ON CERTAIN
SCATTERING PROBLEMS OF RADIATIVE TRANSFER

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BY

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Preface

This thesis is an attempt to study various scattering problems of Radiative Transfer. Solution of scattering problems have been obtained both exactly for the angular distribution of the emergent intensity and approximately for practical purpose. Numerical calculations have also been made for a few cases. The methods applied for the exact solutions are "Laplace-Transform and Wiener-Hopf technique" , " Laplace Transform and Linear singular operators " and "Principle of Invariance ". The methods used for the approximate solutions are Discrete Ordinate method and Eddington's approximations. The problems of scattering isotropically, anisotropically, coherently and noncoherently, both in finite and semi-infinite atmospheres, have been discussed. The problems with Rayleigh's scattering phase function, Combination of Rayleigh and Isotropically scattering phase function and Planetary phase function has been dealt and in most of the cases the planckian source function has been taken as an exponential function of optical depth.

Application of Wiener-Hopf technique to the Time Dependent X- and Y- functions has been dealt. An exact linearization and decoupling of the integral equation satisfied by Time-Dependent X- and Y- functions has been made. The principle of invariance is applied to derive the functional equations for Time-Dependent diffuse reflection and transmission function.

This thesis contains five chapters with computed results presented in the respective chapters.

LIST OF PUBLICATIONS

1. Deb, T. K. , Biswas, G. and Karanjai, S.: 1991, " Solution of the equation of transfer for interlocked multiplets by the method of discrete ordinates with the planck function as a nonlinear function of optical depth " , Astrophys. Space Sci. Vol. 178, pp. 107-117.

2. Karanjai, S and Deb, T. K.: 1991, " Solution of the equation of transfer for coherent scattering in an exponential atmosphere by Eddington's method " , Astrophys.Space Sci. Vol. 178, pp. 299-302.

3. Karanjai, S and Deb, T. K.: 1991, " An exact solution of the equation of transfer for three-term scattering indicatrix in an exponential atmophere ", Astrophys. Space Sci. Vol. 179, pp. 89-96.

4. Karanjai, S and Deb, T. K.: 1991, " Exact solution of the equation of transfer in a finite exponential atmosphere by the method of Laplace transform and linear singular operator " , Astrophys. Space Sci. Vol. 181, pp. 267-275.

5. Karanjai, S and Deb, T. K.: 1991, " Solution of the equation of transfer for interlocked multiplets with Planck function as a nonlinear function of optical depth ", Astrophys. Space Sci. Vol. 184, pp. 57-63.

6. Deb, T. K. and Karanjai, S.: 1992, " Time-dependent scattering and transmission function in an anisotropic two-layered atmosphere ", Astrophys. Space Sci. Vol. 189, pp. 95-117.

7. Karanjai, S and Deb, T. K.: 1992, "An Exact solution of the equation of transfer for coherent scattering in an exponential atmosphere ", Astrophys. Space Sci. Vol.189 , pp. 119-122.

8. Deb, T. K. and Karanjai, S.: 1992, " Solution of the equation of transfer for coherent scattering in an exponential atmosphere by Busbridge's method ", Astrophys. Space Sci. Vol. 192, pp. 127-132.

9. Karanjai, S and Deb, T. K.: 1992, " Solution of the equation of transfer for coherent scattering in an exponential atmosphere by the method of discrete ordinates ", Astrophys. Space Sci. Vol. 192,pp.209-217.

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13. Deb, T. K. and Karanjai, S.: 1994 , " Solution of a radiative transfer problem with a combined Rayleigh and Isotropic phase function ", Proceedings of the National seminar on Mathemetical Modeling, [Lecture Notes in Mathematical Sciences. ", Vol. 2, pp. 70-78].

CONTENTS

CHAPTER 5

SOLUTION OF RADIATIVE TRANSFER PROBLEMS IN A FINITE ATMOSPHERE

KEY WORDS

Anisotropic scattering
Auxiliary function
Albedo
Absorption coefficient
Coherent scattering
Conservative scattering
Discrete ordinate
Doublet
Diffuse reflection
Exponential atmosphere
Emergent intensity
H- function
Interlocking multiplets
Integro-differential equation
Isotropic scattering
Laplace transform
Linear singular operator
Legendre polynomial
Legendre expansion
Matrix transfer equation
Noncoherent scattering
Optical depth
Principle of invariance
Planck function
Phase function
Residual intensity
Radiative transfer
Redistribution
Rayleigh phase function
Specific intensity
Triplet
Wiener-Hopf technique
X- function
Y- function

LIST OF FIGURES AND TABLES

ABSTRACT

The Laplace transform method in combination with the Wiener-Hopf technique for the exact solution of the Radiative Transfer problem has received much less attention in the history of the transport theory. But the method is undoubtedly an interesting alternative to the other existing method for exact solution of the Radiative Transfer problems. Section 2.2, 2.3 and 2.5 of Chapter 2 is devoted to study scattering problems in an atmosphere scattering anisotropically by the method of Laplace transform in combination with the Wiener-Hopf technique. The work in section 2.2 has been published in Earth, Moon and Planet Vol. 59, pp. 1-10, 1992. The work in section 2.3 has been published in Astrophys. Space Sci. Vol. 179, pp.89-96, 1991. The work in section 2.5 has been published in Lecture Notes in Mathematical Sciences , " Proceedings of the National Seminar on Mathematical Modeling " Vol. 2, pp.70-78, 1994.

One of the most important achievement in the field of stationary transport theory is the introduction of H-,X- and Y- functions for problems of semi-infinite and finite media. Time - Dependent X- and Y- functions are discussed in section 2.6. In section 2.4 and 2.6 of Chapter 2 exact solutions has been obtained by the method of Principle of Invariance. The work in section 2.6 has been published in Astrophys. Space Sci. Vol. 189, pp. 95-117, 1992.

In Chapter 3 , the equation of transfer has been solved exactly using "Laplace transform and Wiener-Hopf technique" (sec-3.3) and modified Principle of Invariance (sec-3.4) and approximately by the method of Discrete Ordinates (sec-3.5) and Eddington's approximations (sec-3.2) in an isotropic coherently scattering atmosphere with an

exponential form of Planckian source function. The work in sec-3.2 , sec-3.3, sec-3.4 and sec-3.5 has been published in Astrophys. Space Sci. Vol 178, pp. 299-302, 1991., Vol. 189, pp. 119-122, 1992., Vol. 192, pp. 127-132, 1992 and Vol.192, pp. 209-217, 1992 respectively.

In Chapter 4 , in section 4.2 and 4.3 the equation of transfer for interlocked multiplets which is a noncoherent scattering has been solved approximately by the method of Discrete Ordinates and exactly by the method of modified Principle of Invariance using Planckian source function as an exponential function of optical depth. In the subsequent sections in this chapter the residual intensities for doublets and triplets has been calculated using some approximate forms of H-function and the results are shown in both tables and figures. The work in sec-4.2 and sec-4.3 has been published in Astrophys. Space Sci. Vol. 178, pp. 107-117, 1991 and Vol 184, pp. 57-63, 1991 respectfully.

In section 5.2 of Chapter 5 the one sided Laplace transform together the theory of Linear singular operators has been applied to solve the transport equation which arises in the problem of a finite atmosphere with the Planck's function as an exponential function of optical depth. In section 5.3 of the same chapter , Laplace transform technique is applied to Time-Dependent X- and Y- functions which play a central role in Radiative Transfer problems, to obtain Fredholm equation. An exact linearized and decoupled integral equation satisfied by Time-Dependent X- and Y- functions has been obtained in section 5.4. The work in sec-5.2 , sec-5.3 and sec-5.4 has been published in Astrophys. Space Sci. Vol. 181, pp. 267-275, 1991., Vol. 196, pp. 223-339,1992 and Vol.203, pp. 135-138, 1993 respectively.

A few relations assumed during the solutions have been obtained and discussed in Appendix I, II and III.

CHAPTER - 1

INTRODUCTION

INTRODUCTION :

A pencil of radiation traversing a medium will be weakened by its interaction with matter. If the specific intensity I_ν *therefore becomes* $I_\nu + dI_\nu$ after traversing a thickness ds in the direction of its propagation, then

$$dI_\nu = -k_\nu \rho \, I_\nu \, ds \qquad (1.1)$$

where ρ is the density of the material. The quantity k_ν introduced in this manner defines the *mass absorption coefficient* for radiation of frequency ν. Now, it should not be assumed that this reduction in intensity, which a pencil of radiation experiences while passing through matter, is necessarily lost to the radiation field. For it can very well happen that the energy lost from the incident pencil may all reappear in other directions as *scattered radiation.* In general, I may however expect that only a part of the energy lost from an incident pencil will reappear as scattered radiation in other directions and that the remaining part will have been 'truly 'absorbed in the sense that it represents the transformation of radiation into other forms of energy (or even of radiation of other frequencies). I shall therefore have to distinguish between *true absorption* and *scattering.* Considering first the case of scattering, I say that a material is charecterised by a *mass scattering coefficient* k_ν if from a pencil of radiation incident on an element of mass of cross-section $d\sigma$ and height ds, energy is scattered from it at the rate

$$K_\nu \rho \, ds \times I_\nu \, \text{Cos}\vartheta \, d\nu \, d\sigma \, d\omega \qquad (1.2)$$

in all directions. Since the mass of the element is

$$dm = \rho \, \text{Cos}\vartheta \, d\sigma \, ds \qquad (1.3)$$

I can also write

$$K_\nu \, I_\nu \, dm \, d\nu \, d\omega \qquad (1.4)$$

It is now evident that to formulate quantitatively the concept of *scattering* we must specify in addition the angular distribution of the scattered radiation (1.4).

I shall therefore introduce a *phase function* $P(\cos\theta)$ such that

$$K_\nu \, I_\nu \, P(\text{Cos}\theta) \, \frac{d\omega'}{4\pi} \, dm \, d\nu \, d\omega \qquad (1.5)$$

gives the rate at which energy is being scattered into an element of solid angle $d\omega'$ and in a direction inclined of an angle θ to the direction of incidence of a pencil of radiation on an element of mass dm . Accordingly the rate of loss of energy from the incident pencil due to scattering in all directions is

$$K_\nu \, I_\nu \, dm \, d\nu \, d\omega \int P(\text{Cos}\theta) \, \frac{d\omega'}{4\pi} \; ; \qquad (1.6)$$

this agrees with (1.4) if $\int P(\text{Cos}\theta) \, \frac{d\omega'}{4\pi} = 1 \qquad (1.7)$

i.e. if the phase function is normalised to unity .

In the general case when both scattering and true absorption are present, I shall still write for the scattered energy the same expression (1.5). But in this case the total loss of energy from the incident pencil must be less than (1.5), accordingly

$$\int P(\cos\theta)\ \frac{d\omega'}{4\pi} = \omega_o \le 1 \qquad (1.8)$$

Thus the general case differs from the case of pure scattering only by the fact that the phase function is not normalised to unity.

It is evident from our definitions that ω_o represents the fraction of the radiation lost from an incident pencil due to scattering, while $(1 - \omega_o)$ represents the remaining fraction which has been transformed into other forms of energy. I shall refer to ω_o as the *albedo for single scattering.* A radiation field is said to be isotropic at a point, if the radiation is independent of direction at that point. And if the intensity is the same at all points and in all directions the radiation field is said to be homogeneous and isotropic. Moreover , when $\omega_o = 1$, I shall say that I have a conservative case of perfect scattering. when $\omega_o \neq 1$ I shall say that I have a non conservative case of scattering .

Next to the isotropic scattering greatest interest is attached to Rayleigh's scattering which is an example of conservative anisotropic scattering.

1.1 Introduction to scattering problems

1.11 Coherent and Non-Coherent Scattering.

When the radiation is emitted in the frequency in which it was absorbed the atom is said to scatter coherently. On the other hand, when frequency of the emitted radiation differs from that of the absorbed radiation I call it the case of non-coherent scattering. Non-coherent scattering is sometimes used to mean that the scattering involves not only a change in frequency but also a complete redistribution in frequency i.e. scattering in which the frequency of re-emission is in correlation with the frequency absorbed. From practical point of view, strictly coherent scattering does not exist in astrophysics (vide, Edmonds [1955]). I designate the scattering as Coherent and Non-coherent according to our theoretical consideration of the problem when an atom absorbs energy of certain frequency, ν, the probability that the energy will be re-emitted in the same frequency will be maximum if

(i) the atom is at rest .

(ii) the atom is in the lowest quantum state

(iii) in a weak radiation field.

Departure from any of the above three conditions will cause non-coherent scattering .

1.12 Coherent Scattering Problems.

Chandrasekhar [1960] applied the method of discretes ordinate to solve the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz.,

$$B_\nu(T) = b_o + b_1\tau. \qquad (1.9)$$

The equation of transfer for coherent scattering has also been solved by Eddington's method (where η_ν, the ratio of line to the continuum absorption coefficient, is constant) and Stromgren method (when η_ν , has small but arbitrary variation with optical depth (vide, Woolley and Stibbs, 1953). Dasgupta [1977] applied the method of Laplace transform and Wiener-Hopf technique to find an exact solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions

$$B_\nu(T) = b_o + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau) \qquad (1.10)$$

by use of a new representation of the H-function obtained by Dasgupta [1977]. Extensive study has been made on coherent scattering by various authors thereafter and before.

1.13 Noncoherent Scattering Problems.

In stars having high temperature and high energy density, the induced transition-probabilities at lower frequencies increase sufficiently. The ground state or the lower state in case of a sub ordinate line then possesses a finite width and the frequency of the absorbed and the emitted radiation differ from each other introducing a noncoherency in the formation of absorption lines. Though in a single scattering there is a change in frequency giving rise to either a loss or a gain in energy of the atom, in a number of scattering the total loss of energy balances with the total gain in energy. In the case of interlocking without redistribution, if radiation in one line flows from centre to the wings then then in another line it flows from wings back to the centre. The doppler broadening introduces, another important type of non-coherent scattering. If a moving atom absorbs radiation from one direction and emits it in another, the frequencies of the absorbed and emitted radiation will differ even if the process is coherent in the atom's rest frame. Another type of non-coherent scattering is that due to pressure broadening which is the simplest and at the same time most important case in stellar atmosphere. Let an electron, due to absorption of energy, jumps to a higher level where there is a perturbing atom or ion. Now if the perturbing atom goes away before the electron suffers downward transition, the atom may absorb some amount of energy from the electron and the electron will consequently emit radiation of frequency quite different from that of the absorption. This type of scattering gives rise to the process known as Stark Effect, e.g. Hydron lines in stellar atmosphere are broadened by Stark Effect. Impact broadening becomes important when the velocity of the perturbation is large. In case of lines widened by impact broadening the scattering is partly coherent and partly non-coherent. Domke and Staude (1973) considered the formation of a Zeeman-multiplets by noncoherent scattering and true absorption in a M-E atmosphere. The solution of the line formation problem is obtained (vide, Domke and Staude, 1973) for an exponential form of the Planckian source function.

1.14 Interlocking Problems.

Interlocking of multiplets is another type of non-coherent scattering. When the lower state possesses a common upper state by absorption from any of the lower sub-state, the re-emission will be controlled by the transition probability of the various lines regardless from a certain sub-state of the lower state in a certain frequency ν has a non-zero probability of returning to another lower sub-state emitting in a frequency different from ν giving rise to non-coherent scattering. Similar case will arise when the number of upper sub-states will possess a common lower state. This type of non-coherency has a special name interlocking of lines without redistribution . Woolley and Stibbs [1953] considered the problem of interlocking without redistribution in details and gave an appropriate solution applying Eddington's method. Busbridge and Stibbs [1953] applied the principle of invariance to solve the same problem and calculated three hypothetical line profiles for doublets. However Busbridge and Stibbs [1953] did not attempt calculation of the line profiles for triplets because they feared any such attempt would have involved considerable labour. Karanjai [1968a] profitably applied his approximate form for the H-function [1968b] to minimize to a great extent the labour of such computations. Dasgupta and Karanjai [1972] applied Sobolev's probabilistic method to solve the transfer equation for the case of interlocking without redistribution.

Another exact solution of the equation of transfer has been given by Dasgupta [1956] by his modified form of Wiener-Hopf technique. Karanjai and Barman [1981] applied the extension of the method of discrete ordinate to find an exact solution of the problem of line formation by interlocking in the M-E model. Karanjai and Karanjai [1985] used the method of Laplace transform and Wiener-Hopf technique to solve the equation of interlocked lines taking the Planck function as a nonlinear function of optical depth. Karanjai [1982] has calculated Mg b line contours with the help of the solution obtained by Dasgupta and

Karanjai [1972] and showed that his calculated lines have a good agreement with the observation. Dasgupta [1978] obtained an exact solution of the transfer equation for non-coherent scattering arising from interlocking of principal lines without redistribution of the H-function obtained by Dasgupta [1977]. While solving the transfer equation Dasgupta considered the Planck's function to be linear in τ (Optical depth) (equation (1.9)).
Karanjai and Karanjai [1985] considered two non-linear form of Planck function Viz;

$$\text{(a)} \quad B_\nu(T) = B(t) = b_o + b_1 e^{-\beta\tau} \qquad (1.11)$$

in an exponential atmosphere (vide, Degl 'Innocenti, 1979) where β, b_o and b_1 are positive constants.

$$\text{(b)} \quad B_\nu(T) = B(t) = b_o + b_1\tau + E_2(\tau); \qquad (1.12)$$

in an atmosphere considered by Busbridge [1955]. Roymondal, Biswas and Karanjai [1988] solved the equation of transfer for non-coherent scattering by F_n method. Recently, Basak and Karanjai [1995] solved the transfer equation for interlocked multiplets in anisotropically scattered atmosphere.

1.15. Anisotropic Scattering Problems.

The equation of transfer for plane parallel Rayleigh's scattering phase function can be put in the form

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{3}{16}\left[(3-\mu^2)\int_{-1}^{+1} I(\tau,\mu')d\mu' + \right.$$

$$+ (3\mu'^2 - 1) \int_{-1}^{+1} I(\tau, \mu')\mu'^2 \, d\mu' \Big] \qquad (1.13)$$

According to Chandrasekhar [1960] the solution of the equation of transfer (1.13) for Rayleigh scattering can be put in the form

$$J(\tau) = \frac{3}{16} \Big[\int_0^{\infty} (3E_1 - E_3)_{|t-\tau|} \, J(t)dt + \int_0^{\infty} (3E_3 - E_1)_{|t-\tau|} \times$$

$$\times k(t) \, dt \Big] \qquad (1.14)$$

and $$k(\tau) = \frac{3}{16} \Big[\int_0^{\infty} (3E_3 - E_5)_{|t-\tau|} \, J(t) \, dt +$$

$$+ \int_0^{\infty} (3E_5 - E_3)_{|t-\tau|} \times k(t) \, dt \Big] \qquad (1.15)$$

where $$J(t) = (1/2) \int_{-1}^{+1} I(\tau, \mu) d\mu \qquad (1.16)$$

$$k(t) = (1/2) \int_{-1}^{+1} I(\tau, \mu)\mu^2 \, d\mu \qquad (1.17)$$

$$E_n(y) = \int_1^{\infty} \frac{dx}{x^n} \, e^{-xy} \qquad (1.18)$$

Equation (1.14) and (1.15) represents a pair of integral equations for J and K. The linear integral equation which replace the equation of transfer (1.13) become increasingly of higher order. According to Rayleigh phase function

$$p(\mu,\phi\ ;\ \mu',\phi') = (3/4)\Big[\ 1 + \mu^2\mu'^2 + (1-\mu^2)(1-\mu'^2)\cos^2 \times$$

$$\times\ (\phi - \phi') + 2\mu\mu'(1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi - \phi')\Big] \quad (1.19)$$

the scattering function can be expressed in the form

$$S(\mu,\phi;\mu_o,\phi_o) = \frac{3}{8}\Big[\ S^{(0)}(\mu,\mu_o) - 4\mu\mu_o(1-\mu^2)^{1/2}(1-\mu_o^{\,2})^{1/2} \times$$

$$\times\ S^{(1)}(\mu,\mu_o)\cos(\phi_o - \phi) + (1-\mu^2)(1 - \mu_o^2) \times$$

$$\times\ S^{(2)}(\mu,\mu_o)\cos 2(\phi_o - \phi)\Big] \quad (1.20)$$

(vide, Chandrasekhar, 1960). The law of darkening for the problem with a constant net flux and for Rayleigh phase function has been expressed in the form (vide, Chandrasekhar, 1960)

$$I(0,\mu) = \frac{3}{4}F\left\{\mu + \frac{3}{16}H(\mu)\int_0^1 \mu'^2 H(\mu')\left[\frac{3-\mu'^2}{\mu+\mu'} + \mu' - c\right]d\mu'\right\} \quad (1.21)$$

Consequently the axially symmetric problem in semi-infinite plane parallel atmosphere with a constant net flux in the total intensity ($I_l + I_r$) is one which is physically significant. The transfer of radiation in the atmosphere of early type stars with surface temperature exceeding 15,000 ^{o}K is predominantly controlled by the scattering by free electrons.

Chandrasekhar [1960] discussed the equations of Radiative transfer for an electron scattering atmosphere and gave the solution of the equation by discrete ordinate method (Chandrasekhar, 1960). Sweigert [1970] solved the integral equation of Radiative transfer numerically for both conservative and non- conservative cases in which scattering is governed by the Rayleigh phase function. The polarisation produced by Rayleigh scattering was neglected. Solution were tabulated over a wide range of optical depths and for varying amounts of absorption measured by the albedo for single scattering. These numerical results may prove useful in the interpretation of planetary reflectiveness, particularly in the ultraviolet where the importance of Rayleigh scattering increases appreciably due to the λ^{-4} dependence of the scattering cross-sections. Sweigert [1970] presented numerical solution to the integral equation for both finite and infinite atmosphere according to the Rayleigh phase function with absorption. Abhyankar and Fymat [1970a] discussed the imperfect Rayleigh scattering in a semi-infinite atmosphere. The extinction of radiation in a coherent scattering gaseous medium is caused partly by true absorption, which result in a loss of incident photons from the radiation field and partly by scattering , which simply modifies the paths of the photons without actually removing them from the field . In other words, the medium exhibits imperfect scattering. The reflection matrix $\phi(\mu,\phi,;\mu_0,\phi_0)$ for a semi-infinite plane parallel stratified homogeneous atmosphere, scattering in accordance with the conservative Rayleigh phase matrix was obtained by Chandrasekhar [1960]. The corresponding solution for a non-conservative Rayleigh atmosphere in which the albedo for single scattering Ω is constant, but different from unity, are presented for some representative values of Ω. They showed that the reduction in value of the albedo increases the absolute degree of polarization and brings the Babinet and Brewster neutral points closer to the Sun; the points even coalesce with the Sun for vary small albedo values. Abhyankar and Fymat [1970b] discussed the theory of radiative transfer in inhomogeneous atmospheres. Here in the case where the phase matrix corresponding to azimuth independent term of the radiation field scattered by an inhomogeneous plane-parallel atmosphere, is separable in the form

$$p^{(o)}(\mu,\mu') = M(\mu).\ M^{+}(M) \qquad (1.22)$$

(Where the sign + stands for simple transaction) is simplified matrix equation of the problem are treated by the perturbation method of Fymat and Abhyankar. In this connection they have studied the regions of convergence in the case of a Rayleigh scattering. The regions of convergence in the case of Rayleigh scattering law are delimited when the solution for conservative Rayleigh scattering is taken as the reference. It has been shown that the region of convergence for Rayleigh scattering is slightly smaller than that of convergent for all optical depth when the maximum value of Ω is less than about 0.945 ; for higher values of Ω there is apparently no convergence for large optical depths.

Fuzhong Weng [1992] applied a multi-layer discrete ordinate method for vector Radiative transfer in a vertically inhomogeneous, emitted and scattering atmosphere. In that work , the up welling radiance from the vector radiance transfer model, established is compared with Chandrasekhar's analytical solutions for a conservative Rayleigh Scattering atmosphere.

While the solution for conservative Rayleigh scattering is known in all details of intensity and state of polarization for a wide range of optical thickness, the corresponding solution for non-conservative Rayleigh scattering , often dealt in planetary atmosphere , are not available. The perturbation method, developed by Fymat and Abhyankar [1970a

,1970b] and its present extension enable to derive such solution for homogeneous atmosphere with albedo for single scattering different from unity. Fymat and Abhyankar [1970c] also discussed the theory of radiative transfer of partially polarised radiation through an inhomogeneous semi-infinite atmosphere. They solved it by the application of matrix perturbation method by introducing a matrix N-function to a semi-infinite atmospheres in the form of a Newmann series. The region of convergence of this series solution is delimited for Rayleigh law of scattering. An iteration scheme for computing the solution was discussed and as an illustration, sample computations were presented in which the N-functions for homogeneous Rayleigh non-conservative atmosphere with albedo for single scattering $\Omega = 0.25$ and 0.75 were derived for the N_o-function for a reference homogeneous atmosphere with $\Omega_o=0.5$.

Fymat and Abhyankar [1970a,1970b] linearized the nonlinear singular integral equations for the radiative transfer in inhomogeneous plane-parallel atmosphere of arbitrary stratification by using a perturbation technique (vide , Fymat and Abhyankar,1970a) which has also been applied (vide , Fymat and Abhyankar,1970b) successfully to a semi-infinite plane parallel atmosphere. Fymat and Abhyankar (1970b) also dealt with diffuse reflection by a semi-infinite non-conservative Raleigh atmosphere. Pomraning [1970] consi-

-dered the classical problem of computing the albedo from a half-space and showed that one can derive an appropriate variational principle for this problem and that the variational estimates of the albedo based upon asymptotic trial functions are remarkably accurate . Further , it is shown that the albedo is insensitive for the descriptions namely

(1) An isotropic phase function averaged over polarization
(2) The Rayleigh phase function averaged over polarization.
(3) Rayleigh scattering properly accounting by the Rayleigh scattering law averaged over polarization, the equation of transfer is (Chandrasekhar, 1960)

$$\mu \frac{\partial I(z,\mu)}{\partial z} + I(z,\mu) = (c/2) \left[\int_{-1}^{+1} I(z,\mu')d\mu' + \right.$$

$$\left. + (1/2)\, P_2(\mu) \int_{-1}^{+1} P_2(\mu')\, I(z,\mu')\, d\mu' \right] \qquad (1.23)$$

$$\text{where} \qquad P_2(\mu) = (3\mu^2 - 1)/2 \qquad (1.24)$$

z being the spatial co-ordinate measured in optical distance , μ, the cosine of the angle between the photon flight direction and an inward normal intensity and c the ratio of the scattering coefficient to the collision coefficient. Pomraning suggested that in a certain work on radiative transfer the complexities introducing by accounting for polarization effects and the anisotropy of the Rayleigh phase function can be avoided. It may be sufficient depending upon the accuracy required to assume an isotropic phase function averaged over polarization.

Casti, Kagiwada and Kalaba [1970] discussed about external radiation fields for isotropically scattering finite atmospheres bounded by a Lambert law Reflection. Casti, Kagiwada and Kalaba [1070] provided formulae for obtaining the diffusely transmitted and reflected radiation fields for a planetary isotropically scattering atmosphere of finite thickness in terms of the solution to the problem with no planetary surface .

From numerical result they showed that these reflected and transmitted fluxes are essentially the same whether isotropic or Rayleigh scattering laws are assumed.

Kagiwada and Kalaba [1971] derived all the basic equations of the Cauchy system mathematically from the basic integral equation for the source function $\mathcal{J}$ for the atmospheres bounded by Lambert's law Reflector.

The problem of the determination of radiation fields in finite ,conservative, isotropically scattering media bounded by a Lambert's law Reflector, has been reduced by Kalaba (1970) to a Cauchy system involving auxiliary functions of merely one angular argument.

Buell, Casti, Kalaba, and Ueno [1970] discussed exact solution of a family of matrix integral equations for multiple scattered, partially polarised radiation. In the theory of multiple scattering of partially polarized radiation, a key role is played by the integral equation,

$$J(t,x,z) = I\, e^{-(x-t)/z} + \int_0^x K(|t-y|)\, J(y,x,z)\, dy \quad (1.25)$$

$$0 \le t \le x \le x_1 , \quad 0 \le z \le 1 \quad (1.26)$$

where J and K stand for n $\times$ n square matrices; I is the unit n $\times$ n matrix, and the matrix kernel k can be represented in the form

$$K(r) = \int_0^1 e^{-r/z'}\, W(z')\, dz' , \quad r>0 \quad (1.27)$$

where W is a square n $\times$ n matrix. It is shown that this family of matrix integrals can be transformed into a Cauchy problem. The Cauchy system solves the integral equation for the matrix J. The theory is for general phase function.

Hulst and Grossman [1968] discussed multiple light scattering in planetary atmosphere. The diffuse reflection and transmission by plane , homogeneous atmospheres consisting of particles with an anisotropic scattering was discussed for various phase functions.

It has been shown that "the doubling method" can be purformed most conveniently with great accuracy from very thin to very thick layers. The accuracy obtained with various integration schemes in depth and in angle was discussed in some detail .

Kagiwada and Kalaba [1967] estimated the local anisotropic scattering function on the basis of multiple scattering properties for the general phase function . The phase function is expanded in a series of Legendre polynomials i. e.,

$$p(\cos\alpha) = \sum_{m=0}^{m} C_m \, P_m \, (\mathrm{Cos}\,\alpha) \tag{1.28}$$

and the coefficients are determined so as to best explain diffuse reflection measurements .

Busbridge [1960] discussed the anisotropic scattering with general phase function :

$$p(\mu,\mu') = \sum_{\nu=0}^{N} \omega_\nu \, P_\nu(\mu) \, P_\nu(\mu') \tag{1.29}$$

$$\text{where} \quad -1 \le \mu \le 1 \;, \quad -1 \le \mu' \le 1 \tag{1.30}$$

Busbridge [1960] discussed the solution of the homogeneous equation given by

$$J(\tau,\mu) = \Lambda_{\tau,\mu} \left\{ J(t,\mu') \right\} \tag{1.31}$$

which are atmost $O(\tau)$ as $\tau \longrightarrow \infty$, for conservative and non-conservative cases.

The solution for 'The auxiliary equation' given by

$$(1-\Lambda)_{\tau,\mu} \left\{J(t,\mu',\mu_o)\right\} = p(\mu, -\mu_o) \tag{1.32}$$

$$\exp(-\tau/\mu_o) \quad \text{where} \quad 0<\mu_o<1, \qquad -1\le \mu \le 1 \tag{1.33}$$

has also been discussed (vide, Busbridge, 1960) in terms of H-function. Finally, the law of diffuse reflection has been worked out .

Horak and Chandrasekhar [1970] considered the the problem in radiative transfer, parallel light of flux density πF_o is incident on a plane-parallel, semi-infinite atmosphere which scatters light in accordance with the phase function

$$p(\cos\vartheta) = \omega_o + \omega_1 P_1(\cos\vartheta) + \omega_2 P_2(\cos\vartheta) \tag{1.34}$$

where $\omega_o \le 1$ and ω_o (the albedo), ω_1, ω_2 are constants and P_1 and P_2 are Legendre polynomials. They have found out the exact and the details of the solution for the emergent radiation field by using the invariance principle method.

The diffuse reflection of light by a semi-infinite atmosphere scattering with phase function

$$1 + \omega_1 P_1(\cos\vartheta) + \omega_2 P_2(\cos\vartheta) \qquad (1.35)$$

has been delt by Horak and Janowsek [1965].

Orchard [1967] obtained the reflection and transmission of light by thick atmosphere of pure scattering with the same phase function. To obtain these Orchard (1967) applied exact radiative transfer theory to the case of a parallel light incident from an arbitrary direction on the non-absorbing plane parallel atmosphere of large optical thickness.

Busbridge and Orchard [1968] applied the same theory to find reflection and transmission of light by thick atmospheres of pure scattering with a phase function

$$1 + \sum_{n=1}^{N} \omega_n P_n(\cos\vartheta) \qquad (1.36)$$

Kolesov and Sobolev [1969] and Kolesov and Smoktii [1972] applied the general theory of anisotropic scattering developed by Sobolev to solve the problem of diffuse reflection and transmission of light by a semi-infinite atmosphere with a three and four term scattering indicatrix.

Kolesov [1971] discussed about H-function for some scattering indicatrices with different values of the asymmetry factor.

The asymptotic solution for the phase function $(1 + \omega \cos\vartheta)$ has been found out by Piotrowski [1955, 1956] using the method of discrete ordinates as developed by Chandrasekhar [1960]. At the same time, Piotrowski has found out the asymptotic value of the transmittance in the case of the phase function

$$\sum_{n=0}^{N} \omega_n P_n (\cos \vartheta) \tag{1.37}$$

but he was unable to obtain the limit of this, as the norm of the partition used for the Gauss quadrature tended to zero. Usugi and Irvine [1970a] computed reflection function for conservative isotropic scattering by the method of successive scattering . By the same method, Usugi and Irvine [1970b] derived basic formulae for the computation of line profiles and equivalent width of an absorption line. Usugi and Irvine [1968] showed that the absorption spectra can be computed in a model planetary atmosphere using the Newmann series solutions.

Uesegi, Irvine and Kawata [1971] showed that the diffuse reflection may be computed for arbitrary single scattering albedo if the reflection functions in the conservative case are known.

Mullikin [1964a] studied the transfer of radiation in homogeneous plane parallel atmosphere of finite and semi-infinite thickness for three different types of phase functions and computed the X- , Y- equations by additional linear constraints so that a unique pair of functions is specified by the requirement of analyticity in a half plane and transforemed the linear singular equations and linear constraints into suitable form for numerical computations.

For the semi-infinite atmosphere, Fredholm equations are solved exactly (vide, Mullikin, 1964a) to give a determination of the H-function in terms of simple quadratures.

Burniston and Siewert [1970] discussed a matrix version of the classical Riemann-Hilbert problem defined on an open contour. Finally as an illustration linear integral equation for Chandrasekhar's function $H_l(\mu)$ and $H_r(\mu)$ are established in a form enumarable to solution by numerical iteration. Bond and Siewert [1970] have computed the first twenty two moments of Chandrasekhar's function $H_l(\mu)$ and $H_r(\mu)$ related to the scattering of polarized light.

Carlstedt and Mullikin [1966] obtained equations needed to determine the X- and Y- functions firstly studied by Busbridge. Carlstedt and Mullikin [1966] also obtained asymptotic formulae for thick atmospheres uniformly valid for various Characteristic functions. All these equations are also applicable to Rayleigh phase functions. Domke [1971, 1972] solved radiative transfer equation with conservative Rayleigh scattering for both finite and semi-infinite atmosphere, based on Sobolev's method for arbitrary distribution of primary sourses.

Mullikin [1966a] has studied extensively and analytically and numerically the complete Rayleigh scattered field within a homogeneous plane-parallel atmosphere. The solution to this problem at any optical depth has been expressed in terms of scalar function for which there already exists an efficient and accurate computer programme. Various asymptotic formulae of a relatively simple form have been obtained from this solution.

Steady state multiple scattering problems for homogeneous plane parallel atmospheres have been extensively studied [Mullikin, 1966b] by means of the principle of invariance of Ambertsumian and Chandrasekhar. The purpose of that was to report on the results obtained from a fruitful combination of the linear and nonlinear theories. This analysis is applied to Rayleigh polarization scattering .

A study is made of the existence and uniqueness problems

[Mullikin, 1964b] for Chandrasekhar ψ_l and ϕ_l equations for radiative transfer in homogeneous atmosphere's with anisotropic scattering .

Mullikin [1963] reported on some recent mathematical studies concerning the uniqueness of solutions to Chandrasekhar's mathematical formulation of principle of invariance in the theory of Radiative Transfer . The uniqueness question for his ψ_l^m and ϕ_l^m equations has been studied.

Siewert and Burniston [1972] showed that a solution to the system of singular integral equations and the linear constraint which define mathematically the H-matrix relevant to the scattering of polarizated light can exist and are unique. Finally, Siewert and Burniston (1972) gave an explicit analytical result for the appropriate canonical matrix for conservative Rayleigh scattering. Hulst (1970) reduced the problems of radiative transfer with a general anisotropic phase functions completely to H-functions and two sets of polynomials known as the Kuščer polynomials and the Busbridge polynomials.

Hulst [1969] discussed some problems of anisotropic scattering in planetary atmospheres. Here the similarity rules to compare atmospheres with anisotropic and isotropic scattering were reviewed .

With the aid of the invariant imbedding technique, Bellman , Kagiwada, Kalaba and Ueno [1967] derived a complete set of integro-differential equations for the dissipation functions of an inhomogeneous finite slab with anisotropic scattering.

Siewert [1968] presented a new approach to develope Chandrasekhar's scattering matrix for a semi-infinite Rayleigh scattering atmosphere which can be used to determine the emergent angular distribution for any of the standard half space problems.

Siewert and Fralay [1967] solved the conservative Rayleigh scattering problem in a semi-infinite atmosphere by the application of the singular eigen function expansion technique. Bond and Siewert [1971] have studied the non-conservative equation of transfer for a combination of Rayleigh and isotropic scatter scattering.

Wallance [1972] presented a discussion on Rayleigh and Raman scattering by pure H_2 in a planetary atmosphere. Kuzmina [1970a, 1970b] discussed Milne's problem for polarized radiation scattered according to conservative and non-conservative Rayleigh's law.

Sobolev [1969a] investigated on diffuse reflection and transmission of light by an atmosphere with anisotropic scattering. Sobolev [1969b, 1970] also discussed on anisotropic light scattering in an atmosphere of finite optical thickness.

Kolesov and Sobolev [1969] discussed on some asymptotic formulae in the theory of anisotropic light scattering. Grinin [1971] discussed on the theory of non-stationary radiation transfer for anisotropic scattering by the application of the modified Sobolev's probability method. Pomraning (1969) formulated the modified Eddington's approximation proposed earlier for isotropic scattering for a general scattering law.

Stokes and De Marcus [1971] used variational principle for calculating line profiles of inhomogeneous planetary atmosphere.

Sekera and Ashburn [1953] , and Sekera and Blauch [1954] gave tables relating to Rayleigh scattering of light in the atmosphere. The extensive numerical results based on Chandrasekhar's analysis have been obtained for Rayleigh atmospheres with optical thickness ranging up to 1 (vide, Sekera, 1956, 1967 and vide, Sekera and Viezee, 1961).

Case and Zweifel [1967] treated isotropic scattering and some simple example of anisotropic transfer , based on the work of Mika and others . Formulations for general anisotropic scattering were presented by McCormik and Kuscer [1966] and in practical form by Shultis and Kaper [1969] and in full detail by Kaper, Shultis and Veninga [1970].

Chandrasekhar [1960] has considered the problem of radiative transfer with general anisotropic scattering in the Milne-Eddington model to obtain the exact form of emergent intensity from the bounding face and nth approximate intensity at any optical depth by discrete ordinates procedure assuming Planck's function to be linear in the optical depth. Das [1973] obtained an exact solution of this problem using the Laplace transform and Wiener-Hopf technique.

Das [1978,1980] has solved various problems of radiative transfer in finite and semi-infinite atmosphere using a method involving Laplace transform and linear singular operators.

Sobolev [1956] dealt with the one dimensional problem of time-dependent diffuse reflection and transmission by a probabilistic method.

Diffuse reflection of time-dependent parallel rays by a semi-infinite atmosphere was treated by Ueno [1962] on the basis of the principle of invariance. Bellman et al [1962] obtained an integral equation governing diffuse reflection of time dependent parallel rays from the lower boundary of a finite inhomogeneous atmosphere .

In recent years Karanjai and Talukdar (1991, 1992), Karanjai and Biswas (1992, 1993) and Roy Choudhury and Karanjai (1995a, 1995b) solved radiative transfer problems in anisotropically scattering media by spherical harmonic method using different approximate forms for the intensity. Ueno [1965] also obtained this equation by probabilistic method. Matsumoto [1967a] derived functional equations in the internal radiation field due to time-dependent incident radiation allowing for the time dependence given by Dirac's δ-function and Heaviside unit step function Matsumoto [1967b] also derived a complete set of functional equations for the scattering (S) and transmission (T) functions which govern the laws of diffuse reflection and transmission of time-dependent parallel rays by a finite , inhomogeneous,plane parallel, non-emitting and isotropically scattering atmosphere with incident radiation governed by Dirac's δ-function and Heaviside's unit step-function. A formulation of time-dependent H-function was accomplished by means of the Laplace transform in the time-domain. Numerical evaluation of the H-function based on numerical inversion of the Laplace transform presented by Bellman et.al [1966] was made.

Recently Karanjai and Biswas [1988] derived the time-dependent X- and Y-functions for homogeneous, plane parallel non-emitting and isotropic atmosphere of finite optical thickness using the integral equation method developed by Rybicki [1971]. Biswas and Karanjai [1990a] have derived the time-dependent H-, X- and Y- functions in a homogeneous atmosphere scattering anisotropically with Dirac's δ-function and heaviside unit step-function type time-dependent incidence. Biswas and Karanjai [1990b] have also derived the solution of diffuse reflection and transmission problem for homogeneous isotropic atmosphere of finite optical depth. The problem of the time-independent scattering and transmission of radiation in plane parallel atmosphere of two layers was treated first by Van de Hulst [1963], (vide, Tsujita, 1968). Hawking [1961] dealt with the problem analytically starting with Milne's integral equation. Gutshabad [1957] formulated the problem as solutions of simultaneous integral equations. So far as his equations are solvable , the scattering and transmission functions required are given exactly for two layers of different albedo and different large optical thickness.

In the theory of radiative transfer for homogeneous plane parallel stratified finite atmosphere the X– and Y– functions of Chandrasekhar [1960], play a central role. These equations satisfy a system of coupled non-linear integral equations. Busbridge [1960] has demonstrated the existence of the solutions of these coupled nonlinear integral equations in terms of a particular solution of an auxiliary equation. Busbridge [1960] has obtained two coupled linear integral equations for X(z) and Y(z) which defined the meromorphic extensions to the complex domain |Z| of the real valued solution of the coupled non-liner integral equations for X– and Y– functions are the solutions of the coupled linear integral equations. Mullikin [1964c] has proved that all solutions of coupled nonlinear integral equations are solutions of the coupled linear integral equation but there exists a unique solution of the coupled linear integral equations with some linear constraints. Finally Mullikin (1964c) has obtained the Fredholm equations of X– and Y– functions which are easy for iterative computations. Das [1979] has obtained a pair of the Fredholm equations with Wiener–Hopf technique from the coupled linear integral equations with coupled linear constraints. The transport equation for the intensity of radiation in a semi-infinite atmospheres with no incident radiation and scattering according to the planetary phase function $\omega(1 + x \cos \vartheta)$ has been considered. This equation has been solved by Chandrasekhar [1960] using his principle of invariance to get the emergent radiation.

The singular eigen function approach of Case [1960] is also applied to get the intensity of radiation at any optical depth. Boffi [1970] has also applied the two sided Laplace transform to get the emergent intensity and the intensity at any optical depth. Das [1979] solved exactly the equation of transfer for scattering albedo $\omega < 1$ using Laplace transform and the Wiener-Hopf technique and also deduced the intensity at any optical depth by inversion.

In the study of the time-dependent radiative transfer problem in finite homogeneous plane-parallel atmosphere, it is convenient to introduce X- and Y- functions [1960]. These functions satisfy non-linear coupled integral equations . Due to their important role in solving transport problems , it is advantageous to simplify the equations satisfied by them. Lahoze [1989] did this andobtained exact linear and decoupled integral equations satisfies by the time-independent X- and Y- functions .

1.2 SUMMARY OF WORK DONE.

The present thesis is concerned with the solution of some scattering problems of Radiative Transfer. The work presented in chapter 2 is concerned mainly with the solution of scattering problems by the method based on " Laplace Transform and Wiener-Hopf technique " and " Principle of Invariance " .

The transport equation for the intensity of radiation in a semi-infinite atmosphere with no incident radiation and scattering according to the planetary phase function $\omega(1 + x \cos\theta)$ has been solved exactly by a method based on the use of laplace transform and Wiener-Hopf technique. in section 2.2. The exact solution of the transfer equation with three-term scattering indicatrix in an exponential atmosphere is obtained by the same method in section 2.3. The matrix transform equation for a scattering which scatters radiation in accordance with the phase matrix obtained from a combination of Rayleigh and isotropic scattering in a semi-infinite atmosphere has been solved in section 2.5 by the same method . The basic matrix equation is subject to the Laplace transform to obtain an integral equation for the emergent intensity matrix. On application of the Wiener-Hopf technique this matrix integral equation gives the emergent intensity matrix in terms of a singular H-matrix and an unknown matrix. The unknown matrix has been obtained by equating the asymptotic solution of the boundary condition at infinity.

The equation of transfer for a semi-infinite plane parallel atmosphere with no incident radiation and for the scattering according to the conservative anisotropic phase function has been solved by the method of " Principle of Invariance " and using the law of diffuse reflection in **section 2.4.** In **section 2.6** the nonlinear integral equations for X- and Y-functions (vide , Chandrasekhar, 1960) for anisotropically scattering atmosphere has been derived. The anisotropy is represented by means of a phase function which can be expressed in terms of finite-order Legendre Polynomials.

The principle of invariance is applied to derive the functional equations for time-dependent diffuse reflection and transmission function. Next I consider the time dependent diffuse reflection and transmission of plane parallel rays by a slab consisting of two homogeneous anisotropically scattering layers, whose scattering and transmission functions are known

In chapter 3 the equation of transfer has been solved by different methods Viz.,

(i) Eddington's Method (Sec-3.2).

(ii) Laplace transform and Wiener-Hopf technique (Sec-3.3).

(iii) Busbridge's Method (Sec-3.4).

(iv) Discrete Ordinates (Sec-3.5).

in an isotropic coherently scattering atmosphere with exponential Planck function (equation (1.11)).

In chapter 4 the equation of transfer for interlocked multiplets, has been solved by the discrete ordinate method and by the method used by Busbridge and Stibbs [1954] using Planck function as an exponential function of optical depth in sections 4.2 and 4.3 respectively. Four approximate forms of H-function (vide, Karanjai and Sen, 1970, 1971) has been used to calculate the residual intensities for doublets and triplets in section 4.4. and the concerned results has been shown in both tables and figures.

In chapter 5 the one sided Laplace transform together with the theory of linear singular operators has been applied to solve the transport equation which arises in the problem of a finite atmosphere having ground reflection according to Lambert's Law taking the Planck's function as an exponential function of optical depth (Sec-5.2).

The time-dependent X- and Y- functions (Biswas and Karanjai, 1990) which gives rise to a pair of the Fredholm equations with the application of the Wiener-Hopf technique has been obtained in section 5.3. These Fredholm equations define time-dependent X-functions in terms of time-dependent Y-functions and vice-versa. These representations are unique with respect to the coupled linear constraints defined by Mullikin (1964a). An exact linearized and decoupled integral equation satisfied by Time-Dependent X- and Y- function has been obtained using the method used by Lahoz (1989) in section 5.4.

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CHAPTER - 2

SOLUTION OF RADIATIVE TRANSFER PROBLEMS IN AN ATMOSPHERE SCATTERING ANISOTROPICALLY

2.1. INTRODUCTION .

The transport equation for the intensity of radiation in a semi-infinite atmosphere with no incident radiation and scattering according to the phase function $\omega(1 + x \cos\theta)$ has been solved by Chandrasekhar [1960] using his principle of invariance to get emergent radiation. The singular eigen function approach of case [1960] is also applied to get the intensity of radiation at any optical depth. Boffi [1970] has also applied the two sided Laplace Transform to get the emergent intensity and the intensity at any optical depth. In the present work , the above problem has been solved exactly by a method based on the use of laplace transform and Wiener-Hopf technique.(Section 2.2).

Chandrasekhar [1960] has considered the problem of radiative transfer with general anisotropic scattering in the Milne-Eddington model to obtain the exact form of intensity from the bounding face and nth approximate intensity at any optical depth. Das [1979a] obtained an exact solution of this problem using the same method as applied in section 2.2. Wilson and Sen [1964] solved the same problem by a modified S.H.M.. In the present work, the exact solution for emergent intensity from the boundary surface is obtained using exponential form of Planck function (equation 1.11), in Section. 2.3. by the same method as in

Section 2.2.

In the case of axially symmetric about the normal to the plane of stratification , where the intensity and source function are azimuth independent the equation of transfer in the standard form for a plane-parallel atmosphere with a constant net flux can be written (vide, Chandrasekhar, 1960) as

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - (1/2)\int_{-1}^{+1} p^{(0)}(\mu,\mu')\, I(\tau,\mu')\,d\mu' \qquad (2.1)$$

where $I(\tau,\mu)$ stands for specific intensity at an optical depth τ, and $\mu = \cos\vartheta$, ϑ being the inclination to the outward normal , and $\omega = \int p(\cos\vartheta)\frac{d\omega'}{4\pi}$, being the albedo for single scattering and

$$p^{(0)}(\mu,\mu') = (1/2\pi)\int_{0}^{2\pi} p(\mu,\phi;\mu',\phi')\, I(\tau,\mu')\,d\mu' \qquad (2.2)$$

$p(\mu,\phi;\mu',\phi')$ being the phase function for the angle between the directions (ϑ,ϕ) and (ϑ',ϕ'). In the equation of transfer (equation (2.1)) the normal optical thickness τ is measured from the boundary surface inwards.

Based on the invariance principle method which Chandrasekhar [1960] and Kourganoff [1963] have discussed extensively, Horak and Chandrasekhar [1961] have solved the equation of transfer for a semi-infinite plane parallel atmosphere with a phase function

$$p(\cos) = \omega + \omega_1 p_1(\cos\vartheta) + \omega_2 p_2(\cos\vartheta) \quad (\omega \leq 1) \qquad (2.3)$$

where ω (the albedo) ω_1, and ω_2 are constants and P_1 and P_2 are Legendre polynomials.

The invariance principle method has also been applied by Chandrasekhar [1960] to find the angular distribution of emergent radiation in a semi-infinite atmosphere with no incident radiation and for scattering according to

(i) the isotropic non-conservative case

(Vide, Chandrasekhar , 1960, P. 344) and

(ii) phase function $\omega(1 + x\cos\vartheta)$ (Vide, Chandrasekhar, 1960)

Karanjai and Baraman [1974] solved the same problem for Rayleigh scattering phase function.

In the present work , the same problem for the scattering according to the phase function

$$P(\cos\vartheta) = 1 + \omega_1 p_1(\cos\vartheta) + \omega_2 p_2(\cos\vartheta) \qquad (2.4)$$

has been solved by the method of " Principle of Invariance " and using the law of diffuse reflection (Section 2.4)

Chandrasekhar [1960] has considered the problem of the basic non-conservative matrix equation of radiative transfer for diffuse reflection for a combination of Rayleigh and isotropic scattering in a semi-infinite atmosphere. Schnatz and Siewert [1970] have obtained the exact solution of the basic transport equations for non-conservative Rayleigh phase matrix by the eigen function approach of Case [1960]. Bond and Siewert [1971] have obtained a rigorous general solution of a non-conservative matrix equation of transfer , which appears for consideration of polarization by the eigen function approach of Case [1960]. Das [1979b] solved the basic integro-differential equation for Radiative transfer in diffuse reflection in a combination of Rayleigh and isotropic scattering for a semi-infinite atmosphere exactly for the emergent intensity matrix by the method as in section 2.2.

