

**CERTAIN INVESTIGATIONS ON BITOPOLOGICAL SPACES**

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## DECLARATION

I declare that the thesis entitled ‘**CERTAIN INVESTIGATIONS ON BITOPOLOGICAL SPACES**’ has been prepared by me under the guidance of **Dr. M. K. Bose**, Professor, Department of Mathematics, University of North Bengal. No part of this thesis has formed the basis for the award of any degree or fellowship previously.

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## CERTIFICATE

I certify that **Ajoy Mukharjee** has prepared the thesis entitled ‘**CERTAIN INVESTIGATIONS ON BITOPOLOGICAL SPACES**’, for the award of Ph. D. degree of the University of North Bengal, under my guidance. He has carried out the work at the Department of Mathematics, University of North Bengal.

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## ABSTRACT

J. C. Kelly initiated the systematic study of bitopological spaces in 1963. Since then many contributions to bitopological settings were done by several authors. Some of them attempted to generalize the notion of paracompactness in bitopological spaces. But a suitable concept of paracompactness in bitopological spaces could not be obtained. So it appears that the generalization of paracompactness in bitopological spaces is an intractable problem. The aim of the present research project is to develop a suitable notion of paracompactness in bitopological spaces, and to study some related notions.

Generalizing the notion of pairwise compactness, we introduce a concept of pairwise paracompactness. Along with other results, we show that a pairwise Hausdorff and pairwise paracompact space is pairwise normal. We prove an analogue of Michael's characterization (E. Michael, A Note on Paracompact Spaces, Proc. Amer. Math. Soc. **4** (1953), 831–838) for a strongly pairwise regular pairwise paracompact space.

We introduce and study two notions of bitopological full normality, namely, pairwise full normality and  $a$ -pairwise full normality. Along with other results, we prove that a pairwise Hausdorff and pairwise paracompact space is  $a$ -pairwise fully normal, and conversely, a pairwise Hausdorff and pairwise fully normal space is pairwise paracompact.

We also introduce the notions of pairwise closure-preserving and hereditarily pairwise closure-preserving properties of a collection of subsets of a bitopological space. Then some characterizations of pairwise paracompactness are obtained for a bitopological space having some properties.

Thereafter generalizing the notion of pairwise paracompactness, we introduce and study the notion of countable pairwise paracompactness. A notion stronger than countable pairwise paracompactness is also introduced and we call it countable  $(*)$ pairwise paracompactness. Along with other results, we obtain some equivalent conditions to countable pairwise paracompactness for  $\alpha^*$ -pairwise normal spaces.

We also introduce the notions of a nearly pairwise compact space and a bifilter. The notion of near pairwise compactness is a generalization of both the notions of pairwise compactness and near compactness. We obtain some results on nearly pairwise compact spaces.

Finally, considering a countable number of topologies on a set, we introduce the notion of  $(\aleph_0)$ topological spaces as a generalization of both the notions of bitopological spaces and  $(\omega)$ topological spaces. We study several separation axioms on  $(\aleph_0)$ topological spaces, and explain them with examples. Along with other results, we prove a result analogous to Michael's theorem (E. Michael, A Note on Paracompact Spaces, Proc. Amer. Math. Soc. **4** (1953), 831–838) for  $(\beta\text{-}\aleph_0)$ paracompact

$(\aleph_0)$ regular spaces. From this, we get the Michael's theorem for  $\beta$ -pairwise paracompactness (T. G. Raghavan and I. L. Reilly, A New Bitopological Paracompactness, J. Aust. Math. Soc. (Series A) **41** (1986), 268–274) of a pairwise regular bitopological space, as a particular case.

## PREFACE

The writing of a thesis for Doctor of Philosophy is indeed a significant dream to everyone, especially for those who are engaged in academic ambience. It being the time for me to write a thesis, I would like to take this golden opportunity to express my profound gratitude and deep regards to all who directly or indirectly has helped me to touch this milestone of my life. However, in this regard the following personages deserve special mention indeed.

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# CHAPTER I

## Overviews of Bitopological Spaces

### 1.1. Introduction

Weston [47] initiated an investigation to answer some natural questions those arose when one contemplates a set of points with two topologies. A topology  $\mathcal{P}$  on a nonempty set  $X$  is a collection of subsets of  $X$  with the following properties:

- (i)  $\emptyset, X \in \mathcal{P}$ .
- (ii) The union of arbitrary collection of members of  $\mathcal{P}$  is a member of  $\mathcal{P}$ .
- (iii) The intersection of finitely many members of  $\mathcal{P}$  is a member of  $\mathcal{P}$ .

A nonempty set  $X$  with a topology  $\mathcal{P}$  is called a topological space and it is denoted by  $(X, \mathcal{P})$ . But for brevity, we usually write ‘ $X$  is a topological space’ when no confusion can arise about  $\mathcal{P}$ . For a topological space  $X$ , the members of  $\mathcal{P}$  are called the open sets of  $X$ . A subset  $E$  of  $X$  is called a closed set if  $X - E$  is an open set. A collection  $\mathcal{B}$  of open sets of  $X$  is called a base of the topology  $\mathcal{P}$  if every open set of  $X$  can be expressed as a union of some members of  $\mathcal{B}$ . A collection  $\mathcal{C} \subset \mathcal{P}$  is called a subbase of  $\mathcal{P}$  if the collection of all finite intersections of elements belonging to  $\mathcal{C}$  constitutes a base of  $\mathcal{P}$ . A subset  $G$  of  $X$  is called a neighbourhood of  $x \in X$  if there

exists an open set  $U$  such that  $x \in U \subset G$ . For a subset  $A$  of  $X$ , the intersection of all closed sets containing  $A$  is called the closure of  $A$ , and the union of all open sets contained in  $A$  is called the interior of  $A$ . A subset of a topological space is called an  $F_\sigma$  set if it is a union of countably many closed sets.

The separation axioms of a topological space play vital roles in study of topological spaces, and we recall the following separation axioms.

- (i) A topological space  $X$  is said to be a  $T_1$ -space if for every pair of distinct points  $x, y \in X$ , there exist open sets  $U, V$  such that  $x \in U, y \in V, x \notin V$  and  $y \notin U$ .
- (ii) A topological space  $X$  is said to be a Hausdorff space if for every pair of distinct points  $x, y \in X$ , there exist open sets  $U, V$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ .
- (iii) A topological space  $X$  is said to be a regular space if for each  $x \in X$  and each closed set  $F$  with  $x \notin F$ , there exist open sets  $U, V$  such that  $x \in U, F \subset V$  and  $U \cap V = \emptyset$ .
- (iv) A topological space  $X$  is said to be a normal space if for each pair of closed sets  $E, F$  with  $E \cap F = \emptyset$ , there exist open sets  $U, V$  such that  $E \subset U, F \subset V$  and  $U \cap V = \emptyset$ .

A cover of a topological space  $X$  is a collection  $\mathcal{A}$  of subsets of  $X$  such that  $\bigcup\{A \mid A \in \mathcal{A}\} = X$ . A subcollection  $\mathcal{B} \subset \mathcal{A}$  is called a subcover of  $\mathcal{A}$  if  $\mathcal{B}$  covers  $X$ . A cover of  $X$  consisting of open sets only is called an open cover of  $X$ . A topological

space  $X$  is said to be compact if each open cover of  $X$  has a finite subcover. A cover  $\mathcal{V}$  of  $X$  is said to be a refinement of a cover  $\mathcal{U}$  of  $X$  if for each  $V \in \mathcal{V}$ , there exists a  $U \in \mathcal{U}$  such that  $V \subset U$ . A collection  $\mathcal{U}$  of subsets of  $X$  is said to be locally finite if each point of the space has a neighbourhood which meets only a finite number of elements  $U \in \mathcal{U}$ . A topological space  $X$  is said to be paracompact if each open cover of  $X$  has an open locally finite refinement. It is to be noted that some topologists included Hausdorffness in the definition of paracompactness of a topological space [48] (see also [15]). Kelley [25] included regularity\* in the definition of paracompactness of a topological space.

Weston [47] expressed the relation of inclusion of two topologies  $\mathcal{P}_1$  and  $\mathcal{P}_2$  on  $X$  in terms of neighbourhoods:  $\mathcal{P}_1 \subset \mathcal{P}_2$  if and only if each  $(\mathcal{P}_1)$ nbd of any  $x \in X$  is a  $(\mathcal{P}_2)$ nbd of the point  $x \in X$ . The notations are explained in article 1.3 of this chapter. He also noted that,  $\mathcal{P}_1 \supset \mathcal{P}_2$  if and only if  $(\mathcal{P}_1)\text{cl}E \subset (\mathcal{P}_2)\text{cl}E$  for each  $E \subset X$ . A topology  $\mathcal{P}_1$  is said to be coupled to  $\mathcal{P}_2$  [47] if for any  $G \in \mathcal{P}_1$  we have  $(\mathcal{P}_1)\text{cl}G \subset (\mathcal{P}_2)\text{cl}G$ . He also defined consistency of two topologies as follows: two topologies  $\mathcal{P}_1$  and  $\mathcal{P}_2$  on a set  $X$  are said to be consistent with each other if for any two distinct points  $x, y \in X$ , there exist a  $(\mathcal{P}_1)$ nbd  $U$  and a  $(\mathcal{P}_2)$ nbd  $V$  such that  $x \in U, y \in V$  with  $U \cap V = \emptyset$ .

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\*It is easy to see that a Hausdorff space  $X$  is regular [15, p. 163] if each open cover of the topological space  $X$  has an open locally finite refinement.

‘Weston [47] considered an arbitrary pair  $(\mathcal{P}_1, \mathcal{P}_2)$  of topological structures, but not as a structure on  $X$ , and he studied only relations between two topologies  $\mathcal{P}_1$  and  $\mathcal{P}_2$  for arbitrary pairs of topological structures defined on the same set  $X$ . He introduced no special term for triples  $(X, \mathcal{P}_1, \mathcal{P}_2)$ ’ (Ivanov [22]).

## 1.2. Naturalness of Bitopological Spaces

The systematic study of a space equipped with two topologies on a nonempty set  $X$  was initiated by Kelly [26] and he coined the term ‘Bitopological Space’ for such a space.

**DEFINITION 1.1.** A metric  $d$  on a nonempty set  $X$  is a non-negative real-valued function defined on  $X \times X$  satisfying the following conditions:

- (i)  $d(x, y) = 0$  if and only if  $x = y$ ,  $x, y \in X$ ,
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,
- (iii)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

On omitting ‘ $d(x, y) = 0$  only if  $x = y$ ’ from the first condition of the definition of a metric, we obtain a pseudo metric. There arises no difficulty in generalizing the standard theorems of metric spaces, in particular those concerning metrization of topological spaces on using a pseudo metric. For example, one may consider the following theorem: every metric space is paracompact. On the other hand, if one attempts to omit the second

condition then the appropriate generalizations are not ensured. Such an unsymmetric distance function is called a quasi-metric.

DEFINITION 1.2. A quasi-pseudo-metric  $d$  on a nonempty set  $X$  is a non-negative real-valued function defined on  $X \times X$  satisfying the following conditions:

- (i)  $d(x, x) = 0$  for all  $x \in X$ ,
- (ii)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

Let  $p$  be a quasi-pseudo-metric on  $X$ , and let the function  $q$  be defined on  $X \times X$  by  $q(x, y) = p(y, x)$  for all  $x, y \in X$ . Then  $q$  is also a quasi-pseudo-metric on  $X$ .  $q$  is called conjugate to  $p$  on  $X$ . If  $p$  is a quasi-pseudo-metric on a set  $X$ , then the collection of all the open spheres  $\{s(x, \epsilon)\}$  ( $s(x, \epsilon) = \{y \mid p(x, y) < \epsilon\}$ ) forms a base for a topology on  $X$ . Similarly,  $q$  determines a topology on  $X$ . Thus the natural topological structure associated with a quasi-pseudo-metric on a set  $X$  is that of the set  $X$  with two topologies. A set  $X$  equipped with two topologies  $\mathcal{P}$  and  $\mathcal{Q}$  on  $X$  is called a bitopological space [26] and is denoted by  $(X, \mathcal{P}, \mathcal{Q})$ .

### 1.3. Symbols and Notations

Throughout the thesis, along with conventional symbols and notations, we use the following symbols and notations.

- (i) Sets are denoted by capital alphabets  $A, B, \dots$  and the family of sets are denoted by script alphabets  $\mathcal{A}, \mathcal{B}, \dots$

- (ii) The symbols  $N, Q, Z$  and  $R$  denote the set of natural numbers, the set of rational numbers, the set of integers and the set of real numbers respectively. The empty set is denoted by  $\emptyset$ .
- (iii) For brevity, we write without quotes ‘nbd’ for ‘neighbourhood’, ‘iff’ for ‘if and only if’ and ‘resp.’ for ‘respectively’.
- (iv) A  $(\mathcal{T})$ open set (resp.  $(\mathcal{T})$ closed set,  $(\mathcal{T})\text{cl}A$  etc.) means an open set (resp. a closed set, closure of  $A$  etc.) in the topological space  $(X, \mathcal{T})$ .
- (v) Unless or otherwise mentioned,  $X$  denotes the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  and  $Y$  denotes the bitopological space  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$ .
- (vi)  $\mathcal{P}$  is the topology on  $X$  generated by the subbase  $\mathcal{P}_1 \cup \mathcal{P}_2$  and  $\mathcal{Q}$  is the topology on  $Y$  generated by the subbase  $\mathcal{Q}_1 \cup \mathcal{Q}_2$ .
- (vii) Always  $i, j \in \{1, 2\}$  and whenever they appear together,  $i \neq j$ .
- (viii)  $(a, b)$  stands for the open interval  $\{x \mid a < x < b\}$  and  $[a, b]$  stands for the closed interval  $\{x \mid a \leq x \leq b\}$ .
- (ix) The symbol ‘ $\square$ ’ indicates the end of a proof.

#### 1.4. Some Definitions and Theorems of Kelly

On extending some notions of a topological space, Kelly [26] introduced some notions on a bitopological space, and we recall them for ready references.

DEFINITION 1.3.  $X$  is said to be pairwise Hausdorff if for each pair of distinct points  $x$  and  $y$  of  $X$ , there exist  $U \in \mathcal{P}_i$  and  $V \in \mathcal{P}_j$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ .

This notion of pairwise Hausdorffness may be compared with the notion of consistency of two topologies introduced by Weston [47].

DEFINITION 1.4.  $\mathcal{P}_i$  is said to be regular with respect to  $\mathcal{P}_j$  if for each  $x \in X$  and each  $(\mathcal{P}_i)$ -closed set  $A$  with  $x \notin A$ , there exist  $U \in \mathcal{P}_i$  and  $V \in \mathcal{P}_j$  such that  $x \in U, A \subset V$  and  $U \cap V = \emptyset$ .  $X$  is said to be pairwise regular if  $\mathcal{P}_i$  is regular with respect to  $\mathcal{P}_j$  for each  $i \in \{1, 2\}$ .

DEFINITION 1.5.  $X$  is said to be pairwise normal if for any pair of a  $(\mathcal{P}_i)$ -closed set  $A$  and a  $(\mathcal{P}_j)$ -closed set  $B$  with  $A \cap B = \emptyset$ , there exist  $U \in \mathcal{P}_j$  and  $V \in \mathcal{P}_i$  such that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$ .

DEFINITION 1.6. Let  $\rho$  be the usual topology on  $R$ . A function  $f : (X, \mathcal{P}_1, \mathcal{P}_2) \rightarrow (R, \rho)$  is lower (resp. upper) semi-continuous if  $f^{-1}(r, \infty) \in \mathcal{P}_i$  (resp.  $f^{-1}(-\infty, r) \in \mathcal{P}_i$ ) for all  $r \in R$  and for each  $i \in \{1, 2\}$ .

Kelly [26] agreed to say without referring topologies  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  that the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  possesses a particular property if both the topologies  $\mathcal{P}_1$  and  $\mathcal{P}_2$  possess that property. For instance, a

bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is said to satisfy the second axiom of countability if both the topologies  $\mathcal{P}_1$  and  $\mathcal{P}_2$  satisfy the second axiom of countability.

In this regard, we may recall the following definitions.

DEFINITION 1.7. A topological space  $(X, \mathcal{P})$  is said to be

- (i) first countable (or satisfies the first axiom of countability) if every point of  $X$  has a countable nbd base.
- (ii) second countable (or satisfies the second axiom of countability) if its topology has a countable base.

We note the following results of Kelly [26].

- (i) If  $X$  is pairwise Hausdorff then  $\mathcal{P}_1$  and  $\mathcal{P}_2$  both are  $T_1$ -topologies.
- (ii)  $X$  is pairwise regular iff for each  $x \in X$  and each  $(\mathcal{P}_i)$ open set  $G$  with  $x \in G$ , there exists a  $(\mathcal{P}_i)$ open set  $H$  such that

$$x \in H \subset (\mathcal{P}_j)\text{cl}H \subset G.$$

- (iii)  $X$  is pairwise normal iff for any  $(\mathcal{P}_j)$ closed set  $K$  and  $(\mathcal{P}_i)$ open set  $V$  with  $K \subset V$ , there exists a  $(\mathcal{P}_i)$ open set  $U$  such that

$$K \subset U \subset (\mathcal{P}_j)\text{cl}U \subset V.$$

Kelly [26] proved the following generalization of Urysohn's lemma in a bitopological space.

**THEOREM 1.1.** *If  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise normal, then given a  $(\mathcal{P}_j)$  closed set  $E$  and a  $(\mathcal{P}_i)$  closed set  $F$  with  $E \cap F = \emptyset$ , there exists a real-valued function  $g$  on  $X$  such that*

(i)  $g(x) = 0$  for  $x \in E$ ,  $g(x) = 1$  for  $x \in F$  and  $0 \leq g(x) \leq 1$  for  $x \in X$ ,

(ii)  $g$  is  $(\mathcal{P}_i)$  upper semi-continuous and  $(\mathcal{P}_j)$  lower semi-continuous.

## 1.5. Research Topics

Since the work of Kelly [26] on bitopological spaces, a considerable amount of works were done by several authors: Lane [30], Fletcher et al. [19], Raghavan and Reilly [38], Ganster and Reilly [20], Kovár [28, 29], Mršević [33], Murdeshwar and Naimpally [35], and others. Besides some research articles (e.g. [17], [18]), Dvalishvili wrote a book [16] on bitopological spaces. He discussed the applications of the theory of bitopological spaces in analysis, potential theory and general topology in [17, 18]. Kannan [24] studied the nature and properties of some generalized closed sets in bitopological spaces associated to the digraph.

Fletcher et al. [19] introduced the following definitions.

**DEFINITION 1.8.** A cover  $\mathcal{U}$  of  $X$  is said to be pairwise open if  $\mathcal{U} \subset \mathcal{P}_1 \cup \mathcal{P}_2$ , and for each  $i \in \{1, 2\}$ ,  $\mathcal{U} \cap \mathcal{P}_i$  contains a nonempty set.

DEFINITION 1.9. The bitopological space  $X$  is pairwise compact if every pairwise open cover of  $X$  has a finite subcover.

Fletcher et al. [19] obtained the following results.

- (i) If  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise compact and if the topological spaces  $(X, \mathcal{P}_1)$  and  $(X, \mathcal{P}_2)$  are Hausdorff, then  $\mathcal{P}_1 = \mathcal{P}_2$ .
- (ii) If  $X$  is pairwise Hausdorff and pairwise compact, then  $X$  is pairwise regular.
- (iii) If  $X$  is pairwise compact and either  $\mathcal{P}_1$  is regular with respect to  $\mathcal{P}_2$  or  $\mathcal{P}_2$  is regular with respect to  $\mathcal{P}_1$ , then  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise normal.

Fletcher et al. [19] defined the notion of pairwise paracompactness in a natural way, namely, the bitopological space  $X$  is pairwise paracompact if it is pairwise Hausdorff, and for  $i \neq j$  every  $(\mathcal{P}_i)$ open cover has a  $(\mathcal{P}_j)$ open  $(\mathcal{P}_j)$ locally finite refinement. But with this definition at last they arrived at the conclusion  $\mathcal{P}_1 = \mathcal{P}_2$  and the resulting single topological space is paracompact. Thus the structure of a bitopological space is wiped out. Thereafter they did not proceed further to get an alternative definition of pairwise paracompactness. However, the attempts persist to obtain a definition of pairwise paracompactness to generalize the notion of paracompactness of a space with single topology. Datta [12], and Raghavan and Reilly [38] defined the notion of pairwise paracompactness in different

ways. Datta's definition of pairwise paracompactness is a generalization of pairwise compactness. Datta [12] introduced the following definition.

**DEFINITION 1.10.** A pairwise open cover  $\mathcal{V}$  of  $X$  is called a parallel refinement of a pairwise open cover  $\mathcal{U}$  of  $X$  if every  $(\mathcal{P}_i)$ open set of  $\mathcal{V}$  is contained in a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ .

Datta [12] called a pairwise open cover  $\mathcal{U}$  of  $X$ , pairwise locally finite if for every  $x \in X$ , there exists a  $(\mathcal{P}_i)$ open nbd of  $x$  which meets only a finite number of  $(\mathcal{P}_j)$ open sets of  $\mathcal{U}$  for each  $i \in \{1, 2\}$ . Then he defined pairwise paracompactness as follows: the bitopological space  $X$  is called pairwise paracompact if  $X$  is pairwise Hausdorff and every pairwise open cover of  $X$  has a pairwise locally finite parallel refinement. With this definition of pairwise paracompactness, Datta attempted to get a bitopological version of Michael's theorem [32]. But unfortunately, he did not get a complete analogue of the said theorem.

Raghavan and Reilly [38] defined  $\delta$ -pairwise paracompactness as follows: the bitopological space  $X$  is said to be  $\delta$ -pairwise paracompact if for each  $i \in \{1, 2\}$ , each  $(\mathcal{P}_i)$ open cover of  $X$  has a  $(\mathcal{P})$ open  $(\mathcal{P})$ locally finite refinement. With this definition of  $\delta$ -pairwise paracompactness, Raghavan and Reilly [38] proved an analogue of Michael's theorem [32]. But the proof is not correct which was pointed out by Kovár [28]. Also Raghavan

and Reilly's definition of  $\delta$ -pairwise paracompactness does not generalize the notion of pairwise compactness.

*Thus the generalization of paracompactness in bitopological spaces is an intractable problem.*<sup>†</sup> So our goal is to find a useful definition of paracompactness in a bitopological space which generalizes the notion of pairwise compactness, and to study some related notions in bitopological spaces.

## 1.6. Summary of the Thesis

CHAPTER I: This chapter is basically concerned with bitopological spaces introduced and developed by J. C. Kelly. We recall some definitions and results of bitopological spaces obtained by several authors: P. Fletcher, H. B. Hoyle III and C. W. Patty; M. C. Datta; M. Ganster and I. L. Reilly; and others. We use these results in the sequel to develop the theory of bitopological spaces. In this chapter, we also indicate the main problem of our work.

CHAPTER II: We introduce the notion of  $\mathcal{U}$ -locally finite property for a subcollection of a parallel refinement of a pairwise open cover  $\mathcal{U}$ . Using this notion of local finiteness, we define the concept of pairwise paracompactness: a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is said to be pairwise paracompact if

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<sup>†</sup>Cobzas [11] in his recently published book cited our paper [5] along with [12, 27, 38, 39] to epitomize that paracompactness within the framework of bitopological spaces is a delicate matter.

every pairwise open cover  $\mathcal{U}$  of  $X$  has a  $\mathcal{U}$ -locally finite parallel refinement. This notion of pairwise paracompactness is more general than the notion of pairwise compactness. Along with other results, an analogue of Michael's characterization [32] of a regular paracompact space is proved for a bitopological space.

The contents of this chapter are the subject matter of our paper: *Mat. Vesnik* **60** (4) (2008), 255–259, [5].

CHAPTER III: The notion of bitopological full normality is introduced: a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is said to be pairwise fully normal if for every pairwise open cover  $\mathcal{U}$  of  $X$ , there exists a pairwise open cover  $\mathcal{V}$  of  $X$  such that  $\mathcal{V}$  is a parallel star refinement of  $\mathcal{U}$ . Along with other results, we prove a bitopological version of A. H. Stone's theorem [45] on paracompactness: a Hausdorff topological space is paracompact if and only if it is fully normal.

The subject matter of this chapter forms our paper: *Mat. Vesnik* **62** (1) (2010), 11–18, [4].

CHAPTER IV: The notion of pairwise closure-preserving property of a collection of subsets of a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is introduced: a collection of subsets of  $X$  is said to be pairwise closure-preserving if it is  $(\mathcal{P}_i)$ -closure-preserving for each  $i \in \{1, 2\}$ . A collection  $\mathcal{A}$  of subsets of  $X$  is hereditarily pairwise closure-preserving if any collection  $\mathcal{B}$  containing subsets of sets belonging to  $\mathcal{A}$  such that each set  $A \in \mathcal{A}$  has one and

only one subset belonging to  $\mathcal{B}$ , is pairwise closure-preserving. Then some characterizations of pairwise paracompactness are obtained for a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  satisfying some conditions.

The contents of this chapter are the subject matter of our paper: *Mat. Vesnik* **62** (4) (2010), 299–309, [3].

CHAPTER V: Generalizing the notion of pairwise paracompactness, we introduce and study the notion of countable pairwise paracompactness: a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is said to be countably pairwise paracompact if every countable pairwise open cover  $\mathcal{U}$  of  $X$  has a  $\mathcal{U}$ -locally finite parallel refinement. We also define countable  $(*)$ pairwise paracompactness: a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is said to be countably  $(*)$ pairwise paracompact if each pairwise open cover  $\mathcal{U}$  of  $X$  has the following property: every countable  $(\mathcal{P})$ open refinement of  $\mathcal{U}$  has a  $\mathcal{U}$ -locally finite pairwise open refinement parallel to the cover  $\mathcal{U}$ . The notion of countable  $(*)$ pairwise paracompactness is stronger than the notion of countable pairwise paracompactness. Along with other results, we obtain some equivalent conditions to countable pairwise paracompactness for  $\alpha^*$ -pairwise normal spaces.

The subject matter of this chapter forms our paper: *Int. J. Mod. Math.* **5** (3) (2010), 275–284, [2].

CHAPTER VI: We introduce and study the notion of nearly pairwise compact spaces: a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is said to be nearly pairwise compact if for each pairwise open cover  $\mathcal{U}$  of  $X$ , there exists a finite subcollection  $\mathcal{V} \subset \mathcal{U}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V) \mid V \in \mathcal{V} \cap \mathcal{P}_i, i \in \{1, 2\}\}$  covers  $X$ . The notion of near pairwise compactness is a generalization of both the notions of pairwise compactness and near compactness.

The contents of this chapter mainly form our paper: *Kyungpook Math. J.* **53** (1) (2013), 125–133, [34].

CHAPTER VII: Considering a countable number of topologies on a set  $X$ , we introduce the notion of  $(\aleph_0)$ topological spaces as a generalization of both the notions of bitopological spaces [26] and  $(\omega)$ topological spaces [6], and study some of their properties. Let  $\{\mathcal{P}_n\}$  be a sequence of topologies on a set  $X$ . The sequence  $\{\mathcal{P}_n\}$  is said to satisfy the condition  $(*)$  if for any positive integer  $m$  the union of a finite number of sets belonging to  $\bigcup_{n \neq m} \mathcal{P}_n$  is a set belonging to  $\bigcup_{n \neq m} \mathcal{P}_n$ . If  $\{\mathcal{P}_n\}$  is a sequence of topologies on a set  $X$  satisfying the condition  $(*)$ , then the pair  $(X, \{\mathcal{P}_n\})$  is called an  $(\aleph_0)$ topological space. A set  $X$  equipped with an increasing sequence  $\{\mathcal{T}_n\}$  of topologies is called an  $(\omega)$ topological space. In the  $(\aleph_0)$ topological spaces, we introduce some separation axioms:  $(m)$ Hausdorffness,  $(\aleph_0)$ Hausdorffness,  $(m)$ regularity,  $(\aleph_0)$ regularity,  $(m)$ normality,  $(\aleph_0)$ normality, complete  $(\aleph_0)$ normality. We also introduce the notions of  $(\aleph_0)$ compactness,  $(\aleph_0)$ paracompactness,  $(\beta\text{-}\aleph_0)$ paracompact-

ness. Along with other results, we prove a result analogous to Michael's theorem [32] for  $(\beta\text{-}\aleph_0)$ paracompact  $(\aleph_0)$ regular spaces. From this, we get the Michael's theorem for  $\beta$ -pairwise paracompactness [38] of a pairwise regular bitopological space, as a particular case.

The contents of this chapter form our paper: *Novi Sad J. Math.* **40** (2) (2010), 7–16, [1].

## CHAPTER II

### Bitopological Paracompactness

#### 2.1. Introduction

The study of bitopological spaces was initiated by J. C. Kelly [26]. Since then many works on bitopological spaces were done by several authors: Lane [30], Fletcher et al. [19], Datta [12], Raghavan and Reilly [38], Ganster and Reilly [20], Kovár [28], Srivastava and Bhatia [43], and others. Fletcher et al. [19] defined pairwise paracompactness. But in presence of pairwise Hausdorffness, the two topologies in a bitopological space pairwise paracompact in the sense of [19] coincide, and the bitopological space becomes a single topological space. This was proved by Fletcher et al., and they did not proceed further with this definition. Later Datta [12], and Raghavan and Reilly [38] defined the notion of pairwise paracompactness in different ways. With his definition of pairwise paracompactness, Datta [12], attempted to get a bitopological version of the following theorem of Michael [32] on paracompactness. But a complete analogue of the theorem could not be obtained.

**THEOREM 2.1** (Michael [32], Dugundji [15]). *Let  $X$  be a regular topological space. Then the following statements are equivalent.*

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The subject matter of this chapter forms our research article: *Mat. Vesnik* **60** (4) (2008), 255–259, [5].

- (a)  $X$  is paracompact.
- (b) Each open cover of  $X$  has an open refinement  $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ , where each  $\mathcal{V}_n$  is a locally finite collection of open sets.
- (c) Each open cover of  $X$  has a locally finite refinement.
- (d) Each open cover of  $X$  has a locally finite closed refinement.

Raghavan and Reilly [38] proved an analogue of Michael's theorem with their notion of  $\delta$ -pairwise paracompactness. Unfortunately, the proof of the theorem is not correct (Kovár [28]). Also  $\delta$ -pairwise paracompactness is not a generalization of pairwise compactness. In this chapter, we introduce a definition of pairwise paracompactness (Definition 2.2) as a generalization of pairwise compactness and prove an analogue of Michael's theorem.

Throughout this chapter, for a pairwise open cover  $\mathcal{U}$  of the bitopological space  $X$ ,  $x \in X$  and  $E \subset X$ , by ' $E$  is  $(\mathcal{P}_{\mathcal{U}x})$ open' we mean ' $E$  is  $(\mathcal{P}_1)$ open (resp.  $(\mathcal{P}_2)$ open)' if  $x$  belongs to a  $(\mathcal{P}_1)$ open (resp.  $(\mathcal{P}_2)$ open) set of  $\mathcal{U}$ .

## 2.2. Definitions and Results

We introduce the following definitions.

**DEFINITION 2.1.** A subcollection  $\mathcal{C}$  of a refinement  $\mathcal{V}$  of a pairwise open cover  $\mathcal{U}$  of  $X$  is  $\mathcal{U}$ -locally finite if for each  $x \in X$ , there exists a neighbourhood of  $x$  intersecting a finite number of

members of  $\mathcal{C}$ , the neighbourhood being  $(\mathcal{P}_i)$ open if  $x$  belongs to a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ .

In other words,  $\mathcal{C}$  is said to be  $\mathcal{U}$ -locally finite if for each  $x \in X$ , there exists a  $(\mathcal{P}_{\mathcal{U}_x})$ open nbd of  $x$  intersecting a finite number of members of  $\mathcal{C}$ .

**DEFINITION 2.2.** The bitopological space  $X$  is *pairwise paracompact* if every pairwise open cover  $\mathcal{U}$  of  $X$  has a  $\mathcal{U}$ -locally finite parallel refinement.

If in the above definition, some sets  $U \in \mathcal{U}$  are both  $(\mathcal{P}_1)$ open and  $(\mathcal{P}_2)$ open, then for each such set  $U$ , we select one of  $\mathcal{P}_1$  and  $\mathcal{P}_2$  with respect to which  $U$  is open. For this choice, we have a  $\mathcal{U}$ -locally finite refinement of  $\mathcal{U}$ . Changing the choice, we get a class of  $\mathcal{U}$ -locally finite refinements of  $\mathcal{U}$ . If there are two distinct sets  $U_1, U_2 \in \mathcal{U}$  such that for  $i \in \{1, 2\}$ ,  $U_i$  is  $(\mathcal{P}_i)$ open and  $U_1 \cap U_2 \neq \emptyset$ , then for  $\mathcal{U}$ -local finiteness of a subcollection  $\mathcal{C}$  of the refinement  $\mathcal{V}$  of  $\mathcal{U}$  at the points  $x \in U_1 \cap U_2$ , we must get two nbds  $N_i, i \in \{1, 2\}$  of  $x$  such that  $N_i$  is  $(\mathcal{P}_i)$ open and each intersects a finite number of members of  $\mathcal{C}$ .

**EXAMPLE 2.1** (Fletcher et al. [19], p. 330). Let  $X$  be the set of nonnegative reals,  $\mathcal{P}$  be the usual topology on  $X$  and let  $\mathcal{Q} = \{\emptyset\} \cup \{U \cup (x, \infty) \mid U \in \mathcal{P} \text{ and } x \in X\}$ . The bitopological space  $(X, \mathcal{P}, \mathcal{Q})$  is pairwise compact, and hence pairwise paracompact. It is also pairwise Hausdorff but the topologies do not coincide.

DEFINITION 2.3. The bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is said to be *strongly pairwise regular* if it is pairwise regular, and if both the topological spaces  $(X, \mathcal{P}_1)$  and  $(X, \mathcal{P}_2)$  are regular.

The above notion is substantiated by giving examples (Example 2.2 and Example 2.3).

Now we prove three results on pairwise paracompactness. Theorem 2.3 is a bitopological version of Michael's theorem (Theorem 2.1).

