INTRODUCTION

1. INTRODUCTION

1.1 The equation of radiative transfer

We shall derive the fundamental equation which governs the variation of intensity in a medium characterized by an absorption coefficient k_ν and an emission coefficient j_ν . We consider a small cylindrical element of cross – section $d\sigma$ and length ds in the medium. From the definition of intensity, it follows that the difference in the radiant energy in the frequency interval { $\nu, \nu+d\nu$ } crossing the two faces normally, in a time dt and confined to an element of solid angle $d\omega$, is given by

$$\frac{dI_\nu}{d\sigma} ds\, d \quad d \quad d \quad dt.$$

The difference in energy must arise from the excess of emission over absorption in the frequency interval and element of solid angle considered. The amount absorbed is

$$k_\nu \rho\, ds\, I_\nu d \quad d \quad d \quad dt$$

while the amount emitted is $j_\nu \rho\, d\sigma\, ds\, d\nu\, d\sigma\, d\omega\, dt$ where ρ is the density, of the medium. Counting up the gains and losses in the pencil of radiation during traversal of the cylinder, we have

$$\frac{dI_\nu}{ds} = -k_\nu . \rho I_\nu + j_\nu \rho \qquad (1.1.1)$$

In terms of the source function F_ν we can rewrite this equation in the form

$$-\frac{dI_\nu}{k_\nu \rho\, ds} = I_\nu - F_\nu \qquad (1.1.2)$$

This is the basic equation of radiative transfer for the flow of radiation through the outer layers of the star governing the radiation field in a medium which absorbs, emits and scatters radiation. Due to the functionally dependency of source function on the intensity at a point, the equation of transfer is generally an integro – differential equation.

1.2 Equation of transfer in different media and geometries

(1) Plane – parallel medium

Here medium is considered to be stratified in planes perpendicular to OZ–axis. The radiative properties in each plane is uniform. We define optical depth t by

$$t=\int_{s}^{\infty} k_{\nu}\rho\, ds \qquad (1.2.1)$$

where s is the height of the medium. Below there are some cases in this geometries

(a) Local thermodynamic equilibrium with no scattering.

In this case Kirchoff's law holds and the source function F_{ν} is given by

$$F_{\nu}=B_{\nu}(T) \qquad (1.2.2)$$

where $B_{\nu}(T)$ is Planck function given by

$$B_{\nu}(T)=\frac{2h\nu^{3}}{c^{2}}.\frac{1}{\exp(\frac{h\nu}{kT})-1} \qquad (1.2.3)$$

where h is the Planck constant, k the Boltzmann constant and T is the characteristic temperature. The equation of transfer in this case is

$$-\mu\frac{dI(t_\nu,\mu)}{dt_\nu}=I(t_\nu,\mu)-B_\nu(T) \qquad (1.2.4)$$

where $t_\mu=\int_s^\infty k_\nu \rho\, ds$ is the optical depth and $\mu=\cos\theta,\theta$ being the angle the pencil of incident radiation makes with the onward drawn normal from an element of area $d\sigma$

(b) Medium where the scattering is isotropic :

Here the source function take the form

$$F_\nu=\frac{1}{2}\int_{-1}^{1} I(t,\mu')d\ ' \qquad (1.2.5)$$

and the equation of transfer has the form

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-\frac{1}{2}\int_{-1}^{1} I(t,\mu')d\ ' \qquad (1.2.6)$$

(c) In the case of scattering medium, source function is

$$F_\nu=\frac{\omega}{2}\int_{-1}^{1} p(\mu,\mu')I(t,\mu')d\ ' \qquad (1.2.7)$$

where $p(\mu,\mu')$ is the phase function which governs the directional distribution of intensity, ω is the albedo of single scattering which is equal to 1 for single scattering (conservative case) and equal to zero for the case of true absorption. In nonconservative case $\omega<1$ and in the case of neutron transport $\omega>1$. If the scattering is isotropic

$p(\mu,\mu')=1$ and in anisotropic scattering $p(\mu,\mu')$ has different well known forms e.g.

$$p(\mu,\mu') = (1+x\mu\mu') \text{ Planetary scattering}$$

$$= 1+\frac{1}{2}P_2(\mu)P_2(\mu') \text{ Rayleigh scattering}$$

$$= 1+\frac{\alpha}{3}P_2(\mu)P_2(\mu') \text{ Pomraning phase function,}$$

$$\alpha=\frac{5}{5-3\lambda} \quad = \Sigma_{k=0}^{\infty}\omega_k P_k(\mu)P_k(\mu') \text{ General phase function,}$$

$$= 1+b_0P_4(\mu) \text{ Carlstedt and Mullikin's phase function,}$$

$$= 1+3gP_1(\mu)P_1(\mu')+5g^2P_2(\mu)P_2(\mu')+7g^3P_3(\mu)P_3(\mu')$$

Henyey – Greenstein phase function,

1.3 <u>The Wiener – Hopf Technique</u>

The Wiener – Hopf method was first employed, in a joint study by N. Wiener and E. Eopf (1931) in the solutions of integral equations with a difference Kernel in the case of a semi – infinite interval:

$$U(x)=\lambda\int_0^\infty v(x-s)U(s)ds+f(x) \qquad (1.3.1)$$

Consequently equations of this kind were considered by V. A. Fock (1944) who made a substantial contribution to the development of general methods of their solutions.

The Wiener - Hopf method was designed by its authors as a means of obtaining an explicit formula for the solution of homogeneous equations such as the Milne equation.

$$J(t)=\int_0^\infty J(\tau)k_1(|\tau-t|)\ d\tau \qquad (1.3.2)$$

The method has been used for investigating the solutions of equations whose kernels are subject to less restrictive conditions than $k_1(\tau)$

The general method of solving functional equations which become known as the Wiener - Hopf method or factorization method (Hopf, 1934) has been successively used in the solution of many problems of diffraction and the theory of elasticity, of boundary value problems involving the heat conduction equation, of integral equation of the theory of radiative transport and many other problems of mathematical physics.

By various ingeneous devices, atomic physicists have used the Wiener – Hopf method as a means of solving particular cases of the non – homogeneous equation (Davison B., 1957; Elliot J. P., 1955).

A modified version of the method was given by Titchmarsh (1957) and it is this form which has been widely used by atomic physicists. Titchmarsh employed Fourier transforms of the form

$$F^+(\tau)=\frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\exp(ix\tau)\ dx$$

$$F^-(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 f(x)\exp(ixt)\ dx \qquad (1.3.3)$$

where τ is a complex variable. Since, however, Laplace transforms arise naturally in transport theory, the most satisfactory treatment seems to be to follow Titchmarsh's method.

In the general case a problem solvable by the Wiener – Hopf technique (Sveshnikov and Tikhonov 1973; Parton and Perlin 1984) reduces to the following.

It is required to determine functions $\Psi_+(k)$ and $\Psi_-(k)$ of a complex variable k, which are analytic respectively in the half planes $Imk > \tau_-$ and $Imk < \tau_+ (\tau_- < \tau_+)$ and tend to zero as $|k| \to \infty$ in both domains of analyticity and satisfy in this strip $\tau_- < Imk < \tau_+$ the functional equation

$$A(k)\Psi_+(k) + B(k)\Psi_-(k) + C(k) = 0 \qquad (1.3.4)$$

The functions $A(k), B(k), C(k)$ are given functions of the complex variable k regular in the strip τ_- < Im(k) < τ_+ . For simplicity we assume that $A(k), B(k), C(k)$ are non – zero in the strip.

The following statement must be proved in order to arrive at a solution of the factorization problem.

Let $F(z)$ be an analytic function in the interval, satisfying the estimate

$$|F(x+iy)| < C|x|^{-p} (p > 0 \; for \; x \to \infty) \qquad (1.3.5)$$

We assume that the inequality (1.3.5) is satisfied uniformly for all values of y in the interval $y_- + \epsilon \le y \le y_+ - \epsilon (\epsilon \ge 0)$. Then the following representation is satisfied for all values of y in the interval $y_{-}<c<y<d<y_{+}$;

$$F(z)=F_{+}(z)+F_{-}(z) \qquad (1.3.6)$$

where the functions $F_{+}(z)$ and $F_{-}(z)$ are analytic in the half planes $y>y_{-}$ and $y<y_{+}$ respectively.

