CHAPTER-V

LARGE AMPLITUDE FREE VIBRATIONS OF IRREGULAR PLATES PLACED ON ELASTIC FOUNDATION *

INTRODUCTION

An approximate method for investigating the large deflection of initially flat isotropic plates has been proposed by Berger (1955). An application of this technique has been made to the vibration problems by Nash and Modeer (1960) who found the large amplitude free vibrations of rectangular and circular plates.

In this paper a unified method for determining the lower natural frequency of large amplitude free vibrations of thin elastic plates of any shape and placed on elastic foundation is given. Following Berger's method a simple fourth order differential equation coupled with a second order nonlinear equation is obtained. If the boundary of the plate is a curve natural to any of the common coordinate system, the solution of the differential equation can be expressed in terms of known functions. For more "exotic" boundaries, the natural co-ordinate must first be determined and after this is done, the solution would inevitably involve some unfamiliar functions. The determination of natural frequencies in this case will then be very complicated. Therefore a common co-ordinate system and its associated function is used for the case of plates with complicated boundaries.

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In order to satisfy the prescribed boundary conditions, the domain is conformally transformed onto a unit circle. Once the transformation function is known, the problem is reduced to the solution of the transformed differential system. In this paper Galerkin's method is used to solve the transformed equation.

The ratio of time periods for circular, square and cornered plates placed on elastic foundation have been determined under simply supported and clamped edge boundary conditions. The foundation is assumed to be of the Winkler type. Experimental values are also obtained for circular and square plates under both the boundary conditions. The results are presented in the form of graphs and they are compared with other known results.

THEORY

Let us consider the large amplitude free vibrations of a thin elastic plate placed on an elastic foundation having the reaction, K' per unit area per unit deflection.

By adding the potential energy of the foundation reaction to the energy expression, using Hamilton's principle and Euler's variational equations one gets the following two differential equations after neglecting e_2 and the inertia effects in the plane of the plate [Nash and Modeer (1960)]

$$\nabla^4 W - \alpha^2 F^2(t)\nabla^2 W + \frac{12}{h^2 c_p^2}\cdot\frac{\partial^2 w}{\partial t^2} + \frac{K'}{D} w = 0 \qquad \dots (5.1)$$

$$e_1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h^2}{12} f(t) \qquad \dots (5.2)$$

in which $$C_\rho^{-2} = \frac{\rho h^3}{12 D} \qquad \ldots (5.3)$$

f(t) being equal to $F^2(t)$ ... (5.4)

Let $w = W(x,y)\,F(t)$ ... (5.5)

Combining Eqs.(5.1) and (5.5) one gets

$$F(t)\,\nabla^4 W - \alpha^2 F^3(t)\,\nabla^2 W + \frac{12}{h^2 C_\rho^2}\cdot\frac{d^2F}{dt^2}\,W + \frac{k'}{D}\,W\,F(t) = 0 \qquad \ldots (5.6)$$

Eq.(5.6) may be written as

$$\left(\frac{\nabla^4 W}{W} + \frac{k'}{D}\right) F(t) - \alpha^2 F^3(t)\,\frac{\nabla^2 W}{W} + \frac{12}{h^2 C_\rho^2}\cdot\frac{d^2 F}{dt^2} = 0 \qquad \ldots (5.7)$$

A solution of Eq.(5.7) is possible if

$$\frac{\nabla^4 W}{W} = K^4 \qquad \ldots (5.8)$$

and

$$\frac{\nabla^2 W}{W} = -K^2 \qquad \ldots (5.9)$$

in which k is a constant.