THEOREM 2.2. *If  $X$  is pairwise Hausdorff and pairwise paracompact, then it is pairwise normal.*

PROOF. Let  $x \in X$  and  $F$  be a  $(\mathcal{P}_i)$ -closed set with  $x \notin F$ . For  $\xi \in F$ , there exist  $U_\xi \in \mathcal{P}_i$  and  $V_\xi \in \mathcal{P}_j$ ,  $i \neq j$  such that  $x \in U_\xi$ ,  $\xi \in V_\xi$  and  $U_\xi \cap V_\xi = \emptyset$ . Then the collection  $\mathcal{V} = \{V_\xi \mid \xi \in F\} \cup \{X - F\}$  forms a pairwise open cover of  $X$ . Therefore it has a  $\mathcal{V}$ -locally finite parallel refinement  $\mathcal{W}$ . Let  $H = \bigcup \{W \in \mathcal{W} \mid W \cap F \neq \emptyset\}$ . Since  $\mathcal{W}$  is a parallel refinement of  $\mathcal{V}$ ,  $F \cap W \neq \emptyset$  implies  $W \in \mathcal{P}_j$ . Thus  $H \in \mathcal{P}_j$ . It is clear that  $F \subset H$ . Now  $x \in X - F$  and  $X - F$  is a  $(\mathcal{P}_i)$ -open set belonging to  $\mathcal{V}$ . So there exists a  $(\mathcal{P}_i)$ -open nbd  $D$  of  $x$  intersecting a finite number of  $(\mathcal{P}_j)$ -open sets  $W_1, W_2, \dots, W_n$  of  $\mathcal{W}$  with  $W_k \cap F \neq \emptyset$ ,  $k = 1, 2, \dots, n$ . Let  $W_k \subset V_{\xi_k}$ ,  $\xi_k \in F$ ,  $k = 1, 2, \dots, n$ . Then  $G = D \cap (\bigcap_{k=1}^n U_{\xi_k}) \in \mathcal{P}_i$ . Also  $x \in G$  and  $G \cap H = \emptyset$ . Thus  $X$  is pairwise regular. Therefore, given

two disjoint sets  $A$  and  $B$  which are  $(\mathcal{P}_j)$ closed and  $(\mathcal{P}_i)$ closed respectively, for  $x \in B$ , we get two sets  $U_x \in \mathcal{P}_i$  and  $V_x \in \mathcal{P}_j$  such that  $A \subset U_x$ ,  $x \in V_x$  and  $U_x \cap V_x = \emptyset$ . Then the pairwise open cover  $\mathcal{C} = \{V_x \mid x \in B\} \cup \{X - B\}$  of  $X$  has a  $\mathcal{C}$ -locally finite parallel refinement  $\mathcal{G}$ . If  $V = \bigcup\{G \in \mathcal{G} \mid G \cap B \neq \emptyset\}$ , then  $V \in \mathcal{P}_j$  and  $B \subset V$ . Now consider a point  $y \in A$ . Then  $y$  belongs to the  $(\mathcal{P}_i)$ open set  $X - B$  of the cover  $\mathcal{C}$  and so there exists a  $(\mathcal{P}_i)$ open nbd  $D_y$  of  $y$  intersecting a finite number of elements  $G_1(y), G_2(y), \dots, G_m(y)$  of  $\mathcal{G}$  such that for  $k = 1, 2, \dots, m$ ,  $B \cap G_k(y) \neq \emptyset$ . If  $G_k(y) \subset V_{x_k}$ ,  $x_k \in B$ , then  $U_y \cap V = \emptyset$  and  $y \in U_y$  where  $U_y = D_y \cap (\bigcap_{k=1}^m U_{x_k}) \in \mathcal{P}_i$ . Now if  $U = \bigcup_{y \in A} U_y$ , then  $U \in \mathcal{P}_i$ ,  $A \subset U$  and  $U \cap V = \emptyset$ . Therefore  $X$  is pairwise normal.  $\square$

**THEOREM 2.3.** *If the bitopological space  $X$  is strongly pairwise regular, then the following statements are equivalent.*

- (a)  $X$  is pairwise paracompact.
- (b) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a parallel refinement  $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ , where each  $\mathcal{V}_n$  is  $\mathcal{U}$ -locally finite.
- (c) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a  $\mathcal{U}$ -locally finite refinement.
- (d) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a  $\mathcal{U}$ -locally finite refinement  $\mathcal{B}$  such that if  $B \subset U \in \mathcal{U}$ ,  $B \in \mathcal{B}$ , then  $((\mathcal{P}_1)clB) \cup ((\mathcal{P}_2)clB) \subset U$ .

**PROOF.** (a) $\Rightarrow$ (b): Straightforward.

(b) $\Rightarrow$ (c): Let  $\mathcal{U}$  be a pairwise open cover of  $X$ . Suppose  $\mathcal{H}$  is a parallel refinement of  $\mathcal{U}$  such that  $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ , where each  $\mathcal{H}_n$  is  $\mathcal{U}$ -locally finite. Let  $\mathcal{H}_n = \{H_{n\alpha} \mid \alpha \in A\}$ . If  $W_n = \bigcup_{\alpha} H_{n\alpha}$ , then the collection  $\{W_n \mid n \in N\}$  is a cover of  $X$ . We write  $E_n = W_n - \bigcup_{k < n} W_k$ . Let  $x \in X$ . If  $n(x)$  is the first  $n$  for which  $x \in W_{n(x)}$ , then  $x \in E_{n(x)}$ . So  $\{E_n \mid n \in N\}$  is a refinement of  $\{W_n \mid n \in N\}$ . It is also  $\mathcal{U}$ -locally finite. Indeed, there exists an  $\alpha_0$  such that  $x \in H_{n(x)\alpha_0}$ . Then  $H_{n(x)\alpha_0}$  is a  $(\mathcal{P}_{\mathcal{U}_x})$ open nbd of  $x$ , which intersects only a finite number of members of  $\{E_n \mid n \in N\}$ . Now the collection  $\{E_n \cap H_{n\alpha} \mid n \in N, \alpha \in A\}$  is a refinement of  $\mathcal{U}$ . It is also  $\mathcal{U}$ -locally finite. In fact, each  $x \in X$  has a  $(\mathcal{P}_{\mathcal{U}_x})$ open nbd intersecting a finite number of members of  $\{E_n \mid n \in N\}$ , and for each such  $n$ , the point  $x$  has a  $(\mathcal{P}_{\mathcal{U}_x})$ open nbd intersecting at most a finite number of members of  $\{H_{n\alpha} \mid \alpha \in A\}$ .

(c) $\Rightarrow$ (d): For  $x \in X$ , consider  $U_x \in \mathcal{U}$  with  $x \in U_x$ . Suppose  $U_x$  is  $(\mathcal{P}_i)$ open. By the pairwise regularity of  $X$ , there exists a  $(\mathcal{P}_i)$ open set  $G_x^1$  such that

$$x \in G_x^1 \subset (\mathcal{P}_j)\text{cl}G_x^1 \subset U_x.$$

Again by the regularity of  $(X, \mathcal{P}_i)$ , we obtain a  $(\mathcal{P}_i)$ open set  $G_x^2$  such that

$$x \in G_x^2 \subset (\mathcal{P}_i)\text{cl}G_x^2 \subset U_x.$$

We write  $G_x = G_x^1 \cap G_x^2$ . Then  $G_x$  is  $(\mathcal{P}_i)$ open with  $x \in G_x$ .

Now

$$\begin{aligned}
(\mathcal{P}_i)\text{cl}G_x &= (\mathcal{P}_i)\text{cl}(G_x^1 \cap G_x^2) \\
&\subset ((\mathcal{P}_i)\text{cl}G_x^1) \cap ((\mathcal{P}_i)\text{cl}G_x^2) \\
&\subset (\mathcal{P}_i)\text{cl}G_x^2 \subset U_x.
\end{aligned}$$

Similarly,

$$(\mathcal{P}_j)\text{cl}G_x \subset (\mathcal{P}_j)\text{cl}G_x^1 \subset U_x.$$

It then follows by the strong pairwise regularity of  $X$ , for each  $x \in X$  with  $x \in U_x \in \mathcal{U} \cup \mathcal{P}_i$ , there exists a  $(\mathcal{P}_i)$ open nbd  $G_x$  of  $x$  such that

$$((\mathcal{P}_1)\text{cl}G_x) \cup ((\mathcal{P}_2)\text{cl}G_x) \subset U_x.$$

Then  $\mathcal{G} = \{G_x \mid x \in X\}$  is a pairwise open cover of  $X$ . Therefore by (c), there is a cover  $\mathcal{B}$  of  $X$ , which is a  $\mathcal{G}$ -locally finite refinement of  $\mathcal{G}$ , and hence a  $\mathcal{U}$ -locally finite refinement of  $\mathcal{U}$ . If  $B \in \mathcal{B}$ , then for some  $G_x$ , we have  $B \subset G_x \subset U_x$ , and so

$$((\mathcal{P}_1)\text{cl}B) \cup ((\mathcal{P}_2)\text{cl}B) \subset ((\mathcal{P}_1)\text{cl}G_x) \cup ((\mathcal{P}_2)\text{cl}G_x) \subset U_x.$$

(d) $\Rightarrow$ (a): Let  $\mathcal{U}$  be a pairwise open cover of  $X$ . It is sufficient to consider the case when there are no sets belonging to  $\mathcal{U}$ , open with respect to both  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Let  $\mathcal{A}$  be a  $\mathcal{U}$ -locally finite refinement of  $\mathcal{U}$ . For  $x \in X$ , let  $x \in U \in \mathcal{U}$  and  $U$  be  $(\mathcal{P}_i)$ open. Suppose  $W_x$  is a  $(\mathcal{P}_i)$ open nbd of  $x$  intersecting a finite number of members of  $\mathcal{A}$ . Then  $\mathcal{W} = \{W_x \mid x \in X\}$  is a

pairwise open cover of  $X$ . Let  $\mathcal{E} = \{E_\lambda \mid \lambda \in \Lambda\}$  be a  $\mathcal{W}$ -locally finite refinement of  $\mathcal{W}$  such that if  $E_\lambda \subset W_x$ , then

$$((\mathcal{P}_1)\text{cl}E_\lambda) \cup ((\mathcal{P}_2)\text{cl}E_\lambda) \subset W_x.$$

For  $A \in \mathcal{A}$ , we choose  $U_A \in \mathcal{U}$  such that  $A \subset U_A$ . If  $U_A$  is  $(\mathcal{P}_i)$ open, we write

$$F_A = \bigcup \{(\mathcal{P}_i)\text{cl}E \mid E \in \mathcal{E}, ((\mathcal{P}_i)\text{cl}E) \cap A = \emptyset\}.$$

We define  $G_A = X - F_A$  and  $H_A = U_A \cap G_A$ . For  $x \in X$ , if  $x \in A \in \mathcal{A}$ , then  $x \notin F_A$ . So  $x \in G_A$ . Also  $x \in A \subset U_A$ . Therefore  $x \in H_A$ . So the collection  $\{H_A \mid A \in \mathcal{A}\}$  covers  $X$ .

Now let  $p \in U_A \cap ((\mathcal{P}_i)\text{cl}F_A)$ . Then  $p \in U_A$ . Thus there exists a  $(\mathcal{P}_i)$ open nbd  $G_p$  of  $p$  intersecting finitely many  $E_1, E_2, \dots, E_n \in \mathcal{E}$ . So  $G_p$  intersects a finite number of sets  $(\mathcal{P}_i)\text{cl}E_1, (\mathcal{P}_i)\text{cl}E_2, \dots, (\mathcal{P}_i)\text{cl}E_n$  from the collection  $\{(\mathcal{P}_i)\text{cl}E \mid E \in \mathcal{E}, i \in \{1, 2\}\}$ . Also every  $(\mathcal{P}_i)$ open nbd of  $p$  intersects  $F_A$ . So every  $(\mathcal{P}_i)$ open nbd of  $p$  must then intersect  $((\mathcal{P}_i)\text{cl}E_1) \cup ((\mathcal{P}_i)\text{cl}E_2) \cup \dots \cup ((\mathcal{P}_i)\text{cl}E_n)$ . Therefore we have

$$\begin{aligned} p &\in (\mathcal{P}_i)\text{cl}(((\mathcal{P}_i)\text{cl}E_1) \cup ((\mathcal{P}_i)\text{cl}E_2) \cup \dots \cup ((\mathcal{P}_i)\text{cl}E_n)) \\ &= ((\mathcal{P}_i)\text{cl}E_1) \cup ((\mathcal{P}_i)\text{cl}E_2) \cup \dots \cup ((\mathcal{P}_i)\text{cl}E_n). \end{aligned}$$

Hence for some  $k \in \{1, 2, \dots, n\}$ ,  $p \in (\mathcal{P}_i)\text{cl}E_k$ . Thus

$$\begin{aligned} p &\in \bigcup \{(\mathcal{P}_i)\text{cl}E \mid E \in \mathcal{E}, ((\mathcal{P}_i)\text{cl}E) \cap A = \emptyset\} \\ &= F_A \\ \Rightarrow p &\in U_A \cap F_A. \end{aligned}$$

Therefore

$$\begin{aligned} U_A \cap ((\mathcal{P}_i)\text{cl}F_A) &\subset U_A \cap F_A \\ \Rightarrow U_A \cap ((\mathcal{P}_i)\text{cl}F_A) &= U_A \cap F_A. \end{aligned}$$

Thus

$$\begin{aligned} H_A &= U_A - U_A \cap ((\mathcal{P}_i)\text{cl}F_A) \\ &= U_A \cap (X - (\mathcal{P}_i)\text{cl}F_A). \end{aligned}$$

So  $H_A$  is  $(\mathcal{P}_i)$ open. Also it follows,  $H_A \subset U_A$ . Thus  $\{H_A \mid A \in \mathcal{A}\}$  is a parallel refinement of  $\mathcal{U}$ .

Now we show that  $\{H_A \mid A \in \mathcal{A}\}$  is  $\mathcal{U}$ -locally finite. It is sufficient to show that the collection  $\{G_A \mid A \in \mathcal{A}\}$  is  $\mathcal{U}$ -locally finite. Let  $x \in X$  and  $x$  belongs to a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ . Then there exists a  $(\mathcal{P}_i)$ open nbd  $D_x$  of  $x$  intersecting a finite number of  $E_{\lambda_1}, E_{\lambda_2}, \dots, E_{\lambda_n}$  of  $\mathcal{E}$ . Now

$$\begin{aligned} D_x \cap G_A &\neq \emptyset, \\ \Rightarrow E_{\lambda_k} \cap G_A &\neq \emptyset \text{ for some } k = 1, 2, \dots, n, \\ \Rightarrow ((\mathcal{P}_i)\text{cl}E_{\lambda_k}) \cap A &\neq \emptyset \text{ if } U_A \text{ is } (\mathcal{P}_i)\text{open}. \end{aligned}$$

Since for each  $k$ ,  $(\mathcal{P}_i)\text{cl}E_{\lambda_k}$  is contained in some  $W_{x_k} \in \mathcal{W}$ , and  $W_{x_k}$  intersects a finite number of sets belonging to  $\mathcal{A}$ , it follows that  $(\mathcal{P}_i)\text{cl}E_{\lambda_k}$  intersects a finite number of sets belonging to  $\mathcal{A}$ , and hence  $D_x$  intersects a finite number of sets of  $\{G_A \mid A \in \mathcal{A}\}$ . Thus  $\{G_A \mid A \in \mathcal{A}\}$  is  $\mathcal{U}$ -locally finite.  $\square$

**THEOREM 2.4.** *Let  $X$  be pairwise paracompact. If the topological space  $(X, \mathcal{P}_j)$  is regular, and if for  $i \neq j$ ,  $F$  is a  $(\mathcal{P}_i)F_\sigma$  proper subset of  $X$ , then  $F$  is  $(\mathcal{P}_j)$ paracompact.*

**PROOF.** Let  $F = \bigcup_{n=1}^{\infty} F_n$ , where for each  $n$ ,  $F_n$  is a  $(\mathcal{P}_i)$ closed set. Suppose  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  is a  $(\mathcal{P}_j)$ open cover of  $F$ . Then  $U_\alpha = F \cap V_\alpha$ , where  $V_\alpha$  is  $(\mathcal{P}_j)$ open in  $X$ . For each fixed  $n$ ,  $\mathcal{V} = \{V_\alpha\} \cup \{X - F_n\}$  is a pairwise open cover of  $X$ . Hence it has a  $\mathcal{V}$ -locally finite parallel refinement  $\{W_\alpha^n\}$ . Let  $\mathcal{G}_n = \{W_\alpha^n \cap F \mid W_\alpha^n \cap F_n \neq \emptyset\}$ . Then for each  $n$ ,  $\mathcal{G}_n$  is  $(\mathcal{P}_j)$ locally finite. Also  $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$  is a  $(\mathcal{P}_j)$ open cover of  $F$ , and is a refinement of  $\mathcal{U}$ . Therefore by Theorem 2.1,  $F$  is  $(\mathcal{P}_j)$ paracompact.  $\square$

Now we present two examples of bitopological spaces: one is strongly pairwise regular, and the other is pairwise regular but not strongly pairwise regular.

**EXAMPLE 2.2.** Let  $a, b \in R$  with  $a < b$ , and  $\mathcal{T}$  be the collection of subsets  $G$  of  $R$  for which  $(a, b) \subset R - G$  or  $G \cap (a, b)$  is the union of some open subintervals of  $(a, b)$ . Then  $\mathcal{T}$  is a topology on  $R$ . Firstly, we show  $(R, \mathcal{T})$  is regular. Let  $x \in R$  and  $F$  be  $(\mathcal{T})$ closed with  $x \notin F$ . We may now have the following cases.

Case I:  $(a, b) \subset F$ . Then  $(a, b) \cap F = (a, b)$  and hence  $F$  is  $(\mathcal{T})$ open. Also  $F$  being  $(\mathcal{T})$ closed,  $R - F$  is  $(\mathcal{T})$ open with

$x \in R - F$ . So we obtain ( $\mathcal{T}$ )open sets  $U = R - F$ ,  $V = F$  such that  $x \in U$ ,  $F \subset V$  and  $U \cap V = \emptyset$ .

Case II:  $(a, b) \subset R - F$ . Just like case I, in this case also  $F$  is both ( $\mathcal{T}$ )open and ( $\mathcal{T}$ )closed.

Case III:  $(a, b) \cap F \neq \emptyset$ ,  $(a, b) \cap (R - F) \neq \emptyset$ .

Subcase i:  $x \notin (a, b)$ . Here we choose  $V = F \cup (a, b)$  and  $U = R - V$ . Since  $(a, b) \subset V$ ,  $V$  is ( $\mathcal{T}$ )open. Again  $(a, b) \subset V = R - U$  which means  $U$  is ( $\mathcal{T}$ )open. Thus we get  $x \in U$ ,  $F \subset V$  with  $U \cap V = \emptyset$ .

Subcase ii:  $x \in (a, b)$ . So  $x \in (a, b) \cap (R - F)$ . Since  $R - F$  is ( $\mathcal{T}$ )open, it contains open subintervals of  $(a, b)$ . Let  $(\alpha, \beta)$  be an open subinterval of  $(a, b)$  such that  $x \in (\alpha, \beta) \subset R - F$ . We put  $U = F \cup (a, \alpha) \cup (\beta, b)$  and  $V = (\alpha, \beta)$ . Here  $U \cap (a, b)$  is the union of open subintervals  $(a, \alpha)$  and  $(\beta, b)$  of  $(a, b)$ . So  $U$  is ( $\mathcal{T}$ )open. Since  $V$  is an open subinterval of  $(a, b)$ ,  $V$  is ( $\mathcal{T}$ )open. Also we have  $F \subset U$ ,  $x \in V$  with  $U \cap V = \emptyset$ .

Hence  $(R, \mathcal{T})$  is regular.

If  $\rho$  is the usual topology on  $R$ , then  $(R, \rho)$  is also regular.

We now show that the bitopological space  $(R, \rho, \mathcal{T})$  is pairwise regular. Let  $x \in X$  and  $F$  be  $(\rho)$ closed with  $x \notin F$ . Since  $(R, \rho)$  is regular and  $F$  is  $(\rho)$ closed with  $x \notin F$ , there exists  $(\rho)$ open sets  $U$  and  $V$  such that  $x \in U$ ,  $F \subset V$  with  $U \cap V = \emptyset$ . We note,  $\rho \subset \mathcal{T}$ . Thus  $V$  is also ( $\mathcal{T}$ )open.

Let  $x \in X$  and  $F$  be  $(\mathcal{T})$ closed with  $x \notin F$ . We consider the cases I, II, III.

Case I:  $(a, b) \subset F$ . We choose  $V = R - \{x\}$  and  $U = \{x\}$ . So  $V$  is  $(\rho)$ open and  $U$  is  $(\mathcal{T})$ open with  $F \subset V$ ,  $x \in U$  and  $U \cap V = \emptyset$ .

Case II:  $(a, b) \subset R - F$ .

Subcase i:  $x \notin (a, b)$ . This case is similar to Case I.

Subcase ii: Let  $x \in (a, b)$ . Then there exist reals  $c, d$  such that  $a < c < x < d < b$ . We choose  $U = (c, d)$  and  $V = R - [c, d]$ . So  $U$  is  $(\mathcal{T})$ open,  $V$  is  $(\rho)$ open with  $x \in U$ ,  $F \subset V$  and  $U \cap V = \emptyset$ .

Case III:  $(a, b) \cap F \neq \emptyset$ ,  $(a, b) \cap (R - F) \neq \emptyset$ .

Subcase i:  $x \in (a, b)$ . Hence  $x \in (a, b) \cap (R - F)$ , and  $R - F$  is  $(\mathcal{T})$ open and so contains a subinterval  $(\alpha, \beta)$  of  $(a, b)$  such that  $x \in (\alpha, \beta) \subset (a, b) \cap (R - F)$ . We choose reals  $c, d$  such that  $\alpha < c < x < d < \beta$ . Then as in Case II (Subcase ii), we get the required sets  $U = (c, d)$  and  $V = R - [c, d]$ .

Subcase ii:  $x \notin (a, b)$ . The case is similar to Case I.

Thus the bitopological space  $(R, \rho, \mathcal{T})$  is pairwise regular. Hence the space  $(R, \rho, \mathcal{T})$  is strongly pairwise regular.

**EXAMPLE 2.3.** Suppose  $\mathcal{T} = \{G \subset R \mid R - G \text{ is a bounded set}\} \cup \{\emptyset\}$ . Then  $\mathcal{T}$  forms a topology on  $R$ . Also let  $\rho$  be the usual topology on  $R$ .

Firstly, let  $x \in R$  and  $F$  be  $(\mathcal{T})$ closed such that  $x \notin F$ . Since  $F$  is bounded, we get two real numbers  $a, b$  such that  $a < y < b$  for all  $y \in F$ . We note,  $(a, b)$  is  $(\rho)$ open and  $F \subset (a, b)$ . Even if  $x \in (a, b)$ , we may have some open subintervals  $(c, d)$  of  $(a, b)$  such that  $x \notin (c, d)$  and  $F \subset \bigcup \{(c, d) \mid (c, d) \subset (a, b)\} = G$  (say). We choose,  $H = R - G$ . Here  $G$  is  $(\rho)$ open and  $H$  is  $(\mathcal{T})$ open with  $F \subset G$ ,  $x \in H$  and  $G \cap H = \emptyset$ . Next, let  $x \in R$  and  $F$  be  $(\rho)$ closed such that  $x \notin F$ . Since  $F$  is  $(\rho)$ closed, there exists a bounded  $(\rho)$ open set  $H$  such that  $x \in H$  and  $F \cap H = \emptyset$ . Thus  $G = R - H$  is a  $(\mathcal{T})$ open set with  $F \subset G$ . Thus it follows, the bitopological space  $(R, \rho, \mathcal{T})$  is pairwise regular.

Let  $x \in R$  and  $F$  be  $(\mathcal{T})$ closed with  $x \notin F$ . Then  $F$  is a bounded set. If possible, suppose there exist two  $(\mathcal{T})$ open sets  $G$  and  $H$  such that  $x \in G$ ,  $F \subset H$  and  $G \cap H = \emptyset$ . Since  $G$  is  $(\mathcal{T})$ open,  $R - G$  is bounded. Also  $H \subset R - G$  which implies  $H$  is a bounded set. This is a contradiction to the fact,  $R - H$  is a bounded set. So the space  $(R, \mathcal{T})$  is not regular, and hence  $(R, \rho, \mathcal{T})$  is not strongly pairwise regular.

## CHAPTER III

### Bitopological Full Normality

#### 3.1. Introduction

Since the introduction of bitopological spaces by Kelly [26], a considerable amount of works were done on bitopological spaces by several authors. Generalizing the notion of pairwise compactness, we have introduced a notion of pairwise paracompactness and have obtained an analogue of Michael's theorem (Theorem 2.1) in Chapter II. In this chapter, we introduce the notions of pairwise full normality (Definition 3.5) and  $a$ -pairwise full normality (Definition 3.5). For a pairwise Hausdorff bitopological space  $X$ , we prove that  $X$  is  $a$ -pairwise fully normal if it is pairwise paracompact, and conversely,  $X$  is pairwise paracompact if it is pairwise fully normal. To prove the converse part, we use the above Michael's theorem on pairwise paracompactness.

#### 3.2. Definitions

Let  $X$  denote the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$ .

Throughout this chapter, for each  $i \in \{1, 2\}$ ,  $\mathcal{U}^i$  denotes the class of  $(\mathcal{P}_i)$ open sets belonging to the pairwise open cover  $\mathcal{U}$  of  $X$ . For a point  $x \in X$ , a set  $A \subset X$  and a collection  $\mathcal{C}$  of

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The subject matter of this chapter forms our paper: *Mat. Vesnik* **62** (1) (2010), 11–18, [4].

subsets of  $X$ , we write

$$\begin{aligned} St(x, \mathcal{C}) &= \bigcup \{C \in \mathcal{C} \mid x \in C\}, \\ St(A, \mathcal{C}) &= \bigcup \{C \in \mathcal{C} \mid A \cap C \neq \emptyset\}. \end{aligned}$$

A cover  $\mathcal{U}$  of  $X$  is said to be point finite (Dugundji [15]) if each  $x \in X$  belongs to only finitely many  $U \in \mathcal{U}$ .

In the sequel, we use the following theorem.

**THEOREM 3.1** (Dugundji [15], p. 82). *Let  $\{A_\alpha \mid \alpha \in \Delta\}$  be a locally finite family of subsets in a topological space  $(X, \mathcal{T})$ . Then*

- (i)  $\{(\mathcal{T})clA_\alpha \mid \alpha \in \Delta\}$  is also locally finite.
- (ii) For each  $\Lambda \subset \Delta$ ,  $\bigcup \{(\mathcal{T})clA_\alpha \mid \alpha \in \Lambda\}$  is closed in  $X$ .

We introduce the following definitions.

**DEFINITION 3.1.** Let  $\mathcal{U}$  be a pairwise open cover of  $X$ . A parallel refinement  $\mathcal{V}$  of  $\mathcal{U}$  is said to be a *parallel star* (resp. *barycentric*) *refinement* of  $\mathcal{U}$  whenever it satisfies the following conditions: (1) if there are two distinct sets  $U_1, U_2 \in \mathcal{U}$  such that  $U_i$  is  $(\mathcal{P}_i)$ open ( $i \in \{1, 2\}$ ) and  $U_1 \cap U_2 \neq \emptyset$ , then for  $x \in U_1 \cap U_2$ , there are two sets  $V_1, V_2 \in \mathcal{V}$  such that  $V_i \subset U_i$ ,  $V_i$  is  $(\mathcal{P}_i)$ open ( $i \in \{1, 2\}$ ) and  $x \in V_1 \cap V_2$ ; (2) for any  $V \in \mathcal{V}$  (resp.  $x \in X$ ), there exists a  $U \in \mathcal{U}$  such that  $St(V, \mathcal{V}) \subset U$  (resp.  $St(x, \mathcal{V}) \subset U$ ).

A  $(\mathcal{P})$ open refinement  $\mathcal{V}$  of  $\mathcal{U}$  is said to be a  $(\mathcal{P})$ open barycentric refinement of  $\mathcal{U}$  if for any  $x \in X$ , there exists a  $U \in \mathcal{U}$  such that  $St(x, \mathcal{V}) \subset U$ .

DEFINITION 3.2. A set  $G \in \mathcal{P}$  is said to be  $(\mathcal{P}_j^*)$ open if it is a union of a  $(\mathcal{P}_i)$ open set and a nonempty  $(\mathcal{P}_j)$ open set. The complement of a  $(\mathcal{P}_j^*)$ open set is called a  $(\mathcal{P}_j^*)$ closed set.

DEFINITION 3.3.  $X$  is said to be  $\alpha$ -pairwise normal if for any pair of a  $(\mathcal{P}_i)$ closed set  $A$  and a  $(\mathcal{P}_j^*)$ closed set  $B$  with  $A \cap B = \emptyset$ , there exist a set  $U \in \mathcal{P}$  and a set  $V \in \mathcal{P}_i$  such that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$ .

It is easy to see that  $X$  is  $\alpha$ -pairwise normal iff for any  $(\mathcal{P}_j^*)$ closed set  $K$  and any  $(\mathcal{P}_i)$ open set  $U$  with  $K \subset U$ , there exists a  $(\mathcal{P}_i)$ open set  $V$  such that  $K \subset V \subset (\mathcal{P})\text{cl}V \subset U$ .

DEFINITION 3.4. A pairwise open cover  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  is said to be *shrinkable* if there exists a pairwise open cover  $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$  such that for each  $\alpha \in A$ ,  $(\mathcal{P})\text{cl}V_\alpha \subset U_\alpha$ .  $\mathcal{V}$  is then called a *shrinking* of  $\mathcal{U}$ .

DEFINITION 3.5.  $X$  is said to be *pairwise* (resp. *a-pairwise*) *fully normal* if for every pairwise open cover  $\mathcal{U}$  of  $X$ , there is a pairwise open (resp.  $(\mathcal{P})$ open) cover  $\mathcal{V}$  of  $X$  such that  $\mathcal{V}$  is a parallel (resp.  $(\mathcal{P})$ open) star (resp. barycentric) refinement of  $\mathcal{U}$ .

### 3.3. Theorems

**THEOREM 3.2.**  *$X$  is pairwise fully normal iff for every pairwise open cover  $\mathcal{U}$  of  $X$ , there is a pairwise open cover  $\mathcal{V}$  of  $X$  such that  $\mathcal{V}$  is a parallel barycentric refinement of  $\mathcal{U}$ .*

The above theorem can be proved with standard arguments.

**THEOREM 3.3.** *If  $X$  is pairwise fully normal, then it is  $\alpha$ -pairwise normal and pairwise normal.*

**PROOF.** Let  $A$  and  $B$  be two disjoint subsets of  $X$  which are  $(\mathcal{P}_i)$ closed and  $(\mathcal{P}_j^*)$ closed respectively with  $i \neq j$ . Then there exist a  $(\mathcal{P}_i)$ open set  $G_1$  and a nonempty  $(\mathcal{P}_j)$ open set  $G_2$  such that  $X - B = G_1 \cup G_2$ . So  $\{X - A, G_1, G_2\}$  is a pairwise open cover of  $X$ . Therefore there exists a parallel star refinement  $\mathcal{V}$  of  $\{X - A, G_1, G_2\}$ . Then  $G = St(A, \mathcal{V})$  and  $H = St(B, \mathcal{V})$  are  $(\mathcal{P})$ open and  $(\mathcal{P}_i)$ open respectively with  $A \subset G$  and  $B \subset H$ . We claim  $G \cap H = \emptyset$ . If  $G \cap H \neq \emptyset$ , then there exist  $V', V'' \in \mathcal{V}$  with  $A \cap V' \neq \emptyset$ ,  $B \cap V'' \neq \emptyset$  and  $V' \cap V'' \neq \emptyset$ . Since  $St(V', \mathcal{V}) = \bigcup\{V \in \mathcal{V} \mid V' \cap V \neq \emptyset\}$  and  $V' \cap V'' \neq \emptyset$ , we have  $V'' \subset St(V', \mathcal{V})$ . Also  $V'' \cap B \neq \emptyset$  and hence  $B \cap St(V', \mathcal{V}) \neq \emptyset$ . Again  $V' \cap A \neq \emptyset$  and  $V' \subset St(V', \mathcal{V})$ . Thus  $A \cap St(V', \mathcal{V}) \neq \emptyset$ . So  $St(V', \mathcal{V})$  intersects both  $A$  and  $B$  which is impossible. Thus  $X$  is  $\alpha$ -pairwise normal.

To prove pairwise normality of  $X$ , we suppose that  $A$  is  $(\mathcal{P}_i)$ closed and  $B$  is  $(\mathcal{P}_j)$ closed with  $A \cap B = \emptyset$ . So  $\{X -$

$A, X - B\}$  is a pairwise open cover of  $X$ . Hence by pairwise full normality of  $X$ , there exists a pairwise open cover  $\mathcal{V}$  of  $X$  such that  $\mathcal{V}$  is a parallel star refinement of  $\{X - A, X - B\}$ . We write  $G = St(A, \mathcal{V})$ ,  $H = St(B, \mathcal{V})$ . Then  $G$  is  $(\mathcal{P}_j)$ open and  $H$  is  $(\mathcal{P}_i)$ open with  $A \subset G$ ,  $B \subset H$ . If  $G \cap H \neq \emptyset$ , then there exist  $V_1, V_2 \in \mathcal{V}$  such that  $A \cap V_1 \neq \emptyset$ ,  $B \cap V_2 \neq \emptyset$  and  $V_1 \cap V_2 \neq \emptyset$ . Then proceeding as above, we see  $St(V_1, \mathcal{V})$  intersects both  $A$  and  $B$  which is impossible. So  $G \cap H = \emptyset$  and hence  $X$  is pairwise normal.  $\square$

Now we present an example of a bitopological space which is  $\alpha$ -pairwise normal but not pairwise normal.

EXAMPLE 3.1. For any  $a \in R$ , we define

$$\begin{aligned}\mathcal{P}_1 &= \{\emptyset, R, (-\infty, a], (a, \infty)\}, \\ \text{and } \mathcal{P}_2 &= \{\emptyset, R, (-\infty, a), [a, \infty)\}.\end{aligned}$$

It is easy to see that  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are topologies on  $R$  and hence  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is a bitopological space.

We note that each set belonging to  $\mathcal{P}_i$  ( $i \in \{1, 2\}$ ) is both  $(\mathcal{P}_i)$ closed and  $(\mathcal{P}_i)$ open. Let  $F$  be a  $(\mathcal{P}_1)$ closed set and  $K$  be a  $(\mathcal{P}_2^*)$ closed set with  $F \cap K = \emptyset$ . Then  $F \subset R - F \in \mathcal{P}_1$  and  $K \subset R - F \in \mathcal{P}_1$ . Also  $F \cap (R - F) = \emptyset$ . Next let  $F_1$  be a  $(\mathcal{P}_1^*)$ closed set and  $K_1$  be a  $(\mathcal{P}_2)$ closed set with  $F_1 \cap K_1 = \emptyset$ . Then  $F_1 \subset R - K_1 \in \mathcal{P}_2$  and  $K_1 \subset R - K_1 \in \mathcal{P}_2$ . Also  $(R_1 - K_1) \cap K_1 = \emptyset$ . Thus the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is  $\alpha$ -pairwise normal.

Now we consider the  $(\mathcal{P}_1)$ closed set  $(a, \infty)$  and the  $(\mathcal{P}_2)$ closed set  $(-\infty, a)$ . Obviously,  $(a, \infty) \cap (-\infty, a) = \emptyset$ . The only nontrivial  $(\mathcal{P}_2)$ open set containing  $(a, \infty)$  is  $[a, \infty)$  and the only nontrivial  $(\mathcal{P}_1)$ open set containing  $(-\infty, a)$  is  $(-\infty, a]$ . But  $[a, \infty) \cap (-\infty, a] = \{a\} \neq \emptyset$ . Hence the space is not pairwise normal.

The following is an example of a pairwise normal bitopological space which is not  $\alpha$ -pairwise normal.

EXAMPLE 3.2. Let  $p \in R$ ,  $\mathcal{P}_1 = \{\emptyset, R\} \cup \{E \cup (x, \infty) \mid p \notin E \subset R, x \in R \text{ and } x \geq p + 1\}$  and  $\mathcal{P}_2 =$  the usual topology of  $R$ .

It is easy to see that  $\mathcal{P}_1$  is a topology on  $R$  and hence  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is a bitopological space.

$(R, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise normal since for any  $(\mathcal{P}_1)$ closed set  $A(\neq \emptyset, R)$ , we have

$$A = E \cap (-\infty, x], \quad p \in E \subset R, x \geq p + 1$$

and for any  $(\mathcal{P}_2)$ closed set  $B$  with  $A \cap B = \emptyset$ , we have  $p \notin B$ , one can take for  $y > x$ ,

$$\begin{aligned} U &= (X - B) \cap (-\infty, y) \in \mathcal{P}_2, \\ V &= B \cup (y, \infty) \in \mathcal{P}_1 \end{aligned}$$

so that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$ .

But  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is not  $\alpha$ -pairwise normal, since for the  $(\mathcal{P}_1)$ closed set

$$F = ((p-1, p+1) \cup (\text{the set of rationals})) \cap (-\infty, x], \quad x \geq p+1,$$

and the  $(\mathcal{P}_2^*)$ closed set

$$K = M \cap ((-\infty, p-1] \cup [p+1, \infty))$$

where  $M$  is the  $(\mathcal{P}_1)$ closed set

$$((p-1, p+1) \cup (\text{the set of irrationals})) \cap (-\infty, x], \quad x \geq p+1,$$

we have  $F \cap K = \emptyset$  but there exists no pair of a  $(\mathcal{P})$ open set  $U$  and a  $(\mathcal{P}_1)$ open set  $V$  with  $F \subset U, K \subset V$  and  $U \cap V = \emptyset$ .

From the above two examples, it follows that the notions of pairwise normality and  $\alpha$ -pairwise normality are independent.