In the present work , the matrix transform equation for a scattering which scatters radiation in accordance with the phase matrix obtained from a combination of Rayleigh and isotropic scattering in a semi-infinite atmosphere has been solved (Sec 2.5) by the same method as in sec.2.2. The basic matrix equation is subject to the Laplace transform to obtain an integral equation for the emergent intensity matrix. On application of the Wiener-Hopf technique this matrix integral equation gives the emergent intensity matrix in terms of a singular H-matrix and an unknown matrix. The unknown matrix has been obtained by equating the asymptotic solution of the boundary condition at infinity.

Sobolev [1976] dealt with the one dimensional problem of time-dependent diffuse reflection and transmission by the probabilistic method. Diffuse reflection of time-dependent parallel rays by a semi-infinite atmosphere was treated by Ueno [1962] on the basis of the principle of invariance. Bellman et al [1962] obtained an integral equation governing diffuse reflection of time dependent parallel rays from the lower boundary of a finite inhomogeneous atmosphere . Ueno [1965] also obtained this equation by probabilistic method. Matsumoto [1967a] derived functional equations in the integral radiation allowing for the time dependence given by Dirac's δ-function and Heaviside unit step-function. Matsumoto [1967b] also derived a complete set of functional equations for the scattering (S) and transmission (T) functions which govern the laws of diffuse reflection and transmission of time-dependent parallel rays by a finite , inhomogeneous, plane parallel, non emitting, and isotropic scattering atmosphere where the dependence of the time of the incident radiation is given by Dirac's δ-function and Heaviside's unit step-function . A formulation of time-dependent H-function was accomplished by means of the Laplace transform in the time-domain. Numerical evaluation of the H-function based on numerical inversion of the Laplace transform presented by Bellman et.al [1966] was made.

Recently Karanjai and Biswas [1988] derived the time-dependent X- and Y-functions for homogeneous, plane parallel non-emitting and isotropic atmosphere of finite optical thickness using the integral equation method developed by Rybicki [1971]. Biswas and Karanjai [1990a] have derived the time-dependent H-, X- and Y- functions in a homogeneous atmosphere scattering anisotropically with Dirac's δ-function and heaviside unit step-function type time-dependent incidence. Biswas and Karanjai [1990b] have also derived the solution of diffuse reflection and transmission problem for homogeneous isotropic atmosphere of finite optical depth.

In section (2.6) I derived the nonlinear integral equations for X- and Y-functions (vide, Chandrasekhar, 1960) for anisotropically scattering atmosphere. The anisotropy is represented by means of a phase function which can be expressed in terms of finite-order Legendre Polynomials . The principle of invariance is applied to derive the functional equations for time-dependent diffuse reflection and transmission function. Next I consider the time dependent diffuse reflection and transmission of plane parallel rays by a slab consisting of two homogeneous anisotropically scattering layers, whose scattering and transmission functions are known. The problem of the time-independent scattering and transmission of radiation in plane-parallel atmosphere of two layers was treated first by Van de Hulst (1963; vide, Tsujita, 1968). Hawking [1961] dealt with the problem analytically starting with Milne's integral equation. Gutshabash [1957] formulated the problem as solutions of simultaneous integral equations. So far as his equations are solvable, the scattering and transmission functions required are given exactly for two layers of different albedos and large optical thickness. In the present work , the same problem has been extended (vide, Tsujita, 1968) for the time-dependent transfer of radiation (section 2.6).

2.2 Exact Solution of the Equation of Transfer with Planetary Phase Function.

2.21. Basic Equation and its Solution.

The equation of transfer appropriate to the problem (vide, Chandrasekhar, 1960) is

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\omega\int_{-1}^{+1} I(\tau,\mu')(1 + x\mu\mu')\, d\mu' , \qquad (2.5)$$

where the symbols have their usual meaning.

I shall have the following boundary conditions

$$I(0,-\mu) = 0, \qquad 0 < \mu < 1 \; ; \tag{2.6}$$

$$I(\tau,\mu) \longrightarrow L_o \exp(k\tau)\, \frac{1 + x(1-\omega)(\mu/k)}{1 - k\mu}, \quad \text{as } \tau \longrightarrow \infty \tag{2.7}$$

where Lo is a constant and k is the positive root, less than 1, of the transcendental equation.

$$1 = \frac{\omega}{2k}\left[1 + \frac{x(1-\omega)}{k^2}\right]\log\left(\frac{1+k}{1-k}\right) - \frac{1}{k^2}\, x\omega(1-\omega) \tag{2.8}$$

Let us define

$$f^*(s) = s\int_0^\infty \exp(-s\tau) f(\tau)\, d\tau \quad , \qquad \text{Rl } s > 0 \tag{2.9}$$

Let us set

$$I_m(\tau) = \frac{1}{2}\int_{-1}^{+1} I(\tau,\mu')\mu'^m\, d\mu' \qquad \text{where } m = 0,1. \tag{2.10}$$

which gives

$$I_0^*(s) = \frac{1}{2}\int_{-1}^{+1} I^*(s,\mu')\, d\mu' \tag{2.11}$$

and

$$I_1^*(s) = \frac{1}{2}\int_{-1}^{+1} I^*(s,\mu')\mu'\, d\mu' , \tag{2.12}$$

Equation (2.5) with equation (2.10) takes the form

$$\mu\, \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \omega I_0(\tau) - \omega x\mu I_1(\tau) \tag{2.13}$$

Now, subjecting equation (2.13) to the Laplace transform as define in equation (2.9), I have, using the boundary conditions,

$$(\mu s - 1)\, I^*(s,\mu) = \mu s I(0,\mu) - \omega I_0^*(s) - \omega x\mu I_1^*(s) \tag{2.14}$$

Equation (2.14) gives (on putting $s = 1/\mu$)

$$I(0,\mu) = \omega I_o^*(1/\mu) + \omega x\mu I_1^*(1/\mu) \qquad (2.15)$$

Equation (2.15) with $\mu = 1/s$, s is complex, takes the form

$$I(0,1/s) = \omega I_o^*(s) + \omega x s^{-1} I_1^*(s) \qquad (2.16)$$

I apply the operator

$$\frac{1}{2}\int_{-1}^{+1} \ldots\ldots d\mu \quad , \qquad (2.17)$$

on both sides of equation (2.14) to get

$$I_1^*(s) - (1-\omega)\, s^{-1} I_o^*(s) = \frac{1}{2}\int_0^1 \mu I(0,\mu)d\mu \qquad (2.18)$$

I apply the operator

$$\frac{1}{2}\int_{-1}^{+1} \ldots \frac{d\mu}{\mu s - 1} \qquad (2.19)$$

$$a(1/s) = 1 + \omega t_o(1/s) + \omega x t_1(1/s) I_1^*(s) \qquad (2.20)$$

where

$$a(1/s) = \frac{1}{2}\int_0^1 \frac{\mu s}{\mu s - 1}\, I(0,\mu)d\mu \qquad (2.21)$$

and

$$t_m(1/s) = \frac{1}{2}\int_{-1}^{+1} (\mu s - 1)^{-1}\mu^m d\mu\,, \quad m = 0,1 \qquad (2.22)$$

Eliminating $I_o^*(s)$, $I_1^*(s)$ among equations (2.16), (2.18) and (2.20) and setting $s = 1/z$, I have

$$T(z)I(0,z) = \frac{\omega}{2}\int_0^1 \frac{\mu}{\mu - z}\left[1 + \mu x(1-\omega)z\right] I(0,\mu)\, d\mu\,, \qquad (2.23)$$

where

$$T(z) = 1 + \omega x(1-\omega)z^2 + \omega[1 + x(1-\omega)z^2]t_o(z)\,, \qquad (2.24)$$

where

$$t_0(z) = \frac{z}{2}\int_{-1}^{+1} \frac{d\mu}{\mu - z}, \tag{2.25}$$

Following Chandrasekhar [1960] and considering equation (2.8), I see that T(z) has a pair of roots at $z = \pm k^{-1}$ and

$$T(z) = \frac{1}{H(z)H(-z)}, \qquad z \in (-1,1)^c, \tag{2.26}$$

where H(z) is Chandrasekhar's H-function for planetary scattering. Equation (2.23) with equation (2.26) takes the form

$$\frac{I(0,z)}{H(z)} = H(-z)\frac{\omega}{2}\int_0^1 \frac{\mu}{\mu - z}[1 + \mu x(1 - \omega)z]I(0,\mu)d\mu, \tag{2.27}$$

Equation (2.27) can be written as

$$\frac{I(0,z)}{H(z)} = H(-z)\omega\, G(z), \tag{2.28}$$

where

$$G(z) = \frac{1}{2}\int_0^1 \frac{\mu}{\mu - z} [1 + \mu x(1 - \omega)z]I(0,\mu)d\mu. \tag{2.29}$$

Let us seek solution I(0,z) of equation (2.27) by Wiener-Hopf technique on the assumption that I(0,z) is regular for Rl z>0 and bounded at the origin. equation (2.28) with the above assumption on I(0,z) gives the following properties of G(z); G(z) is regular on $(0,1)^c$, bounded at the origin and a constant as $z \longrightarrow \alpha$. Equation (2.28) then gives

$$\frac{(1 - kz)I(0,z)}{H(z)} = \omega(1 - kz)H(-z)G(z), \tag{2.30}$$

where H(-z), H(z), 1/H(z) has the following properties; H(z) is regular for $z \in (-1,0)^c$, uniformly bounded at the origin has a simple pole at $z = -(1/k)$, $k<1$; k is real on the negative real axis and bounded at infinity and tends to

$$H_0 + H_{-1} z^{-1} + H_{-2} z^{-2} + \ldots \text{ when } z \longrightarrow \alpha .$$

Hence, $1/H(z)$ is regular for z in $(-1,0)^c$ and bounded at the origin. Similarly, $H(-z)$ is regular for $z \in (0,1)^c$, uniformly bounded at the origin, has a simple pole at $z = 1/k$, $k<1$; k is real , on the positive side of the real axis and bounded at infinity and tends to

$$H_0 - H_{-1} z^{-1} - H_{-2} z^{-2} - \ldots \quad \text{when} \quad z \longrightarrow \alpha .$$

Following properties of $H(z)$, $1/H(z)$, $H(-z)$ (vide, Busbridge, 1960) the left hand side of equation (2.30) is regular for Rl $z>0$, bounded at the origin and the right hand side of equation (2.30) is regular for $z \in (0,1)^c$ and bounded at the origin and tends to a polynomial say $A + Bz$, as $z \longrightarrow \alpha$. Hence by a modified form of Liouville's theorem

$$\frac{(1 - kz)I(0,z)}{H(z)} = A + Bz, \quad \text{when } z \in (-1,0)^c \quad (2.31)$$

and

$$A + Bz = \omega(1 - kz)H(-z)G(z), \quad \text{when} \quad z \in (0,1)^c. \quad (2.32)$$

Equation (2.31) gives the emergent radiation as

$$I(0,z) = \frac{(A + Bz)H(z)}{1 - kz}, \quad (2.33)$$

where the constants A and B are two arbitrary constants to be determined later on.

2.22. Intensity at any Optical Depth

The radiation intensity at an optical depth τ is given by the inversion integral as

$$I(\tau,\mu) = (1/2\pi i) \lim_{\delta \to \alpha} \int_{c-i\delta}^{c+i\delta} \exp(s\tau)\, I^{*}(s,\mu)ds, \quad c>0 \qquad (2.34)$$

Equation (2.14) with equation (2.16) takes the form

$$I^{*}(s,\mu)/s = \phi(s,\mu)/(s - 1/\mu), \qquad (2.35)$$

where

$$\phi(s,\mu) = I(0,\mu) - I(0,1/s) + \frac{\omega(s - 1/s)}{s} I_{o}^{*}(s). \qquad (2.36)$$

But

$$\lim_{s \to 1/\mu} (\mu - 1/\mu) I^{*}(s,\mu) \exp(s\tau)/s \longrightarrow 0 \qquad (2.37)$$

Hence the integral of equations (2.34) is regular for $s \in (-\alpha,-1)^{c}$ and has simple pole at $s = \pm k$, $k < 1$.

Hence by Cauchy's residue theorem, equation (2.34) gives

$$I(\tau,\mu) = R_{p} + \lim_{R \to \alpha} (1/2\pi i) \int_{\Gamma} I^{*}(s,\mu) e^{s\tau}\, ds/s, \qquad (2.38)$$

where R_p is the sum of the residues of the poles at $s = \pm k$ and $\Gamma = \Gamma_1 \cup CD \cup \nu \cup EF \cup \Gamma_2$.

Γ_1 and Γ_2 are arcs of the circle of radius R having centre at $s = 0$ (clockwise) and ν is an arc of a small circle of radius r having centre at $s = -1$ (anticlockwise) and CD and EF are the lower edge and upper edge of the singular line $(-R, -1)$ (Figure 2.1). Hence, following Kourganoff (1960) I have

$$\int_{\Gamma_1 \cup \Gamma_2} I^*(s,\mu)\exp(s\tau)ds/s \longrightarrow 0, \text{ when } R \longrightarrow \alpha \tag{2.39}$$

and

$$\int_{\nu} I^*(s,\mu)\exp(s\tau)ds/s \longrightarrow 0, \quad \text{when } r \longrightarrow 0. \tag{2.40}$$

Hence in the limit of $R \longrightarrow \alpha$, $r \longrightarrow 0$, equation (2.38) with equation (2.39) and (2.40) becomes

$$I(\tau,\mu) = R_p + (1/2\pi i)\int_{CD} I^*(s,\mu)e^{s\tau} \; ds/s +$$

$$+ (1/2\pi i)\int_{EF} I^*(s,\mu)e^{s\tau} \; ds/s. \tag{2.41}$$

Here on CD and EF,

$$s = -v, \qquad v \geq 1$$

and on CD,

$$H(1/s) = \frac{X(1/v) + i\pi Y(1/v)}{H(1/v)Z(1/v)} \tag{2.42}$$

and on EF

$$H(1/s) = \frac{X(1/v) - i\pi Y(1/v)}{H(1/v)Z(1/v)} \tag{2.43}$$

where $X(1/v) = 1 + \omega x(1-\omega)v^{-2} - \omega\left[1 + x(1-\omega)v^{-2}\right] \times$

$$\times \; (1/2v) \log\left(\frac{v+1}{v-1}\right) \tag{2.44}$$

$$Y(1/v) = (\omega/2)\; v^{-1} \tag{2.45}$$

$$Z(1/v) = (X^2(1/v) + \pi^2 Y^2(1/v, \mu)) \tag{2.46}$$

Therefore on CD

$$\phi\ (s,\mu) = V(1/v) - i\pi W(1/v,\mu) \tag{2.47}$$

and on EF

$$\phi\ (s,\mu) = V(1/v,\mu) + i\pi W(1/v,\mu) \tag{2.48}$$

where

$$V(1/v,\mu) = I(0,\mu) - \left[\frac{(B - vA)Y(1/v)}{(v + k)\ H(1/k)Z(1/v)}\right] \times$$

$$\times \left[1 + \frac{v + 1/\mu}{1 + x(1-\omega)/v^2}\right] + \frac{(v + 1/\mu)\ \omega\ \alpha_1/2}{1 + x(1-\omega)/v^2} \tag{2.49}$$

$$W(1/v,\mu) = \left[\frac{(B - vA)Y(1/v)}{(v + k)H(1/k)\ Z(1/v)}\right] \times$$

$$\times \left[1 + \frac{v + 1/\mu}{1 + x(1-\omega)/v^2}\right] \tag{2,50}$$

Now, equation (2.39) with equations (2.35), (2.41), (2.47) and (2.48) gives

$$I(\tau,\mu) = R_p - \frac{1}{2\pi i}\int_1^{\alpha} \frac{\{V(1/v,\mu) - i\pi W(1/v,\mu)\}}{v + 1/\mu} e^{-v\tau} dv +$$

$$+ \frac{1}{2\pi i}\int_1^{\alpha} \frac{V(1/v,\mu) + i\pi W(1/v,\mu)}{v + 1/\mu} e^{-v\tau} dv. \tag{2.51}$$

Hence when $\mu > 0$, equation (2.51) give

$$I(\tau,\mu) = R_p + \int_0^{\alpha} W(1/v,\mu) e^{-v\tau} dv/(v + 1/\mu), \tag{2.52}$$

where $\mu < 0$, I shall assume that

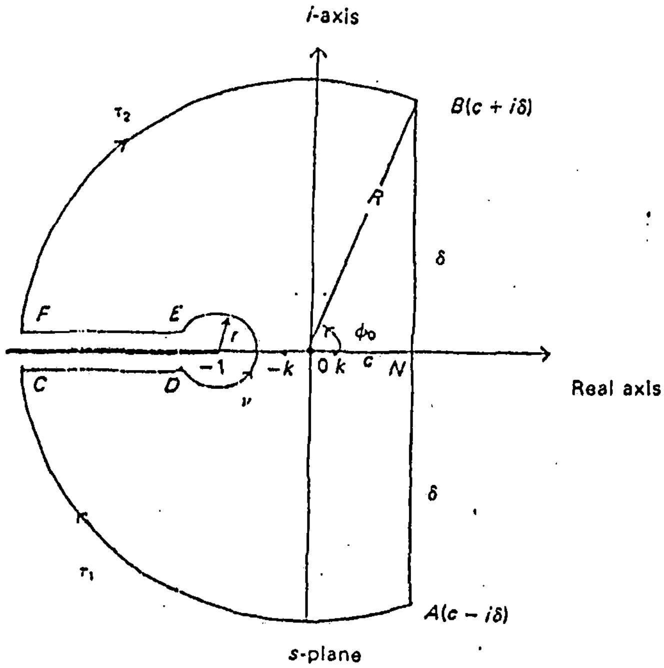

Fig. 2.0. The s-plane

$(V(1/v,\mu) \pm i\pi W(1/v,\mu))e^{-v\tau}$ satisfies Hölder condition on $(1,\alpha)$ and I have by Plemelj's formula (vide, Muskhelishvili, 1946)

$$\frac{1}{2\pi i}\int_1^{\alpha} \frac{V(1/v,\mu) \pm i\pi W(1/v,\mu)}{v + 1/\mu} e^{-v\tau} dv = \pm \frac{1}{2} (V(-\mu,\mu) \pm$$

$$\pm i\pi W(-\mu,\mu)e^{\tau/\mu} + \frac{1}{2\pi i} P\int_1^{\alpha} \frac{V(1/v,\mu) \pm i\pi W(1/v,\mu)}{v + 1/\mu} e^{-v\tau} dv, \quad (2.53)$$

where P denotes the Cauchy principal value of the integral. Hence equation (2.51) with equation (2.53) for $\mu < 0$ gives

$$I(\tau,\mu) = R_p + V(-\mu,\mu)e^{\tau/\mu} + P\int_1^{\alpha} \frac{W(1/v,\mu)e^{-v\tau}}{v + 1/\mu} dv, \quad (2.54)$$

where
$$R_p = R_k + R_{-k}, \quad (2.55)$$

where, $R_{\pm k}$ is the residue of the integral in equation (2.37) at $s = \pm k$, and R_k is given by

$$R_k = \lim_{s \longrightarrow k} (s - k) I^*(s,\mu)e^{s\tau}/s,$$

$$= \lim_{s \longrightarrow k} \frac{H(1/s)(As + B)s}{\{s^2 + x(1 - \omega)\}(1 - s\mu)} [1 + x(1 - \omega)/s]e^{s\tau}$$

$$= \frac{H(1/k)(Ak + B)k}{[k^2 + x(1 - \omega)](1 - k\mu)} [1 + x(1 - \omega)/k]e^{s\tau} \quad (2.56)$$

Similarly, R_{-k} is given by

$$R_{-k} = \lim_{s \longrightarrow (-k)} (s + k)I^*(s,\mu)\, e^{s\tau}/s$$

$$= \lim_{s \to (-k)} \frac{(s + k)H(1/s)(As + B)s}{(s - k)\{s^2 + x(1 - \omega)\}(1 - s\mu)} [1 + x(1-\omega)/s]e^{s\tau}$$

$$= \frac{(B - Ak)k[1 - x(1 - \omega/k)]e^{-k\tau}}{2k\{k^2 + x(1 + \omega)\}(1 - s\mu)} \lim_{s \to (-k)} (s + k)/T(1/s) \quad (2.57)$$

$$= \frac{(B - Ak)[1 - x(1 - \omega)k\} e^{-k\tau}}{2\{k^2 + x(1 - \omega)\}(1 + k\mu)} [dT(1/s)/ds]^{-1}_{s=-k} \quad (2.58)$$

2.23. Determination of constants A and B.

When $z \longrightarrow 0$, from equation (2.32) I get

$$A = (\omega/2)\int_0^1 I(0,\mu)d\mu. \quad (2.58)$$

From equation (2.58) and equation (2.32) I get after simplification

$$A[1 - \frac{\omega}{2}\int_0^1 \frac{H(\mu)d\mu}{1 - k\mu}] = \frac{\omega B}{2k}[-\alpha_0 + \int_0^1 \frac{H(\mu)d\mu}{1 - k\mu}] = m, \quad (2.59)$$

where $\alpha_0 = \int_0^1 H(\mu)d\mu$, m = constant .

$H(z)$ has a simple pole at $z = -(1/k)$ where

$$1/H(z) = 1 - zH(z)\int_0^1 \frac{\psi(z)\ H(\mu)d\mu}{\mu + z}, \quad (2.60)$$

where

$$\psi(\mu) = -\frac{\omega}{2}[1 + x(1 - \omega)\mu^2] \quad (2.61)$$

Equation (2.60) has a zero at $z = -(1/k)$ and so

$$1 + \frac{1}{k}\int_0^1 \frac{\psi(\mu)H(\mu)d\mu}{\mu - 1/k} = 0 \qquad (2.62)$$

In equation (2.62) putting the value of $\psi(\mu)$ and simplifying and using equation (2.59) I get

$$A = \frac{2mN}{kQ} \Big/ \left[\frac{x(1-\omega)}{k} - c \right], \quad B = \frac{2mN}{Q(k+c)}, \qquad (2.63)$$

where

$$N = k^2 + x(1-\omega), \quad Q = 2 - \omega\alpha_0, \quad c = \frac{x\omega(1-\omega)\alpha_1}{Q}$$

then

$$A + B\mu = \frac{2mN}{QR}\left\{ (1 + \frac{c}{k}) + (\frac{x(1-\omega)}{k} - c) \right\} \qquad (2.64)$$

Putting

$$\mu = 1/k \quad \text{I get} \quad kA + B = \frac{2mN^2}{QkR} \qquad (2.65)$$

where

$$R = \left\{ \frac{x(1-\omega)}{k} - c \right\}(k + c) \qquad (2.66)$$

If I use equations (2.65) and (2.66) I get from equation (2.33)

$$I(0,\mu) = \frac{(kA + B)k}{k^2 + x(1-\omega)} \left[\left(1 + \frac{c}{k}\right) + \left\{ \frac{x(1-\omega)}{k} - c \right\} \mu \right] \times$$

$$\times \frac{H(\mu)}{1 - k\mu} \qquad (2.67)$$

when $\tau \longrightarrow \infty$. From equations (2.54), (2.55) and (2.56) I get

$$I(\tau,\mu) \longrightarrow \frac{H(1/k)(Ak + B)k}{[k^2 + x(1-\omega)](1 - k\mu)}[1 + x(1-\omega)/k]e^{k\tau} \qquad (2.68)$$

Hence equation (2.68) with equation (2.7) gives

$$\frac{(Ak + B)k}{k^2 + x(1 - \omega)} = \frac{L_o}{H(1/k)} , \quad (2.69)$$

$$I(0,\mu) = \frac{L_o}{H(1/k)} \left\{1 + \frac{c}{k} + \mu\left[\frac{x(1 - \omega)}{k} - c\right]\right\} \frac{H(\mu)}{1 - k\mu} , \quad (2.70)$$

which is the expression obtained by Chandrasekhar [1960].

2.3. An Exact Solution of the Equation of Transfer with Three-Term Scattering Indicatrix in An Exponential Atmosphere .

2.31. Basic Equation and Boundary Conditions.

The equation of transfer in a stellar atmosphere can be written (vide, Chandrasekhar 1960; Das, 1979c) as

$$\mu \frac{dI_\nu(x,\mu)}{\rho dx} = (k_\nu + \sigma_\nu)I_\nu(x,\mu) - (1/2)\sigma_\nu (1 - \varepsilon_\nu) \int_{-1}^{+1} p(\mu,\mu')$$

$$\times I_\nu(x,\mu')d\mu' - (k_\nu + \varepsilon_\nu\sigma_\nu) B_\nu(T) \quad (2.71)$$

where

$$P(\mu,\mu') = \sum_{l=0}^{2} W_l P_l(\mu)P_l(\mu') \quad (2.72)$$

is the phase function for non-conservative scattering with a three-term indicatrix ; $I_\nu(x,\mu)$, the specific intensity in the direction arc cosμ at a depth x ; k_ν, the absorption coefficient ; arc cosμ is being measured from outward drawn normal to the face $x = 0$; σ_ν , the scattering coefficient ;ρ , the density of the atmosphere ; $B_\nu(T)$, Planck's function

; T, the local temperature at a depth x ; ε_ν , the collision constant and ν, the frequency. I define the optical depth t_ν in terms of the scattering and absorption coefficient and the optical depth τ_ν in terms of the absorption coefficient;

$$t_\nu = \int_x^\infty (k_\nu + \sigma_\nu)\rho \, dx \quad , \tag{2.73}$$

$$\tau_\nu = \int_x^\infty k_\nu \rho \, dx \quad , \tag{2.74}$$

with
$$dt_\nu = -(k_\nu + \sigma_\nu)\, \rho \, dx \; , \tag{2.75}$$

$$d\tau_\nu = -k_\nu \, \rho \, dx \tag{2.76}$$

If I follow Degl'Innocenti [1979] and karanjai and Karanjai [1985] I can take

$$B_\nu(\tau_\nu) = B_\nu^{(0)} + B_\nu^{(1)} e^{-\alpha\tau_\nu} \tag{2.77}$$

where $B_\nu^{(0)}$, $B_\nu^{(1)}$ and α are three positive constants. Hence , equation (2.77) with equations (2.75) and (2.76) becomes

$$B_\nu(t_\nu) = b_0 + b_1 e^{-\beta t_\nu} \tag{2.78}$$

where $b_0 = B_\nu^{(0)}$; $b_1 = B_\nu^{(1)}$ and $\beta = \alpha k_\nu \,/\, (k_\nu + \sigma_\nu)$ (2.79)

In this model I shall assume that

$$\eta_\nu = (k_\nu + \sigma_\nu)^{-1} \tag{2.80}$$

is constant with optical depth. Equation (2.71) with equations (2.73) and (2.78) becomes

$$\mu dI(t,\mu)/dt = I(t,\mu) - (1-c_o/\omega_o)B(t) - (1/2)\int_{-1}^{+1} \times$$

$$\times (c_o + c_1\mu\mu' + \frac{1}{4}c_2(3\mu^2 - 1)(3\mu'^2 - 1)I(t,\mu')d\mu' \quad (2.81)$$

where c_o, c_1, and c_2 are given by

$$c_o/\omega_o = c_1/\omega_1 = c_2/\omega_2 = \sigma(1 - \varepsilon)(k + \sigma) \quad (2.82)$$

and for convenience , I have omitted the subscript ν . For the solution of equation (2.81) I have the boundary conditions

$$I_\nu(0, -\mu') = 0 \quad , \quad 0 \leq \mu' \leq 1 \quad (2.83)$$

$$\text{and} \quad I_\nu(t,\mu')\, e^{-t/\mu'} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \alpha \quad (2.84)$$

2.32. Solution for Emergent Intensity :

The Laplace transform of F(t) is denoted by $F^*(s)$, where $F^*(s)$ is defined by

$$F^*(s) = s\int_0^\alpha \exp(-st)F(t)dt, \quad \text{Re } s>0 \;; \quad (2.85)$$

and I set

$$I_m(t) = (1/2)\int_{-1}^{+1} \mu^m\; I_m^*(s,\mu)d\mu, \quad m = 1,\ 1,\ 2. \quad (2.86)$$

which implies that

$$I_m^*(s) = (1/2)\int_{-1}^{+1} \mu^m\; I_m^*(s,\mu)d\mu, \quad m = 1,\ 1,\ 2, \quad (2.87)$$

Equation (2.81) with equation (2.86) takes the form

$$\mu\,\frac{dI(t,\mu)}{dt} = I(t,\mu) - [c_o I_o(t) + c_1\mu I_1(t) + \frac{1}{4}c_2(3\mu^2 - 1) \times$$

$$\times\ (3I_2(t) - I_0(t))] - (1 - c_0/\omega_0)B(t) \qquad (2.88)$$

Now subjecting equation (2.88) to the Laplace transform as defined in equation (2.85) I have , using the boundary conditions,

$$(\mu s - 1)\ I^*(s,\mu) = \mu s I(0,\mu) - (1 - c_0/\omega_0)B^*(s) - (c_0 I_0^*(s) +$$

$$+ c_1\mu\ I_1^*(s) + \frac{1}{4}c_2(3\mu^2 - 1)(3I_2^*(s) - I_0^*(s))] \qquad (2.89)$$

Equation (2.89) gives

$$I(0,\mu) = c_0 I_0^*(1/\mu) + c_1\mu I_1^*(1/\mu) + \frac{1}{4}c_2(3\mu^2 - 1) +$$

$$+ (3I_2^*(1/\mu) - I_0^*(1/\mu))\ (1 - c_0/\omega_0)\ B^*(1/\mu) \qquad (2.90)$$

Equation (2.90) with $\mu = s^{-1}$, s is complex , takes the form

$$I(0,s^{-1}) = (c_0 - (1/4)c_2(3s^{-2} - 1))I_0^*(s) + c_1 s^{-1} I_1^*(s) +$$

$$+ \frac{3}{4}c_2(3s^{-2} - 1)I_2^*(s) + (1 - c_0/\omega_0)\ B^*(s) \qquad (2.91)$$

I shall apply the operator $(1/2)\int_0^1 \ldots.\ \mu d\mu$ (2.92)

on both sides of equation (2.89) to get

$$-(1 - c_0)\ s^{-1}\ I_0^*(s) + I_1^*(s) = (1/2)\int_0^1 \mu I(0,\mu)\ d\mu -$$

$$- (1 - c_0/\omega_0)s^{-1}\ B^*(s) \qquad (2.93)$$

and

$$-(1 - \frac{1}{3}c_1)s^{-1}\ I_1^*(s) + I_2^*(s) = (1/2)\int_0^1 \mu^2\ I(0,\mu)\ d\mu\ , \qquad (2.94)$$

I shall also apply the operator

$$(1/2)\int_{-1}^{+1} \dots \quad d\mu/(\mu s - 1) \tag{2.95}$$

on both sides of equation (2.90) to get

$$\begin{aligned} as(^{-1}) - (1 - c_o/\omega_o)B^{*}(s)t_o(s^{-1}) &= [1 + c_o t_o(s^{-1}) - \\ &- \frac{1}{4}c_2(3t_2(s^{-1}) - t_o(s^{-1}))]I_o^{*}(s) + c_1 t_1(s^{-1})I_1^{*}(s) + \\ &+ \frac{3}{4}c_2[3t_2(s^{-1}) - t_o(s^{-1})]I_2^{*}(s), \end{aligned} \tag{2.96}$$

where

$$a(s^{-1}) = (1/2)\int_{0}^{1} \mu s(\mu s - 1)^{-1} \; I(0,\mu)\; d\mu \tag{2.97}$$

and

$$t_m(s^{-1}) = (1/2)\int_{-1}^{+1} (\mu s - 1)^{-1}\, \mu^m \, d\mu \tag{2.98}$$

$$, \; m = 0,1,2.$$

If I follow the usual procedure for elimination of $I_o^{*}(s)$, $I_1^{*}(s)$, and $I_2^{*}(s)$ among equations (2.93), (2.94), (2.96) and (2.97), after some calculations setting $s = z^{-1}$, I have

$$\begin{aligned} T(z)I(0,z) &= (1/2)\int_{0}^{1} x(x - z)^{-1}\, L(x,z)I(0,x)dx + \\ &+ (1 - c_o/\omega_o)\; B^{*}(z^{-1}) \end{aligned} \tag{2.99}$$

$$\text{where } T(z) = 1 - 2z^2\int_{0}^{1} \psi(x)(z^2 - x^2)^{-1}\, dx \tag{2.100}$$

$$\psi(x) = (1/2)(A + Bx^2 + Cx^4)\;, \tag{2.101}$$

$$L(x,z) = A - \frac{3}{4}c^2x^2 + (B + C + \frac{3}{4}c_2)xz - (1/3)Cz^2 +$$

$$+ Cx^2z^2 \qquad (2.102)$$

$$B^*(z^{-1}) = b_o + b_1/(1+\beta z) = (d_o + d_1 z)/(1+\beta z), \qquad (2.103)$$

where

$$d_o = b_o + b_1 , \qquad d_1 = b_o\beta \qquad (2.104)$$

$$A = c_o + \frac{1}{4}c_2 , \; B = c_1(1 - c_o) - \frac{3}{4}c_2 - \qquad (2.105)$$

$$- \frac{3}{4}c_2(1 - c_o)(1 - c_1/3) \qquad (2.106)$$

where I shall assume that

$$\psi(x) = \frac{1}{2}(A + Bx^2 + Cx^4) > 0 \qquad (2.107)$$

and
$$\psi_o = \int_0^1 \psi(x)\,dx < 1/2 \qquad (2.108)$$

but for
$$y = k(k + \sigma) < 1 , \qquad (2.109)$$

$B^*(z^{-1})$ is analytic in $(-y^{-1}, o)^c$, bounded at the origin and $0 < y < 1$. According to Busbridge [1960], the equation for $T(z)$ possesses the following properties : $T(z)$ is analytic in z for $(-1,1)^c$, bounded at the origin , has a pair of zeros at $z = \pm K$ $(K > 1)$, K is real and can be expressed as

$$T(z) = [\,H(z)\,H(-z)]^{-1} \qquad (2.110)$$

where $H(z)$ and $H(-z)$ have the following properties : $H(z)$ is analytic for $z \in (-1, 0)^c$, bounded at the origin, has a pole at $z = -K$, $H(-z)$ is analytic for $z \in (0,1)^c$, bounded at the origin , has a pole at $z = K$.

If I follow Busbridge [1960], Das [1979d] and Dasgupta [1977] I have for $\psi_o < 1/2$,

$$H(z) = 1 + zH(z)\int_0^1 \psi(x)H(x)(x + z)^{-1}dx \tag{2.111}$$

or

$$H(z) = (A_o + H_o z)/(z + K) - M(z), \tag{2.112}$$

where

$$M(z) = \int_0^1 P(x)\ dx/(x + z) \tag{2.113}$$

$$P(x) = \phi(x)/H(x) \tag{2.114}$$

$$\phi(x) = \pi^{-1}Y_o(x)/[T_o^2(x) + Y_o^2\ (x)] \tag{2.115}$$

$$T_o(x) = 1 - 2x^2\int_0^1 (\psi(t)- \psi(x))/(x^2 - t^2) -$$

$$- \psi(x)x\log((1 + x)/(1 - x)) , \tag{2.116}$$

$$Y_o(x) = \pi x\psi(x) \tag{2.117}$$

$$A_o = (1 + P_{-1})K , \tag{2.118}$$

$$P_{-1} = \int_0^1 x^{-1}\ P(x)\ dx , \tag{2.119}$$

$$H_o = (1 - 2\psi_o)^{-1/2} \tag{2.120}$$

Equation (2.99) with equation (2.110) takes the form

$$I(0,z)/H(z) = H(-z)G(z) + (1 - C_o/\omega_o)H(-z)B^*(z^{-1}) \tag{2.121}$$

where

$$G(z) = (1/2)\int_0^1 x(x - z)^{-1}\ L(x,z)I(0,x)dx , \tag{2.122}$$

I shall assume that

$$I(0,z) \text{ is regular for } \quad \text{Re } z>0 \qquad (2.123)$$

bounded at the origin. Equation (2.122) with the above assumption on $I(0,z)$ gives the following properties of $G(z)$: $G(z)$ is regular on $(0,1)^c$, bounded at the origin $O(z)$ when $z \longrightarrow \alpha$.

Equation (2.121) with equation (2.103) and (2.122) gives

$$I(0,z)/H(z) = H(-z)\Big[(1/2)\int_0^1 x(x-z)^{-1}L(x,z)I(0,x)dx + $$
$$+ (1 - c_0/\omega_0)(d_0 + d_1 z)/(1 + \beta z)\Big] \qquad (2.124)$$

Equation (2.124) can be put in the form

$$I(0,\mu)/H(z) = H(-z)\Big[(1/2)\int_0^1 x(x-z)^{-1}L(x,z)I(0,x)dx + $$
$$+ (1 - c_0/\omega_0)(d_0/z + d_1)/(z^{-1} + \beta)\Big] \qquad (2.125)$$

Therefore, the left-hand side of equation (2.125) is regular for Re $z>0$ and bounded at the origin and the right-hand side of equation (2.125) is regular for z on $(0,1)^c$ and bounded at the origin and tends to a linear polynomial in z , say $(x_0 + x_1 z)$ when $z \longrightarrow \alpha$. Hence , by a modified form of Liouville's theorem I have

$$I(0,z) = [x_0 + x_1 z]H(z) \qquad (2.126)$$

and

$$(1/2)\int_0^1 x(x-z)^{-1}L(x,z)I(0,x)dx + $$
$$+ (1 - c_0/\omega_0)(d_0 + d_1 z)/(1 + \beta z) = $$

$$= [x_o + x_1 z]/H(-z) \tag{2.127}$$

Equation (2.126) will give emergent intensity from the bounding face if x_o and x_1 are determined. If I set $z = 0$ in equation (2.127), I have

$$(1/2)\int_0^1 L(x,0)I(0,x)dx + d_o(1 - c_o/\omega_o) = x_o \tag{2.128}$$

Equation (2.128) with equation (2.126) gives

$$x_o\, y_1 + x_1\, y_2 + z_1 = 0, \tag{2.129}$$

where

$$y_1 = (1/2)\int_0^1 L(x,0)H(x)dx - 1 \tag{2.130}$$

$$y_2 = (1/2)\int_0^1 xL(x,0)H(x)dx \tag{2.131}$$

$$z_1 = (1 - c_o/\omega_o)d_o \tag{2.132}$$

As T(z) has a zero at $z = K$, equation (2.129) gives

$$(1/2)\int_0^1 x(x - K)^{-1} L(x,K)I(0,x)dx + + (1 - c_o/\omega_o)(d_o + d_1 K)/(1 + \beta K)\Big] = 0 \tag{2.133}$$

Equation (2.133) with equation (2.126) gives

$$x_o y_3 + x_1 y_4 + z_2 = 0\ , \tag{2.134}$$

where

$$y_3 = (1/2)\int_0^1 x(x - K)^{-1} L(x,K)H(x)dx \tag{2.135}$$

$$y_4 = (1/2)\int_0^1 x^2(x - K)^{-1} L(x,K)H(x)dx \qquad (2.136)$$

$$z_2 = (1 - c_o/\omega_o)(d_o + d_1 K)/(1 + \beta K) \qquad (2.137)$$

Equations (2.129) and (2.137) gives

$$x_o = (y_2 z_2 - z_1 y_4)/(y_1 y_4 - y_3 y_2) \qquad (2.138)$$

$$x_1 = (z_1 y_3 - y_1 z_2)/(y_1 y_4 - y_3 y_2) \qquad (2.139)$$

where $$(y_1 y_4 - y_3 y_2) \neq 0 \qquad (2.140)$$

Hence, equation (2.126) with equation (2.138) and (2.139) gives the emergent intensity from the bounding face of the atmosphere.

2.4. Solution of the equation of transfer for conservative anisotropic scattering phase function.

2.41. Formulation of the Problem .

In accordance with the scattering phase function

$$p(\cos\vartheta) = = 1 + \omega_1 P_1(\cos\vartheta) + \omega_2 P_2(\cos\vartheta), \qquad (2.141)$$

the phase function $p(\mu,\phi;\mu',\phi')$ for the angle between the directions specified by (ϑ,ϕ) and (ϑ',ϕ') can be written as

$$p(\mu,\phi;\mu',\phi') = 1 + \omega_1\mu\mu' + (1/4)\omega_2(3\mu^2 - 1)(3\mu'^2 - 1) +$$
$$+ [\omega_1(1-\mu^2)^{1/2}(1-\mu'^2)^{1/2} +$$
$$+ 3\omega_2\mu\mu'(1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}]\cos(\phi - \phi') +$$
$$+ (3/2)\omega_2(1-\mu^2)^{1/2}(1-\mu'^2)^{1/2} \cos 2(\phi - \phi') \qquad (2.142)$$

Hence $$p^{(0)}(\mu,\mu') = 1 - (1/4)\omega_2(3\mu^2 - 1) + \omega_1\mu\mu' +$$
$$+ (3/4)\omega_2(3\mu^2 - 1)\mu'^2 \qquad (2.143)$$

The equation of transfer (equation (2.1)) then becomes

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - (1/2)\int_{-1}^{+1} I(\tau,\mu')d\mu' + (\omega_2/2)(3\mu^2 - 1) \times$$

$$\times \int_{-1}^{+1} I(\tau,\mu')d\mu' - (\omega_1/2)\mu\int_{-1}^{+1} \mu' I(\tau,\mu')d\mu' +$$

$$+ (3\omega_2/8)(3\mu^2 - 1)\int_{-1}^{+1} \mu'^2 I(\tau,\mu')d\mu' \qquad (2.144)$$

This is the equation of transfer for the scattering according to the phase function (2.141), in a Semi-infinite plane-parallel atmosphere. I shall now find the solution to equation (2.144).

2.42. Solution of the Equation of Transfer.

The Equation of transfer appropriate to this problems, according to the phase function considered here (equation (2.143)), given by equation (2.144)
can be written in the form

$$\mu_i \frac{dI(\tau,\mu_i)}{d\tau} = I(\tau,\mu_i) - (1/2)\sum a_j I(\tau,\mu_j) +$$

$$+ (\omega_2/2)(3\mu^2{}_i - 1)\sum_j a_j I(\tau,\mu_j) - (\omega_2/2)\mu_i\sum_j a_j\mu_j I(\tau,\mu_j) -$$

$$- (3\omega_2/8)(3\mu^2{}_i - 1)\sum_j a_j\mu_i^2 I(\tau,\mu_j) \qquad (2.145)$$

The required solutions of equation (2.1) satisfy the boundary conditions

$$(1) \quad I(0, -\mu) = 0 \tag{2.146}$$

$$(2) \quad \mathcal{F}(\tau,\mu)e^{-\tau} \longrightarrow \propto \tag{2.147}$$

I seek a solution of equation (2.145) of the form

$$I(\tau, \mu_i) = g(\mu_i)\, e^{-k\tau} \qquad (i = \pm 1 , \pm 2 , \pm n) \tag{2.148}$$

where k is a constant (unspecified for the present) and $g(\mu_i)$ is a function of μ only.

Substituting for $I(\tau,\mu_i)$ for equation (2.148) into equation (2.145) I find that

$$g(\mu_i)[\, 1 + \mu_i k \,] = (1/2)\sum_j a_j g(\mu_j) -$$

$$-(\omega_2/8)(3\mu_i^2 - 1)\sum_j a_j g(\mu_j) + (\omega_1/2)\, \mu_i \sum_j a_j \mu_j g(\mu_j) +$$

$$+ (3\omega_2/8)\, (3\mu_i^2 - 1)\sum_j a_j \mu_j^2 g(\mu_j) \tag{2.149}$$

Equation (2.149) implies that $g(\mu_i)$ must be expressible in the form

$$g(\mu_i) = \frac{\alpha\mu_i^2 + \beta\mu_i + \gamma}{1 + \mu_i k} \tag{2.150}$$

From equations (2.149) and (2.150)

$$\alpha\mu_i^2 + \beta\mu_i + \gamma = (1/2)\sum_j a_j g(\mu_j) -$$

$$- (\omega_2/8)(3\mu_i^2 - 1)\sum_j a_j g(\mu_j) + (\omega_1/2)\mu_i \sum_j a_j \mu_j g(\mu_j) +$$

$$+ (3\omega_2/8)\, (3\mu_i^2 - 1)\sum_j a_j\, \mu_j^2\, g(\mu_j) \tag{2.151}$$

After some simplification

$$\alpha\mu_i^2 + \beta\mu_i + \gamma = (3\omega_2/8)\left[\left(\frac{4+\omega_2}{3\omega_2} - \mu_i^2\right)\sum_j a_j \frac{\alpha\mu_j^2 + \beta\mu_j + \gamma}{1+\mu_j k} + \right.$$

$$+ (3\mu_i^2 - 1)\sum_j a_j \frac{\alpha\mu_j^2 + \beta\mu_j + \gamma}{1+\mu_j k}\mu_j^2 +$$

$$\left. + (4\omega_1/3\omega_2)\mu_i \sum_j a_j \frac{\alpha\mu_j^2 + \beta\mu_j + \gamma}{1+\mu_j k}\mu_j \right] \qquad (2.152)$$

Since

$$D_m(k) = \sum_j \frac{a_j \mu_j^m}{1+\mu_j k} \qquad (\ m = 0,1,2, \qquad 4n\ ) \qquad (2.153)$$

(vide, Chandrasekhar, 1960, Chapter III, equation (18)).

Equation (2.152) then reduces to

$$(8/3\omega_2)[\alpha\mu_i^2 + \beta\mu_i + \gamma] = \frac{4+\omega_2}{3\omega_2}(\alpha D_2 + \beta D_1 + \gamma D_0) -$$

$$- (\alpha D_4 + \beta D_3 + \gamma D_2) + (4\omega_1/3\omega_2)(\alpha D_3 + \beta D_2 + \gamma D_1)\mu_i +$$

$$+ \Big\{ 3(\alpha D_4 + \beta D_3 + \gamma D_2) - (\alpha D_2 + \beta D_1 + \gamma D_0) \Big\}\mu_i^2 \qquad (2.154)$$

erefore,

$$/3\omega_2)\,\alpha = 3(\alpha D_4 + \beta D_3 + \gamma D_2) - (\alpha D_2 + \beta D_1 + \gamma D_0) \qquad (2.155)$$

$$(8/3\omega_2)\beta = (4\omega_1/3\omega_2)(\alpha D_3 + \beta D_2 + \gamma D_1) \qquad (2.156)$$

$$(8/3\omega_2)\gamma = \frac{4+\omega_2}{3\omega_2}(\alpha D_2 + \beta D_1 + \gamma D_0) -$$

$$- (\alpha D_4 + \beta D_3 + \gamma D_2) \qquad (2.157)$$

Equations (2.155),(2.156) and (2.157) represents a system of *homogeneous linear equations for* α *and* β . *The determinant* of this system must vanish ,therefore,

$$\begin{vmatrix} 3D_4-D_2-(8/3\omega_2) & 3D_3 - D_1 & 3D_2 - D_0 \\ (4\omega_1/3\omega_2)D_3 & (4\omega_1/3\omega_2)D_2-(8/3\omega_2) & (4\omega_1/3\omega_2)D_1 \\ \dfrac{4+\omega_2}{3\omega_2} D_2 - D_4 & \dfrac{4+\omega_2}{3\omega_2} D_1 - D_3 & \dfrac{4+\omega_2}{3\omega_2} D_0 - D_2 - (8/3\omega_2) \end{vmatrix} = 0 \tag{2.158}$$

These D's satisfy the relations (vide, Chandrasekhar , 1960 Chapter III , equations (21) and (22))

$$D_{2j} = \frac{1}{k^2}\left[D_{2j-2} - \frac{2}{2j-1}\right] \tag{2.159}$$

and

$$D_{2j-1} = -kD_{2j} \tag{2.160}$$

In particular ,

$$\left.\begin{aligned} D_1 &= -kD_2 = -\frac{1}{k}(D_0 - 2) \\ D_2 &= \frac{1}{k^2}(D_0 - 2) \quad ; \quad D_4 = \frac{1}{k^4}(D_0 - 2) - \frac{2}{3k^2} \\ D_3 &= \frac{2}{3k} - \frac{1}{k^3}(D_0 - 2) \end{aligned}\right] \tag{2.161}$$

Therefore, from equation (2.158) I can find $g(\mu_i)$ following the same procedure as shown by Barman and Karanjai (1974) in the form

$$g(\mu_i) = L_0\left[\frac{\dfrac{4+\omega_2}{3\omega_2} - \mu_i^2}{1+\mu_i k}\right] \tag{2.162}$$

where L_0 is a constant interms of ω_2 and k

Therefore, from Equation (2.148)

$$I(\tau,\mu_i) = L_o\left[\frac{\frac{4+\omega_2}{3\omega_2} - \mu_i^2}{1+\mu_i k}\right] e^{-k\tau} \qquad (2.163)$$

Omitting i's I get

$$I(\tau,\mu) = L_o\left[\frac{\frac{4+\omega_2}{3\omega_2} - \mu^2}{1+\mu k}\right] e^{-k\tau} \qquad (2.164)$$

Therefore the solution can be written as

$$I(\tau,\mu) = L_o\left[\frac{\frac{4+\omega_2}{3\omega_2} - \mu^2}{1\pm\mu k}\right] e^{\mp k\tau} \qquad (2.165)$$

Inserting the expression for $g(\mu_i)$ from equation (2.162) in equation (2.149) and simplifying (i.e., omitting i's) I get

$$1 = \frac{3\omega_2}{4}\left[\frac{1}{k^2} + \frac{1}{2k}\left\{\frac{4+\omega_2}{3\omega_2} - \frac{1}{k^2}\right\}\log\left(\frac{1+k}{1-k}\right)\right] \qquad (2.166)$$

which gives the characteristic equation, the solutions of which for various ω_2's are given in Table I. I seek, therefore, a solution of equation (2.148) which satisfies the boundary condition

$$I(0,-\mu) = 0 \qquad (0 \le \mu \le 1) \qquad (2.167)$$

for no incident radiation and which behaves as

$$I(\tau,\mu) \longrightarrow L_o\left[\frac{\frac{4+\omega_2}{3\omega_2} - \mu^2}{1+\mu k}\right] e^{-k\tau} \quad \text{as } \tau \longrightarrow \alpha \qquad (2.168)$$

TABLE 2.1

The Characteristic Roots k

ω_2	k
0.0000	0.0000000
0.0125	9.682602E-02
0.0250	0.1369389
0.0500	0.193697
0.1000	0.2741478
0.1500	0.3362519
0.2000	0.389154
0.2500	0.4365286
0.3000	0.4804349
0.3500	0.5223672
0.4000	0.563804
0.4500	0.6069927
0.5000	0.6587551
0.5100	0.6729191
0.5150	0.6817015
0.5200	0.6936500
0.5220	0.7017839
0.5225	0.7057352

L_o being some some assigned constant. This does not satisfy the condition (2.167). I shall therefore let

$$I(\tau,\mu) = L_o\left[\frac{\frac{4+\omega_2}{3\omega_2} - \mu^2}{1+\mu k}\right] e^{-k\tau} + I^*(\tau,\mu) \qquad (2.169)$$

represents the solution of the problem. Note that $I^*(\tau,\mu)$ $(0 \le \mu \le 1)$ must result from the reflection of $I^*(\tau,-\mu)$

$(0 \le \mu \le 1)$ by the semi-infinite atmosphere below τ . Thus ,

$$I(\tau ,\mu) = L_o \left[\frac{\frac{4 + \omega_2}{3\omega_2} - \mu^2}{1 + \mu k} \right] e^{-k\tau} + \frac{3\omega_2}{8\mu} \int_o^1 S^{(o)} (\mu ,\mu') \times$$

$$\times \; I^*(\tau ,\mu')d\mu' \qquad (2.170)$$

where $\frac{3\omega_2}{8\mu} S^{(o)}$ is the azimuth-independent term in the scattering function (Vide, Chandrasekhar, 1960, Chapter VI, equation (5)).