From Eq.(5.8)

$$(\nabla^2 - K^2)(\nabla^2 + K^2)\,W = 0 \qquad \ldots (5.10a)$$

From Eq.(5.9)

$$(\nabla^2 + K^2)\,W = 0 \qquad \ldots (5.10b)$$

Therefore a solution of Eq.(5.7) can be obtained by satisfying Eq.(5.10b). To satisfy the prescribed boundary conditions for plates of any irregular shape let the domain be conformally transformed onto a unit circle. If $z = x + iy$, $\bar{z} = x - iy$, Eq.(5.10b) changes into

$$4\frac{\partial^2 w}{\partial z \partial \bar{z}} + K^2 w = 0 \qquad \text{... (5.11)}$$

Let $z = f(\xi)$ be the analytic function which maps the boundary under consideration in the ξ - plane onto a unit circle. Thus Eq.(5.11) transforms into complex co-ordinates as

$$\left[\nabla^2 + K^2\left(\frac{dz}{d\xi}\right)^2\right] w(\xi\bar{\xi}) = 0 \qquad \text{... (5.12)}$$

in which

$$\xi = re^{i\theta}, \quad \bar{\xi} = re^{-i\theta}$$

The solution of Eq.(5.12) can be expressed in the form

$$W \approx \sum_{n=1}^{\infty} B_n\left[1-(\xi\bar{\xi})^n\right] \qquad \text{... (5.13a)}$$

or

$$W \approx \sum_{n=1}^{\infty} B_n\left[1-(\xi\bar{\xi})^n\right]^2 \qquad \text{... (5.13b)}$$

according to the prescribed boundary conditions. Eq.(5.13a) is an admissible function for the simply supported edge condition in the sense that this satisfies the kinematic boundary condition $w = 0$ at $r = 1$, but does not satisfy the force boundary condition $M_r = 0$.

The form of w in Eq.(5.13b) satisfies $w = 0 = \frac{dw}{dr}$ at $r = 1$ and can be taken as an admissible function for the clamped edge condition. Substituting Eq.(5.13a) or Eq.(5.13b) into Eq.(5.12) results the error function, $\epsilon_{r,\theta}$, which does not vanish, in general, since Eq.(5.13a) or Eq.(5.13b) is not an exact solution. Galerkin's procedure requires that the error function, $\epsilon_{r,\theta}$ to be orthogonal over the domain under consideration : i.e.,

$$\int_C \epsilon_{r,\theta}(\xi\bar{\xi})\, w(\xi\bar{\xi})\, dc = 0 \quad (n = 1,2,3,\ldots N) \qquad \ldots (5.14)$$

From Eq.(5.14) a homogeneous system of linear equations is obtained. Such a system can have nontrivial solutions only if the determinant of the coefficients of the unknowns vanishes identically. From this determintal equation, the values of $k_1^2, k_2^2 \ldots\ldots k_N^2$ can be solved. For the fundamental frequency the lowest value of k^2 is to be taken. Combining Eqs.(5.7), (5.8) and (5.9) the following differential equation for determining $F(t)$ is obtained.

$$\ddot{F}(t) + \lambda_1 F(t) + \mu F^3(t) = 0 \qquad \ldots (5.15)$$

in which

$$\lambda_1 = \frac{1}{12}\left(K^4 + \frac{K'}{D}\right) h^2 C_\rho^2 \qquad \ldots (5.16)$$

$$\mu = \frac{1}{12}\left(\alpha^2 K^2 h^2 C_\rho^2\right) \qquad \ldots (5.17)$$

Eq.(5.15) is to be solved subject to the initial conditions

$$F(0) = 1, \quad \dot{F}(0) = 0 \qquad \ldots (5.18)$$

Solution of Eq.(5.18) can be taken in the form

$$F(t) = C_n(\omega_1 t, \lambda_2) \qquad \ldots (5.19)$$

in which ω_1 and λ_2 are positive constants given by

$$\omega_1^2 = \frac{1}{12}\left(1 + \frac{\alpha^2}{K^2} + \frac{K'}{DK^4}\right) h^2 \varphi^2 K^4 \qquad \ldots (5.20)$$

$$\lambda_2^2 = \frac{1}{2\left(1 + \frac{K^2}{\alpha^2} + \frac{K'}{D\alpha^2 K^2}\right)} \qquad \ldots (5.21)$$

and c_n is Jacobi's elliptic function.

