + \frac{12l^3 + 18l^2 - 2l - 4}{(2l-1)(2l+1)}P_l(2\mu - 1) +$$

$$\left.\left. + 2lP_{l-1}(2\mu - 1) + \frac{l^2 - l}{2l-1}P_{l-2}(2\mu - 1)\right)\right] \qquad (4.2.8a)$$

and

$$\frac{1-\mu^2}{r}\frac{\partial I^+(r,\mu)}{\partial \mu} = I(0,0)\left[\sum_{l=0}^{L} \frac{I_l^{+'}(r)}{4r}\left(- \frac{(l+1)^2(l+2)}{2l+3}P_{l+2}(2\mu - 1) - \right.\right.$$

$$- (l+1)(3l+2)P_{l+1}(2\mu - 1) + \frac{22l^3 + 33l^2 - 5l - 8}{(2l+3)(2l-1)}P_l(2\mu - 1) +$$

$$+ (3l+l)P_{l-1}(2\mu - 1) + \frac{l^3 - l^2}{2l-1}P_{l-2}(2\mu - 1)\Bigg)\Bigg] \qquad (4.2.8b)$$

and from (4.2.6) the other equations are given by

$$\mu\frac{\partial I^-(r,\mu)}{\partial r} = I(0,0)\Bigg[\mu\phi'(r) + \sum_{l=0}^{L}\frac{I_l^{-'}(r)}{4}\Bigg(\frac{l^2 + 3l + 2}{2l+3}P_{l+2}(2\mu + 1) -$$

$$- 2(l+1)P_{l+1}(2\mu + 1) + \frac{12l^3 + 18l^2 - 2l - 4}{(2l-1)(2l+1)}P_l(2\mu + 1) -$$

$$- 2lP_{l-1}(2\mu + 1) + \frac{l^2 - l}{2l-1}P_{l-2}(2\mu + 1)\Bigg)\Bigg] \qquad (4.2.9a)$$

$$\frac{1-\mu^2}{r}\frac{\partial I^-(r,\mu)}{\partial \mu} = I(0,0)\Bigg[\sum_{l=0}^{L}\frac{I_l^-(r)}{4r}\Bigg(\frac{(l+1)^2(l+2)}{2l+3}P_{l+2}(2\mu + 1) -$$

$$- (l+1)(3l+2)P_{l+1}(2\mu + 1) + \frac{22l^3 + 33l^2 - 5l - 8}{(2l+3)(2l-1)}P_l(2\mu + 1) -$$

$$- (3l + l)P_{l-1}(2\mu + 1) + \frac{l^3 - l^2}{2l - 1}P_{l-2}(2\mu + 1)\Big)\Big] \qquad (4.2.9b)$$

Using (4.2.7a) and (4.2.7b) in the equation (4.2.6a) we get the form of equation of transfer as

$$\mu\phi'(r) + \sum_{l=0}^{L}\frac{I_l^{+'}(r)}{4}\Big[\frac{l^2 + 3l + 2}{2l + 3}P_{l+2}(2\mu - 1) + 2(l + 1)P_{l+1}(2\mu - 1) +$$

$$\frac{12l^3 + 18l^2 - 2l - 4}{(2l - 1)(2l + 3)}P_l(2\mu - 1) + 2lP_{l-1}(2\mu - 1) + \frac{l^2 - l}{2l - 1}P_{l-2}(2\mu - 1)\Big] +$$

$$+ \sum_{l=0}^{L}\frac{I_l^{+}(r)}{4r}\Big[- \frac{(l + 1)^2(3l + 2)}{2l + 3}P_{l+2}(2\mu - 1) - (l + 1)(3l + 2)P_{l+1}(2\mu - 1) +$$

$$\frac{22l^3 + 33l^2 - 5l - 8}{(2l + 3)(2l - 1)}P_l(2\mu - 1) + (3l + l)P_{l-1}(2\mu - 1) + \frac{l^3 - l^2}{2l - 1}P_{l-2}(2\mu - 1)\Big] +$$

$$+ \psi(\mu) + \sum_{l=0}^{L}(2l + 1)I_l^{+}(r)\mu P_l(2\mu - 1) = \frac{1}{2} - \frac{1}{4}\Big[I_0^{+} + I_1^{+} - I_0^{-} + I_1^{-}\Big] \qquad (4.2.10)$$