At $\tau = 0$ (from equation (2.169))

$$I^*(\tau , -\mu') = - L_o \left[\frac{\frac{4 + \omega_2}{3\omega_2} - \mu'^2}{1 + \mu' k} \right] e^{-k\tau} \qquad (2.171)$$

by condition (2.167). Therefore , from equations (2.170) and (2.171) I find that

$$I(0,\mu) = L_o \left\{ \frac{\frac{4 + \omega_2}{3\omega_2} - \mu^2}{1 + \mu k} - \frac{3\omega_2}{8\mu} \int_o^1 \frac{S^{(o)} (\mu ,\mu')}{1 - \mu' k} \times \right.$$

$$\left. \times \left(\frac{4 + \omega_2}{3\omega_2} - \mu'^2 \right) \right\} \qquad (2.172)$$

But (Vide, Chandrasekhar, 1960, Chapter VI, equation (11))

$$S^{(o)} (\mu ,\mu') = \frac{\mu\mu'}{\mu + \mu'} H(\mu) H(\mu') \left[\frac{4 + \omega_2}{3\omega_2} - \right.$$

$$\left. - c(\mu + \mu') + \mu\mu' \right] \qquad (2.173)$$

$$\text{where} \quad c = \frac{\alpha_2}{\alpha_1} \qquad (2.174)$$

α_1 and α_2 are the moments of $H(\mu)$ defined by the Characteristic function.

$$\psi(\mu) = \frac{3\omega_2}{8}\left(\frac{4+\omega_2}{3\omega_2} - \mu^2\right) \tag{2.175}$$

Moreover $H(\mu)$ is the unique solution of

$$H(\mu) = 1 + \frac{1}{8}\mu H(\mu)\int_0^1 \frac{(4+\omega_2) - 3\omega_2\mu'^2}{\mu+\mu'} H(\mu')d\mu' \tag{2.176}$$

which is bounded in the interval $(0 \le \mu \le 1)$.

Therefore, from equation (2.172) it follows that

$$I(0,\mu) = L_0\left\{\frac{(4+\omega_2) - 3\omega_2\mu'^2}{\mu+\mu'} - \frac{3\omega_2}{8}H(\mu)\int_0^1 \frac{\mu'}{(\mu+\mu')(1-k\mu')} \times \right.$$

$$\times\left[\frac{4+\omega_2}{3\omega_2} - c(\mu+\mu') + \mu\mu'\right] \times$$

$$\left.\times\left(\frac{4+\omega_2}{3\omega_2} - \mu'^2\right)H(\mu')d\mu'\right\} \tag{2.177}$$

Expressing $\mu'/(\mu+\mu')(\mu - k\mu')$ in partial fractions and rearranging the terms , I obtain

$$I(0,\mu) = \frac{L_0}{1+k\mu}\left\{\left(\frac{4+\omega_2}{3\omega_2} - \mu'^2\right) - \frac{3\omega_2}{8}H(\mu)\int_0^1\left[\frac{1}{1-k\mu'} - \right.\right.$$

$$\left. - \frac{\mu}{\mu+\mu'}\right]\left[\frac{4+\omega_2}{3\omega_2} - c(\mu+\mu') + \mu\mu'\right] \times$$

$$\times \left(\frac{4 + \omega_2}{3\omega_2} - \mu'^2 \right) H(\mu')d\mu' \Bigg\} \qquad (2.178)$$

After some algebra , the second part in the braces $\{ \quad \}$ of equation (2.178) can be written as

$$\frac{3\omega_2}{8} H(\mu) \int_0^1 \left[\frac{1}{1 - k\mu'} - \frac{\mu}{\mu + \mu'} \right] \times$$

$$\times \left[\frac{4 + \omega_2}{3\omega_2} - c(\mu + \mu') + \mu\mu' \right] \left(\frac{4 + \omega_2}{3\omega_2} - \mu'^2 \right) H(\mu')d\mu' =$$

$$= \left\{ H(\mu) \left[\frac{4 + \omega_2}{3\omega_2} - c\mu \right] - \frac{H(\mu)}{H(-1/k)} \left[\frac{\mu - c}{k} + \frac{4 + \omega_2}{3\omega_2} - c\mu \right] \right\} -$$

$$- \left\{ H(\mu) \left[\frac{4 + \omega_2}{3\omega_2} - c\mu \right] - \frac{4 + \omega_2}{3\omega_2} + \mu^2 \right\} \qquad (2.179)$$

Where H(-1/k) is deduced from equation (2.176).

Thus , I obtain

$$I(0,\mu) = \frac{L_o}{1 + k\mu} \frac{H(\mu)}{H(-1/k)} \left[\left(\frac{4 + \omega_2}{3\omega_2} - c\mu \right) + \frac{\mu - c}{k} \right] \qquad (2.180)$$

This is the required solution for the conservative anisotropic phase function considered here (equation (2.141)).

The constant L_o in equation (2.180) can be determined by the condition (Vide, Chandrasekhar, 1960).

$$F = 2\int_0^1 I(0,\mu)\mu d\mu \qquad (2.181)$$

2.43. Applications.

The solution of the equation of transfer for the phase function considered here (equation (2.141)) can be applied to find the laws of darkening. With the aid of Equation (2.180), the laws of darkening i.e. $I(0,\mu)/I(0,1)$ for the phase function (equation (2.141)) can be written as

$$\frac{I(0,\mu)}{I(0,1)} = \frac{\dfrac{H(\mu)}{1+k\mu}\left[\left(\dfrac{4+\omega_2}{3\omega} - c\mu\right) + \dfrac{\mu - c}{k}\right]}{\dfrac{H(1)}{1+k}\left[\left(\dfrac{4+\omega_2}{3\omega_2} - c\right) + \dfrac{1-c}{k}\right]} \qquad (2.182)$$

I calculated the laws of darkening for the phase function considered here from the relations (2.182) for $\omega_2 = .5$ and the results are given in Table II, where the value of c and H(1) are taken from Chandrasekhar (1960), and the value of k is taken from table I.

Table 2.2. Laws of darkening for the phase function

$P(\mu,\mu') = 1 + \omega_1 P_1(\cos\vartheta) + \omega_2 P_2(\cos\vartheta)$

for $\omega_2 = .5$ $c = .71139$, $k = .6587551$, $H(1) = 3.01973$

μ	By Deb & Karanjai $I(0,\mu)/I(0,1)$	By Karanjai & Barman $I(0,\mu)/I(0,1)$
0.00	0.3868091	0.34524
0.10	0.4782439	0.43394
0.20	0.5483665	0.50504
0.30	0.6122809	0.57159
0.40	0.6726329	0.63575
0.50	0.7305679	0.69808
0.60	0.7867261	0.75996
0.70	0.8415158	0.82077
0.80	0.8951063	0.88096
0.90	0.9479748	0.94067
1.00	1.0000000	1.00000

2.44. Conclusion.

Though the results obtained here is in agreement with that of Chandrasekhar (1960) and Karanjai and Barman (1979) only upto one decimal point but here I can calculate the laws of darkening for the phase function from the relations (2.182) for various values of $\omega_2 \leq .5225$. Also as $\mu \longrightarrow 1$, the results are in good agreement with that of Chandrasekhar (1960) and Karanjai and Barman (1979).

2.5 Solution of a Radiative transfer problem with a combined Rayleigh and isotropic phase matrix.

2.51. Basic Matrix Transfer Equation and Boundary Conditions :

The basic integro-differential equation for infinity matrix $I(\tau,\mu)$ can be written in the form

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\omega \int_{-1}^{+1} K(\mu,\mu')I(\tau,\mu')d\mu' \qquad (2.183)$$

where τ is the optical thickness of the atmosphere , μ is the direction parameter, $I(\tau,\mu)$ is a (2x1) matrix , ω ($0< \omega <1$) is the albedo for single scattering. According to Burniston and Siewert [1970] , $K(\mu,\mu)$, a (2x2) matrix , can be written as

$$K(\mu,\mu') = Q(\mu)Q^{T}(\mu') \qquad (2.184)$$

where $Q(\mu)$, a (2x2) matrix , can be defined by

$$Q(\mu) = \frac{3(c+2)^{1/2}}{c+2}\begin{bmatrix} c\mu^2 + \frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{bmatrix} \qquad (2.185)$$

$Q^{T}(\mu)$ is the transpose of $Q(\mu)$, and c is a parameter ($0< c <1$). A solution of equation (2.183) is required with the following boundary conditions

$$I(0,-\mu) = 0 , \qquad 0 \le \mu \le 1 \tag{2.186}$$

$$\text{and} \quad I(\tau,\mu) \longrightarrow \frac{1}{2}\omega L_o \left[\frac{k}{k-\mu} \right] e^{\tau/k} Q(\mu)$$

$$\text{as } \tau \longrightarrow \alpha \tag{2.187}$$

where k is a positive root greater than one and real of the equation

$$T(z) = \det D(z) \tag{2.188}$$

where

$$D(z) = E + z \int_{-1}^{+1} \psi(\mu) \frac{d\mu}{\mu - z} \tag{2.189}$$

$\psi(\mu)$ is a (2x1) matrix and $\psi(\mu)$ is defined by

$$\psi(\mu) = (1/2)\omega\, Q^T(\mu)Q(\mu) \tag{2.190}$$

and E is a unit matrix ,D(z) is a (2x2) matrix and L_o is a specified (2x1) matrix .

2.52. Solution for Emergent Intensity Matrix .

The Laplace transform of the intensity matrix is defined by

$$I^*(s,\mu) = s \int_0^{\alpha} e^{-s\tau} I(\tau,\mu)\, d\mu , \qquad \text{Re } s>0 \tag{2.191}$$

Let us set $I_U(\tau)$, a (2x1) matrix as

$$I_U(\tau) = (1/2) \int_{-1}^{+1} Q^T(\mu')I(\tau,\mu')d\mu' \tag{2.192}$$

$$I_U(s) = (1/2) \int_{-1}^{+1} Q^T(\mu')I^*(s, \mu')d\mu' \tag{2.193}$$

I subject the Laplace transform as defined in equation

(2.191) to equation (2.183) to get (using equations (2.186), (2.192), (2.193))

$$(\mu s - 1)I^*(s,\mu) = \mu s I(0,\mu) - \omega\, Q(\mu) I_U^*(s) \qquad (2.194)$$

The solution for the emergent intensity matrix arrived from equation (2.194)

$$I(0,\mu) = \omega\, Q(\mu)\; I_U^*\; (1/\mu) \qquad (2.195)$$

Equation (2.195) gives for $\mu = 1/s$, s is complex

$$I(0,1/s) = \omega\, Q(1/s) I_U^*(s) \qquad (2.196)$$

I now apply the (2x2) matrix operator

$$(1/2)\int_{-1}^{+1} \ldots \frac{Q^T(\mu)\, d\mu}{(\mu s - 1)} \quad , \qquad (2.197)$$

to equation (2.194) to get $D(1/s) I_U^*(s) = a(1/s)$ (2.198)

where D(1/s) is a (2x2) matrix and a(1/s) is (2x1) matrix defined by

$$D(1/s) = E + \int_{-1}^{+1} \frac{\psi(\mu) d\mu}{(\mu s - 1)} \quad , \qquad (2.199)$$

and

$$a(1/s) = (1/2)\int_0^1 \frac{\mu s Q^T(\mu) I(0,\mu) d\mu}{(\mu s - 1)} \qquad (2.200)$$

respectively where $\psi(\mu)$ is given by equation (2.190), is a (2x2) unit matrix . Eliminating $I_U^*(s)$ between equations (2.196) and (2.198) I get a matrix integral equation as

$$D(z)\; I(0,z) = \omega\; Q(z)\; a(z) \;, \quad \text{where } s = 1/z \qquad (2.201)$$

Following Bond and Siewert [1971] , I have

$$T(z) = \det D(z) = \tfrac{1}{8} c T_1(z) T_2(z) \;+$$

$$+ \left[(1 - c) + \frac{3}{2}c\ (1 - \omega)z^2\right] T_o(z) \qquad (2.202)$$

and
$$T_n(z) = (-1)^n + 3(1 - z^2)T_o(z) - $$
$$- (-1)^n\, 3(1 - \omega)z^2, \quad n = 1 \text{ or } 2 \qquad (2.203)$$

$$T_o(z) = 1 + (1/2)\omega z \int_{-1}^{+1} \frac{d\mu}{\mu - z} \quad , \qquad (2.204)$$

where $T(z)$ is analytic in the complex plane cut from -1 to $+1$ along the real axis with two zeros at $z = \pm k$, k is real ($k > 1$). I consider the (2x2) H-matrix equation (vide, Abhyankar and Fymat, 1970) as

$$H(z) = E + zH(z) \int_0^1 H^T(\mu)\psi(\mu)d\mu/(\mu + z\ ) \qquad (2.205)$$

where $\psi(\mu)$ is given by equation (2.190) .

I shall assume that the (2x2) $H(z)$ matrix is analytic in the complex plane cut from -1 to 0 , bounded at the origin , has a pole at $z = -k$, k is real ($k > 1$) and similarly the $H(-z)$ matrix is analytic in the complex plane cut from 0 to 1 , bounded at the origin , has a pole at $z = k$, k is real , ($k > 1$). Hence , $H^{-1}(z)$, the inverse of the H-matrix , is analytic in the complex plane cut from -1 to 0 and bounded at the origin. If the (2x2) H-matrix is a symmetric matrix ,it can be proved that

$$D(z) = H^{-1}(z)\, H^{-1}(-z) \quad , \quad z \in (-1,\ 1)^c \qquad (2.206)$$

Now Equation (2.201) together with Equation (2.206) takes the form

$$H^{-1}(z)Q^{-1}(z)I(0,z)\left[\frac{k-z}{k}\right] = \omega\left[\frac{k-z}{k}\right]H(-z)a(z) \quad (2.207)$$

where the left hand side of equation (2.207) is regular for Re $z > 0$, bounded at the origin and the right hand side of equation (2.207) is analytic in $(0,1)^c$, bounded at the origin and tends to a constant matrix (2x1) say A, when $z \longrightarrow \infty$ subject to the assumption that $I(0,z)$ is analytic for Re $z>0$ and bounded at the origin. Hence, by a modified form of Liouville's theorem, equation (2.207) gives the emergent intensity matrix $I(0,z)$ as

$$I(0,z) = \left[\frac{k}{k-z}\right]Q(z)H(z)A \quad (2.208)$$

I now determine the matrix A. The inversion integral gives the intensity matrix $I(\tau,\mu)$ as

$$I(\tau,\mu) = (1/2\pi i)\lim_{\nu\to\infty}\int_{\sigma-i\nu}^{\sigma+i\nu} I^{*}(s,\mu)e^{s\tau}\,ds/s\,, \quad \sigma > 0\,, \quad (2.209)$$

where $I^{*}(s,\mu)$ can be obtained as

$$I^{*}(s,\mu)/s = \Big[\, I(0,\mu) - (\mu s)^{-1}Q^{-1}(1/s)\times$$

$$\times\, Q(\mu)I(0,\mu)\Big]/(s - 1/\mu) \quad (2.210)$$

$$I^{*}(s,\mu)/s = \Big[\, I(0,\mu)/(s-1/2) - Q(\mu)\times$$

$$\times\, H(1/s)A/(s-1/k)\mu(s-1/\mu)\Big] \quad (2.211)$$

The integral of equation (2.209) is analytic for s in $(-\alpha\ ,\ -1)^c$, has poles at $s = \pm\ 1/k$, k is real k < 1 , where $s = 1/\mu$ is not a pole as

$$\lim_{s \longrightarrow 1/\mu} (s - 1/\mu) I^*(s, \mu) e^{s\tau}/s \longrightarrow 0 \qquad (2.212)$$

The contribution of pole at $s = 1/k$ will give the asymptotic solution of equation (2.183) as

$$I(\tau, \mu) \longrightarrow \left[\frac{k}{k - \mu}\right] Q(\mu) H(k) e^{s/k} A$$
$$\text{when} \quad \tau \longrightarrow \alpha \qquad (2.213)$$

Equation (2.187) with equation (2.213) gives the matrix A as

$$A = (1/2) \left[\omega\ H^{-1}(k)\right] L_o \qquad (2.214)$$

Equation (2.208) with equation (2.214) gives the emergent intensity in the form

$$I(0, z) = (1/2) \omega L_o H^{-1}(k) H(z) Q(z) \left[\frac{k}{k - z}\right] \qquad (2.215)$$

2.53. Conclusions.

Here I allow the values c $(0 < c < 1)$ and ω $(0 < \omega < 1)$ to study the general mixture of Rayleigh and isotropic scattering.

a. When $\omega = 1$ and c $(0 < c < 1)$ the basic matrix transport equation yields a conservative model for a mixture of Rayleigh and isotropic scattering .

b. when ω $(0 < \omega < 1)$ and $c = 1$, we obtain the general Rayleigh scattering problem .

c. When $c = 1$ and $w = 1$, the problem yields Chandrasekhar's

[1960] Rayleigh scattering model and $Q(\mu)$ reduces to Sekera's [1963] form for factorising the Rayleigh scattering phase matrix (vide, Das, 1979e).

d. In this problem there exists some possibilities for future development such as determination of the H-matrix expression and the values of the D(z) matrix on both sides of the cut etc.

e. There exists some possibilities to determine a characteristic function which is an even function having polynomial expression but has a transcendental form.

2.6. Time-Dependent Scattering and Transmission Function in an Anisotropic Two-Layered Atmosphere.

2.61. Formulation of the problem .

In an anisotropically-scattering medium, the intensity of radiation $I(\tau,\mu,\phi,t)$ at any time t, any optical depth τ, in the direction $\cos^{-1}\mu$, satisfies the equations of transfer

$$\frac{1}{c}\frac{\partial I(\tau,\mu,\phi,t)}{\partial t} + \mu\frac{\partial I(\tau,\mu,\phi,t)}{\partial \tau} + I(\tau,\mu,\phi,t) = J(\tau,\mu,\phi,t), \tag{2.216}$$

in which the source function $J(\tau,\mu,\phi,t)$ is given by

$$I(\tau,\mu,\phi,t) = \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} P(\mu,\phi;\mu',\phi')I(\tau,\mu',\phi',t)d\mu'd\phi' \tag{2.217}$$

where $P(\tau,\mu,\phi,t)$, the general phase function and c represents the velocity of light. In the above, μ and ϕ represent,respectively,the cosine of the zenith distance and the azimuthal angle. I decompose the intensity of radiation field into two components for two directions,viz.,intensity directed towards the lower surface of the atmosphere ($I^{+}(\tau,\mu,\phi,t)$) and intensity directed towards the upper surface of the atmosphere ($I^{-}(\tau,\mu,\phi,t)$).

I consider the initial boundary conditions

$$I(\tau,\mu,\phi,0) = 0. \tag{2.218}$$

$$I^{+}(0,\mu,\phi,t) = I^{*}_{inc}(\mu,\phi,t) \tag{2.219}$$

$$I^{-}(\tau_1,\mu,\phi,t) = I^{*}_{inc}(\mu,\phi,t). \tag{2.220}$$

Equation (2.219) and (2.220) asserts that the lower and the upper surface are illuminated. However, I shall restrict ourselves for the time being to the case of illumination on the upper surface ($\tau = 0$) by means of an instantaneously collimated beam of light at time $t = 0$. The other surface will be free from any incident radiation. I now distinguish between the reduced incident intensity which is incident on boundary surface and penetrates to the depth τ without suffering any collision and diffuse radiation which arises due to different processes (vide, Chandrasekhar, 1960). For the total radiation field, I have

$$I^{+}(\tau,\mu,\phi,t) = I^{+}_{d}(\tau,\mu,\phi,t) +$$

$$+ I_{inc}\left(\mu,\phi,t-\frac{\tau}{c\mu}\right)\exp\left(-\frac{\tau}{\mu}\right), \tag{2.221}$$

$$I^-(\tau,\mu,\phi,t) = I_d^-(\tau,\mu,\phi,t) +$$

$$+ I_{inc}^+\left(\mu,\phi,t-\frac{\tau_1-\tau}{c\mu}\right)\exp\left(-\frac{\tau_1-\tau}{\mu}\right), \tag{2.222}$$

where the subscript 'd' represent diffuse fields. If I substitute these expression for $I^+(\tau,\mu,\phi,t)$ and $I^-(\tau,\mu,\phi,t)$ in equation (2.216) I get two separate equations of transfer for two components

$$\left(c^{-1}\frac{\partial}{\partial t}+\frac{\partial}{\partial\tau}+1\right) I_d^+ (\tau,\mu,\phi,t) = J(\tau,\mu,\phi,t), \tag{2.223}$$

$$\left(c^{-1}\frac{\partial}{\partial t}+\frac{\partial}{\partial\tau}+1\right) I_d^- (\tau,\mu,\phi,t) = J(\tau,\mu,\phi,t), \tag{2.224}$$

where

$$J(\tau,\mu,\phi,t) = \frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} I_d(\tau,\mu',\phi',t)\times$$

$$\times P(\mu,\phi;\mu',\phi')\mu' d\phi' + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(\mu,\phi;\mu',\phi')\times$$

$$\times I_{inc}\left(\mu',\phi',t-\frac{\tau}{c\mu}\right)\exp\left(-\frac{\tau}{\mu}\right)d\mu' d\phi' + \frac{1}{4\pi}\times$$

$$\times\int_0^1\int_0^{2\pi} I_{inc}^*\left(\mu',\phi',t-\frac{\tau_1-\tau}{c\,\mu}\right)\exp\left(-\frac{\tau_1-\tau}{\mu}\right)\times$$

$$\times P(\mu,\phi;\mu',\phi')d\mu' d\phi' \tag{2.225}$$

Let us now put in equation (2.225)

$$I_{inc}(\mu,\phi,t) = F\delta(t)\delta(\mu - \mu_o)\delta(\phi - \phi_o), \quad (2.226)$$

$$I^*_{inc}(\mu,\phi,t) = 0; \quad (2.227)$$

where F is a constant.

Hence, I get

$$J(\tau,\mu,\phi,t) = \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} I_d(\tau,\mu',\phi',t)\ P(\mu,\phi;\mu',\phi')d\mu' d\phi' +$$

$$+ \frac{1}{4} FP(\mu,\phi;\mu_o,\phi_o)\exp\left(-\frac{\tau}{\mu_o}\right)\delta\left(t-\frac{\tau}{c\mu_o}\right). \quad (2.228)$$

The new set of boundary conditions are given by

$$I^+_d(\tau,\mu,\phi,t) = 0, \quad (2.229)$$

$$I^-_d(\tau,\mu,\phi,t) = 0. \quad (2.230)$$

This simplification of boundary conditions are the characteristic of such formulation. Let us now define the scattering and transmission function (vide, Matsumoto,1967a) as

$$S(\tau,\mu,\phi;\mu_o,\phi_o,t) = I^-_d(0,\mu,\phi,t), \quad (2.231)$$

$$I(\tau,\mu,\phi;\mu_o,\phi_o,t) = I^+_d(\tau,\mu,\phi,t). \quad (2.232)$$

2.62. Principle of Invariance.

I shall now derive the functional equations for these two functions. The four principles of invariance (vide,Matsumoto ,1969) for this problem take the following forms: (A) The intensity $I^-_d(\tau,\mu,\phi,t)$ in the upward direction at time t and at depth τ is given by

$$I_d^-(\tau,\mu,\phi,t) = F\mu^{-1} S\left[\tau_1 - \tau ; \mu,\phi : \mu_o,\phi_o, t - \frac{\tau}{c\mu_o}\right] \exp\left[-\frac{\tau}{\mu_o}\right] +$$

$$+ \frac{1}{4\pi\mu} \int_o^t dt' \int_o^1 \int_o^{2\pi} S(\tau_1 - \tau ; \mu,\phi ; \mu',\phi', t-t') I_d^+ \times$$

$$\times (\tau,\mu',\phi',t') d\mu' d\phi' . \qquad (2.233)$$

(B) The intensity $I_d^+(\tau,\mu,\phi,t)$ in the downward direction at time t and at a depth τ is given by

$$I_d^+(\tau,\mu,\phi,t) = F\mu^{-1} T(\tau ; \mu,\mu_o\phi_o, t) + \frac{1}{4\pi\mu} \int_o^1 dt' \times$$

$$\times \int_o^1 \int_o^{2\pi} S(\tau ; \mu,\phi ; \mu',\phi', t-t') I_d^-(\tau,\mu',\phi',t') d\mu' d\phi' \qquad (2.234)$$

(C) The diffuse reflection of the incident radiation by the entire atmosphere is given by

$$F\mu^{-1} S(\tau_1 ; \mu ; \phi ; \mu_o, \phi_o, t) = F\mu^{-1} (\tau ; \mu,\phi,\mu',\phi', t) +$$

$$+ I_d^- \left[\tau,\mu,\phi, t - \frac{\tau}{c\mu}\right] \exp\left[-\frac{\tau}{\mu}\right] + \frac{1}{4\pi\mu} \int_o^t dt' \times$$

$$\times \int_o^1 \int_o^{2\pi} T(\tau ; \mu,\phi ; \mu',\phi', t-t') \; I_d^-(\tau,\mu',\phi',t') \; d\mu' d\phi' \qquad (2.235)$$

(D) The diffuse transmission of incident radiation by the entire atmosphere is given by

$$F\mu^{-1} T(\tau ; \mu,\phi ; \mu_o\phi_o, t) = F\mu^{-1} T\left[\tau_1 - \tau ; \mu,\phi ; \mu_o,\phi_o, t - \frac{\tau}{c\mu_o}\right] \times$$

$$\times \exp\left(-\frac{\tau}{c\mu_o}\right) + I_d^+\left(\tau,\mu,\phi,t-\frac{\tau_1-\tau}{c\mu}\right)\exp\left(-\frac{\tau_1-\tau}{\mu}\right) +$$

$$+\frac{1}{4\pi\mu}\int_0^t dt'\int_0^1\int_0^{2\pi} T(\tau_1-\tau;\mu,\phi,\mu_o\phi_o,t-t')\times$$

$$\times I_d^+(\tau,\mu',\phi',t')d\mu' d\phi' \quad . \qquad (2.236)$$

A derivation of these four equations is based on classical intuitive physical arguments (vide, Ambartsumian, 1943; Chandrasekhar, 1960 ; Presendorfer, 1958). Although these equations do not provide a complete knowledge of radiation intensity at any depth (or neutron distribution in a given medium) but only the reflected and transmitted intensities, it has some real advantages for numerical computations.

2.63. Integral Equations for the Scattering and Transmission Function.

I differentiate equation (2.233) with respect to τ and take the limit as $\tau \longrightarrow 0$

$$\lim_{\tau\to o}\frac{dI_d^-(\tau,\mu,\phi,t)}{d\tau} = -F\mu^{-1}\left[(c\mu_o)^{-1}\frac{\partial}{\partial t} + (\mu_o)^{-1} + \frac{\partial}{\partial\tau_1}\right]\times$$

$$\times S(\tau_1,\mu,\phi;\mu_o,\phi_o,t) + \frac{1}{4\pi\mu}\int_0^t dt'\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu',\phi',t-t')\times$$

$$\times \left[\frac{dI_d^+ \ (\tau, \mu', \phi', t')}{d\tau} \ d\mu' d\phi' \right]_{\tau=0} \qquad (2.237)$$

From equation (2.223), I get by use of equations (2.229), (2.230)

$$\lim_{\tau \longrightarrow 0} \frac{dI_d^+ (\tau, \mu', \phi', t'}{d\tau} = \frac{J(0, \mu', \phi', t')}{\mu'} , \qquad (2.238)$$

where

$$J(0, \mu', \phi', t) = \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} \frac{F}{\mu''} S(\tau_1, \mu'', \phi'', t) \ d\mu'' d\phi'' +$$

$$+ \frac{1}{4} F\delta \ (t') \ P(\mu, \phi; \mu_0, \phi_0) \ . \qquad (2.239)$$

In deriving equation (2.239) I have used the expression for $J(\tau, \mu, \phi, t)$, equation (2.224) now yields, after use of equations (2.229) , (2.230) and (2.231).

$$\lim_{\tau \longrightarrow 0} \frac{dI_1^- (\tau, \mu, \phi, t)}{d\tau} = - \frac{J(0, \mu, \phi, t)}{\mu} +$$

$$+ \left(c^{-1} \frac{\partial}{\partial t} + 1 \right) \mu^{-1} \ F\mu^{-1} \ S(\tau_1 \mu, \phi; \mu_0, \phi_0, t) . \qquad (2.240)$$

If I substitute equations (2.238) and (2.240) in equation (2.233), after cancellation and re-arrangements of terms, I get

$$\frac{\partial S(\tau_1; \mu, \phi, \mu_0, \phi_0, t)}{\partial \tau_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_0} \right) \left(\frac{1}{c} \frac{\partial}{\partial t} + 1 \right) \times$$

$$\times S(\tau_1;\mu,\phi;\mu_0,\phi_0,t) = P(\mu,\phi;\mu_0,\phi_0)\,\delta(t) +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(\mu,\phi;\mu'',\phi'')\; S(\tau_1;\mu'',\phi'';\mu_0,\phi_0 t)\;\frac{d\mu''}{\mu''}\,d\phi'' +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(-\mu',\phi''-\mu_0,\phi_0)\;\frac{d\mu'}{\mu'}\,d\phi' +$$

$$+\frac{1}{16\pi^2}\int_0^t dt'\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu_0',\phi_0',t-t')\times$$

$$\times P(-\mu',\phi';\mu'',\phi'')S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,t)\;\frac{d\mu'}{\mu}\,d\phi'\,\frac{d\mu''}{\mu''}\,d\phi'' \qquad (2.241)$$

Equation (2.241) is the required functional equation of the time-dependent S-function. Again, if I differentiate equations (2.234), (2.235), and (2.236) with respect to τ and taking the limit as $\tau \longrightarrow \tau_1$ and $\tau \longrightarrow 0$, respectively, and following the same procedure I get

$$\frac{\partial T(\tau_1;\mu,\phi;\mu_0,\phi_0,t}{\partial\tau_1} + \mu^{-1}\left[1+\frac{1}{c}\frac{\partial}{\partial t}\right] I(\tau_1;\mu,\phi;\mu_0\phi_0,t) =$$

$$= \exp\left[-\frac{\tau_1}{\mu_0}\right]\delta\left[t-\frac{\tau}{c\mu_0}\right] P(-\mu,\phi;-\mu_0,\phi_0) +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(-\mu,\phi;-\mu'',\phi'')\; T(\tau_1;\mu'',\phi'';\mu_0,\phi_0 t)\;\frac{d\mu''}{\mu''} +$$

$$+ \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} S(\tau_1, \mu, \phi; \mu', \phi', t-t') \, \delta\left[t - \frac{\tau}{c\mu_0}\right] \exp\left[-\frac{\tau_1}{\mu_0}\right] \times$$

$$\times P(\mu, \phi; -\mu_0, \phi_0) \; \frac{d\mu'}{\mu'} \, d\phi' + \frac{1}{16\pi^2} \int_0^t dt' \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} \times$$

$$\times S(\tau_1; \mu, \phi; \mu', \phi', t-t') \; T(\tau_1; \mu'', \phi''; \mu_0, \phi_0, t') \times$$

$$\times P(\mu', \phi'; -\mu'', \phi'') \frac{d\mu'}{\mu'} \, d\phi' \frac{d\mu''}{\mu''} \, d\phi'' \,, \tag{2.242}$$

$$\frac{\partial S(\tau_1; \mu, \phi; \mu_0, \phi_0, t)}{\partial \tau_1} = P(\mu, \phi; -\mu_0, \phi_0) \times$$

$$\times \exp\left[-\tau_1\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)\right] \delta\left[t - \frac{\tau_1}{c\mu} - \frac{\tau_1}{c\mu_0}\right] + \exp\left[-\frac{\tau_1}{\mu}\right] \times$$

$$\times \frac{1}{4\pi} \int_0^t dt' \int_0^1 \int_0^{2\pi} T(\tau_1; \mu'', \phi'', \mu_0, \phi_0, t-t') \; P(\mu, \phi; -\mu'', \phi'') \times$$

$$\times \delta\left[t' - \frac{\tau_1}{c\mu}\right] \frac{d\mu''}{\mu''} \, d\phi'' + \frac{1}{4\pi} \int_0^t dt' \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi', t-t') \times$$

$$\times \delta\left[t' - \frac{\tau_1}{c\mu}\right] \exp\left[-\frac{\tau_1}{\mu_0}\right] P(\mu', \phi'; -\mu_0, \phi_0) \frac{d\mu'}{\mu'} d\phi' + \frac{1}{16\pi^2} \int_0^t dt' \times$$

$$\times \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi', t-t') \; T(\tau_1; \mu'', \phi''; \mu_0, \phi_0, t') \times$$

$$\times P(\mu', \phi'; -\mu'', \phi'') \frac{d\mu'}{\mu'} \, d\phi' \frac{d\mu''}{\mu''} \, d\phi'' \,, \tag{2.243}$$

$$\frac{\partial T(\tau_1;\mu,\phi,\mu_0,\phi_1,t)}{\partial\tau_1} + \frac{1}{\mu_0}\left(\frac{1}{c}\frac{\partial}{\partial t} + 1\right) T(\tau_1,\mu,\phi;\mu_0,\phi_0,t) =$$

$$= P(-\mu,\phi;\ -\mu_0,\phi_0)\ \exp\left(-\frac{\tau_1}{\mu}\right)\delta\left(t-\ \frac{\tau_1}{c\mu}\right) + \frac{1}{4\pi}\exp\left(-\ \frac{\tau_1}{\mu}\right) \times$$

$$\times \int_0^1 \int_0^{2\pi} P(-\mu,\phi;\mu'',\phi'')S\left(\tau_1;\mu'',\phi'',\mu_0,\phi_0,t-\ \frac{\tau_1}{c\mu}\right) \times$$

$$\times\ \frac{d\mu''}{\mu''}\ d\phi'' + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu,\mu',\phi',t)P(-\mu,\phi;\ -\mu_0,\phi_0) \times$$

$$\times\ \frac{d\mu'}{\mu'}\ d\phi' + \frac{1}{16\pi^2}\int_0^1 dt'\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu,\phi;\mu',t-t') \times$$

$$\times\ S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,t) \times$$

$$\times\ P(-\mu',\phi';\mu'',\phi'')\frac{d\mu'}{\mu'}\ d\phi'\ \frac{d\mu''}{\mu''}\ d\phi'' \qquad (2.244)$$

Equation (2.241), (2.242), (2.243), and (2.244) are the required functional equations for 'S' and 'T' functions. Let us now introduce the Laplace transform with respect to the time-variable which enables us to eliminate (at least formally) the time-variable,

$$\frac{\partial S(\tau_1;\mu,\phi;\mu_0,\phi_0,s)}{d\tau_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\left(1 + \frac{s}{c}\right)\ S(\tau_1;\mu,\phi;\mu_0,\phi_0,s) =$$

$$= P(\mu,\phi;\ -\mu_o,\phi_o) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(\mu,\phi;\mu'',\phi'') \times$$

$$\times\ S(\tau_1;\mu'',\phi'';\ \mu_o,\phi_o,s)\ \frac{d\mu''}{\mu''}\ d\phi'' +$$

$$+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu',\phi',s)\ P(-\mu',\phi';\ -\mu_o,\phi_o)\ \frac{d\mu'}{\mu'}\, d\phi' +$$

$$+ \frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi,\mu',\phi',s)\ S(\tau_1;\mu'',\phi'';\mu_o,\phi_o,s) \times$$

$$\times\ P(-\mu',\phi';\mu'',\phi'')\,\frac{d\mu'}{\mu'}\ d\phi'\ \frac{d\mu''}{\mu''}\ d\phi'' \qquad (2.245)$$

$$\frac{\partial T(\tau_1;\mu,\phi;\mu_o,\phi_o,s)}{\partial\tau_1} + \left(1 + \frac{s}{c}\right) T(\tau_1;\mu,\phi;\mu_o,\phi_o,s)\ \mu^{-1} =$$

$$= P(-\mu,\phi;\mu_o,\phi_o)\exp\left(-\frac{\tau_1 s}{c\mu_o}\right) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu'',\phi'';\mu_o,\phi_o,s) \times$$

$$\times\ P(-\mu,\phi;\ -\mu'',\phi'')\ \frac{d\mu''}{\mu''}\ d\phi'' + \frac{1}{4\pi}\ \exp\left(-\frac{\tau_1}{\mu_o}\right)\ \exp\left(-\frac{\tau_1 s}{c\mu_o}\right) \times$$

$$\times\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi,\mu',\phi',s)\ P(\mu,\mu';\ -\mu_o,\phi_o)\ \frac{d\mu'}{\mu'}\ d\phi' +$$

$$+ \frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu',\phi',s)\ P(\mu',\phi';\ -\mu'',\phi'') \times$$

$$\times\ T(\tau_1;\mu'',\phi'';\mu_o\phi_o s)\ \frac{d\mu'}{\mu'}d\phi'\ \frac{d\mu''}{\mu''}d\phi'', \tag{2.246}$$

$$\frac{\partial S(\tau_1;\mu,\phi;\mu_o,\phi_o,s)}{\partial\tau_1} = \exp\left[-\tau_1\left(\frac{1}{\mu} + \frac{1}{\mu_o}\right)\right] \times$$

$$\times \exp\left[-\frac{\tau_1 s}{c}\left(\frac{1}{\mu}+\frac{1}{\mu_o}\right)\right] P(\mu,\phi;\ -\mu_o,\phi_o) + \frac{1}{4\pi}\exp\left[-\frac{\tau_1}{\mu_o}\right] \times$$

$$\times\ \exp\left[-\frac{\tau_1 s}{c\mu}\right]\int_0^1\int_0^{2\pi} T(\tau_1;\mu'',\phi'';\mu_o,\phi_o,s) \times$$

$$P(\mu,\phi,\ -\mu'',\phi'',)\ \frac{d\mu''}{\mu''}\,d\phi'' + \frac{1}{4\pi}\exp\left[-\frac{\tau_1}{\mu_o}\right]\exp\left[\frac{\tau_1 s}{c\mu_o}\right] \times$$

$$\times\int_0^1\int_0^{2\pi} I(\tau_1;\mu,\phi,\mu',\phi,s)\ P(\mu',\phi';\ -\mu_o,\phi_O)\frac{d\mu'}{\mu'}\ d\phi' +$$

$$+\ \frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu,\phi;\mu',\phi',s)\ P(\mu',\phi';\ -\mu'',\phi'') \times$$

$$\times\ T(\tau_1;\mu'',\phi'';\mu_o,\phi_o,s)\ \frac{d\mu''}{\mu''}\,d\phi''\ \frac{d\mu'}{\mu'}\,d\phi', \tag{2.247}$$

$$\frac{\partial T(\tau_1;\mu,\phi,\mu_o,\phi_o,s)}{\partial\tau_1} + \frac{1}{\mu_o}\left[1+\frac{s}{c}\right] T(\tau_1;\mu,\phi;\mu_o,\phi_o,s) =$$

$$= \exp\left[-\frac{\tau_1}{\mu}\right]\ \exp\left[-\frac{\tau_1 s}{\mu}\right]\ P(-\mu,\phi;\ -\mu_o,\phi_o) +$$

$$+ \frac{1}{4\pi} \exp\left[-\frac{\tau_1}{\mu}\right] \exp\left[-\frac{\tau_1 s}{c\mu}\right] \int_0^1 \int_0^{2\pi} P(-\mu,\phi;\mu''\phi'') \times$$

$$\times S(\tau_1;\mu'',\phi'';\mu_o,\phi_o,s) \frac{d\mu''}{\mu''} d\phi'' + \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} T(\tau_1;\mu,\phi;\mu',\phi',s) \times$$

$$\times P(-\mu',\phi',-\mu_o,\phi_o) \frac{d\mu'}{\mu'} d\phi' + \frac{1}{16\pi^2} \times$$

$$\times \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} T(\tau_1;\mu,\phi,\mu',\phi',s) \; P(-\mu',\phi',\mu'',\phi'') \times$$

$$\times S(\tau_1;\mu'',\phi'';\mu_o,\phi_o,s) \frac{d\mu'}{\mu'} d\phi' \frac{d\mu''}{\mu''} d\phi''. \qquad (2.248)$$

2.64. The Reduction of the Integral Equations.

I have

$$P(\mu,\phi;\mu',\phi') = \sum_{m=0}^{N} (2-\delta_{0,m}) \left[\sum_{l=m}^{N} \omega_l^m \; P_l^m(\mu) \; P_l^m(\mu') \right] \times$$
$$\times \cos m(\phi' - \phi). \qquad (2.249)$$

If I follow Chandrasekhar (1960), I obtain

$$S(\tau_1;\mu,\phi;\mu_o,\phi_o,s) = \sum_{m=0}^{N} S^{(m)}(\tau_1;\mu,\mu_o;s) \cos m(\phi_o - \phi) \qquad (2.250)$$

$$T(\tau_1;\mu,\phi;\mu_o,\phi_o,s) = \sum_{m=0}^{N} T^{(m)}(\tau_1;\mu,\mu_o,s) \cos m(\phi_o - \phi) \qquad (2.251)$$

If I substitute these expansions of S and T in Equations (2.245)-(2.248) and after some rearrangements I get

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\left(1+\frac{s}{c}\right) S^{(m)}(\tau_1;\mu,\mu_0;s) + \frac{S^{(m)}(\tau_1;\mu;\mu_0;s)}{\partial\tau_1} =$$

$$= (2-\delta_{0,m})\sum_{m=0}^{N}(-1)^{m+1}\omega_l^m\Big[P_l^m(\mu) + \frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 S^m(\tau_1;\mu,\mu',s)\, P_l^m(\mu')\,\frac{d\mu'}{\mu'}\Big]\Big[P_l^m(\mu) + \frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 P_l^m(\mu'')\, S^{(m)}(\tau_1;\mu'',\mu_0,,s)\frac{d\mu''}{\mu''}, \qquad (2.252)$$

$$\frac{1}{\mu}\left(1+\frac{s}{c}\right) T^{(m)}(\tau_1;\mu,\mu_0,s) + \frac{\partial T^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1} =$$

$$= (2-\delta_{0,m})\sum_{l=m}^{N}\omega_l^m\Big[P_l^m(\mu) + \frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 S^m(\tau_1;\mu,\mu',s)\, P_l^m(\mu)\,\frac{d\mu'}{\mu'}\Big]\times$$

$$\times\Big[\exp\Big[-\frac{\tau_1}{\mu_0}\left(1+\frac{s}{c}\right)\Big]\Big] P_l^m(\mu_0) +$$

$$+ \frac{1}{2(2-\delta_{o,m})}\int_0^1 T^{(m)}(\tau_1;\mu'',\mu_o,s)\, P_l^m(\mu'')\,\frac{d\mu''}{\mu''}\Big] , \quad (2.253)$$

$$\frac{\partial S(\tau_1;\mu,\mu_o,s)}{\partial\tau_1} = (2-\delta_{o,m})\sum_{l=m}^{N}(-1)^{l+m}\,\omega_l^m \times$$

$$\times P_l^m(\mu)\exp\left(-\frac{\tau_1}{\mu}\left(1+\frac{s}{c}\right)\right) + \frac{1}{2(2-\delta_{o,m})}\times$$

$$\times\int_0^1 T^{(m)}(\tau_1;\mu,\mu',s)\,P_l^m(\mu')\,\frac{d\mu'}{\mu'}\Big]\times$$

$$\times\Big[P_l^m(\mu_o)\exp\Big[-\frac{\tau_1}{\mu_o}\left(1+\frac{s}{c}\right)\Big] + \frac{1}{2(2-\delta_{o,m})}\times$$

$$\times\int_0^1 P_l^m(\mu'')\,T^{(m)}(\tau_1;\mu'',\mu_o,s)\,\frac{d\mu''}{\mu''}\Big] , \quad (2.254)$$

$$\frac{1}{\mu_o}\left(1+\frac{s}{c}\right)T^{(m)}(\tau_1;\mu;\mu_o,s) + \frac{\partial T^{(m)}(\tau_1;\mu.\mu_o,s)}{\partial\tau_1} =$$

$$= (2-\delta_{o,m})\sum_{l=m}^{N}\omega_l^m\Big[P_l^m(\mu)\exp\Big[-\frac{\tau_1}{\mu_o}\left(1+\frac{s}{c}\right)\Big) +$$

$$+ \frac{1}{2(2-\delta_{o,m})}\int_0^1 T^{(m)}(\tau_1;\mu;\mu_o,s)\,P_l^m(\mu')\,\frac{d\mu'}{\mu'}\Big]\times$$

$$\times\Big[P_l^m(\mu_o) + \frac{(-1)^{l+m}}{2(2-\delta_{o,m})}\times$$

$$\times \int_0^1 P_l^m(\mu'')\ S^{(m)}(\tau_1;\mu'',\mu_0,s)\ \frac{d\mu''}{\mu''}\Big] \qquad (2.255)$$

If I now let

$$\psi_l^m(\tau_1;\mu,s) = P_l^m(\mu) + \frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\int_0^1 S^m(\tau_1;\mu,\mu',s)$$

$$\times P_l^m(\mu')\ \frac{d\mu'}{\mu'} \qquad (2.256)$$

and

$$\phi_l^m(\tau_1;\mu,s) = \exp\left[-\frac{\tau_1}{\mu_0}\left(1+\frac{s}{c}\right)\right] P_l^m(\mu_0) + \frac{1}{2(2-\delta_{0,m})} \times$$

$$\times \int_0^1 T^{(m)}(\tau_1;\mu,\mu',s)\ P_l^m(\mu)\ \frac{d\mu'}{\mu'} \qquad (2.257)$$

then,in view of principle of reciprocity (vide,Chandrasekhar ,1960) I can rewrite equations (2.252)-(2.255) in the form

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\left(1+\frac{s}{c}\right) S^{(m)}(\tau_1;\mu,\mu_0;s) + \frac{\partial T^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial \tau_1} =$$

$$= (2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{m+1}\omega_l^m\ \psi_l^m(\tau_1;\mu,s)\ \psi_l^m(\tau_1;\mu_0,s), \qquad (2.258)$$

$$\frac{1}{\mu}\left(1+\frac{s}{c}\right) T^{(m)}(\tau_1;\mu,\mu_0,s) + \frac{\partial T^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial \tau_1} =$$

$$= (2 - \delta_{o,m}) \sum_{l=m}^{N} \omega_l^m \psi_l^m (\tau_1 ;\mu , s) \phi_l^m (\tau_1 ;\mu_o s) \qquad (2.259)$$

and

$$\frac{\partial T^{(m)} (\tau_1 ;\mu ,\mu_o , s)}{\partial \tau_1} = (2 - \delta_{o,m}) \sum_{l=m}^{N} (-1)^{m+1} \omega_l^m \times$$

$$\times \phi_l^m (\tau_1 ;\mu , s) \phi_l^m (\tau_1 ;\mu_o s) \qquad (2.260)$$

$$\frac{1}{\mu_o} \left[1 + \frac{s}{c} \right] T^{(m)} (\tau_1 ;\mu ,\mu_o , s) + \frac{\partial T^{(m)} (\tau_1 ;\mu ,\mu_o , s)}{\partial \tau_1} =$$

$$= (2 - \delta_{o,m}) \sum_{l=m}^{N} \omega_l^m \phi_l^m (\tau_1 ;\mu , s) \psi_l^m (\tau_1 ;\mu_o , s) \qquad (2.261)$$

Now by use of equations (2.258) and (2.260) I get

$$\left(\frac{1}{\mu_o} + \frac{1}{\mu} \right) \left[1 + \frac{s}{c} \right] S^{(m)} (\tau_1 ;\mu ,\mu_o ; s) =$$

$$= (2 - \delta_{o,m}) \sum_{l=m}^{N} (-1)^{l+m} \omega_l^m [\psi_l^m (\tau_1 ;\mu , s) \psi_l^m (\tau_1 ;\mu_o , s) -$$

$$- \phi_l^m (\tau_1 ;\mu , s) \phi_l^m (\tau_1 ;\mu_o s)]; \qquad (2.262)$$

and by use of equation (2.259) and (2.261)

$$\left(\frac{1}{\mu} + \frac{1}{\mu} \right) \left[1 + \frac{s}{c} \right] T^{(m)} (\tau_1 ;\mu ,\mu_o , s) = (2 - \delta_{o,m}) \sum_{l=m}^{N} \times$$

$$\times w_l^m [\phi_l^m (\tau_1 ;\mu , s) \psi_l^m (\tau_1 ;\mu_o , s) -$$

$$- \psi_l^m(\tau_1;\mu,s)\ \phi_l^m(\tau_1;\mu_0,s)] \qquad (2.263)$$

Equations (2.262) and (2.263) are the two fundamental equations of our problem.