**THEOREM 3.4.** *If  $X$  is pairwise Hausdorff and pairwise paracompact, then  $X$  is  $\alpha$ -pairwise normal.*

**PROOF.** Let us consider a  $(\mathcal{P}_i)$ closed set  $A$  and a  $(\mathcal{P}_j^*)$ closed set  $B$  with  $A \cap B = \emptyset$  and  $i \neq j$ . Let  $\xi \in B$ . Then  $\xi \notin A$ . Since  $X$  is pairwise Hausdorff and pairwise paracompact, it is pairwise regular (Theorem 2.2). Therefore there exist a set  $U_\xi \in \mathcal{P}_j$  and a set  $V_\xi \in \mathcal{P}_i$  such that  $A \subset U_\xi, \xi \in V_\xi$  and  $U_\xi \cap V_\xi = \emptyset$ . The set  $X - B$  is  $(\mathcal{P}_j^*)$ open, and so there exist a  $(\mathcal{P}_i)$ open set  $G_1$  and a nonempty  $(\mathcal{P}_j)$ open set  $G_2$  such that  $X - B = G_1 \cup G_2$ . Therefore the family  $\mathcal{V} = \{V_\xi \mid \xi \in B\} \cup \{G_1, G_2\}$  is a pairwise

open cover of  $X$ . Since  $X$  is pairwise paracompact, there exists a  $\mathcal{V}$ -locally finite parallel refinement  $\mathcal{D}$  of  $\mathcal{V}$ . Let  $V = \bigcup\{D \in \mathcal{D} \mid D \cap B \neq \emptyset\}$ . Then  $V \in \mathcal{P}_i$  and  $B \subset V$ . Now let  $x \in A \subset X - B$ . Since  $X - B = G_1 \cup G_2$  and  $G_1, G_2 \in \mathcal{V}$ , it follows that there exists a neighbourhood  $W_x$  of  $x$  such that  $W_x \in \mathcal{P}_i$  (resp.  $W_x \in \mathcal{P}_j$ ) if  $x \in G_1$  (resp.  $x \in G_2$ ) and  $W_x$  intersects a finite number of sets  $D_x^1, D_x^2, \dots, D_x^m$  with  $B \cap D_x^k \neq \emptyset$  and  $D_x^k \in \mathcal{D}$  for  $k = 1, 2, \dots, m$ . If  $D_x^k \subset V_{\xi_k}, \xi_k \in B$ , then  $U_x \cap V = \emptyset$  and  $x \in U_x$  where  $U_x = W_x \cap (\bigcap_{k=1}^m U_{\xi_k}) \in \mathcal{P}$ . If  $U = \bigcup_{x \in A} U_x$ , then  $U \in \mathcal{P}, A \subset U$  and  $U \cap V = \emptyset$ . Therefore  $X$  is  $\alpha$ -pairwise normal.  $\square$

**THEOREM 3.5.** *If  $X$  is  $\alpha$ -pairwise normal, then every point finite pairwise open cover is shrinkable.*

**PROOF.** Let  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  be a point finite pairwise open cover of  $X$ . We well-order the index set  $A$ , and write  $A = \{1, 2, \dots, \alpha, \dots\}$ . By transfinite induction, we now construct a pairwise open cover  $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$  which is a shrinking of  $\mathcal{U}$ . We write  $F_1 = X - \bigcup\{U_\alpha \mid \alpha > 1\}$ . Since  $\mathcal{U}$  is a pairwise open cover, it follows that if  $U_1$  is  $(\mathcal{P}_i)$ open, then  $F_1$  is  $(\mathcal{P}_j^*)$ closed and  $F_1 \subset U_1$ . Therefore there exists a  $(\mathcal{P}_i)$ open set  $V_1$  such that  $F_1 \subset V_1 \subset (\mathcal{P})\text{cl}V_1 \subset U_1$ . Assume that  $V_\beta$  is defined for every  $\beta < \alpha$ , and consider the set

$$F_\alpha = X - \left( \left( \bigcup\{V_\beta \mid \beta < \alpha\} \right) \cup \left( \bigcup\{U_\gamma \mid \gamma > \alpha\} \right) \right).$$

If  $U_\alpha$  is  $(\mathcal{P}_i)$ open, then  $F_\alpha$  is  $(\mathcal{P}_j^*)$ closed. Also  $F_\alpha \subset U_\alpha$ . Therefore there exists a set  $V_\alpha \in \mathcal{P}_i$  such that

$$F_\alpha \subset V_\alpha \subset (\mathcal{P})\text{cl}V_\alpha \subset U_\alpha. \quad (3.1)$$

If  $x \in X$ , then there exist a finite number of sets  $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$  such that  $x \in U_{\alpha_i}$  for all  $i = 1, 2, \dots, n$ . If  $\alpha = \max(\alpha_1, \alpha_2, \dots, \alpha_n)$ , then for  $\gamma > \alpha$ ,  $x \notin U_\gamma$ . Therefore  $x \in F_\alpha \subset V_\alpha$  if  $x \notin V_\beta$  for all  $\beta < \alpha$ . So  $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$  is a pairwise open cover of  $X$ . Hence it follows from (3.1) that  $\mathcal{V}$  is a shrinking of  $\mathcal{U}$ .  $\square$

Now we prove an analogue of A. H. Stone's theorem (Stone [45]) on paracompactness.

**THEOREM 3.6.** *Suppose  $X$  is pairwise Hausdorff. If  $X$  is pairwise paracompact, then it is  $\alpha$ -pairwise fully normal. Conversely, if  $X$  is pairwise fully normal, then it is pairwise paracompact.*

**PROOF.** At first we suppose that  $X$  is pairwise Hausdorff and pairwise paracompact.

Let  $\mathcal{U}$  be a pairwise open cover of  $X$ . Then there exists a  $\mathcal{U}$ -locally finite parallel refinement  $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$  of  $\mathcal{U}$ . Since  $\mathcal{V}$  is  $\mathcal{U}$ -locally finite, it is point finite. Again by Theorem 3.4,  $X$  is  $\alpha$ -pairwise normal, and so by Theorem 3.5, there exists a shrinking  $\mathcal{W} = \{W_\alpha \mid \alpha \in A\}$  of  $\mathcal{V}$ .  $\mathcal{W}$  is a pairwise open

cover of  $X$  such that for each  $\alpha$ ,

$$(\mathcal{P})\text{cl}W_\alpha \subset V_\alpha. \quad (3.2)$$

For  $x \in X$ , we write

$$D_x = \bigcap \{V_\alpha \mid x \in (\mathcal{P})\text{cl}W_\alpha\}. \quad (3.3)$$

From (3.2) and point finiteness of  $\mathcal{V}$ , it follows that there are finite number of  $V_\alpha$  in the intersection (3.3). Hence  $D_x \in \mathcal{P}$ .

Now let

$$K_x = \bigcup \{(\mathcal{P})\text{cl}W_\alpha \mid x \notin (\mathcal{P})\text{cl}W_\alpha\}.$$

Since  $\mathcal{V}$  is  $\mathcal{U}$ -locally finite,  $\{(\mathcal{P})\text{cl}W_\alpha\}$  is  $(\mathcal{P})$ locally finite. So by Theorem 3.1,  $K_x$  is a  $(\mathcal{P})$ closed set. Therefore  $G_x = X - K_x$  is a  $(\mathcal{P})$ open set. Hence the collection  $\mathcal{B} = \{D_x \cap G_x \mid x \in X\}$  is a  $(\mathcal{P})$ open cover of  $X$ . For  $y \in X$ , let  $y \in (\mathcal{P})\text{cl}W_\alpha$ . If  $y \in D_x \cap G_x$ , then  $x \in (\mathcal{P})\text{cl}W_\alpha$ , since otherwise  $(\mathcal{P})\text{cl}W_\alpha \subset K_x$  and hence  $y \notin G_x$ . Again if  $x \in (\mathcal{P})\text{cl}W_\alpha$ , then  $D_x \subset V_\alpha \Rightarrow D_x \cap G_x \subset V_\alpha$ . Therefore  $\mathcal{B}$  is a  $(\mathcal{P})$ open barycentric refinement of  $\mathcal{V}$  and hence of  $\mathcal{U}$ . Therefore  $X$  is  $a$ -pairwise fully normal.

Conversely, suppose  $X$  is pairwise Hausdorff and pairwise fully normal. Let  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  be a pairwise open cover of  $X$ . By Theorem 3.2, we can construct a sequence  $\{\mathcal{U}_n\}$  of pairwise open covers of  $X$  such that  $\mathcal{U}_1$  is a parallel barycentric refinement of  $\mathcal{U}$ , and for each  $n \in N$ ,  $\mathcal{U}_{n+1}$  is a parallel barycentric refinement of  $\mathcal{U}_n$ . For  $\alpha \in A$ , let

$$\begin{aligned} V_\alpha^n &= \{x \in U_\alpha \mid St(x, \mathcal{U}_n) \subset U_\alpha\}, \\ V_\alpha &= \bigcup_{n=1}^{\infty} V_\alpha^n. \end{aligned}$$

If  $x \in V_\alpha$ , then  $x \in V_\alpha^n$  for some  $n$ , and so  $St(x, \mathcal{U}_n) \subset U_\alpha$ . Now let  $y \in St(x, \mathcal{U}_{n+1})$ , then  $x \in St(y, \mathcal{U}_{n+1})$ . Since  $\mathcal{U}_{n+1}$  is a barycentric refinement of  $\mathcal{U}_n$ , it follows that,  $St(y, \mathcal{U}_{n+1}) \subset St(x, \mathcal{U}_n) \subset U_\alpha$ . So  $y \in V_\alpha^{n+1} \subset V_\alpha$ . Thus  $St(x, \mathcal{U}_{n+1}) \subset V_\alpha$ . Since  $\mathcal{U}_1$  is a barycentric refinement of  $\mathcal{U}$ , for any  $x \in X$ , there exists a  $U_\alpha$  such that  $St(x, \mathcal{U}_1) \subset U_\alpha$  and so  $x \in V_\alpha^1 \subset V_\alpha$ . Therefore  $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$  is a refinement of  $\mathcal{U}$ . We now well-order  $\mathcal{V}$  as  $V_1, V_2, \dots, V_\alpha, \dots$ . For a fixed  $n \in N$ , we define

$$\begin{aligned} B_1^n &= X - St(X - V_1, \mathcal{U}_n), \\ B_\alpha^n &= X - St\left((X - V_\alpha) \cup \left(\bigcup_{\beta < \alpha} B_\beta^n\right), \mathcal{U}_n\right) \quad \text{if } \alpha > 1. \end{aligned}$$

Therefore for all  $\alpha$ , we have

$$\begin{aligned} &B_\alpha^n \cap St(X - V_\alpha, \mathcal{U}_n) = \emptyset \\ \Rightarrow &\text{Any set } \in \mathcal{U}_n \text{ meeting } X - V_\alpha \text{ does not meet } B_\alpha^n \\ \Rightarrow &St(B_\alpha^n, \mathcal{U}_n) \cap (X - V_\alpha) = \emptyset \\ \Rightarrow &St(B_\alpha^n, \mathcal{U}_n) \subset V_\alpha. \end{aligned}$$

Also we have

$$\begin{aligned} &B_\alpha^n \cap St\left(\bigcup_{\beta < \alpha} B_\beta^n, \mathcal{U}_n\right) = \emptyset \\ \Rightarrow &B_\alpha^n \cap St(B_\beta^n, \mathcal{U}_n) = \emptyset, \quad \beta < \alpha \\ \Rightarrow &St(B_\alpha^n, \mathcal{U}_n) \cap B_\beta^n = \emptyset, \quad \beta < \alpha \\ \Rightarrow &St(B_\alpha^n, \mathcal{U}_n) \cap B_\beta^n = \emptyset \text{ for all } \beta \neq \alpha. \end{aligned} \tag{3.4}$$

Let  $x \in X$ . Since  $\{V_\alpha \mid \alpha \in A\}$  is a cover of  $X$ , there is a first index  $\alpha$  such that  $x \in V_\alpha$ . Then  $St(x, \mathcal{U}_m) \subset V_\alpha$  for some  $m$ . We now show  $x \in B_\alpha^m$ . If possible, suppose  $x \notin B_\alpha^m$ . Then

$$\begin{aligned}
& x \in St\left((X - V_\alpha) \cup \left(\bigcup_{\beta < \alpha} B_\beta^m\right), \mathcal{U}_m\right) \\
\Rightarrow & St(x, \mathcal{U}_m) \cap \left((X - V_\alpha) \cup \left(\bigcup_{\beta < \alpha} B_\beta^m\right)\right) \neq \emptyset \\
\Rightarrow & St(x, \mathcal{U}_m) \cap B_\beta^m \neq \emptyset \text{ for some } \beta < \alpha \\
& \hspace{15em} (\text{since } St(x, \mathcal{U}_m) \subset V_\alpha) \\
\Rightarrow & x \in St(B_\beta^m, \mathcal{U}_m) \subset V_\beta.
\end{aligned}$$

This contradicts the fact that  $\alpha$  is the first index for which  $x \in V_\alpha$ . Therefore  $x \in B_\alpha^m$ . Hence  $\{B_\alpha^n \mid n \in N, \alpha \in A\}$  is a cover of  $X$ . We now define

$$G_\alpha^n = St(B_\alpha^n, \mathcal{U}_{n+2}^i), n \in N, \alpha \in A \text{ if } U_\alpha \text{ is } (\mathcal{P}_i)\text{open}.$$

Then  $G_\alpha^n$  is  $(\mathcal{P}_i)$ open. Since  $St(B_\alpha^n, \mathcal{U}_n) \subset V_\alpha$ , we have  $St(B_\alpha^n, \mathcal{U}_{n+2}^i) \subset V_\alpha$  and hence  $G_\alpha^n \subset V_\alpha$ . Now let  $x \in X$ . Then  $x \in B_\alpha^n$  for some pair of  $n$  and  $\alpha$ , and so  $x \in U_\alpha$  since  $B_\alpha^n \subset St(B_\alpha^n, \mathcal{U}_n) \subset V_\alpha \subset U_\alpha$ . If  $U_\alpha$  is  $(\mathcal{P}_i)$ open, then by the definition of parallel barycentric refinement,  $x \in U$  for some  $U \in \mathcal{U}_{n+2}^i$ . So  $x \in St(B_\alpha^n, \mathcal{U}_{n+2}^i) = G_\alpha^n$ . Therefore  $\mathcal{G} = \{G_\alpha^n \mid n \in N, \alpha \in A\}$  is a cover of  $X$  and hence a parallel refinement of  $\mathcal{U}$ . We now show that there exists no  $U \in \mathcal{U}_{n+2}$  intersecting both  $G_\alpha^n$  and  $G_\beta^n$  for  $\alpha \neq \beta$ , whenever both  $U_\alpha$  and  $U_\beta$  are  $(\mathcal{P}_i)$ open. Suppose if possible,  $U \in \mathcal{U}_{n+2}$  intersects both  $G_\alpha^n$  and  $G_\beta^n$  for  $\alpha \neq \beta$  with  $U_\alpha, U_\beta \in \mathcal{P}_i$ . Then there exist  $H_1, H_2 \in \mathcal{U}_{n+2}^i$  such that  $H_1$  intersects both  $B_\alpha^n$  and  $U$ , and  $H_2$  intersects both  $B_\beta^n$  and  $U$ . Hence  $St(U, \mathcal{U}_{n+2}^i)$  intersects both  $B_\alpha^n$  and  $B_\beta^n$ . Since  $\mathcal{U}_{n+2}$  is a star refinement of  $\mathcal{U}_n$ , it follows that some  $W \in \mathcal{U}_n$  intersects

both  $B_\alpha^n$  and  $B_\beta^n$ . Therefore  $St(B_\alpha^n, \mathcal{U}_n)$  intersects  $B_\beta^n$  which contradicts (3.4).

Since  $\mathcal{U}_{n+2}$  is a parallel refinement of  $\mathcal{U}$ , it thus follows that for each  $n \in N$ ,  $\mathcal{G}_n = \{G_\alpha^n \mid \alpha \in A\}$  is  $\mathcal{U}$ -locally finite. Also we have  $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ .

Since  $X$  is pairwise Hausdorff, any singleton subset of  $X$  is  $(\mathcal{P}_i)$ -closed for each  $i \in \{1, 2\}$ . Therefore by Theorem 3.3,  $X$  is pairwise regular. Next we show that both  $(X, \mathcal{P}_1)$  and  $(X, \mathcal{P}_2)$  are regular topological spaces. Let  $x \in X$  and for  $i \in \{1, 2\}$ , let  $F$  be a  $(\mathcal{P}_i)$ -closed subset of  $X$  with  $x \notin F$ . Considering  $\{x\}$  as a  $(\mathcal{P}_i)$ -closed set, we get a parallel star refinement  $\mathcal{V}$  of  $\{X - \{x\}, X - F\}$ . Then  $G = St(\{x\}, \mathcal{V})$  and  $H = St(F, \mathcal{V})$  are  $(\mathcal{P}_i)$ -open sets with  $x \in G$ ,  $F \subset H$  and  $G \cap H = \emptyset$ . So  $(X, \mathcal{P}_i)$  is regular. Hence  $X$  is strongly pairwise regular. Therefore by Theorem 2.3,  $X$  is pairwise paracompact.  $\square$

## CHAPTER IV

### Pairwise Closure-Preserving Collections and Pairwise Paracompactness

#### 4.1. Introduction

The notion of pairwise paracompactness in a bitopological space have been introduced and studied in Chapters II and III. We have obtained some characterizations of pairwise paracompactness there. In this chapter, we introduce the notion of pairwise closure-preserving collection (Definition 4.2) of sets. Then we obtain some new characterizations of pairwise paracompactness which are analogous to the characterizations of paracompactness obtained by Michael [31].

#### 4.2. Preliminaries

A collection  $\mathcal{B}$  of subsets of a topological space  $(X, \mathcal{T})$  is called a  $(\mathcal{T})$ closure-preserving collection if for any subcollection  $\mathcal{D}$  of  $\mathcal{B}$ ,  $(\mathcal{T})\text{cl}(\bigcup_{D \in \mathcal{D}} D) = \bigcup_{D \in \mathcal{D}} (\mathcal{T})\text{cl}D$ .

We introduce the following definitions.

DEFINITION 4.1.  $X$  is said to be  $(*)$ pairwise normal if  $X$  is pairwise normal and if for every pair of a  $(\mathcal{P}_j)$ closed set  $A$  and a  $(\mathcal{P}_i)$ closed set  $B$  with  $A \cap B = \emptyset$ , there exist  $U, V \in \mathcal{P}_i$  such

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The subject matter of this chapter forms our paper: *Mat. Vesnik* **62** (4) (2010), 299–309, [3].

that

$$A \subset U, B \subset V \text{ and } U \cap V = \emptyset,$$

and there exist  $G, H \in \mathcal{P}_j$  such that

$$A \subset G, B \subset H \text{ and } G \cap H = \emptyset.$$

It is easy to see that  $X$  is  $(*)$ pairwise normal iff it satisfies the following conditions:

For any  $(\mathcal{P}_j)$ closed set  $A$  and  $(\mathcal{P}_i)$ open set  $W$  with  $A \subset W$ ,

- (1) there exist  $U \in \mathcal{P}_i$  such that  $A \subset U \subset (\mathcal{P}_i)\text{cl}U \subset W$ ,
- (2) there exist  $V \in \mathcal{P}_j$  such that  $A \subset V \subset (\mathcal{P}_j)\text{cl}V \subset W$ ,
- (3) there exist  $G \in \mathcal{P}_i$  such that  $A \subset G \subset (\mathcal{P}_j)\text{cl}G \subset W$ .

Now we give an example of a  $(*)$ pairwise normal space.

EXAMPLE 4.1. For  $a \in R$ , we define two topologies  $\mathcal{P}_1$  and  $\mathcal{P}_2$  on  $R$  by

$$\mathcal{P}_1 = \{R, \emptyset, (-\infty, a], (a, \infty)\},$$

$$\mathcal{P}_2 = \{R, \emptyset, R - \{a\}, (-\infty, a), (-\infty, a], (a, \infty)\}.$$

We now consider the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$ .

Here the nontrivial  $(\mathcal{P}_1)$ closed sets are  $(a, \infty), (-\infty, a]$  and the nontrivial  $(\mathcal{P}_2)$ closed sets are  $\{a\}, [a, \infty), (a, \infty), (-\infty, a]$ .

- (i) For the  $(\mathcal{P}_1)$ closed set  $A = (a, \infty)$  and the  $(\mathcal{P}_2)$ closed set  $B = \{a\}$ , we have  $A \cap B = \emptyset$ . Then we obtain a  $(\mathcal{P}_2)$ open set  $U = (a, \infty)$  and a  $(\mathcal{P}_1)$ open set  $V = (-\infty, a]$  such that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$ .

- (ii) For the  $(\mathcal{P}_1)$ closed set  $A = (a, \infty)$  and the  $(\mathcal{P}_2)$ closed set  $B = (-\infty, a]$ , we have  $A \cap B = \emptyset$ . Then we obtain a  $(\mathcal{P}_2)$ open set  $U = (a, \infty)$  and a  $(\mathcal{P}_1)$ open set  $V = (-\infty, a]$  such that  $A \subset U$ ,  $B \subset V$  and  $U \cap V = \emptyset$ .
- (iii) For the  $(\mathcal{P}_1)$ closed set  $A = (-\infty, a]$  and the  $(\mathcal{P}_2)$ closed set  $B = (a, \infty)$ , we have  $A \cap B = \emptyset$ . Then we obtain a  $(\mathcal{P}_2)$ open set  $U = (-\infty, a]$  and a  $(\mathcal{P}_1)$ open set  $V = (a, \infty)$  such that  $A \subset U$ ,  $B \subset V$  and  $U \cap V = \emptyset$ .

There do not exist any more pairs of a  $(\mathcal{P}_1)$ closed set  $A$  and a  $(\mathcal{P}_2)$ closed set  $B$  with  $A \cap B = \emptyset$ . In all the above cases,  $U$  and  $V$  are  $(\mathcal{P}_1)$ open as well as  $(\mathcal{P}_2)$ open. Hence the space is  $(*)$ pairwise normal.

Next we show that there exists a pairwise normal space which is not  $(*)$ pairwise normal.

EXAMPLE 4.2. Let  $X$  be any set with  $a, b \in X$ . Suppose

$$\begin{aligned}\mathcal{P}_1 &= \{\emptyset, X\} \cup \{A \subset X \mid a \in A\}, \\ \mathcal{P}_2 &= \{\emptyset, X\} \cup \{A \subset X \mid a \notin A, b \in A\}.\end{aligned}$$

Obviously,  $\mathcal{P}_1$  and  $\mathcal{P}_2$  form topologies on  $X$ . We consider the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$ .

Let  $E$  be  $(\mathcal{P}_1)$ closed and  $F$  be  $(\mathcal{P}_2)$ closed with  $E \cap F = \emptyset$ . Since  $E$  is  $(\mathcal{P}_1)$ closed and  $F$  is  $(\mathcal{P}_2)$ closed, we have  $a \notin E$  and  $a \in F$ ,  $b \notin F$ . So  $F$  is  $(\mathcal{P}_1)$ open.

We write  $V = F \in \mathcal{P}_1$ . If  $b \in E$  then  $E$  is  $(\mathcal{P}_2)$ open. In this case, we choose  $U = E$ . If  $b \notin E$ , we put  $U = E \cup \{b\}$ . So in both the cases,  $U$  is  $(\mathcal{P}_2)$ open. Also we have  $E \subset U$ ,  $F \subset V$  with  $U \cap V = \emptyset$ . Hence  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise normal.

We note, for each  $i \in \{1, 2\}$  no two sets belonging to  $\mathcal{P}_i$  are disjoint, and hence the space is not  $(*)$ pairwise normal.

DEFINITION 4.2. A collection of subsets of  $X$  is said to be *pairwise closure-preserving* if it is  $(\mathcal{P}_i)$ closure-preserving for both  $i = 1$  and  $i = 2$ .

DEFINITION 4.3. (*cf.* Burke et al. [9]) A collection  $\mathcal{A}$  of subsets of  $X$  is *hereditarily pairwise closure-preserving* if any collection  $\mathcal{B}$  containing subsets of sets belonging to  $\mathcal{A}$  such that each set  $A \in \mathcal{A}$  has one and only one subset belonging to  $\mathcal{B}$ , is pairwise closure-preserving.

DEFINITION 4.4. Let  $\mathcal{U}$  be a pairwise open cover of  $X$ . A collection  $\mathcal{C}$  of subsets of  $X$  is  *$\mathcal{U}$ -discrete* if for each  $x \in X$ , there exists a neighbourhood of  $x$  intersecting at most one set of  $\mathcal{C}$ , the neighbourhood being  $(\mathcal{P}_i)$ open if  $x$  belongs to a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ .

In this chapter, for a subcollection  $\mathcal{A}$  of a refinement of a pairwise open cover  $\mathcal{U}$  of  $X$ , we denote by  $\mathcal{A}_i$  ( $i \in \{1, 2\}$ ), the collection of sets in  $\mathcal{A}$  which are subsets of  $(\mathcal{P}_i)$ open sets of  $\mathcal{U}$ . So throughout this chapter,  $\mathcal{U}_i$  stands for the collection of all

$(\mathcal{P}_i)$ open sets belonging to  $\mathcal{U}$ . If a set  $A$  belonging to  $\mathcal{A}$  is a subset of a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ , then  $\text{cl}A$  denotes the  $(\mathcal{P}_i)$ closure of  $A$ . The collection  $\{\text{cl}A \mid A \in \mathcal{A}\}$  is denoted by  $\overline{\mathcal{A}}$ .

Throughout Section 4.3, we assume that the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  satisfies the following two conditions:

(\*) For any pairwise open cover  $\mathcal{U}$  of  $X$

$$A \subset \bigcup \{E \mid E \in \mathcal{U}_i\} \Rightarrow (\mathcal{P}_i)\text{cl}A \subset \bigcup \{E \mid E \in \mathcal{U}_i\}. \quad (4.1)$$

(\*\*) If  $\mathcal{D}$  is  $(\mathcal{P}_i)$ closure-preserving, then  $\mathcal{D}$  is  $(\mathcal{P}_j)$ closure-preserving, when  $\mathcal{D}$  is a collection of subsets of a set belonging to  $\mathcal{P}_1 \cup \mathcal{P}_2 - \{X\}$ .

### 4.3. Lemmas

To prove the desired characterizations as anticipated in introduction, we require the following lemmas.

LEMMA 4.1. *Suppose  $\mathcal{V}$  is a refinement of a pairwise open cover  $\mathcal{U}$  of the bitopological space  $X$ . If a collection  $\mathcal{A} \subset \mathcal{V}$  is  $\mathcal{U}$ -locally finite, then  $\mathcal{A}$  is pairwise closure-preserving.*

PROOF. Let  $\mathcal{B}$  be a subcollection of  $\mathcal{A}$  and let

$$x \in (\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_i} B \right). \quad (4.2)$$

By the condition (\*),  $x$  belongs to a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ . Therefore there exists a  $(\mathcal{P}_i)$ open neighbourhood of  $x$ , which intersects a finite number of sets in  $\mathcal{B}_i$ , say  $B_1, B_2, \dots, B_n$ . Again by

(4.2), every  $(\mathcal{P}_i)$ open neighbourhood of  $x$  intersects  $\bigcup_{B \in \mathcal{B}_i} B$ . Hence it follows that every  $(\mathcal{P}_i)$ open neighbourhood of  $x$  intersects  $B_1 \cup B_2 \cup \dots \cup B_n$ . So  $x \in (\mathcal{P}_i)\text{cl}(B_1 \cup B_2 \cup \dots \cup B_n) = ((\mathcal{P}_i)\text{cl}B_1) \cup ((\mathcal{P}_i)\text{cl}B_2) \cup \dots \cup ((\mathcal{P}_i)\text{cl}B_n)$ . Therefore  $x \in \bigcup_{B \in \mathcal{B}_i} (\mathcal{P}_i)\text{cl}B$ . Hence

$$\begin{aligned} & (\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_i} B \right) \subset \bigcup_{B \in \mathcal{B}_i} (\mathcal{P}_i)\text{cl}B \\ \Rightarrow & (\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_i} B \right) = \bigcup_{B \in \mathcal{B}_i} (\mathcal{P}_i)\text{cl}B. \end{aligned} \quad (4.3)$$

Therefore by the condition (\*\*),

$$(\mathcal{P}_j)\text{cl} \left( \bigcup_{B \in \mathcal{B}_i} B \right) = \bigcup_{B \in \mathcal{B}_i} (\mathcal{P}_j)\text{cl}B.$$

Similarly, we get

$$(\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_j} B \right) = \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_i)\text{cl}B. \quad (4.4)$$

Now

$$\begin{aligned} (\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}} B \right) &= (\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_i} B \right) \cup (\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_j} B \right) \\ &= \left( \bigcup_{B \in \mathcal{B}_i} (\mathcal{P}_i)\text{cl}B \right) \cup \left( \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_i)\text{cl}B \right) \\ & \hspace{15em} \text{(by (4.3) and (4.4))} \\ &= \bigcup_{B \in \mathcal{B}} (\mathcal{P}_i)\text{cl}B. \end{aligned}$$

□

LEMMA 4.2. *Let  $\mathcal{V}$  be a refinement of a pairwise open cover  $\mathcal{U}$  of  $X$ . Then a collection  $\mathcal{A} \subset \mathcal{V}$  is pairwise closure-preserving iff  $\overline{\mathcal{A}}$  is pairwise closure-preserving.*

PROOF. Firstly, let  $\mathcal{A}$  be pairwise closure-preserving. For any subcollection  $\mathcal{D}$  of  $\overline{\mathcal{A}}$ , there exists a subcollection  $\mathcal{B}$  of  $\mathcal{A}$  such that  $\mathcal{D} = \overline{\mathcal{B}}$ . Since  $\mathcal{A}$  is pairwise closure-preserving, we have

$$(\mathcal{P}_j)\text{cl} \left( \bigcup_{B \in \mathcal{B}_j} B \right) = \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_j)\text{cl}B.$$

Thus

$$\begin{aligned} (\mathcal{P}_j)\text{cl} \left( \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_j)\text{cl}B \right) &= (\mathcal{P}_j)\text{cl} \left( (\mathcal{P}_j)\text{cl} \left( \bigcup_{B \in \mathcal{B}_j} B \right) \right) \\ &= (\mathcal{P}_j)\text{cl} \left( \bigcup_{B \in \mathcal{B}_j} B \right) \\ &= \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_j)\text{cl}B \\ &= \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_j)\text{cl}((\mathcal{P}_j)\text{cl}B). \end{aligned}$$

So the collection  $\{(\mathcal{P}_j)\text{cl}A \mid A \in \mathcal{A}_j\}$  is  $(\mathcal{P}_j)$ closure-preserving. Hence by the condition (\*\*),  $\{(\mathcal{P}_j)\text{cl}A \mid A \in \mathcal{A}_j\}$  is  $(\mathcal{P}_i)$ closure-preserving. So we have

$$(\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_j)\text{cl}B \right) = \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{cl}B).$$

Then we have

$$\begin{aligned} (\mathcal{P}_i)\text{cl} \left( \bigcup_{D \in \mathcal{D}} D \right) &= (\mathcal{P}_i)\text{cl} \left( \left( \bigcup_{B \in \mathcal{B}_i} (\mathcal{P}_i)\text{cl}B \right) \cup \right. \\ &\quad \left. \left( \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_j)\text{cl}B \right) \right) \\ &= (\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_i} (\mathcal{P}_i)\text{cl}B \right) \cup \\ &\quad (\mathcal{P}_i)\text{cl} \left( \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_j)\text{cl}B \right) \end{aligned}$$

$$\begin{aligned}
 &= \left( \bigcup_{B \in \mathcal{B}_i} (\mathcal{P}_i) \text{cl}((\mathcal{P}_i) \text{cl} B) \right) \cup \\
 &\quad \left( \bigcup_{B \in \mathcal{B}_j} (\mathcal{P}_i) \text{cl}((\mathcal{P}_j) \text{cl} B) \right) \\
 &= \bigcup_{D \in \mathcal{D}} (\mathcal{P}_i) \text{cl} D.
 \end{aligned}$$

Therefore  $\overline{\mathcal{A}}$  is pairwise closure-preserving.

Similarly, we can prove that  $\mathcal{A}$  is pairwise closure-preserving if  $\overline{\mathcal{A}}$  is pairwise closure-preserving.  $\square$

LEMMA 4.3. *If the pairwise open cover  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  of  $X$  has a pairwise closure-preserving refinement  $\mathcal{B}$  such that*

$$((\mathcal{P}_2) \text{cl}((\mathcal{P}_1) \text{cl} B)) \cup ((\mathcal{P}_1) \text{cl}((\mathcal{P}_2) \text{cl} B)) \subset U_\alpha \quad (4.5)$$

where  $B \subset U_\alpha, B \in \mathcal{B}$ , then there exists a pairwise closure-preserving refinement  $\mathcal{E} = \{E_\alpha \mid \alpha \in A\}$  of  $\mathcal{U}$  such that

$$((\mathcal{P}_2) \text{cl}((\mathcal{P}_1) \text{cl} E_\alpha)) \cup ((\mathcal{P}_1) \text{cl}((\mathcal{P}_2) \text{cl} E_\alpha)) \subset U_\alpha \text{ for each } \alpha \in A.$$

PROOF. For each  $\alpha$ , we write  $E_\alpha = \bigcup \{B \in \mathcal{B} \mid B \subset U_\alpha\}$ . Then

$$\begin{aligned}
 &((\mathcal{P}_2) \text{cl}((\mathcal{P}_1) \text{cl} E_\alpha)) \cup ((\mathcal{P}_1) \text{cl}((\mathcal{P}_2) \text{cl} E_\alpha)) \\
 &= \left( (\mathcal{P}_2) \text{cl} \left( (\mathcal{P}_1) \text{cl} \left( \bigcup_{B \subset U_\alpha} B \right) \right) \right) \cup \\
 &\quad \left( (\mathcal{P}_1) \text{cl} \left( (\mathcal{P}_2) \text{cl} \left( \bigcup_{B \subset U_\alpha} B \right) \right) \right) \\
 &= \left( (\mathcal{P}_2) \text{cl} \left( \bigcup_{B \subset U_\alpha} (\mathcal{P}_1) \text{cl} B \right) \right) \cup \\
 &\quad \left( (\mathcal{P}_1) \text{cl} \left( \bigcup_{B \subset U_\alpha} (\mathcal{P}_2) \text{cl} B \right) \right) \\
 &\quad \text{(since } \mathcal{B} \text{ is pairwise closure-preserving)}.
 \end{aligned}$$

Again we have for  $i \in \{1, 2\}$ ,

$$(\mathcal{P}_i)\text{cl} \left( \bigcup_{B \subset U_\alpha} (\mathcal{P}_i)\text{cl} B \right) = \bigcup_{B \subset U_\alpha} (\mathcal{P}_i)\text{cl} ((\mathcal{P}_i)\text{cl} B).$$

Therefore by the condition (\*\*), we get

$$(\mathcal{P}_j)\text{cl} \left( \bigcup_{B \subset U_\alpha} (\mathcal{P}_i)\text{cl} B \right) = \bigcup_{B \subset U_\alpha} (\mathcal{P}_j)\text{cl} ((\mathcal{P}_i)\text{cl} B).$$

Hence

$$\begin{aligned} & ((\mathcal{P}_2)\text{cl} ((\mathcal{P}_1)\text{cl} E_\alpha)) \cup ((\mathcal{P}_1)\text{cl} ((\mathcal{P}_2)\text{cl} E_\alpha)) \\ &= \left( \bigcup_{B \subset U_\alpha} (\mathcal{P}_2)\text{cl} ((\mathcal{P}_1)\text{cl} B) \right) \cup \left( \bigcup_{B \subset U_\alpha} (\mathcal{P}_1)\text{cl} ((\mathcal{P}_2)\text{cl} B) \right) \\ &\subset U_\alpha \quad (\text{by (4.5)}). \end{aligned}$$

Let us now consider a subcollection  $\mathcal{D}$  of  $\mathcal{E} = \{E_\alpha \mid \alpha \in A\}$ . For  $D \in \mathcal{D}$ , there exists an  $\alpha(D) \in A$  such that  $D = E_{\alpha(D)}$ . We write  $\mathcal{C}_D = \{B \in \mathcal{B} \mid B \subset U_{\alpha(D)}\}$ . Thus

$$\begin{aligned} D &= E_{\alpha(D)} \\ &= \bigcup \{B \in \mathcal{B} \mid B \subset U_{\alpha(D)}\} \quad (\text{by the definition of } E_{\alpha(D)}) \\ &= \bigcup_{C \in \mathcal{C}_D} C. \end{aligned}$$

Then  $\mathcal{C} = \bigcup_{D \in \mathcal{D}} \mathcal{C}_D$  is a subcollection of  $\mathcal{B}$ , and

$$\bigcup_{C \in \mathcal{C}} C = \bigcup_{D \in \mathcal{D}} \left( \bigcup_{C \in \mathcal{C}_D} C \right) = \bigcup_{D \in \mathcal{D}} D. \quad (4.6)$$

Now

$$\begin{aligned}
(\mathcal{P}_i)\text{cl}\left(\bigcup_{D \in \mathcal{D}} D\right) &= (\mathcal{P}_i)\text{cl}\left(\bigcup_{C \in \mathcal{C}} C\right) \quad (\text{by (4.6)}) \\
&= \bigcup_{C \in \mathcal{C}} (\mathcal{P}_i)\text{cl}C \\
&\quad (\text{since } \mathcal{B} \text{ is } (\mathcal{P}_i)\text{closure-preserving}) \\
&= \bigcup_{D \in \mathcal{D}} \left(\bigcup_{C \in \mathcal{C}_D} (\mathcal{P}_i)\text{cl}C\right) \\
&= \bigcup_{D \in \mathcal{D}} (\mathcal{P}_i)\text{cl}\left(\bigcup_{C \in \mathcal{C}_D} C\right) \\
&= \bigcup_{D \in \mathcal{D}} (\mathcal{P}_i)\text{cl}D.
\end{aligned}$$

□

LEMMA 4.4. *If any pairwise open cover  $\mathcal{U}$  of  $X$  has a pairwise closure-preserving refinement  $\mathcal{B}$  satisfying (4.5), then  $X$  is (\*)pairwise normal.*

PROOF. Let  $A$  be a  $(\mathcal{P}_i)$ closed set and  $B$  be a  $(\mathcal{P}_j)$ closed set with  $A \cap B = \emptyset$ ,  $i \neq j$ . Then  $\{X - A, X - B\}$  is a pairwise open cover of  $X$ . So by Lemma 4.3, there exists a refinement  $\{C, D\}$  of  $\{X - A, X - B\}$  such that

$$\begin{aligned}
((\mathcal{P}_1)\text{cl}C) \cup ((\mathcal{P}_2)\text{cl}C) &\subset X - A \\
\text{and } ((\mathcal{P}_1)\text{cl}D) \cup ((\mathcal{P}_2)\text{cl}D) &\subset X - B.
\end{aligned}$$

Then  $A \subset X - (\mathcal{P}_i)\text{cl}C$ ,  $B \subset X - (\mathcal{P}_i)\text{cl}D$ ,  $X - (\mathcal{P}_i)\text{cl}C$ ,  $X - (\mathcal{P}_i)\text{cl}D \in \mathcal{P}_i$  and  $(X - (\mathcal{P}_i)\text{cl}C) \cap (X - (\mathcal{P}_i)\text{cl}D) = \emptyset$ .