To determine α Eq.(5.2) is transformed into complex co-ordinates by the transformation $z = x + iy$, $\bar{z} = x - iy$, Thus one gets

$$\frac{\alpha^2 h^2}{12} f(t) = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) u + i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) v + 2\frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}} \qquad \ldots (5.22)$$

If the mapping function $z = f(\zeta)$ be introduced, Eq.(5.22) reduces to

$$\frac{\alpha^2 h^2}{12} \cdot \frac{dz}{d\zeta} \cdot \frac{d\bar{z}}{d\bar{\zeta}} f(t) = \frac{\partial u}{\partial \zeta}\frac{d\bar{z}}{d\bar{\zeta}} + \frac{\partial u}{\partial \bar{\zeta}}\frac{dz}{d\zeta} +$$

$$+\left\{\frac{\partial v}{\partial z} \cdot \frac{d\bar{z}}{d\bar{\zeta}} - \frac{\partial v}{\partial \bar{z}} \cdot \frac{dz}{d\zeta}\right\} i + 2\frac{\partial w}{\partial z} \cdot \frac{\partial w}{\partial \bar{z}} \qquad \ldots (5.23)$$

Now the normalised constant α can be determined from Eq.(5.13a) or (5.13b) and (5.23) by integrating Eq.(5.23) over the cycle 2π. The terms involving u and v can be easily eliminated (since u and v are of little importance in the case of large amplitude vibration) by considering suitable expressions for u and v compatible with the boundary conditions. Finally the following integral will determine α

$$\iint_S \frac{\alpha^2 h^2}{12} \frac{dz}{d\xi} \frac{d\bar{z}}{d\bar{\xi}} ds = 2 \iint_S \frac{\partial w}{\partial \xi} \cdot \frac{\partial w}{\partial \bar{\xi}} \cdot ds \qquad \text{... (5.24)}$$

Thus having determined k and α, the nonlinear frequency, ω_1, is completely determined. The nonlinear period, T_1, is given by

$$T_1 = \frac{4K}{\omega_1} \qquad \text{... (5.25)}$$

K being the complete elliptic integral of the first kind. The linear period, T_2, is given by

$$T_2 = \frac{2\pi}{\omega_2} \qquad \text{... (5.26)}$$

in which ω_2 is to be determined from the equation

$$\ddot{F}(t) + \lambda_1 F(t) = 0 \qquad \text{... (5.27)}$$

in the form $\omega_2^2 = \lambda_1$

Thus the ratio of the periods, $\frac{T_1}{T_2}$, is obtained as

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \frac{1}{\sqrt{\left\{1 + \frac{K^2\alpha^2}{K^4 + \frac{K'}{D}}\right\}}} \qquad \text{... (5.28)}$$

APPLICATIONS

a. Let us apply the procedure explained above to the case of a clamped cornered plate. The mapping function is given by

$$Z = \frac{25}{43} a \left(\xi - \frac{1}{25} \xi^5 \right) \quad \text{... (5.29)}$$

Using Eq.(5.13b) with $n = 1$ an approximate value of k^2 is obtained from Eq.(5.14)

$$k^2 = \frac{24{\cdot}55}{a^2} \quad \text{... (5.30)}$$

With $n = 2$, an improved lower value of k^2 is obtained

$$k^2 = \frac{21{\cdot}71}{a^2} \quad \text{... (5.31)}$$

To determine α the following functions for u and v are taken,

$$u = \sum_{m=1,3,5,\ldots}^{\infty} U_m(r) \cos m\theta \, F^2(t) \quad \text{... (5.32)}$$

$$v = \sum_{m=1,3,5,\ldots}^{\infty} V_m(r) \sin m\theta \, F^2(t) \quad \text{... (5.33)}$$

Substituting Eqs.(5.32) and (5.33) in Eq.(5.23) one gets Eq.(5.24) determining α. To determine the value of α for the fundamental frequency, the value of n in Eq.(5.13b) is taken to be 1. Substituting Eq.(5.13b) with $n = 1$, and Eq.(5.29) in Eq.(5.24) the following value of α corresponding to the lowest frequency is obtained,

$$\alpha^2 = 29{\cdot}28 \frac{B_1^2}{a^2 h^2} \quad \text{... (5.34)}$$

Thus $\dfrac{T_1}{T_2}$ is obtained from Eq.(5.23)