2.65. Legendre Expansion of the phase function and the principle of invariance.

Let us now consider that the atmosphere consists of two different layers. Denoting the quantities in the upper layer by subscript '1' and the quantities in the lower by subscript '2' and if I use equations (2.262) and (2.263) I have

$$S_1^{(m)}(\tau_i;\mu,\mu_0;s) = \frac{\mu\mu_0}{\mu+\mu_0}(2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}\frac{\omega_{i,l}^{(m)}}{Q}\times$$

$$\times\ \psi_l^m(\tau_i;\mu,s)\ \psi_l^m(\tau_i;\mu_0,s) -$$

$$- \phi_l^m(\tau_i;\mu,s)\ \phi_l^m(\tau_i;\mu_0,s) \qquad (2.264)$$

$$T_i^{(m)}(\tau_i;\mu,\mu_0;s) = \frac{\mu\mu_0}{\mu+\mu_0}(2-\delta_{0,m})\sum_{l=m}^{N}\frac{\omega_{i,l}^{(m)}}{Q}\times$$

$$\times\ [\phi_l^m(\tau_i;\mu,s)\ \psi_l^m(\tau_i;\mu_0,s) - \phi_l^m(\tau_i;\mu,s)\ \phi_l^m(\tau_i;\mu_0,s)], \qquad (2.265)$$

$$\psi_l^m(\tau_i;\mu_0,s) = P_l^m(\mu) + \frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\int_0^1 S^m(\tau_i;\mu,\mu',s)$$

$$\times\ P_l^m(\mu')\ \frac{d\mu'}{\mu'} \qquad (2.266)$$

and $\quad \phi_l^m(\tau_i;\mu,s) = P_l^m(\mu)\ \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{1}{2(2-\delta_{0,m})}\times$

$$\times \int_0^1 T_i^{(m)}(\tau_i;\mu,\mu'_0,s)P_l^m(\mu')\frac{d\mu'}{\mu'} , \quad (2.267)$$

where

$$Q = 1 + \frac{s}{c} \quad \text{and} \quad i = 1,2. \quad (2.268)$$

If I use the above representations and again if I use Equations (2.250) and (2.251) I can write the scattering and transmission function in each layer as

$$S_i(\tau_i;\mu,\phi;\mu_0,\phi_0,s) =$$

$$= \sum_{m=0}^{N} S_i^{(m)}(\tau_i;\mu,\mu_0,s) \cos m(\phi_0 - \phi); \quad (2.269)$$

$$T_i(\tau_i;\mu,\phi;\mu_0,\phi_0,s) = \sum_{m=0}^{N} T_i^{(m)}(\tau_i;\mu,\mu_0,s) \cos m(\phi_0 - \phi)$$

$$(i = 1,2). \quad (2.270)$$

In what follows I inquire into how represent the scattering and transmission functions in the whole atmosphere. If I follow Tsujita, I introduce diffuse radiation intensities $I_1(\tau_i;\mu,\phi;\mu_0,\phi_0,s)$ and $I_2(\tau_i,\mu,\phi;\mu_0,\phi_0;s)$ which leave the upper and lower layers in the direction (μ,ϕ) with respect to the boundary between the two layers, where (μ_0,ϕ_0) denotes the direction of the incident radiation at the upper surface $\tau = 0$. $I_1(\tau_i;\mu,\phi;\mu_0,\phi_0,s)$ and $I_2(\tau_i;\mu,\phi;\mu_0,\phi_0,s)$ must satisfy the conditions

$$I_1(\tau_i;\mu,\phi;\mu_0,\phi_0,s) = 0 \qquad \text{for} \quad 0<\mu<1 \quad (2.271)$$

$$I_2(\tau_i;\mu,\phi;\mu_0,\phi_0,s) = 0 \qquad \text{for} \quad -1<\mu<0 \quad (2.272)$$

Then from the principle of invariance (A) – (B) I have after the laplace transform with respect to time variable

$$I_2^{(m)}(\tau_1;\mu,\mu_0,s) = F\mu^{-1}\; S_2^{(m)}(\tau_2;\mu,\mu_0,s)\; \exp\left(-\frac{Q\tau_1}{\mu_0}\right) + \\ + \frac{1}{2(2-\delta_{0,m})\mu}\int_0^1 S_2^{(m)}(\tau_2;\mu,\mu',s)\times \\ \times\; I_1^{(m)}(\tau_1;\mu',\mu_0,s)\; d\mu'\, d\phi' , \tag{2.273}$$

$$I_1^{(m)}(\tau_1;\mu,\mu_0,s) = F\mu^{-1}\; T_1^{(m)}(\tau_1;\mu,\mu_0,s) + \frac{1}{2(2-\delta_{0,m})\mu}\times \\ \times\int_0^1 S_1^{(m)}(\tau_1;\mu,\mu',s)\, I_2^{(m)}(\tau_1;\mu',\mu_0,s)\; d\mu'\, d\phi' \tag{2.274}$$

From (C)-(D),

$$F\mu^{-1}S(\tau_0;\mu,\phi;\mu_0,\phi_0,s) = F\mu^{-1}S_1(\tau_1;\mu,\phi;\mu_0,\phi_0,s) + \\ + I_2(\tau_1;\mu,\phi;\mu_0,\phi_0,s)\exp\left(-\frac{\tau_1\phi}{\mu}\right) + \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi}\times \\ \times\; T_1(\tau_1;\mu,\phi;\mu_0',\phi,s)\; I_2(\tau_1;\mu',\phi';\mu_0,\phi_0;s)\; d\mu'\, d\phi' \tag{2.275}$$

and

$$F\mu^{-1}T(\tau_0;\mu,\phi;\mu_0,\phi_0,s) = F\mu^{-1}T_2(\tau_2;\mu,\phi;\mu_0,\phi_0,s)\times \\ \exp\left(-\frac{\tau_1 Q}{\mu_0}\right)\; I_1(\tau_1;\mu,\phi;\mu_0,\phi_0,s)\; \exp\left(-\frac{\tau_2 Q}{\mu}\right) + \\ + \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T_2(\tau_2;\mu,\phi;\mu_0',\phi_0';s)\times \\ \times\; I_1(\tau_1;\mu',\phi';\mu_0,\phi_0;s)\; d\mu'\, d\phi' \tag{2.276}$$

where τ_0, τ_1, and τ_2 are the optical thickness of the whole atmosphere, the upper and the lower layer, respectively. Furthermore, I assume that $I_1(\tau_1, \mu, \phi, \mu', \phi', s)$ can be expanded in the form

$$I_i(\tau_1; \mu, \phi; \mu_0', \phi_0', s) = \sum_{m=0}^{N} I_i^{(m)}(\tau_1; \mu, \mu_0', s) \cos m(\phi' - \phi), \qquad (i = 1,2) \tag{2.277}$$

If I substitute this expansion in equations (2.274) and (2.273) and taking account of equations (2.269) and (2.270) and allowing for

$$\int_0^{2\pi} \cos m(\phi'' - \phi) \cos n(\phi' - \phi'') d\phi'' =$$

$$= \delta_{m,n} \pi \cos m(\phi' - \phi) \; (m \neq 0, n \neq 0) = 2\pi \qquad (m = n = 0) \tag{2.278}$$

I obtain

$$I_1^{(m)}(\tau_1; \mu, \mu_0, s) = F\mu^{-1} T_1^{(m)}(\tau_2; \mu, \mu_0, s) +$$

$$+ \frac{1}{2(2 - \delta_{0,m})\mu} \int_0^1 S_1^{(m)}(\tau_2; \mu, \mu', s) \, I_2^{(m)}(\tau_1; \mu', \mu_0, s) d\mu \tag{2.279}$$

$$I_2^{(m)}(\tau_1; \mu, \mu_0, s) = F\mu^{-1} S_2^{(m)}(\tau_1; \mu, \mu_0, s) \exp\left[-\frac{\tau_1 Q}{\mu_0} \right] +$$

$$+ \frac{1}{2(2 - \delta_{0,m})\mu} \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s) \, I_1^{(m)}(\tau_1; \mu', \mu_0, s) d\mu' \tag{2.280}$$

2.66. Auxiliary Functions and their Functional Relations.

Let us now consider some auxiliary functions in term of which $I_1(\tau_1; \mu, \phi; \mu_0, \phi_0, s)$ and $I_2(\tau_1, \mu, \phi; \mu_0, \phi_0; s)$ are formed. If I assume that they depend on only one argument, I seek functional relations satisfied by them and then solve the system of equations. For convenience, I put

$$I_1^{(m)}(\tau_1;\mu,\mu_0,s) = F\,\frac{\mu_0}{\mu-\mu_0}\sum_{l=m}^{N}\omega_{1,l}^{(m)}\,A_l^{(m)}(\mu,\mu_0,s), \tag{2.281}$$

$$I_2^{(m)}(\tau_1;\mu,\mu_0,s) = F\,\frac{\mu_0}{\mu+\mu_0}\sum_{l=m}^{N}\omega_{2,l}^{(m)}\,B_l^{(m)}(\mu,\mu_0,s) \tag{2.282}$$

If I insert equations (2.281),(2.282),(2.264),and (2.265) into equations (2.279) and (2.280) and rearrange them appropriately, I have

$$\sum_{l=m}^{N}\omega_{2.l}^{(m)}\,A_l^{(m)}(\mu,\mu_0,s) = (2-\delta_{0,m})\sum_{l=m}^{N}\frac{\omega_{i,l}^{(m)}}{Q}\,\phi_l^{(m)}(\tau_1,\mu,s)\times$$

$$\times\ \psi_l^m(\tau_1;\mu_0,s) - \psi_l^m(\tau_1;\mu,s)\,\phi_l^m(\tau_1;\mu_0,s) +$$

$$+\ \frac{1}{2}\int_0^1\left\{\sum_{l=m}^{N}(-1)^{l+m}\frac{\omega_{i,l}^{(m)}}{Q}\left[\psi_1^m(\tau_1,\mu,s)\,\psi_1^m(\tau_1,\mu',s)\ -\right.\right.$$

$$\left.\left.-\ \phi_1^m(\tau_1,\mu,s)\,\phi_1^m(\tau_1,\mu',s)\right]\right\}\left[\sum_{l=m}^{N}\omega_{2.l}^{(m)}\,B_l^{(m)}(\mu',\mu_0,s)\right]\times$$

$$\times\left[\frac{\mu}{\mu-\mu'}-\frac{\mu_0}{\mu'+\mu_0}\right]d\mu', \tag{2.283}$$

$$\sum_{l=m}^{N}\omega_{2.l}^{(m)}\,B_l^{(m)}(\mu,\mu_0,s) = (2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}\frac{\omega_{2,l}^{(m)}}{Q}\ \times$$

$$\times\ [\psi_l^m(\tau_2,\mu,s)\,\psi_l^m(\tau_2,\mu_0,s) - \phi_l^m(\tau_2,\mu,s)\,\phi_l^m(\tau_2,\mu_0,s)]\ \times$$

$$\times \exp\left(-\frac{\tau_1 Q}{\mu_0} \right) + \frac{1}{2}\int_0^1 \left\{ \sum_{l=m}^{N} (-1)^{l+m} \frac{\omega_{i,l}^{(m)}}{Q} \times \right.$$

$$\left. \times \ [\psi_l^m(\tau_2, \mu, s)\ \psi_l^m(\tau_2, \mu_0', s) - \phi_l^m(\tau_2, \mu, s)\ \phi_l^m(\tau_2, \mu_0', s)] \right\} \times$$

$$\times \left[\sum_{l=m}^{N} \omega_{1,l}^{(m)} \ A_l^{(m)}(\mu, \mu_0, s)\right] \left[\frac{\mu}{\mu + \mu'} + \frac{\mu_0}{\mu' - \mu_0} \right] d\mu' , \qquad (2.284)$$

I rewrite equation (67) as

$$\sum_{l=m}^{N} \omega_{1,l}^{(m)} \ A_l^{(m)}(\mu, \mu_0, s) = \sum_{l=m}^{N} \frac{\omega_{i,l}^{(m)}}{Q} \phi_l^{(m)}(\tau_1, \mu, s) \times$$

$$\times \left[(2 - \delta_{0,m})\ \psi_l^m(\tau_1, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \int_0^1 \phi_l^m(\tau_1, \mu', s) \times \right.$$

$$\left. \times \frac{\sum_{l=m}^{N} \omega_{2,l}^{(m)} \ B_l^{(m)}(\mu', \mu_0, s)}{\mu' + \mu_0} d\mu' \right] - \sum_{l=m}^{N} \frac{\omega_{i,}^{(m)}}{Q} \psi_l^{(m)}(\tau_1, \mu, s) \times$$

$$\times \left[(2 - \delta_{0,m})\ \phi_l^m(\tau_1, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \int_0^1 \psi_l^m(\tau_1, \mu', s) \times \right.$$

$$\left. \times \frac{\sum_{l=m}^{N} \omega_{2,l}^{(m)} \ B_l^{(m)}(\mu', \mu_0, s)}{\mu' + \mu_0} d\mu' \right] + \frac{\mu}{2} \sum_{l=m}^{N} (-1)^{l+m} \frac{\omega_{i,l}^{(m)}}{Q} \times$$

$$\left[\int_0^1 \frac{\psi_l^m(\tau_1, \mu, s)\ \psi_l^m(\tau_1, \mu', s) - \phi_l^m(\tau_1, \mu, s)\ \phi_l^m(\tau_1, \mu', s)}{\mu + \mu'} \right] \times$$

$$\times \left[\sum_{l=m}^{N} \omega_{2.l}^{(m)} \; B_l^{(m)} (\mu', \mu_o, s) \; d\mu' \right], \tag{2.285}$$

If I take account of equations (2.264), we write the third term of the right-hand side of the above equation as

$$\frac{1}{2(2-\delta_{o,m})} \int_0^1 S_1^{(m)} (\tau_1; \mu, \mu', s) \times$$
$$\times \left[\sum_{l=m}^{N} \omega_{2,l}^{(m)} \; B_l^{(m)} (\mu', \mu_o, s) \right] \frac{d\mu'}{\mu'} \tag{2.286}$$

Then I put

$$\alpha_{1,l}^{m} (\mu_o, s) = (2 - \delta_{o,m}) \, \psi_l^{m} (\tau_1, \mu_o, s) + \frac{(-1)^{l+m}}{2} \mu_o \times$$
$$\times \int_0^1 \phi_1^{(m)} (\tau_1, \mu', s) \, \frac{\sum_{l=m}^{N} \omega_{2,l}^{(m)} \; B_l^{(m)} (\mu', \mu_o, s)}{\mu' + \mu_o} \, d\mu', \tag{2.287}$$

$$\alpha_{2,l}^{m} (\mu_o, s) = (2 - \delta_{o,m}) \, \phi_l^{m} (\tau_1, \mu_o, s) + \frac{(-1)^{l+m}}{2} \mu_o \times$$

$$\times \int_0^1 \psi_1^{(m)} (\tau_1, \mu', s) \, \frac{\sum_{l=m}^{N} \omega_{2,l}^{(m)} \; B_l^{(m)} (\mu', \mu_o, s)}{\mu' + \mu_o} \, d\mu'. \tag{2.288}$$

If I make use of equations (2.286),(2.287),(3.288) and rewrite equation (2.285) once more, I have

$$A_l^{(m)} (\mu, \mu_o, s) = \alpha_{1,l}^{m} (\mu_o, s) \, \phi_l^{(m)} (\tau_1, \mu, s) - \alpha_{2,l}^{m} (\mu_o, s) \times$$

$$\times\, \psi_l^{(m)}(\tau_1,\mu,s) + \frac{1}{2(2-\delta_{o,m})}\left[\frac{\omega_{i,l}^{(m)}}{\omega_{1,l}^{(m)}}\; Q\right] \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1,\mu,\mu',s)\; B_l^{(m)}(\mu',\mu_o,s)\, \frac{d\mu'}{\mu'}\;. \qquad (2.289)$$

On the other hand, by rewriting equation (2.284), I have

$$\sum_{l=m}^{N} \omega_{2.l}^{(m)}\; B_l^{(m)}(\mu,\mu_o,s) = \sum_{l=m}^{N} \frac{\omega_{2,l}^{(m)}}{Q}\,(-1)^{l+m}\, \psi_l^{(m)}(\tau_2,\mu,s) \times$$

$$\times \Bigg[(2-\delta_{o,m})\, \psi_l^{m}(\tau_2,\mu_o,\; s)\, \exp\left(-\frac{\tau_1 Q}{\mu_o}\right) + \frac{\mu_o}{2} \times$$

$$\times \int_0^1 \psi_1^{(m)}(\tau_2,\mu',s)\; \frac{\sum_{l=m}^{N} \omega_{1,l}^{(m)}\; A_l^{(m)}(\mu',\mu_o,s)}{\mu'-\mu_o}\; d\mu' \Bigg] -$$

$$-\sum_{l=m}^{N} \frac{\omega_{2,l}^{(m)}}{Q}\,(-1)^{l+m}\, \phi_l^{(m)}(\tau_2,\mu,s) \Bigg[(2-\delta_{o,m})\phi_l^{m}(\tau_2,\mu_o,\; s) \times$$

$$\times \exp\left(-\frac{\tau_1 Q}{\mu_o}\right) + \frac{\mu_o}{2}\int_0^1 \phi_1^{(m)}(\tau_2,\mu',s) \times$$

$$\times \frac{\sum_{l=m} \omega_{1,l}^{(m)}\; A_l^{(m)}(\mu',\mu_o,s)}{\mu'-\mu_o}\; d\mu' \Bigg] + \frac{1}{2(2\delta - \delta_{o,m})} \times$$

$$\times \int_0^1 S_2^{(m)}(\tau_2,\mu,\mu',s)\left[\sum_{l=m}^{N} \omega_{1.l}^{(m)}\; B_1^{(m)}(\mu',\mu_o,s)\right] \frac{d\mu'}{\mu'} \qquad (2.290)$$

Then I write $\alpha_{3,l}^{(m)}(\mu_o,s)$ and $\alpha_{4,l}^{(m)}(\mu_o,s)$ as

$$\alpha_{3,l}^{(m)}(\mu_o,s) = (2-\delta_{o,m})\, \psi_l^{m}(\tau_2,\mu_o,\; s)\, \exp\left(-\frac{\tau_1 Q}{\mu_o}\right) +$$

$$+ \frac{\mu_o}{2} \int_0^1 \psi_1^{(m)} (\tau_2 ,\mu' ,s) \frac{\sum_{l=m}^{N} \omega_{1,l}^{(m)} A_l^{(m)} (\mu' ,\mu_o ,s)}{\mu' - \mu_o} d\mu' \tag{2.291}$$

$$\alpha_{4,l}^{(m)} (\mu_o ,s) = (2 - \delta_{o,m}) \phi_l^m (\tau_2 ,\mu_o , s) \exp\left[-\frac{\tau_1 Q}{\mu_o} \right] +$$

$$+ \frac{\mu_o}{2} \int_0^1 \phi_1^{(m)} (\tau_2 ,\mu' ,s) \frac{\sum_{l=m}^{N} \omega_{1,l}^{(m)} A_l^{(m)} (\mu' ,\mu_o ,s)}{\mu' - \mu_o} d\mu' \tag{2.292}$$

If I make use of equations (2.291) and (2.292) and rewrite equation (2.290) once more, I have

$$B_l^{(m)} (\mu ,\mu_o ,s) = \alpha_{3,l}^{(m)} (\mu_o ,s) \psi_1^{(m)} (\tau_2 ,\mu ,s) - \alpha_{4,l}^{(m)} \times$$

$$\times (\mu_o ,s) \phi_1^{(m)} (\tau_2 ,\mu ,s) + \frac{1}{2 (2 - \delta_{o,m})} \left[\frac{\omega_{1,l}^{(m)}}{\omega_{2,l}^{(m)}} \right] Q \times$$

$$\times \int_0^1 S_2^{(m)} (\tau_2 ,\mu ,\mu' ,s) A_l^{(m)} (\mu' ,\mu_o ,s) \frac{d\mu'}{\mu'} . \tag{2.293}$$

From equations (2.289) and (2.293) I get

$$A_l^{(m)} (\mu ,\mu_o ,s) = \alpha_{1.l}^m (\mu_o ,s) \phi_l^{(m)} (\tau_1 ,\mu ,s) - \alpha_{2.l}^m (\mu_o ,s) \times$$

$$\times \psi_l^{(m)} (\tau_1 ,\mu ,s) + \alpha_{9,l}^{(m)} (\mu_o ,s) \frac{1}{2 (2 - \delta_{o,m})} \left[\frac{\omega_{2,l}^{(m)}}{\omega_{1.l}^{(m)}} \right] Q \times$$

$$\times \int_0^1 S_1^{(m)} (\tau_1 ,\mu ,\mu' ,s) \phi_l^{(m)} (\tau_2 ,\mu' ,s) - \alpha_{4,l}^{(m)} (\mu_o ,s) \frac{1}{2(2 - \delta_{o,m})} \times$$

$$\times \left[\frac{\omega_{2,l}^{(m)}}{\omega_{1.l}^{(m)}} \right] Q \int_0^1 S_1^{(m)} (\tau_1 ,\mu ,\mu_o ,s) \phi_l^{(m)} (\tau_2 ,\mu' ,s) \frac{d\mu'}{\mu'} +$$

$$+ \frac{1}{4 (2 - \delta_{o,m})} \int_0^1 S_1^{(m)} (\tau_1 ,\mu ,\mu' ,s) \int_0^1 S_2^{(m)} (\tau_1 ,\mu ,\mu'' ,s)$$

$$A_1^{(m)} \times (\mu'' ,\mu_o ,s) \frac{d\mu''}{\mu''} \frac{d\mu'}{\mu'} , \tag{2.294}$$

and

$$B_l^{(m)}(\mu,\mu_o,s) = \alpha_{1,l}^{m}(\mu_o,s)\,\frac{1}{2(2-\delta_{o,m})}\left[\frac{\omega_{1,l}^{(m)}}{\omega_{2,l}^{(m)}}\right] Q \times$$

$$\times \int_0^1 S_2^{(m)}(\tau_2,\mu,\mu',s)\,\phi_l^{(m)}(\tau_1,\mu',s)\,\frac{d\mu'}{\mu'} - \alpha_{2.l}^{m}(\mu_o,s) \times$$

$$\times \frac{1}{2(2-\delta_{o,m})}\left[\frac{\omega_{1,l}^{(m)}}{\omega_{2.l}^{(m)}}\right] Q \int_0^1 S_2^{(m)}(\tau_2,\mu,\mu',s)\,\psi_l^{(m)}(\tau_1,\mu',s) \times$$

$$\times \frac{d\mu'}{\mu'}\,\alpha_{3.l}^{m}(\mu_o,s)\,\psi_l^{(m)}(\tau_2,\mu',s) - \alpha_{4.l}^{m}(\mu_o,s) \times$$

$$\times \phi_l^{(m)}(\tau_2,\mu',s) + \frac{1}{4(2-\delta_{o,m})}\int_0^1 S_2^{(m)}(\tau_2,\mu,\mu',s) \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1;\mu,\mu'',s)\,B_l^{(m)}(\mu'',\mu_o,s)\,\frac{d\mu''}{\mu''}\,\frac{d\mu'}{\mu'} \; . \qquad (2.295)$$

Again from equations (2.294) and (2.295), if I use equations (2.289) and (2.293) I get

$$A_l^{(m)}(\mu,\mu_o,s) = \alpha_{1,l}^{(m)}(\mu_o,s)\;\beta_{1,l}^{(m)}(\mu,,s) - \alpha_{2,l}^{(m)}(\mu_o,s) \times$$

$$\times \beta_{2,l}^{(m)}(\mu,s) + \frac{1}{2(2-\delta_{o,m})}\left[\frac{\omega_{2,l}^{(2)}}{\omega_{1,l}^{(m)}}\right] Q\,\alpha_{3,l}^{(m)}(\mu_o,s) \times$$

$$\times \beta_{3,l}^{(m)}(\mu,s) - \alpha_{4,l}^{(m)}(\mu_o,s)\,\beta_{4,l}^{(m)}(\mu,s) \qquad (2.296)$$

$$B_l^{(m)}(\mu,\mu_o,s) = \frac{1}{2\,(2-\delta_{o,m})}\left[\frac{\omega_{1,l}^{(2)}}{\omega_{2,l}^{(m)}}\right] Q\,\alpha_{1,l}^{(m)}(\mu_o,s) \times$$

$$\times \gamma_{1,l}^{(m)}(\mu,s) - \alpha_{2,l}^{(m)}(\mu_o,s)\,\gamma_{2,l}^{(m)}(\mu,s) + \alpha_{3,l}^{(m)}(\mu_o,s) \times$$

$$\times \gamma_{3,l}^{(m)}(\mu_o,s) - \alpha_{4,l}^{(m)}(\mu_o,s)\,\gamma_{4,l}^{(m)}(\mu,s), \qquad (2.297)$$

$$B_{1,l}^{(m)}(\mu,s) = \phi_l^m(\tau_1,\mu,s)\frac{1}{4(2-\delta_{o,m})^2}\times$$

$$\times\int_0^1 S_l^{(m)}(\tau_1;\mu,\mu',s)\,\gamma_{1,l}^{(m)}(\mu',s)\frac{d\mu'}{\mu'}, \quad (2.298)$$

$$\beta_{2,l}^{(m)}(\mu,s) = \psi_l^{(m)}(\tau_1,\mu,s) + \frac{1}{4(2-\delta_{o,m})^2}\times$$

$$\times\int_0^1 S_l^{(m)}(\tau_1;\mu,\mu',s)\,\gamma_{2,l}^{(m)}(\mu',s)\frac{d\mu'}{\mu'}, \quad (2.299)$$

$$\beta_{3,l}^{(m)}(\mu,s) = \int_0^1 S_l^{(m)}(\tau_1;\mu,\mu',s)\,\gamma_{3,l}^{(m)}(\mu',s)\frac{d\mu'}{\mu'}, \quad (2.300)$$

$$\beta_{4,l}^{(m)}(\mu,s) = \int_0^1 S_l^{(m)}(\tau_1;\mu,\mu',s)\,\gamma_{4,l}^{(m)}(\mu',s)\frac{d\mu'}{\mu'}, \quad (2.301)$$

$$\gamma_{1,l}^{(m)}(\mu,s) = \int_0^1 S_2^{(m)}(\tau_2;\mu,\mu',s)\,\beta_{1,l}^{(m)}(\mu',s)\frac{d\mu'}{\mu'}, \quad (2.302)$$

$$\gamma_{2,l}^{(m)}(\mu,s) = \int_0^1 S_2^{(m)}(\tau_2;\mu,\mu',s)\,\beta_{2,l}^{(m)}(\mu',s)\frac{d\mu'}{\mu'}, \quad (2.303)$$

$$\gamma_{3,l}^{(m)}(\mu,s) = \psi_l^{(m)}(\tau_2,\mu,s) + \frac{1}{4(2-\delta_{o,m})^2}\times$$

$$\times\int_0^1 S_2^{(m)}(\tau_2;\mu,\mu',s)\,\beta_{3,l}^{(m)}(\mu',s)\frac{d\mu'}{\mu'}, \quad (2.304)$$

$$\gamma_{4,l}^{(m)}(\mu,s) = \phi_l^{(m)}(\tau_2,\mu,s) + \frac{1}{4(2-\delta_{o,m})^2} \times$$

$$\times \int_0^1 S_2^{(m)}(\tau_2;\mu,\mu',s)\,\beta_{4,l}^{(m)}(\mu',s)\frac{d\mu'}{\mu'} \qquad (2.305)$$

If I combine equation (2.298) with equation (2.302), equation (2.299) with equation (2.303), equation (2.300) with equation (2.304), and equation (2.301) with equation (2.305), I can determine $\beta_{i,l}^{(m)}(\mu,s)$ and $\gamma_{i,l}^{(m)}(\mu,s)$ (i=1,2,3,4) numerically. From equations (2.287), (2.288), (2.291), (2.292), (2.296), and (2.297) $\gamma_{i,l}^{(m)}(\mu_o,s)$, $A_l^{(m)}(\mu,\mu_o,s)$ and $\beta_l^{(m)}(\mu;\mu_o,s)$ can be calculated and then from equations (2.281) and (2.282), $I_1^{(m)}(\tau_2,\mu,\mu_o,s)$ and $I_2^{(m)}(\tau_2,\mu,\mu_o,s)$ are determined .Thus I obtained $S(\tau_o,\mu,\phi,\mu_o,\phi_o,s)$ and $T(\tau_o,\mu,\phi;\mu_o,\phi_o,s)$ from equations (2.275) and (2.276).

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CHAPTER - 3

SOLUTION OF RADIATIVE TRANSFER PROBLEMS IN AN ATMOSPHERE SCATTERING COHERENTLY

3.1. Introduction.

Chandrasekhar [1960] applied the method of discrete ordinates to solve the transfer equation for coherent scattering in stellar atmosphere with planck's function as a linear function of optical depth (equation (1.9)).

The equation of transfer for coherent scattering has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient , is constant) and by Strömgren's method (when η_ν has small but arbitrary variation with optical depth, (vide, Woolley and Stibbs, 1953)). Dasgupta [1977a] applied the method of Laplace transform and Wiener-Hopf technique to find an exact solution of transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions (equation (1.10) by use of a new representation of the H-function obtained by Dasgupta [1977b].

In the present work, the equation of transfer has been solved by different methods Viz.,

(i) Eddington's Method (Sec-3.2).

(ii) Laplace transform and Wiener-Hopf technique (Sec-3.3).

(iii) Busbridge's Method (Sec-3.4).

(iv) Discrete Ordinates (Sec-3.5).

in an isotropic coherently scattering atmosphere with exponential Planck function (equation (1.11)).

3.2. An approximate solution of the equation of transfer for coherent isotropic scattering by the method used by Eddington.

3.21. Equation of Transfer.

The equation of transfer for coherent scattering can be written (vide, Woolley and Stibbs, 1953) in the form

$$\cos\vartheta \, dI_\nu(\vartheta)/\rho dx = -(k + l_\nu)\, I_\nu(\vartheta) +$$

$$+ (1 - \varepsilon) l_\nu J_\nu + (k + \varepsilon l_\nu)\, B_\nu(T) \qquad (3.1)$$

To find an approximate solution of equation (3.1) we proceed as follows : let

$$J_\nu = (1/4\pi) \int I_\nu(\vartheta)\, d\omega \qquad (3.2)$$

$$H_\nu = (1/4\pi) \int I_\nu(\vartheta) \cos\vartheta \, d\omega \qquad (3.3)$$

$$K_\nu = (1/4\pi)\int I_\nu(\theta)\cos^2\vartheta\, d\omega \quad , \qquad (3.4)$$

in which the integration is made over all directions.

By multiplying equation (3.1) by ($d\omega/4\pi$) and ($d\omega \cos\vartheta/4\pi$) and integrating I obtain

$$dH_\nu/\rho dx = -(k + \varepsilon l_\nu)(J_\nu - B_\nu) \qquad (3.5)$$

$$dK_\nu/\rho dx = -(k + l_\nu)H_\nu \qquad (3.6)$$

where $B_\nu(T) = B_\nu$. If I measure the optical depth in the continuous spectrum outside the line so that $d\tau = -k\rho\, dx$ and set $l_\nu/k = \eta_\nu$, then (3.5) and (3.6) becomes

$$dH_\nu/d\tau = (1 + \varepsilon\eta_\nu)(J_\nu - B_\nu) \qquad (3.7)$$

$$dK_\nu/d\tau = (1 + \eta_\nu)H_\nu \qquad (3.8)$$

If , moreover ,I assume that η_ν is independent of τ , the equation can be readily integrated. Introducing Eddington's approximation

$$K_\nu = (1/3)J_\nu \qquad (3.9)$$

Equations (3.7) and (3.8) can be combined to give

$$\frac{d^2 J_\nu}{d\tau^2} = q_\nu^2\,(J_\nu - B_\nu) \qquad (3.10)$$

where

$$q_\nu^2 = 3(1 + \varepsilon\eta_\nu)(1 + \eta_\nu) \qquad (3.11)$$

Equation (3.10) is to be solved subject to the boundary conditions :

(A) $J_{\nu} = 2H_{\nu}$ at $\tau = 0$ and

(B) the requirement that $(J_{\nu} - B_{\nu})$ shall not increase exponentially as $\tau \longrightarrow \alpha$.

3.22. Solution of the Equation.

Using the exponential form of the planck function, (equation (1.11)) the equation (3.10) can be written in the form

$$\frac{d^2 J_{\nu}}{d\tau^2} = q_{\nu}^2 J_{\nu} - b_o q_{\nu}^2 \left[1 + (b_1/b_o) e^{-\beta\tau} \right] \quad (3.12)$$

which is a second order differential equation .

If I solve equation (3.12) and use the boundary condition (B) I get

$$J_{\nu} = b_o + b_1 e^{-\beta\tau} + b_2 e^{-q_{\nu}\tau} + \left[b_1 \beta^2 / (q_{\nu}^2 - \beta^2) \right] e^{-\beta\tau} \quad (3.13)$$

where b_2 is a constant to be determined from the boundary condition (A), where $\beta \neq q_{\nu}$.

From equation (3.13) I get

$$(dJ_\nu/d\tau)_{\tau=0} = -\left[\, b_1\beta + b_2 q_\nu + b_1\beta^3/(q_\nu^2 - \beta^2)\,\right] \qquad (3.14)$$

From equations (3.8) and (3.9) I find that

$$H_\nu = \left[\, 1/(3(1+\eta_\nu))\right]\left[\,(dJ_\nu/d\tau)\right] \qquad (3.15)$$

Hence , $b_2 =$

$$= -\frac{\left[(1+\eta_\nu)(b_0+b_1) + \frac{2}{3}\beta b_1 + (1+\eta_\nu+\frac{2}{3}\beta)\frac{b_1\beta^2}{q_\nu^2-\beta^2}\right]}{1+\eta_\nu+\frac{2}{3}q_\nu} \qquad (3.16)$$

Finally , I get

$$J_\nu = b_0 + b_1 e^{-\beta\tau} + \left[\, b_1\beta^2/(q_\nu^2 - \beta^2)\,\right]e^{-\beta\tau} -$$

$$-\frac{\left[(1+\eta_\nu)(b_0+b_1) + \frac{2}{3}\beta b_1 + (1+\eta_\nu+\frac{2}{3}\beta)\frac{b_1\beta^2}{q_\nu^2-\beta^2}\right]e^{-q_\nu\tau}}{1+\eta_\nu+\frac{2}{3}q_\nu} \qquad (3.17)$$

Now , J_ν (the average intensity) enables us to find the intensity within the absorption line at any optical depth and in any direction by solving the fundamental equation of the line formation ,

$$\cos\theta \; dI_\nu/d\tau = (1+\eta_\nu)I_\nu(\theta) - (1-\varepsilon)\eta_\nu J_\nu -$$

$$- (1 + \varepsilon\eta_\nu)B_\nu \qquad (3.18)$$

J_ν and b_ν being known function of τ. The solution for $I_\nu(\vartheta)$ can be written down immediately since equation (3.18) is a linear differential equation with constant coefficients.

3.23. Residual Intensity

The residual intensity in the mean contours is given (vide, Woolley and Stibbs, 1953) by

$$r_\nu = (H_\nu/H)_{\tau = 0}, \qquad (3.19)$$

where the omission of the suffix ν means outside the line. By virtu of the boundary condition $J_\nu = 2H_\nu$ at $\tau = 0$ we have

$$r_\nu = (J_\nu/J)_{\tau = 0}, \qquad (3.20)$$

Also, outside the line

$$\eta_\nu = 0 \text{ and } q_\nu = \sqrt{3}, \qquad (3.21)$$

equation (3.17) with $\tau = 0$, gives

$$J_\nu(0) = b_0 + b_1 + \left[b_1\beta^2/(q_\nu^2 - \beta^2) \right] -$$

$$- \frac{\left[(1 + \eta_\nu)(b_0 + b_1) + \frac{2}{3}\beta b_1 + (1 + \eta_\nu + \frac{2}{3}\beta)\frac{b_1\beta^2}{q_\nu^2 - \beta^2}\right]}{1 + \eta_\nu + \frac{2}{3}q_\nu} \qquad (3.22)$$

Hence , by equations (3.20), (3.21) and (3.22) I have

$$r_\nu = \frac{\frac{2}{3} q_\nu (\beta^2 - q_\nu^2) b_0 + \frac{2}{3} q_\nu^2 (\beta - q_\nu) b_1}{2\sqrt{3}(\beta^2 - 3) b_0 + 6(\beta - \sqrt{3}) b_1} \times$$

$$\times \frac{(\beta^2 - 3)(3 + 2\sqrt{3})}{(\beta^2 - q_\nu)(1 + \eta_\nu + \frac{2}{3} q_\nu)} \tag{3.23}$$

3.3. An Exact Solution of the Equation of Transfer for Coherent Scattering in an Exponential Atmosphere by the method of Laplace Transform and Wiener-Hopf technique.

3.31. Equation of transfer.

The equation of transfer considered here is of the form

$$\frac{dI_\nu(\tau,\mu)}{d\tau} = I_\nu(\tau,\mu) - \omega J_\nu(\tau) - (1-\omega) B_\nu(T) \tag{3.24}$$

where $B_\nu(T)$ is given by the equation (1.11) of Chapter 1,

where $\quad 0 < (1 - \varepsilon_\nu)/(1 + \eta_\nu) = \omega < 1 \quad (3.25)$

$$l_\nu/k = \eta_\nu , \qquad 0 < \varepsilon_\nu < 1 \; ; \tag{3.26}$$

l_ν , k being the line and continuous absorption coefficient; τ , the optical depth in the total absorption coeffient ; ε_ν , the collision constant ; and $I_\nu(\tau,\mu)$ is the intensity in the frequency , in the direction $\cos^{-1}\mu$, $J_\nu(\tau)$ is the average intensity

$$J_\nu(\tau) = (1/2)\int_{-1}^{+1} I_\nu(\tau,\mu)\, d\mu \;, \qquad (3.27)$$

For the solution of equation (3.24) I have the boundary conditions

$$I_\nu(0, -\mu) = 0 \;, \qquad 0 \le \mu \le 1 \qquad (3.28)$$

and $$I_\nu(\tau,\mu)\, e^{-\tau/\mu} \longrightarrow 0 \quad \text{as} \quad \tau \longrightarrow \infty \qquad (3.29)$$

3.32. Solution for Emergent Intensity.

The Laplace transform of $F(\tau)$ is denoted by $F^*(s)$, where $F^*(s)$ is defined by

$$F^*(s) = s\int_0^{\infty} \exp(-s\tau)\, F(\tau)\, d\tau \;, \qquad \mathrm{Rl}\; s>0 \qquad (3.30)$$

The formal solution of equation (3.24) (vide, Dasgupta, 1977a) is

$$I_\nu(0,\mu) = \omega\, J_\nu^*(1/\mu) + (1-\omega)\, B_\nu^*(1/\mu) \qquad (3.31)$$

The Laplace transformation of equation (3.24) with necessary rearrangement (vide, Dasgupta, 1977a) yields

$$T(z)\, I_\nu(0,z) = \omega\, G_\nu(z) + (1-\omega)\, B_\nu^*(1/z) \qquad (3.32)$$

where $$T(z) = 1 - (\omega/2)z \log\left[(z+1)/(z-1)\right] \tag{3.33}$$

and $$G_\nu(z) = (1/2)\int_0^1 \frac{x\, I_\nu(0,x)\, dx}{x-z} \tag{3.34}$$

T(z) has its roots $\pm K$, real for $0 < \omega \le 1$

$$k(>1) \longrightarrow \alpha \quad \text{as} \quad \omega \longrightarrow 1$$

According to Dasgupta [1974] I have

$$H(z) \longrightarrow H_0 + H_{-1}/z + \ldots \text{ as } z \longrightarrow \alpha \tag{3.35}$$

where $$H_0 = (1-\omega)^{-1/2} \tag{3.36}$$

and

$$H_{-1} = -(\omega H_0^2/2)\int_0^1 xH(x)\, dx \tag{3.37}$$

By the well-known relation (vide, Busbridge, 1960)

$$1/T(z) = H(z)H(-z) \qquad \text{on} \qquad [-1,1]^c \tag{3.38}$$

I rewrite equation (3.32) as

$$I_\nu(0,z)/H(z) = H(-z)\left[\omega\, G_\nu(z) + (1-\omega)\, B_\nu^*(1/z)\right] \tag{3.39}$$

If I use the Laplace transformation of equation (1.11) by equation (3.30) I have

$$B_\nu^*(s) = b_0 + sb_1/(s+\beta) \tag{3.40}$$

For $s = z^{-1}$

$$B_\nu^*(1/z) = b_0 + b_1/(1+\beta z) = (d_0 + d_1 z)/(1+\beta z) \text{ (say)} \tag{3.41}$$

where $d_1 = b_0\beta$ and $d_0 = b_0 + b_1$ (3.42)

If I insert equation (3.41) in equation (3.39) I have

$$I_\nu(0,z)/H(z) = H(-z)\left[\omega G_\nu(z) + (1-\omega)\times\right.$$
$$\left.\times\,(d_0 + d_1 z)/(1+\beta z)\right] \quad (3.43)$$

which can be written as

$$I_\nu(0,z)/H(z) = H(-z)\left[\omega G_\nu(z) +\right.$$
$$\left.+\,(1-\omega)\,(d_0/z + d_1)/(1/z+\beta)\right] \quad (3.44)$$

Now as $z \longrightarrow \infty$, $G_\nu(z) \longrightarrow O(1/z)$, since I seek a solution $I_\nu(0,z)$ regular for Re $z > 0$ and continuous on $[0,1]^c$ and since $H(z)$ is regular on $[-1,0]^c/[-k]$, $-k$ is a simple pole of $H(z)$, $1/H(z)$ being regular on $[-1,0]^c$.

I see that the left-hand side of equation (3.44) is regular at least for Re $z > 0$ except perhaps at ∞, and the right-hand side of equation (3.44) is regular at on $[0,1]^c$ except at ∞, both sides being bounded at the origin. The right-hand side of equation (3.44) is

$$C_0 \quad \text{as} \quad z \longrightarrow \infty \quad (3.45)$$

where $C_0 = H_0\,(1-\omega)\,d_1/\beta$ (3.46)

Hence , by a modified Lioville's theorem both sides of equation (3.44) can be equated to C_o, so that the left-hand side of equation (3.44) is C_o as z ——————> α , the right hand side of (3.44) is C_o as z ——————> α . Equation (3.44) can be put in the form

$$I(0,z)/H(z) = C_o = H_o(1 - \omega)\, d_1\beta \qquad (3.47)$$

If I use the relationship $d_1 = b_o\beta$ in (3.47) I get when z

$$I(0,z) = H(z)\,(1 - \omega)\, H_o b_o \qquad (3.48)$$

Since I have $H_o = (1 - \omega)^{-1/2}$. (3.49)

Hence , from equation (3.49) I get

$$I(0,z) = H(z)\,(1 - \omega)^{1/2}\, b_o \qquad (3.50)$$

Which is the same as deducted by Karanjai and Karanjai [1985].

3.4. Solution of the Equation of Transfer for Coherent Scattering in an Exponential Atmosphere by Busbridge's Method.

3.41. Equation of Transfer.

With the usual notation of transfer for the Milne-Eddington Model can be written (vide, Busbridge, 1953; Chandrasekhar , 1960) as

$$\mu \frac{dI_\nu}{\rho dz} = (k_\nu + \sigma_\nu)I_\nu - (1/2)\sigma_\nu \int_{-1}^{+1} I_\nu d\mu' - K_\nu B_\nu(T) \qquad (3.51)$$

where z is the depth below the surface ; k_ν, the continuous absorption coefficient ; and σ_ν is the line scattering coefficient . I assume that k_ν and σ_ν are independent of depth and I write

$$t = \int_0^z \rho(k_\nu + \sigma_\nu)\, dz \,, \qquad (3.52)$$

$$\tau = \int_0^z \rho k_\nu\, dz \,, \qquad (3.53)$$

$$\eta_\nu = \frac{\rho_\nu}{k_\nu} \,, \quad \lambda_\nu = \frac{1}{1+\eta_\nu} = \frac{k_\nu}{k_\nu + \rho_\nu} \qquad (3.54)$$

Then $\tau = \lambda_\nu t$ and $B_\nu(T) = b_0 + b_1 e^{-\beta\lambda_\nu t}$ (3.55)

where $B_\nu(T)$ is the Planck's function. Substituting into equation (3.51), I get

$$\mu \frac{dI_\nu}{dt} = I_\nu(t,\mu) - (1/2)(1-\lambda_\nu) \times$$

$$\times \int_{-1}^{+1} I_\nu(t,\mu')\, d\mu' - \lambda_\nu(b_0 + b_1 e^{-\beta\lambda_\nu t}) \qquad (3.56)$$

Equation (3.56) has to be solved subject to the boundary conditions

$$I_\nu(0, -\mu') = 0 \quad , \qquad 0 \le \mu' \le 1 \tag{3.57}$$

and $$I_\nu(t,\mu')\, e^{-t/\mu'} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \alpha \tag{3.58}$$

3.42. Solution for Emergent Intensity.

For convenience I suppress the subscript ν to the various quantities and consider a particular solution of equation (3.56) , which does not satisfy equation (3.57) in the form (vide, Busbridge , 1953)

$$I(\tau,\mu) = b_o + \frac{T_1 b_1}{1 + \beta\lambda\mu}\, e^{-\beta\lambda t} \tag{3.59}$$

where

$$T_1 = \frac{\lambda}{1 - \frac{1}{2\lambda\beta}\,(1-\lambda)\,\log\frac{1+\lambda\beta}{1-\lambda\beta}} \tag{3.60}$$

as readily verified by substitution . I therefore write (vide, Busbridge, 1953)

$$I(\tau,\mu) = b_o + \frac{T_1 b_1}{1 + \beta\lambda\mu}\, e^{-\beta\lambda t} + I^*(t,\mu) \tag{3.61}$$

Then $I^*(t,\mu)$ satisfied the integro-differential equation

$$\mu\,\frac{dI^*(t,\mu)}{dt} = I^*(t,\mu) - (1/2)(1-\lambda)\int_{-1}^{+1} I^*(t,\mu')\,d\mu' \tag{3.62}$$

together with the boundary conditions

$$I^{*}(\tau, -\mu') = -b_o - \frac{T_1 b_1}{1 - \beta\lambda\mu} \quad (0 \le \mu' \le 1) \qquad (3.63)$$

and $\quad I^{*}(t,\mu)\, e^{-t/\mu} \longrightarrow 0 \quad$ as $\quad t \longrightarrow \infty \qquad (3.64)$

I require the emergent intensity $I^{*}(0,\mu)$. This is the sum of $I_1^{*}(0,\mu)$, where $I_1^{*}(t,\mu)$ is the solution of equation (3.62) subject to boundary condition

$$I_1^{*}(0,\mu') = 0 \quad , \quad (0 < \mu' < 1) \qquad (3.65)$$

and $I_2(0,\mu)$ which is the diffusely reflected intensity corresponding to the incident intensity given by equation (3.63). It can be shown (See Appendix I) that unless $\lambda_\nu = 0$ (which is not so) ,

$$I_1^{*}(t,\mu) = 0 \qquad (3.66)$$

Hence

$$I^{*}(0,\mu) = I_2^{*}(0,\mu) = \frac{1}{2\mu}\int_0^1 S(\mu,\mu')\left[\frac{T_1 b_1}{\beta\lambda\mu - 1} - b_o\right] d\mu' \quad , \qquad (3.67)$$

where (vide, Chandrasekhar, 1960)

$$S(\mu,\mu') = (1 - \lambda) \frac{\mu\mu'}{\mu + \mu'} H(\mu)\, H(\mu') \qquad (3.68)$$

and $\quad H(\mu)$ is the solution of

$$H(\mu) = 1 + \frac{1}{2}(1 - \lambda)\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu + \mu'} d\mu' \qquad (3.69)$$

From equation (3.67) and (3.68) I have

$$I(0,\mu) = \frac{1}{2}(1-\lambda)H(\mu)\int_0^1 \left[\frac{T_1 b_1}{\beta\lambda\mu' - 1} - b_o \right] \frac{H(\mu')}{\mu + \mu'} \mu\mu' \, d\mu' =$$

$$= \frac{1}{2}(1-\lambda)H(\mu)T_1 b_1 \int_0^1 \frac{H(\mu')\mu'}{(\mu+\mu')(\beta\lambda\mu' - 1)} d\mu' -$$

$$- \frac{1}{2}(1-\lambda)H(\mu)b_o \int_0^1 \frac{H(\mu')\mu'}{\mu+\mu'} d\mu' =$$

$$= \frac{1}{2}(1-\lambda)H(\mu)\frac{T_1 b_1}{\beta\lambda}\int_0^1 \frac{H(\mu')}{\mu+\mu'} d\mu' +$$

$$+ \frac{1}{2}(1-\lambda)H(\mu)\frac{T_1 b_1}{\beta\lambda}\int_0^1 \frac{H(\mu')}{(\mu+\mu')(\beta\lambda\mu' - 1)} d\mu' -$$

$$- \frac{1}{2}(1-\lambda)H(\mu)b_o \int_0^1 \left[1 - \frac{\mu}{\mu+\mu'} \right] H(\mu')d\mu' \qquad (3.70)$$

After some rearrangement and with equation (3.69) , this gives

$$I^*(0,\mu) = \frac{H(\mu)\ T_1 b_1}{1+\beta\lambda\mu} \frac{1}{H(-1/\beta\lambda)} - \frac{T_1 b_1}{1+\beta\lambda\mu} + (H(\mu) - 1)\ b_o -$$

$$- \frac{1}{2}(1-\lambda)\ b_o\ H(\mu)\ b_o \alpha_o \qquad (3.71)$$

where

$$\alpha_n = \int_0^1 H(\mu)\ \mu^n\ d\mu \qquad (3.72)$$

Following Chandrasekhar [1960]

$$1 - \frac{1}{2}(1 - \lambda)\,\alpha_o = \lambda^{1/2} \tag{3.73}$$

I have from equations (3.61) and (3.71)

$$I(0,\mu) = \frac{H(\mu)\;T_1 b_1}{1 + \beta\lambda\mu}\;\frac{1}{H(-1/\beta\lambda)} + H(\mu)\,\lambda^{1/2}\;b_o \tag{3.74}$$

which represents our solution.

3.5. Solution of the Equation of Transfer for Coherent Scattering in an Exponential Atmosphere by the Method of Discrete Ordinates.

3.51. Equation of transfer.

The equation of transfer considered here is of the same form as in section 3.4. Following the same procedure as in section 3.4.1., equation (3.56) has to be solved using discrete ordinate method subject to the boundary conditions (3.57) and (3.58)

3.52. Solution for Emergent Intensity.

For convenience I suppress the subscript ν to the various quantities, assume $\alpha_\nu = \beta\lambda_\nu$ and in the n-th approximation , I replace equation (3.62) by the system of 2n linear equations

$$\mu_i \frac{dI_i^*}{dt} = I_i^* - (1/2)(1-\lambda) \sum_j a_j I_j^*, \quad i = \pm 1, \pm 2, \dots \pm n \qquad (3.75)$$

where the μ_i 's ($i = \pm 1$, ± 2, ... $\pm n$) and μ_{-i} $-\mu_i$) are the zeros of the Legendre polynomial $P_{2n}(\mu)$, a_j' s (j = ± 1, ... $\pm n$ and $a_{-j} = a_j$) are corresponding Gaussian weights. However , it is to be noted that there is no term with j = 0. For simplicity , in equation (3.75) I write

$$I_i^* \quad \text{for} \quad I_i^*(t,\mu_i) \qquad (3.76)$$

The system of equations (3.75) admits of integral of the form

$$I_i^* = g_i e^{-kt} \qquad (i = \pm 1 , \pm \quad \pm n) \qquad (3.77)$$

where the g_i's and k are constants.