In order to prove this let us define a sufficiently large number A and consider the Catchy integral

$$F(z)=\frac{1}{2\pi i}\int_{-A+ic}^{A+ic}\frac{F(t)}{(t-z)}dt+\frac{1}{2\pi i}\int_{A+ic}^{A+id}\frac{F(t)}{(t-z)}dt$$
$$+\frac{1}{2\pi i}\int_{A+id}^{-A+id}\frac{F(t)}{(t-z)}dt+\frac{1}{2\pi i}\int_{-A+id}^{-A+ic}\frac{F(t)}{(t-z)}dt \qquad (1.3.7)$$

It follows from the estimate (1.3.5) that with increasing A, the integrals over the vertical parts tend to zero.

In the limit we get

$$F(z)=\frac{1}{2\pi i}\int_{-\alpha+ic}^{\alpha+ic}\frac{F(t)}{(t-z)}dt+\frac{1}{2\pi i}\int_{\alpha+id}^{-\alpha+id}\frac{F(t)}{(t-z)}dt \qquad (1.3.8)$$
$$=F_{-}(z)+F_{+}(z)$$

which is the required representation, since c and d are arbitrarily chosen.

Let us assume that the function $M(z)=\frac{\phi(z)}{N(z)}$ which is to be factorized in the form

$$M(z)=M_+(z)M_-(z) \qquad (1.3.9)$$

does not have any zeros in the interval $\tau_- < Im\,(z) < \tau_+$ and tends to unity as $x \to \infty$. In this case, neither of the functions $M_+(z)$ and $M_-(z)$ will have any zero, and we can take the logarithm of both sides of the equation (1.3.9).

$$\log M(z)=\log M_+(z)+\log M_-(z) \qquad (1.3.10)$$

The function $F(z)=\log M(z)$ satisfies the condition (1.3.5) and hence the relation (1.3.10) can always be solved with the help of the representation (1.3.8).

Finally, we get

$$M(z)=e^{F_+(z)}.e^{F_-(z)} \qquad (1.3.11)$$

If the function $M(z)$ has zeros in the interval, we must consider a new function:-

$$M_1(z)=(z^2+b^2)^{\frac{N}{2}}\frac{M(z)}{\Pi_{i=1}^{N}(z-z_i)^{\alpha_i}} \qquad (1.3.12)$$

where z_i and α_i are the zeros, their multiplicity in the interval is given by $N_1 \leq N$, where N is the total number of zeros, $b > max\ (\tau_-, \tau_+)$. The factor in the numerator of (1.3.12) ensures that the properties of the auxiliary function were conserved at infinity.

Let us now consider the relation (1.3.6) and carry out the factorization into $L_+(k)$ and $\frac{1}{L_-(k)}$ for same interval of the ratio $\frac{A(k)}{B(k)}$.

Sometimes $L_-(k)$ and $L_+(k)$ can be found by inspection but in any case for the $A(k)$ and $B(k)$ which occur in our problem can always be found. Using the relation $\frac{A(k)}{B(k)}=\frac{L_+(k)}{L_-(k)}$ (1.3.13)

we arrange (1.3.4) in the form,

$$L_+(k)\Psi_+(k)+L_-(k)\Psi_-(k)+L_-(k)\frac{C(k)}{B(k)}=0 \quad (1.3.14)$$

The expression $L_-(k)\frac{C(k)}{B(k)}$ can be decomposed in the following form in accordance with (1.3.6):

$$L_-(k)\frac{C(k)}{B(k)}=D_+(k)+D_-(k) \quad (1.3.15)$$

where the functions $D_-(k)$ and $D_+(k)$ are regular in the half planes $\tau_-'' < Im\,k$ and $Imk < \tau_+''$ respectively and all three strips $\tau_- < Im\,k < \tau_+$ $\tau_-' < Im\,k < \tau_+'$ and $\tau_-'' < Im\,k < \tau_+''$ have a common portion – the strip $\tau_-^0 < Im\,(k) < \tau_+^0$.

With the help of (1.3.15) we can put (1.3.14) in the form

$$L_+(k)\Psi_+(k)+D_+(k)=-L_-(k)\Psi_-(k)-D_-(k) \quad (1.3.16)$$

It follows from the generalized Liouville's theorem that the left as well as the right hand sides of (16) represent the same polynomial J(k). We have

$$L_+(k)\Psi_+(k)+D_+(k)=-L_-(z)\Psi_-(k)-D_-(z)=J(k) \quad (1.3.17)$$

So far this equation defines J(k) only in the strip $\tau_-^0<Im\,k<\tau_+^0$. But the first part of the equation is defined and is regular in $\tau_-^0<Im\,k$,and the second part is defined in the half plane $Im\,k<\tau_+^0$. Hence by analytic continuation we can define $J(k)$ over the whole k – plane and $J(k)$ is regular in the whole k – plane. Now suppose that it can be shown that

$$|L_+(k)\Psi_+(k)+D_+(k)|<|k|^p \quad as\, k\to\infty,\, Im\,k>\tau_-^0$$

and

$$|L_-(k)\Psi_-(k)+D_-(k)|<|k|^q \quad as\, k\to\infty,\, Im\,k<\tau_+^0$$

Then by extended form of Liouville's theorem $J(k)$ is a polynomial $P(k)$ of degree less than or equal to the integral part of min (p,q) i.e.

$$\left.\begin{aligned} L_+(k)\Psi_+(k)+D_+(k)&=P(k) \\ L_-(k)\Psi_-(k)+D_-(k)&=-P(k) \end{aligned}\right\} \quad (1.3.18)$$

Equations (1.3.18) determine $\Psi_+(k)$ and $\Psi_-(k)$ to within the arbitrary polynomial $P(k)$ i.e. to within a finite number of arbitrary constants which must be determined from supplementary conditions of the problem.

Thus the use of Wiener – Hopf technique is based on the representations (1.3.13) and (1.3.15). Numerous examples involving the Wiener – Hopf technique have been given by B. Noble (1958).

Although the possibility of factorization has in principle been established above, the general method leads to quite cumbersome results. At the same time factorization in some cases is directly considered on account of the relative simplicity of the initial problem. This forms the basis for the construction of effective algorithms by means of approximate but explicit factorization.

Factorization may be accomplished when the method of integral transforms is used for solving boundary value problems of mathematical physics for a half plane in which the boundary conditions are different in different regions of the boundary. Also the method is quite effectively used for solving a certain class of integral equations called the Wiener – Hopf equations.

The method of integral transforms is one of the most effective methods for solving differential and integral equations.

The application of the method of integral transform to differential equations results in the lowering of number of variables of an equation by one, while equations can be directly related to algebraic equations with the help of this method. There are several forms of integral transforms and one form may be obtained from the other by a transformation of the co-ordinate and the functions. The choice of an integral transform depends on the structure of the equation and the geometry of the domain.

1.4 Previous works

DasGupta (1957 considered the anisotropic scattering of mono – energetic neutrons diffusing without change of energy through a large slab of non–capturing medium having a uniformly distributed source of constant strength and took the transport equation in the form

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-[1-c(3\mu^2-1)]J(t)-b\mu F(t)-3c(3\mu^2-1)L(t)-p_0 \quad (1.4.1)$$

where

$$J(t)=\frac{1}{2}\int_{-1}^{1} I(t,\mu)d \quad (1.4.2)$$

$$F(t)=\frac{1}{2}\int_{-1}^{1} \mu I(t,\mu)d \quad (1.4.3)$$

$$L(t)=\frac{1}{2}\int_{-1}^{1} \mu^2 I(t,\mu)d \quad (1.4.4)$$

with boundary condition

$$I(0,\mu')=0,0<\mu'\leq 1 \quad (1.4.5)$$

and $$I(t,\mu)e^{-t/\mu}\to 0 \quad as \quad t\to\infty \quad (1.4.5a)$$

DasGupta solved the problem for emergent neutron distribution $I(0,\mu)$ by Laplace transform and Wiener Hopf technique.