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2}\left(\frac{636}{471 + K_F}\right)\right]^{\frac{1}{2}}} \qquad \ldots (5.35)$$

in which the nondimensional foundation modulus, K_F , is given by

$$K_F = \frac{k'}{D}\, a^4$$

The mapping function of a square plate is given by

$$Z = 1{\cdot}08a \left[\zeta - \frac{1}{10}\, \zeta^5 + \ldots\ldots \right] \qquad \ldots (5.36)$$

Using Eq.(5.13b) with $n = 1$ and proceeding in the same manner as before one gets for a clamped square plate

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2}\left(\frac{457}{400 + 12{\cdot}3\, K_F}\right)\right]^{\frac{1}{2}}} \qquad \ldots (5.37)$$

The mapping function for a circular plate is given by

$$Z = a\,\zeta \qquad \ldots (5.38)$$

and for a clamped circular plate one gets

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2}\left(\frac{480}{400 + 9\, K_F}\right)\right]^{\frac{1}{2}}} \qquad \ldots (5.39)$$

b. Let us consider the case of a simply supported circular plate. Using Eq.(5.13a) with $n = 1$ and proceeding in the same manner as before one gets the ratio $\frac{T_1}{T_2}$,

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2}\left(\frac{72}{36 + K_F}\right)\right]^{\frac{1}{2}}} \qquad \ldots (5.40)$$

For $K_F = 0$, Eq. (5.40) becomes

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + 2\frac{B_1^2}{h^2}\right]^{\frac{1}{2}}} \qquad \ldots (5.41)$$

The corresponding result for the circular plate obtained by Nash and Modeer (1960) is

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + 6\frac{A^2}{h^2} J_1^2(KR)\right]^{\frac{1}{2}}} \qquad \ldots (5.42)$$

where $J_o(KR) = 0$, R being the radius of the circle.

For a simply supported square plate one gets

$$\frac{T_1}{T_2} = \frac{2K}{\pi} \cdot \frac{1}{\left[1 + \frac{B_1^2}{h^2}\left(\frac{67.5}{36 + 1.37\ K_F}\right)\right]^{\frac{1}{2}}} \qquad \ldots (5.43)$$

EXPERIMENTAL VERIFICATION

Experimental verifications were made with circular and square plates under simply supported and clamped edge boundary conditions. The circular plates were 150 mm diameter and the square plates with 150 mm side. Plate material was mild steel of 0.75 mm thickness. Free transverse vibrations of different amplitudes and frequencies were initiated in the apparatus shown in Fig.5.1. The test piece, T, was statically deflected by the load spindle, L and the central deflection was measured by the dial indicator, D. After giving a predetermined central deflection the spindle, L, was lifted quickly by the release spring, R, and the corresponding frequency was measured in a vibration meter, M, with the help of a noncontact type of vibration pick up, P. Simply supported edge conditions were realised by placing the edges of the plates over a knife edge placed around the periphery of the cavity, C, the shape of which conformed the shape of the plates used. Clamped edge conditions were achieved by clamping the edges of the plates rigidly by means of eight bolts, B, with the base of the apparatus. Experiments were carried out first with the cavity, empty and next by placing the plates over eight free helical springs, S, each spring being located at the centre of eight equal areas of the plates. The combined reaction of the springs used was determined experimentally to be $K_F = 6.2$. Care was taken in selecting the stiffness of the spring, R, so that the spindle, L was released quickly from the plate without obstructing the upward motion of the plates.

R E S U L T S

Numerical as well as experimental results for the case of simply supported circular and square plates without foundation have been presented in Fig.5.2 and 5.3 respectively. The corresponding results obtained by Nash and Modeer (1960) for the circular and square plates and the results obtained by Chu and Herrmann (1956) for the square plates have also been presented for comparison. Numerical and experimental results for clamped circular and square plates both with and without foundation have been presented in Fig.5.4 and 5.5 respectively.