Similarly the corresponding equation for $I^{-}(\tau,\mu)$ is

$$\mu\phi'(r) + \sum_{l=0}^{L}\frac{I_l^{-'}(r)}{4}\left[\frac{l^2+3l+2}{2l+3}P_{l+2}(2\mu-1) - 2(l+1)P_{l+1}(2\mu-1) + \right.$$

$$\left.\frac{12l^3+18l^2-2l-4}{(2l-1)(2l+3)}P_l(2\mu+1) - 2lP_{l-1}(2\mu+1) + \frac{l^2-l}{2l-1}P_{l-2}(2\mu+1)\right] +$$

$$+ \sum_{l=0}^{L}\frac{I_l^{-}(r)}{4r}\left[\frac{(l+1)^2(3l+2)}{2l+3}P_{l+2}(2\mu+1) + (l+1)(3l+2)P_{l+1}(2\mu+1) + \right.$$

$$\left.\frac{22l^3+33l^2-5l-8}{(2l+3)(2l-1)}P_l(2\mu+1) - (3l+l)P_{l-1}(2\mu+1) - \frac{l^3-l^2}{2l-1}P_{l-2}(2\mu+1)\right] +$$

$$+ \psi(\mu) + \sum_{l=0}^{L}(2l+1)I_l^{-}(r)\mu P_l(2\mu+1) = \frac{1}{2} - \frac{1}{4}\left[I_0^{+} + I_1^{+} - I_0^{-} + I_1^{-}\right] \qquad (4.2.11)$$

Next, we multiply (4.2.10) by $P_l(2\mu - 1)$ integrating over $[0,1]$ and using the orthogonal property of Legendre polynomials in $[0,1]$ we find,

$$\phi'(r)\int_0^1 \mu P_l(2\mu-1)d\mu + \frac{1}{4(2l+1)}\left[\frac{l^2-l}{2l-1}I_{l-2}^{+'} + 2lI_{l-1}^{+'} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_l^{+'} + \right.$$

$$\left. + 2(l+1)I_{l+1}^{+'} + \frac{l^2+3l+2}{2l+3}I_{l+2}^{+'}\right] + \frac{1}{4r(2l+1)}\left[-\frac{l^3-2l^2+1}{2l-1}I_{l-2}^{+} - (3l^2-l)I_{l+1}^{+} + \right.$$

$$\frac{22l^3+33l^2-5l-8}{(2l+3)(2l-1)}I_l^{+} + (3l^2+7l+4)I_{l+1}^{+} + \frac{l^3+5l^2+8l+4}{2l+3}I_{l+2}^{+}\Bigg] + \delta_{0,l} +$$

$$+ \frac{1}{2(2l+1)}\left[lI_{l-1}^{+} + (2l+1)I_l^{+} + (l+1)I_{l-1}^{+}\right] =$$

$$= \delta_{0,l}\left[\frac{1}{2} + \frac{1}{4}\left(I_0^{+} + I_1^{+} - I_0^{-} + I_1^{-}\right)\right] \qquad (4.2.12)$$

Now we multiply (4.2.11) by $P_l(2\mu + 1)$ integrating over [-1,0] and using the orthogonal property of Legendre polynomials in [-1,0] we find,

$$\phi'(r)\int_{-1}^{0}\mu P_l(2\mu+1)d\mu + \frac{1}{4(2l+1)}\Bigg[\frac{l^2-l}{2l-1}I_{l-2}^{-'} - 2lI_{l-1}^{-'} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I_l^{-'} -$$

$$- 2(l+1)I_{l+1}^{-'} + \frac{l^2+3l+2}{2l+3}I_{l+2}^{-'}\Bigg] + \frac{1}{4r(2l+1)}\Bigg[\frac{l^3-2l^2+l}{2l-1}I_{l-2}^{-} - (3l^2-l)I_{l+1}^{-} +$$

$$\frac{22l^3+33l^2-5l-8}{(2l+3)(2l-1)}I_l^{-} - (3l^2+7l+4)I_{l+1}^{-} - \frac{l^3+5l^2+8l+4}{2l+3}I_{l+2}^{-}\Bigg] + \delta_{0,l} +$$

$$+ \frac{1}{2(2l+1)}\left[lI_{l-1}^{-} - (2l+1)I_{l}^{-} + (l+1)I_{l-1}^{-}\right] =$$

$$= \delta_{0,l}\left[\frac{1}{2} + \frac{1}{4}\left(I_0^{+} + I_1^{+} - I_0^{-} + I_1^{-}\right)\right] \qquad (4.2.13)$$

The equations (4.2.12) and (4.2.13) are to be solved subject to the boundary conditions.