The formal solution of (1.4.1) was found in the form

$$I(0,\mu)=[I-c(3\mu^2-1)]J(\frac{1}{\mu})+b\mu F(\frac{1}{\mu})+3c(3\mu^2-1)L(\frac{1}{\mu})+p_0 \quad (1.4.6)$$

The Laplace transform of $f(t)$ is defined by

$$f^*(s)=s\int_0^{\infty} e^{-st}f(t)dt, \quad Res>0 \quad (1.4.7)$$

DasGupta applied Laplace transform to (1.4.1) and put $s=\frac{1}{z}$ and obtained

$$P(z)I(0,z)=Q(z)-G(z)+P_0Q(z)-cp_0-abz[-p_0z+n_0+p_0h]$$
$$-3c[-(\frac{p_0}{3})(3-b)z^2+\frac{1}{3}(3-b)(n_0+p_0h)z-p_0z+L(0)] \quad (1.4.8)$$

where

$$P(z)=T(z)[1-c(3z^2-1)]-c \quad (1.4.9)$$

$$T(z)=1-\frac{1}{2}z\log\frac{z+1}{z-1} \quad (1.4.10)$$

$$Q(z)=[1-c(3z^2-1)]-c \quad (1.4.11)$$

$$G(z)=\frac{1}{2}\int_0^1 \frac{xI(0,x)dx}{x-z} \quad (1.4.12)$$

T (z) satisfies the relation (Chandrasekhar, 1960)

$$T(z)=\frac{1}{H(z)H(-z)} \quad (1.4.13)$$

Splitting (1.4.8) he obtained

$$3c(Y^2-z^2)\frac{I(o,z)}{H(z)}=H(-z)3c(Y^2-z^2)G(z)+p_0-3c[L(0)+(n_o+p_o)z] \quad (1.4.14)$$

The right hand side of (1.4.14) tends to a polynomial of degree two as $z\to\infty$. Hence by a modified form of Liouville's theorem both sides of (1.4.14) were equal to a polynomial $K(z)=K_0+K_1z+K_2z^2$

Thus, $I(0,z)=\frac{H(z)K(z)}{3c(Y^2-z^2)}$ (1.4.15)

and

$$3c(Y^2-z^2)G(z)H(-z)+P_0H(-z)-3cL(0)H(-z)-3c(n_0+p_0)zH(-z)=K(z)$$

(1.4.16)

The constants $K_{0,}\ K_{1,}\ K_2$ were found from boundary conditions.

DasGupta (1958a) considered the transfer equation in the Milne – Eddington model for the conservative case of isotropic scattering of radiation in the outer layers of a star in the form

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-J(t) \quad (1.4.17)$$

with boundary conditions

$$I(0,\mu')s=0, 0<\mu'\leq 1 \quad (1.4.18)$$

and $I(t,\mu)e^{-t/\mu}\to 0 \quad as\ t\to\infty$ (1.4.19)

where $J(t)=\frac{1}{2}\int_{-1}^{1}I(t,\mu)d\mu$ (1.4.20)

DasGupta obtained the formal solution of (1.4.17) in the form

$$I(0,\mu) = J(\frac{1}{\mu}) \quad (1.4.21)$$

$$= CH(\mu) \quad (1.4.22)$$

where C is Constant

Subjecting (1.4.17) to Laplace transform and putting $s=\frac{1}{z}$ DasGupta obtained

$$T(z)H(z) = G(z) \tag{1.4.23}$$

where

$$T(z)=1-\frac{1}{2}z\log\frac{z+1}{z-1} \tag{1.4.24}$$

and

$$G(z)=\frac{1}{2}\int_0^1 \frac{xI(0,x)}{x-z}dx \tag{1.4.25}$$

The splitting of equation (1.4.23) resulted in

$$\frac{2}{\sqrt{3}}L_-(z)D_-(z)H(z)=\frac{\sqrt{3}}{2}G(z)D_+(z)L_+(z) \tag{1.4.26}$$

where
$$\log D_+(z)=\frac{1}{2\pi i}\int_{C_+}\log\frac{D(\omega)}{(\omega-z)}d\omega \tag{1.4.27}$$

$$\log D_-(z)=\frac{1}{2\pi i}\int_{C_-}\log\frac{D(\omega)}{(\omega-z)}d\omega \tag{1.4.28}$$

$$D(z)=\frac{4}{3}\frac{D_-(z)}{D_+(z)} \tag{1.4.29}$$

$$L_+(z)=1+z\log\frac{(z-1)}{z} \tag{1.4.30}$$

$$L_{-}(z)=1-z\log\frac{(z+1)}{z} \qquad (1.4.31)$$

Integration along c_+ is anticlockwise and that along c_- is clockwise.

The right hand side of (1.4.26) tends to a constant as $z\rightarrow\infty$. Hence by the principle of analytic continuation and by a modified form of Liouville's theorem, both sides of (1.4.26) was equated to a constant K.

Thus Dasgupta obtained

$$H(z)=(\frac{2}{\sqrt{3}})\frac{K}{D_{-}(z)L_{-}(z)} \qquad (1.4.32)$$

and

$$G(z)=(\frac{2}{\sqrt{3}})\frac{K}{D_{+}(z)L_{+}(z)} \qquad (1.4.33)$$

Finally, the results $D_{-}(0)=\frac{\sqrt{3}}{2}$ and $D_{-}(0)=\frac{\sqrt{3}}{2}K$ gave $K=1$

DasGupta (1958 b) considered the transfer equation for the conservative case of grey scattering according to Rayleigh Phase function in outer layers of a star in the Milne – Eddington model in the form

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-(\frac{3}{8})(3\mu^2-1)J(t)-\frac{3}{8}(3\mu^2-1)K(t) \qquad (1.4.34)$$

where

$$J(t)=\frac{1}{2}\int_{-1}^{1}I(t,\mu)d\mu \qquad (1.4.35)$$

$$K(t)=\frac{1}{2}\int_{-1}^{1}\mu^2 I(t,\mu)\,d\mu \qquad (1.4.36)$$

with boundary conditions

$$I(0,\mu')=0, 0<\mu'\leq I \qquad (1.4.37)$$

and $$I(t,\mu)e^{-t/\mu}\to 0 \quad as \ t\to\infty \qquad (1.4.38)$$

DasGupta applied Laplace transform and Wiener – Hopf technique to solve the problem.

The formal solution of (1.4.34) was found in the form

$$I(0,\mu)=\frac{3}{8}(3-\mu^2)J(\frac{1}{\mu})+(3\mu^2-1)[F(0)\mu+K(0)] \quad (1.4.39)$$

The Laplace transform of $f(t)$ is defined by

$$f^*(s)=s\int_0^\infty e^{-st}f(t)\,dt, \ Res>0 \qquad (1.4.40)$$

DasGupta applied Laplace transform to (1.4..4) and obtained

$$P(z)I(0,z)=\frac{3}{8}(3-z^2)G(z)-\frac{3}{8}[F(0)+K(0)] \qquad (1.4.41)$$

where

$$P(z)=1-\frac{3}{16}[2z^2+(3z-z^3)\log\frac{z+1}{z-1}] \qquad (1.4.42)$$

$$G(z)=\frac{1}{2}\int_0^1 \frac{I(o,x)}{(x-z)}dx \tag{1.4.43}$$

The splitting of the equation (1.4.43) resulted in

$$\frac{6}{5}D_-(z)L_-(z)I(0,z)=\frac{3}{8}\frac{D(z)}{D_+(z)}[(3-z^2)G(z)-F(0)z-K(0)] \tag{1.4.44}$$

where

$$\log D_-(z)=\frac{1}{2\pi i}\int_{c+}\log\frac{D(\omega)}{(\omega-z)}d\omega \tag{1.4.45}$$

$$\log D_+(z)=\frac{1}{2\pi i}\int_{c-}\log\frac{D(\omega)}{(\omega-z)}d\omega \tag{1.4.46}$$

$$D(z)=\frac{P(z)}{L_+(z)L_-(z)}=\frac{6}{5}\frac{D_-(z)}{D_+(z)} \tag{1.4.47}$$

$$L_+(z)=1+z\log\frac{(z-1)}{z} \tag{1.4.48}$$

$$L_-(z)=1-z\log\frac{(z+1)}{z} \tag{1.4.49}$$

By the principle of analytic continuation and modified form of Liouville's theorem both sides of equation (1.4.44) was equated to a polynomial

$$A(z)=A_0+A_1z+A_2z^2$$

Hence

$$I(0,z)=(\frac{5}{6})^{\frac{1}{2}}\frac{A(z)}{D_-(z)L_-(z)} \tag{1.4.50}$$

where the constants were obtained by substituting (1.4.50) in (1.4.46) and equating the coefficients of equal powers of z.