Lowest natural frequencies of large amplitude free vibrations of thin plates of any shape can readily be calculated by the conformal mapping technique used in this study, if the mapping functions are known. From Figs.5.4 and 5.5 it is observed that the results obtained with one term approximation of the trial function Eq.(5.13b) for the clamped edge boundary conditions are in excellent agreement with the practical values. For the simply supported edge conditions the theoretical results given in Fig.5.2 and 5.3 are in somewhat poorer agreement with the values obtained experimentally. By using higher term approximations of the trial functions Eq.(5.13a) and (5.13b) and with smoothened mapping functions the results for both simply supported and clamped edge boundary conditions will be refined. But this will involve additional numerical computations.

The periods for rectangular plates obtained by Chu and Herrmann is dependent on the aspect ratio of the plate, whereas the corresponding results obtained by Nash and Modeer are independent of that ratio. The mapping functions for rectangular plates with different aspect ratio will be different and therefore the present study indicates that the periods will depend on the aspect ratio. It should be pointed out that the theory used in this study allows the solution of the eigenvalue problem under consideration from a unified point of view since the trial functions used are the same for all shapes. Considering the fact that one term approximation reduces considerably the computational effort, the results obtained in this study are considered excellent for practical purposes.

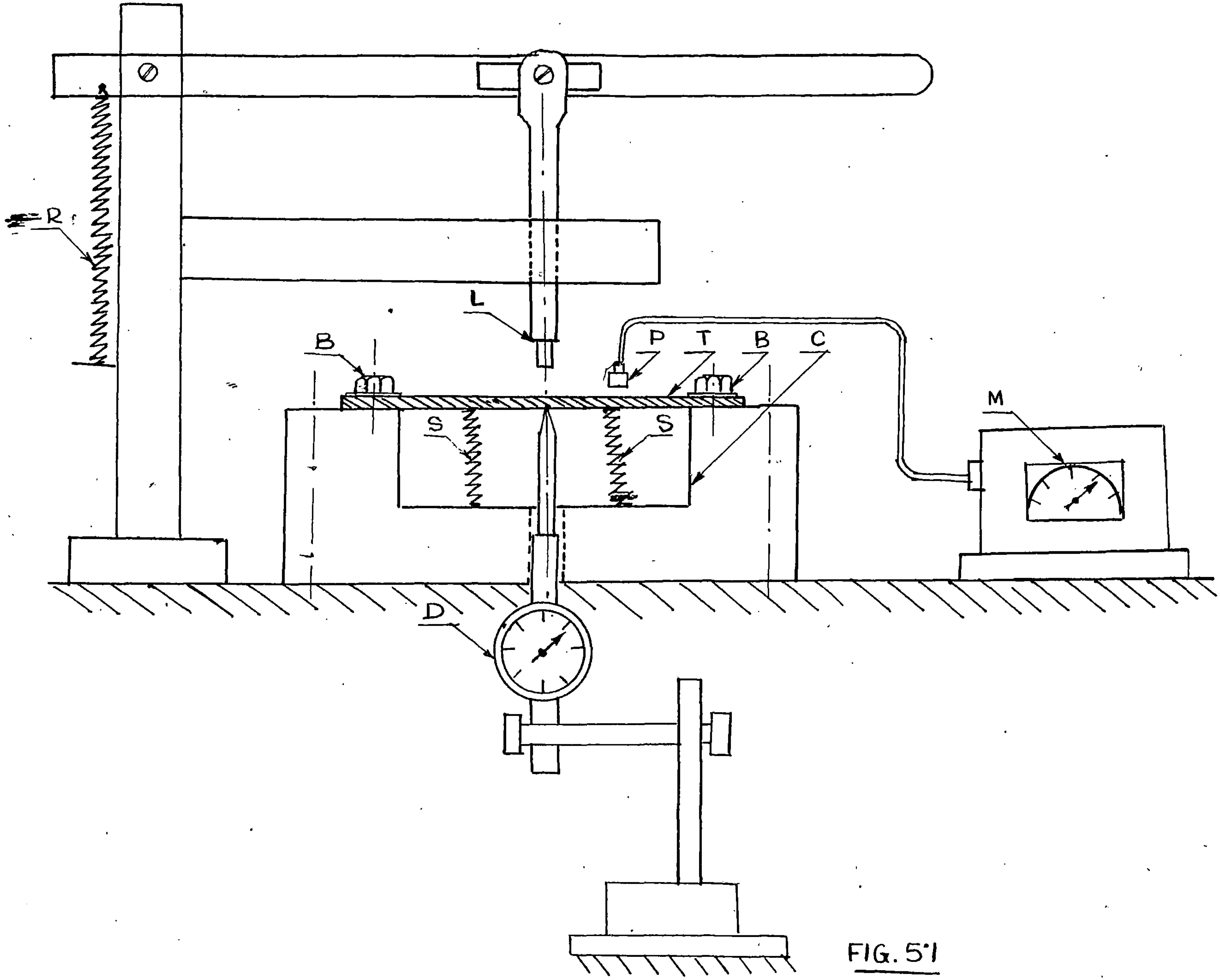

FIG. 5·1

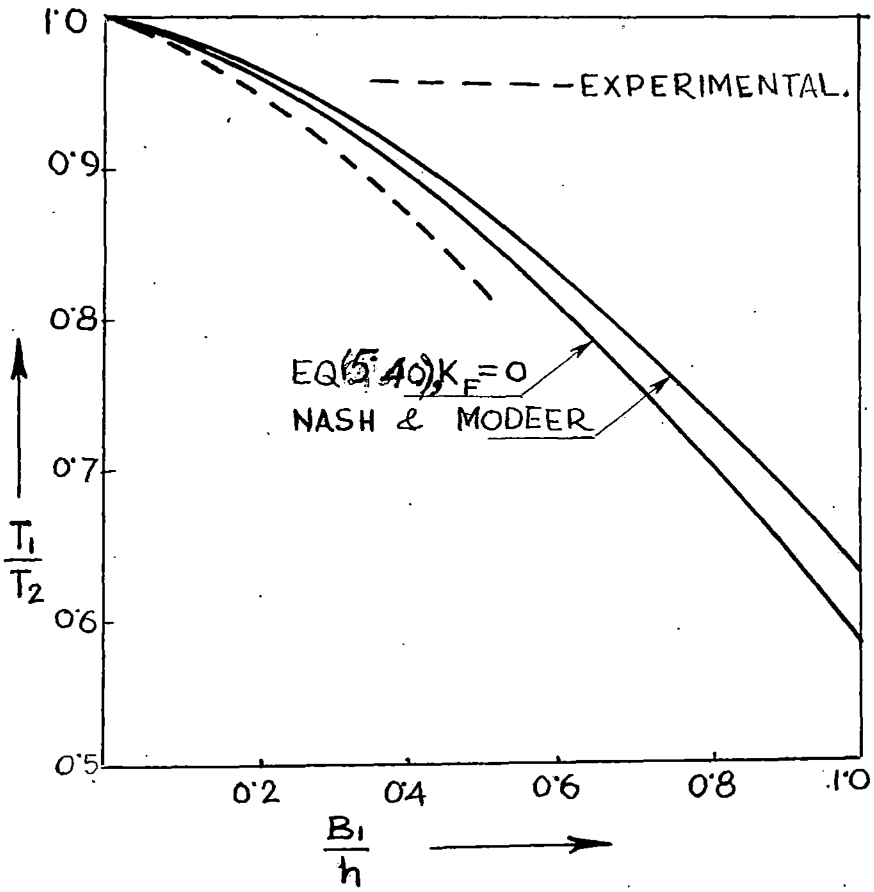

FIG. 5·2 SIMPLY SUPPORTED CIRCULAR PLATE.