Now if I insert this form for I_i^* in equation (3.75) I have

$$g_i (1 + \mu_i k) = \frac{1}{2}(1-\lambda) \sum_j a_j g_j \qquad (3.78)$$

Therefore

$$g_i = (1-\lambda) \frac{\text{constant}}{1 + \mu_i k} \qquad (3.79)$$

If I insert for g_i from equation (3.79) back into equation (3.78) I obtain the characteristic equation in the form

$$1 = \frac{1}{2}(1-\lambda)\sum_{j}\frac{a_j}{1+\mu_j k} \qquad (3.80)$$

If I remember that $a_j = a_{-j}$ and $\mu_{-j} = -\mu_j$ I can rewrite the characteristic equation in the form

$$1 = (1-\lambda)\sum_{j=1}^{n}\frac{a_j}{1-\mu_j^2 k^2} \qquad (3.81)$$

This is the characteristic equation which gives the values of k. If $\lambda > 0$, the characteristic equation (3.81) gives distinct non-zero roots which occur in pairs as $\pm k_r$ $(1,2,..,n)$ Therefore, equation (3.75) admits the 2n independent integrals of the form

$$I_i^* = (1-\lambda)\frac{\text{constant}}{1 \pm \mu_i k_r} e^{\pm k_r t} \qquad (3.82)$$

According to Chandrasekhar [1960], the solutions (3.77) satisfying our requirements that the solutions bounded by

$$I_i^* = (1-\lambda)\, b_1 \sum_{r=1}^{n}\frac{L_r e^{-k_r t}}{1+\mu_i k_r} \qquad (3.83)$$

together with the boundary condition

$$I_{-i}^* = -\frac{b_1 T}{1-\alpha\,\mu_{-i}} - b_0 \quad \text{at} \quad t = 0 \qquad (3.84)$$

3.53. The Elimination of the Constant and Expression of the Law of Diffuse Reflection in Closed Form.

The boundary condition and at the emergent intensity can be expressed in the form

$$S(\mu_i) = 0 \qquad (i = 1,2, \quad \ldots . \quad n) \tag{3.85}$$

and

$$I^*(0,\mu) = (1-\lambda)b_1\left[S(-\mu) - \frac{T/(1-\lambda)}{1+\alpha\mu} - \frac{b_0}{(1-\lambda)b_1}\right] \tag{3.86}$$

where

$$S(\mu) = \sum_{r=1}^{n} \frac{L_r}{1-k_r\mu} + \frac{T/(1-\lambda)}{1-\alpha\mu} + \frac{b_0}{(1-\lambda)b_1} \tag{3.87}$$

Next I observe that the function

$$(1-\alpha\mu)\prod_{r=1}^{n}(1-k_r\mu)\ S(\mu) \tag{3.88}$$

is a polynomial of degree $n+1$ in μ which vanishes for $\mu = \mu_i$, $i = 1,2, \ldots . n$. There must accordingly exist a relation of the form

$$(1-\alpha\mu)\prod_{r=1}^{n}(1-k_r\mu)S(\mu) \propto (\mu - C)\prod_{i=1}^{n}(\mu-\mu_i) \tag{3.89}$$

where C is a constant. The constant of proportionality can be found by comparing the coefficients of the highest power of μ (Viz., μ^{n+1}). Thus from equation (3.89) I have

$$S(\mu) = \frac{(-1)^{n+1}}{b_1(1-\lambda)}\ b_0\ k_1\ k_2 \ldots k_n \cdot \alpha\,\frac{P(\mu)(\mu-C)}{R(\mu)(1-\alpha\mu)}, \tag{3.90}$$

where

$$P(\mu) = \prod_{i=1}^{n} (\mu - \mu_i) \quad , \quad i = 1,2, \ldots n \tag{3.91}$$

and

$$R(\mu) = \prod_{r=1}^{n} (1 - k_r \mu) \quad , \quad r = 1,2 \quad ..n \tag{3.92}$$

Moreover, combining equations (3.90) and (3.91) I obtain

$$L_r = (-1)^n \frac{b_1}{b_o(1-\lambda)} k_1 \ldots\ldots k_n \, \alpha \, \frac{p(1/k_r)(1/k_r - C)}{R_r(1/k_r)(1 - \alpha/k_r)} \tag{3.93}$$

where $$R_r(x) = \prod_{h \neq r} (1 - k_r x) \tag{3.94}$$

and $$\alpha \neq k_r \tag{3.95}$$

The roots of the characteristic equation (3.80) can be written in the form

$$k_1 k_2 \ldots\ldots k_n \; \mu_1 \mu_2 \; \ldots\ldots\ldots\ldots \mu_n \quad = \lambda^{1/2} \tag{3.96}$$

Now by use of equation (3.96) equation (3.90) becomes

$$S(\mu) = - \frac{b_o \alpha \lambda^{1/2} H(-\mu)(\mu - C)}{(1-\lambda) \; b_1 \; (1 - \alpha\mu)} \tag{3.97}$$

where

$$H(\mu) = \frac{1}{\mu_1 \mu_2 \qquad \mu_n} \; \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{r=1}^{n} (1 + k_r \mu)} \tag{3.98}$$

and the characteristic roots are evaluated from Equation (3.87)

If I put $\mu = 0$ in equations (3.87) and (3.97) I have

$$\sum_{r=1}^{n} L_r + \frac{T}{1-\lambda} + \frac{b_o}{(1-\lambda)b_1} = \frac{b_o\lambda^{1/2}C\alpha}{(1-\lambda)b_1} \qquad (3.99)$$

I can next evaluate

$\sum_{r=1}^{n} L_r$ from equation (3.93) . Then

$$\sum_{r=1}^{n} L_r = (-1)^{n+1} \frac{b_o}{(1-\lambda)b_1} k_1 k_2 \ldots k_n \alpha f(0), \qquad (3.100)$$

where

$$f(x) = \sum_{r=1}^{n} \frac{P(1/k_r)(1/k_r - C)}{R_r(1/k_r)(1-\alpha/k_r)} \qquad (3.101)$$

Now $f(x)$ defined in this manner is a polynomial of degree $(n-1)$ in x which takes the values

$$\frac{P(1/k_r)(1/k_r - C)}{(1-\alpha/k_r)} \qquad (3.102)$$

for $x = 1/k_r$ $(r = 1, 2, \ldots\ldots, n)$ (3.103)

In other words ,

$$(1-\alpha x)f(x) - P(x)(x - C) = 0 \qquad (3.104)$$

Therefore, I must accordingly have a relation of the form

$$(1-\alpha x)f(x) - P(x)(x - C) = R(x)(Ax + B), \qquad (3.105)$$

where A and B are certain constants to be determined. The constant A follows from the comparison of the coefficients of x^{n+1}. Thus

$$A = \frac{(-1)^{n+1}}{k_1 k_2 \ldots k_n} \tag{3.106}$$

Next, if I put $x = \alpha^{-1}$ in equation (3.109) I have

$$B = \frac{(-1)^n}{\alpha k_1 k_2 \ldots k_n} + \frac{(C - 1/\alpha)P(1/\alpha)}{R(\alpha^{-1})} \tag{3.107}$$

$$\text{i.e., } B = \frac{(-1)^n}{\alpha k_1 k_2 \ldots k_n} + (-1)^n \mu_1, \mu_2 \ldots \mu_n H(-1/\alpha)(C - 1/\alpha) \tag{3.108}$$

Now by use of the relations (3.106), (3.107), and (3.109) I get

$$f(0) = -CP(0) + BR(0) = -C(-1)^n \mu_1 \mu_2 \ldots \mu_n +$$

$$+ \frac{(-1)^n}{\alpha k_1 k_2 \ldots k_n} + (-1)^n \mu_1, \mu_2, \ldots \mu_n H(-1/\alpha)(C - \alpha^{-1}) \tag{3.109}$$

From the equation (3.101) using equation (3.109) I have

$$\sum_{r=1}^{n} L_r = \frac{b_0 \lambda^{1/2} C\alpha}{(1-\lambda)b_1} - \frac{b_0}{(1-\lambda)b_1} +$$

$$+ \frac{b_0 \alpha \lambda^{1/2} H(-1/\alpha)(1/\alpha - C)}{(1-\lambda)b_1} \tag{3.110}$$

By use of equation (3.110) in equation (3.104) I get

$$C = \frac{1}{\alpha} + \frac{Tb_1}{b_0 \alpha \lambda^{1/2} H(-1/\alpha)} \tag{3.111}$$

If moreover, we combine equation (3.110) , the diffusely reflected intensity $I^*(0,\mu)$ in equation (3.86) takes the form

$$I^{*}(0,\mu) = \frac{b_o \alpha \lambda^{1/2} H(\mu)[\mu + C]}{1 + \alpha\mu} - \frac{T b_o}{1 + \alpha\mu} - b_o \qquad (3.112)$$

This is the required solution in closed form. If I combine equation (3.61) at t = 0 and equation (3.112) I have

$$I(0,\mu) = \frac{b_o \alpha \lambda^{1/2} H(\mu)[\mu + C]}{1 + \alpha\mu} \qquad (3.113)$$

which is the required solution of equation (3.56) in the nth appoximation by the discrete ordinate method .

On putting C from equation (3.111) I get the solution in the form

$$I(0.\mu) = b_o \lambda^{1/2} H(\mu) + \frac{b_1 T H(\mu)}{1 + \alpha\mu} \frac{1}{H(-1/\alpha)} \qquad (3.114)$$

Chandrasekhar's (1960) solution for $I(0,\mu)$ in the case of coherent scattering is given by (for $B_\nu(T) = b_o + b_1 \tau$)

$$I(0,\mu) = b_o \lambda^{1/2} H(\mu) + b_1 \lambda^{3/2} H(\mu)\mu +$$

$$+ (1/2) b_1 \lambda (1 - \lambda) H(\mu) \alpha_1 \qquad (3.115)$$

$$\text{where } \alpha_n = \int_0^1 H(\mu)\mu^n d\mu \qquad (3.116)$$

3.54. Conclusion.

If I compare equations (3.114) and (3.115) I see that by putting $b_1 = 0$ I have the same solution for both the cases.

Moreover for large values of β (i.e., $\beta \longrightarrow \alpha$), since $\alpha = \beta\lambda$, the solutions (3.114) takes the form

$$I(0,\mu) = b_o \lambda^{1/2} H(\mu) \qquad (3.117)$$

i.e. $B_\nu(T)$ then behaves like a constant or independent of τ. This fact can also be explained from the point of view that

$$B_\nu(T) = b_o + b_1 e^{-\beta\tau} \longrightarrow b_o \text{ as } \beta \longrightarrow \alpha$$

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CHAPTER - 4

SOLUTION OF RADIATIVE TRANSFER PROBLEMS IN AN ATMOSPHERE SCATTERING NONCOHERENTLY

4.1. Introduction.

When there is no redistribution in frequencies the line is said to be formed by coherent scattering. Eddington noticed another departure from the simple case , which he called interlocking of lines (without redistribution).

If two or more lines in a spectrum have a common upper state , the atom can be excited to that state by absorption in either line but the re-imission will take place according to the transmission probability regardless of the path by which the excitation was made.

The equations of formation of the lines are not independent but contain cross terms. The equation for the intensity in a particular frequency of a spectral line might then, in general , contain an infinite set of terms involving the intensities of the other frequencies in the same line , as well as term involving the intensities in a finite number of other lines in the same spectrum.

Fortunately, these difficulties do not arise in some important cases , namely principal lines in spectra, in which the ground state is sharp. The reason for this is that the distribution of energy levels within a state depends on the life of the state.

The fundamental equation for the formation of a line , under any circumstances is given by equation

$$\cos\vartheta \frac{dI_\nu(\vartheta)}{\rho dx} = -(k_\nu + l_\nu)\, I_\nu(\vartheta) - j_\nu \qquad (4.1)$$

In the case of coherent scattering I had

$$j_\nu = (k_\nu + \varepsilon l_\nu)\, B(\nu,T) + (1-\varepsilon) l_\nu J_\nu \qquad (4.2)$$

which I can replace by more general expression when $p(\nu,\nu') = 1$ for $\nu' = \nu$ and is zero for all other ν, I recover the case of coherent formation. Otherwise $p(\nu,\nu')$ must have non-zero values for all frequencies ν' which are connected with the frequencies ν , either by interlocking or by redistribution.

Interlocking without redistribution :

(a) Lines with common upper state .

After coherent scattering , the next simplest case is that of interlocking of principal lines , for then $p(\nu,\nu')$ takes a small number only of non-zero values. Examples of this are the principal lines, of Al, $^2S_{1/2} - {}^2P_{3/2}$ at λ 3,962A , and $^2S_{1/2} - {}^2P_{1/2}$ at λ 3,944A, in which $^2P_{1/2}$ is the ground state and $^2P_{3/2}$ metastable: and the principal triplet of Mg, $^3S_1 — ^3P_2$ at λ 5,184A, $^3S_1 — ^3P_1$ at λ 5,173A , and $^3S_1 — ^3P_0$ at λ 5,167A. In this case $^3P_2$ and $^3P_0$ are metastable , and $^3P_1$ is linked by an inter combination line to the ground state $^1S_0$.

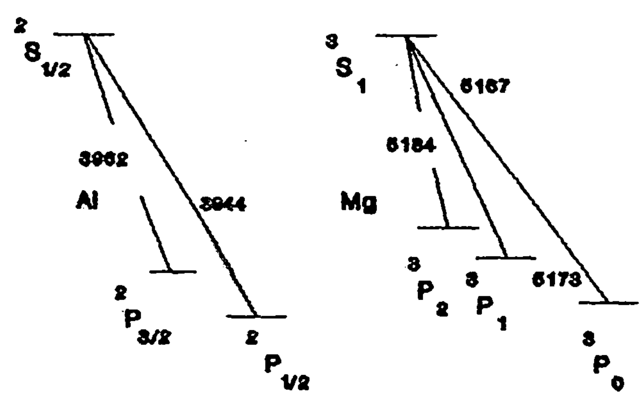

Fig.4.0. Interlocked principal lines of Al and Mg.

Taking the Planck function to be linear in optical depth Woolley and Stibbs [1953] have obtained a solution by means of Eddington's approximation and calculated the residual intensities and the total absorption in the emergent flux for doublet and triplet lines. Busbridge and Stibbs [1954] applied the principle of invariance governing the law of diffuse reflection with a slight modification to solve exactly the equation of transfer in the M–E Model. Dasgupta and Karanjai [1972] applied Sobolev's Probabilistic method to solve the transfer equation for the case of interlocking without redistribution. Dasgupta [1978] obtained an exact solution of the transfer equation for non-coherent scattering arising from interlocking without redistribution by Laplace transform and the Wiener–Hopf technique using a new representation of the H–function obtained by Dasgupta [1977]. Dasgupta considered the planck function to be linear in τ (optical depth) (equation (1.9)). Karanjai and Barman [1981] solved the same problem using discrete ordinate method taking Planck function as linear in τ.

Karanjai and Karanjai [1985] solved the equation of transfer for interlocked multiplets with the Planck function as a non–linear function of optical depth following the method used by Dasgupta [1978]. They considered two non–linear forms of Planck function Viz;,

a) an exponential atmosphere ,(vide, Delg'Innocenti, 1979) (equation (1.11)),

b) an atmosphere (vide, Busbridge ,1955) in which

$$B_\nu (T) = B(t) = b_0 + b_1\tau + E_2 (\tau) ; \tag{4.3}$$

In the present work, the same problem, has been solved by the discrete ordinate method and by the method used by Busbridge and Stibbs [1954] using Planck function as an exponential function of optical depth in section 4.2. and 4.3. respectively.

Busbridge and Stibbs (1954) applied the principle of invariance govening the law of diffuse reflection with a slight modification to solve the same problem. The expression for emergent intensity thus obtained involves Chandrasekhar's H-function within and outside the integral sign. They were afraid that the computional labour in the calculation of H-function would be great and so avoided the calculation of residual intensities for triplets and higher multiplets.

Karanjai (1968a) used his approximate form of H-function (1968b) in the calculation of residual intensities from the expression obtained by Busbridge and Stibbs (1954) for a doublet and triplet.

In the prsent work, other various approximate forms of H-function (vide, Karanjai and Sen, 1970, 1971) have been used to calculate the residual intensities for doublets and triplets (Sec-4.4).

4.2. Solution of the equation of transfer for interlocked multiplets by the method of discrete ordinates with the planck function as a nonlinear function of optical depth.

4.21. The Equation of Transfer.

Woolley and Stibbs [1953] made certain assumptions ,Viz (i) that the lines are so close together that variations of the continuous absorption coefficient k and of the Planck function $B_\nu(T)$ with wavelength may be neglected . This also means that the lower states are nearly equal in excitation potential and that they have the same classical damping constant. Then the values of $\eta_1, \eta_2, \eta_3,$ $,\eta_k$ (the ratios of the line absorption coefficients to k)are proportional to the transition probabilities for spontaneous emission from the upper state to the respective lower states ; (ii) that $\eta_1, \eta_2, \eta_3,$ $,\eta_k$ are independent of depth ; (iii) that the coefficient ε , which is independent of both frequency and depth .

The equation of transfer considered here is of the form (vide, Woolley and Stibbs, 1953)

$$\frac{\mu dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)\, B_\nu(T) -$$

$$- (1/2)(1-\varepsilon)\alpha_r \sum_{p=1}^{k} \eta_p \int_{-1}^{+1} I_p(\tau,\mu')d\mu' , \ (r = 1,2, \ \ k) \qquad (4.4)$$

where

$$\alpha_r = \eta_r \,/\, (\eta_1 + \eta_2 + \ldots + \eta_r) \quad , \tag{4.5}$$

and $$\alpha_1 + \alpha_2 + \;\ldots\; + \alpha_k = 1 \tag{4.6}$$

In equation (4.4) the subscript r denotes the quantity corresponding to the line of frequency ν_r. The equation (4.4) have to be solved subject to the boundary conditions

$$I_r(0,-\mu') = 0 \qquad (0< \mu'<1) \quad , \tag{4.7}$$

together with a condition limiting $I_r(\tau,\mu)$ for large τ. I shall assume that $I_r(\tau,\mu)$ is at most linear in τ for large τ.

4.22. Solution.

If I consider the planck function ,i.e., the thermal source function to be exponential (equation (1.11) (vide, Degl'Innocenti , 1979) then following Stibbs [1953] and Busbridge [1953] I have

$$I_r(\tau,\mu) = b_0 + \left[\frac{b_1 T_r}{1 + \beta\mu\zeta_r} \right] e^{-\beta\tau} + I_r^*(\tau , \mu) \tag{4.8}$$

represents the solution of equation (4.4). where T_r can be expressed as

$$T_r = (1 + \varepsilon\, \eta_r) + \alpha_r \sum_{p=1}^{k} T_p \left[\frac{1}{2\beta}\, (1 - \varepsilon) \times \right.$$

$$\times \log \frac{1 + \eta_p + \beta}{1 + \eta_p - \beta} \Bigg] \tag{4.9}$$

and

$$\zeta_r = \frac{1}{1 + \eta_r} \tag{4.10}$$

This consists of two parts. The first part consists of the solution for a bounded atmosphere as τ tends to infinity. The second part viz., $I_r^*(\tau,\mu)$, represents the departure of the asymptotic solution from the value $I_r(\tau,\mu)$ as I approach the boundary.

Now inserting $I_r(\tau,\mu)$ from equation (4.8) in equation (4.4) and taking

$$\omega_r = \frac{(1 - \varepsilon)\, \eta_r}{1 + \eta_r} \tag{4.11}$$

I have the equation

$$\zeta_r \mu \frac{dI_r^*(\tau,\mu)}{d\tau} = I_r^*(\tau,\mu) - \frac{\omega_r}{2} \frac{1}{\sum_{p=1}^{k} \eta_p} \Bigg[\sum_{p=1}^{k} \eta_p \times$$

$$\times \int_{-1}^{+1} I_p^*(\tau,\mu')\, d\mu' \Bigg] \tag{4.12}$$

together with the boundary conditions

$$I_r^*(0,-\mu') = -\ b_0 - \frac{b_1 T_r}{1-\beta\zeta_r\mu'} \tag{4.13}$$

and $$I_r^*(\tau,\mu)\ e^{-\tau/\mu} \longrightarrow 0 \quad \text{as} \quad \tau \longrightarrow \alpha \tag{4.14}$$

Further $I_r^*(\tau,\mu)$ should be at most linear in τ as $\tau \longrightarrow \alpha$
For convenience , equation (4.12) is written in the form

$$\zeta_r\mu \frac{dI^*_{(r)}(\tau,\mu)}{d\tau} = I^*_{(r)}(\tau,\mu) - \frac{\omega_r}{2}\frac{1}{\sum_{p=1}^{k}\eta_p}\left[\sum_{p=1}^{k}\eta_p \times \right.$$

$$\left. \times \int_{-1}^{+1} I^*_{(p)}(\tau,\mu')\ d\mu'\right] \tag{4.15}$$

together with the boundary conditions

$$I_r^*(0,-\mu') = -\ b_0 - \frac{b_1 T_r}{1-\beta\zeta_r\mu'} \tag{4.16}$$

and $$I_r^*(\tau,\mu)\ e^{-\tau/\mu} \longrightarrow 0 \quad \text{as} \quad \tau \longrightarrow \alpha \tag{4.17}$$

Equation (4.15) can be replaced by the system of 2n linear equations

$$\zeta_r\ \mu_{(r)i}\frac{dI^*_{(r)i}}{d\tau} = I^*_{(r)i} - \frac{\omega_r}{2}\frac{1}{\sum_{p=1}^{k}\eta_p}\left[\times\right.$$

$$\left. \times \sum_{p=1}^{k}\eta_p \sum_j a_j\ I^*_{(p)j}\right] \tag{4.18}$$

where $(i = \pm 1, \pm 2, \ldots, \pm n)$

where the $\mu_{(r)i}$'s ($i = 1, \ldots, n$ and $\mu_{(r)-i} = -\mu_{(r)i}$) are the zeros of the Legendre polynomials $P_{2n}(\mu)$ which are dependent on the lines of interlocking and a_j's ($j = \pm 1, \ldots, \pm n$ and $a_{-j} = a_j$) are corresponding Gaussian weights. However, it is to be noted that there is no term with $j = 0$. For simplicity, I write

$$I^{*}_{(r)i} \text{ for } I^{*}_{(r)}(\tau, \mu_{(r)i}) \tag{4.19}$$

in equation (4.18)

The system of equations admits of integrals of the form

$$I^{*}_{(r)i} = g_{(r)i} \; e^{-K\tau} \quad (i = \pm 1, \ldots \pm n) \tag{4.20}$$

where the $g_{(r)i}$'s and K are constants.

Now inserting this form for $I^{*}_{(r)i}$ in equation (4.18) I have

$$g_{(r)i} \; [1 + \zeta_r \mu_{(r)i} K] = \frac{\omega_r}{2} \frac{1}{\sum_{p=1}^{k} \eta_p} \times$$

$$\times \left[\sum_{p=1}^{k} \eta_p \sum_{j} a_j g_{(p)j} \right] \tag{4.21}$$

$$\therefore \quad g_{(r)i} = \omega_r \frac{\text{Constant}}{1 + \zeta_r \mu_{(r)i} K} \tag{4.22}$$

Inserting for $g_{(r)i}$ from equation (4.22) back into equation (4.21) I are left with

$$1 = \frac{1}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \omega_p \sum_j \frac{a_j}{1 - \zeta_p^2 \mu_{(p)j}^2 K^2} \right] \qquad (4.23)$$

This is the characteristic equation which gives the values of $K_{(r)}$. If $\omega_r < 1$,($r = 1,2,3, \ldots n$), the characteristic equation (4.23) gives distinct non-zero roots which occur in pairs as $\pm K_{(r)\alpha}$ ($\alpha = 1,2, \ldots,n$). Therefore , equation (4.18) admits the 2n independent integrals of the form

$$I^*_{(r)i} = \omega_r \frac{\text{Constant}}{1 \pm \zeta_r \mu_{(r)i} K_{(r)\alpha}} e^{\mp K_{(r)\alpha}\tau} \qquad (4.24)$$

According to Chandrasekhar [1960] , the solutions (4.20) satisfying our requirements of the boundedness of the solutions are

$$I^*_{(r)i} = \omega_r b_1 \sum_{\alpha=1}^{n} \frac{L_{(r)\alpha} e^{-K_{(r)\alpha}\tau}}{1 + \zeta_r K_{(r)\alpha} \mu_{(r)i}} , \qquad (4.25)$$

together with the boundary condition

$$I^*_{(r)-1} = - b_o - \frac{b_1 T_r}{1 - \zeta_r \beta \, \mu_{(r)-i}} \text{ at } \tau = 0 \qquad (4.26)$$

4.23. The elimination of the constants and the expression of the law of diffuse reflection in closed form.

The boundary conditions and the emergent intensity can be expressed in the form

$$S_r(\mu_{(r)i}) = 0 \;, (i = 1,2, \; ... \; n \;) \qquad (4.27)$$

and

$$I^*_{(r)} (0,\mu) = \omega_r b_1 \left[S_r \; (-\mu) - \frac{T_r/\omega_r}{1 + \zeta_r \beta\mu} - \frac{b_o}{\omega_r b_1} \right] \qquad (4.28)$$

where

$$S_r (\mu) = \sum_{\alpha=1}^{n} \frac{L_{(r)\alpha}}{1 - \zeta_r K_{(r)\alpha}\mu} + \frac{T_r/\omega_r}{1 - \zeta_r \beta\mu} + \frac{b_o}{\omega_r b_1} \qquad (4.29)$$

Next I observe that the function

$$(1 - \zeta_r \beta\mu) \prod_{\alpha=1}^{n} (1 - \zeta_r K_{(r)\alpha} \mu) \; S_r (\mu) \qquad (4.30)$$

is a polynomial of degree (n+1) in μ which vanishes for μ = μ_i. (i = 1,2, n). There must accordingly exist a relation of the form

$$(1 - \zeta_r \beta\mu) \prod_{\alpha=1}^{n} (1 - \zeta_r K_{(r)\alpha} \mu) \times$$

$$\times S_r(\mu) \,..(\mu - C_r)\prod_{i=1}^{n} (\mu - \mu_i) \tag{4.31}$$

where C_r is a constant.

The constant of proportionality can be found by comparing the coefficient of the highest power of μ (namely μ^{n+1}) So I have , from equation (4.31)

$$S_r(\mu) = (-1)^{n+1}\frac{\zeta_r\beta}{b_1\omega_r}\zeta_r K_{(r)1} \cdots \zeta_r K_{(r)n} \times$$

$$\times \frac{P_r(\mu)(\mu - C_r)}{R_{(r)}(\mu)(1-\beta\zeta_r\mu)} \tag{4.32}$$

where C_r is a constant and $P_r(\mu)$ and $R_r(\mu)$ can be defined in a manner similar to Chandrasekhar's [1960] formulae

$$P_r(\mu) = \prod_{i=1}^{n} (\mu - \mu_{(r)i}) \quad (i = 1,2,...n) \tag{4.33}$$

and $$R_r(\mu) = \prod_{\alpha=1}^{n} (1 - \zeta_r K_{(r)\alpha}\,\mu) \quad (\alpha = 1,2,...n) \tag{4.34}$$

Moreover, combining equations (4.32) and (4.33)

$$L_{(r)\alpha} = (-1)^n\frac{b_0}{b_1\omega_r}\zeta_r K_{(r)1} \cdot \zeta_r K_{(r)n}\zeta_r\beta\frac{P_r[(1/\zeta_r K_{(r)\alpha}\ ]}{R_{(r)\alpha}[\ 1/\zeta_r K_{(r)\alpha}]} \times$$

$$\times \frac{\left[\dfrac{1}{\zeta\ K_{(r)\alpha}} - C_r \right]}{\left[1 - \dfrac{\beta\ \zeta_r}{K_{(r)\alpha}\zeta_r} \right]} \tag{4.35}$$

where

$$R_{(r)\alpha}(x) = \prod_{\gamma \neq \alpha} \left[1 - \zeta_r K_{(r)\gamma}\, x \right] \tag{4.36}$$

$$\text{and} \qquad \beta \neq K_{(r)\alpha} \tag{4.37}$$

The roots of the characteristic equation (4.23) can be written in the form

$$\zeta_r K_{(r)1} \ldots \zeta_r K_{(r)n}\ \mu_{(r)1}\mu_{(r)2} \ldots \mu_{(r)n} = \left[1 - \frac{M}{N} \right]^{1/2} \tag{4.38}$$

where

$$M = \sum_{r=1}^{k} \eta_r \omega_r \qquad \text{and} \qquad N = \sum_{r=1}^{k} \eta_r \tag{4.39}$$

Now by use of (4.38), the equation (4.32) becomes

$$S_r(\mu) = - \frac{b_0 \zeta_r (1 - M/N)^{1/2} \beta H_r(-\mu) \left(\mu - C_r \right)}{\omega_r b_1 (1 - \beta\zeta_r \mu)}, \tag{4.40}$$

where

$$H_r(\mu) = \frac{1}{\mu_{(r)1}\ \mu_{(r)2} \qquad \mu_{(r)n}} \frac{\prod_{i=1}^{n} (\mu + \mu_{(r)i})}{\prod_{\alpha=1}^{n} (1 + \zeta_r K_{(r)\alpha}\mu)} \tag{4.41}$$

and the characteristic roots are evaluated from equation

(4.23). Putting $\mu = 0$ in equation (4.29) and in the expression of $H_r(\mu)$ I have

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} + \frac{b_o}{\omega_r b_1} + \frac{T_r}{\omega_r} = \frac{b_o}{\omega_r b_1} \zeta_r \beta C_r \left[1 - M/N \right]^{1/2} \qquad (4.41)$$

I can next evaluate $\sum_{\alpha=1}^{n} L_{(r)\alpha}$ from equation (4.35). Then

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} = (-1)^{n+1} - \frac{b_o}{b_1 \omega_r} \times$$

$$\times [\zeta_r K_{(r)1} \ldots \zeta_r K_{(r)n} \zeta_r \beta . f_r (0),] \qquad (4.42)$$

where

$$f_r(x) = \sum_{\alpha=1}^{n} \frac{P_r[1/\zeta_r K_{(r)\alpha}][1/\zeta_r K_{(r)\alpha} - C_r]}{R_{(r)\alpha}[1/\zeta_r K_{(r)\alpha}] \ [1 - \beta/K_{(r)\alpha}]} \times$$

$$\times \ R_{(r)\alpha}(x) \qquad (4.43)$$

Now since $f_r(x)$, defined in this manner , is a polynomial of degree (n – 1) in x, which takes the value

$$\frac{P_r(1/\zeta_r K_{(r)\alpha})\left[\dfrac{1}{\zeta_r K_{(r)\alpha}} - C_r\right]}{\left[1 - \dfrac{\zeta_r \beta}{K_{(r)\alpha} \zeta_r}\right]} \qquad (4.44)$$

for $x = 1/\zeta_r K_{(r)\alpha}$, ($\alpha = 1, 2, \ldots n$) . In other words

$$(1 - \zeta_r \beta x) f_r(x) - P_r(x)(x - C_r) = 0 \qquad (4.45)$$

Therefore, I must accordingly have a relation of the form

$$(1-\zeta_r\beta x)f_r(x) - (x - C_r)\ P_r(x) = R_r(x)\ (A_r x + B_r), \quad (4.46)$$

where A_r and B_r are constants to be determined from the condition that the coefficients x^{n+1} and x^n must vanish on the right-hand side. Thus I have

$$A_r = \frac{(-1)^{n+1}}{\zeta_r K_{(r)1}\ldots\ldots\zeta_r K_{(r)n}} \quad (4.47)$$

Next, putting $x = 1/\zeta_r\beta$ in equation (4.46) (Vide, Chandrasekhar, 1960), I have

$$B_r = \frac{(-1)^n}{\zeta_r\beta\,\zeta_r k_{(r)1}\ldots\ldots\zeta_r K_{(r)n}} + (-1)^n\mu_{(r)1}\ \ldots\ldots\ \mu_{(r)n} \times$$

$$\times\ H_r\left(-\frac{1}{\beta\zeta_r}\right)\ \left(C_r - \frac{1}{\beta\zeta_r}\right) \quad (4.48)$$

Now using the relations (4.48),(4.47) and (4.46) for $x = 0$ I have, $f_r(0) = -\ (-\ 1)^n\ C_r\ \mu_{(r)1}\ldots\ \mu_{(r)n}\ +$

$$+\ \frac{(-1)^n}{\zeta_r\beta\,\zeta_r K_{(r)1}\ldots\zeta_r K_{(r)n}}\ +$$

$$+\ (-1)^n\mu_{(r)1}\ \ldots\ldots\ \mu_{(r)n}\ \ H_r\left(-\frac{1}{\beta\zeta_r}\right)\left(C_r - \frac{1}{\beta\zeta_r}\right) \quad (4.49)$$

From equations (4.43) and (4.49) it follows that

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} = \frac{b_0}{\omega_r b_1}\Big[C_r(1-M/N)^{1/2}\zeta_r\beta - 1 +$$

$$\zeta_r \beta (1 - M/N)^{1/2} H_r\left(-\frac{1}{\beta\zeta_r}\right)\left(C_r - \frac{1}{\beta\zeta_r}\right) \qquad (4.50)$$

Substituting the value of $\sum_{\alpha=1}^{n} L_{(r)\alpha}$ from equation (4.50) in equation (4.42) I have

$$C_r = \frac{1}{\zeta_r\beta} + \frac{T_r b_1}{b_o \zeta_r (1 - M/N)^{1/2} \; H_r(- 1/\beta\zeta_r)} \qquad (4.51)$$

and if I combine equation (4.40) , the diffusely reflected intensity $I_r^*(0,\mu)$ in (4.28) takes the form

$$I_r^*(0,\mu) = \frac{b_o \zeta_r \beta \; (1 - M/N)^{1/2} H_r(\mu) \; (\mu + C_r)}{1 + \beta\zeta_r \mu} - $$

$$- \frac{T_r b_1}{1 + \beta\zeta_r \mu} - b_o \qquad (4.52)$$

This is the required solution in closed form .

If I combine equations (4.8) at $\tau = 0$ and (4.52), I have,

$$I_r(0,\mu) = \frac{b_o \zeta_r \beta (1 - M/N)^{1/2} H_r(\mu)(\mu + C_r)}{1 + \beta\zeta_r \mu} \qquad (4.53)$$

This is the required solution, in the n-th approximation by the discrete ordinate method.

If I put C_r from equation (4.51), I get the solution in the form

$$I_r(0,\mu) = b_o(1 - M/N)^{1/2} H_r(\mu) +$$

$$+ \frac{b_1 T_r}{H_r(-1/\zeta_r\beta)} \frac{H_r(\mu)}{(1+\beta\zeta_r\mu)} \quad (4.54)$$

4.24. Conclusion.

Chandrasekhar's [1960] equation for $I_r(0,\mu)$ in case of coherent scattering is given by (vide, Karanjai and Barman, 1981),

$$I_r(0,\mu) = b_1\zeta_r(1 - M/N)^{1/2} \mu H_r(\mu) + b_o(1 - M/N)^{1/2} H_r(\mu) +$$

$$b_1(1-M/N)^{1/2}\zeta_r \left[\sum_{\alpha=1}^{n} \frac{1}{\zeta_r K_{(r)\alpha}} - \sum_{j=1}^{n} \mu_{(r)j} \right] \quad (4.55)$$

If I compare equations (4.54) and (4.55) I see that if I put $b_1 = 0$ I have the same solution for both the cases. Moreover, for large values of β, i.e., $\beta \longrightarrow \alpha$. The solution (4.54) takes the form

$$I_r(o,\mu) = b_o(1 - M/N)^{1/2} H_r(\mu), \quad (4.56)$$

i.e., $B_\nu(T)$ then behaves like a constant or independent of τ. This fact can also be explained from the point of view that

$$B_\nu(T) = b_o + b_1 e^{-\beta\tau} \longrightarrow b_o \quad \text{as} \quad \beta \longrightarrow \alpha \quad . \quad (4.57)$$

4.3. Solution of the equation of transfer for interlocked multiplets with planck function as a nonlinear function of optical depth.

4.31. Equation of Transfer.

I have considered here the same equation as in section 4.2.1. with the same boundary conditions. Here I write the formal solution as

$$I_r(\tau,\mu) = b_o + \frac{b_1 T_r}{1 + \zeta_r\beta\mu} e^{-\beta\tau} + I_r^*(\tau,\mu), \quad (r = 1,2..k), \quad (4.58)$$

where T_r can be expressed as

$$T_r = (\lambda_r/\zeta_r) + \alpha_r \sum_{p=1}^{k} T_p\left[\frac{1}{2\beta}(1-\varepsilon)\log\frac{1+\zeta_p\beta}{1-\zeta_p\beta}\right] \quad (4.59)$$

and $\lambda_r = (1+\varepsilon\eta_r)/(1+\eta_r)$, and $\zeta_r = 1/(1+\eta_r)$ (4.60)

Then $I_r^*(\tau,\mu)$ satisfies the equation

$$\mu\frac{dI_r^*(\tau,\mu)}{d\tau} = (1+\eta_r)I_r^*(\tau,\mu) - (1-\varepsilon)\alpha_r\sum_{p=1}^{k}(1/2)\,\eta_p \times$$

$$\times \int_{-1}^{+1} I_p^*(\tau,\mu')\, d\mu' \tag{4.61}$$

$$(r = 1,2, \;\ldots.\; k\,)$$

together with the boundary condition

$$I_r^*(0,-\mu') = \frac{b_1 T_r}{\zeta_r \beta\, \mu' - 1} - b_o \tag{4.62}$$

$$(0< \mu' \leq 1 \;;\; r = 1,2,..k)$$

Moreover, $I_r(\tau,\mu)$ must be at most linear in τ as $\tau \longrightarrow \alpha$.

Now I have the problem of a scattering atmosphere (exponential) subject to external radiation whose intensity is given by equation (4.62). We want to find the emergent intensity $I_r^*(0, \mu)$ of frequency ν_r. This will be the intensity of the diffusely reflected radiation and can be calculated when the appropriate scattering function is known.

In the present problem the scattering function splits up into k^2 functions $S_{rs}(\mu.\mu')$ $(r = 1,2,..k;\; s= 1,2, \;..\; k)$ but it is convenient to reunite them temporarily in the function

$$P(\nu,\nu')\; S(\nu,\nu';\mu,\mu'),$$

where ν is any one of $\nu_1, \nu_2, \ldots\ldots \nu_k$.

$$P(\nu,\nu') = \alpha_\nu \sum_{p=1}^{k} \delta(\nu_p - \nu') \tag{4.63}$$

δ denoting Dirac's δ-function ,and

$$S(\nu_r,\nu_s ; \mu,\mu') = S_{rs}(\mu,\mu'). \qquad (4.64)$$

Then the law of diffuse reflection for the atmosphere can be written as (vide, Stibbs, 1953; Busbridge, 1953),

$$I_r^{ref}(0,\mu) = \frac{1}{2\mu}\int_0 P(\nu,\nu')\, d\nu' \times$$

$$\times \int_0^1 S(\nu,\nu' ; \mu,\mu')\, I_{\nu'}^{inc}(0,-\mu')\, d\mu' , \qquad (4.65)$$

The equivalent form in terms of the functions $S_{rs}(\mu,\mu')$ is

$$I_r^{ref}(0,\mu) = \alpha_r \frac{1}{2\mu}\sum_{p=1}^{k}\int_0^1 S_{rp}(\mu,\mu')\, I_p^{inc}(0,-\mu')\, d\mu' \qquad (4.66)$$

4.32. Scattering Function.

If I follow Busbridge and Stibbs [1954] I have the scattering function from frequency ν_s and direction $-\mu'$ into frequency ν_r and direction μ, in the form

$$S_{rs}(\mu,\mu') = \eta_r(1-\lambda_s)\frac{\mu\,\mu'}{\zeta_r\mu + \zeta_s\mu'} H(\zeta_r\mu)H(\zeta_r\mu'), \qquad (4.67)$$

where

$$H(\zeta_r\mu) = 1 + (1/2)\zeta_r\mu\, H(\zeta_r\mu)\sum_{p=1}^{k}\alpha_p(1-\lambda_p)\times$$

$$\times \int_0^1 \frac{H(\zeta_r\mu')\, d\mu'}{\zeta_r\mu + \zeta_r\mu'} \tag{4.68}$$

4.33. H-function.

Following Busbridge and Stibbs [1954] , equation (4.68) can be written as

$$1/H(\zeta_r\mu) = \left(\sum_{p=1}^{k}\alpha_p\ \lambda_p\right)^{1/2} + (1/2)\sum_{p=1}^{k}\alpha_p\,(1-\lambda_p)\times$$
$$\times \int_0^1 \frac{\zeta_p\mu' H(\zeta_p\mu')}{\zeta_r\mu + \zeta_p\mu'} \tag{4.69}$$

4.34. Emergent Intensity.

From equations (4.60),(4.62) and (4.66) I have

$$I_r^*(0,\mu) = \frac{\alpha_r}{2\mu}\sum_{p=1}^{k}\int_0^1 S_{rp}(\mu,\mu')\left(\frac{b_1 T_p}{\zeta_p\beta\ \mu' - 1} - b_o\right) \tag{4.70}$$

If I substitute from equation (4.67) I get

$$I_r^*(0,\mu) = (1/2)\alpha_r H(\zeta_p\mu)\sum_{p=1}^{k}\zeta_r\,(1-\lambda_p)\int_0^1 \frac{\mu'}{\zeta_r\mu + \zeta_r\mu'}\times$$

$$\times\left(\frac{b_1 T_p}{\zeta_p\beta\mu' - 1} - b_o\right)\times H(\zeta_p\mu')d\mu' = (1/2)\alpha_r b_1 H(\zeta_r\mu)\times$$

$$\sum_{p=1}^{k}\zeta_r (1-\lambda_p) T_p \int_0^1 \frac{\mu' H(\zeta_p \mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')(\zeta_p\beta\mu' - 1)} -$$

$$- (1/2)\alpha_r b_0 H(\zeta_p\mu) \sum_{p=1}^{k}\zeta_r (1-\lambda_p) \int_0^1 \frac{\mu' H(\zeta_p \mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} \quad (4.71)$$

If I use the relations

$$\frac{1}{(\zeta_p\beta\mu' - 1)(\zeta_p\mu + \zeta_p\mu')} =$$

$$= \frac{1}{(\zeta_p\beta\mu + 1)}\left[\frac{\beta}{\zeta_p\beta\mu - 1} - \frac{1}{\zeta_r\mu + \zeta_p\mu'}\right] \quad (4.72)$$

I get from equation (4.71)

$$I_p^*(0,\mu) = (1/2)\alpha_r b_1 H(\zeta_r\mu) \sum_{p=1}^{k}\zeta_r (1-\lambda_p) T_p \int_0^1 \frac{\mu'}{(\zeta_r\beta\mu' + 1)} \times$$

$$\times \left[\frac{\beta}{\zeta_p\beta\mu' - 1} - \frac{1}{\zeta_r\mu + \zeta_r\mu'}\right] H(\zeta_p\mu')d\mu' -$$

$$- (1/2)\alpha_r b_0 H(\zeta_r\mu) \sum_{p=1}^{k}\zeta_r (1-\lambda_p) \times \int_0^1 \frac{\mu' H(\zeta_p \mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} =$$

$$= (1/2)\alpha_r b_1 H(\zeta_r\mu) \sum_{p=1}^{k}\zeta_r (1-\lambda_p) T_p \left(\frac{\beta}{\zeta_r\beta\mu + 1}\right) \times$$

$$\times \int_0^1 \frac{\mu' H(\zeta_p \mu')d\mu'}{(\zeta_p\beta\mu' - 1)} - (1/2)\alpha_r b_1 H(\zeta_r\mu) \sum_{p=1}^{k}\zeta_r (1-\lambda_p) T_p \times$$

$$\times \left(\frac{1}{\zeta_r\beta\mu + 1}\right) \int_0^1 \frac{\mu' H(\zeta_p\mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} - (1/2)\alpha_r b_o H(\zeta_r\mu) \times$$

$$\times \sum_{p=1}^{k} \zeta_r (1 - \lambda_p) \int_0^1 \frac{\mu' H(\zeta_p\mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} \tag{4.73}$$

From equation (4.58)

$$I_r(0,\mu) = b_o + \frac{b_1 T_r}{1 + \zeta_r\beta\mu} + I_r^*(0,\mu) \tag{4.74}$$

If I use equations (4.69) , (4.72) , (4.73) I get

$$I_r(0,\mu) = \left(b_o + \frac{b_1 T_r}{1 + \zeta_r\beta\mu}\right) \left\{ \left(\sum_{p=1}^{k} \alpha_p\lambda_p \right)^{1/2} + \right.$$

$$\left. + \sum_{p=1}^{k} \alpha_p (1 - \lambda_p) \int_0^1 \frac{\zeta_p\mu' H(\zeta_p\mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} \right\} +$$

$$+ (1/2)\alpha_r b_1 H(\zeta_r\mu) \sum_{p=1}^{k} \zeta_r (1 - \lambda_p)\left(\frac{T_p\beta}{\zeta_r\beta\mu + 1}\right) \times$$

$$\times \int_0^1 \frac{\mu' H(\zeta_p\mu')d\mu'}{(\zeta_p\beta\mu' - 1)} - (1/2)\alpha_r b_1 H(\zeta_r\mu) \times$$

$$\times \sum_{p=1}^{k} \zeta_r (1 - \lambda_p)\left(\frac{T_p}{\zeta_r\beta\mu + 1}\right)\int_0^1 \frac{\mu' H(\zeta_p\mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} -$$

$$- (1/2)\alpha_r b_o H(\zeta_r\mu) \sum_{p=1}^{k} \zeta_r (1-\lambda_p) \int_o^1 \frac{\mu' H(\zeta_p\mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} \qquad (4.75)$$

and thus

$$I_r(0,\mu) = b_o H(\zeta_r\mu) \left\{ \left(\sum_{p=1}^{k} \alpha_p\lambda_p \right)^{1/2} + \sum_{p=1}^{k} (\alpha_p\zeta_p - \alpha_r\zeta_r) \times \right.$$

$$\left. \times (1-\lambda_p) \int_o^1 \frac{\mu' H(\zeta_p\mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} \right\} + b_1 \frac{H(\zeta_r\mu)}{(1 + \zeta_r\beta\mu)} \left\{ T_r \left(\sum_{p=1}^{k} \alpha_p\lambda_p \right)^{1/2} + \right.$$

$$\left. + (1/2) \sum_{p=1}^{k} (\alpha_p\zeta_p T_r - \alpha_r\zeta_r T_p)(1-\lambda_p) \int_o^1 \frac{\mu' H(\zeta_p\mu')d\mu'}{(\zeta_r\mu + \zeta_p\mu')} \right\} +$$

$$+ (1/2)\, b_1 \zeta_r\mu \frac{H(\zeta_r\mu)}{1 + \zeta_r\beta\mu} \sum_{p=1}^{k} T_p (1-\lambda_p) \int_o^1 \frac{\mu' H(\zeta_p\mu')d\mu'}{(\zeta_p\beta\mu' - 1)} \qquad (4.76)$$

which is the final form of the emergent intensity in the rth line.

4.4. On Calculation of Interlocked Multiplets Lines In M-E Model.

4.41. The Equation of Transfer.

The equation of transfer for the rth line of multiplets in the case of interlocking without redistribution is

$$\frac{\mu \ dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\alpha_r)(a+b\tau) -$$

$$- (1-\varepsilon)\alpha_r \sum \frac{1}{2}\eta_p \int_{-1}^{+1} I_p(\tau,\mu')\ d\mu' \qquad (4.77)$$

where $\qquad r = 1, 2, \ldots \ k \qquad$ (4.78)

and $\qquad \alpha_r = \eta_r/(\eta_1 + \eta_2 + \ \ldots \ \eta_k) \qquad$ (4.79)

so that $\qquad \alpha_1 + \alpha_2 + \qquad \alpha_k = 1 \qquad$ (4.80)

and the ratio of line to the continuum absorption coefficient for the rth line is independent of depth but is a function of frequency . The coefficient of thermal emission , ε , is independent of both frequency and depth.

The expression for emergent intensity in the rth line obtained by Busbridge and Stibbs [1954] by solving equation (4.77) is

$$I_r(0,\mu) = (a + bn_r\mu)\ H(n_r\mu) \left\{ \left(\sum_{p=1}^{k} \alpha_p \lambda_p\right)^{1/2} + - \right.$$

$$+ \frac{1}{2}\sum_{p=1}^{k} (\alpha_p n_p - \alpha_r n_r)(1-\lambda_p) \int_0^1 \frac{\mu' H(n_p\mu')}{n_r\mu + n_p\mu'}\ d\mu' \left.\right\} +$$

$$+ \frac{1}{2}\ b\alpha_r n_r H(n_r\mu) \sum_{p=1}^{k} (1-\lambda_p) \int_0^1 \mu' H(n_p\mu')\,d\mu' , \qquad (4.81)$$

where $\lambda_r = \dfrac{1 + \varepsilon\eta_r}{1 + \eta_r}$ and $n_r = \dfrac{1}{1 + \eta_r}$ (4.82)

The H-function for the rth line , $H(n_r\mu)$, is the solution of the integral equation

$$\frac{1}{H(n_r\mu)} = \left[\sum_{p=1}^{k} \alpha_p \lambda_p \right]^{1/2} + \frac{1}{2}\sum_{p=1}^{k} \alpha_p (1-\lambda_p) \times$$
$$\times \int_0^1 \frac{\mu' H(n_p\mu')}{n_r\mu + n_p\mu'} d\mu' \quad (4.83)$$

Following Chandrasekhar [1960] ,Busbridge and Stibbs [1954] approximated $H(n_1\mu)$, the H-function for the first line in the multiplet, as

$$H(n_1\mu) = H(1-\alpha_1\lambda_1-\alpha_2\lambda_2-k_1 , \mu) \quad (4.84)$$

where, $k_1 = (1/2)\alpha_2 (1 - \lambda_2)(1 - n_1/n_2)$ (4.85)

$H(n_2\mu)$ and $H(n_3\mu)$ were calculated from $H(n_1\mu)$ using the relation

$$H(n_r\mu) = H(n_1 .(n_r/n_1)\mu) \qquad r = 2,3 \quad (4.86)$$

I have used four approximate forms of the H-function

<u>Form (1) :</u> $H(\omega,\mu) = 1 + a\mu + b\mu^2 + c\mu^3$ (4.87)

(vide, Karanjai and Sen, 1971, eqn.(2.1)) where a, b, c, are the functions of albedo ω

<u>Form (2) :</u> $H(\omega,\mu) = 1 + (a\mu + b\mu^2)/(A + 2\mu)$ (4.88)

(vide, Karanjai and Sen, 1970) where a and b are the functions of ω and

$$A = (1 - \omega)^{1/2} \quad (4.89)$$

<u>Form (3) :</u> $H(\omega,\mu) = 1 + \dfrac{a\mu + b\mu^2 + c\mu^3}{A + 2\mu}$ (4.90)

(vide, Karanjai and Sen, 1971) where a, b, c, are the functions of ω and A is given by the equation (4.87)

<u>Form (4) :</u> $H(\omega,\mu) = 1 + \dfrac{a\mu + b\mu^2 + c\mu^3}{1 + KM}$ (4.91)

(vide, Karanjai and Sen 1971 eqn.3.1) where a, b, c are the functions of ω and K is a root of the transcendental equation

$$(\omega/2K) \log \left[(1+K)/(1-K) \right] = 1 \quad (4.92)$$

The cases considered here are

I. $\eta_1 = 1, \eta_2 = .5, \varepsilon = 0, \lambda_1 = n_1 = 1/2, \lambda_2 = n_2 = 2/3$
II. $\eta_1 = 2, \eta_2 = 1, \varepsilon = 0, \lambda_1 = n_1 = 1/3, \lambda_2 = n_2 = 1/2$
III. $\eta_1 = 4, \eta_2 = 2, \varepsilon = 0, \lambda_1 = n_1 = 1/5, \lambda_2 = n_2 = 1/3$
IV. $\eta_1 = 6, \eta_2 = 3, \varepsilon = 0, \lambda_1 = n_1 = 1/7, \lambda_2 = n_2 = 1/4$
V. $\eta_1 = 8, \eta_2 = 4, \varepsilon = 0, \lambda_1 = n_1 = 1/9, \lambda_2 = n_2 = 1/5$
VI. $\eta_1 = 10, \eta_2 = 5, \varepsilon = 0, \lambda_1 = n_1 = 1/11, \lambda_2 = n_2 = 1/6$

(4.93)

Out of the six cases considered here , three cases (case

I,III,VI) are identical with the cases considered by Busbridge and Stibbs [1954].