Also  $A \subset X - (\mathcal{P}_j)\text{cl}C$ ,  $B \subset X - (\mathcal{P}_j)\text{cl}D$ ,  $X - (\mathcal{P}_j)\text{cl}C$ ,  $X - (\mathcal{P}_j)\text{cl}D \in \mathcal{P}_j$  and  $(X - (\mathcal{P}_j)\text{cl}C) \cap (X - (\mathcal{P}_j)\text{cl}D) = \emptyset$ .

Moreover,  $A \subset X - (\mathcal{P}_j)\text{cl}C$ ,  $B \subset X - (\mathcal{P}_i)\text{cl}D$  and  $(X - (\mathcal{P}_j)\text{cl}C) \cap (X - (\mathcal{P}_i)\text{cl}D) = \emptyset$ .  $\square$

LEMMA 4.5. *Let the space  $X$  be  $(*)$ pairwise normal,  $\mathcal{U}$  be a pairwise open cover of  $X$  and  $\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}$  be a disjoint collection of sets belonging to  $\mathcal{P}_1 \cup \mathcal{P}_2$  such that if  $V_\gamma$  is  $(\mathcal{P}_i)$ open, then it is a subset of a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$  and let  $\mathcal{D} = \{D_\gamma \mid \gamma \in \Gamma\}$  be a collection of subsets of  $X$ , which is pairwise closure-preserving and*

$$((\mathcal{P}_1)\text{cl}D_\gamma) \cup ((\mathcal{P}_2)\text{cl}D_\gamma) \subset V_\gamma. \quad (4.7)$$

*Then there exists a  $\mathcal{U}$ -discrete collection  $\{W_\gamma \mid \gamma \in \Gamma\}$  of subsets of  $X$  such that  $D_\gamma \subset W_\gamma \subset V_\gamma$  and  $W_\gamma$  is  $(\mathcal{P}_i)$ open if  $V_\gamma$  is  $(\mathcal{P}_i)$ open.*

PROOF. We write  $U_i = \bigcup_{U \in \mathcal{U}_i} U$ . Let  $S_i = \{x \in U_i \mid \text{some } (\mathcal{P}_i)\text{open neighbourhood of } x \text{ intersects at most one } V_\gamma\}$ . Then  $S_i$  is  $(\mathcal{P}_i)$ open and contains all  $V \in \mathcal{V}_i$ . By (4.7), we get

$$\begin{aligned} \bigcup_{D \in \mathcal{D}_i} (\mathcal{P}_j)\text{cl}D &\subset S_i \\ \Rightarrow (\mathcal{P}_j)\text{cl}\left(\bigcup_{D \in \mathcal{D}_i} D\right) &\subset S_i. \end{aligned}$$

Therefore by the  $(*)$ pairwise normality of  $X$ , there exist sets  $G_i^1, G_i^2 \in \mathcal{P}_i$  such that

$$\begin{aligned} (\mathcal{P}_j)\text{cl}\left(\bigcup_{D \in \mathcal{D}_i} D\right) &\subset G_i^1 \subset (\mathcal{P}_i)\text{cl}G_i^1 \subset S_i, \\ (\mathcal{P}_j)\text{cl}\left(\bigcup_{D \in \mathcal{D}_i} D\right) &\subset G_i^2 \subset (\mathcal{P}_j)\text{cl}G_i^2 \subset S_i, \end{aligned}$$

and there exist sets  $H_j^1, H_j^2 \in \mathcal{P}_j$  such that

$$\begin{aligned} (\mathcal{P}_i)\text{cl} \left( \bigcup_{D \in \mathcal{D}_j} D \right) &\subset H_j^1 \subset (\mathcal{P}_i)\text{cl} H_j^1 \subset S_j, \\ (\mathcal{P}_i)\text{cl} \left( \bigcup_{D \in \mathcal{D}_j} D \right) &\subset H_j^2 \subset (\mathcal{P}_j)\text{cl} H_j^2 \subset S_j. \end{aligned}$$

We now have

$$(\mathcal{P}_i)\text{cl} (G_i^1 \cup H_j^1) \cup (\mathcal{P}_j)\text{cl} (G_i^2 \cup H_j^2) \subset S_i \cup S_j. \quad (4.8)$$

We write  $G_i = G_i^1 \cap G_i^2$ ,  $H_j = H_j^1 \cap H_j^2$  and,

$$\begin{aligned} W_\gamma &= V_\gamma \cap G_i \text{ if } V_\gamma \in \mathcal{P}_i, \\ &= V_\gamma \cap H_j \text{ if } V_\gamma \in \mathcal{P}_j. \end{aligned}$$

So  $D_\gamma \subset W_\gamma \subset V_\gamma$ . Next we show that  $\{W_\gamma \mid \gamma \in \Gamma\}$  is  $\mathcal{U}$ -discrete. Let  $x$  belong to some  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ . Then  $x \in U_i$ . If  $x \in S_i$ , then there exists a  $(\mathcal{P}_i)$ open neighbourhood of  $x$ , intersecting at most one  $V_\gamma$  and hence intersecting at most one  $W_\gamma$ . If  $x \notin S_i \cup S_j$ , then by (4.8),  $x \notin (\mathcal{P}_i)\text{cl}(G_i^1 \cup H_j^1)$ . Again since  $G_i \subset G_i^1$  and  $H_j \subset H_j^1$ , we have  $\bigcup_\gamma W_\gamma \subset G_i^1 \cup H_j^1$ . Therefore there exists a  $(\mathcal{P}_i)$ open neighbourhood of  $x$  intersecting none of  $\{W_\gamma \mid \gamma \in \Gamma\}$ . Also we have  $G_i \subset G_i^2$  and  $H_j \subset H_j^2$ , and so  $\bigcup_\gamma W_\gamma \subset G_i^2 \cup H_j^2$ . Thus if  $x \in U_i \cap U_j$ , and  $x \notin S_i \cup S_j$ , then considering  $x \notin (\mathcal{P}_j)\text{cl}(G_i^2 \cup H_j^2)$ , we also get a  $(\mathcal{P}_j)$ open neighbourhood of  $x$  intersecting none of  $\{W_\gamma \mid \gamma \in \Gamma\}$ .  $\square$

LEMMA 4.6. *Suppose  $\mathcal{U}$  is a pairwise open cover of the space  $X$  and  $\{K_\alpha \mid \alpha \in A\}$  is a  $\mathcal{U}$ -locally finite collection of subsets of  $X$  and suppose for each  $\alpha \in A$ ,  $\mathcal{B}_\alpha$  is a pairwise closure-preserving collection of subsets of  $K_\alpha$  such that each member of*

$\mathcal{B}_\alpha$  is a subset of some set in  $\mathcal{U}$ . Then  $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha \mid \alpha \in A\}$  is also pairwise closure-preserving.

PROOF. Let  $\mathcal{C}$  be a subcollection of  $\mathcal{B}$ . We have to show

$$(\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}} C \right) = \bigcup_{C \in \mathcal{C}} (\mathcal{P}_i)\text{cl} C, \quad i \in \{1, 2\}.$$

Now

$$\begin{aligned} & (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}} C \right) \\ &= (\mathcal{P}_i)\text{cl} \left( \left( \bigcup_{C \in \mathcal{C}_i} C \right) \cup \left( \bigcup_{C \in \mathcal{C}_j} C \right) \right) \quad i \neq j \\ &= (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_i} C \right) \cup (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_j} C \right). \end{aligned} \quad (4.9)$$

Also

$$(\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_i} C \right) = (\mathcal{P}_i)\text{cl} \left( \bigcup_{\gamma \in \Gamma} \left( \bigcup_{C \in \mathcal{C}_{\gamma i}} C \right) \right).$$

We suppose

$$x \in (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_i} C \right). \quad (4.10)$$

Since each member of  $\mathcal{B}_\alpha$  is a subset of some sets belonging to  $\mathcal{U}$ , we get  $\bigcup_{C \in \mathcal{C}_i} C \subset \bigcup\{U \mid C \subset U \in \mathcal{U}_i\}$ . Hence by the condition (\*), we have

$$(\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_i} C \right) \subset \bigcup\{U \mid C \subset U \in \mathcal{U}_i\}.$$

So  $x$  belongs to a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ .

Let  $\mathcal{C} = \bigcup_{\gamma \in \Gamma} \{\mathcal{C}_\gamma \mid \gamma \in \Gamma \subset A \text{ and } \mathcal{C}_\gamma \subset \mathcal{B}_\gamma\}$ . Since  $\{K_\alpha \mid \alpha \in A\}$  is  $\mathcal{U}$ -locally finite,  $\{K_\gamma \mid \gamma \in \Gamma\}$  is also  $\mathcal{U}$ -locally finite. Hence there exists a  $(\mathcal{P}_i)$ open nbd of  $x$  intersecting finitely many  $K_{\gamma_1}, K_{\gamma_2}, \dots, K_{\gamma_n}$  of  $\{K_\gamma \mid \gamma \in \Gamma\}$ . This nbd of  $x$  can

only intersects at most the sets belonging to  $\mathcal{C}_{\gamma_1 i}, \mathcal{C}_{\gamma_2 i}, \dots, \mathcal{C}_{\gamma_n i}$ . Again by (4.10), every  $(\mathcal{P}_i)$ open nbd of  $x$  intersects  $\bigcup_{C \in \mathcal{C}_i} C$ . So every  $(\mathcal{P}_i)$ open nbd of  $x$  intersects  $\left(\bigcup_{C \in \mathcal{C}_{\gamma_1 i}} C\right) \cup \left(\bigcup_{C \in \mathcal{C}_{\gamma_2 i}} C\right) \cup \dots \cup \left(\bigcup_{C \in \mathcal{C}_{\gamma_n i}} C\right)$ . Thus

$$\begin{aligned} x &\in (\mathcal{P}_i)\text{cl} \left( \left( \bigcup_{C \in \mathcal{C}_{\gamma_1 i}} C \right) \cup \left( \bigcup_{C \in \mathcal{C}_{\gamma_2 i}} C \right) \right. \\ &\quad \left. \cup \dots \cup \left( \bigcup_{C \in \mathcal{C}_{\gamma_n i}} C \right) \right) \\ &= (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_{\gamma_1 i}} C \right) \cup (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_{\gamma_2 i}} C \right) \\ &\quad \cup \dots \cup (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_{\gamma_n i}} C \right) \\ &= \left( \bigcup_{C \in \mathcal{C}_{\gamma_1 i}} (\mathcal{P}_i)\text{cl} C \right) \cup \left( \bigcup_{C \in \mathcal{C}_{\gamma_2 i}} (\mathcal{P}_i)\text{cl} C \right) \\ &\quad \cup \dots \cup \left( \bigcup_{C \in \mathcal{C}_{\gamma_n i}} (\mathcal{P}_i)\text{cl} C \right). \end{aligned}$$

(since each  $\mathcal{B}_\alpha$ ,  $\alpha \in A$  is pairwise closure-preserving)

Thus for some  $\gamma_k$  ( $k = 1, 2, \dots, n$ ), there exists a  $C \in \mathcal{C}_{\gamma_k i}$  such that  $x \in (\mathcal{P}_i)\text{cl} C$ . Hence we get

$$\begin{aligned} &(\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_i} C \right) \subset \left( \bigcup_{C \in \mathcal{C}_i} (\mathcal{P}_i)\text{cl} C \right) \\ \Rightarrow &(\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_i} C \right) = \left( \bigcup_{C \in \mathcal{C}_i} (\mathcal{P}_i)\text{cl} C \right). \quad (4.11) \end{aligned}$$

Also  $\bigcup_{C \in \mathcal{C}_i} C \subset \bigcup \{U \mid C \subset U \in \mathcal{U}_i\}$  and  $\bigcup \{U \mid C \subset U \in \mathcal{U}_i\}$  is a subset of  $\mathcal{P}_1 \cup \mathcal{P}_2 - \{X\}$ . Thus by the condition (\*\*),

$$(\mathcal{P}_j)\text{cl} \left( \bigcup_{C \in \mathcal{C}_i} C \right) = \left( \bigcup_{C \in \mathcal{C}_i} (\mathcal{P}_j)\text{cl} C \right).$$

Similarly, we have

$$(\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_j} C \right) = \left( \bigcup_{C \in \mathcal{C}_j} (\mathcal{P}_i)\text{cl} C \right).$$

Hence we get from (4.9),

$$\begin{aligned} (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}} C \right) &= (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_i} C \right) \cup (\mathcal{P}_i)\text{cl} \left( \bigcup_{C \in \mathcal{C}_j} C \right) \\ &= \left( \bigcup_{C \in \mathcal{C}_i} (\mathcal{P}_i)\text{cl} C \right) \cup \left( \bigcup_{C \in \mathcal{C}_j} (\mathcal{P}_i)\text{cl} C \right) \\ &= \bigcup_{C \in \mathcal{C}} (\mathcal{P}_i)\text{cl} C. \end{aligned}$$

□

#### 4.4. The characterizations of pairwise paracompactness

**THEOREM 4.1.** *If the bitopological space  $X$  is strongly pairwise regular, and satisfies the conditions  $(*)$  and  $(**)$ , then the following statements are equivalent.*

- (a)  $X$  is pairwise paracompact.
- (b) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a hereditarily pairwise closure-preserving parallel refinement.
- (c) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a parallel refinement  $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ , where each  $\mathcal{V}_n$  is hereditarily pairwise closure-preserving.
- (d) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a pairwise closure-preserving refinement.
- (e) Each pairwise open cover  $\mathcal{U}$  of  $X$  has a pairwise closure-preserving refinement  $\mathcal{B}$  such that if  $B \subset U \in \mathcal{U}$ ,  $B \in$

$\mathcal{B}$ , then

$$((\mathcal{P}_2)\text{cl}((\mathcal{P}_1)\text{cl}B)) \cup ((\mathcal{P}_1)\text{cl}((\mathcal{P}_2)\text{cl}B)) \subset U.$$

PROOF. (a)  $\Rightarrow$  (b): Follows from Lemma 4.1.

(b)  $\Rightarrow$  (c): Obvious.

(c)  $\Rightarrow$  (d): Let  $\mathcal{U}$  be a pairwise open cover of  $X$ . By (c),  $\mathcal{U}$  has a parallel refinement  $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ , where each  $\mathcal{V}_n$  is hereditarily pairwise closure-preserving. Let

$$\begin{aligned} V_n &= \bigcup \{V \mid V \in \mathcal{V}_n\}, \quad n \in N, \\ K_1 &= X, \\ K_n &= X - \bigcup_{m=1}^{n-1} V_m, \quad n = 2, 3, \dots \end{aligned}$$

Then the class  $\{K_n \mid n \in N\}$  is  $\mathcal{U}$ -locally finite.

We write

$$\mathcal{B}_n = \{V \cap K_n \mid V \in \mathcal{V}_n\},$$

and

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n.$$

Then  $\mathcal{B}$  is a refinement of  $\mathcal{U}$ . Since  $\mathcal{V}_n$  is hereditarily pairwise closure-preserving, each  $\mathcal{B}_n$  is pairwise closure-preserving. Since  $\{K_n \mid n \in N\}$  is  $\mathcal{U}$ -locally finite, from Lemma 4.6, it follows that  $\mathcal{B}$  is pairwise closure-preserving.

(d)  $\Rightarrow$  (e): By strong pairwise regularity of  $X$ , there is a parallel refinement  $\mathcal{V}$  of  $\mathcal{U}$  such that for  $V \in \mathcal{V}$ , we have a  $U \in \mathcal{U}$  with

$$((\mathcal{P}_2)\text{cl}((\mathcal{P}_1)\text{cl}V)) \cup ((\mathcal{P}_1)\text{cl}((\mathcal{P}_2)\text{cl}V)) \subset U. \quad (4.12)$$

By (d), there is a pairwise closure-preserving refinement  $\mathcal{B}$  of  $\mathcal{V}$ , and hence of  $\mathcal{U}$ . If  $B \in \mathcal{B}$ , then for some  $V \in \mathcal{V}$  and  $U \in \mathcal{U}$  satisfying (4.12), we have  $B \subset V$  and so

$$((\mathcal{P}_2)\text{cl}((\mathcal{P}_1)\text{cl}B)) \cup ((\mathcal{P}_1)\text{cl}((\mathcal{P}_2)\text{cl}B)) \subset U.$$

(e)  $\Rightarrow$  (a): Let  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  be a pairwise open cover of  $X$  and let the index set  $A$  be well-ordered. For each positive integer  $n$ , we construct a family  $\mathcal{B}_n = \{B_{\alpha,n} \mid \alpha \in A\}$  of subsets of  $X$  satisfying the following conditions:

(I)  $\mathcal{B}_n = \{B_{\alpha,n} \mid \alpha \in A\}$  is a pairwise closure-preserving cover of  $X$ , and

$$((\mathcal{P}_2)\text{cl}((\mathcal{P}_1)\text{cl}B_{\alpha,n})) \cup ((\mathcal{P}_1)\text{cl}((\mathcal{P}_2)\text{cl}B_{\alpha,n})) \subset U_\alpha \text{ for all } \alpha.$$

(II)  $((\mathcal{P}_i)\text{cl}(\text{cl}B_{\alpha,n+1})) \cap ((\mathcal{P}_i)\text{cl}(\text{cl}B_{\beta,n})) = \emptyset$  for all  $\alpha > \beta$  if  $U_\alpha \in \mathcal{P}_i$ .

For  $n = 1$ , the cover can be obtained from Lemma 4.3.

Suppose for  $n = 1, 2, \dots, m$ , the covers  $\mathcal{B}_n$  have been constructed. For  $U_\alpha \in \mathcal{P}_i$ , we write

$$K_{\alpha,m} = \bigcup_{\beta < \alpha} \{(\mathcal{P}_i)\text{cl}(\text{cl}B_{\beta,m})\}.$$

Since  $\mathcal{B}_m$  is pairwise closure-preserving, by Lemma 4.2, it follows that the set  $K_{\alpha,m}$  is  $(\mathcal{P}_i)$ closed. So the set  $U_{\alpha,m+1} = U_\alpha - K_{\alpha,m}$  is  $(\mathcal{P}_i)$ open. If  $x \in X$ , then  $x \in U_{\alpha,m+1}$  for the first  $\alpha$  for which  $x \in U_\alpha$ . Therefore the collection  $\mathcal{U}_{m+1} = \{U_{\alpha,m+1} \mid \alpha \in A\}$  forms a refinement of  $\mathcal{U}$ . By Lemma 4.3, it has a pairwise closure-preserving refinement  $\{B_{\alpha,m+1} \mid \alpha \in A\}$

such that

$$\begin{aligned} & ((\mathcal{P}_2)\text{cl}((\mathcal{P}_1)\text{cl}B_{\alpha,m+1})) \cup ((\mathcal{P}_1)\text{cl}((\mathcal{P}_2)\text{cl}B_{\alpha,m+1})) \\ & \subset U_{\alpha,m+1} \quad \text{for all } \alpha. \end{aligned} \quad (4.13)$$

Therefore the condition (I) is satisfied for  $n = m + 1$ . From (4.13) and the definition of  $U_{\alpha,m+1}$ , it follows that (II) is satisfied for  $n = m$ . If  $U_\alpha \in \mathcal{P}_i$ , we define

$$V_{\alpha,n} = X - \bigcup_{\beta \neq \alpha} \{(\mathcal{P}_i)\text{cl}(\text{cl}B_{\beta,n})\}.$$

We show that

(III)  $\{V_{\alpha,n} \mid \alpha \in A, n \in N\}$  is a pairwise open cover of  $X$  such that for all  $\alpha \in A$  and  $n \in N$ ,  $V_{\alpha,n} \subset U_\alpha$  and  $V_{\alpha,n}$  is  $(\mathcal{P}_i)$ open if  $U_\alpha$  is  $(\mathcal{P}_i)$ open.

(IV)  $V_{\alpha,n} \cap V_{\beta,n} = \emptyset$  whenever  $\alpha \neq \beta$ .

Since  $\mathcal{B}_n$  is pairwise closure-preserving, it follows that  $V_{\alpha,n}$  is  $(\mathcal{P}_i)$ open. Also we have

$$V_{\alpha,n} \subset B_{\alpha,n} \subset U_\alpha \quad \text{for all } \alpha \in A \text{ and } n \in N.$$

Therefore from the definition of  $V_{\alpha,n}$ , (IV) follows. We consider a point  $x \in X$ . If  $x \in U \in \mathcal{U} \cap \mathcal{P}_i$ , we define

$$\alpha_n = \min\{\alpha \in A \mid x \in (\mathcal{P}_i)\text{cl}(\text{cl}B_{\alpha,n}), n \in N\},$$

and

$$\alpha_l = \min\{\alpha_n \mid n \in N\}.$$

If  $\alpha > \alpha_l$ , from (II) we get

$$((\mathcal{P}_i)\text{cl}(\text{cl}B_{\alpha,l+1})) \cap ((\mathcal{P}_i)\text{cl}(\text{cl}B_{\alpha_l,l})) = \emptyset,$$

and therefore  $x \notin (\mathcal{P}_i)\text{cl}(\text{cl}B_{\alpha,l+1})$ , since  $x \in (\mathcal{P}_i)\text{cl}(\text{cl}B_{\alpha,l})$ . Also by the definition of  $\alpha_l$ ,  $x \notin (\mathcal{P}_i)\text{cl}(\text{cl}B_{\alpha,l+1})$  for  $\alpha < \alpha_l$ . Therefore  $x \in V_{\alpha_l,l+1}$ . Thus the collection  $\mathcal{V} = \{V_{\alpha,n} \mid \alpha \in A, n \in N\}$  forms a pairwise open cover of  $X$ . By Lemma 4.3, we get a pairwise closure-preserving cover  $\mathcal{D} = \{D_{\alpha,n} \mid \alpha \in A, n \in N\}$  of  $X$  such that

$$((\mathcal{P}_1)\text{cl}D_{\alpha,n}) \cup ((\mathcal{P}_2)\text{cl}D_{\alpha,n}) \subset V_{\alpha,n} \text{ for all } \alpha \text{ and } n.$$

By Lemma 4.4,  $X$  is  $(*)$ pairwise normal and so applying Lemma 4.5, for each  $n$ , we get a  $\mathcal{U}$ -discrete collection  $\mathcal{W}_n = \{W_{\alpha,n} \mid \alpha \in A\}$  such that  $W_{\alpha,n}$  is  $(\mathcal{P}_i)$ open if  $V_{\alpha,n}$  is  $(\mathcal{P}_i)$ open and

$$D_{\alpha,n} \subset W_{\alpha,n} \subset V_{\alpha,n} \text{ for all } \alpha.$$

Then the collection  $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$  is a parallel refinement of  $\mathcal{U}$  where each  $\mathcal{W}_n$  is  $\mathcal{U}$ -discrete and hence  $\mathcal{U}$ -locally finite. Hence by Theorem 2.3,  $X$  is pairwise paracompact.  $\square$

#### 4.5. Some examples

In this section,  $\mathcal{T}$  denotes the usual topology on  $R$ , and for a set  $A \subset R$ ,  $\mathcal{T}_A$  denotes the subspace topology on  $A$  in  $(R, \mathcal{T})$ . Firstly, we give an example of a strongly pairwise regular pairwise paracompact space.

**EXAMPLE 4.3.** Let  $Q$  be the set of rational numbers. If  $\mathcal{E}^1$  is the collection of the singleton sets  $\{r\}$ ,  $r \in Q$  and  $\mathcal{E}^2$  is the collection of the singleton sets  $\{r\}$ ,  $r \in R - Q$ , then for  $i \in \{1, 2\}$ , we define  $\mathcal{P}_i$  to be the topology generated by the base

$\mathcal{T} \cup \mathcal{E}^i$ . Then the topological spaces  $(R, \mathcal{P}_i)$  are regular (Steen and Seebach [44, p. 90]). We now consider the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$ . Let  $F$  be a  $(\mathcal{P}_i)$ closed set and  $x \in R - F \in \mathcal{P}_i$ . If  $x$  belongs to some  $(\mathcal{T})$ open set, then there exist a  $(\mathcal{T})$ open set  $U$  and a  $(\mathcal{T})$ open set  $V$  such that  $x \in U, F \subset V$  and  $U \cap V = \emptyset$ . If  $x \in \{r\} = U \in \mathcal{E}^i$ , then  $F \subset R - \{r\} = V$ . So in any case  $x \in U \in \mathcal{P}_i$  and  $F \subset V \in \mathcal{P}_j$  and  $U \cap V = \emptyset$ . Thus  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise regular and hence strongly pairwise regular. Now let  $\mathcal{U}$  be a pairwise open cover of  $R$ . Then there exists a parallel refinement  $\mathcal{V}$  containing sets belonging to  $\mathcal{T}$  and sets belonging to  $\mathcal{E}^1 \cup \mathcal{E}^2$ . We may assume that no element of  $\mathcal{V} \cap (\mathcal{E}^1 \cup \mathcal{E}^2)$  belongs to any element of  $\mathcal{V} \cap \mathcal{T}$ , since otherwise we can delete the corresponding singleton sets from  $\mathcal{V}$ . Let  $V = \bigcup \{G \in \mathcal{V} \cap \mathcal{T}\}$ . The  $(\mathcal{T}_V)$ open cover  $\{G \in \mathcal{V} \cap \mathcal{T}\}$  of the subspace  $(V, \mathcal{T}_V)$  has a  $(\mathcal{T}_V)$ locally finite  $(\mathcal{T}_V)$ open refinement  $\mathcal{W}$ . Let  $\mathcal{E}_{\mathcal{V}}^i = \{\{r\} \in \mathcal{E}^i \cap \mathcal{V}\}$ . If  $x \notin V$ , then  $x \in \bigcup \{\{r\} \in \mathcal{E}_{\mathcal{V}}^1 \cup \mathcal{E}_{\mathcal{V}}^2\}$  and  $\{x\}$  can intersect only  $\{x\} \in \mathcal{V}$ . Again no element of  $\mathcal{W}$  can intersect any element of  $\mathcal{E}_{\mathcal{V}}^1 \cup \mathcal{E}_{\mathcal{V}}^2$ . Thus it follows that  $\mathcal{W} \cup \mathcal{E}_{\mathcal{V}}^1 \cup \mathcal{E}_{\mathcal{V}}^2$  is a  $\mathcal{U}$ -locally finite parallel refinement of  $\mathcal{U}$ . Therefore  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise paracompact.

Now we give an example of a bitopological space satisfying both the conditions  $(*)$  and  $(**)$ .

**EXAMPLE 4.4.** For each  $i \in \{1, 2\}$ , let  $\{a_n^i\}_{n=1}^{\infty}$  be a strictly

decreasing sequence of real numbers with  $\lim a_n^i = -\infty$  and  $\{b_n^i\}_{n=1}^\infty$  be a strictly increasing sequence of real numbers with  $\lim b_n^i = \infty$  such that  $a_1^i < b_1^i$ . Let  $\mathcal{P}_i$  be the topology on  $R$  generated by the base

$$\mathcal{B}_i = \{\emptyset\} \cup \{(a_1^i, b_1^i)\} \cup \{(a_{n+1}^i, a_n^i), (b_n^i, b_{n+1}^i) \mid n \in N\} \cup \{\{a_n^i\}, \{b_n^i\} \mid n \in N\}.$$

Then each  $(\mathcal{P}_i)$ open set is  $(\mathcal{P}_i)$ closed. Therefore the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  satisfies both the conditions  $(*)$  and  $(**)$ . Obviously, the space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise paracompact.

In the space considered above, for  $i \in \{1, 2\}$ , any collection of subsets of  $R$  is  $(\mathcal{P}_i)$ closure-preserving.

Next we give an example of a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  in which for  $i \in \{1, 2\}$ , there are collections of sets which are not  $(\mathcal{P}_i)$ closure-preserving, but if a collection is  $(\mathcal{P}_1)$ closure-preserving, then it is  $(\mathcal{P}_2)$ closure-preserving, and conversely.

**EXAMPLE 4.5.** Let  $a \in R$  and let us consider the infinite intervals  $(-\infty, a]$  and  $(a, \infty)$ . We write  $A = (-\infty, a]$ . Suppose  $\{b_n^1\}_{n=1}^\infty$  and  $\{b_n^2\}_{n=1}^\infty$  are two strictly increasing sequences of real numbers with  $a = b_1^i$  and  $\lim b_n^i = \infty$  for  $i \in \{1, 2\}$ . Let  $\mathcal{P}_i$  be the topology on  $R$  generated by the base

$$\mathcal{B}_i = \mathcal{I}_A \cup \{(b_n^i, b_{n+1}^i] \mid n \in N\}.$$

We now consider a  $(\mathcal{P}_i)$ closure-preserving collection  $\mathcal{A}$  of subsets of  $R$ . Let  $\mathcal{D}$  be a subcollection of  $\mathcal{A}$ . We write

$$\begin{aligned}\mathcal{D}_1 &= \{D \in \mathcal{D} \mid D \subset (-\infty, a]\}, \\ \mathcal{D}_2 &= \{D \in \mathcal{D} \mid D \subset (a, \infty)\}, \\ \mathcal{D}^a &= \{D \in \mathcal{D} \mid D \cap (-\infty, a] \neq \emptyset, D \cap (a, \infty) \neq \emptyset\}, \\ \mathcal{D}_1^a &= \{D \cap (-\infty, a] \mid D \in \mathcal{D}^a\}, \\ \mathcal{D}_2^a &= \{D \cap (a, \infty) \mid D \in \mathcal{D}^a\}.\end{aligned}$$

Then

$$\begin{aligned}& (\mathcal{P}_j)\text{cl} \left( \bigcup \{D \in \mathcal{D}\} \right) \\ &= (\mathcal{P}_j)\text{cl} \left( \bigcup \{D \in \mathcal{D}_1\} \right) \cup (\mathcal{P}_j)\text{cl} \left( \bigcup \{D \in \mathcal{D}_2\} \right) \\ & \quad \cup (\mathcal{P}_j)\text{cl} \left( \bigcup \{D \in \mathcal{D}^a\} \right) \\ &= (\mathcal{P}_j)\text{cl} \left( \bigcup \{D \in \mathcal{D}_1\} \right) \cup (\mathcal{P}_j)\text{cl} \left( \bigcup \{D \in \mathcal{D}_2\} \right) \\ & \quad \cup (\mathcal{P}_j)\text{cl} \left( \bigcup \{D \in \mathcal{D}_1^a\} \right) \cup (\mathcal{P}_j)\text{cl} \left( \bigcup \{D \in \mathcal{D}_2^a\} \right). \quad (4.14)\end{aligned}$$

The  $(\mathcal{P}_j)$ closure of any set contained in  $(-\infty, a]$ , is identical with its  $(\mathcal{P}_i)$ closure, and any collection of sets contained in  $(a, \infty)$  are both  $(\mathcal{P}_1)$  and  $(\mathcal{P}_2)$ closure-preserving. Since  $\mathcal{A}$  is  $(\mathcal{P}_i)$ closure-preserving, it follows from (4.14) that  $\mathcal{A}$  is  $(\mathcal{P}_j)$ closure-preserving. So the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  satisfies the condition (\*\*). It is also clear that the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise paracompact.

NOTE 4.1. The space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  of Example 4.5, does not satisfy the condition (\*) but satisfies a slightly weaker condition.

To explain this, let us consider a pairwise open cover  $\mathcal{U}$  of  $R$  containing only one  $(\mathcal{P}_1)$ open set and suppose it is of the form  $(\alpha, \beta) \cup (b_n^1, b_{n+1}^1]$  such that  $(\alpha_1, \alpha_2) \cup (b_m^1, b_{m+1}^1]$  is the only  $(\mathcal{P}_2)$ open set belonging to  $\mathcal{U}$  with  $\alpha_1 < \alpha < \alpha_2 \leq a$ . Then  $\mathcal{U}$  does not satisfy (4.1). Hence the space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  does not satisfy the condition (\*). But replacing the sets of type  $G \cup (\bigcup_{n \in N_0} (b_n^i, b_{n+1}^i])$  by the sets  $G$  and  $\bigcup_{n \in N_0} (b_n^i, b_{n+1}^i]$ , where  $G \in \mathcal{T}_A$  and  $N_0 \subset N$ , we can have a parallel refinement  $\mathcal{U}_0$  of  $\mathcal{U}$  such that  $\mathcal{U}_0$  satisfies (4.1). It is clear from the context that it is sufficient to have (4.1) satisfied by a parallel refinement of  $\mathcal{U}$ . So we can relax the condition (\*) in this manner. In that case the Lemma 4.1 is required to change slightly according to our requirements.

## CHAPTER V

### Countable Pairwise Paracompactness

#### 5.1. Introduction

Dieudonné [13] introduced the notion of paracompact topological spaces. Later Dowker [14] generalized this concept and introduced the class of countably paracompact spaces: a topological space is countably paracompact if each countable open cover of the space has a locally finite open refinement. Here we introduce the notion of bitopological version of countable paracompactness and we call it countable pairwise paracompactness (Definition 5.2). This notion generalizes the concept of pairwise paracompactness introduced in Chapter II. Along with other results, we obtain some equivalent conditions to countable pairwise paracompactness for  $\alpha^*$ -pairwise normal spaces (Definition 5.4). We also introduce a stronger notion called countable  $(*)$ pairwise paracompactness (Definition 5.3) and prove that the product of a countably  $(*)$ pairwise paracompact  $\alpha^*$ -pairwise normal space and a pairwise compact second countable quasi-metric space is  $\pi$ -pairwise normal (Definition 5.5).

#### 5.2. Preliminaries

We recall here the following known definitions.