DasGupta (1977b) considered the equation of transfer for coherent scattering in stellar atmosphere in the form

$$\mu \frac{dI_\nu(t,\mu)}{dt} = I_\nu(t,\mu) - \omega J_\nu(t) - (1-\omega) B_\nu(t) \qquad (1.4.51)$$

where the Planck's function $B_\nu(t)$ is given (Kourganoff, 1963) by

$$B_\nu(t) = b_0 + b_1 t + \Sigma_{r=2}^{n} a_r E_r(t) \qquad (1.4.52)$$

$$E_r(t) = \int_1^\infty \left(\frac{e^{tx}}{x^r}\right) dx \qquad (1.4.53)$$

$$0 < (1-\epsilon_\nu)\eta_\nu(1+\eta_\nu) = w < 1, \qquad (1.4.54)$$

$$\eta_\nu = \frac{l_\nu}{k}, 0 < \epsilon_\nu < 1 \qquad (1.4.55)$$

I_ν, k are the line and continuous absorption coefficients, t the optical depth in the total absorption coefficient, ϵ_ν the collision constant, $I_\nu(t,\mu)$ is the intensity in the frequency ν, in the direction $\cos^{-1}\mu$. $J_\nu(t)$ is the average intensity and

$$J_\upsilon(t) = \frac{1}{2}\int_{-1}^{1} I_\upsilon(t,\mu) d\mu \qquad (1.4.56)$$

DasGupta solved the problem for emergent intensity $I_\nu(0,\mu)$ by the method of Laplace transform and Wiener – Hopf technique. He used the new form of the H – function (DasGupta., 1974) and obtained

$$I_\nu(0,z)=H(z)[K\frac{c(z)-N^-(z)}{(K-z)}-(1-\omega)\Sigma^n_{r=2}\ a_r F^-_r(z)] \qquad (1.4.57)$$

Woolley and Stibbs (1953) have discussed the transfer equation for non – coherent scattering arising form interlocking of principal lines without redistribution and they have obtained solution based on Eddington's approximation. Bus-bridge and Stibbs (1954a) have solved the equations for the emergent intensities by the Principle of invariance. DasGupta (1956) have solved the same problem by L.T and W.H.T. technique. Siewert and Ozisik (1969) solved the problem for the line intensities at any optical depth by the method of singular eigenfunction expansion, developed by Case (1960). DasGupta and Karanjai (1972) have applied Sobolev's method of 'Probability of quantum exit from the medium to get an exact alternative form of emergent intensities in different lines.

DasGupta (1978a) considered the equation of transfer (Woolley and Stdbbs, 1953) for non – coherent scattering arising from interlocking of principle lines without redistribution, in the form

$$\mu\frac{dI_r(t,\mu)}{dt}=n_r\ I_r\ (t,\mu)-\delta_r B(t)-a_r\ \Sigma^N_{p=1}\ \eta_p J_p(t) \qquad (1.4.58)$$

where $\eta_r=\frac{k_r}{k}$, k_r is the absorption coefficient for the rth line, k is the continuous absorption coefficient which is assumed to be same for each line and t is the optical depth,

$$n_r = 1+\eta_r = \frac{1}{\lambda_r}, 1+\epsilon\eta_r = \delta_r, \frac{(1-\epsilon)\eta_r}{\Sigma_{p=1}^{N}\eta_p} = a_r$$

DasGupta have considered ϵ and the Planck's function $B(t)$ to be Linear i.e.

$$B(t) = b_0 + b_1 t$$

where $b_{0,}b_1$ are known constants.

He has defined Laplace transform of $f(t)$ by

$$f^*(s) = s\int_0^{\alpha} f(t)e^{-st}\,dt, Re\, s>0$$

The boundary conditions are

$I_v(0,\mu)=0, -1\leqslant\mu<0$ and $I_v(t,\mu)e^{-\frac{t}{\mu}}\to 0 \;\; as \;\; t\to\infty$

DasGupta applied the Laplace transform and Wiener – Hopf technique to solve the problem and deduced the emergent intensity $I_v(0,z)$ in the form

$$I_v(0,z) = H(\lambda_r z) \;\; [\{ \; kC(\lambda_r z)+\lambda_r b(\lambda_r z)(A_0-\lambda_r H_0 z).\frac{F_r^-(\lambda_r z)}{(k-\lambda_r z)} \; \} \; -D_r^-(\lambda_r z) \;]$$

(1.4.59)

He obtained the emergent intensity $I_v(0,z)$ in the case of coherent scattering in the form

$$I_v(0,z)=H(z,w)b_1[\frac{(1-w)^{\frac{1}{2}}}{(1+\epsilon\eta_r)}].[z+(1+\eta_r)\frac{b_0}{b_1}+\frac{(a_1 w)}{2}(1-w)^{-\frac{1}{2}}] \quad (1.4.60)$$

which is the same as Chandrasekhar's solution (Chandrasekhar, 1960).

DasGupta (1978b) developed a new method, combined with Laplace Transformation and Wiener – Hopf technique to obtain unique solution of transport equation in finite media. He developed a new technique for obtaining coupled linear singular integral equations peculiar to finite atmosphere problems of radiative transfer or diffusion. He considered the problem of diffuse reflection by a plane parallel atmosphere with axial symmetry scattering radiation with moderate anisotropy. He considered the transport equation (Chandrasekhar, 1960) in the form

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-(\frac{1}{2})\int_{-1}^{1}p^0(\mu,\mu')I(t,\mu')d\mu'-\frac{F}{4}e^{-\frac{t}{\mu_0}}p^0(\mu,-\mu_0) \quad (1.4.61)$$

where

$$p^0(\mu,\mu')=\omega_0+\omega_1\mu\mu' \quad (1.4.62)$$

with

$$0<\omega_0\leq 1, |\omega_1|<3\omega_0 \quad (1.4.63)$$

$\omega_0=1$ means that the atmosphere is purely scattering and $w_0=0$ means that there is no scattering. ω_1 is a measure of the deviation of the scattering function from isotropy.

Boundary conditions are

$$I(0,-\mu)=0,\ 0<\mu\leq 1 \quad (1.4.64)$$

$$I(t_{0,}+\mu)=0,\ 0<\mu\leq 1 \tag{1.4.65}$$

Equation (1.4.61) takes the form

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-\omega_0 J_0(t)-\omega_1 J_1(t)-P(\mu)h(t,\mu_0) \tag{1.4.66}$$

where

$$J_0(t)=(\frac{1}{2})\int_{-1}^{1} I(t,\mu)d\mu \tag{1.4.67}$$

$$J_1(t)=(\frac{1}{2})\int_{-1}^{1} \mu(0,\mu)d \tag{1.4.68}$$

$$p(\mu)=\frac{F}{4}(w_0-w_1\mu_0\mu) \tag{1.4.69}$$

$$h(t,\mu_0)=e^{-\frac{t}{\mu_0}} \tag{1.4.70}$$

DasGupta obtained the following three coupled integral equations

$$\begin{aligned}T(z)U(z)=&Q(z)g_\mu^+(z)+h(t_{0,}z)Q(z)g_\nu^-(z)+\frac{\mu_0 f(z)}{(\mu_0+z)}\\ &-q\,f(z)\frac{h(t_0,z)}{(\mu_0+z)}+\frac{1}{2}[u_1-\nu_1 h(t_0,z)]z\,\omega_1(1-\omega_0)\end{aligned} \tag{1.4.71}$$

$$\begin{aligned}T(z)V(z)=&Q(z)g_\nu^+(z)+h(t_{0,}z)Q(z)g_\mu^-(z)+\mu_0 f(-z)\frac{h(t_0,z)}{(\mu_0-z)}\\ &-\frac{q f(-z)}{(\mu_0-z)}-\frac{1}{2}[u_1 h(t_0,z)-\nu_1]z\,\omega_1(1-\omega_0)\end{aligned} \tag{1.4.72}$$