4.3. Solution

We consider the case L = 1. Putting successively $l = 0, 1$ we obtain from (4.2.12) and (4.2.13) the following equations

$$\left[\frac{4}{3}I_0^{+'} + 2I_1^{+'}\right] + \frac{2}{r}\left[\frac{4}{3}I_0^{+} + 2I_1^{+}\right] + \left(I_0^{+} + I_1^{+} - I_0^{-} + I_1^{-}\right) = -2 - 2\phi'(r) \qquad (4.3.1)$$

$$\left[I_0^{+'} + \frac{12}{5}I_1^{+'}\right] + \frac{1}{r}\left[-I_0^{+} + \frac{21}{5}I_1^{+}\right] + \left(I_0^{+} + 3I_1^{+}\right) = -\phi'(r) \qquad (4.3.2)$$

$$\left[\frac{4}{3}I_0^{-'} - 2I_1^{-'}\right] + \frac{2}{r}\left[\frac{4}{3}I_0^{-} - 2I_1^{-}\right] - \left(I_0^{+} + I_1^{+} - I_0^{-} + I_1^{-}\right) = 2 + 2\phi'(r) \qquad (4.3.3)$$

$$\left[-I_0^{-'} + \frac{12}{5}I_1^{-'}\right] + \frac{1}{r}\left[I_0^{-} - \frac{21}{5}I_1^{-}\right] + \left(I_0^{-} - 3I_1^{-}\right) = -\phi'(r) \qquad (4.3.4)$$

The boundary condition now becomes

$$\left.\begin{array}{l} I_0^-(R) = 0 \\ I_1^-(R) = 0 \end{array}\right\} \tag{4.3.5}$$

$$\phi(R) = 0 \tag{4.3.6}$$

Let us take the form of $\phi(r)$ as

$$\phi(r) = \alpha + \frac{\beta}{r} + \frac{\gamma}{r^2} + \frac{\delta}{r^3} + e^{kr}\left[\frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3}\right] \tag{4.3.7}$$

where A, B, C, α, β, γ, δ are constants and they will have to be evaluated by means of boundary conditions. Again we know that the net flux is defined by

$$F = 2\left[\int_{-1}^{0} \mu I^-(r,\mu)d\mu + \int_{0}^{1} \mu I^+(r,\mu)d\mu\right]$$

Let us now carry out simple calculations

$$\int_0^1 \mu I^+(r,\mu)d\mu = I(0,0)\left[\frac{\phi(r)}{2} + \frac{1}{2} + \frac{1}{3}I_0^+ + \frac{1}{2}I_1^+\right]$$

and

$$\int_0^1 \mu I^-(r,\mu)d\mu = I(0,0)\left[-\frac{\phi(r)}{2} + \frac{1}{3}I_0^+ - \frac{1}{2}I_1^+\right]$$

Therefore, we find that the flux integral in this case is

$$F = \frac{1}{2}\left[\frac{4}{3}\left(I_0^+ + I_0^-\right) + 2\left(I_1^+ - I_1^-\right) + 2\right]$$

Let us the from of I_l^+, I_l^- as

$$I_i^+ = \frac{\gamma_i^+}{r^2} + \frac{\delta_i^+}{r^3} + e^{kr}\left[f_i^+ + \frac{q_i^+}{r} + \frac{h_i^+}{r^2} + \frac{l_i^+}{r^3}\right] \tag{4.3.8}$$

$$I_i^- = \frac{\gamma_i^-}{r^2} + \frac{\delta_i^-}{r^3} + e^{kr}\left[f_i^- + \frac{q_i^-}{r} + \frac{h_i^-}{r^2} + \frac{l_i^-}{r^3}\right] \tag{4.3.9}$$

Using (4.3.8) and (4.3.9) in (4.3.1) to (4.3.4) and comparing the coefficients of e^{kr} on each side we get following linear equations

$$\left(\frac{4k}{3} + 1\right)f_0^+ + (2k+1)f_1^+ + f_0^- - f_1^- = 0 \tag{4.3.10a}$$

$$(2k+2)f_0^+ + \left(\frac{24k}{5} + 6\right)f_1^+ = 0 \tag{4.3.10b}$$

$$f_0^+ + f_1^+ + \left(-\frac{4k}{3} + 1\right)f_0^- + (2k-1)f_1^- = 0 \tag{4.3.10c}$$

$$(2k-2)f_0^- + \left(-\frac{24k}{5}+6\right)f_1^- = 0 \tag{4.3.10d}$$

The unknowns $f_0^+, f_1^+, f_0^-, f_1^-$ appearing in equations (4.3.10a) to (4.3.10d) will have a nontrivial solution if the determinant of the coefficients is zero. Thus we have D(k) = 0 where

$$D(k) = \begin{vmatrix} \frac{4k}{3}+1 & 2k+1 & 1 & -1 \\ 2k+2 & \frac{24k}{5}+6 & 0 & 0 \\ 1 & 1 & -\frac{4k}{3}+1 & 2k-1 \\ 0 & 0 & 2k-2 & -\frac{24k}{5}+6 \end{vmatrix}$$

This gives $k = 0, 0, \pm 1.8257$. To satisfy the boundary condition we take $k = k_0 = 1.8257$.