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The subject matter of this chapter forms our paper: *Int. J. Mod. Math.* **5** (3) (2010), 275–284, [2].

Let  $(X, \mathcal{P}_1, \mathcal{P}_2)$  be a bitopological space. A cover  $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$  of  $X$  is said to be pairwise closed (Pahk and Choi [37]) if  $\{X - F_\alpha \mid \alpha \in A\}$  is pairwise open. A pairwise closed cover  $\mathcal{E}$  of  $X$  is said to be a parallel closed refinement of a pairwise open (resp. closed) cover  $\mathcal{U}$  (resp.  $\mathcal{F}$ ) of  $X$  if each  $(\mathcal{P}_i)$ -closed set of  $\mathcal{E}$  is contained in some  $(\mathcal{P}_i)$ -open (resp.  $(\mathcal{P}_i)$ -closed) set of  $\mathcal{U}$  (resp.  $\mathcal{F}$ ). A pairwise open cover  $\mathcal{U}$  of  $X$  is said to be point finite if each  $x \in X$  belongs to only finitely many  $U \in \mathcal{U}$ .

Swart [46] introduced the notion of product bitopological space: let  $(X, \mathcal{P}_1, \mathcal{P}_2)$  and  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  be two bitopological spaces and let  $\mathcal{P}_i \times \mathcal{Q}_i$  be the product topology on  $X \times Y$  of the topologies  $\mathcal{P}_i$  and  $\mathcal{Q}_i$  on  $X$  and  $Y$  respectively. Then  $(X \times Y, \mathcal{P}_1 \times \mathcal{Q}_1, \mathcal{P}_2 \times \mathcal{Q}_2)$  is a bitopological space and it is called the product bitopological space of the spaces  $(X, \mathcal{P}_1, \mathcal{P}_2)$  and  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$ .

**DEFINITION 5.1** (Swart [46], p.136). Let  $(X, \mathcal{P}_1, \mathcal{P}_2)$  and  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  be two bitopological spaces. A function  $f : (X, \mathcal{P}_1, \mathcal{P}_2) \rightarrow (Y, \mathcal{Q}_1, \mathcal{Q}_2)$  is said to be continuous if  $f : (X, \mathcal{P}_1) \rightarrow (Y, \mathcal{Q}_1)$  and  $f : (X, \mathcal{P}_2) \rightarrow (Y, \mathcal{Q}_2)$  are continuous.

We need the following theorems.

**THEOREM 5.1** (Fletcher et al. [19]). *If the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise compact and  $K$  is a  $(\mathcal{P}_i)$ -closed subset of  $X$ , then  $K$  is  $(\mathcal{P}_j)$ -compact.*

**THEOREM 5.2** (Kelly [26]). *Let  $q_1$  and  $q_2$  be conjugate quasi-metrics on  $X$ . If  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  are the topologies on  $X$  generated by  $q_1$  and  $q_2$  respectively, then the bitopological space  $(X, \mathcal{Q}_1, \mathcal{Q}_2)$  is pairwise normal.*

Throughout this chapter, for a pairwise open cover  $\mathcal{U}$  of  $X$ ,  $\mathcal{U}^i$  denotes the class of  $(\mathcal{P}_i)$ open sets belonging to  $\mathcal{U}$ . Likewise, for a pairwise closed cover  $\mathcal{F}$  of  $X$ ,  $\mathcal{F}^i$  denotes the class of  $(\mathcal{P}_i)$ closed sets belonging to  $\mathcal{F}$ . The usual topology on  $R$  is denoted by  $\rho$  here.

### 5.3. Countably Pairwise Paracompact and Bitopological Product Spaces

We introduce the following definitions.

**DEFINITION 5.2.** The bitopological space  $X$  is said to be *countably pairwise paracompact* if every countable pairwise open cover  $\mathcal{U}$  of  $X$  has a  $\mathcal{U}$ -locally finite parallel refinement.

**DEFINITION 5.3.** The bitopological space  $X$  is said to be *countably (\*)pairwise paracompact* if each pairwise open cover  $\mathcal{U}$  of  $X$  has the following property: every countable  $(\mathcal{P})$ open refinement of  $\mathcal{U}$  has a  $\mathcal{U}$ -locally finite pairwise open refinement parallel to the cover  $\mathcal{U}$ .

Clearly, a countably (\*)pairwise paracompact space is countably pairwise paracompact. But the converse is not true:

We consider the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  of Example 4.3. This space is pairwise paracompact and hence countably pairwise paracompact.

But it is not countably  $(*)$ pairwise paracompact. For, let  $\mathcal{U}$  be a pairwise open cover of  $R$  containing  $(a, b)$ . We choose  $\alpha \in Q$  and  $\beta \in R - Q$  such that  $a < \alpha < \beta < b$ . Let  $\mathcal{U}_0$  be a  $(\mathcal{P})$ open refinement of  $\mathcal{U}$  containing  $(a, \alpha), [\alpha, \beta], (\beta, b)$ . We show any refinement of  $\mathcal{U}_0$  parallel to  $\mathcal{U}$  cannot cover  $R$ . We note,  $(a, b)$  is both  $(\mathcal{P}_1)$ open and  $(\mathcal{P}_2)$ open. So  $R$  should be countably  $(*)$ pairwise paracompact provided we get a refinement of  $\mathcal{U}_0$  parallel to  $\mathcal{U}$  considering both the cases. First we suppose,  $(a, b)$  is  $(\mathcal{P}_1)$ open. So the refinement of  $[\alpha, \beta]$  should be  $(\mathcal{P}_1)$ open. But no  $(\mathcal{P}_1)$ open refinement of  $[\alpha, \beta]$  can cover  $[\alpha, \beta]$  as  $\{\beta\} \notin \mathcal{P}_1$ . Therefore the space is not countably  $(*)$ pairwise paracompact.

**DEFINITION 5.4.** The bitopological space  $X$  is said to be  $\alpha^*$ -pairwise normal if for any pair of a  $(\mathcal{P})$ closed set  $A$  and a  $(\mathcal{P}_i)$ closed set  $B$  with  $A \cap B = \emptyset$ , there exist sets  $U, V \in \mathcal{P}_i$  such that

$$A \subset U, B \subset V \text{ and } U \cap V = \emptyset.$$

Equivalently,  $X$  is  $\alpha^*$ -pairwise normal iff for any  $(\mathcal{P})$ closed set  $K$  and any  $(\mathcal{P}_i)$ open set  $U$  with  $K \subset U$ , there exists a  $(\mathcal{P}_i)$ open set  $V$  such that

$$K \subset V \subset (\mathcal{P}_i)\text{cl}V \subset U.$$

Obviously,  $\alpha^*$ -pairwise normality implies  $\alpha$ -pairwise normality. The following example shows that the converse is not true.

EXAMPLE 5.1. Let  $a, b \in R$  and  $a \neq b$ . Suppose  $\mathcal{P}_1 = \{\emptyset\} \cup \{E \subset R \mid a \in E\}$  and  $\mathcal{P}_2 = \{\emptyset\} \cup \{E \subset R \mid b \in E\}$ . Obviously,  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are topologies of  $R$ . We consider the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$ .

Let  $A$  be a  $(\mathcal{P}_i)$ closed set and  $B$  be a  $(\mathcal{P}_j^*)$ closed set with  $A \cap B = \emptyset$ . Let

$$U = \begin{cases} A \cup \{b\} & \text{if } i = 1 \\ A \cup \{a\} & \text{if } i = 2, \end{cases}$$

and

$$V = \begin{cases} B \cup \{a\} & \text{if } i = 1 \\ B \cup \{b\} & \text{if } i = 2. \end{cases}$$

Then  $U \in \mathcal{P}_j \subset \mathcal{P}$ ,  $V \in \mathcal{P}_i$  and  $U \cap V = \emptyset$ . Also  $A \subset U$  and  $B \subset V$ . Therefore the space is  $\alpha$ -pairwise normal.

Since there are no disjoint pair of  $(\mathcal{P}_i)$ open sets for both  $i = 1$  and  $i = 2$ , it follows that the space is not  $\alpha^*$ -pairwise normal.

DEFINITION 5.5. The product space  $(X \times Y, \mathcal{P}_1 \times \mathcal{Q}_1, \mathcal{P}_2 \times \mathcal{Q}_2)$  of the bitopological spaces  $(X, \mathcal{P}_1, \mathcal{P}_2)$  and  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  is said to be  $\pi$ -pairwise normal if for every pair of a  $(\mathcal{P}_i \times \mathcal{Q}_i)$ closed set  $A$  and a  $(\mathcal{P}_j \times \mathcal{Q}_j)$ closed set  $B$  with  $A \cap B = \emptyset$ ,  $i \neq j$ , there exist  $(\mathcal{P} \times \mathcal{Q})$ open sets  $U, V$  such that

$$A \subset U, B \subset V \text{ and } U \cap V = \emptyset.$$

Swart [46] proved that the product bitopological space  $(X \times Y, \mathcal{P}_1 \times \mathcal{Q}_1, \mathcal{P}_2 \times \mathcal{Q}_2)$  of two pairwise compact bitopological spaces  $(X, \mathcal{P}_1, \mathcal{P}_2)$  and  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  may not be pairwise compact. However the following theorem can be proved easily.

**THEOREM 5.3.** *If the topological space  $(X, \mathcal{T})$  is compact and the bitopological space  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  is pairwise compact, then the product space  $(X \times Y, \mathcal{T} \times \mathcal{Q}_1, \mathcal{T} \times \mathcal{Q}_2)$  is pairwise compact.*

We prove the following theorems.

**THEOREM 5.4.** *If the topological space  $(X, \mathcal{T})$  is paracompact and the bitopological space  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  is pairwise compact, then the bitopological space  $(X \times Y, \mathcal{T} \times \mathcal{Q}_1, \mathcal{T} \times \mathcal{Q}_2)$  is pairwise paracompact.*

**PROOF.** Let  $\mathcal{W}$  be a pairwise open cover of  $X \times Y$ . We replace each element of  $\mathcal{W}$  by the basis elements  $U \times V_i$ ,  $U \in \mathcal{T}$ ,  $V_i \in \mathcal{Q}_i$ . For a fixed  $x \in X$ , the space  $\{x\} \times Y$  is pairwise compact. Hence we can cover  $\{x\} \times Y$  by a finite number of basis elements

$$U_1^x \times V_{11}^x, \dots, U_{m_1(x)}^x \times V_{1m_1(x)}^x, U_{m_1(x)+1}^x \times V_{21}^x, \dots, U_{m_1(x)+m_2(x)}^x \times V_{2m_2(x)}^x \quad (5.1)$$

where  $U_p^x \in \mathcal{T}$ ,  $p = 1, 2, \dots, m_1(x) + m_2(x)$  and  $V_{ip}^x \in \mathcal{Q}_i$ ,  $p = 1, 2, \dots, m_i(x)$ .

We assume that each of the above sets intersect  $\{x\} \times Y$ . Then

$$G^x = \bigcap_{p=1}^{m(x)} U_p^x \in \mathcal{T} \text{ and } x \in G^x,$$

where  $m(x) = m_1(x) + m_2(x)$ . The sets in (5.1) cover  $G^x \times Y$ . It then follows that there exists a finite subcollection  $\mathcal{W}^x = \{W_1^x, W_2^x, \dots, W_{n(x)}^x\}$  of  $\mathcal{W}$  such that  $G^x \times Y \subset \bigcup_{k=1}^{n(x)} W_k^x$ . Since  $(X, \mathcal{T})$  is paracompact, there exists a  $(\mathcal{T})$ -locally finite  $(\mathcal{T})$ -open refinement  $\mathcal{U}$  of the  $(\mathcal{T})$ -open cover  $\{G^x \mid x \in X\}$  of  $X$ . Then the collection  $\mathcal{D} = \{(U \times Y) \cap W_k^x \mid U \in \mathcal{U}, U \subset G_x, k = 1, 2, \dots, n(x), x \in X\}$  is a parallel refinement of  $\mathcal{W}$ . Also for  $(x, y) \in X \times Y$ , there is a  $(\mathcal{T})$ -open nbd  $M$  of  $x$  intersecting a finite number of sets belonging to  $\mathcal{U}$ . Therefore the nbd  $M \times Y$  of  $(x, y)$  intersects only a finite number of sets belonging to  $\mathcal{D}$ . Since  $M \in \mathcal{T}$  and  $Y \in \mathcal{Q}_1 \cap \mathcal{Q}_2$ , it follows that  $\mathcal{D}$  is a  $\mathcal{W}$ -locally finite parallel refinement of  $\mathcal{W}$ .  $\square$

The following example shows that the product of a pairwise paracompact space and a pairwise compact space may not be pairwise paracompact.

EXAMPLE 5.2 (Swart [46], p.142). Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two topologies on  $R$  generated by the bases  $\mathcal{B}_1 = \{\emptyset\} \cup \{(-\infty, a) \mid a \in R\}$  and  $\mathcal{B}_2 = \{\emptyset\} \cup \{(a, \infty) \mid a \in R\}$  respectively. Then the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise compact but the product space  $(R \times R, \mathcal{P}_1 \times \mathcal{P}_1, \mathcal{P}_2 \times \mathcal{P}_2)$  is not pairwise paracompact.

**THEOREM 5.5.** *If the topological space  $(X, \mathcal{T})$  is countably paracompact and the bitopological space  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  is pairwise compact, then the bitopological space  $(X \times Y, \mathcal{T} \times \mathcal{Q}_1, \mathcal{T} \times \mathcal{Q}_2)$  is countably pairwise paracompact.*

**PROOF.** Let  $\mathcal{W} = \{W_n \mid n \in N\}$  be a countable pairwise open cover of  $X \times Y$  and  $G_n$  be the set of points  $x \in X$  such that  $\{x\} \times Y \subset \bigcup_{k \leq n} W_k$ . Proceeding as in Theorem 5.4, for any  $x \in G_n$ , we get an  $N_x \in \mathcal{T}$  such that  $x \in N_x$  and  $N_x \times Y \subset \bigcup_{k \leq n} W_k$ . Therefore  $N_x \subset G_n$  and so  $G_n \in \mathcal{T}$ . Since for any  $x \in X$ ,  $\{x\} \times Y$  is pairwise compact, it is contained in some finite number of sets belonging to  $\{W_n \mid n \in N\}$ . Hence  $x \in G_n$  for some  $n$ . Therefore  $\{G_n \mid n \in N\}$  is a countable  $(\mathcal{T})$ open cover of  $X$ . By the countable paracompactness of  $X$ ,  $\{G_n \mid n \in N\}$  has a  $(\mathcal{T})$ locally finite  $(\mathcal{T})$ open refinement  $\mathcal{H}$ . For each  $H \in \mathcal{H}$ , let  $g(H)$  be the first  $G_n$  containing  $H$  and  $H_n = \bigcup \{H \in \mathcal{H} \mid g(H) = G_n\}$ . Then  $\{H_n \mid n \in N\}$  is a  $(\mathcal{T})$ locally finite  $(\mathcal{T})$ open cover of  $X$  with  $H_n \subset G_n$  for all  $n \in N$ . For  $k \leq n$ ,  $D_{nk} = (H_n \times Y) \cap W_k \in \mathcal{T} \times \mathcal{Q}_i$  if  $W_k \in \mathcal{T} \times \mathcal{Q}_i$ . If  $(x, y) \in X \times Y$ , then  $x \in H_n$  for some  $n$ , and so  $(x, y) \in H_n \times Y$ . Again since  $H_n \subset G_n$ ,  $\{x\} \times Y \subset \bigcup_{k \leq n} W_k$ . Therefore  $(x, y) \in W_k$  for some  $k \leq n$ . Hence  $(x, y) \in D_{nk}$ . Also  $D_{nk} \subset W_k$ . It follows that  $\{D_{nk}\}$  is a parallel refinement of  $\mathcal{W}$ . If  $(x, y) \in X \times Y$ , then  $x \in X$  and so there exists a  $(\mathcal{T})$ open set  $M_x$  intersecting a finite number of sets belonging to  $\{H_n \mid n \in N\}$ . Hence  $M_x \times Y$  intersects only a finite number

of sets belonging to  $\{D_{nk}\}$ . It now follows that  $\{D_{nk}\}$  is a  $\mathcal{W}$ -locally finite parallel refinement of  $\mathcal{W}$ .  $\square$

The product space of Example 5.2 is not even countably pairwise paracompact. Therefore the product space of a countably pairwise paracompact space and a pairwise compact space may not be countably pairwise paracompact.

**THEOREM 5.6.** *Let the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  be  $\alpha^*$ -pairwise normal. Then the following statements are equivalent.*

- (i)  *$X$  is countably pairwise paracompact.*
- (ii) *Every countable pairwise open cover of  $X$  has a point finite parallel refinement.*
- (iii) *Every countable pairwise open cover of  $X$  is shrinkable.*
- (iv) *Every countable pairwise open cover  $\mathcal{G} = \{G_k \mid k \in N\}$  has a parallel closed refinement  $\mathcal{F} = \{F_k \mid k \in N\}$  such that  $F_k \subset G_k$ .*

**PROOF.** (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): Let  $\mathcal{U} = \{U_k \mid k \in N\}$  be a countable pairwise open cover of  $X$  and  $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$  be a point finite parallel refinement of  $\mathcal{U}$ . If  $U_k \in \mathcal{P}_i$ , we write

$$W_k = \bigcup \{V_\alpha \mid V_\alpha \subset U_k, V_\alpha \not\subset U_m \text{ if } m < k \text{ and } V_\alpha, U_m \in \mathcal{P}_i\}.$$

Then  $W_k \subset U_k$  for all  $k$  and  $\{W_k \mid k \in N\}$  is also a point finite parallel refinement of  $\mathcal{U}$ . Since  $X$  is  $\alpha^*$ -pairwise normal, it is

$\alpha$ -pairwise normal and hence by Theorem 3.5,  $\{W_k \mid k \in N\}$  is shrinkable. So  $\mathcal{U}$  is shrinkable.

(iii) $\Rightarrow$ (iv): Let  $\mathcal{G} = \{G_k \mid k \in N\}$  be a countable pairwise open cover of  $X$ . By (iii), there exists a countable parallel refinement  $\{U_k \mid k \in N\}$  of  $\mathcal{G}$  such that  $(\mathcal{P})\text{cl}U_k \subset G_k$  for all  $k \in N$ . If  $G_k \in \mathcal{P}_i$ , by  $\alpha^*$ -pairwise normality of  $X$ , there exists a  $V_k \in \mathcal{P}_i$  such that

$$(\mathcal{P})\text{cl}U_k \subset V_k \subset (\mathcal{P}_i)\text{cl}V_k \subset G_k.$$

Therefore  $\{(\mathcal{P}_i)\text{cl}V_k \mid k \in N\}$  is the required cover.

(iv) $\Rightarrow$ (i): Let  $\mathcal{U} = \{U_k \mid k \in N\}$  be a countable pairwise open cover of  $X$ . Then by (iv), there exists a parallel closed cover  $\mathcal{F} = \{F_k \mid k \in N\}$  such that  $F_k \subset U_k$  for all  $k \in N$ . We denote by  $U_1^i$  (resp.  $F_1^i$ ), the first  $(\mathcal{P}_i)$ open (resp.  $(\mathcal{P}_i)$ closed) set of  $\mathcal{U}$  (resp.  $\mathcal{F}$ ); by  $U_2^i$  (resp.  $F_2^i$ ), the second  $(\mathcal{P}_i)$ open (resp.  $(\mathcal{P}_i)$ closed) set of  $\mathcal{U}$  (resp.  $\mathcal{F}$ ) and so on. Therefore  $\mathcal{U}^i = \{U_k^i \mid k \in N\}$  and  $\mathcal{F}^i = \{F_k^i \mid k \in N\}$ . By  $\alpha^*$ -pairwise normality of  $X$ , there exists a  $(\mathcal{P}_i)$ open set  $V_{kl}^i$ ,  $l \in N$  such that

$$F_k^i \subset V_{kl}^i \subset (\mathcal{P}_i)\text{cl}V_{kl}^i \subset V_{kl+1}^i \subset U_k^i.$$

Let  $W_{1l}^i = V_{1l}^i$  and  $W_{kl}^i = V_{kl}^i - \bigcup\{(\mathcal{P}_i)\text{cl}V_{ml+1}^i \mid m = 1, 2, \dots, k-1\}$ ,  $l \in N$  and  $k = 2, 3, \dots$ , where

$$V_{pq} = V_{pq}^1 \cup V_{pq}^2, \quad p, q \in N.$$

We write

$$W_k^i = \bigcup_{l=k}^{\infty} W_{kl}^i, \quad k \in N.$$

We have

$$W_{kl}^i \cap W_{mn}^i = \emptyset \quad \text{if } k > m, \quad l > n,$$

$$W_{kl}^j \cap W_{mn}^i = \emptyset \quad \text{if } k > m, \quad l > n.$$

Therefore

$$W_k^i \cap W_{mn}^i = \emptyset \quad \text{if } k > m, \quad k > n, \quad (5.2)$$

$$W_k^j \cap W_{mn}^i = \emptyset \quad \text{if } k > m, \quad k > n. \quad (5.3)$$

We now show that the collection  $\{W_k^i \mid k \in N, i \in \{1, 2\}\}$  is a  $\mathcal{U}$ -locally finite pairwise open cover of  $X$ . Let us consider a point  $x \in X$ . Suppose  $x$  belongs to a  $(\mathcal{P}_i)$ open set of  $\mathcal{U}$ . We write

$$V_k^i = \bigcup_{l=1}^{\infty} V_{kl}^i,$$

$$V_k = V_k^1 \cup V_k^2.$$

Let  $k_0 = \min\{k \mid x \in V_k\}$  and  $l_0 = \min\{l \mid x \in V_{k_0 l}\}$ . Then  $x \in W_{k_0 l_0}^i$ . If  $k_0 \leq l_0$ , then  $x \in W_{k_0}^i$ . Again by (5.2) and (5.3), the  $(\mathcal{P}_i)$ open set  $W_{k_0 l_0}^i$  does not intersect  $W_k^1$  or  $W_k^2$  for  $k > l_0$ . Now suppose  $k_0 > l_0$ . We have

$$x \in W_{k_0 l_0}^i \subset V_{k_0 l_0}^i \subset V_{k_0 k_0}^i$$

and  $x \notin (\mathcal{P}_i)\text{cl}V_{mk_0+1}$ ,  $m = 1, 2, \dots, k_0 - 1$ . Therefore  $x \in V_{k_0 k_0}^i$  and hence  $x \in W_{k_0}^i$ . By (5.2) and (5.3), the  $(\mathcal{P}_i)$ open nbd  $W_{k_0 k_0}^i$  of  $x$  does not intersect  $W_k^1$  or  $W_k^2$  for  $k > k_0$ . So  $\{W_k^i \mid$

$k \in N, i \in \{1, 2\}$  is a  $\mathcal{U}$ -locally finite parallel refinement of  $\mathcal{U}$ . Thus  $X$  is countably pairwise paracompact.  $\square$

**THEOREM 5.7.** *If the bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is countably  $(*)$ pairwise paracompact and  $\alpha^*$ -pairwise normal, and  $(Y, q_1, q_2)$  is a pairwise compact second countable quasi-metric space then the product space  $X \times Y$  is  $\pi$ -pairwise normal.*

**PROOF.** Let  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  be the topologies on  $Y$  determined by  $q_1$  and  $q_2$  respectively. Suppose  $\mathcal{T}_i$  be the product topology  $\mathcal{P}_i \times \mathcal{Q}_i$  on  $X \times Y$ . We consider two sets  $A, B \subset X \times Y$  such that for  $i \neq j$ ,  $A$  is  $(\mathcal{T}_i)$ closed,  $B$  is  $(\mathcal{T}_j)$ closed and  $A \cap B = \emptyset$ . Let  $\{D_n^i \mid n \in N\}$  be a countable base for  $\mathcal{Q}_i$ . For any nonempty finite subset  $\gamma$  of  $N$ , we write  $H_\gamma^i = \bigcup_{n \in \gamma} D_n^i$ . We denote the set of all nonempty finite subsets  $\gamma$  of  $N$  by  $\Gamma$ . Then  $\Gamma$  is countable.

For each  $x \in X$ , let  $A_x = \{y \in Y \mid (x, y) \in A\}$  and  $B_x = \{y \in Y \mid (x, y) \in B\}$ . Then  $A_x$  and  $B_x$  are  $(\mathcal{Q}_i)$ closed and  $(\mathcal{Q}_j)$ closed respectively. For, let  $y \in X - A_x$ . Then  $(x, y) \in X - A$ . But  $X - A$  is  $(\mathcal{T}_i)$ open. Therefore there exists a  $G \in \mathcal{T}_i$  such that  $(x, y) \in G \subset X - A$ . Since  $G \in \mathcal{T}_i$ , there exist sets  $U_\xi \in \mathcal{P}_i, V_\xi \in \mathcal{Q}_i$  such that  $G = \bigcup_\xi (U_\xi \times V_\xi)$ . So for some  $\xi$ ,  $(x, y) \in U_\xi \times V_\xi$ . So  $x \in U_\xi, y \in V_\xi$ . Now let  $\beta$  be any point of  $V_\xi$ . Then  $(x, \beta) \in U_\xi \times V_\xi$ . Hence  $(x, \beta) \in G \subset X - A \Rightarrow \beta \notin A_x$  i.e.  $\beta \in X - A_x$ . So  $V_\xi \subset X - A_x$ . Thus  $X - A_x$  is  $(\mathcal{Q}_i)$ open i.e.  $A_x$  is  $(\mathcal{Q}_i)$ closed. Similarly,  $B_x$  is  $(\mathcal{Q}_j)$ closed.

Therefore by Theorem 5.1,  $A_x$  and  $B_x$  are  $(\mathcal{Q}_j)$ compact and  $(\mathcal{Q}_i)$ compact respectively. Also we have  $A_x \cap B_x = \emptyset$ . For each  $\gamma \in \Gamma$ , we write

$$\begin{aligned} U_\gamma^i &= \{x \in X \mid A_x \subset H_\gamma^j\}, \\ U_\gamma^j &= \{x \in X \mid B_x \subset Y - (\mathcal{Q}_i)\text{cl}H_\gamma^j\}, \end{aligned}$$

and

$$U_\gamma = U_\gamma^1 \cap U_\gamma^2.$$

Let  $x_0 \in U_\gamma^i$ . Then  $A_{x_0} \subset H_\gamma^j$ . So if  $y \in Y - H_\gamma^j$ , then  $y \notin A_{x_0}$  and hence  $(x_0, y) \notin A$ . Since  $A$  is  $(\mathcal{T}_i)$ closed, there is a  $(\mathcal{T}_i)$ open nbd  $L_y \times M_y$  of  $(x_0, y)$  which does not intersect  $A$ . As  $Y - H_\gamma^j$  is a  $(\mathcal{Q}_j)$ closed subset of  $Y$ , by Theorem 5.1,  $Y - H_\gamma^j$  is  $(\mathcal{Q}_i)$ compact. So the  $(\mathcal{Q}_i)$ open cover  $\{M_y \mid y \in Y - H_\gamma^j\}$  of  $Y - H_\gamma^j$  has a finite subcover  $M_{y_1}, M_{y_2}, \dots, M_{y_n}$ . If  $L_{x_0} = \bigcap_{k=1}^n L_{y_k}$ , then  $L_{x_0}$  is a  $(\mathcal{P}_i)$ open nbd of  $x_0$  and  $L_{x_0} \times (Y - H_\gamma^j)$  does not intersect  $A$ . Therefore if  $x \in L_{x_0}$ , then  $A_x \subset H_\gamma^j$ . Thus  $U_\gamma^i$  is  $(\mathcal{P}_i)$ open. Similarly, the set  $U_\gamma^j$  is  $(\mathcal{P}_j)$ open. Hence  $U_\gamma \in \mathcal{P}$ .

Let  $x \in X$ . By Theorem 5.2,  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  is pairwise normal. So there exists a  $D \in \mathcal{Q}_j$  such that  $A_x \subset D \subset (\mathcal{Q}_i)\text{cl}D \subset Y - B_x$ . Since  $A_x \subset D$  and  $A_x$  is  $(\mathcal{Q}_j)$ compact, there is a set  $\gamma_1 \in \Gamma$  such that  $A_x \subset H_{\gamma_1}^j \subset D$ . Therefore  $A_x \subset H_{\gamma_1}^j \subset (\mathcal{Q}_i)\text{cl}H_{\gamma_1}^j \subset Y - B_x$ . So  $x \in U_{\gamma_1}$ . Hence  $\{U_\gamma \mid \gamma \in \Gamma\}$  forms a countable  $(\mathcal{P})$ open cover of  $X$ , which is a refinement of the countable pairwise open cover  $\mathcal{U} = \{U_\gamma^i \mid \gamma \in \Gamma, i \in \{1, 2\}\}$ . Since  $X$  is countably  $(*)$ pairwise paracompact,  $\{U_\gamma \mid \gamma \in \Gamma\}$

has a  $\mathcal{U}$ -locally finite pairwise open refinement  $\{V_\alpha \mid \alpha \in A\}$  parallel to  $\mathcal{U}$ . We enumerate  $\Gamma$  as  $\{\gamma_1, \gamma_2, \dots\}$  and we write

$$V_{\gamma_k}^i = \bigcup \{V_\alpha \mid V_\alpha \subset U_{\gamma_k}^i, V_\alpha \not\subset U_{\gamma_m}^i \text{ if } m < k \text{ and } V_\alpha \in \mathcal{P}_i\}.$$

Then  $\mathcal{V} = \{V_\gamma^i \mid \gamma \in \Gamma, i \in \{1, 2\}\}$  is a  $\mathcal{U}$ -locally finite pairwise open parallel refinement of  $\mathcal{U}$  with  $V_\gamma^i \subset U_\gamma$ ,  $\gamma \in \Gamma$  and  $i \in \{1, 2\}$ . Again by Theorem 5.6,  $\mathcal{V}$  has a pairwise open parallel refinement  $\mathcal{W} = \{W_\gamma^i \mid \gamma \in \Gamma, i \in \{1, 2\}\}$  with  $(\mathcal{P})\text{cl}W_\gamma^i \subset V_\gamma^i$ . Let

$$U = \bigcup \{(W_\gamma^1 \cup W_\gamma^2) \times H_\gamma^j\}.$$

Then  $U \in \mathcal{P} \times \mathcal{Q}$ . Let  $(x, y)$  be any point of  $A$ . For some  $\gamma \in \Gamma$  and  $i \in \{1, 2\}$ , we have  $x \in W_\gamma^i \subset U_\gamma$ . So  $y \in A_x \subset H_\gamma^j$ . Hence  $(x, y) \in (W_\gamma^1 \cup W_\gamma^2) \times H_\gamma^j$ . Therefore  $A \subset U$ . Since  $\mathcal{V}$  is  $\mathcal{U}$ -locally finite,  $\mathcal{W}$  is also  $\mathcal{U}$ -locally finite. Also for any  $x \in X$ , there is a  $\gamma$  with  $x \in U_\gamma = U_\gamma^1 \cap U_\gamma^2$ . Hence each point  $x \in X$  has for both  $i = 1, 2$ , a  $(\mathcal{P}_i)$ open nbd  $N_x$  intersecting only a finite number of elements of  $\{(W_\gamma^1 \cup W_\gamma^2) \mid \gamma \in \Gamma\}$ . Therefore  $N_x \times Y$  is  $(\mathcal{P} \times \mathcal{Q})$ open nbd of  $(x, y)$ ,  $y \in Y$ , which intersects only a finite number of elements of  $\{(W_\gamma^1 \cup W_\gamma^2) \times H_\gamma^j \mid \gamma \in \Gamma\}$ . It thus follows that

$$\begin{aligned} (\mathcal{P} \times \mathcal{Q})\text{cl}U &= \bigcup \{(\mathcal{P} \times \mathcal{Q})\text{cl}((W_\gamma^1 \cup W_\gamma^2) \times H_\gamma^j) \mid \gamma \in \Gamma\} \\ &= \bigcup \{(\mathcal{P})\text{cl}(W_\gamma^1 \cup W_\gamma^2) \times (\mathcal{Q})\text{cl}H_\gamma^j \mid \gamma \in \Gamma\} \\ &\subset \bigcup \{U_\gamma \times (\mathcal{Q})\text{cl}H_\gamma^j \mid \gamma \in \Gamma\}. \end{aligned}$$

So  $((\mathcal{P} \times \mathcal{Q})\text{cl}U) \cap B = \emptyset$ . Therefore  $X \times Y$  is  $\pi$ -pairwise normal.  $\square$

**THEOREM 5.8.** *Let  $(X, \mathcal{P}_1, \mathcal{P}_2)$  be a bitopological space. If  $X$  is countably pairwise paracompact and  $\alpha^*$ -pairwise normal, then for a lower semi-continuous function  $g : (X, \mathcal{P}_1, \mathcal{P}_2) \rightarrow (R, \rho)$  and a upper semi-continuous function  $h : (X, \mathcal{P}_1, \mathcal{P}_2) \rightarrow (R, \rho)$  with  $h(x) < g(x)$ ,  $x \in X$ , there exists a continuous function  $f^i : (X, \mathcal{P}_i) \rightarrow (R, \rho)$  such that  $h(x) < f^i(x) < g(x)$ ,  $x \in X$ ,  $i \in \{1, 2\}$ .*

**PROOF.** Suppose  $g, h : (X, \mathcal{P}_1, \mathcal{P}_2) \rightarrow (R, \rho)$  be lower semi-continuous and upper semi-continuous functions respectively such that  $h(x) < g(x)$  for all  $x \in X$ . For a rational number  $r$ , let  $G_r = \{x \in X \mid h(x) < r < g(x)\}$ . Since  $g$  is lower semi-continuous, the set  $\{x \in X \mid g(x) > r\}$  is  $(\mathcal{P}_i)$ open. Similarly, by the upper semi-continuity of  $h$ ,  $\{x \in X \mid h(x) < r\}$  is  $(\mathcal{P}_i)$ open. Thus  $\mathcal{G} = \{G_r \mid r \in Q\}$  is a countable pairwise open cover of  $X$ , where each  $G_r$  is  $(\mathcal{P}_i)$ open for both  $i = 1, 2$ . Since  $X$  is countably pairwise paracompact, there exists, for  $i \in \{1, 2\}$ , a  $\mathcal{G}$ -locally finite  $(\mathcal{P}_i)$ open refinement  $\{U_r^i \mid r \in Q\}$  of  $\mathcal{G}$  such that  $U_r^i \subset G_r$ . Now by Theorem 5.6, there exists a countable parallel refinement  $\{V_r^i \mid r \in Q, i \in \{1, 2\}\}$  of  $\{U_r^i \mid r \in Q, i \in \{1, 2\}\}$  satisfying  $(\mathcal{P})\text{cl}V_r^i \subset U_r^i$ . So by  $\alpha^*$ -pairwise normality of  $X$ , there exists a  $W_r^i \in \mathcal{P}_i$  with  $(\mathcal{P})\text{cl}V_r^i \subset W_r^i \subset (\mathcal{P}_i)\text{cl}W_r^i \subset U_r^i$ . Then proceeding as the proof of  $((\alpha) \rightarrow (\beta))$  [14, Theorem 4], we get a continuous function  $f^i : (X, \mathcal{P}_i) \rightarrow (R, \rho)$  such that  $h(x) < f^i(x) < g(x)$ ,  $x \in X$ ,  $i \in \{1, 2\}$ .  $\square$

## CHAPTER VI

### Near Pairwise Compactness

#### 6.1. Introduction

Singal and Mathur [42] introduced and studied the notion of near compactness by generalizing the concept of compactness of a topological space. Later the notion of near compactness was studied and developed considerably by Carnahan [10], Singal and Mathur [40], Herrington [21], Joseph [23] and others. The notion of near compactness became an important meadow to topologists. Following these trends, Nandi [36] introduced the notion of near compactness in bitopological spaces: a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$  is said to be  $(\mathcal{P}_i, \mathcal{P}_j)$ nearly compact if for each  $(\mathcal{P}_i)$ open cover  $\mathcal{U}$  of  $X$ , there exists a finite subcollection  $\mathcal{V} \subset \mathcal{U}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V) \mid V \in \mathcal{V}\}$  covers  $X$ .  $X$  is said to be pairwise nearly compact if it is  $(\mathcal{P}_1, \mathcal{P}_2)$ nearly compact and  $(\mathcal{P}_2, \mathcal{P}_1)$ nearly compact. The notion of pairwise near compactness is defined considering only  $(\mathcal{P}_i)$ open covers. As such, this notion of pairwise near compactness cannot be a generalization of pairwise compactness. Here we introduce a generalized notion of pairwise compactness and we call it nearly pairwise compact (Definition 6.8). It is also a generalization of near compactness.