Das (1978b) considered the radiation transport equation for an axially symmetric Raleigh scattering problem in a semi – infinite planetary atmosphere. Four stake's parameters $I_l(t,\mu), I_r(t,\mu), U(t,\mu)$ an $V(t,\mu)$, the four components of a column matrix $I(t,\mu)$, characterize the radiation field. As the radiation field axially symmetric, the two components $U(t,\mu)$ and $V(t,\mu)$ vanish. Das considered the scalar equations for $I_l(t,\mu)$ and $I_r(t,\mu)$, (Chandrasekhar, 1960) in the forms

$$\frac{\mu dI_l(t,\mu)}{dt}=I_1(t,\mu)-(\frac{1}{4})\ [2\{\ I_{10}(t)-I_{12}(t)\ \}+\ U_2\ \{\ 3I_{r2}(t)-I_{10}(t)\ \}] \tag{1.4.73}$$

$$\mu\frac{dI_r(t,\mu)}{dt}=I_r(t,\mu)-(\frac{1}{4})\ \{\ I_{r2}(t)-I_{r0}(t)\ \} \tag{1.4.74}$$

where t is the optical depth and u is the direction parameter and

$$I_{\xi m}(t)=\int_{-1}^{1} u^m I_\xi(t,u)\,du;\qquad \xi=l,m;m=0,1,2 \tag{1.4.75}$$

The boundary conditions for solving (1.4.73) and (1.4.74) are

$$I_\xi(0,-u)=0,0<u\leq 1 \tag{1.4.76}$$

$$I_\xi(t,u)e^{-\frac{t}{u}}\rightarrow 0\ as,t\rightarrow\infty \tag{1.4.77}$$

Chandrasekhar (1946, 1947) solved the problem by his discrete ordinate method and obtained the n th approximated emergent intensity and the intensity at any optical depth in a semi – infinite plane – parallel atmosphere. He obtained an exact expression of emergent intensity by letting n tend to infinity, Siewert and Frayley (1967) solved the same problem by using Case's eigen function expansion method (Case, 1960). Domke (1971) considered this problem of conservative Rayleigh scattering in a semi – infinite atmosphere. He expanded the scattering matrix suitable to introduce scalar source functions which depend only on the optical depth and obtained Wiener –Hopf equations for these source functions which were solved by Sobolev's method. Das applied the Laplace transform and Wiener –Hopf technique to solve this problem and obtained.

$$I_l(0,u)=3\,Fq\frac{H_l(u)}{8\sqrt{2}}.\,u\epsilon(-1,0)^C \tag{1.4.78}$$

$$I_r(0,u)=3\,FH_r(u)\frac{(u+c)}{8\sqrt{2}}.\,u\epsilon(-1,0)^C \tag{1.4.79}$$

$$I_k(t,u)=R_0+R_{-\frac{1}{k_r}}+\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\Gamma}\frac{I_k^{\infty}(s,u)}{s}e^{s\mu}\,ds \tag{1.4.80}$$

where R_0 is the residue at the pole $s=0$ and $R_{-\frac{1}{k_r}}$ is the residue at the pole $s=-\frac{1}{k}$.

Das (1979d considered the transport equation for imperfect Rayleigh scattering in a semi – infinite atmosphere in the form

$$\mu\frac{dI(t,\mu)}{dt}=I_1(t,\mu)-\left(\frac{1}{2}\right)w\int_{-1}^{1}k(\mu,\mu')I(\mu,\mu')d \tag{1.4.81}$$

where t is the optional thickness of the atmosphere, μ is the direction parameter, $I(t,\mu)$ is a (2x1) matrix, w is the albedo for single scattering such that $0<w<1, k(\mu,\mu')$ is the Rayleigh phase matrix having the form

$$k(\mu,\mu')=\frac{3}{4}\begin{pmatrix} 2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{pmatrix} = Q(\mu)Q^T(\mu')$$

where $Q(\mu)$ is a (2 x 2) matrix defined by

$$Q(\mu)=\frac{\sqrt{3}}{2}\begin{pmatrix} \mu^2 & \sqrt{2(1-\mu^2)} \\ 1 & 0 \end{pmatrix}$$

The boundary conditions are

$$I(0,-\mu)=0, 0<\mu\leq 1$$

$$I(t,\mu)\to\frac{1}{2}w\frac{k}{(k-\mu)}e^{-\frac{t}{k}}Q(\mu)L_0, as \quad t\to\infty$$

where k is the positive real root $(k>1)$ of the equation $T(z)=det\,D(z)$

and $$D(z)=E+2\int_{-1}^{1}U(\mu)\frac{d\mu}{\mu-z}, \qquad (1.4.82)$$

E is a (2 x 2) matrix, $U(\mu)$ is a (2 x 2) matrix defined by

$$U(\mu)=\frac{1}{2}wQ(\mu')Q(\mu), \qquad (1.4.83)$$

$D(z)$ is a (2 x 2) matrix and L_0 is a specified (2 x 1) matrix. The formal solution of (1.4.81) obtained by Das was

$$I(0,\mu)=wQ(\mu)I_{\mu}^{*}(\frac{I}{\mu}) \qquad (1.4.84)$$

where I^{*} is the Laplace transform of $I(t,\mu)$,

Das obtained $$I(0,z)=\frac{k}{(k-D)}G(z)H(z)A \qquad (1.4.85)$$

where A is a matrix whose elements were determined from boundary conditions of the problem.

Das (1978a) considered a problem of diffuse reflection (Chandrasekhar 1960) by a semi - infinite plane – parallel atmosphere scattering radiation isotropically with an albedo $0<\omega_0\leq 1$. The equation considered by him is

$$\frac{dI(t,\mu,-\mu_0)}{dt}=I(t,\mu,-\mu_0)-(\frac{1}{2})\omega_0\int_{-1}^{1}I(t,\mu,-\mu_0)d\mu-(\omega_0\frac{F}{4})e^{-\frac{t}{\mu_0}} \qquad (1.4.86)$$

with boundary conditions,

$$I(o,u,-u_0)=0, 0<u\leq 1, 0\leq u_0<1 \qquad (1.4.87)$$

$$I(t,u,-u_0)e^{-\frac{t}{u}}\rightarrow 0 \ as\ t\rightarrow\infty, |\mu|<1, 0\leq u_0<1 \qquad (1.4.88)$$

This problem was solved by Chandrasekhar (1960) and obtained the emergent distribution of the reflected radiation in terms of his H- function which satisfies the integral equation (Chandrasekhar, 1960).

$$\frac{1}{H(z)}=1+\frac{1}{2}\omega_0 H(z)\int_{-1}^{1}u(x)\frac{H(z)}{(x+z)}dx \qquad (1.4.89)$$

Das applied the Laplace Transform and the Wiener Hopf Technique to solve the problem and obtained the emergent intensity in the form

$$I(o,z,-u_0)=A\frac{H(z)}{(z+u_0)} \tag{1.4.90}$$

where A is a constant determined from boundary conditions and he also obtained the intensity $I(t,-u,u_0)$ at any optical depth in the form

$$I(t,u,-u_0)=R_{-\frac{1}{k}}+\frac{1}{2\pi i}\lim_{R\to\omega}\int_C I^*(s,u,-u_0)e^{st}\frac{ds}{s} \tag{1.4.91}$$

where $$I^*(s,u,-u_0)=s\int_c^{\infty} I(t,u,-u_0)e^{-st}\,dt.$$

Chandrasekhar (1960) solved the problem of diffuse reflection and transmission of light scattering in terms of the phase function $\omega_o(1+x\cos\theta)$ by the principle of invariance and obtained the angular distribution of the emergent radiation from the bounding faces of the finite atmosphere in terms of $X-Y$ functions. Das (1978c) considered the same problem and solved it exactly by the method of Laplace Transform and Wiener Hopf technique.