4.42. Calculation for a Doublet.

Form (1):

After Single iteration , the expression for $H(\omega,\mu)$ in the form (1) is obtained as

$$\frac{1}{H(\omega,\mu)} = 1 - \frac{\mu\omega}{2}\Big[\,(1 - a\mu + b\mu^2 - c\mu^3)\log\,((\mu + 1)/\mu)\,) + + a + b/2 + c/3 + - b\mu - c\mu/2 + c\mu^2\,\Big] \qquad (4.94)$$

for a doublet $k = 2$

$$\eta_1 : \eta_2 = 2:1 \quad , \quad \alpha_1 = 2/3 \quad , \alpha_2 = 1/3 \qquad (4.95)$$

I write $$\omega = 1 - \alpha_1\lambda_1 - \alpha_2\lambda_2 - k_1 \ , \qquad (4.96)$$

so that ω is now a function of the parameters α_i and η_i (i = 1,2). I then calculate $H(n_1\mu)$ with the help of equation (4.84) and (4.85) and tabulate it in Table 4.1. Values for $H(n_2\mu)$ calculated from relation (4.86) are also given in the same table.

In calculating the residual intensities from equation (4.81), the H-function within the integral sign have been replaced by the right hand side of equation (4.88) and the values of H-function outside the integral sign have been taken from

Table 4.1. Values of $r_1(\mu)$ and $r_2(\mu)$ were calculated for Cases I to VI for a region of the spectrum where $b = (3/2)a$

The results are also given in Table 4.1 and are shown by the curves of Fig.4.1.a.(for case I,II and III) and Fig.4.1.b. (for case IV,V and VI), in which the residual intensities are plotted as functions of μ . Since in the wings of the lines , η is proportional to $(\Delta\lambda)^{-2}$, where $\Delta\lambda$ is the distance from the line centre,the cases I-VI of (4.93) lead, respectfully to cross-sections of the line profiles at increasing distances from the centres of the lines. This representation of the results is similar to that adopted by Houtgast [1942] where cross-sections curves for multiplets in the solar spectrum are given.

Form 2:

After single iteration , the expression for $H(\omega,\mu)$ is obtained as

$$\frac{1}{H(\omega,\mu)} = 1 - \frac{\mu\omega}{2(2\mu-A)}\left[(2\mu - A + a\mu - b\mu^2)\log((\mu + 1)/\mu)) - (a/2 - bA/4)A\log((A+2)/A)\right] - (b\mu\omega/4) \quad (4.97)$$

Similarly as in form (1) I have calculated the residual intensities and H-function for doublets and given in table 4.2.

In calculating the residual intensities from equation (4.81) ,the H-function within the integral sign have been replaced by the right hand side of equation (4.88) and the values of H-function outside the integral sign have been taken from Table 4.2. The results are also given in Table 4.2 and are shown by curves of Fig.4.2.a. (for case I,II and III) and Fig.4.2.b. (for case IV,V and VI).

<u>Form 3:</u>

After single iteration , the expression for $H(\omega,\mu)$ is obtained as

$$\frac{1}{H(\omega,\mu)} = 1 - \frac{\mu\omega}{2(2\mu - A)} \Big[(2\mu - A + a\mu - b\mu^2 + c\mu^3) \log((\mu + 1)/\mu)) - (a/2 - bA/4 + cA^2/8)A \log((A + 2)/A) \Big] - (\mu\omega/2)\big[b/2 + c/4 (1 - 2\mu - A) \big] \qquad (4.98)$$

Similarly as in form (1) we have calculated the residual intensities and H-function for doublets and given in table 4.3.

In calculating the residual intensities from equation (4.81) ,the H-function within the integral sign have been replaced by the right hand side of equation (4.90) and the values of H-function outside the integral sign have been taken from Table 4.3. The results are also given in Table 4.3 and are shown by curves of Fig.4.3.a. (for case I,II and III) and Fig.4.3.b. (for case IV,V and VI).

<u>Form 4:</u>

After single iteration , the expression for $H(\omega,\mu)$ is obtained as

$$\frac{1}{H(\omega,\mu)} = \frac{\mu\omega}{2} \left[(1-T_1\mu + T_2\mu^2)\log((\mu + 1)/\mu)) + c/2K + (1/K)(T_1 - T_2/K)\log(1+K) + \frac{T_2(1 - K\mu)}{K} \right] \quad (4.99)$$

where $T_1 = (a - c\mu/K) \left[1/(1 - K\mu)\right]$ (4.100)

$$T_2 = \frac{bK - c(1 + K\mu)}{K} \left[1/(1 - k\mu)\right] \quad (4.101)$$

Similarlyas in form (1) we have calculatedthe residual intensities and H–function for doublets and given in table 4.4.

In calculating the residual intensities from equation (4.79) ,the H–function within the integral sign have been replaced by the right hand side of equation (4.91)and the values of H–function outsidethe integral sign have been taken from Table 4.4. The results are also given in Table 4.4 and are shown by curves of Fig.4.4.a. (for case I,II and III) and Fig.4.4.b. (for case IV,V and VI).

4.43. Calculation for a Triplet.

For a triplet $k = 3$

$$\eta_1 : \eta_2 : \eta_3 = 5:3:1 \;,\quad \alpha_1 = 5/9,\ \alpha_2 = 1/3,\ \alpha_3 = 1/9 \qquad (4.102)$$

Three cases were considered :

$$\text{I.}\quad \left.\begin{array}{l} \eta_1 = 5/9, \quad \eta_2 = 1/3, \quad \eta_3 = 1/9, \quad \varepsilon = 0, \\ \lambda_1 = n_1 = 9/14,\ \lambda_2 = n_2 = 3/4,\ \lambda_3 = n_3 = 9/10 \end{array}\right] \quad (4.103)$$

$$\text{II.}\quad \left.\begin{array}{l} \eta_1 = 5/3, \quad \eta_2 = 1, \quad \eta_3 = 1/3, \quad \varepsilon = 0, \\ \lambda_1 = n_1 = 3/8,\ \lambda_2 = n_2 = 1/2,\ \lambda_3 = n_3 = 3/4 \end{array}\right] \quad (4.104)$$

$$\text{III.}\quad \left.\begin{array}{l} \eta_1 = 5/9, \quad \eta_2 = 1/3, \quad \eta_3 = 1/9, \quad \varepsilon = 0, \\ \lambda_1 = n_1 = 1/6,\ \lambda_2 = n_2 = 1/4,\ \lambda_3 = n_3 = 1/2 \end{array}\right] \quad (4.105)$$

Out of three cases , case I is identical with the case considered by Karanjai [1968b].

Following Busbridge and Stibbs , $H(n_r\mu)$ in the case of a triplet can be approximated by

$$H(n_1\mu) = H(1 - \alpha_1\lambda_1 - \alpha_2\lambda_2 - \alpha_3\lambda_3 - k_2 \;, \mu\ ) \qquad (4.106)$$

where,

$$k_2 = (1/2)[\alpha_2 (1 - \lambda_2)(1 - n_1/n_2) +$$

$$+ \alpha_3(1 - \lambda_3)(1 - n_1/n_3) \qquad (4.107)$$

Here I have considered the four approximate forms of H-function as the case of a doublet (Form (1) to Form (4), equations (4.86) - (4.89)). Writting

$$\omega = 1 - \alpha_1\lambda_1 - \alpha_2\lambda_2 - \alpha_3\lambda_3 - k_2 \qquad (4.108)$$

in equation (4.106) I calculate H-functions and residual intensities for the cases considered and tabulated in the following manner.

Forms	Tables	Cases
(1)	4.5,4.6,4.7	I, II, III
(2)	4.8,4.9,4.10	I, II, III
(3)	4.11,4.12,4.13	I, II, III
(4)	4.14,4.15,4.16	I, II, III

Results are also shown by the curves , in which the residual intensities are plotted as the functions of μ in the fillowing manner .

Forms	Figures	Cases
(1)	4.5.a, 4.5.b, 4.5.c	I, II, III
(2)	4.6.a, 4.6.b, 4.6.c	I, II, III
(3)	4.7.a, 4.7.b, 4.7.c	I, II, III
(4)	4.8.a, 4.8.b, 4.8.c	I, II, III

From tables I-IV and Fig. 4.5.a -4.8.b it will be seen that the effect of interlocking is to increase $r_1(\mu)$ and to decrease $r_2(\mu)$ for all μ. Let $d(\mu)$ denote the differences $r_2(\mu) - r_1(\mu)$ in the interlocked case for a doublet. Similarly for a triplet let

$$d_1(\mu) = r_2(\mu) - r_1(\mu) \tag{4.109}$$

$$d_2(\mu) = r_3(\mu) - r_2(\mu) \tag{4.110}$$

$$d_3(\mu) = r_3(\mu) - r_1(\mu) \tag{4.111}$$

Values of $d(\mu)$ for the cases I-VI are shown in Fig.4.9. to Fig.4.12. and $d_1(\mu)$, $d_2(\mu)$ and $d_3(\mu)$ are shown as follows :

Forms	Figures	Cases
(1)	4.13.a,4.13.b,4.13.c	I, II, III
(2)	4.14.a,4.14.b,4.14.c	I, II, III
(3)	4.15.a,4.15.b,4.15.c	I, II, III
(4)	4.16.a,4.16.b,4.16.c	I, II, III

Remarks :

Residual intensities for doublets and triplets can be calculated in a similar way as the procedure described here in section 4.4. using the exponential form of planck function.

Table 4.1. Values of the H-functions and residual intensities for a doublet with interlocking

	$\eta_1 = 1$, $\eta_2 = 1/2$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.8349	0.8894
0.1	1.0602	1.0728	0.8251	0.8872
0.2	1.0937	1.1107	0.8016	0.8718
0.3	1.1182	1.1374	0.7789	0.8562
0.4	1.1374	1.1577	0.7584	0.8420
0.5	1.1531	1.1737	0.7402	0.8293
0.6	1.1661	1.1869	0.7242	0.8181
0.7	1.1773	1.1978	0.7100	0.8081
0.8	1.1869	1.2071	0.6973	0.7993
0.9	1.1953	1.2151	0.6861	0.7914
1.0	1.2027	1.2221	0.6759	0.7843

Table 4.1. (continued)

	$\eta_1 = 2$, $\eta_2 = 1$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.7108	0.7815
0.1	1.0877	1.1156	0.7061	0.7880
0.2	1.1392	1.1781	0.6836	0.7723
0.3	1.1781	1.2227	0.6604	0.7544
0.4	1.2093	1.2570	0.6388	0.7372
0.5	1.2351	1.2844	0.6192	0.7213
0.6	1.2570	1.3069	0.6016	0.7070
0.7	1.2759	1.3258	0.5858	0.6940
0.8	1.2924	1.3419	0.5716	0.6823
0.9	1.3069	1.3557	0.5587	0.6718
1.0	1.3199	1.3678	0.5471	0.6622

Table 4.1. (continued)

	$\eta_1 = 4$, $\eta_2 = 2$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.5632	0.6331
0.1	1.1153	1.1651	0.5644	0.6506
0.2	1.1867	1.2585	0.5468	0.6391
0.3	1.2424	1.3269	0.5272	0.6228
0.4	1.2881	1.3802	0.5083	0.6058
0.5	1.3269	1.4233	0.4907	0.5894
0.6	1.3603	1.4590	0.4746	0.5741
0.7	1.3895	1.4892	0.4598	0.5601
0.8	1.4153	1.5150	0.4464	0.5472
0.9	1.4383	1.5375	0.4341	0.5354
1.0	1.4590	1.5571	0.4222	0.5245

Table 4.1. (continued)

	$\eta_1 = 6$, $\eta_2 = 3$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.4779	0.5398
0.1	1.1295	1.1934	0.4815	0.5623
0.2	1.2120	1.3059	0.4673	0.5542
0.3	1.2775	1.3895	0.4508	0.5402
0.4	1.3321	1.4554	0.4344	0.5248
0.5	1.3788	1.5091	0.4189	0.5095
0.6	1.4195	1.5540	0.4046	0.4949
0.7	1.4554	1.5921	0.3913	0.4813
0.8	1.4873	1.6248	0.3792	0.4687
0.9	1.5160	1.6534	0.3680	0.4570
1.0	1.5419	1.6785	0.3576	0.4462

Table 4.1. (continued)

	$\eta_1 = 8$, $\eta_2 = 4$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.4213	0.4756
0.1	1.1383	1.2118	0.4260	0.5004
0.2	1.2280	1.2118	0.4140	0.4947
0.3	1.3000	1.3375	0.3996	0.4827
0.4	1.3606	1.4318	0.3852	0.4687
0.5	1.4128	1.5068	0.3713	0.4546
0.6	1.4586	1.5683	0.3584	0.4409
0.7	1.4992	1.6199	0.3464	0.4280
0.8	1.5355	1.6639	0.3353	0.4159
0.9	1.5683	1.7018	0.3251	0.4047
1.0	1.5980	1.7350	0.3156	0.3944

Table 4.1. (continued)

	$\eta_1 = 10$, $\eta_2 = 5$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.3804	0.4285
0.1	1.1443	1.2249	0.3856	0.4542
0.2	1.2392	1.3602	0.3752	0.4502
0.3	1.3158	1.4625	0.3623	0.4396
0.4	1.3807	1.5444	0.3493	0.4270
0.5	1.4370	1.6118	0.3368	0.4138
0.6	1.4866	1.6686	0.3250	0.4010
0.7	1.5307	1.7172	0.3140	0.3889
0.8	1.5703	1.7592	0.3038	0.3774
0.9	1.6062	1.7961	0.2944	0.3667
1.0	1.6388	1.8286	0.2856	0.3568

Table 4.2. Values of the H-functions and residual intensities for a doublet with interlocking

	$\eta_1 = 1$, $\eta_2 = 1/2$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.8218	0.8703
0.1	1.0589	1.0712	0.8117	0.8691
0.2	1.0918	1.1086	0.7887	0.8552
0.3	1.1160	1.1351	0.7667	0.8411
0.4	1.1351	1.1553	0.7470	0.8281
0.5	1.1507	1.1714	0.7295	0.8165
0.6	1.1637	1.1845	0.7141	0.8062
0.7	1.1749	1.1955	0.7005	0.7971
0.8	1.1845	1.2049	0.6883	0.7890
0.9	1.1930	1.2130	0.6775	0.7817
1.0	1.2004	1.2200	0.6678	0.7751

Table 4.2. (continued)

	$\eta_1 = 2$, $\eta_2 = 1$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.8218	0.8703
0.1	1.0849	1.1120	0.6870	0.7588
0.2	1.1351	1.1733	0.6648	0.7453
0.3	1.1733	1.2174	0.6424	0.7296
0.4	1.2040	1.2515	0.6216	0.7144
0.5	1.2297	1.2789	0.6029	0.7003
0.6	1.2515	1.3014	0.5862	0.6874
0.7	1.2703	1.3204	0.5711	0.6758
0.8	1.2869	1.3366	0.5576	0.6652
0.9	1.3014	1.3507	0.5454	0.6557
1.0	1.3144	1.3629	0.5344	0.6471

Table 4.2. (continued)

	$\eta_1 = 4$, $\eta_2 = 2$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.5456	0.5999
0.1	1.1104	1.1586	0.5442	0.6163
0.2	1.1795	1.2496	0.5264	0.6068
0.3	1.2337	1.3168	0.5073	0.5928
0.4	1.2786	1.3697	0.4891	0.5780
0.5	1.3168	1.4127	0.4722	0.5636
0.6	1.3499	1.4484	0.4569	0.5501
0.7	1.3790	1.4788	0.4429	0.5376
0.8	1.4047	1.5049	0.4302	0.5261
0.9	1.4277	1.5276	0.4186	0.5155
1.0	1.4484	1.5475	0.4080	0.5058

Table 4.2. (continued)

	$\eta_1 = 6$, $\eta_2 = 3$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.4621	0.5085
0.1	1.1235	1.1849	0.4629	0.5289
0.2	1.2030	1.2942	0.4482	0.5224
0.3	1.2665	1.3763	0.4318	0.5105
0.4	1.3198	1.4415	0.4160	0.4971
0.5	1.3657	1.4950	0.4011	0.4836
0.6	1.4059	1.5398	0.3874	0.4708
0.7	1.4415	1.5781	0.3749	0.4587
0.8	1.4732	1.6111	0.3633	0.4474
0.9	1.5018	1.6399	0.3528	0.4370
1.0	1.5277	1.6654	0.3431	0.4274

Table 4.2. (continued)

	$\eta_1 = 8$, $\eta_2 = 4$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.4073	0.4470
0.1	1.1315	1.2021	0.4090	0.4691
0.2	1.2177	1.3239	0.3964	0.4647
0.3	1.2874	1.4763	0.3821	0.4544
0.4	1.3465	1.4904	0.3680	0.4422
0.5	1.3976	1.5515	0.3547	0.4298
0.6	1.4427	1.6030	0.3423	0.4177
0.7	1.4829	1.6472	0.3309	0.4063
0.8	1.5189	1.6854	0.3204	0.3955
0.9	1.5515	1.7189	0.3107	0.3855
1.0	1.5812	1.7485	0.3018	0.3761

Table 4.2. (continued)

	$\eta_1 = 10$, $\eta_2 = 5$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.3678	0.4023
0.1	1.1370	1.2142	0.3701	0.4252
0.2	1.2279	1.3452	0.3590	0.4221
0.3	1.3020	1.4454	0.3461	0.4130
0.4	1.3652	1.5261	0.3334	0.4020
0.5	1.4203	1.5931	0.3213	0.3905
0.6	1.4690	1.6497	0.3100	0.3791
0.7	1.5126	1.6984	0.2995	0.3683
0.8	1.5518	1.7407	0.2898	0.3580
0.9	1.5874	1.7779	0.2809	0.3484
1.0	1.6199	1.8108	0.2726	0.3395

Table 4.3. Values of the H-functions and residual intensities for a doublet with interlocking

	$\eta_1 = 1$, $\eta_2 = 1/2$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.8356	0.89901
0.1	1.0603	1.0729	0.8259	0.8880
0.2	1.0938	1.1108	0.8023	0.8725
0.3	1.1183	1.1375	0.7795	0.8569
0.4	1.1375	1.1577	0.7590	0.8426
0.5	1.1531	1.1738	0.7408	0.8299
0.6	1.1662	1.1869	0.7247	0.8186
0.7	1.1773	1.1979	0.7104	0.8086
0.8	1.1869	1.2072	0.6978	0.7997
0.9	1.1953	1.2152	0.6865	0.7918
1.0	1.2027	1.2221	0.6763	0.7847

Table 4.3. (continued)

	$\eta_1 = 2$, $\eta_2 = 1$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.7129	0.7840
0.1	1.0879	1.1157	0.7082	0.7904
0.2	1.1394	1.1782	0.6855	0.7745
0.3	1.1782	1.2228	0.6622	0.7565
0.4	1.2093	1.2571	0.6405	0.7391
0.5	1.2352	1.2844	0.6208	0.7231
0.6	1.2571	1.3069	0.6031	0.7086
0.7	1.2759	1.3258	0.5872	0.6956
0.8	1.2924	1.3418	0.5729	0.6838
0.9	1.3069	1.3557	0.5600	0.6732
1.0	1.3198	1.3678	0.5483	0.6635

Table 4.3. (continued)

	$\eta_1 = 4$, $\eta_2 = 2$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.5661	0.6369
0.1	1.1154	1.1652	0.5673	0.6545
0.2	1.1868	1.2586	0.5495	0.6428
0.3	1.2424	1.3269	0.5299	0.6263
0.4	1.2882	1.3801	0.5108	0.6091
0.5	1.3269	1.4232	0.4931	0.5925
0.6	1.3602	1.4589	0.4768	0.5771
0.7	1.3894	1.4890	0.4620	0.5628
0.8	1.4552	1.5149	0.4484	0.5498
0.9	1.4382	1.5373	0.4360	0.5378
1.0	1.4589	1.5570	0.4247	0.5269

Table 4.3. (continued)

	$\eta_1 = 6$, $\eta_2 = 3$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.5661	0.6369
0.1	1.1154	1.1652	0.5673	0.6545
0.2	1.1868	1.2586	0.5495	0.6428
0.3	1.2424	1.3269	0.5299	0.6263
0.4	1.2882	1.3801	0.5108	0.6091
0.5	1.3269	1.4232	0.4931	0.5925
0.6	1.3602	1.4589	0.4768	0.5771
0.7	1.3894	1.4890	0.4620	0.5628
0.8	1.4152	1.5149	0.4484	0.5498
0.9	1.4382	1.5373	0.4360	0.5378
1.0	1.4589	1.5570	0.4247	0.5269

Table 4.3. (continued)

	$\eta_1 = 8$, $\eta_2 = 4$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.4233	0.4785
0.1	1.1383	1.2119	0.4280	0.5033
0.2	1.2281	1.3374	0.4159	0.4976
0.3	1.3000	1.4317	0.4015	0.4854
0.4	1.3605	1.5066	0.3869	0.4713
0.5	1.4127	1.5680	0.3730	0.4570
0.6	1.4584	1.6196	0.3601	0.4432
0.7	1.4990	1.6636	0.3480	0.4302
0.8	1.5353	1.7015	0.3369	0.4180
0.9	1.5680	1.7347	0.3266	0.4067
1.0	1.5977	1.7640	0.3170	0.3963

Table 4.3. (continued)

	$\eta_1 = 10$, $\eta_2 = 5$			
μ	$H(n_2\mu)$	$H(n_2\mu)$	$r_1(\mu)$	$r_2(\mu)$
0.0	1.0000	1.0000	0.3818	0.4306
0.1	1.1443	1.2249	0.3870	0.4563
0.2	1.2391	1.3600	0.3766	0.4523
0.3	1.3157	1.4623	0.3637	0.4417
0.4	1.3806	1.5441	0.5506	0.4289
0.5	1.4368	1.6115	0.3380	0.4157
0.6	1.4863	1.6683	0.3262	0.4028
0.7	1.5304	1.7168	0.3151	0.3905
0.8	1.5700	1.7589	0.3049	0.3790
0.9	1.6059	1.7957	0.2954	0.3682
1.0	1.6385	1.8282	0.2866	0.3582

Table 4.4. Values of the H-functions and residual intensities for a doublet with interlocking

μ	$\eta_1 = 1$, $\eta_2 = 1/2$			
	$H(n_1\mu)$	$H(n_2\mu)$	$r(n_1\mu)$	$r(n_2\mu)$
0.0	1.0000	1.0000	0.8359	0.8904
0.1	1.0603	1.0729	0.8261	0.8883
0.2	1.0938	1.1108	0.8025	0.8728
0.3	1.1183	1.1375	0.7797	0.8571
0.4	1.1375	1.1577	0.7592	0.8428
0.5	1.1531	1.1738	0.7410	0.8301
0.6	1.1662	1.1869	0.7249	0.8188
0.7	1.1773	1.1979	0.7106	0.8088
0.8	1.1869	1.2072	0.6979	0.7999
0.9	1.1953	1.2152	0.6866	0.7919
1.0	1.2027	1.2221	0.6765	0.7848

Table 4.4. (continued)

μ	$\eta_1 = 2$, $\eta_2 = 1$			
	$H(n_1\mu)$	$H(n_2\mu)$	$r(n_1\mu)$	$r(n_2\mu)$
0.0	1.0000	1.0000	0.7139	0.7852
0.1	1.0879	1.1158	0.7092	0.7916
0.2	1.1394	1.1782	0.6864	0.7757
0.3	1.1782	1.2229	0.6631	0.7576
0.4	1.2094	1.2571	0.6413	0.7401
0.5	1.2352	1.2845	0.6216	0.7241
0.6	1.2571	1.3070	0.6038	0.7095
0.7	1.2760	1.3258	0.5879	0.6964
0.8	1.2924	1.3419	0.5736	0.6846
0.9	1.3070	1.3557	0.5606	0.6739
1.0	1.3199	1.3678	0.5489	0.6642

Table 4.4. (continued)

μ	$\eta_1 = 4$, $\eta_2 = 2$			
	$H(n_1\mu)$	$H(n_2\mu)$	$r(n_1\mu)$	$r(n_2\mu)$
0.0	1.0000	1.0000	0.5484	0.6400
0.1	1.1155	1.1653	0.5696	0.6578
0.2	1.1869	1.2587	0.5518	0.6459
0.3	1.2426	1.3270	0.5320	0.6292
0.4	1.2883	1.3802	0.5129	0.6118
0.5	1.3270	1.4233	0.4951	0.5951
0.6	1.3604	1.4590	0.4787	0.5797
0.7	1.3895	1.4892	0.4638	0.5651
0.8	1.4153	1.5150	0.4502	0.5520
0.9	1.4383	1.5374	0.4377	0.5399
1.0	1.4590	1.5570	0.4263	0.5289

Table 4.4. (continued)

μ	$\eta_1 = 6$, $\eta_2 = 3$			
	$H(n_1\mu)$	$H(n_2\mu)$	$r(n_1\mu)$	$r(n_2\mu)$
0.0	1.0000	1.0000	0.4835	0.5477
0.1	1.1297	1.1937	0.4873	0.5706
0.2	1.2123	1.3062	0.4729	0.5623
0.3	1.2778	1.3897	0.4561	0.5479
0.4	1.3323	1.4555	0.4395	0.5320
0.5	1.3790	1.5092	0.4238	0.5163
0.6	1.4196	1.5540	0.4093	0.5014
0.7	1.4555	1.5920	0.3958	0.4874
0.8	1.4874	1.6248	0.3835	0.4745
0.9	1.5161	1.6533	0.3721	0.4625
1.0	1.5419	1.6784	0.3716	0.4514

Table 4.4. (continued)

μ	$\eta_1 = 8$, $\eta_2 = 4$			
	$H(n_1\mu)$	$H(n_2\mu)$	$r(n_1\mu)$	$r(n_2\mu)$
0.0	1.0000	1.0000	0.4267	0.4835
0.1	1.1386	1.2122	0.4316	0.5088
0.2	1.2284	1.3378	0.4195	0.5029
0.3	1.3004	1.4320	0.4049	0.4905
0.4	1.3609	1.5069	0.3903	0.4762
0.5	1.4131	1.5684	0.3763	0.4617
0.6	1.4588	1.6199	0.3632	0.4476
0.7	1.4994	1.6639	0.3510	0.4344
0.8	1.5357	1.7019	0.3397	0.4220
0.9	1.5684	1.7350	0.3293	0.4106
1.0	1.5981	1.7642	0.3197	0.3999

Table 4.4. (continued)

μ	$\eta_1 = 10$, $\eta_2 = 5$			
	$H(n_1\mu)$	$H(n_2\mu)$	$r(n_1\mu)$	$r(n_2\mu)$
0.0	1.0000	1.0000	0.3855	0.4361
0.1	1.1447	1.2253	0.3909	0.4623
0.2	1.2396	1.3605	0.3804	0.4582
0.3	1.3162	1.4628	0.3674	0.4473
0.4	1.3811	1.5446	0.3542	0.4343
0.5	1.4373	1.6120	0.3415	0.4208
0.6	1.4868	1.6687	0.3296	0.4077
0.7	1.5309	1.7172	0.3184	0.3952
0.8	1.5705	1.7589	0.3081	0.3834
0.9	1.6063	1.7961	0.2985	0.3725
1.0	1.6389	1.8286	0.2896	0.3623

Table 4.5. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.8947	0.9279	1.0016
0.1	1.0376	1.0416	1.0466	0.8859	0.9230	0.9934
0.2	1.0577	1.0629	1.0694	0.8668	0.9095	0.9841
0.3	1.0720	1.0778	1.0849	0.8488	0.8965	0.9761
0.4	1.0830	1.0891	1.0964	0.8327	0.8849	0.9692
0.5	1.0919	1.0980	1.1054	0.8187	0.8748	0.9634
0.6	1.0992	1.1054	1.1126	0.8064	0.8659	0.9584
0.7	1.1054	1.1115	1.1185	0.7956	0.8581	0.9540
0.8	1.1106	1.1166	1.1235	0.7861	0.8512	0.9502
0.9	1.1152	1.1211	1.1277	0.7776	0.8451	0.9469
1.0	1.1193	1.1250	1.1314	0.7700	0.8396	0.9439

Table 4.6. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.7334	0.7887	0.9485
0.1	1.0746	1.0906	1.1173	0.7287	0.7913	0.9425
0.2	1.1173	1.1394	1.1742	0.7066	0.7739	0.9261
0.3	1.1491	1.1742	1.2124	0.6839	0.7552	0.9103
0.4	1.1742	1.2011	1.2403	0.6629	0.7374	0.8962
0.5	1.1949	1.2226	1.2617	0.6439	0.7213	0.8839
0.6	1.2124	1.2403	1.2788	0.6269	0.7068	0.8730
0.7	1.2273	1.2551	1.2927	0.6117	0.6937	0.8635
0.8	1.2403	1.2678	1.3043	0.5981	0.6820	0.8552
0.9	1.2516	1.2788	1.3141	0.5858	0.6715	0.8477
1.0	1.2617	1.2883	1.3226	0.5747	0.6620	0.8411

Table 4.7. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.5162	0.5630	0.7576
0.1	1.1145	1.1526	1.2406	0.5183	0.5783	0.7727
0.2	1.1854	1.2406	1.3575	0.5020	0.5664	0.7560
0.3	1.2406	1.3059	1.4347	0.4836	0.5499	0.7357
0.4	1.2860	1.3575	1.4905	0.4657	0.5328	0.7160
0.5	1.3244	1.3996	1.5328	0.4489	0.5163	0.6979
0.6	1.3575	1.4347	1.5661	0.4334	0.5009	0.6816
0.7	1.3864	1.4647	1.5931	0.4192	0.4866	0.6669
0.8	1.4120	1.4905	1.6153	0.4062	0.4736	0.6539
0.9	1.4347	1.5130	1.6341	0.3943	0.4615	0.6421
1.0	1.4552	1.5328	1.6500	0.3834	0.4505	0.6316

Table 4.8. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.000	1.0000	1.0000	0.8893	0.9203	0.9897
0.1	1.0373	1.0412	1.0463	0.8805	0.9160	0.9824
0.2	1.0573	1.0625	1.0690	0.8617	0.9031	0.9745
0.3	1.0716	1.0775	1.0846	0.8441	0.8907	0.9676
0.4	1.0827	1.0888	1.0962	0.8284	0.8797	0.9617
0.5	1.0916	1.0978	1.1052	0.8147	0.8700	0.9567
0.6	1.0990	1.1052	1.1125	0.8027	0.8615	0.9523
0.7	1.1052	1.1114	1.1185	0.7922	0.8541	0.9485
0.8	1.1105	1.1166	1.1235	0.7829	0.8475	0.9452
0.9	1.1152	1.1211	1.1279	0.7746	0.8417	0.9422
1.0	1.1193	1.1251	1.1316	0.7672	0.8364	0.9396

Table 4.9. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.7220	0.7699	0.9071
0.1	1.0738	1.0896	1.1164	0.7167	0.7732	0.9075
0.2	1.1164	1.1385	1.1738	0.6949	0.7575	0.8967
0.3	1.1483	1.1738	1.2126	0.6730	0.7404	0.8853
0.4	1.1738	1.2011	1.2412	0.6527	0.7241	0.8746
0.5	1.1948	1.2230	1.2633	0.6344	0.7093	0.8650
0.6	1.2126	1.2412	1.2810	0.6181	0.6958	0.8564
0.7	1.2279	1.2565	1.2954	0.6035	0.6838	0.8487
0.8	1.2412	1.2696	1.3075	0.5904	0.6729	0.8419
0.9	1.2529	1.2810	1.3178	0.5786	0.6632	0.8358
1.0	1.2633	1.2909	1.3266	0.5680	0.6543	0.8303

Table 4.10. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.5026	0.5416	0.7001
0.1	1.1139	1.1523	1.2424	0.5039	0.5576	0.7280
0.2	1.1857	1.2424	1.3645	0.4883	0.5481	0.7225
0.3	1.2424	1.3103	1.4468	0.4710	0.5341	0.7104
0.4	1.2894	1.3645	1.5070	0.4541	0.5192	0.6967
0.5	1.3296	1.4092	1.5532	0.4384	0.5047	0.6833
0.6	1.3645	1.4468	1.5898	0.4239	0.4909	0.6706
0.7	1.3952	1.4791	1.6196	0.4107	0.4781	0.6589
0.8	1.4224	1.5070	1.6443	0.3986	0.4663	0.6482
0.9	1.4468	1.5315	1.6652	0.3874	0.4554	0.6384
1.0	1.4688	1.5532	1.6830	0.3772	0.4454	0.6295

Table 4.11. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.8946	0.9275	1.0004
0.1	1.0374	1.0413	1.0464	0.8855	0.9225	0.9923
0.2	1.0574	1.0626	1.0692	0.8665	0.9089	0.9831
0.3	1.0717	1.0776	1.0847	0.8484	0.8960	0.9751
0.4	1.0828	1.0889	1.0962	0.8324	0.8845	0.9684
0.5	1.0916	1.0978	1.1052	0.8184	0.8744	0.9626
0.6	1.0990	1.1052	1.1124	0.8062	0.8655	0.9576
0.7	1.1052	1.1113	1.1183	0.7954	0.8578	0.9534
0.8	1.1105	1.1165	1.1233	0.7859	0.8509	0.9496
0.9	1.1151	1.1209	1.1276	0.7774	0.8448	0.9463
1.0	1.1191	1.1248	1.1313	0.7699	0.8393	0.9434

Table 4.12. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.7334	0.7896	0.9521
0.1	1.0748	1.0907	1.1175	0.7288	0.7922	0.9459
0.2	1.1175	1.1395	1.1744	0.7067	0.7748	0.9292
0.3	1.1492	1.1744	1.2125	0.6840	0.7560	0.9131
0.4	1.1744	1.2012	1.2403	0.6630	0.7382	0.8988
0.5	1.1950	1.2226	1.2618	0.6441	0.7220	0.8863
0.6	1.2125	1.2403	1.2788	0.6271	0.7074	0.8753
0.7	1.2274	1.2552	1.2927	0.6118	0.6943	0.8656
0.8	1.2403	1.2678	1.3043	0.5982	0.6826	0.8571
0.9	1.2517	1.2788	1.3142	0.5859	0.6720	0.8495
1.0	1.2618	1.2884	1.3226	0.5748	0.6625	0.8428

Table 4.13. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.5131	0.5636	0.7710
0.1	1.1147	1.1528	1.2408	0.5158	0.5793	0.7867
0.2	1.1857	1.2408	1.3576	0.4999	0.5676	0.7692
0.3	1.2408	1.3061	1.4348	0.4819	0.5513	0.7480
0.4	1.2862	1.3576	1.4905	0.4642	0.5343	0.7274
0.5	1.3245	1.3997	1.5328	0.4477	0.5179	0.7084
0.6	1.3576	1.4348	1.5661	0.4324	0.5025	0.6914
0.7	1.3865	1.4647	1.5930	0.4184	0.4883	0.6761
0.8	1.4120	1.4905	1.6153	0.4056	0.4753	0.6625
0.9	1.4348	1.5130	1.6340	0.3939	0.4633	0.6503
1.0	1.4552	1.5328	1.6500	0.3831	0.4522	0.6393

Table 4.14. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.8948	0.9281	1.0019
0.1	1.0377	1.0416	1.0467	0.8860	0.9232	0.9938
0.2	1.0577	1.0629	1.0695	0.8669	0.9096	0.9844
0.3	1.0720	1.0779	1.0850	0.8489	0.8966	0.9763
0.4	1.0830	1.0891	1.0964	0.8328	0.8850	0.9695
0.5	1.0919	1.0981	1.1054	0.8188	0.8749	0.9636
0.6	1.0992	1.1054	1.1126	0.8065	0.8660	0.9586
0.7	1.1054	1.1115	1.1185	0.7957	0.8582	0.9542
0.8	1.1107	1.1167	1.1235	0.7861	0.8513	0.9504
0.9	1.1153	1.1211	1.1278	0.7776	0.8452	0.9471
1.0	1.1193	1.1250	1.1314	0.7701	0.8397	0.9441

Table 4.15. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.7332	0.7900	0.9539
0.1	1.0748	1.0907	1.1175	0.7287	0.7926	0.9477
0.2	1.1175	1.1395	1.1744	0.7066	0.7752	0.9308
0.3	1.1492	1.1744	1.2125	0.6839	0.7563	0.9146
0.4	1.1744	1.2012	1.2403	0.6630	0.7385	0.9002
0.5	1.1951	1.2227	1.2618	0.6440	0.7223	0.8875
0.6	1.2125	1.2403	1.2788	0.6270	0.7077	0.8765
0.7	1.2274	1.2552	1.2928	0.6118	0.6946	0.8667
0.8	1.2403	1.2679	1.3044	0.5981	0.6829	0.8581
0.9	1.2517	1.2788	1.3142	0.5858	0.6723	0.8505
1.0	1.2618	1.2884	1.3226	0.5747	0.6627	0.8437

Table 4.16. Values of the H-functions and residual intensities for a triplet.

μ	$H(n_1\mu)$	$H(n_2\mu)$	$H(n_3\mu)$	$r(n_1\mu)$	$r(n_2\mu)$	$r(n_3\mu)$
0.0	1.0000	1.0000	1.0000	0.5088	0.5638	0.7862
0.1	1.1148	1.1529	1.2410	0.5120	0.5800	0.8028
0.2	1.1858	1.2410	1.3577	0.4967	0.5686	0.7845
0.3	1.2410	1.3062	1.4349	0.4792	0.5526	0.7623
0.4	1.2863	1.3577	1.4906	0.4620	0.5358	0.7407
0.5	1.3247	1.3998	1.5329	0.4458	0.5195	0.7209
0.6	1.3577	1.4349	1.5662	0.4308	0.5042	0.7030
0.7	1.3866	1.4648	1.5931	0.4171	0.4901	0.6870
0.8	1.4122	1.4906	1.6154	0.4045	0.4770	0.6727
0.9	1.4349	1.5131	1.6341	0.3929	0.4651	0.6599
1.0	1.4554	1.5329	1.6500	0.3823	0.4540	0.6484

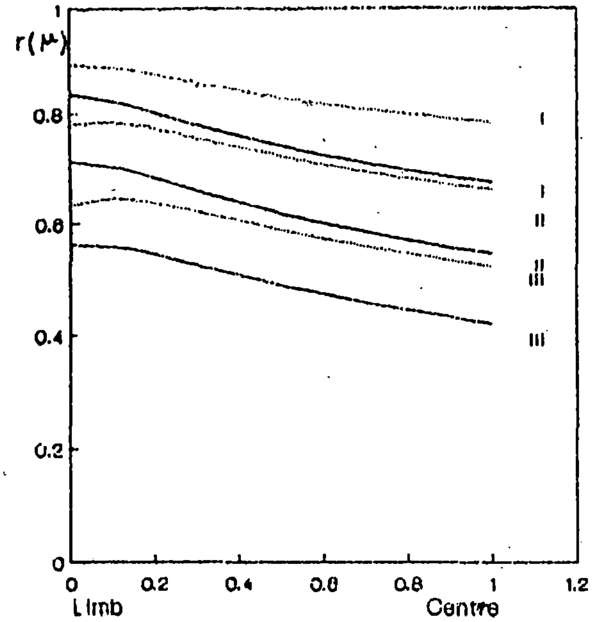

Fig. 4.1.a. The residual intensities for a doublet with interlocking.

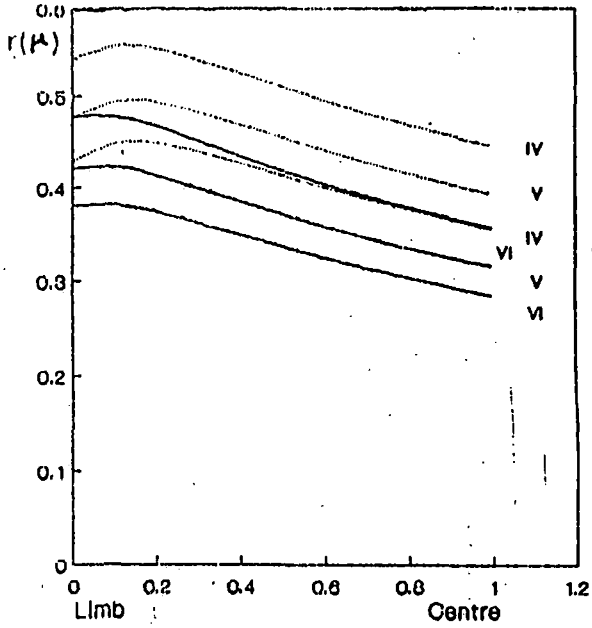

Fig. 4.1.b. The residual intensities for a doublet with interlocking.

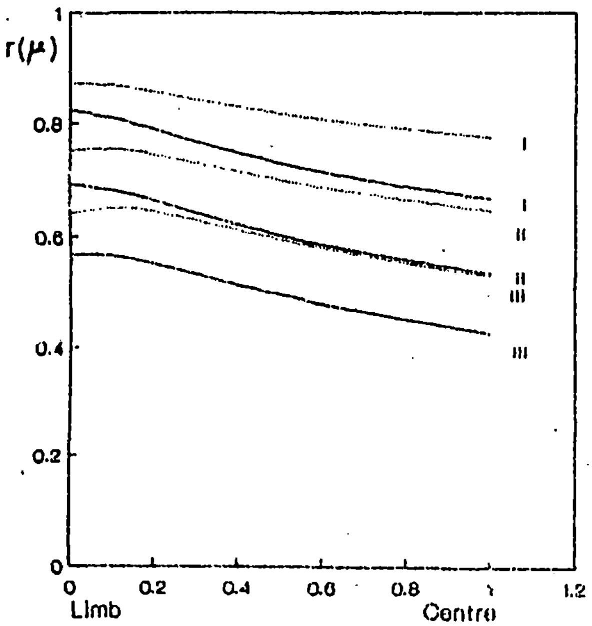

Fig. 4.2.a. The residual intensities for a doublet with interlocking.

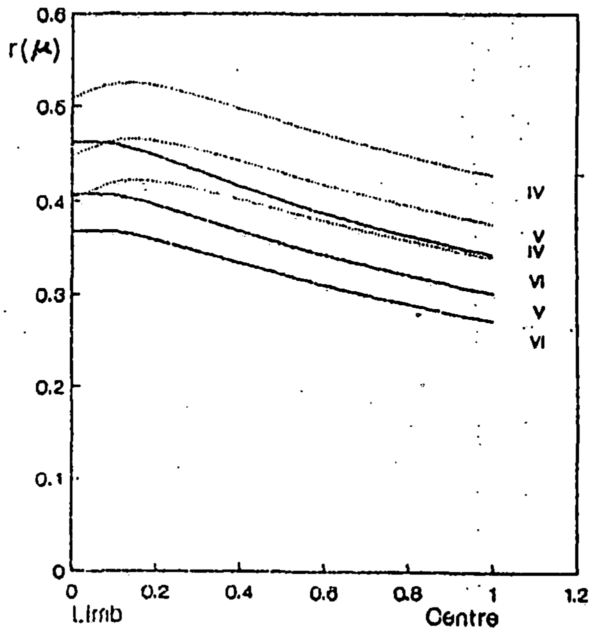

Fig. 4.2.b. The residual intensities for a doublet with interlocking.

All figures show cross-sections of the line profiles at three distances from there centres , the full curves representing the stronger component and dotted curves the weaker component.

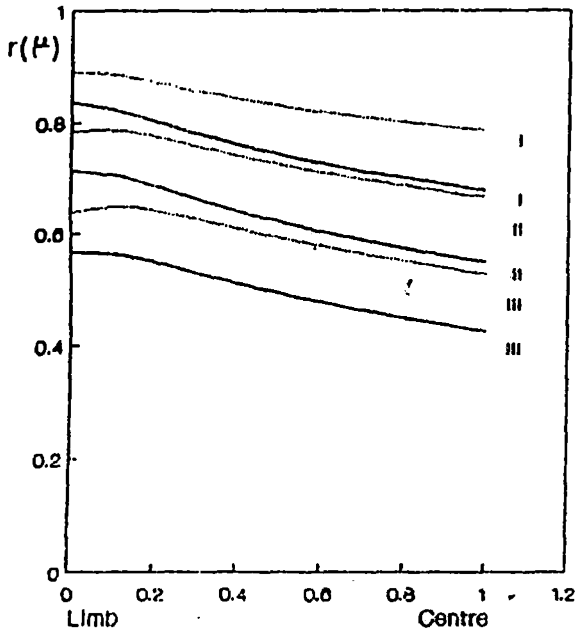

Fig. 4.3.a. The residual intensities for a doublet with interlocking.

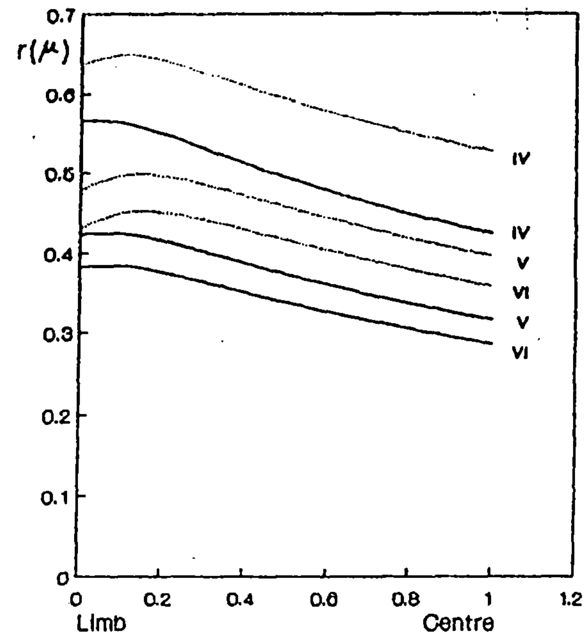

Fig. 4.3.b. The residual intensities for a doublet with interlocking.

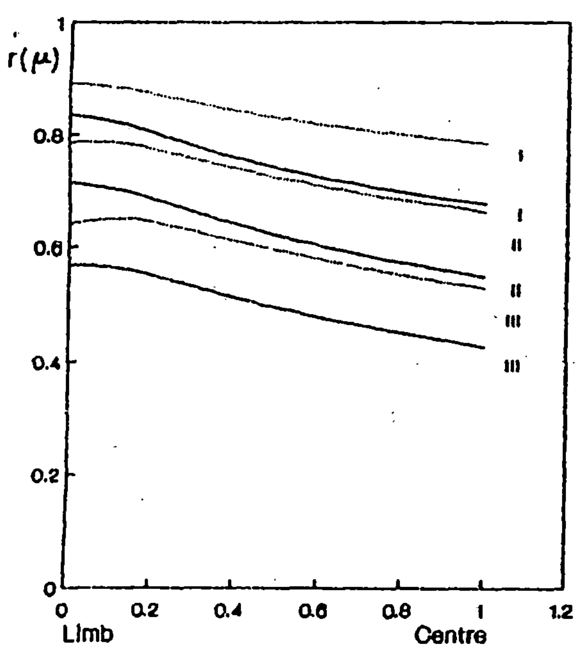

Fig. 4.4.a. The residual intensities for a doublet with interlocking.

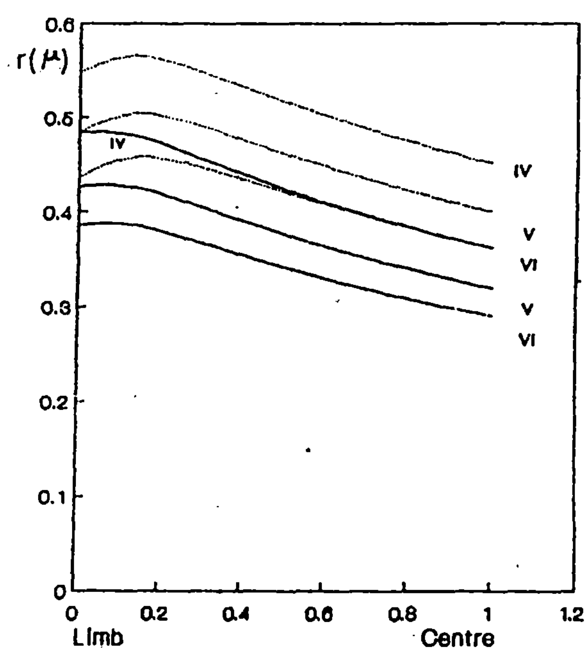

Fig. 4.4.b. The residual intensities for a doublet witn interlocking.

All figures show cross-sections of the line profiles at three distances from there centres , the full curves representing the stronger component and dotted curves the weaker component.

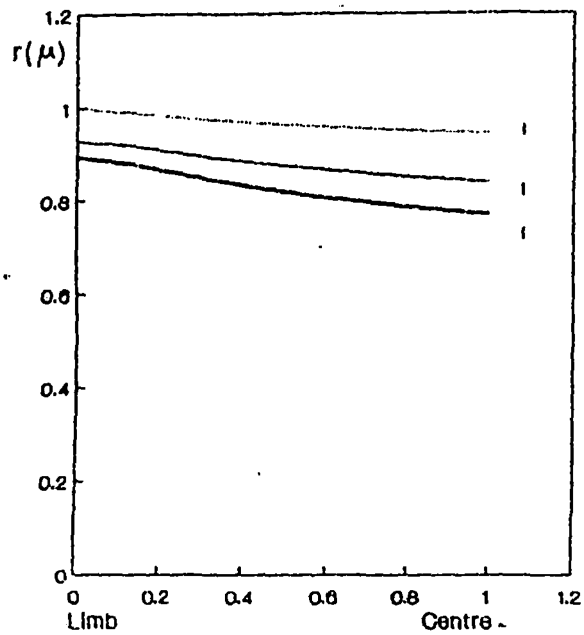

Fig.4.5.a. The residual intensities for a triplet with interlocking.

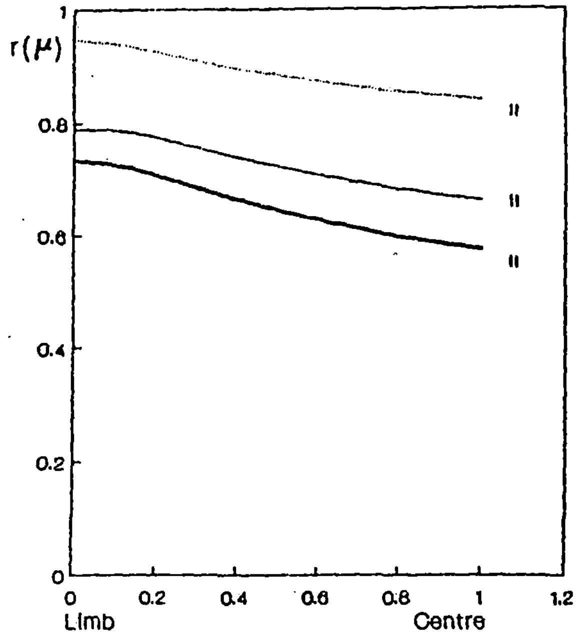

Fig.4.5.b. The residual intensities for a triplet with interlocking.

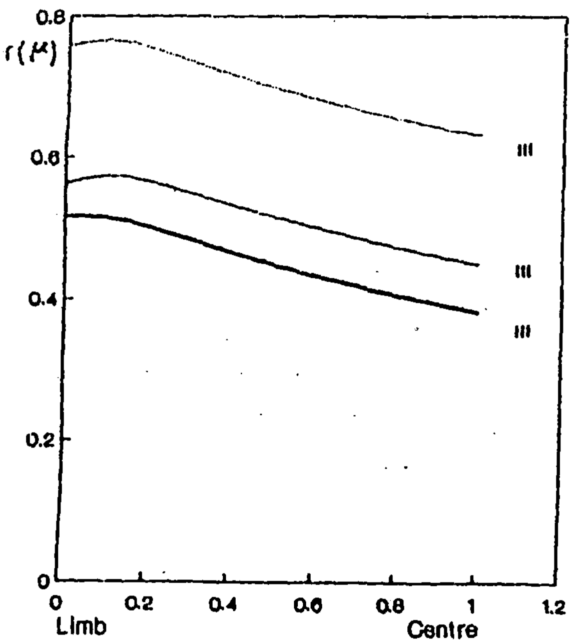

Fig.4.5.c. The residual intensities for a Triplet with interlocking.