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The subject matter of this chapter is mainly the contents of our paper: *Kyungpook Math. J.* **53** (1) (2013), 125–133, [34].

## 6.2. Preliminaries

Let  $(X, \mathcal{P}_1, \mathcal{P}_2)$  and  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  be two bitopological spaces.

We require the following definitions.

**DEFINITION 6.1.** A collection  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  is said to be pairwise open if for each  $\alpha \in A$ ,  $U_\alpha$  is  $(\mathcal{P}_i)$ open for some  $i \in \{1, 2\}$  and for each  $i \in \{1, 2\}$ ,  $\mathcal{U} \cap \mathcal{P}_i \neq \emptyset$ . So a pairwise open collection covering  $X$  is a pairwise open cover.

**DEFINITION 6.2** (Pahk and Choi [37]). A collection  $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$  of subsets of  $X$  is said to be pairwise closed if  $\{X - F_\alpha \mid \alpha \in A\}$  is pairwise open.

**DEFINITION 6.3** (Singal and Singal [41]). A set  $A \subset X$  is said to be  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open if  $A = (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}A)$ .

A subset of  $X$  is said to be  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly closed if its complement is  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open. In other words, a set  $A \subset X$  is  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly closed iff  $A = (\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}A)$ .

**DEFINITION 6.4** (Singal and Singal [41]). The bitopological space  $X$  is said to be pairwise semiregular if for each  $x \in X$  and each  $(\mathcal{P}_i)$ open set  $U$  with  $x \in U$ , there exists a  $(\mathcal{P}_i)$ open set  $V$  such that  $x \in V \subset (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V) \subset U$ .

Obviously, a pairwise regular space is pairwise semiregular.

**DEFINITION 6.5** (Singal and Singal [41]). The bitopological space  $X$  is said to be pairwise almost regular if for each  $x \in X$

and each  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly closed set  $F$  with  $x \notin F$ , there exist a  $(\mathcal{P}_i)$ open set  $U$  and a  $(\mathcal{P}_j)$ open set  $V$  such that  $x \in U$ ,  $F \subset V$  and  $U \cap V = \emptyset$ .

Equivalently,  $X$  is pairwise almost regular iff for each  $x \in X$  and each  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open set  $U$  with  $x \in U$ , there exists a  $(\mathcal{P}_i)$ open set  $V$  such that  $x \in V \subset (\mathcal{P}_j)\text{cl}V \subset U$ .

DEFINITION 6.6 (Bose and Sinha [8]). A mapping  $f : X \rightarrow Y$  is pairwise almost open if the image of each  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open set in  $X$  is  $(\mathcal{Q}_i)$ open in  $Y$ .

DEFINITION 6.7 (Bose and Sinha [7]). A mapping  $f : X \rightarrow Y$  is said to be pairwise almost continuous if the inverse image of each  $(\mathcal{Q}_i, \mathcal{Q}_j)$ regularly open set in  $Y$  is  $(\mathcal{P}_i)$ open in  $X$ .

In the sequel, we use the following theorem.

THEOREM 6.1 (Bose and Sinha [7]). *For a map  $f : X \rightarrow Y$ , the following statements are equivalent:*

- (i)  $f$  is pairwise almost continuous.
- (ii)  $f^{-1}(A) \subset (\mathcal{P}_i)\text{int}(f^{-1}((\mathcal{Q}_i)\text{int}((\mathcal{Q}_j)\text{cl}A)))$  where  $A$  is a  $(\mathcal{Q}_i)$ open subset of  $Y$ .
- (iii)  $(\mathcal{P}_i)\text{cl}(f^{-1}((\mathcal{Q}_i)\text{cl}((\mathcal{Q}_j)\text{int}B))) \subset f^{-1}(B)$  where  $B$  is a  $(\mathcal{Q}_i)$ closed subset of  $Y$ .

We introduce the following definitions.

DEFINITION 6.8. The bitopological space  $X$  is said to be *nearly pairwise compact* if for each pairwise open cover  $\mathcal{U}$  of  $X$ ,

there exists a finite subcollection  $\mathcal{V} \subset \mathcal{U}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V) \mid V \in \mathcal{V} \cap \mathcal{P}_i, i \in \{1, 2\}\}$  covers  $X$ .

Obviously, a pairwise compact space is nearly pairwise compact. The following examples show that the notion of pairwise near compactness and near pairwise compactness are independent.

Firstly, we give an example of a pairwise nearly compact bitopological space which is not nearly pairwise compact.

EXAMPLE 6.1. Let  $b$  be a fixed real number. We define

$$\begin{aligned}\mathcal{P}_1 &= \{\emptyset, R\} \cup \left\{ \left( b - \frac{1}{n}, \infty \right) \mid n \in N \right\} \cup \{[b, \infty)\}, \\ \mathcal{P}_2 &= \{\emptyset, R\} \cup \left\{ \left( -\infty, b - \frac{1}{n} \right) \mid n \in N \right\} \cup \{(-\infty, b), \\ &\quad [b, \infty)\} \cup \left\{ R - \left[ b - \frac{1}{n}, b \right) \mid n \in N \right\}.\end{aligned}$$

Then  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is a bitopological space. Obviously, the space  $(X, \mathcal{P}_1)$  is compact which implies  $X$  is  $(\mathcal{P}_1, \mathcal{P}_2)$ nearly compact. To show  $X$  is  $(\mathcal{P}_2, \mathcal{P}_1)$ nearly compact, we need only to consider the  $(\mathcal{P}_2)$ open cover  $\{(-\infty, b - \frac{1}{n}) \mid n \in N\} \cup \{[b, \infty)\}$ . Here

$$\begin{aligned}(\mathcal{P}_2)\text{int} \left( (\mathcal{P}_1)\text{cl} \left( -\infty, b - \frac{1}{n} \right) \right) &= (\mathcal{P}_2)\text{int} \left( -\infty, b - \frac{1}{n} \right] \\ &= \left( -\infty, b - \frac{1}{n} \right),\end{aligned}$$

and

$$(\mathcal{P}_2)\text{int}((\mathcal{P}_1)\text{cl}([b, \infty)) = (\mathcal{P}_2)\text{int}R = R.$$

So  $X$  is  $(\mathcal{P}_2, \mathcal{P}_1)$ nearly compact. Thus  $X$  is pairwise nearly compact.

Now we consider a pairwise open cover  $\bigcup\{(-\infty, b - \frac{1}{n}) \mid n \in \mathbb{N}\} \cup \{[b, \infty)\}$  where we consider  $[b, \infty)$  as a  $(\mathcal{P}_1)$ open set. Then

$$\begin{aligned} (\mathcal{P}_2)\text{int} \left( (\mathcal{P}_1)\text{cl} \left( -\infty, b - \frac{1}{n} \right) \right) &= (\mathcal{P}_2)\text{int} \left( -\infty, b - \frac{1}{n} \right] \\ &= \left( -\infty, b - \frac{1}{n} \right), \end{aligned}$$

and

$$(\mathcal{P}_1)\text{int}((\mathcal{P}_2)\text{cl}([b, \infty))) = (\mathcal{P}_1)\text{int}([b, \infty)) = [b, \infty).$$

So  $X$  is not nearly pairwise compact.

The following is an example of a nearly pairwise compact space but not pairwise nearly compact.

EXAMPLE 6.2 (cf. Swart [46], p. 142). Let

$$\begin{aligned} \mathcal{P}_1 &= \{\emptyset, R\} \bigcup \{(-\infty, n) \mid n \in Z\}, \\ \mathcal{P}_2 &= \{\emptyset, R\} \bigcup \{(n, \infty) \mid n \in Z\} \end{aligned}$$

where  $Z$  is the set of integers. The bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is not  $(\mathcal{P}_i, \mathcal{P}_j)$ nearly compact for any  $i \in \{1, 2\}$ . Hence the space is not pairwise nearly compact. The space is pairwise compact and hence it is also nearly pairwise compact.

Finally, we present an example of a nearly pairwise compact space which is not pairwise compact.

EXAMPLE 6.3. Let  $b$  be a fixed real number. We define

$$\begin{aligned}\mathcal{P}_1 &= \{\emptyset, R\} \cup \{(-\infty, b], (b, \infty)\}, \\ \mathcal{P}_2 &= \{\emptyset, R\} \cup \{(b, \infty)\} \cup \left\{ \left( b + \frac{1}{n}, \infty \right) \mid n \in N \right\}.\end{aligned}$$

Then  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is a bitopological space.

Now we show that  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is not pairwise compact. We see here that  $\bigcup_{n=1}^{\infty} (b + \frac{1}{n}, \infty) \cup (-\infty, b] = R$ . So  $\mathcal{U} = \{(b + \frac{1}{n}, \infty) \mid n \in N\} \cup \{(-\infty, b]\}$  is a pairwise open cover of  $X$ . But finitely many members of this pairwise open cover cannot cover  $R$ . So  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is not a pairwise compact space.

Now we show that  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is nearly pairwise compact. For  $R$ , we have only two pairwise open cover  $\{(-\infty, b], (b, \infty)\}$  and  $\mathcal{U}$ . For the pairwise open cover  $\mathcal{U}$ ,

$$\begin{aligned}(\mathcal{P}_2)\text{int}((\mathcal{P}_1)\text{cl}(b + \frac{1}{n}, \infty)) &= (\mathcal{P}_2)\text{int}(b, \infty) = (b, \infty), \\ (\mathcal{P}_1)\text{int}((\mathcal{P}_2)\text{cl}(-\infty, b]) &= (\mathcal{P}_1)\text{int}(-\infty, b] = (-\infty, b].\end{aligned}$$

Hence  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is nearly pairwise compact.

DEFINITION 6.9. The bitopological space  $X$  is said to be *almost pairwise compact* if for each pairwise open cover  $\mathcal{U}$  of  $X$ , there exists a finite subcollection  $\mathcal{V} \subset \mathcal{U}$  such that  $\{(\mathcal{P}_j)\text{cl}V \mid V \in \mathcal{V} \cap \mathcal{P}_i, i \in \{1, 2\}\}$  covers  $X$ .

It readily follows from the definitions, a nearly pairwise compact space is an almost pairwise compact space.

DEFINITION 6.10. A cover  $\mathcal{C}$  of  $X$  is said to be a *pairwise basic cover* if there exist two bases  $\mathcal{B}_1$  and  $\mathcal{B}_2$  of the topologies  $\mathcal{P}_1$  and  $\mathcal{P}_2$  respectively such that  $\mathcal{C} \subset \mathcal{B}_1 \cup \mathcal{B}_2$  and for each  $i \in \{1, 2\}$ ,  $\mathcal{C} \cap \mathcal{B}_i$  contains a nonempty set.

DEFINITION 6.11. A collection  $\mathcal{U}$  (resp.  $\mathcal{F}$ ) of subsets of  $X$  is said to be *pairwise regularly open* (resp. *pairwise regularly closed*) if each member of  $\mathcal{U}$  (resp.  $\mathcal{F}$ ) is  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open (resp.  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly closed) for some  $i \in \{1, 2\}$  and contains at least one  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open (resp.  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly closed) set for each  $i \in \{1, 2\}$ .  $\mathcal{U}$  (resp.  $\mathcal{F}$ ) is said to be a *pairwise regularly open* (resp. *pairwise regularly closed*) cover if it covers  $X$ .

DEFINITION 6.12. A *bifilter* is a collection  $\mathcal{F}$  of nonempty subsets of  $X$  with the following properties:

- (a)  $\mathcal{F} \subset \mathcal{P}_1 \cup \mathcal{P}_2$  and  $\mathcal{F} \cap \mathcal{P}_i \neq \emptyset$  for each  $i \in \{1, 2\}$ .
- (b) If  $E, F \in \mathcal{F}$  with  $E, F \in \mathcal{P}_i$  for some  $i \in \{1, 2\}$  then  $E \cap F \in \mathcal{F}$ .
- (c) If  $G \in \mathcal{F}$  and  $H \supset G$  with  $G, H \in \mathcal{P}_i$  for some  $i \in \{1, 2\}$  then  $H \in \mathcal{F}$ .

DEFINITION 6.13. A bifilter  $\mathcal{F}$  on the bitopological space  $X$  is said to be *maximal* if it is not properly contained in any bifilter on  $X$ .

DEFINITION 6.14. A point  $p \in X$  is said to be a *bicluster point* of a bifilter  $\mathcal{F}$  if for each  $F \in \mathcal{F}$ ,  $p \in (\mathcal{P}_i)\text{cl}F$  whenever  $F$  is  $(\mathcal{P}_i)$ open for some  $i \in \{1, 2\}$ .

DEFINITION 6.15. A point  $p$  is said to be a *biconvergent point* of a bifilter  $\mathcal{F}$  if each  $(\mathcal{P}_i)$ open nbd of  $p$  is a member of  $\mathcal{F}$ .

In this chapter, for a pairwise open collection  $\mathcal{U}$  of subsets of a bitopological space  $(X, \mathcal{P}_1, \mathcal{P}_2)$ , we write  $\mathcal{U}^i$  to denote the collection of all  $(\mathcal{P}_i)$ open sets in  $\mathcal{U}$ . For the topological space  $(X, \mathcal{T})$  and  $A \subset X$ , we write  $(A, \mathcal{T}_A)$  to denote the subspace on  $A$  of  $(X, \mathcal{T})$ .

### 6.3. Results

We now establish some theorems on nearly pairwise compact spaces.

THEOREM 6.2. *In the bitopological space  $X$ , the following statements are equivalent:*

- (a)  *$X$  is nearly pairwise compact.*
- (b) *Each pairwise basic cover  $\mathcal{U}$  of  $X$  possesses a finite subcollection  $\mathcal{V} \subset \mathcal{U}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V) \mid V \in \mathcal{V} \cap \mathcal{P}_i, i \in \{1, 2\}\}$  covers  $X$ .*
- (c) *Each pairwise regularly open cover of  $X$  has a finite subcover.*
- (d) *Each pairwise regularly closed collection of subsets of  $X$  with finite intersection property has nonempty intersection.*

(e) Each pairwise closed collection  $\mathcal{F} = \{F_\alpha \mid \alpha \in B\}$  of subsets of  $X$  with the property that for any finite subcollection  $\mathcal{E} \subset \mathcal{F}$ ,  $\bigcap\{(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \mid F_\alpha \in \mathcal{E}, X - F_\alpha \in \mathcal{P}_i, i \in \{1, 2\}\} \neq \emptyset$ , has nonempty intersection.

PROOF. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Let  $\mathcal{G} = \{G_\alpha \mid \alpha \in A\}$  be a pairwise regularly open cover of  $X$  and let  $\mathcal{B}_i$  be a base of the topology  $\mathcal{P}_i$ . For each  $G_\alpha \in \mathcal{G}$  with  $G_\alpha \in \mathcal{P}_i$ , there exist  $\mathcal{H}_{i\alpha} = \{H_\lambda \mid \lambda \in \Lambda_\alpha, H_\lambda \in \mathcal{B}_i\}$  such that  $G_\alpha = \bigcup\{H_\lambda \mid H_\lambda \in \mathcal{H}_{i\alpha}\}$ . Then  $\mathcal{U} = \{H_\lambda \mid \lambda \in \Lambda_\alpha, \alpha \in A\}$  is a pairwise basic cover of  $X$ . So by (b), we obtain a finite subcollection  $\mathcal{V} = \{H_{\lambda_k} \mid k = 1, 2, \dots, m\}$  of  $\mathcal{U}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}H_{\lambda_k}) \mid H_{\lambda_k} \in \mathcal{V} \cap \mathcal{P}_i, k = 1, 2, \dots, m\}$  covers  $X$ . For each  $H_{\lambda_k} \in \mathcal{P}_i$ , there exists a  $G_{\alpha_k} \in \mathcal{P}_i$ ,  $\alpha_k \in A$  such that  $H_{\lambda_k} \subset G_{\alpha_k}$  which implies  $(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}H_{\lambda_k}) \subset (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_{\alpha_k}) = G_{\alpha_k}$ . Then  $\{G_{\alpha_k} \mid k = 1, 2, \dots, m\}$  is a finite subcover of  $\mathcal{G}$ .

(c) $\Rightarrow$ (d): We suppose that  $\mathcal{F} = \{F_\alpha \mid \alpha \in I\}$  is a pairwise regularly closed collection of subsets of  $X$  with finite intersection property i.e. for each  $n \in \mathbb{N}$ ,  $\bigcap\{F_{\alpha_k} \mid k = 1, 2, \dots, n\} \neq \emptyset$ . If possible, let  $\bigcap\{F_\alpha \mid \alpha \in I\} = \emptyset$ . Then  $\{X - F_\alpha \mid \alpha \in I\}$  is a pairwise regularly open cover of  $X$ . So by (c),  $\{X - F_\alpha \mid \alpha \in I\}$  has a finite subcover  $\{X - F_{\alpha_k} \mid k = 1, 2, \dots, m\}$  which in turn implies  $\bigcap\{F_{\alpha_k} \mid k = 1, 2, \dots, m\} = \emptyset$ . This is a contradiction to our assumption. Thus we have  $\bigcap\{F_\alpha \mid \alpha \in I\} \neq \emptyset$ .

(d) $\Rightarrow$ (e): We suppose that  $\mathcal{F} = \{F_\alpha \mid \alpha \in B\}$  is a pairwise closed collection of subsets of  $X$  such that for any finite subcollection  $\mathcal{E}$  of  $\mathcal{F}$ ,  $\bigcap\{(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \mid F_\alpha \in \mathcal{E}, X - F_\alpha \in \mathcal{P}_i, i \in \{1, 2\}\} \neq \emptyset$ . Thus  $\{(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \mid \alpha \in B\}$  is a pairwise regularly closed collection of subsets of  $X$  with finite intersection property. So by (d), we have  $\bigcap\{(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \mid \alpha \in B\} \neq \emptyset$ . Since  $F_\alpha$  is  $(\mathcal{P}_i)$ closed,  $(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \subset F_\alpha$ . Thus it follows that  $\bigcap\{F_\alpha \mid \alpha \in B\} \neq \emptyset$ .

(e) $\Rightarrow$ (a): Suppose  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  is a pairwise open cover of  $X$ . If possible, suppose  $X$  is not nearly pairwise compact. So for any finite subcollection  $\{U_{\alpha_k} \mid \alpha_k \in A, k = 1, 2, \dots, m\}$  of  $\mathcal{U}$ ,  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}) \mid \alpha_k \in A, k = 1, 2, \dots, m; U_{\alpha_k} \in \mathcal{P}_i, i \in \{1, 2\}\}$  is not a cover of  $X$ . Thus  $\bigcap\{X - (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}) \mid \alpha_k \in A, k = 1, 2, \dots, m; U_{\alpha_k} \in \mathcal{P}_i, i \in \{1, 2\}\} \neq \emptyset$ . Now  $(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k})$  is a  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open set and  $U_{\alpha_k} \subset (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k})$ . Then

$$\begin{aligned} X - (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}) &\subset X - U_{\alpha_k} \\ \Rightarrow (\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}(X - (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}))) & \\ &\subset (\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}(X - U_{\alpha_k})) \\ \Rightarrow X - (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}) &\subset (\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}(X - U_{\alpha_k})). \end{aligned}$$

So  $\bigcap\{(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}(X - U_{\alpha_k})) \mid \alpha_k \in A, k = 1, 2, \dots, m; U_{\alpha_k} \in \mathcal{P}_i, i \in \{1, 2\}\} \neq \emptyset$ . Thus  $\{X - U_\alpha \mid \alpha \in A\}$  is a pairwise closed collection of subsets of  $X$  satisfying the properties of

(e). Hence  $\bigcap\{X - U_\alpha \mid \alpha \in A\} \neq \emptyset$  which in turn implies  $\bigcup\{U_\alpha \mid \alpha \in A\} \neq X$ , which is a contradiction.  $\square$

**THEOREM 6.3.** *A pairwise semiregular nearly pairwise compact space is a pairwise compact space.*

**PROOF.** Suppose  $X$  is pairwise semiregular and nearly pairwise compact. Let  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  be a pairwise open cover of  $X$ . For each  $x \in X$ , there exists a  $U_{\alpha(x)} \in \mathcal{U}$ ,  $\alpha(x) \in A$  with  $x \in U_{\alpha(x)}$ . Suppose  $U_{\alpha(x)} \in \mathcal{P}_i$ . So by pairwise semiregularity of  $X$ , there exists a  $(\mathcal{P}_i)$ open set  $G_x$  such that  $x \in G_x \subset (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_x) \subset U_{\alpha(x)}$ . Here  $\mathcal{G} = \{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_x) \mid x \in X\}$  is a pairwise regularly open cover of  $X$ . Using (c) of Theorem 6.2, we obtain a finite subcover  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_{x_k}) \mid k = 1, 2, \dots, n\}$  of  $\mathcal{G}$  which in turn implies  $\{U_{\alpha(x_k)} \mid k = 1, 2, \dots, n\}$  is a finite subcover of  $\mathcal{U}$ .  $\square$

**THEOREM 6.4.** *A pairwise almost regular space is nearly pairwise compact if it is almost pairwise compact.*

**PROOF.** Let  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  be a pairwise regularly open cover of a pairwise almost regular and almost pairwise compact space  $X$ . For each  $x \in X$ , we have a  $U_{\alpha(x)} \in \mathcal{U}$ ,  $\alpha(x) \in A$  such that  $x \in U_{\alpha(x)}$ . Suppose  $U_{\alpha(x)} \in \mathcal{P}_i$ . Hence using the notion of pairwise almost regularity, we obtain a  $(\mathcal{P}_i)$ open set  $G_x$  such that  $x \in G_x \subset (\mathcal{P}_j)\text{cl}G_x \subset U_{\alpha(x)}$ . Obviously,  $\mathcal{G} = \{G_x \mid x \in X\}$  is a pairwise open cover of  $X$ . So there exists

a finite subcollection  $\{G_{x_k} \mid k = 1, 2, \dots, n\}$  of  $\mathcal{G}$  such that  $\{(\mathcal{P}_j)\text{cl}G_{x_k} \mid k = 1, 2, \dots, n\}$  covers  $X$ . Thus  $\{U_{\alpha(x_k)} \mid k = 1, 2, \dots, n\}$  is a finite subcover of  $\mathcal{U}$  for  $X$ . Hence by (c) of Theorem 6.2,  $X$  is nearly pairwise compact.  $\square$

LEMMA 6.1. *Each  $(\mathcal{P}_i)$ open cover  $\mathcal{U}$  of a  $(\mathcal{P}_j, \mathcal{P}_i)$ regularly closed subset  $F$  of a nearly pairwise compact space  $X$  has a finite subfamily  $\mathcal{V}$  of  $\mathcal{U}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}A) \mid A \in \mathcal{V}\}$  covers  $F$ .*

PROOF. The proof is straightforward and hence omitted.  $\square$

THEOREM 6.5. *Every pairwise Hausdorff, nearly pairwise compact space is pairwise almost regular.*

PROOF. Suppose  $X$  is a pairwise Hausdorff and nearly pairwise compact space. Let  $G$  be a  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open set and  $x$  be a point of  $X$  with  $x \in G$ . For each  $y \in X - G$ , we obtain a  $(\mathcal{P}_i)$ open set  $U_y$  and a  $(\mathcal{P}_j)$ open set  $V_y$  such that  $x \in U_y$ ,  $y \in V_y$  and  $U_y \cap V_y = \emptyset$ . Then  $\mathcal{G} = \{V_y \mid y \in X - G\}$  is a  $(\mathcal{P}_j)$ open cover of the  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly closed set  $X - G$ . So by Lemma 6.1,  $\mathcal{G}$  has a finite subcollection  $\mathcal{H} = \{V_{y_k} \mid k = 1, 2, \dots, n\}$  with  $X - G \subset \bigcup\{(\mathcal{P}_j)\text{int}((\mathcal{P}_i)\text{cl}V_{y_k}) \mid k = 1, 2, \dots, n\}$ . We write  $U = \bigcap_{k=1}^n U_{y_k}$  and  $V = \bigcup_{k=1}^n (\mathcal{P}_j)\text{int}((\mathcal{P}_i)\text{cl}V_{y_k})$ . Then  $U$  is  $(\mathcal{P}_i)$ open with  $x \in U$  and  $V$  is  $(\mathcal{P}_j)$ open with  $X - G \subset V$ . Also  $U \cap V = \emptyset$ . For, if  $x \in U \cap V$ , then  $x \in U_{y_k}$  for all  $k = 1, 2, \dots, n$ , and  $x \in (\mathcal{P}_j)\text{int}((\mathcal{P}_i)\text{cl}V_{y_k})$  for some  $k = 1, 2, \dots, n$ . Suppose

$x \in (\mathcal{P}_j)\text{int}((\mathcal{P}_i)\text{cl}V_{y_l}), l \in \{1, 2, \dots, n\}$ . Again since  $x \in U_{y_l}$  and  $U_{y_l}$  is  $(\mathcal{P}_i)$ open with  $U_{y_l} \cap V_{y_l} = \emptyset$ , we have

$$\begin{aligned} x &\notin (\mathcal{P}_i)\text{cl}V_{y_l} \\ \Rightarrow x &\notin (\mathcal{P}_j)\text{int}((\mathcal{P}_i)\text{cl}V_{y_l}), \end{aligned}$$

which is a contradiction. So  $U \cap V = \emptyset$ . Thus  $(\mathcal{P}_j)\text{cl}U \subset X - V$ . Therefore it follows that  $x \in U \subset (\mathcal{P}_j)\text{cl}U \subset G$ .  $\square$

**THEOREM 6.6.** *If the topological space  $(X, \mathcal{T})$  is nearly compact and the bitopological space  $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$  is nearly pairwise compact, then the product space  $(X \times Y, \mathcal{T} \times \mathcal{Q}_1, \mathcal{T} \times \mathcal{Q}_2)$  is nearly pairwise compact.*

**PROOF.** Let  $\mathcal{U}$  be a pairwise basic cover of  $X \times Y$ . For each  $U \in \mathcal{U}$ , we have  $U = G \times H$ ,  $G \in \mathcal{T}$  and  $H \in \mathcal{Q}_i, i \in \{1, 2\}$ . For each  $x \in X$ , the space  $\{x\} \times Y$  is nearly pairwise compact. Hence we get a finite number of elements  $G_x^k \times H_x^k, k = 1, 2, \dots, n$  of  $\mathcal{U}$  such that  $\{x\} \times Y \subset \bigcup_{k=1}^n (\mathcal{T} \times \mathcal{Q}_i)\text{int}((\mathcal{T} \times \mathcal{Q}_j)\text{cl}(G_x^k \times H_x^k))$  where we assume  $H_x^k \in \mathcal{Q}_i$ . We suppose that all the sets  $G_x^k \times H_x^k$  intersects  $\{x\} \times Y$ . Then  $x \in G_x$  where  $G_x = \bigcap_{k=1}^n G_x^k \in \mathcal{T}$ . The  $(\mathcal{T})$ open cover  $\{G_x \mid x \in X\}$  of  $X$  has a finite subfamily  $G_{x_1}, G_{x_2}, \dots, G_{x_m}$  such that  $X = \bigcup_{l=1}^m (\mathcal{T})\text{int}((\mathcal{T})\text{cl}G_{x_l})$ . Hence the collection  $\{(\mathcal{T} \times \mathcal{Q}_i)\text{int}((\mathcal{T} \times \mathcal{Q}_j)\text{cl}(G_{x_l}^k \times H_{x_l}^k)) \mid k = 1, 2, \dots, n; l = 1, 2, \dots, m\}$  covers  $X \times Y$  and  $\{G_{x_l}^k \times H_{x_l}^k \mid k = 1, 2, \dots, n; l = 1, 2, \dots, m\}$  is a finite subcollection of  $\mathcal{U}$ .  $\square$

But the product of two nearly pairwise compact space need

not be nearly pairwise compact. For, we consider Example 6.2. The space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is nearly pairwise compact, but the product space  $(R \times R, \mathcal{P}_1 \times \mathcal{P}_1, \mathcal{P}_2 \times \mathcal{P}_2)$  is not nearly pairwise compact.

LEMMA 6.2. *A bifilter  $\mathcal{F}$  is maximal iff for some  $i \in \{1, 2\}$ , each  $(\mathcal{P}_i)$ open set  $A$  intersecting every member of  $\mathcal{F}^i$  belongs to  $\mathcal{F}$ .*

PROOF. Firstly, suppose  $\mathcal{F}$  is maximal. We write  $\mathcal{G} = \{G \mid G \supset A \cap B \text{ for some } B \in \mathcal{F}^i \text{ and } G \text{ is } (\mathcal{P}_i)\text{open}\} \cup \mathcal{F}^j$ . Obviously,  $\mathcal{G}$  is a bifilter with  $\mathcal{G} \supset \mathcal{F}$  and  $A \in \mathcal{G}$ . Since  $\mathcal{F}$  is a maximal bifilter, we have  $\mathcal{G} = \mathcal{F}$ .

Conversely, suppose the condition holds. If  $\mathcal{F}$  is not maximal, there exists a bifilter  $\mathcal{H}$  such that  $\mathcal{H} \supset \mathcal{F}$ . Let  $H \in \mathcal{H}$  and  $H$  be  $(\mathcal{P}_i)$ open. Then by the definition of a bifilter,  $H$  intersects every member of  $\mathcal{H}^i$  and hence every member of  $\mathcal{F}^i$ . Thus  $H \in \mathcal{F}$  and hence we have  $\mathcal{H} = \mathcal{F}$ .  $\square$

LEMMA 6.3. *A bicluster point of a bifilter is a biconvergent point if it is a maximal bifilter.*

PROOF. Suppose the maximal bifilter  $\mathcal{F}$  has a bicluster point  $p$ . Then for each  $F \in \mathcal{F}$ ,  $p \in (\mathcal{P}_i)\text{cl}F$  whenever  $F$  is  $(\mathcal{P}_i)$ open for some  $i \in \{1, 2\}$ . So each  $(\mathcal{P}_i)$ open nbd  $V$  of  $p$  intersects every  $F \in \mathcal{F}^i$ . Thus by Lemma 6.2,  $V \in \mathcal{F}$  which implies  $p$  is a biconvergent point of  $\mathcal{F}$ .  $\square$

LEMMA 6.4. *Each pairwise open collection of subsets of  $X$  with finite intersection property is contained in a maximal bifilter.*

PROOF. The proof is straightforward and hence omitted.  $\square$

THEOREM 6.7. *Let  $X$  be pairwise almost regular and each bifilter  $\mathcal{A}$  in  $X$  has the following property: for  $A, B \in \mathcal{A}$  with  $A \in \mathcal{P}_1$  and  $B \in \mathcal{P}_2$ ,  $A \cap B$  is nonempty  $(\mathcal{P}_i)$ open for each  $i \in \{1, 2\}$ . Then the following statements are equivalent:*

- (a)  *$X$  is nearly pairwise compact.*
- (b) *Each bifilter in  $X$  has a bicluster point.*
- (c) *Each maximal bifilter in  $X$  has a biconvergent point.*

PROOF. (a)  $\Rightarrow$  (b): Let  $\mathcal{G} = \{G_\alpha \mid \alpha \in A\}$  be a bifilter. For each  $\alpha \in A$ , we write  $F_\alpha = (\mathcal{P}_i)\text{cl}G_\alpha$  if  $G_\alpha \in \mathcal{P}_i$ . Then  $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$  is a pairwise closed collection of subsets of  $X$  with the following property: for any finite subcollection  $\mathcal{E} \subset \mathcal{F}$ ,  $\bigcap\{(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}F) \mid F \in \mathcal{E}, X - F_\alpha \in \mathcal{P}_i, i \in \{1, 2\}\} \neq \emptyset$ . Hence by Theorem 6.2(e),  $\bigcap\{F_\alpha \mid \alpha \in A\} \neq \emptyset$ . Thus there exists a  $p \in X$  with  $p \in F_\alpha$  for each  $\alpha \in A$ . So  $p$  is a bicluster point of  $\mathcal{G}$ .

(b)  $\Rightarrow$  (c): A maximal bifilter is of course a bifilter. So by (b), each maximal bifilter has a bicluster point and hence by Lemma 6.3, it has a biconvergent point.

(c)  $\Rightarrow$  (a): Let  $\mathcal{U}$  be a pairwise regularly open cover of  $X$ . Suppose  $\mathcal{U}$  has no finite subcollection covering  $X$ . Again for each  $x \in X$ , there exists a  $U_x \in \mathcal{U}$  such that  $x \in U_x$ . Suppose  $U_x$  is  $(\mathcal{P}_i, \mathcal{P}_j)$ regularly open. Since  $X$  is pairwise almost regular, we obtain a  $(\mathcal{P}_i)$ open set  $G_x$  such that  $x \in G_x \subset (\mathcal{P}_j)\text{cl}G_x \subset U_x$ . We note here that  $\mathcal{G} = \{G_x \mid x \in X\}$  is a pairwise open cover of  $X$ . Also  $\mathcal{H} = \{X - (\mathcal{P}_j)\text{cl}G_x \mid G_x \in \mathcal{G}\}$  is a pairwise open collection of subsets of  $X$  with finite intersection property. Then by Lemma 6.4, we obtain a maximal bifilter  $\mathcal{E}$  which contains  $\mathcal{H}$ . So by (c),  $\mathcal{E}$  has a biconvergent point  $p$ . A biconvergent point of a maximal bifilter is also a bicluster point. So if  $E \in \mathcal{E}$  is  $(\mathcal{P}_i)$ open then  $p \in (\mathcal{P}_i)\text{cl}E$  for each  $E \in \mathcal{E}$ . Hence  $p \in (\mathcal{P}_j)\text{cl}(X - (\mathcal{P}_j)\text{cl}G_x)$  for each  $G_x \in \mathcal{G}$ . Now we show  $p \notin G_x$  for any  $G_x \in \mathcal{G}$ . We need only to prove the case when  $p \notin X - (\mathcal{P}_j)\text{cl}G_x$  but  $p$  is a  $(\mathcal{P}_j)$ limit point of  $X - (\mathcal{P}_j)\text{cl}G_x$ . If possible, let  $p \in G_z$  for some  $G_z \in \mathcal{G}$ . For definiteness suppose,  $G_z$  is  $(\mathcal{P}_i)$ open. Now each  $(\mathcal{P}_j)$ open set  $A$  with  $p \in A$  intersects each  $E \in \mathcal{E}$  whenever  $E$  is  $(\mathcal{P}_j)$ open. Again  $G_z$  intersects each  $E \in \mathcal{E}$  whenever  $E$  is  $(\mathcal{P}_i)$ open. Therefore by Lemma 6.2,  $A, G_z \in \mathcal{E}$ . So by the given condition,  $A \cap G_z$  is  $(\mathcal{P}_i)$ open for each  $i \in \{1, 2\}$  and  $p \in A \cap G_z \subset G_z$ . Since  $p$  is a  $(\mathcal{P}_j)$ limit point of  $X - (\mathcal{P}_j)\text{cl}G_z$  we have  $(A \cap G_z) \cap (X - (\mathcal{P}_j)\text{cl}G_z) \neq \emptyset$  which is not possible, since  $G_z \cap (X - (\mathcal{P}_j)\text{cl}G_z) = \emptyset$ . Thus our anticipation  $p \notin G_x$  for any  $G_x \in \mathcal{G}$  is true. This contradicts the fact that  $\mathcal{G}$  is a

pairwise open cover of  $X$ . So  $\mathcal{U}$  must have a finite subcover. Hence  $X$  is nearly pairwise compact.  $\square$

REMARK 6.1. *Theorem 6.7 also holds good if the expression ‘ $X$  be pairwise almost regular’ of the theorem is replaced by ‘ $X$  be the bitopological space with each  $(X, \mathcal{P}_i)$  being regular’.*

We now give an example of a bitopological space which satisfies the conditions of Theorem 6.7.