The integro-differential equation for diffuse intensity $I(t,\mu,\theta;-\mu_0,\theta_0)$ considered by Das is

$$\mu\frac{dI(t,\mu,\theta,-\mu_0,\theta_0)}{dt}=I(t,\mu,\theta;-\mu_0,\theta_0)-J(t,\mu,\theta;-\mu_0,\theta_0) \tag{1.4.92}$$

where

$$J(t,\mu,\theta;-\mu_0,\theta_0)$$
$$=\frac{1}{4}FP(\mu,\theta;-\mu_0,\theta_0)e^{-\frac{t}{\mu_0}}+\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}P(t,\mu',\theta';-\mu_0,\theta_0)d\mu'\ d\theta' \tag{1.4.93}$$

and

$$P(\mu,\theta;-\mu',\theta')=\omega_0(1+x\cos\theta)$$
$$=\omega_0[1+x\mu\mu'+x(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos(\theta'-\theta)] \tag{1.4.94}$$

Boundary conditions are

$$I_k(0,-\mu,-\mu_0)=0,\ 0<\mu\le 1,\ 0<\mu_0\le 1 \tag{1.4.95}$$

$$I_k(t_0,-\mu,-\mu_0)=0,\ 0<\mu\le 1,\ 0<\mu_0\le 1 \tag{1.4.96}$$

$k=0,1$. Das deduced the following results,

$$I(0,z,-\mu_0)=\frac{1}{4}\mu_0 F[I_0(\theta,z,-\mu_0)+x(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)\frac{1}{2}\cos(\theta-\theta_0)I_1(0,z,-\mu_0)] \tag{1.4.97}$$

and

$$I(t_0,-z,-\mu_0)=\frac{1}{2}\mu_0 F[I_0(t_0,-z,-\mu_0)+(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)\frac{1}{2}\cos(\theta-\theta_0)I_1(t_0,-z,-\mu_0)] \tag{1.4.98}$$

Das (1979a) considered the radiative transfer equation in a semi-infinite non-conservative atmosphere with no incident radiation and scattering albedo $0<\omega_0\le 1$.

The equation considered was

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-\frac{1}{2}\omega_0\int_{-1}^{1}I(t,\mu')d\mu' \qquad (1.4.99)$$

with boundary conditions

$$I(0,-\mu)=0, 0<\mu\leq 1 \qquad (1.4.100)$$

$$I(t,\mu)\to L_0\exp(kt)(1-k\mu)\ when\ t\to\infty \qquad (1.4.101)$$

where L_0 is a constant, k is the positive root of

$$2k=\omega_0\log[\frac{(1-k)}{(1+k)}] \qquad (1.4.102)$$

Das applied the Laplace transform and Wiener Hopf technique and obtained the emergent intensity $I(0,z)$ and the intensity $I(t,\mu)$ at any optical depth in the forms

$$I(0,z)=\frac{AH(z)}{(1-kz)} \qquad (1.4.103)$$

and

$$I(t,\mu)=R_0+\lim_{R\to\infty}\frac{1}{2\pi i}\int_C\frac{I^*(s,\mu)}{s}e^{st}ds \qquad (1.4.104)$$

where R_0 is the sum of the residues at the poles of the integrand of (1.4.104) within C of infinite radius. $I^*(s,\mu)$ is the Laplace transform of $I(t,\mu)$ defined by

$$I^*(s,\mu)=s\int_0^{\infty}e^{-st}I(t,\mu)d \qquad (1.4.105)$$

He obtained the constant A from boundary conditions.

Das (1979b) considered the transport equations in the forms Chandrasekhar (1960)

$$\mu \frac{dI_l(t,\mu,-\mu_0)}{dt} = I_l(t,\mu,-\mu_0) - \frac{3}{8}\int_{-1}^{1} [\{ 2(1-\mu^2)+\mu'^2 \} I_l(t,\mu',-\mu_0) + \mu^2 I_r(t,\mu',-\mu_0)] d - \frac{3}{16} [F_l \{ 2(1-\mu'^2)+\mu^2(3\mu'^2-2) \} F_r\mu^2] e^{\frac{-t}{\mu_0}}$$

(1.4.105a)

$$\mu \frac{dI_r(t,\mu,-\mu_0)}{dt} = I_r(t,\mu,-\mu_0) - \frac{3}{16}(\mu^2 F_l + F_r) e^{-t/\mu_0} - \frac{3}{8}\int_{-1}^{1} [\mu'^2 F(t,\mu',-\mu_0) + I_r(t,\mu',-\mu_0)] d\mu'$$

(1.4.105b)

$$\mu \frac{df^{(i)}(t,\mu,-\mu_0)}{dt} = f'(t,\mu,-\mu_0) - \exp(-t/\mu_0) - \int_{-1}^{1} U^{(i)}(\mu') f^{(i)}(t,\mu',-\mu_0) d\mu'$$

(1.4.105c)

where $i=1, 2, 3, 4$ and $f^{(i)}(t,\mu,-\mu_0), i=1,2,3,4$ are to be used in conjuncture with $I_l(t,\mu,-\mu_0), I_r(t,\mu,-\mu_0)$ to construct the stoke's parameters in the manner described by Chandrasekhar (Chandrasekhar, 1960) and

$$U^{(1)} = \frac{3}{8}(1-\mu^2)\ (1+2\mu^2)$$

$$U^{(2)}(\mu) = \frac{3}{16}(1+\mu^2)$$

$$U^{(3)}(\mu) = \frac{3}{4}\mu^2$$

$$U^{(4)}(\mu) = \frac{3}{8}(1-\mu^2)$$

Boundary conditions are

$$I_k(0,\mu,-\mu_0)=0, 0<\mu\leqslant 1, 0<\mu_0\leqslant 1$$

$$I_k(t,\mu,-\mu_0)e^{-t/\mu_0}\to 0, |\mu|<1, 0<\mu_0\leqslant 1, t\to\infty$$

$$f^{(i)}(0,\mu,-\mu_0)=0, 0<\mu\leqslant 1, 0<\mu_0\leqslant 1$$

$$f^{(i)}(t,\mu,-\mu_0)e^{-t/\mu_0}\to 0, |\mu|<1, 0<\mu\leqslant 1, t\to\infty$$

where $i=1, 2, 3, 4$ and $k=l, r$.

Das (1979b) obtained by Laplace Transform and Wiener – Hopf technique,

$$I_l(0,z,-\mu_0)=\frac{A_l+B_l(z+z_0)}{z+z_0}.H(z)$$

and $$I_r(0,z,-\mu_0)=\frac{c_r+D_R(z+z_0)+E_r(z+z_0)^2}{z+z_0}.H(z)$$

and obtained the constants

A_l, B_l, C_r, D_r, E_r using boundary conditions and obtained the intensities $I_l(t,z,-\mu_0)$ and $I_r(t,z,-\mu_0)$ by inversion.

Karanjai and Deb (1972) considered the equation of radiative transfer (Chandrasekhar, 1960) in the form

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-\frac{\omega}{2}\int_{-1}^{1}(1+x\mu\mu')I(t,\mu')d \qquad (1.4.105d)$$

with boundary conditions

$$I(0,-\mu_0)=0, \quad 0<\mu_0\leqslant 1$$

and

$$I(t,\mu)\to L_0\left[\frac{1+\frac{x(1-\omega)\mu}{k}}{1-k\mu}\right]\exp(kt)$$

where L_0 is a constant and k is a root of the equation

$$1=\frac{\omega}{2k}\left[1+\frac{x(1-\omega)}{k^2}\right]\log\frac{1+k}{1-k}-\frac{x\omega(1-\omega)}{k^2}$$

Karanjai and Deb (1972) used Laplace Transform and Wiener – Hopf technique and obtained emergent intensity in the form

$$I(0,z)=\frac{(A+Bz)H(z)}{1-kz} \qquad (1.4.105e)$$

The constants A and B were found from boundary conditions. The intensity at any optical depth was obtained by inversion.