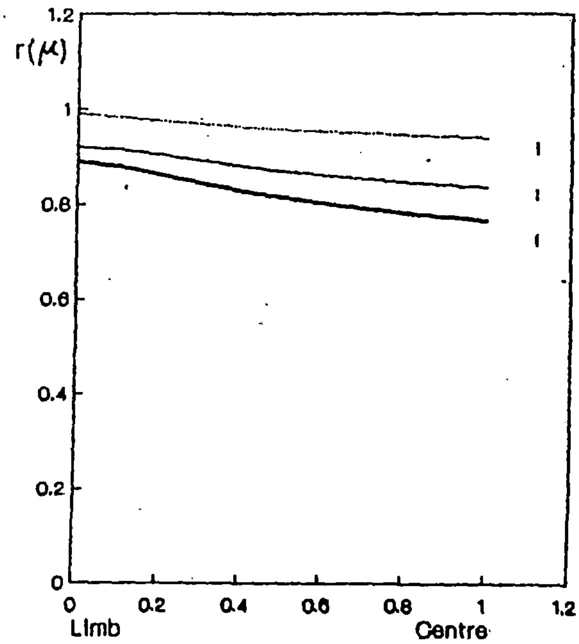

Fig.4.6.a. The residual intensities for a triplet with interlocking.

All figures show cross-sections of the line profiles at three distances from their centres, the thick curves representing the r_1- components (stronger components), the full curves representing the r_2- components and the dotted curves the r_3- components (weaker components).

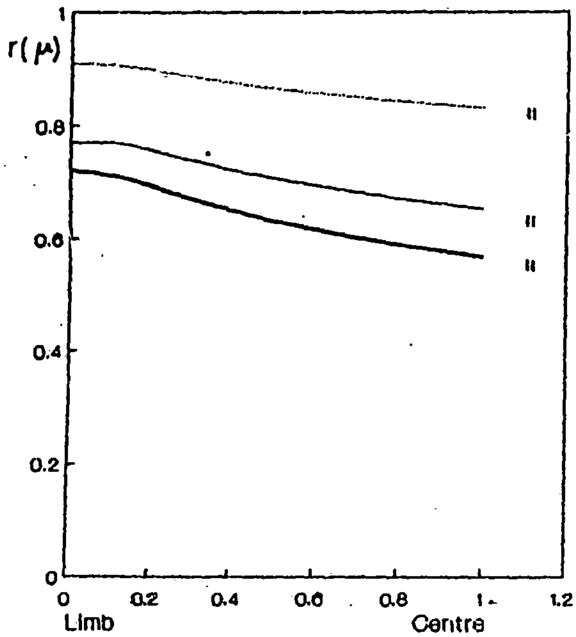

Fig.4.6.b. The residual intensities for a triplet with interlocking.

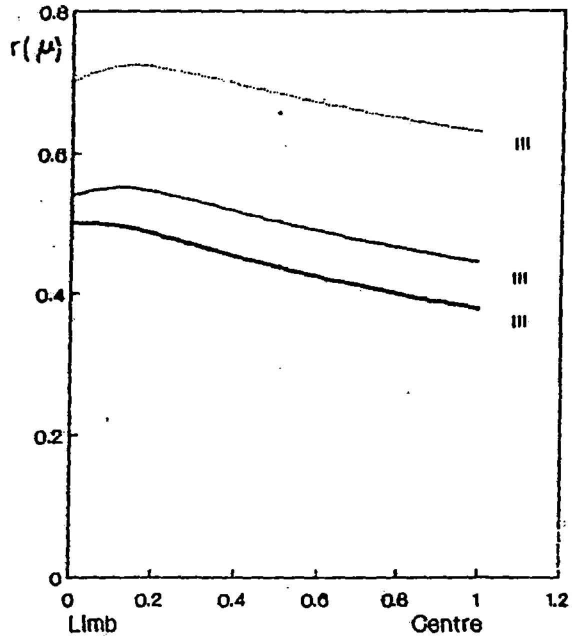

Fig.4.6.c. The residual intensities for a triplet with interlocking.

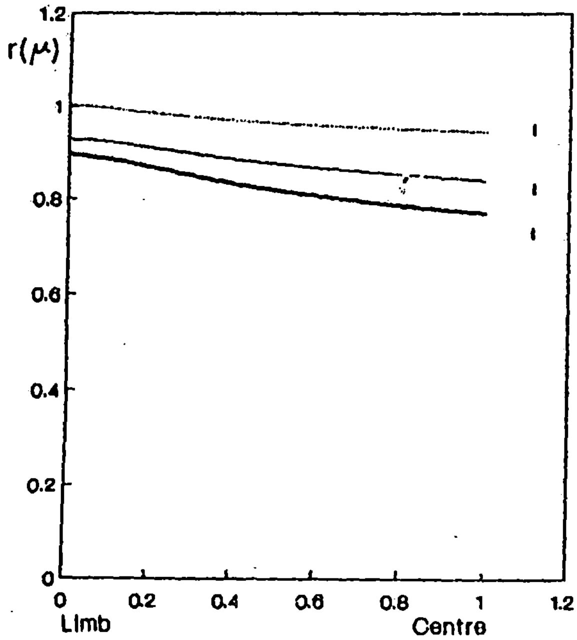

Fig.4.7.a. The residual intensities for a triplet with interlocking.

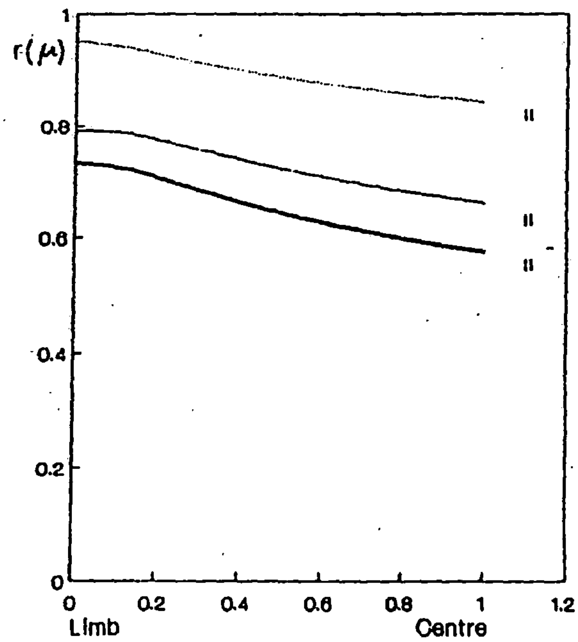

Fig.4.7.b. The residual intensities for a triplet with interlocking.

All figures show cross-sections of the line profiles at three distances from their centres, the thick curves representing the r_1- components (stronger components), the full curves representing the r_2- components and the dotted curves the r_3- components (weaker components).

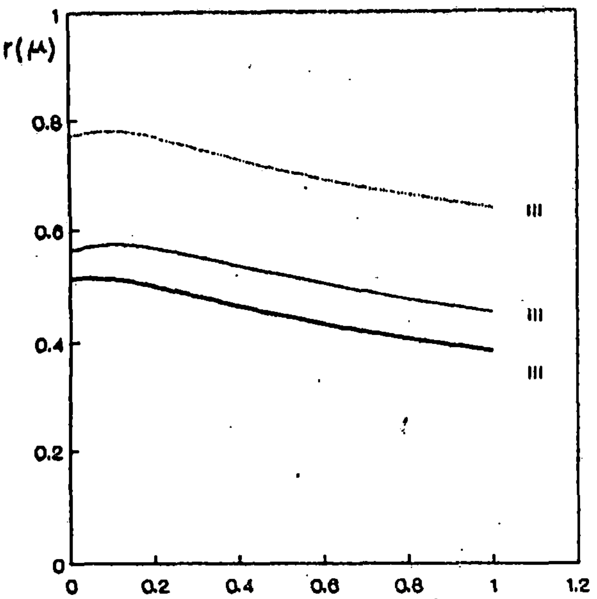

Fig.4.7.c. The residual intensities for a triplet with interlocking.

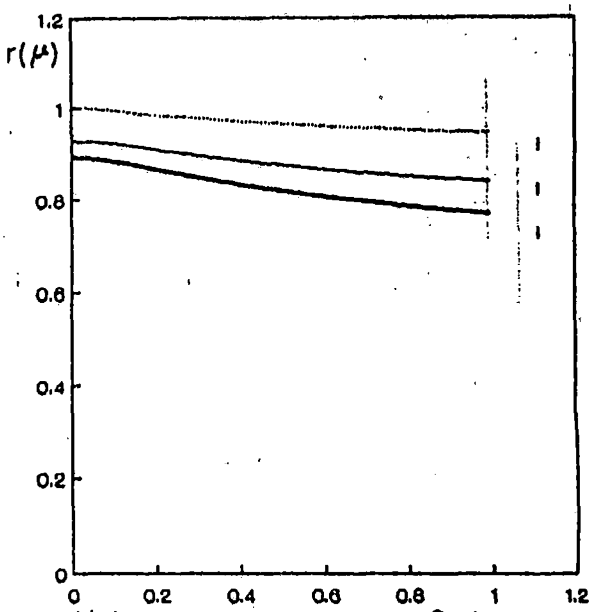

Fig.4.8.a. The residual intensities for a triplet with interlocking.

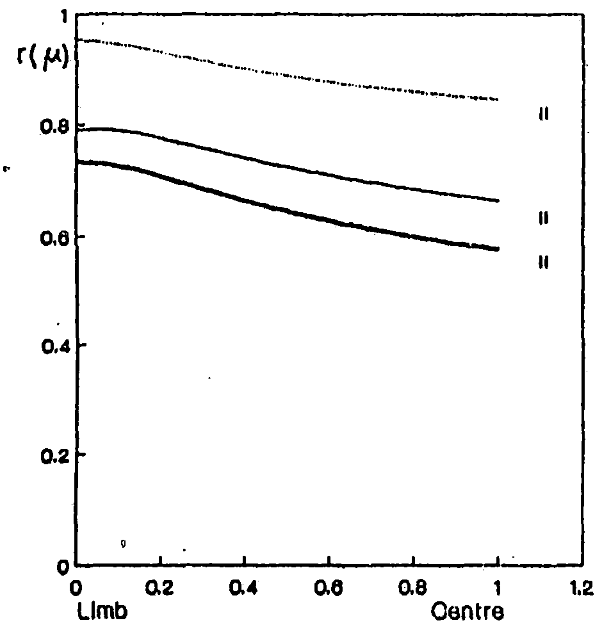

Fig.4.8.b. The residual intensities for a triplet with interlocking.

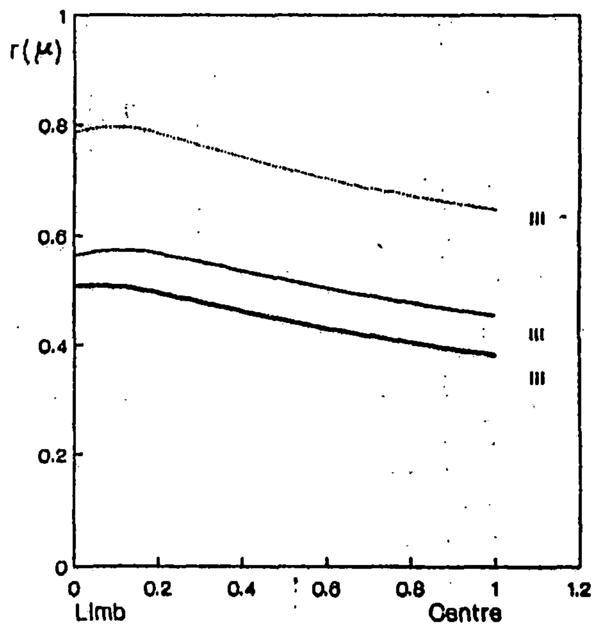

Fig.4.8.c. The residual intensities for a triplet with interlocking.

All figures show cross-sections of the line profiles at three distances from their centres, the thick curves representing the r_1- components (stronger components), the full curves representing the r_2- components and the dotted curves the r_3- components (weaker components).

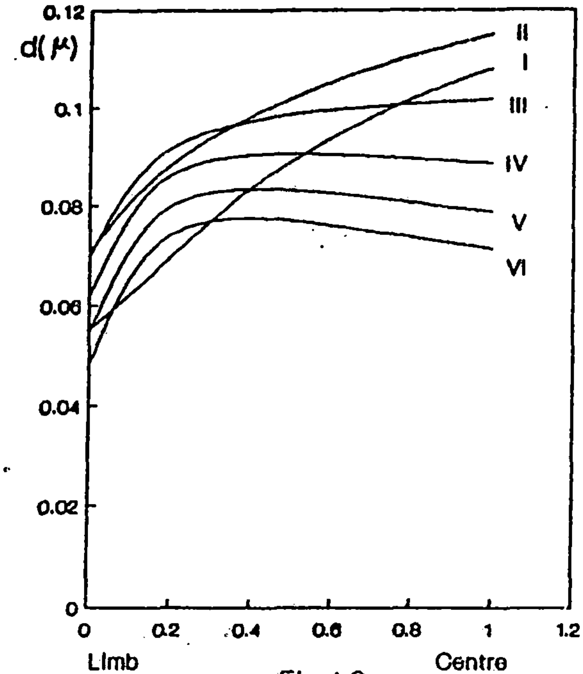

Fig.4.9.

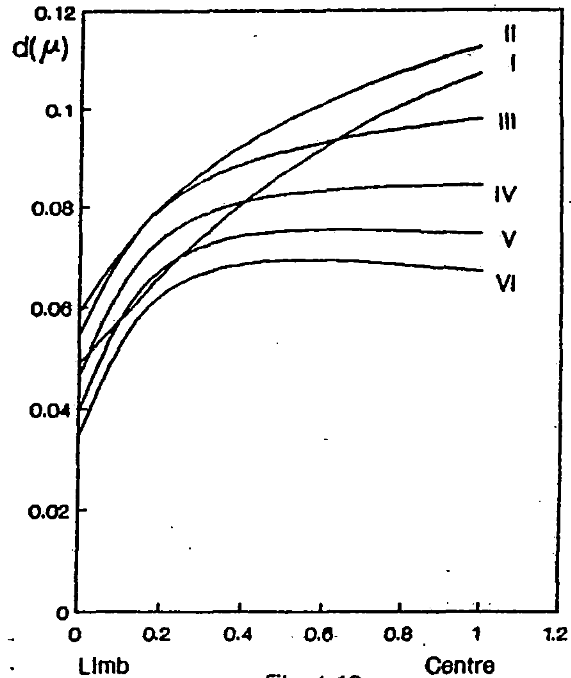

Fig.4.10.

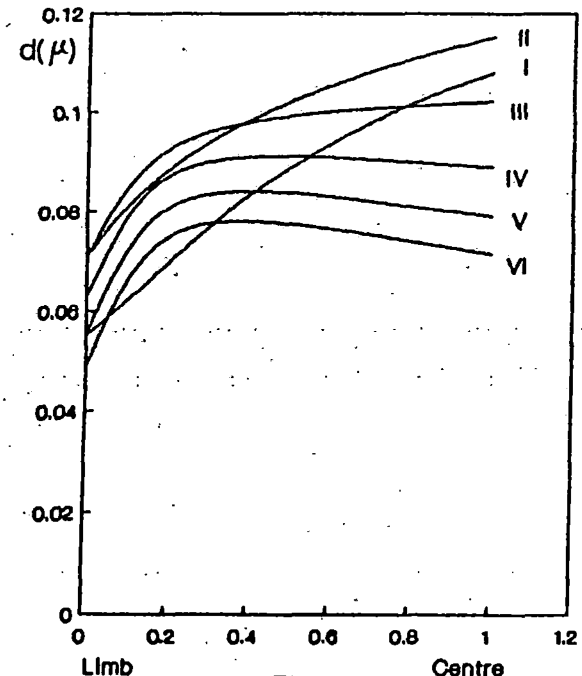

Fig.4.11.

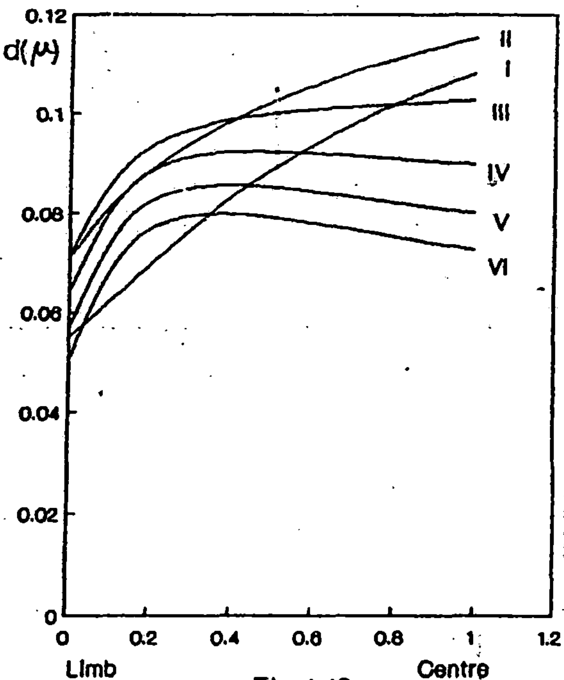

Fig.4.12.

All figures shows the variation of $d(\mu) = r_2(\mu) - r_1(\mu)$ for a doublet from centre to limb for cases I to VI.(Form (1) to (4)).

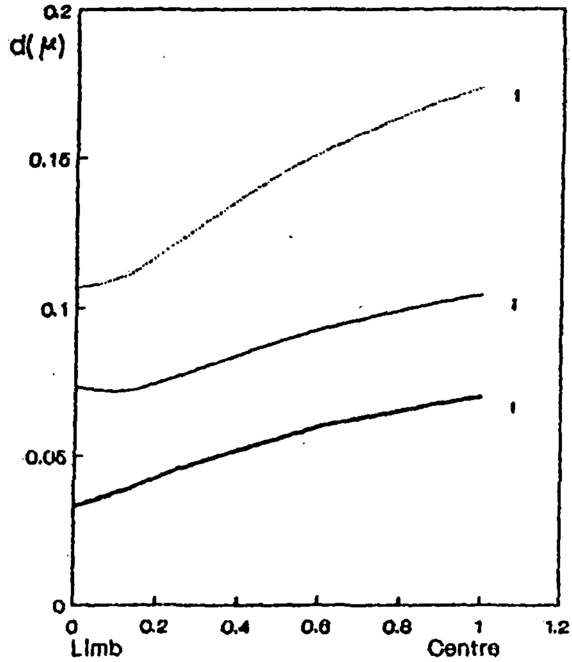

Fig.4.13.a.

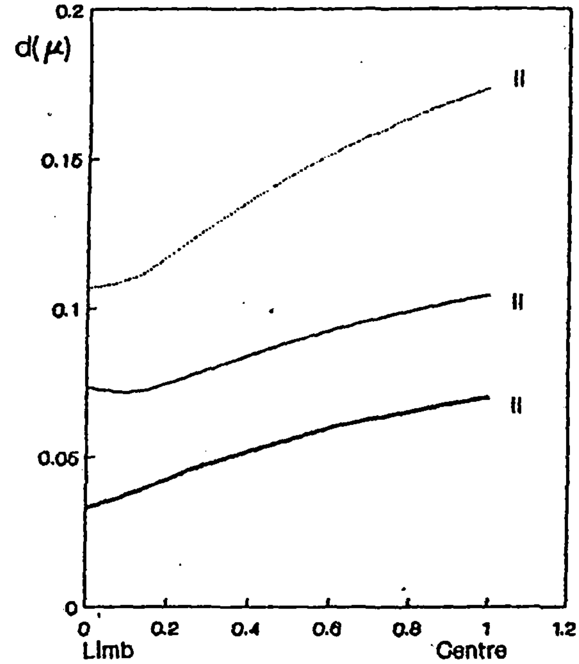

Fig.4.13.b.

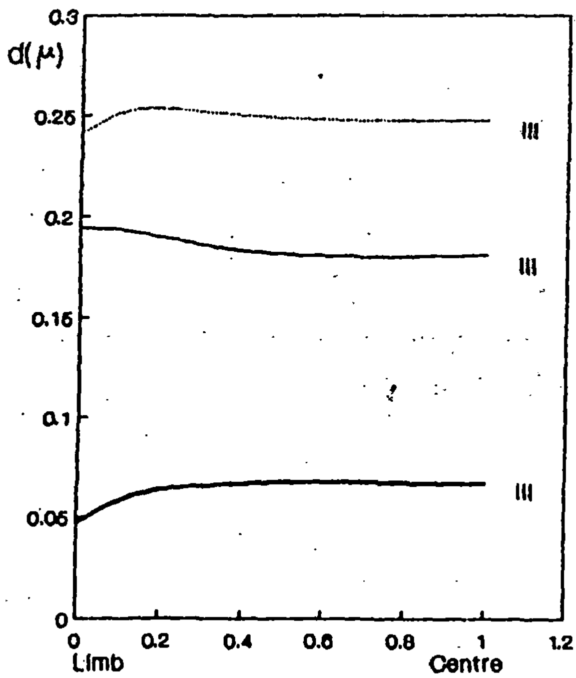

Fig.4.13.c.

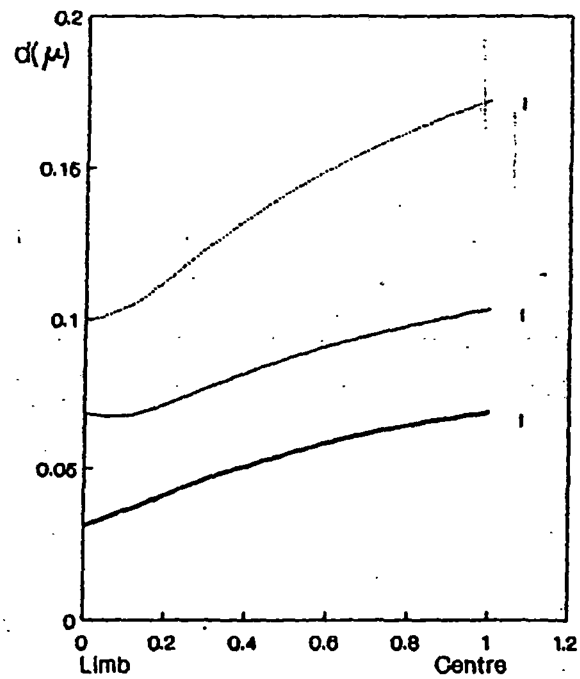

Fig.4.14.a.

All figures shows the variation of $d_1(\mu) = r_2(\mu) - r_1(\mu)$, $d_2(\mu) = r_3(\mu) - r_2(\mu)$ $d_3(\mu) = r_3(\mu) - r_1(\mu)$ for a triplet from centre to limb.

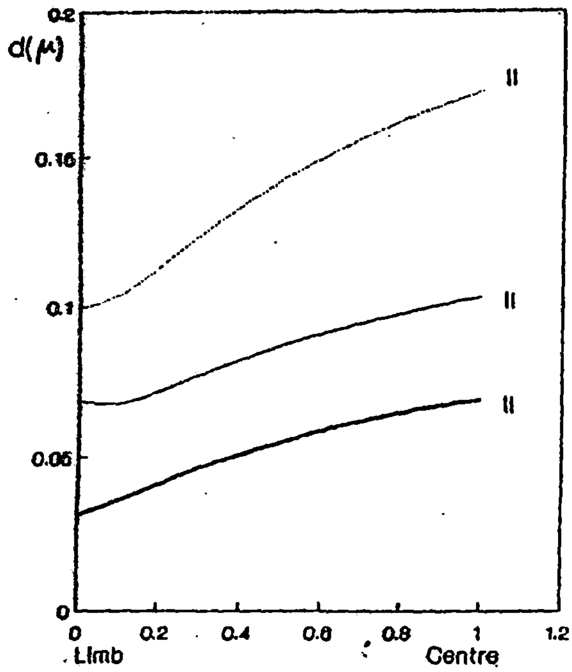

Fig.4.14.b.

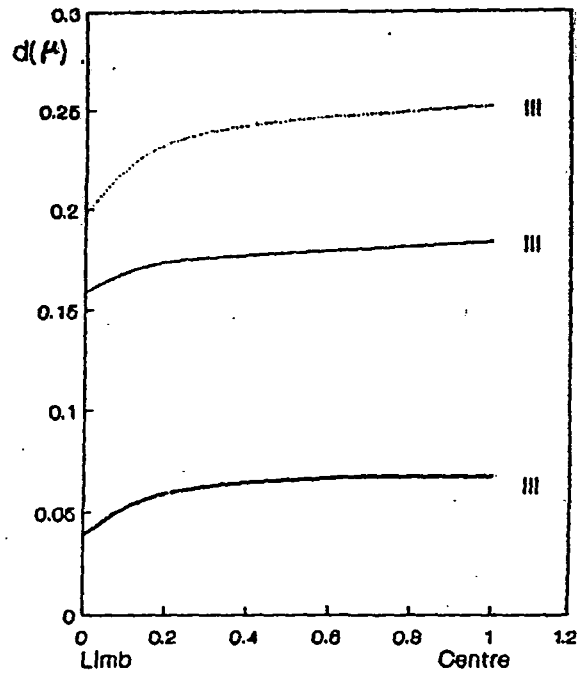

Fig.4.14.c

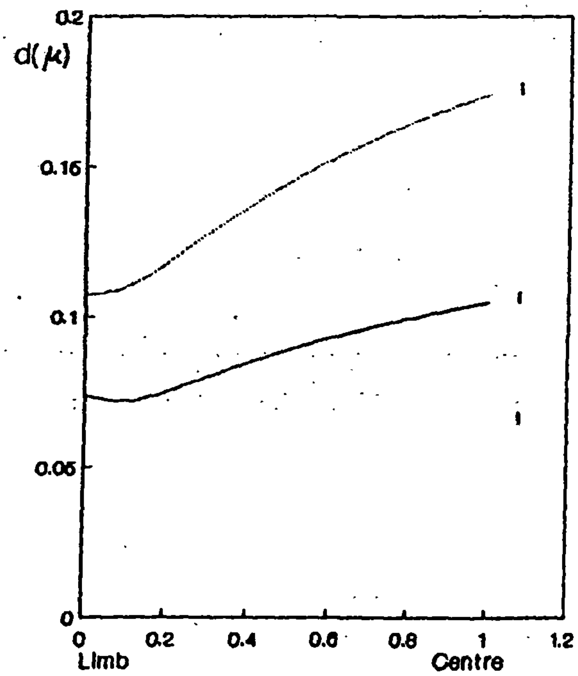

Fig.4.16.a.

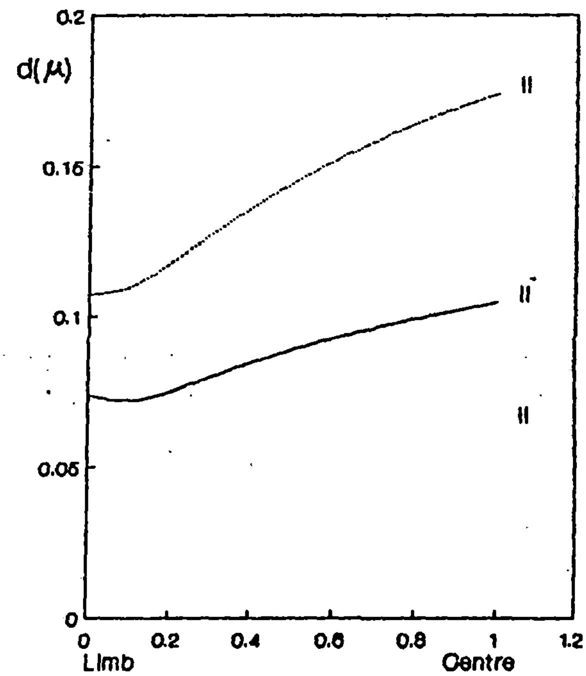

Fig.4.16.b.

All figures shows the variation of $d_1(\mu) = r_2(\mu) - r_1(\mu)$, $d_2(\mu) = r_3(\mu) - r_2(\mu)$ $d_3(\mu) = r_3(\mu) - r_1(\mu)$ for a triplet from centre to limb.

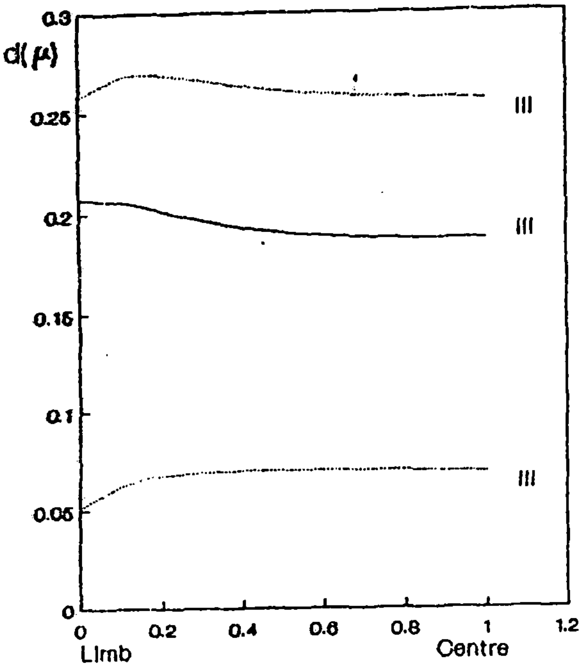

Fig.4.15.c.

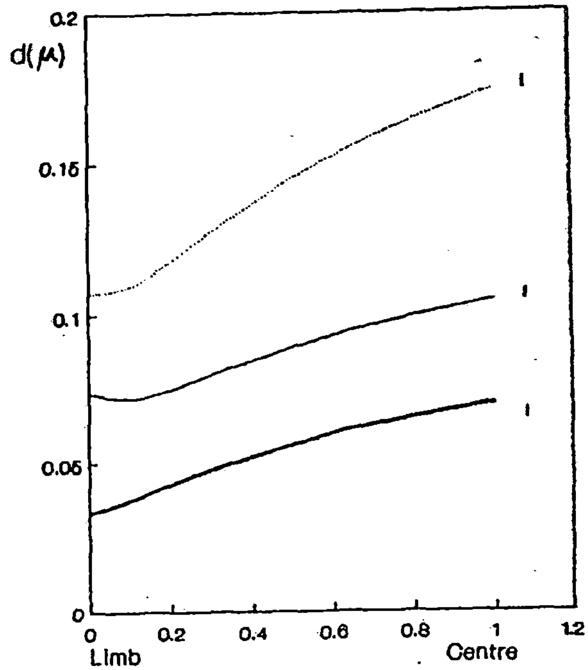

Fig.4.16.a.

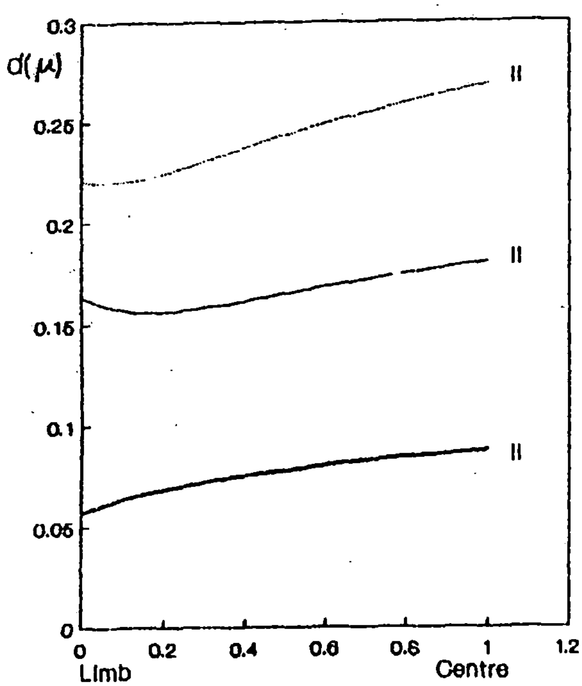

Fig.4.16.b.

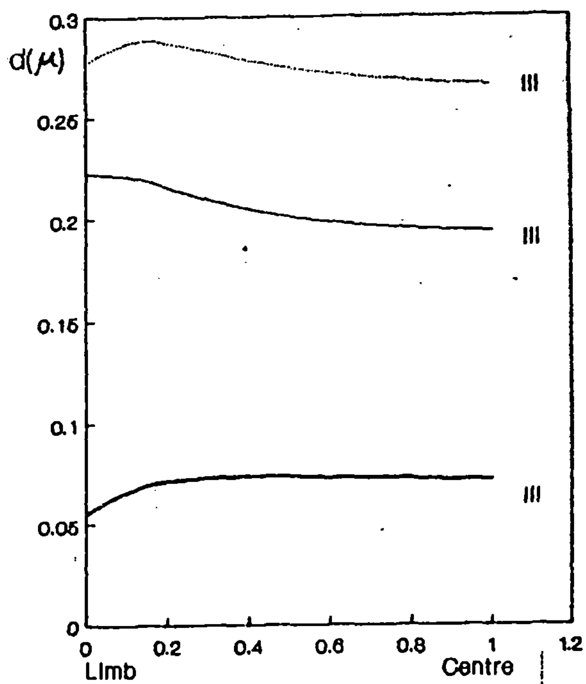

Fig.4.16.c.

All figures shows the variation of $d_1(\mu) = r_2(\mu) - r_1(\mu)$, $d_2(\mu) = r_3(\mu) - r_2(\mu)$ $d_3(\mu) = r_3(\mu) - r_1(\mu)$ for a triplet from centre to limb.

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CHAPTER - 5

SOLUTION OF RADIATIVE TRANSFER PROBLEMS IN A FINITE ATMOSPHERE

5.1. Introduction.

Das (1978, 1980) has solved various problems of radiative transfer in finite and semi-infinite atmosphere using a method involving Laplace transform and linear singular operators.

In the present work , the one sided Laplace transform together with the theory of linear singular operators has been applied to solve the transport equation which arises in the problem of a finite atmosphere having ground reflection according to Lambert's Law taking the Planck's function as an exponential function of optical depth (Sec-5.2).

In the theory of radiative transfer for homogeneous plane-parallel stratified finite atmosphere the X- and Y-functions of Chandrasekhar (1960) play a central role . The equations satisfy a system of coupled nonlinear integral equations. Busbridge (1960) has demonstrated the existence of the solution of these coupled non-linear integral equations in terms of a particular solution of an auxiliary equation. Busbridge (1960) has obtained two coupled linear integral equations for $X(z)$ and $Y(z)$ which defined the meromorfic extensions to the complex domain $|Z|$ of the real valued solution of the coupled non-linear equations of X-and Y- functions.

Busbridge (1960) concluded that all solutions of non-linear coupled integral equations for X- and Y- functions are the solutions of the coupled linear integral equations to the extended complex plane but all solutions of the coupled linear integral equations are not solutions of the coupled non-linear integral equations. Mullikin (1964) has proved that all solution of coupled non-linear integral equations are solutions of the coupled linear integral equations but there exists a unique solution of the coupled linear integral equations with some linear constraints. Finally he has obtained the Fredholm equations of X- and Y- functions which are easy for iterative computations. Das (1979) has obtained a pair of Fredholm equations with the Wiener-Hopf technique from the coupled linear integral equations with coupled linear constraints.

In the present work, the time-dependent X- and Y- functions (Biswas and Karanjai, 1990) which gives rise to a pair of the Fredholm equations with the application of the Wiener-Hopf technique has been obtained (Sec-5.3.). These Fredholm equations define time-dependent X-functions in terms of time-dependent Y-functions and vice-versa. These representations are unique with respect to the coupled linear constraints defined by Mullikin (1964).

In the study of time-dependent radiative transfer problems in finite homogeneous plane-parallel atmospheres it is convenient to introduce X- and Y- functions (vide, Chandrasekhar , 1960). These functions satisfy non-linear coupled integral equations. Due to their important role in solving transport problems, it is advantageous to simplify the equations satisfied by them. Lahoz (1989) did this and obtained exact linear and decoupled integral equations satisfied by the time-independent X- and Y- functions.

In the present work, the same method has extended to the time-dependent radiative transfer problem (Sec-5.4). However, the equations obtained , although linear, are singular and not solvable by the standard methods applicable to Fredholm equations instead they have to be solvable by the theory of singular integral equations (vide, Muskhelishvili, 1946).

5.2. Exact Solution of the Equation of Transfer in a Finite Exponential Atmosphere by the Method of Laplace Transform and Linear Singular Operator.

5.21. Basic Equation and Boundary Conditions.

The integro-differential equation for the intensity of radiation $I(\tau,\mu)$, at an optical depth τ for the problem of diffuse reflection and transmission in a finite atmosphere can be written in the form (vide, Das, 1980) as

$$\mu \frac{dI_\nu(\tau,\mu)}{d\tau} = I_\nu(\tau,\mu) - \int_{-1}^{+1} \psi(\mu')I_\nu(\tau,\mu')d\mu' - B_\nu(T) \qquad (5.1)$$

where $I_\nu(\tau,\mu)$ is the intensity in the direction $\cos^{-1}\mu$ at a depth τ, the angle $\cos^{-1}\mu$ is measured from outside drawn normal to the face $\tau = 0$, $\psi(\mu)$ is the characteristic function for non-conservative scattering which satisfies the condition

$$\psi_0 = \int_0^1 \psi(\mu')\, d\mu' \;;\quad \psi(\mu') \text{ is even }, \qquad (5.2)$$

ν is the frequency and $B_\nu(T)$ is the Planck function at any depth (form is same as in equation (1.11)). Then equation (5.1) becomes

$$\mu \frac{dI_\nu(\tau,\mu)}{d\tau} = I_\nu(\tau,\mu) - \int_{-1}^{+1} \psi(\mu') I_\nu(\tau,\mu') d\mu' -$$

$$- \quad (b_o + b_1 e^{-\beta\tau}) \qquad (5.3)$$

where for convenience I have omitted the subscript ν.

The boundary conditions associated with the equation (5.3) are

$$I(0, -\mu) = 0, \quad 0 < \mu \le 1 \qquad (5.4)$$

$$I(\tau_o,\mu) = I_g, \quad 0 < \mu \le 1 \quad , \tau_o > 0 \qquad (5.5)$$

where τ_o is the thickness of the finite atmosphere and the bounding face $\tau = \tau_o$ is having ground reflection according to Lambert's law is a constant.

5.22. Integral Equations for Surface Quantities.

Let us define

$$f^*(s,\mu) = s\int_0^{\tau_o} f(\tau,\mu)\, e^{-s\tau}\, d\tau \;, \quad \text{Re } s>0 \qquad (5.6)$$

$$f(\tau,\mu) = 0 \;, \text{ when } \; \tau > \tau_o \qquad (5.7)$$

Let us now apply the Laplace transform defined in equation (5.6) to equation (5.7) to obtain the equation satisfying the boundary condition as

$$(\mu s - 1)\ I^*(s,\mu) = \mu s I(0,\mu) - \mu s e^{-\tau_o s} - S^*(s) \qquad (5.8)$$

where

$$S(\tau) = \int_{-1}^{+1} \psi(\mu')I(\tau,\mu')d\mu' + (b_o + b_1 e^{-\beta\tau}) \qquad (5.9)$$

i.e.,

$$S^*(s) = \int_{-1}^{+1} \psi(\mu')I^*(\tau,\mu')d\mu' + b_o(1 - e^{-s\tau o}) +$$

$$+ \frac{sb_1}{s+\beta}(1 - e^{-(s+\beta)\tau_o}) \qquad (5.10)$$

Let us apply the operator

$$\int_{-1}^{+1} \psi(\mu)d\mu/(\mu s - 1)\ , \qquad (5.11)$$

on both sides of equation (5.8) and I obtain , with equation (5.10)

$$T(1/s)\ S^*(s) = \int_{-1}^{+1} \psi(\mu)I(0,\mu)d\mu/(\mu s - 1) -$$

$$- e^{\tau_o s}\int_{-1}^{+1} \mu s\psi(\mu)I(0,\mu)d\mu/(\mu s - 1) + b_o(1 - e^{-s\tau o}) +$$

$$+ \frac{sb_1}{s+\beta}(1 - e^{-(s+\beta)\tau_o}) \qquad (5.12)$$

where $$T(1/s) = 1 + \int_{-1}^{+1} \psi(\mu)d\mu/(\mu s - 1) \;, \tag{5.13}$$

Equation (5.8) gives

$$I(0,\mu) - e^{-\tau_o/\mu} I(\tau_o,\mu) = S^*(1/\mu) \;\Rightarrow \tag{5.14}$$

$$\Rightarrow\; I(0,1/s) - e^{-\tau_o s} I(\tau_o,1/s) = S^*(s) \tag{5.15}$$

Equation (5.12) , together with equation (5.14) , gives for complex z , where z = 1/s,

$$[I(0,z) - e^{-\tau_o/z} I(\tau_o,z)]\; T(z) = \int_{-1}^{+1} \mu\psi(\mu)I(0,\mu)d\mu/(\mu - z) -$$

$$- e^{-\tau_o/z} \int_{-1}^{+1} \mu\psi(\mu)I(\tau_o,\mu)d\mu/(\mu - z) + b_o(1 - e^{-\tau_o/z}) +$$

$$+ \frac{b_1}{1 + \beta z}\,(1 - e^{-(1/z+\beta)\tau_o}) \tag{5.16}$$

Let us put $\alpha_o = \beta^{-1}$, then equation (16) becomes

$$[I(0,z) - e^{-\tau_o/z} I(\tau_o,z)]\; T(z) = \int_{-1}^{+1} \mu\psi(\mu)I(0,\mu)d\mu/(\mu - z) -$$

$$- e^{-\tau_o/z} \int_{-1}^{+1} \mu\psi(\mu) I(\tau_o,\mu) d\mu/(\mu - z) + b_o(1 - e^{-\tau_o/z}) +$$

$$+ \frac{b_1\alpha_o}{z + \alpha_o} (1 - e^{-(1/z + 1/\alpha_o)\tau_o}) \qquad (5.17)$$

Let us put $z = -z$ in equation (5.17) and multiply the resulting equation by $e^{-\tau_o/z}$ on both sides to obtain , for complex z,

$$[I(\tau_o,-z) - e^{-\tau_o/z} I(0,-z)]T(z) = \int_{-1}^{+1} \mu\psi(\mu) I(\tau_o,\mu) d\mu/(\mu + z) -$$

$$- e^{-\tau_o/z} \int_{-1}^{+1} \mu\psi(\mu) I(0,\mu) d\mu/(\mu + z) + b_o(1 - e^{-\tau_o/z}) -$$

$$- \frac{b_1\alpha_o}{\alpha_o - z} (e^{-\tau_o/z} - e^{-\tau_o/\alpha_o}) \qquad (5.18)$$

Equations (5.17) and (5.18) are the linear integral equations for the surface quantities under consideration.

5.23. Linear Singular Integral Equations.

Equation (5.17) and (5.18) are the equations defined for complex z , where does not lie between -1 and 1 . When z lies between -1 and 1 , equation (5.17) and (5.18) will give the linear singular integral equations by the application of Plemelj's formulae (vide, Mushkelishvili,1946) with boundary conditions (4.4) and (5.5) as

$$[I(0,z) - e^{-\tau_o/z} I_g]\, T_o(z) = P\int_o^1 \mu\psi(\mu) I(0,\mu) d\mu/(\mu - z) -$$

$$- e^{-\tau_o/z} \int_o^1 \mu\psi(\mu) I(\tau_o,-\mu) d\mu/(\mu + z) -$$

$$- e^{-\tau_o/z} P\int_o^1 \mu\psi(\mu) I_g d\mu/(\mu - z) + + b_o(1 - e^{-\tau_o/z}) +$$

$$+ \frac{b_1\alpha_o}{z + \alpha_o}(1 - e^{-(1/z + 1/\alpha_o)\tau_o}) \qquad (5.19)$$

and

$$I(\tau_o,-z)\, T_o(z) = P\int_o^1 \mu\psi(\mu) I(\tau_o,-\mu) d\mu/(\mu - z) -$$

$$- e^{-\tau_o/z} \int_o^1 \mu\psi(\mu) I(0,\mu) d\mu/(\mu + z) +$$

$$+ e^{-\tau_o/z} \int_o^1 \mu\psi(\mu) I_g d\mu/(\mu + z) + b_o(1 - e^{-\tau_o/z}) -$$

$$- \frac{b_1\alpha_o}{\alpha_o - z}(e^{\tau_o/z} - e^{-\tau_o/\alpha_o}) \qquad (5.20)$$

where

$$T_o(z) = 1 - 2z^2 \int_0^1 d\mu\, [\psi(\mu) - \psi(z)] \;/\; (z^2 - \mu^2) -$$

$$- 2z^2\psi(z)\; P\int_0^1 d\mu/(z^2 - \mu^2) \qquad (5.21)$$

in which P denotes the Cauchy principal value of the integral.

Equations (5.19) and (5.20) are the linear singular integral equations from which I shall determine the surface quantities $I(0,z)$ and $I(\tau_o, -z)$ by the application of the theory of linear singular operators.

5.24. Theory of Linear Singular Operators.

Following Das [1978,1980] I can write the following theorems.

THEOREM 1.

The linear integral equations for $z \in (0,1)$,

$$L_+[R(z,-x_o)] = l(z,-x_o) \; , \qquad (5.22)$$

$$L_-[R(z,-x_o)] = m(z,-x_o) \; , \qquad (5.23)$$

where

$$L_+ [f(z,-x_o) = f(\mu,-x_o)T_o(z) - P\int_0^1 \mu\psi(\mu)f(\mu,-x_o)d\mu/(\mu - z) +$$

$$+ e^{-\tau_o/z}\int_0^1 \mu\psi(\mu)f(\mu,-x_o)d\mu/(\mu + z) \tag{5.24}$$

$$L_- [f(z,-x_o) = f(\mu,-x_o)T_o(z) - P\int_0^1 \mu\psi(\mu)f(\mu,-x_o)d\mu/(\mu - z) -$$

$$+ e^{-\tau_o/z}\int_0^1 \mu\psi(\mu)f(\mu,-x_o)d\mu/(\mu + z) \tag{5.25}$$

where

$$l(z,-x_o) = \frac{x_o}{z + x_o}[1 - e^{-(1/z + 1/x_o)\tau_o}] +$$

$$+ \frac{x_o}{z - x_o}[e^{-\tau_o/z} - e^{-\tau_o/x_o}] \tag{5.26}$$

$$m(z,-x_o) = \frac{x_o}{z + x_o}[1 - e^{-(1/z + 1/x_o)\tau_o}] +$$

$$- \frac{x_o}{z - x_o}[e^{-\tau_o/z} - e^{-\tau_o/x_o}] \tag{5.27}$$

admit of solutions of the form

$$R(z,-x_o) = S(z,-x_o) + T(z,-x_o) \tag{5.28}$$

$$Q(z,-x_o) = S(z,-x_o) + T(z,-x_o) \tag{5.29}$$

where

$$S(z,-x_o) = x_o[X(z)X(x_o) - Y(z)\ Y(x_o)]/(z + x_o) \qquad (5.30)$$

$$T(z,-x_o) = x_o[X(z)Y(x_o) - Y(z)\ X(x_o)]/(x_o - z) \qquad (5.31)$$

With constraints on X(z) and Y(z) as

(i) when ψ_o <1/2

$$1 = K\int_0^1 X(\mu)\psi(\mu)d\mu/(K - \mu) +$$

$$+ e^{-\tau_o/K}K\int_0^1 Y(\mu)\psi(\mu)d\mu/(K + \mu) \qquad (5.32)$$

$$e^{-\tau_o/K} = K\int_0^1 Y(\mu)\psi(\mu)d\mu/(K - \mu) +$$

$$+ e^{-\tau_o/K}K\int_0^1 X(\mu)\psi(\mu)d\mu/(K + \mu) \qquad (5.33)$$

(ii) when ψ_o = 1/2

$$1 = \int_0^1 [X(\mu) + Y(\mu)]\psi(\mu)d\mu \qquad (5.34)$$

$$+ \tau_o\int_0^1 Y(\mu)\psi(\mu)d\mu = \int_0^1 [X(\mu) - Y(\mu)]\mu\psi(\mu)d\mu \qquad (5.35)$$

and K is the positive root of the function T(z), when ψ_o <1/2 , defined by

$$T(z) = 1 + \int_{-1}^{+1} z\ \psi(\mu)d\mu/(\mu - z) \qquad (5.36)$$

and where $[X(\mu) - Y(\mu)]$ and $[X(\mu) + Y(\mu)]$ are the respective solutions of

$$L_+\ [f(z)] = (1 - e^{-\tau_o/z}) \left[1 - \int_0^1 f(\mu)\psi(\mu)d\mu \right] \qquad (5.37)$$

$$L_-\ [f(z)] = (1 + e^{-\tau_o/z}) \left[1 - \int_0^1 f(\mu)\psi(\mu)d\mu \right] \qquad (5.38)$$

THEOREM 2.

As the operators L_+ and L_- are linear for $z \varepsilon (0,1)$, then for any constant C, I have

$$L_{\pm}\ (CF(z,-x_o)) = CL_{\pm}\ (F(z,-x_o) \qquad (5.39)$$

and

$$L_{\pm}(zf(z)) = zL_{\pm}(f(z) - (1 \mp e^{-\tau_o/z})\int_0^1 \mu\psi(\mu)f(\mu)d\mu \qquad (5.40)$$

THEOREM 3.