EXAMPLE 6.4. For any  $a \in R$ , we define

$$\begin{aligned}\mathcal{P}_1 &= \{\emptyset, R, (-\infty, a), (-\infty, a], (a, \infty), R - \{a\}\}, \\ \mathcal{P}_2 &= \{\emptyset, R, (-\infty, a), (-\infty, a]\}.\end{aligned}$$

In the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$ , the nontrivial  $(\mathcal{P}_1)$ closed sets are  $[a, \infty)$ ,  $(a, \infty)$ ,  $(-\infty, a]$  and  $\{a\}$ , and the nontrivial  $(\mathcal{P}_2)$ closed sets are  $[a, \infty)$  and  $(a, \infty)$ . Hence it follows that the only nontrivial  $(\mathcal{P}_1, \mathcal{P}_2)$ regularly closed set is  $(-\infty, a]$ , and the only nontrivial  $(\mathcal{P}_2, \mathcal{P}_1)$ regularly closed set is  $(a, \infty)$ . Now it is easy to see that the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is pairwise almost regular. The possible bifilters of this space are  $\{(-\infty, a], R\}$  and  $\{(-\infty, a), (-\infty, a], R - \{a\}, R\}$ . Clearly, they satisfy the conditions of Theorem 6.7. It also follows, the bitopological space  $(R, \mathcal{P}_1, \mathcal{P}_2)$  is not pairwise regular.

THEOREM 6.8. *Let  $f : X \rightarrow Y$  be a pairwise almost continuous, pairwise almost open mapping onto the space  $Y$ . Then  $Y$  is nearly pairwise compact if  $X$  is so.*

PROOF. Suppose  $X$  is nearly pairwise compact. Let  $\mathcal{U}^{(Y)} = \{U_\alpha \mid \alpha \in A\}$  be a pairwise regularly open cover of  $Y$ . Then using pairwise almost continuity of  $f$ , we see  $\mathcal{U}^{(X)} = \{f^{-1}(U_\alpha) \mid \alpha \in A\}$  is a pairwise open cover of  $X$ . Since  $X$  is nearly pairwise compact, there exists a finite subcollection  $\mathcal{V}^{(X)} = \{f^{-1}(U_{\alpha_k}) \mid \alpha_k \in A, k = 1, 2, \dots, n\}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k}))) \mid U_{\alpha_k} \in \mathcal{U}^{(Y)} \cap \mathcal{Q}_i, k = 1, 2, \dots, n\}$  covers  $X$ . Now we have

$$\begin{aligned}
Y &= f(X) \quad (\text{since } f \text{ is onto}) \\
&= f\left(\bigcup_{k=1}^n \left\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k}))) \mid \right. \right. \\
&\quad \left. \left. f^{-1}(U_{\alpha_k}) \in \mathcal{V}^{(X)} \cap \mathcal{P}_i, i \in \{1, 2\}\right\}\right) \\
&= \bigcup_{k=1}^n \left\{f\left((\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right) \mid \right. \\
&\quad \left. f^{-1}(U_{\alpha_k}) \in \mathcal{V}^{(X)} \cap \mathcal{P}_i, i \in \{1, 2\}\right\}. \quad (6.1)
\end{aligned}$$

Also  $f\left((\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right) \subset f\left((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k}))\right)$  and  $f\left((\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right)$  is  $(\mathcal{Q}_i)$ open in  $Y$ . Thus we have

$$f\left((\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right) \subset (\mathcal{Q}_i)\text{int}\left(f\left((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k}))\right)\right). \quad (6.2)$$

Now we put  $V_{\alpha_k} = (\mathcal{Q}_j)\text{cl}U_{\alpha_k}$ . Hence we have

$$U_{\alpha_k} \subset V_{\alpha_k} \Rightarrow U_{\alpha_k} \subset (\mathcal{Q}_j)\text{cl}\left((\mathcal{Q}_i)\text{int}V_{\alpha_k}\right).$$

Therefore from (6.2) we obtain

$$\begin{aligned}
&f\left((\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right) \\
&\quad \subset (\mathcal{Q}_i)\text{int}\left(f\left((\mathcal{P}_j)\text{cl}(f^{-1}((\mathcal{Q}_j)\text{cl}((\mathcal{Q}_i)\text{int}V_{\alpha_k})))\right)\right)
\end{aligned}$$



**THEOREM 6.9.** *Let  $f : X \rightarrow Y$  be pairwise almost continuous onto the space  $Y$ . Then  $Y$  is almost pairwise compact if  $X$  is almost pairwise compact.*

**PROOF.** Suppose  $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$  is a pairwise regularly open cover of  $Y$ . Then  $\mathcal{G} = \{f^{-1}(U_\alpha) \mid \alpha \in A\}$  is a pairwise open cover of  $X$ . Therefore  $\mathcal{G}$  has a finite subfamily  $\mathcal{H} = \{f^{-1}(U_{\alpha_k}) \mid k = 1, 2, \dots, n\}$  such that  $\{(\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})) \mid f^{-1}(U_{\alpha_k}) \in \mathcal{H} \cap \mathcal{P}_i, k = 1, 2, \dots, n\}$  covers  $X$ . Since  $f$  is pairwise almost continuous, on using the scheme we adopt to establish (6.3) of Theorem 6.8, we obtain  $(\mathcal{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})) \subset f^{-1}((\mathcal{Q}_j)\text{cl}U_{\alpha_k})$ . Hence the finite subcollection  $\{(\mathcal{Q}_j)\text{cl}U_{\alpha_k} \mid U_{\alpha_k} \in \mathcal{Q}_i, k = 1, 2, \dots, n\}$  of  $\mathcal{U}$  covers  $Y$ . So by Lemma 6.5,  $Y$  is almost pairwise compact.  $\square$

**THEOREM 6.10.** *Let  $A$  be  $(\mathcal{P}_i)$  open in  $X$  for each  $i \in \{1, 2\}$ . Then  $A$  is nearly pairwise compact iff each pairwise open cover  $\mathcal{U}$  with respect to  $(X, \mathcal{P}_1, \mathcal{P}_2)$  of  $A$  has a finite subcollection  $\mathcal{V}$  of  $\mathcal{U}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V) \mid V \in \mathcal{V} \cap \mathcal{P}_i, i \in \{1, 2\}\}$  covers  $A$ .*

**PROOF.** Firstly, suppose  $A$  is nearly pairwise compact. Let  $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$  be a pairwise open cover of  $A$  with respect to  $(X, \mathcal{P}_1, \mathcal{P}_2)$ . Now  $A \cap U_\alpha$  is  $(\mathcal{P}_{iA})$  open if  $U_\alpha$  is  $(\mathcal{P}_i)$  open. So  $\mathcal{U}^{(A)} = \{A \cap U_\alpha \mid \alpha \in I\}$  is a pairwise open cover of  $A$  with respect to  $(A, \mathcal{P}_{1A}, \mathcal{P}_{2A})$ . Hence we obtain a finite subcollection  $\mathcal{V}^{(A)} = \{A \cap U_{\alpha_k} \mid \alpha_k \in I, k = 1, 2, \dots, n\}$  such that

$\{(\mathcal{P}_{iA})\text{int}((\mathcal{P}_{jA})\text{cl}(A \cap U_{\alpha_k})) \mid U_{\alpha_k} \in \mathcal{P}_i, k = 1, 2, \dots, n\}$  covers  $A$ . The ‘only if’ part now follows from the following relations.

$$\begin{aligned} A &= \bigcup_{k=1}^n \{(\mathcal{P}_{iA})\text{int}((\mathcal{P}_{jA})\text{cl}(A \cap U_{\alpha_k}))\} \\ &= \bigcup_{k=1}^n \{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}(A \cap U_{\alpha_k}))\} \text{ (since a } (\mathcal{P}_{iA})\text{open} \\ &\quad \text{set in } (A, \mathcal{P}_{1A}, \mathcal{P}_{2A}) \text{ is a } (\mathcal{P}_i)\text{open set in } (X, \mathcal{P}_1, \mathcal{P}_2)) \\ &\subset \bigcup_{k=1}^n (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}). \end{aligned}$$

To prove the ‘if’ part, let  $\mathcal{G} = \{G_\alpha \mid \alpha \in \Gamma\}$  be a pairwise open cover of  $A$  with respect to  $(A, \mathcal{P}_{1A}, \mathcal{P}_{2A})$ . Then  $\mathcal{G}$  is also a pairwise open cover of  $A$  with respect to  $(X, \mathcal{P}_1, \mathcal{P}_2)$ , since  $A$  is  $(\mathcal{P}_i)$ open for each  $i \in \{1, 2\}$ . Hence we have a finite subcollection  $\mathcal{H} = \{G_{\alpha_k} \mid k = 1, 2, \dots, m\}$  such that  $\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_{\alpha_k}) \mid G_{\alpha_k} \in \mathcal{H} \cap \mathcal{P}_i\}$  covers  $A$ . Now

$$\begin{aligned} A &= \bigcup_{k=1}^m \{A \cap (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_{\alpha_k})\} \\ &= \bigcup_{k=1}^m \{(\mathcal{P}_{iA})\text{int}(A \cap (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_{\alpha_k}))\} \\ &\subset \bigcup_{k=1}^m \{(\mathcal{P}_{iA})\text{int}((\mathcal{P}_{jA})\text{cl}G_{\alpha_k})\}. \end{aligned}$$

Hence  $A$  is nearly pairwise compact.  $\square$

LEMMA 6.6. *Each proper subset of a pairwise Hausdorff space  $X$  has a  $(\mathcal{P}_i)$ open cover.*

PROOF. Obvious.  $\square$

THEOREM 6.11. *Let  $X$  be pairwise Hausdorff and nearly pairwise compact.  $A$  and  $B$  are  $(\mathcal{P}_1, \mathcal{P}_2)$ regularly closed and  $(\mathcal{P}_2, \mathcal{P}_1)$  regularly closed subsets of  $X$  respectively with  $A \cap B =$*

$\emptyset$ . Then there exist a  $(\mathcal{P}_2)$  open set  $U$  and a  $(\mathcal{P}_1, \mathcal{P}_2)$  regularly open set  $V$  such that  $A \subset U, B \subset V$  with  $U \cap V = \emptyset$ .

PROOF. Firstly, we fixed a  $y \in B$ . Then for each  $x \in A$ , on using pairwise Hausdorffness of  $X$ , we obtain a  $(\mathcal{P}_2)$  open set  $U_x$  and a  $(\mathcal{P}_1)$  open set  $V_x$  such that  $x \in U_x, y \in V_x$  with  $U_x \cap V_x = \emptyset$ . So  $\mathcal{U} = \{U_x \mid x \in A\}$  is a  $(\mathcal{P}_2)$  open cover of  $A$  which in turn implies  $\mathcal{U} \cup \{X - A\}$  is a pairwise open cover of  $X$ . So near pairwise compactness of  $X$  ensure the existence of a finite subcollection  $\mathcal{G}$  of  $\mathcal{U} \cup \{X - A\}$  such that  $A \subset \bigcup\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V) \mid V \in \mathcal{G} \cap \mathcal{P}_i, i \in \{1, 2\}\}$ . Now we have  $(\mathcal{P}_1)\text{int}((\mathcal{P}_2)\text{cl}(X - A)) = X - A$ . Hence  $\mathcal{G}_A = \mathcal{G} - \{X - A\}$  is a finite subcollection of  $\mathcal{U}$  such that  $A \subset \bigcup\{(\mathcal{P}_2)\text{int}((\mathcal{P}_1)\text{cl}V \mid V \in \mathcal{G}_A\}$ . Let  $\mathcal{G}_A = \{U_{x_k} \mid k = 1, 2, \dots, n\}$ . Thus  $y \in \bigcap_{k=1}^n V_{x_k}$  and

$$\begin{aligned} A &\subset \bigcup_{k=1}^n (\mathcal{P}_2)\text{int}((\mathcal{P}_1)\text{cl}U_{x_k}) \\ &\subset (\mathcal{P}_2)\text{int}\left(\bigcup_{k=1}^n (\mathcal{P}_1)\text{cl}U_{x_k}\right) \\ &= (\mathcal{P}_2)\text{int}\left((\mathcal{P}_1)\text{cl}\left(\bigcup_{k=1}^n U_{x_k}\right)\right). \end{aligned}$$

We write  $U_y = (\mathcal{P}_2)\text{int}((\mathcal{P}_1)\text{cl}(\bigcup_{k=1}^n U_{x_k}))$  and  $V_y = \bigcap_{k=1}^n V_{x_k}$ . Then  $U_y$  is  $(\mathcal{P}_2, \mathcal{P}_1)$  regularly open with  $A \subset U_y$  and  $V_y$  is  $(\mathcal{P}_1)$  open with  $y \in V_y$ . Now we show  $U_y \cap V_y = \emptyset$ . If possible, let  $z \in U_y \cap V_y$ . We note,  $z \in U_y = (\mathcal{P}_2)\text{int}((\mathcal{P}_1)\text{cl}(\bigcup_{k=1}^n U_{x_k})) \subset (\mathcal{P}_1)\text{cl}(\bigcup_{k=1}^n U_{x_k}) = \bigcup_{k=1}^n (\mathcal{P}_1)\text{cl}U_{x_k}$ . Hence for some  $l \in \{1, 2, \dots, n\}$ ,  $z \in (\mathcal{P}_1)\text{cl}U_{x_l}$ . Since  $z \in V_{x_k}$  for all  $k = 1, 2, \dots, n$  and  $U_{x_l} \cap V_{x_l} = \emptyset$ , we have  $z \notin U_{x_l}$ . Also  $z \in V_{x_l}$  and  $V_{x_l}$  is  $(\mathcal{P}_1)$  open

with  $U_{x_l} \cap V_{x_l} = \emptyset$ . Thus  $z$  cannot be a  $(\mathcal{P}_1)$ limit point of  $U_{x_l}$  which contradicts to the fact,  $z \in (\mathcal{P}_1)\text{cl}U_{x_l}$ .

Now we vary  $y \in B$  over  $B$ . For each  $y \in B$ , we obtain a  $(\mathcal{P}_2, \mathcal{P}_1)$ regularly open set  $U_y$  and a  $(\mathcal{P}_1)$ open set  $V_y$  such that  $A \subset U_y, y \in V_y$  and  $U_y \cap V_y = \emptyset$ . So using the steps describe above, we obtain a finite family  $\{V_{y_k} \mid k = 1, 2, \dots, m\}$  of  $(\mathcal{P}_1)$ open sets with

$$\begin{aligned} B &\subset \bigcup_{k=1}^m (\mathcal{P}_1)\text{int}((\mathcal{P}_2)\text{cl}V_{y_k}) \\ &\subset (\mathcal{P}_1)\text{int}\left((\mathcal{P}_2)\text{cl}\left(\bigcup_{k=1}^m V_{y_k}\right)\right). \end{aligned}$$

We put  $U = \bigcap_{k=1}^m U_{y_k}$  and  $V = (\mathcal{P}_1)\text{int}((\mathcal{P}_2)\text{cl}(\bigcup_{k=1}^m V_{y_k}))$ . Thus  $U$  is  $(\mathcal{P}_2)$ open and  $V$  is  $(\mathcal{P}_1, \mathcal{P}_2)$ regularly open with  $A \subset U, B \subset V$ . To show  $U \cap V = \emptyset$ , let  $z \in U \cap V$ . Then  $z \in U_{y_k}$  for each  $k = 1, 2, \dots, m$ . Also  $z \in V = (\mathcal{P}_1)\text{int}((\mathcal{P}_2)\text{cl}(\bigcup_{k=1}^m V_{y_k})) \subset (\mathcal{P}_2)\text{cl}(\bigcup_{k=1}^m V_{y_k}) = \bigcup_{k=1}^m (\mathcal{P}_2)\text{cl}V_{y_k}$ . Thus  $z \in (\mathcal{P}_2)\text{cl}V_{y_l}$  for some  $l \in \{1, 2, \dots, m\}$ . But  $U_{y_l} \cap V_{y_l} = \emptyset$  which implies  $z \notin V_{y_l}$ . Also  $z \in U_{y_l}$  is  $(\mathcal{P}_2)$ open and  $U_{y_l} \cap V_{y_l} = \emptyset$ . So  $z$  is not a  $(\mathcal{P}_2)$ limit point of  $V_{y_l}$ . Thus  $z \notin (\mathcal{P}_1)\text{cl}V_{y_l}$ , a contradiction. Hence we conclude  $U \cap V = \emptyset$ . □

#### 6.4. Concluding Remarks

The results of bitopological spaces are mainly the generalizations of the results of topological spaces. Several authors contributed many works to bitopological spaces by generalizing

many results of topological spaces. At the same time, due to absence of a suitable definition of paracompactness in a bitopological setting, the generalizations of some notions like near paracompactness of topological spaces in bitopological settings are clogged. Since we have now the definition of pairwise paracompactness, it may be generalized to get concepts like near pairwise paracompactness and almost pairwise paracompactness.

In this thesis, we have so far developed and obtained some characterizations of the notions of pairwise paracompactness, countable pairwise paracompactness and near pairwise compactness of a bitopological space. We hope these notions can be further developed. There also remains a lot of similar notions which may be undertaken in future.

## CHAPTER VII

### A Generalization of Bitopological Spaces

#### 7.1. Introduction

The purpose of this chapter is to study a space equipped with countable number of topologies. The notion of bitopological spaces was introduced by Kelly [26] and the notion of  $(\omega)$ topological spaces was introduced by Bose and Tiwari [6]. Generalizing both of the above notions, we introduce here the notion of  $(\aleph_0)$ topological spaces (Definition 7.1). In the  $(\aleph_0)$ topological spaces, we define  $(\aleph_0)$ Hausdorffness,  $(\aleph_0)$ regularity,  $(\aleph_0)$ normality and  $(\aleph_0)$ compactness. We also define complete  $(\aleph_0)$ regularity and complete  $(\aleph_0)$ normality. We prove some results on these notions.

In the  $(\aleph_0)$ topological spaces, we introduce  $(\aleph_0)$ paracompactness. A space which is  $(\aleph_0)$ Hausdorff and  $(\aleph_0)$ paracompact is  $(\aleph_0)$ regular (Theorem 7.14). Raghavan and Reilly [38] introduced  $\alpha$ -,  $\beta$ -,  $\gamma$ - and  $\delta$ -pairwise paracompactness. We introduce here  $(\beta\text{-}\aleph_0)$ paracompactness and prove the Michael's theorem for  $(\beta\text{-}\aleph_0)$ paracompactness of the space  $X$  when it is  $(\aleph_0)$ regular (Theorem 7.15). From this, we obtain an analogue of Michael's theorem for  $\beta$ -pairwise paracompactness of a pairwise regular bitopological space, as a particular case.

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The subject matter of this chapter forms our paper: *Novi Sad J. Math.* **40** (2) (2010), 7–16, [1].

A set  $X$  equipped with an increasing sequence  $\{\mathcal{T}_n\}$  of topologies is called an  $(\omega)$ topological space. If  $\mathcal{T}_n = \mathcal{T}_{n'}$  for all  $n, n'$ , then the  $(\omega)$ topological space  $X$  becomes a topological space. Thus the notion of  $(\omega)$ topological spaces generalizes the notion of topological spaces. But it does not generalize the notion of bitopological spaces.

## 7.2. Definitions

Let  $\{\mathcal{P}_n\}$  be a sequence of topologies on a set  $X$ . The sequence  $\{\mathcal{P}_n\}$  is said to satisfy the condition  $(*)$  (resp. condition  $(a)$ ) if for any positive integer  $m$ , the union (resp. intersection) of a finite number of sets belonging to  $\bigcup_{n \neq m} \mathcal{P}_n$  is a set belonging to  $\bigcup_{n \neq m} \mathcal{P}_n$ .

We introduce the following definitions.

**DEFINITION 7.1.** If  $\{\mathcal{P}_n\}$  is a sequence of topologies on a set  $X$  satisfying the condition  $(*)$ , then the pair  $(X, \{\mathcal{P}_n\})$  is called an  $(\aleph_0)$ topological space. A set  $G \in \bigcup_n \mathcal{P}_n$  is called an  $(\aleph_0)$ open set.

Unless otherwise mentioned,  $X$  denotes the  $(\aleph_0)$ topological space  $(X, \{\mathcal{P}_n\})$ . Elements of  $X$  are denoted by  $x, y$  etc. All the sets considered here are subsets of  $X$ . The members of  $N$  are generally denoted by  $k, l, m, n$  etc.

**EXAMPLE 7.1.** We enumerate all the rational numbers:  $r_1, r_2, r_3, \dots$ . Let  $\mathcal{Q}_n$  denote the particular point topology (Steen

and Seebach, Jr. [44]) on  $R$ , where each nonempty set belonging to  $\mathcal{Q}_n$  contains  $r_n$ .

Then  $(R, \{\mathcal{Q}_n\})$  is an  $(\aleph_0)$ topological space, where the sequence  $\{\mathcal{Q}_n\}$  does not satisfy the condition (a). In the  $(\aleph_0)$ topological space considered in Example 7.2,  $\{\mathcal{P}_n\}$  satisfies the condition (a).

DEFINITION 7.2. Suppose  $Y \subset X$  and  $\mathcal{P}_n | Y$  denotes the subspace topology on  $Y$  induced by  $\mathcal{P}_n$ . Then  $(Y, \{\mathcal{P}_n | Y\})$  is called a *subspace* of  $(X, \{\mathcal{P}_n\})$ .

DEFINITION 7.3. A function  $f : X \rightarrow R$  is said to be  $(\bigcup_{n \in M} \mathcal{P}_n)$  *upper semi-continuous* (resp.  $(\bigcup_{n \in M} \mathcal{P}_n)$  *lower semi-continuous*) if for every  $a \in R$ ,  $f^{-1}((-\infty, a)) \in \bigcup_{n \in M} \mathcal{P}_n$  (resp.  $f^{-1}((a, \infty)) \in \bigcup_{n \in M} \mathcal{P}_n$ ) where  $M \subset N$ .

DEFINITION 7.4.  $X$  is said to be an  $(\aleph_0)$ *compact space* if every  $(\aleph_0)$ open cover  $\mathcal{C}$  of  $X$  with  $\mathcal{C} \cap \mathcal{P}_n \neq \emptyset$  for at least two values of  $n$  has a finite subcover. Let  $M \subset N$ .  $X$  is said to be  $(M)$ *compact* if every cover of  $X$  consisting of sets belonging to  $\bigcup_{n \in M} \mathcal{P}_n$  has a finite subcover.

DEFINITION 7.5. An  $(\aleph_0)$ open cover  $\{U_\alpha | U_\alpha \in \mathcal{P}_{n_\alpha}\}$  of  $X$  is said to be  $(\aleph_0)$ *shrinkable* if there is another  $(\aleph_0)$ open cover  $\{V_\alpha | V_\alpha \in \mathcal{P}_{n_\alpha}\}$  of  $X$  with  $(\mathcal{P}_n)\text{cl}V_\alpha \subset U_\alpha$  for some  $n \neq n_\alpha$ .

DEFINITION 7.6. A set  $D$  is called a  $(\mathcal{P}_n)$ *neighbourhood*, or in short a  $(\mathcal{P}_n)$ *nbhd*, of a set  $A$  if  $A \subset (\mathcal{P}_n)\text{int}D$ .

DEFINITION 7.7.  $X$  is said to be  $(m)$ Hausdorff if for any pair of distinct points  $x, y \in X$ , there exist  $U \in \mathcal{P}_m$  and  $V \in \bigcup_{n \neq m} \mathcal{P}_n$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ . If  $X$  is  $(m)$ Hausdorff for all  $m \in N$ , then  $X$  is said to be an  $(\aleph_0)$ Hausdorff space.

The  $(\aleph_0)$ topological space  $(R, \{\mathcal{Q}_n\})$  considered in Example 7.1, is not  $(n)$ Hausdorff for any  $n$ . Also for any  $n$ , the topological space  $(R, \mathcal{Q}_n)$  is not Hausdorff and for any pair of integers  $m, n$ , the bitopological space  $(R, \mathcal{Q}_n, \mathcal{Q}_m)$  is not pairwise Hausdorff.

EXAMPLE 7.2. Let the sequence  $\{\mathcal{P}_n\}$  of topologies on  $R$  be defined by

$$\mathcal{P}_1 = \mathcal{U},$$

$$\mathcal{P}_n = \{\emptyset\} \cup \{G \cup (n, \infty) \mid G \in \mathcal{U}\} \text{ for all } n > 1,$$

where  $\mathcal{U}$  is the usual topology on  $R$ .

Then the  $(\aleph_0)$ topological space  $(R, \{\mathcal{P}_n\})$  is  $(1)$ Hausdorff but it is not  $(n)$ Hausdorff for any  $n \neq 1$ , and so it is not  $(\aleph_0)$ Hausdorff. For, let  $x, y \in R$  with  $x \neq y$ . Without loss of generality, we suppose that  $x < y$ . Now we choose  $a, b, c \in R$  such that  $a < x < b < y < c$ . Then  $(a, b), (b, c) \in \mathcal{P}_1$  with  $x \in (a, b)$  and  $y \in (b, c)$ . We choose  $n \in N$  such that  $n > c$ . Hence  $y \in (b, c) \cup (n, \infty) \in \mathcal{P}_n$  and  $x \in (a, b) \cup (n, \infty) \in \mathcal{P}_n$ . Thus  $(R, \{\mathcal{P}_n\})$  is  $(1)$ Hausdorff.

Now we choose  $x, y \in R$  such that  $x, y > m \in N$  and  $x \neq y$ . Hence for any  $(\mathcal{P}_m)$ open set  $U_m$ , we have  $x, y \in U_m$ . Thus

it is not possible to obtain a  $U \in \mathcal{P}_m, V \in \bigcup_{n \neq m} \mathcal{P}_n$  such that  $x \in U, y \in V$  with  $U \cap V = \emptyset$ . Thus  $(R, \{\mathcal{P}_n\})$  is not  $(n)$ Hausdorff for any  $n \neq 1$  and so it is not  $(\aleph_0)$ Hausdorff.

DEFINITION 7.8.  $X$  is said to be  $(m)$ regular if for any point  $x \in X$  and any  $(\mathcal{P}_n)$ closed set  $A$  with  $n \neq m$  and  $x \notin A$ , there exist  $U \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $V \in \mathcal{P}_m$  such that  $x \in U, A \subset V, U \cap V = \emptyset$ .  $X$  is said to be  $(\aleph_0)$ regular if it is  $(m)$ regular for all  $m \in N$ .

It is easy to see that  $X$  is  $(m)$ regular iff for any point  $x \in X$  and any  $(\mathcal{P}_n)$ open set  $G$  containing  $x$  with  $n \neq m$ , there exists a  $U \in \bigcup_{n \neq m} \mathcal{P}_n$  containing  $x$  such that  $(\mathcal{P}_m)\text{cl}U \subset G$ .

DEFINITION 7.9.  $X$  is said to be *completely*  $(m)$ regular if for any point  $x \in X$  and any  $(\mathcal{P}_n)$ closed set  $A$  with  $n \neq m$  and  $x \notin A$ , there exists a function  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0, f(y) = 1$  for all  $y \in A$ ,  $f$  is  $(\bigcup_{n \neq m} \mathcal{P}_n)$ upper semi-continuous and  $(\mathcal{P}_m)$ lower semi-continuous.  $X$  is *completely*  $(\aleph_0)$ regular if it is completely  $(m)$ regular for all  $m \in N$ .

DEFINITION 7.10.  $X$  is said to be  $(m)$ normal if given a  $(\mathcal{P}_m)$ closed set  $A$  and a  $(\mathcal{P}_n)$ closed set  $B$  with  $n \neq m$  and  $A \cap B = \emptyset$ , there exist  $U \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $V \in \mathcal{P}_m$  such that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$ . If  $X$  is  $(m)$ normal for all  $m \in N$ , then it is said to be  $(\aleph_0)$ normal.

From the definition, it follows that  $X$  is  $(m)$ normal iff for any  $(\mathcal{P}_m)$ closed set  $A$  and any  $(\mathcal{P}_n)$ open set  $W$  containing  $A$  with  $n \neq m$ , there exists a  $U \in \bigcup_{n \neq m} \mathcal{P}_n$  such that  $A \subset U \subset (\mathcal{P}_m)\text{cl}U \subset W$ .

DEFINITION 7.11.  $X$  is said to be *completely  $(m)$ normal* if for each pair  $A, B$  of subsets of  $X$  satisfying  $(A \cap (\mathcal{P}_n)\text{cl}B) \cup ((\mathcal{P}_m)\text{cl}A \cap B) = \emptyset, n \neq m$ , there exist  $U \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $V \in \mathcal{P}_m$  such that  $A \subset U, B \subset V$  and  $U \cap V = \emptyset$ .

*Complete  $(\aleph_0)$ normality* is defined in the obvious way.

The  $(\aleph_0)$ topological space  $(R, \{\mathcal{Q}_n\})$  considered in Example 7.1, is completely  $(\aleph_0)$ normal but for no  $n$ , the topological space  $(R, \mathcal{Q}_n)$  is normal. Also for no  $n$ ,  $(R, \{\mathcal{Q}_n\})$  is  $(n)$ regular.

EXAMPLE 7.3. Let  $X$  be an infinite set and  $a, b, c \in X$ . Suppose  $\mathcal{G}$  (resp.  $\mathcal{H}$ ) is a collection of subsets of  $X$ , which contains, in addition to  $\emptyset$  and  $X$ , all those subsets  $E$  of  $X$  for which  $a \notin E$  and  $c \in E$  (resp.  $a, b \notin E$  and  $c \in E$ ). Then  $\mathcal{G}$  and  $\mathcal{H}$  form topologies on  $X$ . Let  $\mathcal{P}_n = \mathcal{G}$  for odd  $n$ , and  $= \mathcal{H}$  for even  $n$ .

Then the  $(\aleph_0)$ topological space  $(X, \{\mathcal{P}_n\})$  is  $(\aleph_0)$ normal but not completely  $(\aleph_0)$ normal. In fact, it is completely  $(m)$ normal for no  $m$ .

DEFINITION 7.12. A collection  $\mathcal{C}$  of subsets of  $X$  is said to be  $(\aleph_0)$ locally finite if each  $x \in X$  has a  $(\mathcal{P}_n)$ open nbd, for at

least two values of  $n$ , intersecting at most finitely many  $U \in \mathcal{C}$ . For any  $m \in N$ ,  $\mathcal{C}$  is said to be  $(\bigcup_{n \neq m} \mathcal{P}_n)$  locally finite if for each  $x \in X$ , there exists a set  $U \in \bigcup_{n \neq m} \mathcal{P}_n$  containing  $x$  and intersecting at most finitely many  $U \in \mathcal{C}$ .

DEFINITION 7.13. A refinement  $\mathcal{V}$  of an  $(\aleph_0)$ open cover  $\mathcal{U}$  is said to be a *parallel refinement* if  $V \in \mathcal{V}$  is  $(\mathcal{P}_n)$ open whenever  $V \subset U \in \mathcal{U}$  and  $U$  is  $(\mathcal{P}_n)$ open.

DEFINITION 7.14.  $X$  is said to be  $(\aleph_0)$ paracompact if every  $(\aleph_0)$ open cover  $\mathcal{U}$  of  $X$  with  $\mathcal{U} \cap \mathcal{P}_n \neq \emptyset$  for at least two values of  $n$  has an  $(\aleph_0)$ locally finite parallel refinement.

It follows from the definitions that an  $(\aleph_0)$ compact space is  $(\aleph_0)$ paracompact. The converse is not true:

EXAMPLE 7.4. Let  $\mathcal{T}$  be the indiscrete topology on  $R$ , and  $\mathcal{U}_n$  be the subspace topology  $\mathcal{U} \upharpoonright I_n$  on  $I_n = (-n, n)$ , of the usual topology  $\mathcal{U}$  on  $R$ . If  $\mathcal{P}_n = \mathcal{T} \cup \mathcal{U}_n$ , then  $(R, \{\mathcal{P}_n\})$  is an  $(\aleph_0)$ topological space, which is  $(\aleph_0)$ paracompact but not  $(\aleph_0)$ compact.

Let  $\mathcal{P}$  denote the smallest topology containing all the topologies  $\mathcal{P}_n, n \in N$ . We call a cover  $\mathcal{C}$  of  $X$ , a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ cover if  $\mathcal{C} \subset \bigcup_{n \neq m} \mathcal{P}_n$ .

DEFINITION 7.15.  $X$  is said to be  $(\beta\text{-}\aleph_0)$ paracompact if for all  $m \in N$ , every  $(\bigcup_{n \neq m} \mathcal{P}_n)$ cover of  $X$  has a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite  $(\mathcal{P})$ open refinement.

The following example shows that a  $(\beta\text{-}\aleph_0)$ paracompact space may not be  $(\aleph_0)$ paracompact.

EXAMPLE 7.5. Suppose  $X = [0, \infty)$  and  $\mathcal{U}$  is the usual topology on  $X$ . Let  $\mathcal{P}_1 =$  discrete topology on  $X$ ,  $\mathcal{P}_2 = \mathcal{U}$  and for  $n > 2$ ,  $\mathcal{P}_n = \{\emptyset\} \cup \{G \cup (n, \infty) \mid G \in \mathcal{U}\}$ . Then the  $(\aleph_0)$ topological space  $(X, \{\mathcal{P}_n\})$  is  $(\beta\text{-}\aleph_0)$ paracompact but not  $(\aleph_0)$ paracompact.

It remains to see whether  $(\aleph_0)$ paracompactness of a space implies  $(\beta\text{-}\aleph_0)$ paracompactness of the space.

### 7.3. Results

THEOREM 7.1. *If  $X$  is  $(\aleph_0)$ compact, and  $K$  is  $(\mathcal{P}_m)$ closed for some  $m$ , then  $K$  is  $(\aleph_0)$ compact. If  $K$  is a proper subset of  $X$ , then  $K$  is  $(\mathcal{P}_n)$ compact for all  $n \neq m$ .*

PROOF. We prove only the second part. Choose  $n_0 \neq m$ . Let  $\mathcal{U}$  be a  $(\mathcal{P}_{n_0})$ open cover of  $K$ . Then  $\mathcal{U}_1 = \mathcal{U} \cup \{X - K\}$  is an  $(\aleph_0)$ open cover of  $X$  with  $\mathcal{U}_1 \cap \mathcal{P}_n \neq \emptyset$  for two values of  $n$ . Therefore  $\mathcal{U}_1$  has a finite subcover. Hence  $\mathcal{U}$  has a finite subcover.  $\square$

THEOREM 7.2. *If  $(X, \mathcal{P}_n)$  is a Hausdorff topological space for each  $n$ , and  $(X, \{\mathcal{P}_n\})$  is  $(\aleph_0)$ compact, then all  $\mathcal{P}_n$  are identical.*

PROOF. Consider a topology  $\mathcal{P}_m$  and let  $G$  be a nonempty set belonging to  $\mathcal{P}_m$ . Then  $A = X - G$  is  $(\mathcal{P}_m)$ closed and hence by Theorem 7.1,  $A$  is  $(\mathcal{P}_n)$ compact for all  $n \neq m$ . Since for all  $n$ ,  $(X, \mathcal{P}_n)$  are Hausdorff, it follows that for all  $n \neq m$ ,  $A$  is  $(\mathcal{P}_n)$ closed and so  $G$  is  $(\mathcal{P}_n)$ open. Therefore  $\mathcal{P}_m \subset \mathcal{P}_n$  for all  $n$ .  $\square$

THEOREM 7.3. *Let  $M = \{n \in N \mid n \neq m\}$ . If  $X$  is  $(m)$ Hausdorff and  $(M)$ compact, then  $\mathcal{P}_n \subset \mathcal{P}_m$  for all  $n$ .*

PROOF. Let  $n \neq m$ . If  $\mathcal{P}_n \not\subset \mathcal{P}_m$ , then there exists a set  $U \in \mathcal{P}_n$  such that  $U \notin \mathcal{P}_m$ . Then  $X - U$  is  $(\mathcal{P}_n)$ closed, and hence  $(M)$ compact. Since  $X - U$  is not  $(\mathcal{P}_m)$ closed, there is a point  $p \in U$  which is a  $(\mathcal{P}_m)$ limit point of  $X - U$ . Again since  $X$  is  $(m)$ Hausdorff, for each  $x \in X - U$ , there exist  $U_x \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $V_x \in \mathcal{P}_m$  such that  $x \in U_x, p \in V_x$  and  $U_x \cap V_x = \emptyset$ . Then  $\{U_x \mid x \in X - U\}$  is a cover of  $X - U$  by sets belonging to  $\bigcup_{n \in M} \mathcal{P}_n$ . Therefore it has a finite subcover  $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ . If  $V = \bigcap_{i=1}^n V_{x_i}$ , then  $V \in \mathcal{P}_m, p \in V$  and  $V \cap (X - U) = \emptyset$ , a contradiction, since  $p$  is a  $(\mathcal{P}_m)$ limit point of  $X - U$ .  $\square$

THEOREM 7.4.  *$X$  is  $(m)$ Hausdorff iff for all  $x$ ,*

$$\{x\} = \bigcap \{(\mathcal{P}_n)clU \mid n \neq m, U \in \mathcal{P}_m, x \in U\}.$$

PROOF. Let the space be  $(m)$ Hausdorff. Then for any  $y \in X$  with  $x \neq y$ , there exist a  $(\mathcal{P}_m)$ open set  $U$  and a  $(\mathcal{P}_n)$ open set  $V$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ . So  $y \notin (\mathcal{P}_n)clU$ .