Das (1979c) considered the coupled nonlinear integral equations satisfied by the X and Y functions of Chandrasekhar (1960) in the forms

$$X(u)=I-u\int_0^1 U(x)\left[X(x)X(\mu)-\frac{Y(x)Y(u)}{(x+u)}\right]dx\,;0\le u<1 \quad (1.4.106)$$

$$Y(u)=e^{-t_0/\mu}+u\int_0^1 U(x)\left[Y(x)X(u)-\frac{X(x)Y(u)}{(x-u)}\right]dx\,;\ 0\le u<1 \quad (1.4.107)$$

where t_0 is the optical thickness of the finite atmosphere, u is the direction parameter, $U(x)$ is the Characteristic function, satisfying the Holder Condition on $0\le x\le 1$, is non – negative and satisfies the equation

$$u_0=\int_0^1 U(x)\,dx \le \frac{1}{2} \qquad (1.4.108)$$

Busbridge (1960) and Mullikin (1962) have demonstrated the existence of the solutions of these coupled non – linear equations.

Das applied Wiener–Hopf technique and obtained X and Y in the following forms for $u_0<\frac{1}{2}$

$$X(z)=H(z)\ [\ ze^{-\frac{t_0}{z}}\ \{\ D(Y,P_0)(z)-\frac{(A_0-H_0z)}{K-z}V(y)(z)\ \}\ +A\ ] \qquad (1.4.109)$$

and

$$Y(z)=H(z)\ [\ ze^{-\frac{t_0}{z}}\ \{\ D(X,P_0)(z)-\frac{(A_0-H_0z)}{K-z}V(x)(z)\ \}\ +B\ ] \qquad (1.4.110)$$

From boundary conditions Das obtained A and B.

For $u_o=\frac{1}{2}$, he found $A=1, B=0$ and

$$X(z)=H(z)\ [\ 1+ze^{-\frac{t_0}{z}}\ \{\ D(Y,P_0)(z)-(-h_1z+h_0)V(y)(z)\ \}] \qquad (1.4.111)$$

and

$$Y(z)=H(z)ze^{-\frac{t_0}{z}}[D(X,P_0)(z)-(h_1z+h_0)V(x)(z)] \qquad (1.4.112)$$

Das (1979e) considered the problem of determining the emergent intensity from the boundary face of a semi–infinite atmosphere having conservative scattering and intensely at any optical depth.

He considered the equation of radiative transfer in the form

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-\int_{-1}^{1}U(\mu')I(t,\mu')d\mu' \qquad (1.4.113)$$

where t is the optical depth. $U(z)$ is assumed to be finite and even function of x and

$$u_0=\int_0^1 u(x)dx=\frac{1}{2}$$

Boundary conditions are

$$I(0,-\mu)=I_0(\mu), 0<\mu\leq 1 \qquad (1.4.114)$$

$$I(t,\mu)e^{-\frac{t}{\mu}}\rightarrow 0 \ as\ t\rightarrow\infty, |\mu|\leq 1 \qquad (1.4.115)$$

He solved the problem by the Laplace transform and Wiener Hopf technique. He obtained emergent intensity in the form

$$I(0,z)=[C+\int_0^1 xU(x)I_0(x)L(x,z)\frac{dx}{(x+z)}]H(z) \quad (1.4.116)$$

where C is a constant determined from boundary conditions of the problem.

He also deduced intensity at any optical depth in the form

$$I(t,u)=R_0+\frac{1}{2\pi i}\lim_{R\rightarrow\infty}\int_c\frac{I^*(s,u)e^{st}ds}{s} \qquad (1.4.117)$$

where R_0 is the residue of the integrand of (1.4.117) at the simple pole $s=0$.

Schuster (1905) introduced an idealization of a stellar atmosphere to interpret the absorption lines in stellar spectra. A corrected theory was initiated by Eddington (1926, 1929) and Milne (1928, 1929) for stellar absorption lines, including the effects of scattering, absorption and thermal emission. Later on Planck's intensity function (Kourganoff, 1963) was brought into use to consider temperature distribution.

Chandrasekhar (1960) considered this problem in Milne-Eddington model and obtained exact form of emergent intensity by the Wick-Chandrasekhar discrete ordinate procedure. Code(1950) has extended the discrete ordinate procedure to solve a problem which considered the effect of electron scattering and grey absorption. Busbridge and Busbridge (1952) considered the problem which was a combination of coherent scattering and cyclic transition. DasGupta (1965, 1967) solved the equation of transfer for a partly coherent and partly non – coherent formation of absorption lines by the method of Laplace transform and Wiener-Hopf Technique and obtained emergent intensity from the boundary face with new boundary conditions and unit scattering function. Siewert and McCormick (1967) used Case's (1960) eigen function approach and obtained exact solution for emergent intensity and compared the results obtained by McCormick and Kuscar (1965), Mourad (1974) has derived an accurate numerical method to solve the azimuth independent polarized radiation in a stellar atmosphere. Lin and Chain (1975) have obtained an analytical solution for a problem of energy transfer through a non – grey absorbing and emitting medium bounded by two block surfaces. Yen et al (1976) have obtained the general solution of the equation of radiative transfer in a two–group picket–fence model for a plane–parallel, emitting, absorbing and isotropically scattering medium containing a uniform source.

Das (1979f) considered a problem for radiative transfer in getting the monochromatic line scattering intensity at any optical depth in a semi-infinite stellar atmosphere in the Milne – Eddington model. General continuous absorption and monochromatic line scattering are considered in this problem. He considered scattering quadratically anisotropic in the cosine of the scattering angle.

The equation considered by Das (Chandrasekhar, 1960) is

$$-\mu\rho^{-1}\frac{dI_\nu(t,\mu)}{dt}+(k_\nu+\sigma_\nu)I_\nu(t,\mu) \\ =(k_\nu+\sigma_\nu\epsilon_\nu)B_\nu(T(t))+\frac{1}{2}\sigma_\nu(1-\epsilon_\nu)\int_{-1}^{1}p(\mu,\mu')I_\nu(t,\mu')d\mu' \qquad (1.4.118)$$

where

$$p(\mu,\mu')=\omega_0+\omega_1\mu\mu'+\frac{1}{2}\omega_2(3\mu^2-1)(3\mu'^2-1) \qquad (1.4.119)$$

$I_\nu(t,\mu)$ is the specific intensity in the direction of arc cos u at a depth t, k_ν is the absorption coefficient, δ_ν is the scattering coefficient, ρ is the density of the atmosphere, $B(T(t))$ is Planck's function, $T(t)$ is the local temperature at depth t, ϵ_ν is the collision coefficient and u is the frequency.

Boundary conditions are $I(0,-\mu)=0, 0<\mu\leq 1$ and

$I(t,u)e^{-\frac{t}{\mu}}\rightarrow 0 \text{ as } t\rightarrow\infty, |\mu|\leq 1$

Das applied the Laplace transform and Wiener–Hopf technique to solve the problem(1.4.118). He obtained the emergent intensity in the form

$$I(0,z)=[x_0+x_1 z+(1-\frac{c_0}{\omega_0})c\int_0^1 \frac{S(\mu)\mu R(\mu,z)d\mu}{(\mu+kz)(k-z)}]H(z) \qquad (1.4.120)$$

and deduced intensity at any optical depth in the form

$$I(t,u)=R_0+R_{-\frac{1}{k}}+R_{\frac{1}{k}}+\frac{1}{2\pi i}\int_{CD}\frac{I^*(s,\mu)e^{st}ds}{s}+\frac{1}{2\pi i}\int_{EF}\frac{I^*(s,\mu)e^{st}ds}{s} \qquad (1.4.121)$$

where $$I^*(s,\mu)=s\int_0^\infty I(t,\mu)e^{-st}dt \qquad (1.4.122)$$

and $R_0, R_{-\frac{1}{k}}, R_{\frac{1}{k}}$ are residues of the integrand in (1.4.120), at the poles $s=0,\ s=-\frac{1}{k}, s=\frac{1}{k}$ respectively.