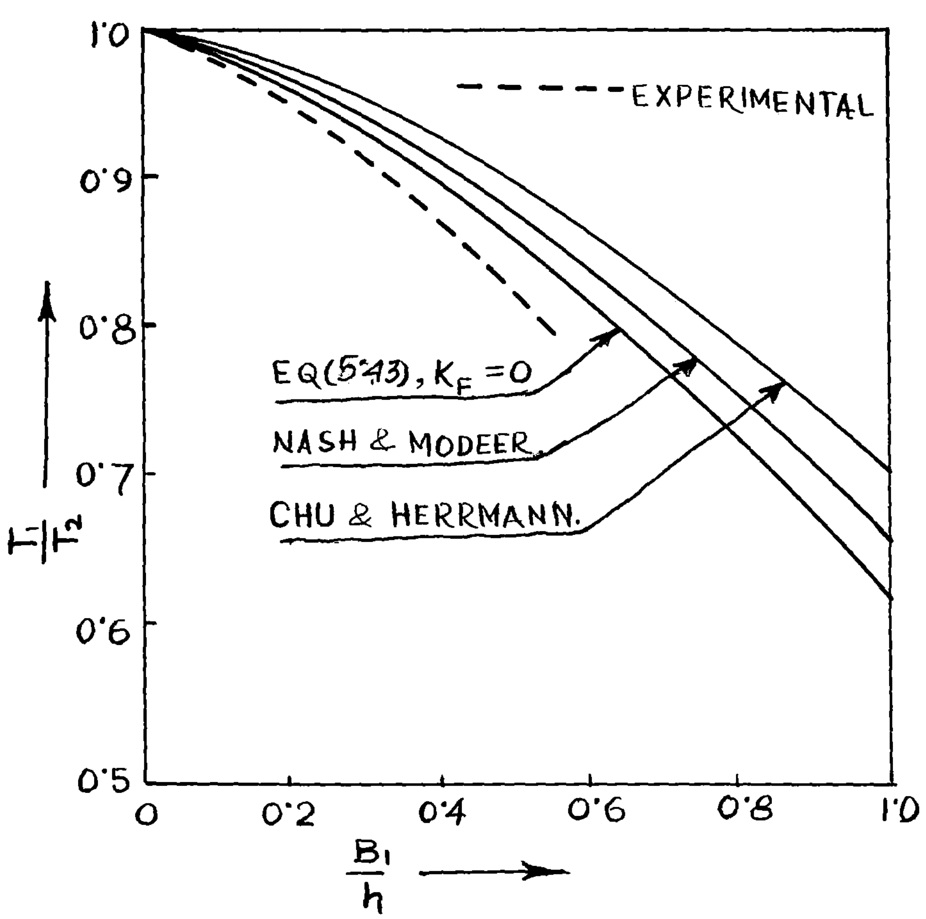

FIG. 5·3 SIMPLY SUPPORTED SQUARE PLATE.