We assume

$$\phi(r) = \alpha + \frac{\beta}{r} + \frac{\gamma}{r^2} + \frac{\delta}{r^3} + e^{-k_0 r}\left[\frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3}\right] +$$

$$+ e^{k_0 r}\left[\frac{A'}{r} + \frac{B'}{r^2} + \frac{C'}{r^3}\right] \tag{4.3.11}$$

$$I_i^+(r) = \frac{\gamma_i^+}{r^2} + \frac{\delta_i^+}{r^3} + e^{-k_0 r}\left[f_i^+ + \frac{q_i^+}{r} + \frac{h_i^+}{r^2} + \frac{l_i^+}{r^3}\right] +$$

$$+ e^{k_0 r}\left[f_i^{+'} + \frac{q_i^{+'}}{r} + \frac{h_i^{+'}}{r^2} + \frac{l_i^{+'}}{r^3}\right] \qquad (4.3.12)$$

and

$$I_i^- = \frac{\gamma_i^-}{r^2} + \frac{\delta_i^-}{r^3} + e^{k_0 r}\left[f_i^- + \frac{q_i^-}{r} + \frac{h_i^-}{r^2} + \frac{l_i^-}{r^3}\right] +$$

$$+ e^{k_0 r}\left[f_i^{-'} + \frac{q_i^{-'}}{r} + \frac{h_i^{-'}}{r^2} + \frac{l_i^{-'}}{r^3}\right] \qquad (4.3.13)$$

We will now determine various constants based upon the boundary conditions. We will compare different parts,for example, if we compare $\frac{1}{r^2}$ we get following equations

$$\gamma_0^+ + \gamma_1^+ + \gamma_0^- - \gamma_1^- = 2\beta \qquad (4.3.14a)$$

$$\gamma_0^+ + 3\gamma_1^+ = \beta \qquad (4.3.14b)$$

$$\gamma_0^- - 3\gamma_1^- = \beta \qquad (4.3.14c)$$

Similarly if we compare different powers such as $\frac{1}{r^3}, \frac{1}{r^4}$ we get different constants. The results are given below.

$$\beta = \frac{3}{4}F, \qquad \gamma = 0, \qquad \delta = -\frac{3}{16}F$$

$$\gamma_0^+ = \gamma_0^- = \frac{3}{4}F, \qquad \gamma_1^+ = \gamma_1^- = 0$$

$$\delta_0^+ = -\frac{9}{16}F = -\delta_0^-, \qquad \delta_1^+ = \frac{15}{16}F = \delta_1^-$$

$$f_0^+ = .3289C, \quad f_0^- = .6130C, \quad f_1^+ = -.5492C, \quad f_1^- = 1.5983C$$

$$g_0^+ = -3.9676C, \quad g_0^- = 13.8767C, \quad g_1^+ = -21.5651C, \quad g_1^- = 6.5733C$$

$$h_0^+ = -.2620C, \quad h_0^- = -6.1635C, \quad h_1^+ = -.2620C, \quad h_1^- = 7.0764C$$

$$f_0^{+'} = -27.0989B', \quad f_0^{-'} = -31.8869B', \quad f_1^{+'} = 26.3152B', \quad f_1^{-'} = 2.5503B'$$

$$g_0^{+'} = -9.5739B', \quad g_0^{-'} = -4.9256B', \quad g_1^{+'} = -12.6921B', \quad g_1^{-'} = -12.1232B'$$

$$h_0^{+'} = 12.7900B', \quad h_0^{-'} = .3637B', \quad h_1^{+'} = -5.8900B', \quad h_1^{-'} = 0.8637B'$$

$$l_0^{+} = l_0^{-} = 3C, \qquad l_1^{+} = l_1^{-} = -5C$$

$$l_0^{+'} = -l_0^{-'} = -1.6431B', \qquad l_1^{+'} = 1_1^{-'} = -2.7385B'$$

$$A' = -9.2467B', \qquad C' = -.5477B'$$

The rest of the constants α, A, B$'$ can be determined form the boundary conditions.