Thus we have

$$\bigcap \{(\mathcal{P}_n)\text{cl}U \mid n \neq m, U \in \mathcal{P}_m, x \in U\} = \{x\}.$$

For the converse, we consider  $x, y \in X$  with  $x \neq y$ . There exists a  $U \in \mathcal{P}_m$  such that  $x \in U$  and  $y \notin (\mathcal{P}_n)\text{cl}U$ ,  $n \neq m$ . Then  $y \in V = X - (\mathcal{P}_n)\text{cl}U$  and  $U \cap V = \emptyset$ .  $\square$

**THEOREM 7.5.** *If  $X$  is  $(m)$ Hausdorff, then for each filterbase  $\mathcal{F}$  on  $X$  such that  $\mathcal{F}$  is  $(\mathcal{P}_m)$ convergent to  $x$  and  $(\mathcal{P}_n)$ convergent to  $y$  for some  $n \neq m$ ,  $x = y$ . The converse is also true if  $\{\mathcal{P}_n\}$  satisfies the condition (a).*

**PROOF.** Let  $X$  be  $(m)$ Hausdorff, and  $\mathcal{F}$  be a filterbase on  $X$ , which is  $(\mathcal{P}_m)$ convergent to  $x$ . Let  $y \neq x$ . Then there exist  $U \in \mathcal{P}_m$  and  $V \in \bigcup_{n \neq m} \mathcal{P}_n$  such that  $x \in U, y \in V, U \cap V = \emptyset$ . There is an  $A \in \mathcal{F}$  with  $A \subset U$ . If  $V \in \mathcal{P}_n, n \neq m$ , then  $\mathcal{F}$  cannot be  $(\mathcal{P}_n)$ convergent to  $y$ . For, if it is so, there must be a  $B \in \mathcal{F}$  with  $B \subset V$ , which is not possible, since any two elements of  $\mathcal{F}$  have nonempty intersection.

To prove the converse, suppose  $X$  is not  $(m)$ Hausdorff. Then there exists a pair  $x, y$  ( $x \neq y$ ) such that for any  $U \in \mathcal{P}_m$  and  $V \in \bigcup_{n \neq m} \mathcal{P}_n$  with  $x \in U, y \in V$ , we have  $U \cap V \neq \emptyset$ . So the class  $\mathcal{C} = \{U \cap V \mid U \in \mathcal{P}_m, V \in \bigcup_{n \neq m} \mathcal{P}_n, x \in U, y \in V\}$  is a filterbase on  $X$ , which is  $(\mathcal{P}_m)$ convergent to  $x$  and  $(\mathcal{P}_n)$ convergent to  $y$  for all  $n \neq m$ .  $\square$

If  $X$  is  $(\aleph_0)$ Hausdorff, then Theorem 7.5 is true for any pair of elements  $m, n \in N$ .

Likewise, each of the following theorems with  $(m)$  has a corresponding result with  $(\aleph_0)$ .

**THEOREM 7.6.** *If  $X$  is  $(m)$ Hausdorff and  $M = \{n \mid n \neq m\}$ , then  $(M)$ compact subsets are  $(\mathcal{P}_m)$ closed.*

**PROOF.** Let  $K$  be an  $(M)$ compact subset of  $X$  and  $x \in X - K$ . If  $y \in K$ , then there exist sets  $U_{xy} \in \mathcal{P}_m$  and  $V_{xy} \in \bigcup_{n \neq m} \mathcal{P}_n$  such that  $x \in U_{xy}, y \in V_{xy}$  and  $U_{xy} \cap V_{xy} = \emptyset$ . Then  $\mathcal{V} = \{V_{xy} \mid y \in K\}$  forms a cover of  $K$  by sets belonging to  $\bigcup_{n \in M} \mathcal{P}_n$ . Therefore  $\mathcal{V}$  has a finite subcover  $V_{xy_1}, V_{xy_2}, \dots, V_{xy_k}$ . If  $V_x = \bigcup_{i=1}^k V_{xy_i}$  and  $U_x = \bigcap_{i=1}^k U_{xy_i}$ , then  $x \in U_x \in \mathcal{P}_m$ ,  $K \subset V_x \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $U_x \cap V_x = \emptyset$ . Now

$$\begin{aligned} X - K &\subset \bigcup \{U_x \mid x \in X - K\} \\ &\subset \bigcup \{X - V_x \mid x \in X - K\} \\ &\subset X - K \text{ since } K \subset V_x \\ \Rightarrow X - K &= \bigcup \{U_x \mid x \in X - K\}. \end{aligned}$$

Thus  $K$  is  $(\mathcal{P}_m)$ closed. □

**THEOREM 7.7.**  *$X$  is  $(m)$ regular if and only if for  $n_0 \neq m$ , any  $(\mathcal{P}_{n_0})$ closed set  $A$  is the intersection of all  $(\mathcal{P}_n)$ closed  $(\mathcal{P}_m)$ nbds of  $A$ ,  $n \neq m$ .*

**PROOF.** Let  $X$  be  $(m)$ regular. Suppose  $x$  belongs to the complement  $X - A$  of the  $(\mathcal{P}_{n_0})$ closed set  $A$  with  $n_0 \neq m$ .

Since  $X - A$  is  $(\mathcal{P}_{n_0})$ open,  $n_0 \neq m$ , there exists a set  $G \in \bigcup_{n \neq m} \mathcal{P}_n$  such that  $x \in G \subset (\mathcal{P}_m)\text{cl}G \subset X - A$ . Then  $A \subset X - (\mathcal{P}_m)\text{cl}G \subset X - G$ . Thus  $X - G$  is a  $(\mathcal{P}_n)$ closed  $(\mathcal{P}_m)$ nbd of  $A$  with  $n \neq m$  and  $x \notin X - G$ . Hence the intersection of all  $(\mathcal{P}_n)$ closed  $(\mathcal{P}_m)$ nbd's of  $A$  with  $n \neq m$ , is the set  $A$ .

To prove the converse, suppose for  $n_0 \neq m$ ,  $A$  is a  $(\mathcal{P}_{n_0})$ closed set not containing the point  $x$ . Then there exists a  $(\mathcal{P}_n)$ closed  $(\mathcal{P}_m)$ nbd  $F$  of  $A$  with  $n \neq m$  and  $x \notin F$ . If  $U = X - F$  and  $V = (\mathcal{P}_m)\text{int}F$ , then  $U \in \bigcup_{n \neq m} \mathcal{P}_n, V \in \mathcal{P}_m, x \in U, A \subset V$  and  $U \cap V = \emptyset$ .  $\square$

**THEOREM 7.8.** *If  $X$  is  $(\aleph_0)$ compact and  $(m)$ Hausdorff, and if  $\{\mathcal{P}_n\}$  satisfies the condition (a), then  $X$  is  $(m)$ regular.*

**PROOF.** Let  $x \in X$  and  $A$  be a  $(\mathcal{P}_n)$ closed set with  $n \neq m$  and  $x \notin A$ . If  $y \in A$ , then by  $(m)$ Hausdorffness of  $X$ , there exist  $U_y \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $V_y \in \mathcal{P}_m$  such that  $x \in U_y, y \in V_y$  and  $U_y \cap V_y = \emptyset$ . Then  $\mathcal{V} = \{V_y \mid y \in A\}$  forms a  $(\mathcal{P}_m)$ open cover of  $A$ . Since  $A$  is  $(\mathcal{P}_n)$ closed, by Theorem 7.1, it is  $(\mathcal{P}_m)$ compact, and hence  $\mathcal{V}$  has a finite subcover  $V_{y_1}, \dots, V_{y_k}$ . If  $U = \bigcap_{i=1}^k U_{y_i}$  and  $V = \bigcup_{i=1}^k V_{y_i}$ , then  $U \in \bigcup_{n \neq m} \mathcal{P}_n, V \in \mathcal{P}_m, x \in U, A \subset V$  and  $U \cap V = \emptyset$ .  $\square$

**THEOREM 7.9.** *Every completely  $(m)$ regular space is  $(m)$ regular.*

**PROOF.** Let  $X$  be a completely  $(m)$ regular space. Also let  $x \in X$  and  $A$  be a  $(\mathcal{P}_n)$ closed set with  $n \neq m$  and  $x \notin A$ .

Then by complete  $(m)$ regularity of  $X$ , there exists a function  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$ ,  $f(y) = 1$  for all  $y \in A$ . Also  $f$  is  $(\bigcup_{n \neq m} \mathcal{P}_n)$  upper semicontinuous and  $(\mathcal{P}_m)$  lower semicontinuous. Hence  $U = f^{-1}((-\infty, \frac{1}{2})) \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $V = f^{-1}((\frac{1}{2}, \infty)) \in \mathcal{P}_m$ . Then we have  $x \in U$ ,  $A \subset V$  and  $U \cap V = \emptyset$ . Hence  $X$  is  $(m)$ regular.  $\square$

**THEOREM 7.10.** *If  $X$  is  $(\aleph_0)$ compact and  $(m)$ regular, then  $X$  is  $(m)$ normal.*

**PROOF.** Suppose  $A$  is a  $(\mathcal{P}_m)$ closed set and  $B$  is a  $(\mathcal{P}_n)$ closed set with  $n \neq m$  and  $A \cap B = \emptyset$ . For  $x \in A$ , there is a  $U_x \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $V_x \in \mathcal{P}_m$  such that  $x \in U_x$ ,  $B \subset V_x$ ,  $U_x \cap V_x = \emptyset$ . Since  $\mathcal{U} = \{U_x \mid x \in A\}$  is a cover of  $A$  with sets belonging to  $\bigcup_{n \neq m} \mathcal{P}_n$ , and  $X$  is  $(\aleph_0)$ compact, it follows that there is a finite subcollection  $U_{x_1}, U_{x_2}, \dots, U_{x_k}$  of  $\mathcal{U}$  covering  $A$ . If  $U = \bigcup_{i=1}^k U_{x_i}$  and  $V = \bigcap_{i=1}^k V_{x_i}$ , then  $U \in \bigcup_{n \neq m} \mathcal{P}_n$ ,  $V \in \mathcal{P}_m$ ,  $U \cap V = \emptyset$ ,  $A \subset U$ ,  $B \subset V$ .  $\square$

**COROLLARY 7.1.** *If  $X$  is  $(\aleph_0)$ compact and  $(m)$ Hausdorff, and if  $\{\mathcal{P}_n\}$  satisfies the condition (a), then  $X$  is  $(m)$ normal.*

**PROOF.** Given  $X$  is  $(\aleph_0)$ compact and  $(m)$ Hausdorff. Then by Theorem 7.8,  $X$  is  $(m)$ regular and so the result follows from Theorem 7.10.  $\square$

From the definitions, it is clear that completely  $(m)$ normal space is  $(m)$ normal. Also it can be shown that complete  $(m)$ nor-

mality is a hereditary property but  $(m)$ normality is not hereditary.  $(m)$ Hausdorff property,  $(m)$ regularity and complete  $(m)$ regularity are hereditary.

**THEOREM 7.11.**  *$X$  is completely  $(m)$ normal iff every subspace of  $X$  is  $(m)$ normal.*

**PROOF.** Since complete  $(m)$ normality is hereditary, and complete  $(m)$ normality implies  $(m)$ normality, the necessary part follows.

Conversely, suppose every subspace of  $X$  is  $(m)$ normal. Let  $A$  and  $B$  be two subsets of  $X$  with

$$(A \cap (\mathcal{P}_n)\text{cl}B) \cup ((\mathcal{P}_m)\text{cl}A \cap B) = \emptyset, n \neq m. \quad (7.1)$$

Let

$$S = (X - (\mathcal{P}_m)\text{cl}A) \cup (X - (\mathcal{P}_n)\text{cl}B). \quad (7.2)$$

Since  $S \cap (\mathcal{P}_m)\text{cl}A$  is  $(\mathcal{P}_m | S)$ closed,  $S \cap (\mathcal{P}_n)\text{cl}B$  is  $(\mathcal{P}_n | S)$ closed and

$$(S \cap (\mathcal{P}_m)\text{cl}A) \cap (S \cap (\mathcal{P}_n)\text{cl}B) = \emptyset$$

and since by hypothesis,  $(S, \{\mathcal{P}_n | S\})$  is  $(m)$ normal, there exist  $G \in \bigcup_{n \neq m} \mathcal{P}_n | S$  and  $H \in \mathcal{P}_m | S$  such that

$$S \cap (\mathcal{P}_m)\text{cl}A \subset G, \quad (7.3)$$

$$S \cap (\mathcal{P}_n)\text{cl}B \subset H, \quad (7.4)$$

$$G \cap H = \emptyset. \quad (7.5)$$

Now

$$\begin{aligned} S \cap (\mathcal{P}_m)\text{cl}A &= (X - (\mathcal{P}_n)\text{cl}B) \cap (\mathcal{P}_m)\text{cl}A \\ &\supset A \cap (\mathcal{P}_m)\text{cl}A = A \text{ (by (7.1)).} \end{aligned}$$

Similarly,

$$S \cap (\mathcal{P}_n)\text{cl}B \supset B.$$

From (7.4) we get

$$\begin{aligned} &(S \cap (\mathcal{P}_n)\text{cl}B) \cap (X - (\mathcal{P}_m)\text{cl}A) \\ &\subset H \cap (X - (\mathcal{P}_m)\text{cl}A) = V \text{ (say)} \\ \Rightarrow &S \cap (\mathcal{P}_n)\text{cl}B \subset V \Rightarrow B \subset V. \end{aligned}$$

Since  $H \in \mathcal{P}_m \mid S$  and  $X - (\mathcal{P}_m)\text{cl}A \in \mathcal{P}_m$ , from (7.2) it follows that  $V \in \mathcal{P}_m$ . Again since  $G \in \bigcup_{n \neq m} \mathcal{P}_n \mid S$ , there exists  $U \in \bigcup_{n \neq m} \mathcal{P}_n$  such that  $U \cap S = G$ . Then

$$\begin{aligned} U \cap V &\subset U \cap H = (U \cap S) \cap H \text{ (since } H \subset S) \\ &= G \cap H = \emptyset \text{ (by (7.5)).} \end{aligned}$$

Also from (7.3) and (7.4), we have  $A \subset U$  and  $B \subset V$ .  $\square$

**THEOREM 7.12.** *Every finite  $(\aleph_0)$ open cover  $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$  of an  $(\aleph_0)$ normal space  $X$  with  $U_i \in \mathcal{P}_{n_i}$  and  $n_i \neq n_{i'}$  if  $i \neq i'$ , is  $(\aleph_0)$ shrinkable.*

**PROOF.** Let  $X$  be  $(\aleph_0)$ normal. We choose a finite  $(\aleph_0)$ open cover  $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$  of  $X$ , satisfying the above mentioned conditions. Then  $(X - U_1) \cap (X - U_2) \cap \dots \cap (X - U_k) = \emptyset$ , and so  $(X - U_1)$  and  $(X - U_2) \cap \dots \cap (X - U_k)$  are disjoint. Again

$X - U_1$  is  $(\mathcal{P}_{n_1})$ closed and  $(X - U_2) \cap \dots \cap (X - U_k)$  is  $(\mathcal{P}_{n_0})$ closed for some  $n_0 \neq n_1$ . So there exist  $V_1 \in \mathcal{P}_n$  for some  $n \neq n_1$  and  $W_1 \in \mathcal{P}_{n_1}$  such that  $X - U_1 \subset V_1$ ,  $(X - U_2) \cap \dots \cap (X - U_k) \subset W_1$  and  $V_1 \cap W_1 = \emptyset$ . Then  $(\mathcal{P}_n)\text{cl}W_1 \subset X - V_1 \subset U_1$ ,  $n \neq n_1$ . Therefore it follows that  $\{W_1, U_2, \dots, U_k\}$  forms an  $(\aleph_0)$ open cover of  $X$  with  $(\mathcal{P}_n)\text{cl}W_1 \subset U_1$ ,  $n \neq n_1$ . If we apply the process  $k$  times, we obtain an  $(\aleph_0)$ open cover  $\{W_1, W_2, \dots, W_k\}$  such that  $W_i \in \mathcal{P}_{n_i}$  and  $(\mathcal{P}_n)\text{cl}W_i \subset U_i$ ,  $n \neq n_i$ .  $\square$

**THEOREM 7.13.** *If  $X$  is  $(\aleph_0)$ paracompact, and if  $K$  is a  $(\mathcal{P}_m)$ closed subset of  $X$  for some  $m$ , then  $K$  is  $(\aleph_0)$ paracompact.*

**PROOF.** Let  $\mathcal{U}$  be an  $(\aleph_0)$ open cover of  $K$  with  $\mathcal{U} \cap \mathcal{P}_n \neq \emptyset$  for atleast two values of  $n$ . Hence  $\mathcal{V} = \mathcal{U} \cup \{X - K\}$  is also an  $(\aleph_0)$ open cover of  $X$  with  $\mathcal{U} \cap \mathcal{P}_n \neq \emptyset$  for atleast two values of  $n$ . Then by the definition of  $(\aleph_0)$ paracompactness of  $X$ ,  $\mathcal{V}$  has an  $(\aleph_0)$ locally finite parallel refinement and hence  $\mathcal{U}$  also has an  $(\aleph_0)$ locally finite parallel refinement. Hence  $K$  is  $(\aleph_0)$ paracompact.  $\square$

**THEOREM 7.14.** *If  $X$  is  $(m)$ Hausdorff and  $(\aleph_0)$ paracompact, and if  $\{\mathcal{P}_n\}$  satisfies the condition (a), then  $X$  is  $(m)$ regular.*

**PROOF.** We choose any  $m \in N$ . Let  $x \in X$  and  $F$  be a  $(\mathcal{P}_n)$ closed set with  $n \neq m$  and  $x \notin F$ . Since  $X$  is  $(m)$ Hausdorff, for each  $y \in F$  there exist  $U_y \in \bigcup_{n \neq m} \mathcal{P}_n$  and  $V_y \in \mathcal{P}_m$  such that  $x \in U_y$ ,  $y \in V_y$  and  $U_y \cap V_y = \emptyset$ . The collection

$\mathcal{V} = \{V_y \mid y \in F\} \cup \{X - F\}$  is an  $(\aleph_0)$ open cover of  $X$  with  $\mathcal{V} \cap \mathcal{P}_n$  for atleast two values of  $n$ . Therefore there exists an  $(\aleph_0)$ locally finite parallel refinement  $\mathcal{G}$  of  $\mathcal{V}$ . We write  $V = \bigcup\{G \in \mathcal{G} \mid G \cap F \neq \emptyset\}$ . Then  $V \in \mathcal{P}_m$ . We choose a  $(\mathcal{P}_n)$ open nbd  $H$  of  $x$  with  $n \neq m$ , meeting only a finite number of sets  $G_1, G_2, \dots, G_k$  of  $\mathcal{G}$ . Let  $G_i \subset V_{y_i}$ ,  $y_i \in F, i = 1, 2, \dots, k$ . Then  $U = H \cap (\bigcap_{i=1}^k U_{y_i}) \in \bigcup_{n \neq m} \mathcal{P}_n, x \in U, F \subset V$  and  $U \cap V = \emptyset$ . Therefore  $X$  is  $(m)$ regular.  $\square$

**THEOREM 7.15.** *Let  $X$  be  $(\aleph_0)$ regular, and let  $\{\mathcal{P}_n\}$  satisfy the condition (a). Then  $X$  is  $(\beta\text{-}\aleph_0)$ paracompact iff for all  $m \in N$ , every  $(\bigcup_{n \neq m} \mathcal{P}_n)$ cover of  $X$  has a  $(\mathcal{P})$ open refinement  $\mathcal{V} = \bigcup_{k=1}^{\infty} \mathcal{V}_k$ , where each  $\mathcal{V}_k$  is  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite.*

**PROOF.** Since the ‘only if’ part of the theorem is obviously true, we prove the ‘if’ part. It is done in three steps.

**Step I.** For  $m \in N$ , let  $\mathcal{G}$  be a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ cover of  $X$ . Then it has a  $(\mathcal{P})$ open refinement  $\mathcal{V} = \bigcup_{k=1}^{\infty} \mathcal{V}_k$ , where each  $\mathcal{V}_k$  is  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite. Suppose  $\mathcal{V}_k = \{V_{k\alpha} \mid \alpha \in A\}$ . Let  $V_{k\alpha} \subset G_{k\alpha} \in \mathcal{G}$ . We write  $W_k = \bigcup_{\alpha} G_{k\alpha}$ ,  $W^i = \bigcup_{k=1}^i W_k$ ,  $A_1 = W_1$ , and for  $i > 1, A_i = W_i - W^{i-1}$ . Then  $\{W_k \mid k \in N\}$  is a cover of  $X$ , and so  $\{A_i \mid i \in N\}$  is a cover of  $X$ . This cover is  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite. In fact, if  $k(x)$  is the first  $k$  for which  $x \in W_{k(x)}$ , then  $x \in G_{k(x)\alpha}$  for some  $\alpha$ ,  $G_{k(x)\alpha} \in \bigcup_{n \neq m} \mathcal{P}_n$ , and  $G_{k(x)\alpha}$  does not intersect any  $A_i$  for  $i > k(x)$ .

Thus  $\mathcal{E}(\mathcal{G}) = \{E_{k\alpha} \mid k \in N, \alpha \in A\}$ , where  $E_{k\alpha} = A_k \cap V_{k\alpha}$ , is a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite refinement of  $\mathcal{G}$ .

Step II. For  $m \in N$ , let  $\mathcal{H}$  be a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ cover of  $X$ . For  $x \in X$ , we choose an  $H_x \in \mathcal{H}$  such that  $x \in H_x$ . Since  $X$  is  $(\aleph_0)$ regular, and since  $H_x$  is  $(\mathcal{P}_n)$ open with  $n \neq m$ , there exists a set  $H_x^1 \in \bigcup_{n \neq m} \mathcal{P}_n$  such that  $x \in H_x^1 \subset (\mathcal{P}_m)\text{cl}H_x^1 \subset H_x$ . Since  $\mathcal{P}_m \subset \mathcal{P}$ , we have

$$(\mathcal{P})\text{cl}H_x^1 \subset (\mathcal{P}_m)\text{cl}H_x^1 \subset H_x. \quad (7.6)$$

Now  $\mathcal{H}^1 = \{H_x^1 \mid x \in X\}$  is also a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ cover of  $X$ . Hence by Step I, we get a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite refinement  $\mathcal{E}(\mathcal{H}^1)$  of  $\mathcal{H}^1$ . For  $E \in \mathcal{E}(\mathcal{H}^1)$ , we get an  $H_x^1 \in \mathcal{H}^1$  with  $E \subset H_x^1$  and hence by (7.6),  $(\mathcal{P})\text{cl}E \subset H_x$ . Thus  $\mathcal{F}(\mathcal{H}) = \{(\mathcal{P})\text{cl}E \mid E \in \mathcal{E}(\mathcal{H}^1)\}$  is a  $(\mathcal{P})$ closed refinement of  $\mathcal{H}$ . Also  $\mathcal{F}(\mathcal{H})$  is  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite: since  $\mathcal{E}(\mathcal{H}^1)$  is  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite, there exists, for  $x \in X$ , a  $G \in \mathcal{P}_n$ ,  $n \neq m$  such that  $x \in G$ , and for all but finitely many  $E \in \mathcal{E}(\mathcal{H}^1)$ ,  $G \cap E = \emptyset \Rightarrow G \cap (\mathcal{P}_n)\text{cl}E = \emptyset \Rightarrow G \cap (\mathcal{P})\text{cl}E = \emptyset$ .

Step III. We choose an  $m \in N$ . Let  $\mathcal{U}$  be a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ cover of  $X$ . By Step I,  $\mathcal{U}$  has a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite refinement  $\mathcal{E}(\mathcal{U})$ . For  $x \in X$ , suppose  $W_x \in \bigcup_{n \neq m} \mathcal{P}_n$  is a set containing  $x$  and intersecting a finite number of members of  $\mathcal{E}(\mathcal{U})$ . Then  $\mathcal{W} = \{W_x \mid x \in X\}$  forms a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ cover of  $X$ . Therefore by Step II,  $\mathcal{W}$  has a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite  $(\mathcal{P})$ closed refinement

$\mathcal{F}(\mathcal{W})$ . For  $E \in \mathcal{E}(\mathcal{U})$ , let

$$S_E = X - \bigcup \{F \in \mathcal{F}(\mathcal{W}) \mid F \cap E = \emptyset\}.$$

Since the collection  $\mathcal{F}(\mathcal{W})$  of  $(\mathcal{P})$ closed sets is  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite, and since  $\bigcup_{n \neq m} \mathcal{P}_n \subset \mathcal{P}$ , by Theorem 3.1, it follows that the set  $\bigcup \{F \in \mathcal{F}(\mathcal{W}) \mid F \cap E = \emptyset\}$  is  $(\mathcal{P})$ closed, and hence  $S_E$  is  $(\mathcal{P})$ open. Since  $\mathcal{E}(\mathcal{U})$  is a cover of  $X$ ,  $\mathcal{S} = \{S_E \mid E \in \mathcal{E}(\mathcal{U})\}$  is a  $(\mathcal{P})$ open cover of  $X$ .  $\mathcal{S}$  is also  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite. For, consider a set  $D_x \in \bigcup_{n \neq m} \mathcal{P}_n$  containing  $x$  and intersecting  $F_1, F_2, \dots, F_k \in \mathcal{F}(\mathcal{W})$ . Now

$$\begin{aligned} D_x \cap S_E &\neq \emptyset, \\ \Rightarrow F_i \cap S_E &\neq \emptyset \text{ for some } i = 1, 2, \dots, k, \\ \Rightarrow F_i \cap E &\neq \emptyset \text{ for some } i = 1, 2, \dots, k. \end{aligned}$$

Since each  $F_i$  is contained in some  $W_x$ , it can intersect at most finitely many  $E \in \mathcal{E}(\mathcal{U})$ . Therefore  $D_x$  can intersect at most finitely many  $S_E \in \mathcal{S}$ .

For every  $E \in \mathcal{E}(\mathcal{U})$ , we take  $U_E \in \mathcal{U}$  such that  $E \subset U_E$ . Then the collection  $\{U_E \cap S_E \mid E \in \mathcal{E}(\mathcal{U})\}$  is a  $(\bigcup_{n \neq m} \mathcal{P}_n)$ locally finite  $(\mathcal{P})$ open refinement of  $\mathcal{U}$ . Therefore it follows that  $X$  is  $(\beta\text{-}\aleph_0)$ paracompact.  $\square$

## LIST OF PUBLICATIONS

The thesis is prepared on the basis of the following publications of ours.

- (1) **Ajoy Mukharjee** and M. K. Bose, *On Nearly Pairwise Compact Spaces*, Kyungpook Math. J. **53** (1) (2013), 125–133.
- (2) M. K. Bose and **Ajoy Mukharjee**, *On Countable Families of Topologies on a Set*, Novi Sad J. Math. **40** (2) (2010), 7–16.
- (3) M. K. Bose and **Ajoy Mukharjee**, *On Countable Pairwise Paracompactness*, Int. J. Mod. Math. **5** (3) (2010), 275–284.
- (4) M. K. Bose and **Ajoy Mukharjee**, *Pairwise Closure-preserving Collections and Pairwise Paracompactness*, Mat. Vesnik **62** (4) (2010), 299–309.
- (5) M. K. Bose and **Ajoy Mukharjee**, *On Bitopological Full Normality*, Mat. Vesnik **62** (1) (2010), 11–18.
- (6) M. K. Bose, A. Roy Choudhury and **Ajoy Mukharjee**, *On Bitopological Paracompactness*, Mat. Vesnik **60** (4) (2008), 255–259.

## BIBLIOGRAPHY

- [1] Bose, M. K. and Mukharjee, Ajoy. *On Countable Families of Topologies on a Set*, Novi Sad J. Math. **40** (2) (2010), 7–16.
- [2] Bose, M. K. and Mukharjee, Ajoy. *On Countable Pairwise Paracompactness*, Int. J. Mod. Math. **5** (3) (2010), 275–284.
- [3] Bose, M. K. and Mukharjee, Ajoy. *Pairwise Closure-preserving Collections and Pairwise Paracompactness*, Mat. Vesnik **62** (4) (2010), 299–309.
- [4] Bose, M. K. and Mukharjee, Ajoy. *On Bitopological Full Normality*, Mat. Vesnik **62** (1) (2010), 11–18.
- [5] Bose, M. K., Roy Choudhury, A. and Mukharjee, Ajoy. *On Bitopological Paracompactness*, Mat. Vesnik **60** (4) (2008), 255–259.
- [6] Bose, M. K. and Tiwari, R. *On Increasing Sequences of Topologies on a Set*, Riv. Mat. Univ. Parma (7) **7** (2007), 173–183.
- [7] Bose, S. and Sinha, D. *Pairwise Almost Continuous Map and Weakly Continuous Map in Bitopological Spaces*, Bull. Calcutta Math. Soc. **74** (1982), 195–206.
- [8] Bose, S. and Sinha, D. *Almost Open, Almost Closed,  $\theta$ -continuous and Almost Quasi Compact Mappings in Bitopological Spaces*, Bull. Calcutta Math. Soc. **73** (1981), 345–354.

- [9] Burke, D., Engelking, R. and Lutzer, D. *Hereditarily Closure-preserving Collections and Metrization*, Proc. Amer. Math. Soc. **51** (1975), 483–488.
- [10] Carnahan, D. *Locally Nearly-compact Spaces*, Boll. Unione Mat. Ital. (4) **6** (1972), 146–153.
- [11] Cobzas, S. *Functional Analysis in Asymmetric Normed Spaces*, Frontiers in Mathematics, Springer Basel, (2013).
- [12] Datta, M. C. *Paracompactness in Bitopological Spaces and an Application to Quasi-metric Spaces*, Indian J. Pure Appl. Math. **8** (6) (1977), 685–690.
- [13] Dieudonné, J. *Une Généralisation des Espaces Compacts*, J. Math. Pures Appl. **23** (1944), 65–76.
- [14] Dowker, C. H. *On Countably Paracompact Spaces*, Canad. J. Math. **3** (1951), 219–224.
- [15] Dugundji, J. *Topology*, Allyn and Bacon, Boston, (1966).
- [16] Dvalishvili, B. P. *Bitopological Spaces: Theory, Relations with Generalized Algebraic Structures, and Applications*, North-Holland, (2005).
- [17] Dvalishvili, B. P. *Relative Compactness, Cotopology and Some Other Notions from the Bitopological Point of View*, Topology Appl. **140** (2004), 37–55.

- [18] Dvalishvili, B. P. *On Some Bitopological Applications*, Mat. Vesnik **42** (1990), 155–156.
- [19] Fletcher, P., Hoyle III, H. B. and Patty, C. W. *The Comparison of Topologies*, Duke Math. J. **36** (1969), 325–331.
- [20] Ganster, M. and Reilly, I. L. *On Pairwise Paracompactness*, J. Aust. Math. Soc. (Series A) **53** (1992), 281–285.
- [21] Herrington, L. L. *Properties of Nearly-compact Spaces*, Proc. Amer. Math. Soc. **45** (1974), 431–436.
- [22] Ivanov, A. A. *Bitopological Spaces*, J. Math. Sci. (N. Y.) **98** (5) (2000), 509–616.
- [23] Joseph, J. E. *Characterizations of Nearly-compact Spaces*, Boll. Unione Mat. Ital. (5) **13-B** (1976), 311–321.
- [24] Kannan, K. *A Note on Some Generalized Closed Sets in Bitopological Spaces Associated to Digraphs*, J. Appl. Math. **2012** (2012), Article ID 508580, 5 Pages, doi: 10.1155/2012/508580.
- [25] Kelley, J. L. *General Topology*, Van Nostrand, New York, (1955).
- [26] Kelly, J. C. *Bitopological Spaces*, Proc. Lond. Math. Soc. (3) **13** (1963), 71–89.

- [27] Konstadilaki-Savopoulou, Ch. and Reilly, I. L. *On Datta's Bitopological Paracompactness*, Indian J. Pure Appl. Math. **12** (7) (1981), 799–803.
- [28] Kovár, M. M. *A Note on Raghavan-Reilly's Pairwise Paracompactness*, Int. J. Math. Math. Sci. **24** (2) (2000), 139–143.
- [29] Kovár, M. M. *A Generalization of  $\theta$ -regularity and Bitopological Covering Properties*, Acta Math. Hungar. **88** (1-2) (2000), 95–104.
- [30] Lane, E. P. *Bitopological Spaces and Quasi-uniform Spaces*, Proc. Lond. Math. Soc. (3) **17** (1967), 241–256.
- [31] Michael, E. *Another Note on Paracompact Spaces*, Proc. Amer. Math. Soc. **8** (1957), 822–828.
- [32] Michael, E. *A Note on Paracompact Spaces*, Proc. Amer. Math. Soc. **4** (1953), 831–838.
- [33] Mršević, M. *On Bitopological Compactness*, Mat. Vesnik **31** (1979), 187–196.
- [34] Mukharjee, Ajoy and Bose, M. K. *On Nearly Pairwise Compact Spaces*, Kyungpook Math. J. **53** (1) (2013), 125–133.
- [35] Murdeshwar, M. G. and Naimpally, S. A. *Quasi-uniform Topological Spaces*, Groninge, (1966).
- [36] Nandi, J. N. *Bitopological Near Compactness I: Characterizations*, Bull. Calcutta Math. Soc. **85** (1993), 337–344.

- [37] Pahk, D. H. and Choi, B. D. *Notes on Pairwise Compactness*, Kyungpook Math. J. **11** (1971), 45–52.
- [38] Raghavan, T. G. and Reilly, I. L. *A New Bitopological Paracompactness*, J. Aust. Math. Soc. (Series A) **41** (1986), 268–274.
- [39] Raghavan, T. G. and Reilly, I. L. *Metrizability of Quasimetric Spaces*, J. Lond. Math. Soc. (2) **15** (1977), 169–172.
- [40] Singal, M. K. and Mathur, A. *On Nearly Compact Spaces - II*, Boll. Unione Mat. Ital. (4) **9** (1974), 670–678.
- [41] Singal, M. K. and Singal, A. R. *Some more Separation Axioms in Bitopological Spaces*, Ann. Soc. Sci. Brux. **84** (1970), 207–230.
- [42] Singal, M. K. and Mathur, A. *On Nearly Compact Spaces*, Boll. Unione Mat. Ital. (4) **6** (1969), 702–710.
- [43] Srivastava, A. and Bhatia, T. *On Pairwise R-compact Bitopological Spaces*, Bull. Calcutta Math. Soc. **98** (2) (2006), 93–96.
- [44] Steen, L. A. and Seebach (Jr.), J. A. *Counterexamples in Topology*, Holt, Rinehart and Winston, New York (1970).
- [45] Stone, A. H. *Paracompactness and Product Spaces*, Bull. Amer. Math. Soc. **54** (1948), 977–982.

- [46] Swart, J. *Total Disconnectedness in Bitopological Spaces and Product Bitopological Spaces*, Indag. Math. **33** (1971), 135–145.
- [47] Weston, J. D. *On the Comparison of Topologies*, J. Lond. Math. Soc. (2) **32** (1957), 342–354.
- [48] Willard, S. *General Topology*, Addison-Wesley Publishing Company, Reading, Massachusetts, (1970).

# INDEX

## Symbols

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 $(\aleph_0)$ compact space, 107  
 $(\aleph_0)$ locally finite, 110  
 $(\aleph_0)$ normal, 109  
 $(\aleph_0)$ paracompact, 111  
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