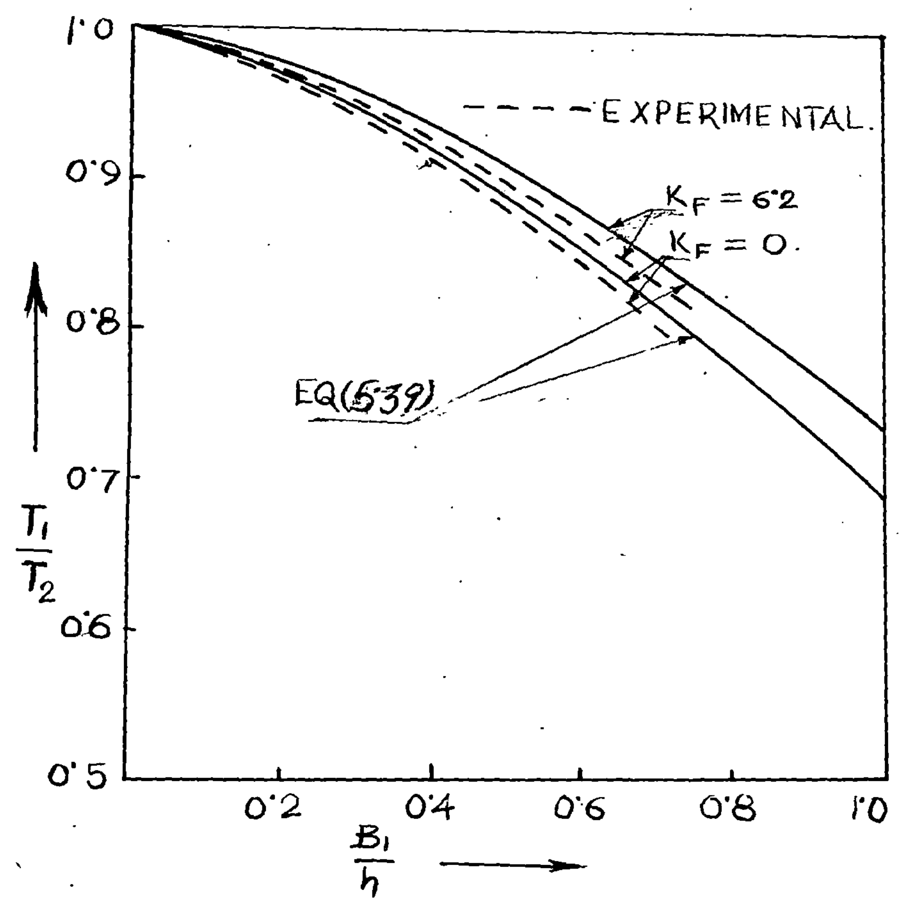

FIG.5·4 CLAMPED CIRCULAR PLATE.

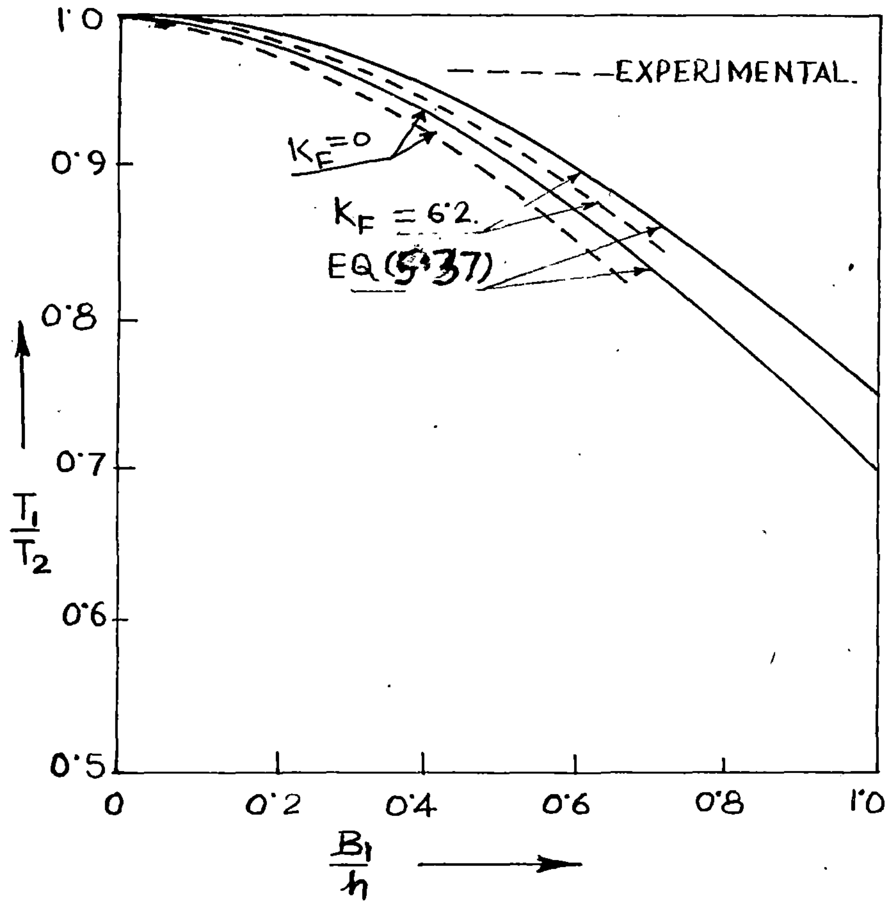

FIG. 5·5 CLAMPED SQUARE PLATE.