DasGupta and Bishnu (1981) obtained exact solution of mono – energetic neutron transport equation in a finite uniform, plane – parallel isotropically scattering multiplying slab having supply of neutrons only through fission. They used Laplace transform and Wiener – Hopf technique to solve the problem. They considered the equation (Case and Zweifel, 1967; Kobayashi et al, 1968)

$$\mu\frac{d\Psi(t,\mu)}{dt}=\Psi(t,\mu)-\omega\Psi_0(t),\ \omega>1 \qquad (1.4.123)$$

with boundary conditions

$$\Psi(t,\mu)=\Psi(b-t,\mu) \qquad (1.4.124)$$

and $$\Psi(0,\mu)=0, 0<\mu\le 1 \qquad (1.4.125)$$

where $$\Psi_0(t)=\frac{1}{2}\int_{-1}^{1}\Psi(t,\mu)d\mu \tag{1.4.126}$$

The following results were obtained.

$$\frac{\eta(z)M(z)}{H^0(z)}+\eta(z)F^-(z)G^-(z)+\eta(z)E_0^-(z)=C_0+C_1z \tag{1.4.127}$$

$$H^0(-z)G^+(z)-\eta(z)E_0^+(z)=C_0+C_1z \tag{1.4.128}$$

and $$M(\mu)=\Psi(0,x)\,on[0,1] \tag{1.4.129}$$

Woolley and Stibbs (1953) applied the theory of formation of absorption lines by coherent scattering to the case of interlocking without redistribution and deduced the equation of transfer in Milne – Eddington model. They have solved the problem for the case of triplets by Eddington's approximate method. Busbridge and stibbs (1954b) have solved the problem by the principle of invariance. An exact solution of the equation of transfer has been given by DasGupta (1956) by his form of the Wiener – Hopf technique. DasGupta and Karanjai (1972) applied Sobolev's probabistic method to solve the transfer equation for the case of interlocking without redistribution. Karanjai and Barman (1981) applied the extension of the method of discrete ordinates to find an exact solution of the problem of line formation by interlocking in the M – E model.

DasGupta (1978a) obtained an exact solution of the transfer equation for non – coherent scattering arising from interlocking of principal lines without redistribution by Laplace transform and Wiener – Hopf technique using a new representation of the H – function obtained by DasGupta (1977a). DasGupta considered the Planck function B (t) linear, i.e. $B(t)=b_0+b_1t$.

Karanjai and Karanjai (1985) considered the equation of transfer for interlocked multiplets in

1. an exponential atmosphere (Degl' Innocenti, 1979) in which

   $B_\nu(T)=B(t)=b_\nu+b_1 e^{-kt}$

2. an atmosphere (Busbridge, 1955) in which

$$B_\nu(T)=B(t)=b_0+b_1 t+b_2 E_2(t)$$

where

$$E_2(t)=\int_0^1 \frac{e^{-tx}}{x^2}dx$$

The equation of transfer considered by them is of the form (Woolley & Stibbs, 1953)

$$\mu\frac{dI_\nu(t,\mu)}{dt}=a_\nu I_\nu(t,\mu)-\delta_r B_\nu(t)-\delta_r \Sigma_{p=1}^{N}\eta_p J_p(t) \qquad (1.4.130)$$

with boundary conditions

$$I_\nu(t,\mu)e^{-\frac{t}{\mu}}\to 0 \text{ as } t\to\infty$$

and

$$I_\nu(0,\mu)=0 \text{ for } -1\le\mu\le 0$$

They applied Laplace transform and Wiener – Hopf technique and obtained the emergent intensity $I_\nu(0,z)$ form

$$(1+\alpha z)I_r(0,n,z)-H(-z)\lambda_r a_r G^-(z)\alpha z$$

$$= H(+z)\ [\ -\lambda_r d_1(KH_0+H_{-1})(\delta r-2\nu_r)+H_0\lambda r(\alpha_0(\delta r-2\nu_r))-d_1 f_1$$

$$+d_1\lambda_r P_{r0}+H_0\lambda_r d_1\frac{(\delta r-2\nu_r)}{(K-z)}+\frac{g_r(k)-g_r(z)}{k-z}-\lambda r(d_0+d_1 z)D_r(z)\ ] \qquad (1.4.131)$$

For $N=1,(2)$ takes the form

$$(1+\alpha z)I_r(0,n_r z)-H(-z)\lambda_r a_r G^+(z)$$

$$=H(-z)[-\lambda_r d_1 H_{-1}\delta_r+H_0\lambda_r d_1\delta_r z+H_0\lambda_r d_0\delta_r+H_0\alpha_r d_1 f_{r1}+d_1\lambda_r P_{r0}] \tag{1.4.132}$$

For $r=1$ and $z\to\infty$, (1.4.132) takes the form

$$I_r(0,z)=H(z,w)b_1(I-w)^{\frac{1}{2}} \tag{1.4.133}$$

Replacing r by v in (1.4.133), they obtained the emergent intensity $I_v(0,z)$ in case of coherent scattering in an exponential atmosphere, in the form

$$I_v(0,z)=H(z,w)b_0(1-w)^{\frac{1}{2}} \tag{1.4.134}$$

For $B_v(T)=b_0+b_1 t+b_2 E_2(t)$, they obtained

$$I(0,z)=[H(z,w)\frac{b_1(1-w)^{\frac{1}{2}}}{(1+\epsilon\eta_r)}].[Z+\frac{(1+\eta r)b_0}{b_1}+\frac{\alpha_1}{2}\omega(1-\omega)^{-\frac{1}{2}}] \tag{1.4.135}$$

Equation (1.4.135) is the sane as Chandrasekhar's (1960) solution.

Bishnu and DasGupta (1991) considered the transport equation for neutron diffusion in an isotropically scattering plane – parallel medium of finite thickness bounded by the planes $x=0$ and $x=2x_0$ in which are situated a plane source of strength δ'_0 represented by $\delta'_0\delta(x-x_0)$ at the middle $x=x_0$ and a uniformly distributed point source of strength P_0 . They took the equation appropriate to the problem (Case and Zweifel, 1967) in the form

$$\mu\frac{d\Psi(x,\mu)}{dx}=\sigma\Psi(x,\mu)+\sigma_s\frac{1}{2}\int_{-1}^{1}\Psi(x,\mu)d\mu+\frac{1}{2}\delta'_0\delta(x-x_0)+\frac{1}{2}P_0^1 \quad (1.4.136)$$

where $\Psi(x,\mu)$ is the neutron density distribution, σ, total macroscopic cross – section for the removal of neutrons; σ_s , total macroscopic scattering cross – section. They solved the problem by the Laplace transform and Wiener – Hopf technique. They made the following transformations in (1.4.136) :

$$t=-\sigma\int_0^x ds \quad => \quad dt=-\sigma dx$$

$$c=\frac{\sigma_s}{\sigma}, \quad \delta_0=\frac{\delta_0'}{\sigma}, \quad \rho_0=\frac{\rho_0'}{\sigma}$$

$$\Psi_0(t)=\frac{1}{2}\int_{-1}^{1}\Psi(x,t)dx$$

and obtained

$$\mu\frac{d\Psi(t,\mu)}{dt}=\Psi(t,\mu)-c\Psi_0(t)-S_0(t) \qquad (1.4.137)$$

where $$S_0(t)=\frac{1}{2}S_0(t-t_0)+\frac{1}{2}\rho_0 \qquad (1.4.138)$$

The boundary conditions are

$$\Psi(0,-\mu)=I_0\delta(\mu-1),\mu>o \qquad (1.4.139)$$

$$\Psi(2t_0,\mu)=I_0\delta(\mu-1),\mu>o \qquad (1.4.140)$$

They obtained the solution of (1.4.137) using Wiener–Hopf and Laplace Transform Technique.