If $R(z,-x_o)$ and $Q(z,-x_o)$ are the solutions of

$$L_+[R(z,-x_o)] = l(z,-x_o)\ , \qquad (5.41)$$

$$L_-[R(z,-x_o)] = m(z,-x_o)\ , \qquad (5.42)$$

then

$$L_+[M(z)] = \int_0^1 \psi(-x_o)\ l(z,-x_o)\ dx_o\ , \qquad (5.43)$$

$$L_-[N(z)] = \int_0^1 \psi(-x_o)\ m(z,-x_o)\ dx_o\ , \qquad (5.44)$$

admit the solution of

$$M(z) = \int_0^1 \psi(-x_o)\ R(z,-x_o)\ dx_o\ , \qquad (5.45)$$

$$N(z) = \int_0^1 \psi(-x_o)\ Q(z,-x_o)\ dx_o\ , \qquad (5.46)$$

5.25. Solution for Surface Quantities.

Linear singular integral equations (5.19) and (5.20) are the required integral equations from which I will have to determine $I(0,\mu)$ and $I(\tau_o,-z)$, the quantities under consideration, by the application of the theory of linear singular operators indicated in section 5.2.4. Equations (5.19) and (5.20) on addition and after some rearrangement give

$$L_+[I(0,z) + I(\tau_o, -z) - e^{-\tau_o/z} I_g] = 2b_o(1 - e^{-\tau_o/z}) +$$

$$+ b_1 l(z , -\alpha_o) + I_g \int_o^1 \psi(\mu)\ l(z , -\mu)d\mu \qquad (5.47)$$

Equations (5.19) and (5.20) on subtraction and after manipulation give

$$L_-[I(0,z) - I(\tau_o, -z) - e^{-\tau_o/z} I_g] =$$

$$= b_1 m(z , -\alpha_o) + I_g \int_o^1 \psi(\mu)\ m(z , -\mu)d\mu \qquad (5.48)$$

where $l(z , -\mu)$ and $m(z , -\mu)$ are given by equations (5.26) and (5.27). Equations (5.47) and (5.48) with Theorems 1,2 and 3 of section 5.2.4. will give us the desired quantities $I(0,z)$ and $I(\tau_o,-z)$. The solution of equation (5.47) is given by

$$[I(0,z) + I(\tau_o, -z) - e^{-\tau_o/z} I_g] = \frac{2b_o}{1 - G_o} [X(z) - Y(z)] +$$

$$+ b_1 R(z , -\alpha_o) + I_g \int_o^1 \psi(\mu)\ R(z , -\mu)d\mu \qquad (5.49)$$

where $$G_o = \int_o^1 [X(\mu) - Y(\mu)]\psi(\mu)d\mu \qquad (5.50)$$

The solution of equation (5.48) is given by

$$[I(0,z) + I(\tau_o, -z) - e^{-\tau_o/z} I_g] =$$

$$= b_1 Q(z\ ,\ -\alpha_o) - I_g \int_0^1 \psi(\mu)\ Q(z\ ,\ -\mu)d\mu \qquad (5.51)$$

Equation (5.50) and (5.51) on addition give $I(0,z)$ and equations (5.47) and (5.51) on subtraction give $I(\tau_o,-z)$ as

$$I(0,z) = I_g\ e^{-\tau_o/z} + I_g \int_0^1 \psi(\mu)\ T(z\ ,\ -\mu)d\mu\ +$$

$$+ \frac{b_o}{1 - G_o} [X(z) - Y(z)) + b_1 S(z\ ,\ -\mu) \qquad (5.52)$$

and

$$I(\tau_o,-z) = I_g \int_0^1 \psi(\mu)\ S(z\ ,\ -\mu)d\mu\ +$$

$$+ \frac{b_o}{1 - G_o} [X(z) - Y(z)] + b_1 T(z\ ,\ -\alpha_o) \qquad (5.53)$$

where $S(z, -\mu)$ and $T(z, -\mu)$ are given by equations (5.30) and (5.31).

5.3. The Time-Dependent X- and Y- Functions.

5.31. Basic Equation.

The coupled nonlinear integral equations satisfied by the time-dependent X- and Y- function (vide, Biswas and Karanjai, 1990) are of the form

$$X(\tau_1,\mu,s) = 1 + \frac{\omega}{2Q}\mu \times$$

$$\times \int_0^1 \frac{X(\tau_1,\mu,s)X(\tau_1,x,s)-Y(\tau_1,\mu,s)Y(\tau_1,x,s)}{\mu + x}\, dx \qquad (5.54)$$

$$0 \le \mu \le 1 \quad .$$

$$Y(\tau_1,\mu,s) = \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{\omega}{2Q}\mu \times$$

$$\times \int_0^1 \frac{Y(\tau_1,\mu,s)X(\tau_1,x,s)-X(\tau_1,\mu,s)Y(\tau_1,x,s)}{\mu - x}\, dx \qquad (5.55)$$

$$0 \le \mu \le 1 \quad .$$

where $$Q = 1 + s/c \qquad (5.56)$$

τ_1 is the thickness of the atmosphere ; c, the velocity of light ; and s, Laplace transform parameter.

Following Chandrasekhar (1960) equations (5.54) and (5.55) can be written as

$$X(\tau_1,\mu,s) = 1 + \frac{\mu}{Q} \times$$

$$\times \int_0^1 \frac{X(\tau_1,\mu,s)X(\tau_1,x,s)-Y(\tau_1,\mu,s)Y(\tau_1,x,s)}{\mu + x}\, \psi(x)dx \qquad (5.57)$$

$$0 \le \mu \le 1 \quad .$$

$$Y(\tau_1,\mu,s) = \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{\mu}{Q} \times$$

$$\times \int_0^1 \frac{Y(\tau_1,\mu,s)X(\tau_1,x,s)-X(\tau_1,\mu,s)Y(\tau_1,x,s)}{\mu - x}\,\psi(\mu)dx \qquad (5.58)$$

$$0 \le \mu \le 1 \quad .$$

where $\psi(x)$, the characteristic function satisfying the Hölder condition on $0 \le x \le 1$, is non-negative and satisfies the condition

$$\psi_o = \int_0^1 \psi(x)\,dx \le 1/2 \qquad (5.59)$$

The atmosphere is said to be conservative when $\psi_o = 1/2$ and non-negative otherwise .

The dispersion function $T(z,s)$, $z \in (-1,1)^c$ can be defined by

$$T(z,s) = 1 - \frac{2z^2}{Q} \int_0^1 \frac{\psi(x)dx}{z^2 - x^2} \qquad (5.60)$$

and

$$T(z,s) = (H(z,s)H(-z,s))^{-1} \qquad (5.61)$$

where

$$H(z,s) = 1 + zH(z,s) \int_0^1 \frac{\psi(x)H(x,s)dx}{x + z} \qquad (5.62)$$

According to Busbridge (1960) , the only zeros of T(z,s) are

at $z = \pm K$, $K>1$, when $\psi_o < 1/2$ and when $\psi_o = 1/2$.

Following Busbridge (1960), Dasgupta (1977), and Das (1978) H(z,s) is meromorfic on $(-1,0)^c$ having a simple pole at $z = -K$ and tend to 1 as $z \longrightarrow 0_+$. It can be represented by

$$H(z,s) = \frac{A_o + H_o z}{K + z} - \int_0^1 \frac{P(x,s)dx}{x + z} \qquad (5.63)$$

$$K > 1, \quad \psi_o < 1/2$$

$$H(z,s) = h_1 z + h_o - \int_0^1 \frac{P(x,s)dx}{x + z} \qquad (5.64)$$

$$, K \longrightarrow \infty , \psi_o = 1/2$$

where

$$A_o = (1 + P_{-1})K, \qquad P_{-1} = \int_0^1 P(x,s)\ dx/x, \qquad (5.65)$$

$$H_o = \left[1 - 2 \int_0^1 \psi(x)\ dx \right]^{-1/2} \qquad (5.66)$$

$$h_1 = \left[2 \int_0^1 x^2\psi(x)\ dx \right]^{-1/2} \qquad (5.67)$$

$$h_o = (1 + P_{-1}) \qquad (5.68)$$

$$P(x,s) = \phi(x,s)/H(x,s) \tag{5.69}$$

$$\phi(x,s) = x\psi(x)/(T_o^2(x,s) + \pi^2 x^2 \psi^2(x)) \tag{5.70}$$

$$T_o(x,s) = 1 - \frac{2x^2}{Q}\int_0^1 \frac{\psi(t) - \psi(x)}{x^2 - t^2} -$$

$$- \frac{x\psi(x)}{Q}\log((1 + x)/(1 - x)) \tag{5.71}$$

where $\phi(x,s)$ is non-negative and continuous on $(0,1)$, tends to $\psi(0)$, as $x \longrightarrow 0_+$, tends to $O((\log(1 - x)^{-2})$ when $x \longrightarrow 1_-$, and $1/H(z,s)$ is regular on $(-1,0)^c$.

Following Busbridge (1960) and Mullikin (1964) I find that the coupled linear equations satisfied by $X(z,s)$ and $Y(z,s)$ for $z \varepsilon (-1,1)^c$ are of the form

$$X(z,s)T(z,s) = 1 + zU(X)(z,s) -$$

$$- \exp(-(\tau_1/z)Q)V(Y)(z,s) \tag{5.72}$$

$$Y(z,s)T(z,s) = \exp(-(\tau_1/z)Q) + zU(Y)(z,s) -$$

$$- z\exp(-(\tau_1/z)Q)V(Y)(z,s) \tag{5.73}$$

with constraints for $\psi_o < 1/2$,

$$0 = 1 + KU(X)(K,s) - K\exp(-(\tau_1/K)QV(Y)(K,s) \tag{5.74}$$

$$0 = (\exp(-(\tau_1/K)Q) + KU(Y)(K,s)) -$$

$$- K\exp(-(\tau_1/K)V(X)(K,s) \tag{5.75}$$

for $\psi_o = 1/2$

$$1 = \int_0^1 \psi(x)(X(x,s) + Y(x,s)\ dx \tag{5.76}$$

$$\tau_1 \int_0^1 Y(x,s)\psi(x)\ dx = \int_0^1 X\psi(x)(X(x,s) - Y(x,s))dx \tag{5.77}$$

The other conditions for which $X(z,s)$ and $Y(x,s)$ hold are

$$X(s,s) \longrightarrow H(z,s) \quad \text{when} \quad \tau_1 \longrightarrow \alpha \tag{5.78}$$

$$Y(z,s) \longrightarrow 0 \text{ when } \quad \tau_1 \longrightarrow \alpha \tag{5.79}$$

where for $M = X$ or Y

$$V(M)(z,s) = \int_0^1 \psi(x)M(x,s)dx/(x + z) \tag{5.80}$$

is analytic for $z \varepsilon (-1,1)$ bounded at the origin $O(z^{-1})$ when $z \longrightarrow \alpha$ and

$$U(M)(z,s) = \int_0^1 \psi(x)M(x,s)dx/(x - z) \tag{5.81}$$

is analytic for $z \varepsilon (0,1)^c$, bounded at the origin $O(z^{-1})$ when $z \longrightarrow \alpha$.

5.32. Fredholm equations.

Equations (5.72)and (5.73) with equation (5.61) can be written in the form

$$X(z,s)/H(z,s) = H(-z,s)(1 + zU(X)(z,s) -$$
$$- \exp(-(\tau_1/z)Q)H(-z,s)V(Y)(z,s) \qquad (5.82)$$

$$Y(z,s)/H(z,s) = H(-z,s)(\exp(-(\tau_1/z)Q) + zU(Y)(z,s) -$$

$$- z\exp(-(\tau_1/z)Q)H(-z,s)V(Y)(z,s) \qquad (5.83)$$

I shall assume that X(z,s) and Y(z,s) are regular for Re z>0 and bounded at the origin. Equation (5.63) gives

$$H(-z,s) = \frac{A_o - H_o z}{K - z} - \int_o^1 \frac{P(x,s)dx}{x - z} \qquad (5.84)$$

for $\psi_o < 1/2$

Hence

$$V(M)(z,s)\int_o^1 \frac{P(x,s)}{x - z}dx = D(M,P_o)(z,s) + D(P,M_o)(z,s) \qquad (5.85)$$

where $$D(M,P_o)(z,s) = \int_o^1 \frac{\psi(x)M(x,s)P_o(x,s)dx}{x + z} \qquad (5.86)$$

and

$$D(P,M_o)(z,s) = \int_o^1 \frac{\psi(x)P(x,s)M_o(x,s)dx}{x - z} \qquad (5.87)$$

where $$P_o(z,s) = \int_o^1 \frac{P(x,s)}{x + z}dx \qquad (5.88)$$

is regular on $(-1,0)^c$, bounded at the origin and $O(z^{-1})$ when $z \longrightarrow \alpha$ and $D(M,P_o)(z,s)$ is regular for z on $(-1,0)^c$, bounded at the origin and $O(z^{-1})$ when $z \longrightarrow \alpha$. and $D(P,M_o)(z,s)$ is regular for z, on $(0,1)^c$ bounded at the origin , and $O(z^{-1})$ when $z \longrightarrow \alpha$.

Hence , equation (5.82) and (5.83) can for $\psi_o < 1/2$ be written in the form

$$X(z,s)/H(z,s) + \exp(-(\tau_1/z)Q)\left[\frac{A_o - H_o z}{K - z} V(Y)(z,s) - D(Y,P_o)(z,s)\right] =$$

$$= H(-z,s)(1 + zU(X)(z,s) + \exp(-(\tau_1/z)Q)(P,Y_o)(z,s)) \quad (5.89)$$

$$Y(z,s)/H(z,s) + z\exp(-(\tau_1/z)Q)\left[\frac{A_o - H_o z}{K - z} V(X)(z,s) - \right.$$

$$\left. - D(X,P_o)(z,s)\right] = H(-z,s)(\exp(-(\tau_1/z)z) + zU(Y)(z,s) +$$

$$+ z\exp(-(\tau_1/z)Q)D(P,X_o)(z,s)) \quad (5.90)$$

The left-hand side of equation (5.89) and (5.90) are regular for Re $z>0$ and bounded at the origin; the right-hand side of equations (5.89) and (5.90) are regular for z, on $(0,1)^c$, bounded at the origin and tends to constants , say A and B, respectively, when $z \longrightarrow \alpha$.

Hence, by modified form of Liouville's theorem I have

$$X(z,s) = H(z,s) \Bigg[z \exp(-(\tau_1 /z)Q \Big[D(Y,P_o(z,s) - $$

$$- \frac{A_o - H_o z}{K - z} V(X)(z,s)\Big] + A \Bigg] , \qquad (5.91)$$

$$Y(z,s) = H(z,s)\Bigg[z \exp(-(\tau_1 /z)Q \Big[D(X,P_o(z,s) - $$

$$- \frac{A_o - H_o z}{K - z} V(X)(z,s)\Big] + B \Bigg] , \qquad (5.92)$$

Equations (5.91) and (5.92) together with Equations (5.78) and (5.79) gives

$$A = 1, \quad B = 0 \qquad (5.93)$$

Hence, for $\psi_o = 1/2$, the expression of X(z,s) and Y(z,s) are

$$X(z,s) = H(z,s)[1 + z \exp(-(\tau_1 /z)Q)(D(Y,P_o)(z,s) - $$

$$- (- h_1 z + h_o) V(Y)(z,s))] \qquad (5.94)$$

$$Y(z,s) = H(z,s) z \exp(-(\tau_1 /z)Q)(D(Y,P_o)(z,s)) - $$

$$- (- h_1 z + h_o) V(Y)(z,s)) \qquad (5.95)$$

Hence, following Mullikin (1964) equations (5.91) and (5.92) together with equations (5.74) and (5.75) give unique representation of time-dependent X- and Y- functions for $\psi_o < 1/2$ and equations (5.94) and (5.95) together with equations (5.76) and (5.77) give unique representations of X- and Y- functions for $\psi_o = 1/2$.

5.4. *An* Exact Linearization *and* Decoupling *of the* Integral Equations Satisfied by Time-Dependent X- and Y-Functions.

5.41. Analysis.

The integral equations incorporating the various invariances of the time-dependent problem of diffuse reflection and transmission can be reduced to one or more pairs of integral equations of the following form (vide, Biswas and Karanjai, 1990)

$X(\mu,s) = 1 +$

$$+ \frac{\omega}{2}\,\frac{\mu}{\theta}\int_{0}^{1} \frac{X(\mu,s)X(\mu',s) - Y(\mu,s)Y(\mu',s)}{\mu + \mu'}\, d\mu' \qquad (5.96)$$

$Y(\mu,s) = \exp[(-\tau_1/\mu)] +$

$$+ \frac{\omega}{2}\,\frac{\mu}{\theta}\int_{0}^{1} \frac{Y(\mu,s)X(\mu',s) - X(\mu,s)Y(\mu',s)}{\mu - \mu'}\, d\mu' \qquad (5.97)$$

Following Chandrasekhar (1960) , I can write the above equations in the form

$X(\mu,s) = 1 +$

$$+ \frac{\mu}{Q}\int_0^1 \frac{X(\mu,s)X(\mu',s) - Y(\mu,s)Y(\mu',s)}{\mu + \mu'}\, \psi(\mu')\, d\mu' \qquad (5.98)$$

$$Y(\mu,s) = \exp[(-\tau_1/\mu)] +$$

$$+ \frac{\mu}{Q}\int_0^1 \frac{Y(\mu,s)X(\mu',s) - X(\mu,s)Y(\mu',s)}{\mu - \mu'}\, \psi(\mu')\, d\mu' \qquad (5.99)$$

where τ_1 is the optical thickness of the atmosphere and $Q = 1 + s/c$, where c is the velocity of light, s is the Laplace invariant of the time variable and the characteristic function $\psi(\mu)$ is an even polynomial in μ satisfying

$$\psi_0 = \int_0^1 \psi(\mu)\, d\mu \le 1/2 \qquad (5.100)$$

where $\psi_0 = 1/2$ holds, $\psi(\mu)$ is said to be conservative ; and non-conservative otherwise.

Clearly , equations (5.98) and (5.99) are non-linear and coupled. These equations have been linearized in an exact manner (vide, Mullikin, 1964) . The results are

$$X(\mu,s)K(\mu,s) = 1 + \frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu - \mu'}\, \psi(\mu')\, d\mu' -$$

$$- \exp[(-\tau_1/\mu)Q]\frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu + \mu'}\, \psi(\mu')\, d\mu' \qquad (5.101)$$

and

$$Y(\mu,s)K(\mu,s) = \exp[(-\tau_1/\mu)Q] + \frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu-\mu'}\psi(\mu')\,d\mu' -$$

$$- \exp[(-\tau_1/\mu)Q]\frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu+\mu'}\psi(\mu')\,d\mu' \qquad (5.102)$$

where $K(\mu,s)$ is defined by

$$K(\mu,s) \equiv 1 - \frac{\mu}{Q}\int_0^1 \left[\frac{1}{\mu+\mu'} - \frac{1}{\mu'-\mu}\right]\psi(\mu')d\mu' \qquad (5.103)$$

I now proceed to decouple equations (5.101) and (5.102) in an exact manner (vide, Lahoz, 1989). I introduce the following singular integral equation, which is linear in $1/T(\mu,s)$:

$$\frac{1}{T(\mu,s)} = 1 - \frac{\mu}{Q}\int_0^1 \left[\frac{1}{T(\mu',s)K(\mu',s)}\right]\frac{\psi(\mu')}{\mu'-\mu}\,d\mu' \qquad (5.104)$$

which in principle , is solvable for $T(\mu,s)$ as $\psi(\mu)$ and $K(\mu,s)$ are known functions.

Next, I multiply equation (5.101) by

$$\frac{(\mu'/Q)\psi(\mu)}{T(\mu,s)\;K(\mu,s)(\mu'-\mu)} \;, \qquad (5.105)$$

which I assume is well defined in $\mu \in [0,1]^c$ and integrate with respect to μ from 0 to 1 to obtain

$$\frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu+\mu'}\,\psi(\mu')\,d\mu' = 1 -$$

$$- T(-\mu,s)\left[1 - P(\mu,s)\,\frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu'-\mu}\,\psi(\mu')\,d\mu' + \right.$$

$$\left. + \frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu'-\mu}\,P(\mu',s)\,\psi(\mu')\,d\mu' \right] \qquad (5.106)$$

where I have used equation (5.104) and defined the function $P(\mu,s)$ (in principle known) by

$$P(\mu,s) \equiv \frac{\mu}{Q}\int_0^1 \frac{1}{\mu'+\mu}\,\frac{\exp(-\tau_1/\mu)}{T(\mu',s)K(\mu',s)}\,\psi(\mu')\,d\mu' \qquad (5.107)$$

Substituting equation (5.106) in equation (5.102) I get the decoupled equation for $Y(\mu,s)$ as follows:

$$Y(\mu,s)K(\mu,s) = T(-\mu,s)\,\exp[(-\tau_1/\mu)Q\,] + T(-\mu,s)\,P(\mu,s)$$

$$[1 - \exp[(-\tau_1/\mu)Q]]\,\frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu'-\mu}\,\psi(\mu')\,d\mu' +$$

$$T(-\mu,s)\,\exp[(-\tau_1/\mu)Q\,]\frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu'-\mu}\times$$

$$\times\,\psi(\mu',s)\,\psi(\mu')\,d\mu' \qquad (5.108)$$

A similar analysis yields the decoupled equation for $X(\mu,s)$:

$$X(\mu,s)K(\mu,s) = [1 - T(-\mu,s)\,P(\mu,s)\exp[(-\tau_1/\mu)Q\,]] \times$$

$$\times \left[\, 1 + \frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu'-\mu}\,\psi(\mu')\,d\mu' \,\right] +$$

$$T(-\mu,s)\,\exp[(-\tau_1/\mu)Q\,]\frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu'-\mu}\times$$

$$\times\,\psi(\mu',s)\quad d\mu' \tag{5.109}$$

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APPENDIX I

6.1 The relation (3.66) of chapter 3

I have to show that (equation (3.66) chapter 3)

$$I_1(t,\mu) = 0 \tag{6.1}$$

For this , with usual notation (vide, Chandrasekhar, 1960) I have

$$I_1^*(t,\mu) \simeq \frac{1}{2}(1-\lambda)\sum_{\alpha=1}^{n}\left\{ L_\alpha e^{-k_\alpha t} \Big/ (1+\mu k_\alpha)\right\} \tag{6.2}$$

where the constants L_α are determined by the equations

$$\sum_{\alpha=1}^{n}\left\{ L_\alpha \Big/ (1-\mu_i k_\alpha)\right\} = 0 \ , \ (i = 1,2,3 \ldots . n) \tag{6.3}$$

Since

$$\prod_{\alpha=1}^{n}(1-\mu k_\alpha)\sum_{\alpha=1}^{n} L_\alpha / (1-\mu k_\alpha) \tag{6.4}$$

is a polynomial of degree (n − 1) with n distinct zero , it is identically zero.

Hence , every $L_\alpha = 0$, and in the limit , as $n \longrightarrow \alpha$

$$I_1^*(t,\mu) = 0 \ . \tag{6.5}$$

which is the required relation.

APPENDIX II

6.2. The relation (3.96) of chapter 3

To establish the relation, (3.96), of Chapter 3, I consider

$$D_m(x) = (1 - \lambda)\sum_i \frac{a_i \mu_i^m}{1 + \mu_i x} =$$

$$= (-1)^m \quad (1 - \lambda)\sum_i \frac{a_i \mu_i^m}{1 - \mu_i x} \tag{6.6}$$

I can derive a single recursion formula for $D_m(x)$ .Then

$$D_m(x) = \frac{1}{x}\left[(1 - \lambda)\sum_i a_i \mu_i^{m-1} \left(1 - \frac{1}{1 + \mu_i x} \right) \right] =$$

$$= \frac{1}{x}\left[\psi_{m-1} - D_{m-1} \right] \tag{6.7}$$

where
$$\psi_m = (1 - \lambda) - \sum_i a_i \mu_i^m \tag{6.8}$$

From this formula I have

$$D_m(x) = \frac{\psi_{m-1}}{x} - \frac{\psi_{m-2}}{x^2} + \ldots\ldots + (-1)^{m-2}\frac{\psi_1}{x^{m-1}} +$$

$$+ \frac{(-1)^{m-1}}{x^m}\left[\psi_o - D_o(x) \right] \qquad (m = 0,1,\ldots 4n) \tag{6.9}$$

and

$$\psi_o = 2(1 - \lambda) \tag{6.10}$$

Moreover, let p_{2j} be the coefficient of μ^{2j} in the Legendre polynomial $P_{2n}(\mu)$, then

$$\sum_{j=o}^{n} P_{2j} \; D_{2j}\left(k_r \right) = (1-\lambda) \sum_{i} \frac{a_i}{1+\mu_i k_r} \left[\sum_{j=o}^{n} P_{2j}\mu_i^{2j} \right] \qquad (6.11)$$

Since μ_i's are the zeros of $P_{2n}(\mu)$, Equation (6.11) reduces to

$$\sum_{j=o}^{n} P_{2j} D_{2j} \left(k_r \right) = 0 \qquad (6.12)$$

Substituting for $D_{2j}\left(k_r \right)$ into equation (6.12) I get the characteristic equation as

$$\frac{P_{2n} \; \lambda}{2n} + \ldots\ldots\ldots\ldots + P_o = 0 \qquad (6.13)$$

From this equation it follows that

$$\frac{1}{\left(k_1 \ldots\ldots k_n \right)^2} = \frac{(-1)^n \; P_o}{\lambda \; P_{2n}} =$$

$$= \frac{\left(\mu_1 \ldots\ldots \mu_n \right)^2}{\lambda} \qquad (6.14)$$

and $\mu_1 . \mu_2 \ldots . \mu_n \quad k_1 \ldots . \; k_n = (\lambda)^{1/2}$ (6.15)

APPENDIX III

6.3 The relation (4.38) of chapter 4

To establish the relation (4.38) , of Chapter 4 , I consider

$$D_m(x) = \sum_{r=1}^{k} \eta_r \omega_r \sum_i \frac{a_i \mu_{(r)i}^m}{1 + \mu_{(r)i}\, x} =$$

$$= (-1)^m \sum_{r=1}^{k} \eta_r \omega_r \sum_i \frac{a_i \mu_{(r)i}^m}{1 - \mu_{(r)i} x} \tag{6.16}$$

I can derive a single recursion formula for $D_m(x)$ .Then

$$D_m(x) = \frac{1}{x} \left[\sum_{r=1}^{k} \eta_r \omega_r \sum_i a_i \mu_{(r)}^{m-1} \left(1 - \frac{1}{1 + \mu_{(r)}^{x}} \right) \right] =$$

$$= \frac{1}{x} \left[\psi_{m-1} - D_{m-1} \right] \tag{6.17}$$

where $\psi_m = \sum_r \eta_r \omega_r \sum_i a_i \mu_{(r)i}^m$ (6.18)

From this formula I have

$$D_m(x) = \frac{\psi_{m-1}}{x} - \frac{\psi_{m-2}}{x^2} + \ldots\ldots + (-1)^{m-2} \frac{\psi_1}{x^{m-1}} +$$

$$+ \frac{(-1)^{m-1}}{x^m} \left[\psi_0 - D_0(x) \right] \qquad (m = 0,1,\ldots 4n) \tag{6.19}$$

and

$$\psi_0 = 2 \sum_{r=1}^{k} \eta_r \omega_r \tag{6.20}$$

Let $P_{2n}(\mu) = \sum_{j=0}^{n} P_{2n}\mu^{2n}$ i.e. p_{2j} be the coefficient of μ^{2j}

in the Legendre polynomial $P_{2n}(\mu)$,

then

$$\sum_{j=0}^{n} P_{2j} \; D_{2j}\left(\zeta_r K_{(r)\alpha}\right) = \sum_{r=1}^{k} \eta_r \omega_r \sum_i \frac{a_i}{1+\mu_{(r)}K_{(r)\alpha}} \times$$

$$\times \left[\sum_{j=0}^{n} P_{2j}\mu^{2j}\right] \qquad (6.21)$$

Since $\mu_{(r)i}$'s are the zeros of $P_{2n}(\mu)$, Equation (6.21) reduces to

$$\sum_{j=0}^{n} P_{2j} D_{2j} \left(\zeta_r K_{(r)\alpha}\right) = 0 \qquad (6.22)$$

Substituting for $D_{2j}\left(\zeta_r K_{(r)\alpha}\right)$ from Equation (6.20) into Equation (6.22) I get required form of the characteristic equation as

$$\frac{P_{2n}\left(1 - M/N\right)}{\zeta_r^{2n} \; K_{(r)\alpha}^{2n}} + \ldots\ldots\ldots\ldots + P_o = 0 \qquad (6.23)$$

where M and N are given by the equation (4.39).

From this equation it follows that

$$\frac{1}{\left(\zeta_r K_{(r)1} \ldots\ldots \zeta_r k_{(r)n}\right)^2} = \frac{(-1)^n \; P_o}{(1 - M/N)\; P_{2n}} =$$

$$= \frac{(\mu_{(r)1} \cdots\cdots \mu_{(r)n})^2}{(1 - M/N)} \qquad (6.24)$$

and $\mu_{(r)1} \cdot \mu_{(r)2} \cdots \mu_{(r)n} \cdot \zeta_r K_{(r)1} \cdots \zeta_r K_{(r)n} =$

$$= (1 - M/N)^{1/2} \qquad (6.25)$$

which is the required relation.

SOLUTION OF THE EQUATION OF TRANSFER FOR INTERLOCKED MULTIPLETS BY THE METHOD OF DISCRETE ORDINATES WITH THE PLANCK FUNCTION AS A NONLINEAR FUNCTION OF OPTICAL DEPTH

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Abstract. The equation of transfer for interlocked multiplets has been solved by the method of discrete ordinates, originally due to Chandrasekhar, considering nonlinear form of the Planck function to be

$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$.

1. Introduction

Woolley and Stibbs (1953) applied the theory of formation of absorption lines by coherent scattering to the case of interlocking without redistribution and deduced the equation of transfer in the Milne–Eddington model. They have also obtained a solution for the case of triplets by Eddington's approximate method. Busbridge and Stibbs (1954) applied the principle of invariance governing the law of diffuse reflection with a slight modification to solve exactly the equation of transfer in the M–E model. Dasgupta and Karanjai (1972) applied Sobolev's probabilistic method to solve the transfer equation for the case of interlocking without redistribution. Another exact solution of the equation of transfer has been given by Dasgupta (1956) by his form of the Wiener–Hopf technique. Karanjai and Barman (1981) applied the extension of the method of discrete ordinates to find an exact solution of the problem of line formation by interlocking in the M–E model. Dasgupta (1978) obtained an exact solution of the transfer equation for non-coherent scattering arising from interlocking of principal lines without redistribution by Laplace transformation and the Wiener–Hopf technique using a new representation of the H-function obtained by Dasgupta (1977). While solving the transfer equation, Dasgupta considered the Planck function to be linear in t (optical depth), i.e., $B_\nu(T) = B(t) = b_0 + b_1\tau$. Karanjai and Karanjai (1985) solved the equation of transfer for interlocked multiplets with the Planck function as a nonlinear function of optical depth following the method used by Dasgupta (1978). They considered two nonlinear forms of $B_\nu(T)$, viz.:

(1) an exponential atmosphere (Delg'Innocenti, 1979) in which

$$B_\nu(T) = B(t) = b_0 + b_1 e^{-\beta t};$$

(2) an atmosphere (Busbridge, 1955) in which

$$B_\nu(T) = B(t) = b_0 + b_1 t + E_2(t)\,.$$

In this paper, we have obtained the solution of the equation of transfer for interlocked multiplets by discrete ordinate method in an exponential atmosphere in which

$$B_\nu(T) = b_0 + b_1\, e^{-\beta\tau}\,,$$

where τ is the optical depth.

2. The Equation of Transfer

The equation of transfer considered here is of the form (Woolley and Stibbs, 1953)

$$\mu\,\frac{\mathrm{d}I_r(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)B_\nu(T) - -\tfrac{1}{2}(1-\varepsilon)\alpha_r \sum_{p=1}^{n} \eta_p \int_{-1}^{+1} I_p(\tau,\mu')\,\mathrm{d}\mu'\,, \tag{1}$$

where τ denotes the optical depth and $\eta_r = k_r/k$ denoting the line absorption coefficient for the rth line and k the continuous absorption coefficient which is assumed to be constant for each line. In the present case we consider that the collision constant ε and Planck's function remain constant for each line. We also consider an exponential atmosphere for which Planck's function, i.e., the thermal source function is given (Degl'Innocenti, 1979) by

$$B_\nu(T) = b_0 + b_1\, e^{-\beta\tau}\,, \tag{2}$$

where b_0, b_1, and β are three positive constants.

Now, if we use Equation (2) in Equation (1) we have the transfer equation for the rth interlocked line in the form

$$\mu\,\frac{\mathrm{d}I_r(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)(b_0 + b_1\, e^{-\beta\tau}) - -\tfrac{1}{2}(1-\varepsilon)\alpha_r \sum_{p=1}^{n} \int_{-1}^{+1} I_p(\tau,\mu')\,\mathrm{d}\mu'\,, \tag{3}$$

where

$$\alpha_r = \eta_r/(\eta_1 + \eta_2 + \cdots + \eta_k)\,,\quad r = 1, 2, \ldots, k\,; \tag{4}$$

so that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1\,. \tag{5}$$

Equation (3) is to be solved subject to the boundary conditions

$$I_r(0, -\mu') = 0\,, \quad (0 < \mu \leq 1) \tag{6}$$

and

$$I_r(\tau, \mu)\, e^{-\tau/\mu} \to 0 \quad \text{as} \quad \tau \to \infty\,. \tag{7}$$

3. Solution of Equation (3)

Following Busbridge (1953) and Stibbs (1953), let

$$I_r(\tau, \mu) = b_0 + \frac{b_1 T_r}{1 + \beta\mu\xi_r}\, e^{-\beta\tau} + I_r^*(\tau, \mu) \tag{8}$$

represent the solution of Equation (3), where

$$T_r = \frac{\xi_r(1 + \varepsilon\eta_r)}{1 - \dfrac{1}{2\beta}(1 - \varepsilon)\eta_r \log \dfrac{1 + \eta_r + \beta}{1 + \eta_r - \beta}} \tag{9}$$

and

$$\xi_r = \frac{1}{1 + \eta_r}\,. \tag{10}$$

This consists of two parts. The first part consists of the solution for a bounded atmosphere as τ tends to infinity. The second part: viz., $I_r^*(\tau, \mu)$ represents the departure of the asymptotic solution from the value $I_r(\tau, \mu)$ as we approach the boundary.

Now if we insert $I_r(\tau, \mu)$ from Equation (8) in Equation (3) and taking

$$w_r = \frac{(1 - \varepsilon)\eta_r}{1 + \eta_r}\,, \tag{11}$$

we have the equation

$$\xi_r\mu\, \frac{\mathrm{d}I_r^*(\tau, \mu)}{\mathrm{d}\tau} = I_r^*(\tau, \mu) - \frac{w_r}{2}\, \frac{1}{\sum\limits_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \int_{-1}^{+1} I_p^*(\tau, \mu')\, \mathrm{d}\mu'\right] \tag{12}$$

together with the boundary conditions

$$I_r^*(0, -\mu') = -b_0 - \frac{b_1 T_r}{1 - \beta\xi_r\mu'} \tag{13}$$

and

$$I_r^*(\tau, \mu)\, e^{-\tau/\mu} \to 0 \quad \text{as} \quad \tau \to \infty\,. \tag{14}$$

For the sake of convenience, Equation (12) can be rewritten in the form

$$\xi_r \mu \frac{\mathrm{d}I^*_{(r)}(\tau, \mu)}{\mathrm{d}\tau} = I^*_{(r)}(\tau, \mu) - \frac{W_r/2}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \int_{-1}^{+1} I^*_{(p)}(\tau, \mu')\, \mathrm{d}\mu' \right], \tag{15}$$

together with the boundary conditions

$$I^*_{(r)}(0, -\mu) = -\frac{b_1 T_r}{1 - \xi_r \beta\mu} - b_0 \tag{16}$$

and

$$I^*_{(r)}(\tau, \mu)\, e^{-\tau/\mu} \to 0 \quad \text{as} \quad \tau \to \infty\,. \tag{17}$$

Equation (15) can be replaced by the system of $2n$ linear equations

$$\xi_r \mu_{(r)i} \frac{\mathrm{d}I^*_{(r)i}}{\mathrm{d}\tau} = I^*_{(r)i} - \frac{w_r}{2} \frac{1}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \sum_j a_j I^*_{(p)_i} \right], \tag{18}$$

$$(i = \pm 1, \pm 2, \ldots, \pm n)\,,$$

where the $\mu_{(r)i}$'s ($i = \pm 1, \ldots, \pm n$ and $\mu_{(r)_{-i}} = -\mu_{(r)_i}$) are the zeros of the Legendre polynomials $P_{2n}(\mu)$ which are dependent on the lines of interlocking and a_j's ($j = \pm 1, \ldots, \pm n$) and ($a_{-j} = a_j$) are corresponding Gaussian weights. However, it is to be noted that there is no term with $j = 0$. For simplicity, we write

$$I^*_{(r)i} \quad \text{for} \quad I^*_{(r)i}(\tau, \mu_{(r)i}) \tag{19}$$

in Equation (18).

The system of Equations (18) admits of integrals of the form

$$I^*_{(r)i} = g_{(r)i}\, e^{-K\tau}, \quad (i = \pm 1, \ldots, \pm n)\,, \tag{20}$$

where $g_{(r)i}$'s and K are constants.

Now if we insert this form for $I^*_{(r)i}$ in Equation (18) we have

$$g_{(r)i}[1 + \xi_r \mu_{(r)i} K] = \frac{w_r}{2} \frac{1}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \sum_j a_j g_{(p)_i} \right], \tag{21}$$

$$\therefore g_{(r)i} = w_r \frac{\text{constant}}{1 + \xi_r \mu_{(r)i} K}\,. \tag{22}$$

If we insert for $g_{(r)i}$ from Equation (22) back into Equation (21) we obtain the characteristic equation in the form

$$1 = \frac{1}{2}\frac{1}{\sum_{p=1}^{k}\eta_p}\left[\sum_{p=1}^{k}\eta_p w_p \sum_j \frac{a_j}{1+\xi_p\mu_{(p)_j}K}\right], \tag{23}$$

in which $a_j = a_{-j}$ and $\mu_{(r)_{-j}} = -\mu_{(r)_j}$.

We can rewrite the characteristic equation in the form

$$1 = \frac{1}{\sum_{p=1}^{k}\eta_p}\left[\sum_{p=1}^{k}\eta_p w_p \sum_{j=1}^{n}\frac{a_j}{1-\xi_p^2\mu_{(p)_j}^2K^2}\right]. \tag{24}$$

This is the characteristic equation which gives the values of $K_{(r)}$. If $w_r < 1$ $(r = 1, 2, \ldots, k)$, the characteristic equation (24) gives distinct non-zero roots which occur in pairs as $\pm K_{(r)\alpha}$ $(\alpha = 1, 2, \ldots, n)$.

Therefore, Equation (18) admits the $2n$ independent integrals of the form

$$I^*_{(r)i} = w_r \frac{\text{constant}}{1 \pm \xi_r\mu_{(r)i}K_{(r)\alpha}} e^{\mp K_{(r)\alpha}\tau}. \tag{25}$$

According to Chandrasekhar (1960), the solutions (20) satisfying our requirements of the boundedness of the solutions are

$$I^*_{(r)i} = w_r b_1 \sum_{\alpha=1}^{n} \frac{L_{(r)\alpha}e^{-K_{(r)\alpha}\tau}}{1+\xi_r K_{(r)\alpha}\mu_{(r)i}}, \tag{26}$$

together with the boundary condition

$$I^*_{(r)-i} = -b_0 - \frac{b_1 T_r}{1-\xi_r\beta\mu_{(r)}} \quad \text{at} \quad \tau = 0\,. \tag{27}$$

4. The Elimination of the Constant and the Expression of the Law of Diffuse Reflection in Closed Form

The boundary conditions and the emergent intensity can be expressed in the form

$$S_r(\mu_{(r)i}) = 0\,, \quad (i = 1, 2, \ldots, n) \tag{28}$$

and

$$I^*_{(r)}(0,\mu) = w_r b_1 S_r(-\mu) - \frac{T_r/w_r}{1+\xi_r\beta\mu} - \frac{b_0}{w_r b_1}, \tag{29}$$

where

$$S_r(\mu) = \sum_{\alpha=1}^{n}\frac{L_{(r)\alpha}}{1-\xi_r K_{(r)}\mu} + \frac{T_r/w_r}{1-\xi_r\beta\mu} + \frac{b_0}{w_r b_1}. \tag{30}$$

Next we observe that the function

$$(1 - \xi_r \beta\mu) \prod_{\alpha=1}^{n} (1 - \xi_r K_{(r)\alpha}\mu) S_r(\mu)$$

in a polynomial of degree $(n + 1)$ in μ which vanishes for $\mu = \mu_i$, $i = 1, 2, \ldots, n$. There must accordingly exist a relation of the form

$$(1 - \xi_r \beta\mu) \prod_{\alpha=1}^{n} (1 - \xi_r K_{(r)\alpha}\mu) S_r(\mu) \sim (\mu - C_r) \prod_{i=1}^{n} (\mu - \mu_i), \tag{31}$$

where C_r is a constant.

The constant of proportionality can be found by comparing the coefficient of the highest power of μ (namely, μ^{n+1}).

So we have, from Equation (31)

$$S_r(\mu) = \frac{(-1)^{n+1}}{b_1 w_r} \xi_r K_{(r)1} \cdots \xi_r K_{(r)n} \xi_r \beta \frac{P_r(\mu)(\mu - C_r)}{R_r(\mu)(1 - \beta\xi_r\mu)}, \tag{32}$$

where

$$P_r(\mu) = \prod_{i=1}^{n} (\mu - \mu_i), \quad (i = 1, 2, \ldots, n) \tag{33}$$

and

$$R_r(\mu) = \prod_{\alpha=1}^{n} (1 - \xi_r K_{(r)\alpha}\mu), \quad (\alpha = 1, 2, \ldots, n). \tag{34}$$

Moreover, if we combine Equations (32) and (33), we obtain

$$L_{(r)\alpha} = (-1)^n \frac{b_0}{w_r b_1} \xi_r K_{(r)1} \cdots \xi_r K_{(r)n} \xi_r \beta \frac{P_r(1/\xi_r K_{(r)\alpha})}{R_{(r)\alpha}(1/\xi_r K_{(r)\alpha})} \times$$

$$\times \frac{\left(\dfrac{1}{\xi K_{(r)\alpha}} - C_r\right)}{\left(1 - \dfrac{\beta\xi_r}{K_{(r)\alpha}\xi_r}\right)}, \tag{35}$$

where

$$R_{(r)\alpha}(x) = \prod_{\gamma \neq \alpha} (1 - \xi_r K_{(r)\gamma} x) \tag{36}$$

and

$$\beta \neq K_{(r)\alpha}. \tag{37}$$

The roots of the characteristic equation (17) can be written in the form

$$\xi_r K_{(r)1} \cdots \xi_r K_{(r)n} \mu_{(r)1} \cdots \mu_{(r)n} = (1 - w_r)^{1/2}. \tag{38}$$

Now by use of Equation (38), Equation (32) becomes

$$S_r(\mu) = -\frac{b_0 \xi_r \beta (1 - w_r)^{1/2} H_r(-\mu)\,[\mu - C_r]}{w_r b_1 (1 - \beta \xi_r \mu)}, \tag{39}$$

where

$$H_r(\mu) = \frac{1}{\mu_{(r)1} \cdots \mu_{(r)n}} \frac{\prod_{i=1}^{n} (\mu + \mu_{(r)i})}{\prod_{\alpha=1}^{n} (1 + \xi_r K_{(r)\alpha} \mu)}; \tag{40}$$

and the characteristic roots are evaluated from Equation (24). If we put $\mu = 0$ in Equations (30) and (40) we have

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} + \frac{T_r}{w_r} + \frac{b_0}{w_r b_1} = \frac{b_0}{w_r b_1} \xi_r \beta (1 - w_r)^{1/2} C_r, \tag{41}$$

and we can next evaluate $\Sigma_{\alpha=1}^{n} L_{(r)\alpha}$ from Equation (35). Then

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} = (-1)^{n+1} \frac{b_0}{b_1 w_r} [\xi_r K_{(r)1} \cdots \xi_r K_{(r)n} \xi_r \beta f_r(0)], \tag{42}$$

where

$$f_r(x) = \sum_{\alpha=1}^{n} \frac{P_r(1/\xi_r K_{(r)\alpha}) \left[\dfrac{1}{\xi_r K_{(r)\alpha}} - C_r\right]}{R_{(r)\alpha}(1/\xi_r K_{(r)\alpha}) \left(1 - \dfrac{\xi_r \beta}{K_{(r)\alpha} \xi_r}\right)} R_{(r)\alpha}(x). \tag{43}$$

Now $f_r(x)$ defined in this manner is a polynomial of degree $n - 1$ in x, which takes the values

$$\frac{P_r(1/\xi_r K_{(r)\alpha}) \left[\dfrac{1}{\xi_r K_{(r)\alpha}} - C_r\right]}{\left(1 - \dfrac{\xi_r \beta}{K_{(r)\alpha} \xi_r}\right)},$$

for

$$x = 1/\xi_r K_{(r)\alpha}, \quad (\alpha = 1, 2, \ldots, n).$$

In other words,

$$(1 - \xi_r \beta x) f_r(x) - P_r(x) \qquad (x - C_r) = 0. \tag{44}$$

Therefore, we must accordingly have a relation of the form

$$(1 - \xi_r \beta x) f_r(x) - P_r(x)(x - C_r) = R_r(x)(A_r x + B_r), \tag{45}$$

where A_r and B_r are certain constants to be determined.

The constant A_r follows from the comparison of the coefficients of x^{n+1}. Thus

$$A_r = \frac{(-1)^{n+1}}{\xi_r K_{(r)1} \cdots \xi_r K_{(r)n}}. \tag{46}$$

Next, if we put $x = (\xi_r \beta)^{-1}$ in Equation (46) (cf. Chandrasekhar, 1960) we have

$$B_r = \frac{(-1)^n}{\xi_r \beta \xi_r K_{(r)1} \cdots \xi_r K_{(r)n}} + \frac{\left(C_r - \frac{1}{\beta \xi_r}\right) P_r\left(\frac{1}{\xi_r \beta}\right)}{R_r\left(\frac{1}{\xi_r \beta}\right)}, \tag{47}$$

i.e.,

$$B_r = \frac{(-1)^n}{\xi_r \beta \xi_r K_{(r)1} \cdots \xi_r K_{(r)n}} + (-1)^n \mu_{(r)1} \cdots \mu_{(r)n} \times$$

$$\times H_r\left(-\frac{1}{\beta \xi_r}\right)\left(C_r - \frac{1}{\xi_r \beta}\right). \tag{48}$$

Now if we use the relations (48), (47), and (46) we get

$$f_r(0) = -C_r P_r(0) + B_r R_r(0),$$

i.e.,

$$f_r(0) = -C_r(-1)^n \mu_{(r)1} \cdots \mu_{(r)n} + \frac{(-1)^n}{\xi_r \beta \xi_r K_{(r)1} \cdots \xi_r K_{(r)n}} +$$

$$+ (-1)^n \mu_{(r)1} \cdots \mu_{(r)n} H_r\left(-\frac{1}{\beta \xi_r}\right)\left(C_r - \frac{1}{\beta \xi_r}\right). \tag{49}$$

From Equation (43) using Equation (49) we have

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} = \frac{b_0}{w_r b_1} C_r (1 - w_r)^{1/2} \xi_r \beta - \frac{b_0}{w_r b_1} +$$

$$+ \frac{b_0}{w_r b_1} \xi_r \beta (1 - w_r)^{1/2} H\left(-\frac{1}{\beta \xi_r}\right)\left(\frac{1}{\xi_r \beta} - C_r\right). \tag{50}$$

Now if we use Equation (50) in Equation (42) we get

$$C_r = \frac{1}{\xi_r\beta} + \frac{T_r b_1}{b_0 \xi_r \beta(1 - w_r)^{1/2} H_r\left(-\frac{1}{\beta\xi_r}\right)} ; \tag{51}$$

and if we combine Equation (40), the diffusely reflected intensity $I^*_{(r)}(0, \mu)$ in Equation (29) takes the form

$$I_r^*(0, \mu) = \frac{b_0\xi_r\beta(1 - w_r)^{1/2} H_r(\mu)\,[\mu + C_r]}{1 + \beta\xi_r\mu} - \frac{T_r b_1}{1 + \beta\xi_r\mu} - b_0 . \tag{52}$$

This is the required solution in a closed form. If we combine Equation (8) at $\tau = 0$ and Equation (52) we have

$$I_r(0, \mu) = \frac{b_0\xi_r\beta(1 - w_r)^{1/2} H_r(\mu)\,[\mu + C_r]}{1 + \beta\xi_r\mu} , \tag{53}$$

which is the required solution of Equation (3) in the nth approximation by the discrete-ordinates method.

If we put C_r from Equation (51), we get the solution in the form

$$I_r(0, \mu) = b_0(1 - w_r)^{1/2} H_r(\mu) + \frac{b_1 T_r}{H_r(-1/\xi_r\beta)} \frac{H_r(\mu)}{(1 + \beta\xi_r\mu)} . \tag{54}$$

Chandrasekhar's (1960) equation for $I_r(0, \mu)$ in the case of coherent scattering is given by $(B_\nu(T) = b_0 + b_1\tau)$ (see also Karanjai and Barman, 1981), and

$$I_r(0, \mu) = b_1\xi_r(1 - w_r)^{1/2} \mu H_r(\mu) + b_0(1 - w_r)^{1/2} H_r(\mu) + \\ + b_1(1 - w_r)^{1/2} \xi_r \left[\sum_{\alpha=1}^{n} \frac{1}{\xi_r K_{(r)\alpha}} - \sum_{j=1}^{n} \mu_{(r)_j} \right] . \tag{55}$$

If we compare Equations (54) and (55) we see that if we put $b_1 = 0$, we have the same solution for both cases. Moreover, for large values of β, i.e., $\beta \to \infty$. The solution (54) takes the form

$$I_r(0, \mu) = b_0(1 - w_r)^{1/2} H_r(\mu) , \tag{56}$$

i.e., $B_\nu(T)$ then behaves like a constant or independent of τ. This fact can also be explained from the point of view that

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau} \to b_0 \quad \text{as} \quad \beta \to \infty .$$

Appendix

To establish the relation (38) we consider

$$D_m(x) = \sum_{r=1}^{k} \eta_r w_r \sum_i \frac{a_i \mu_{(r)i}^m}{1 + \mu_{(r)i} x} =$$

$$= (-1)^m \sum_{r=1}^{k} \eta_r w_r \sum_i \frac{a_i \mu_{(r)i}^m}{1 - \mu_{(r)i} x}, \tag{57}$$

we can derive a single recursion formula for $D_m x$. Then

$$D_m(x) = \frac{1}{x}\left[\sum_{r=1}^{k} \eta_r w_r \sum_i a_i \mu_{(r)i}^{m-1}\left(1 - \frac{1}{1 + \mu_{(r)i} x}\right)\right] =$$

$$= \frac{1}{x}\left[\psi_{m-1} - D_{m-1}\right], \tag{58}$$

where

$$\psi_m = \sum_r \eta_r w_r - \sum_i a_i \mu_{(r)i}^m . \tag{59}$$

From this formula we have

$$D_m(x) = \frac{\psi_{m-1}}{x} - \frac{\psi_{m-2}}{x^2} + \cdots + (-1)^{m-2} \frac{\psi_1}{x^{m-1}} +$$

$$+ \frac{(-1)^{m-1}}{x^m}\left[\psi_0 - D_0(x)\right], \quad (m = 0, 1, \ldots, 4n) \tag{60}$$

and

$$\psi_0 = 2 \sum_{r=1}^{k} \eta_r w_r . \tag{61}$$

Moreover, let P_{2j} be the coefficient of μ^{2j} in the Legendre polynomial $P_{2n}(\mu)$.
Then

$$\sum_{j=0}^{n} P_{2j} D_{2j}(\xi_r K_{(r)\alpha}) =$$

$$= \sum_{r=1}^{k} \eta_r w_r \sum_i \frac{a_i}{1 + \mu_{(r)i} \xi_r K_{(r)\alpha}} \sum_{j=0}^{k} P_{z_j} \mu_{(r)i}^{z_j} . \tag{62}$$

Since the $\mu_{(r)i}$'s are the zeros $P_{2n}(\mu)$. Equation (62) reduces to

$$\sum_{j=0}^{n} P_{2j} D_{z_j}(\xi_r K_{(r)\alpha}) = 0 . \tag{63}$$

If we substitute for $D_{2j}(\xi_r K_{(r)\alpha})$ from Equation (61) into Equation (63) we get the required form of the characteristic equation as

$$\frac{P_{2n}}{\xi_r^{2n} K_{(r)\alpha}^{2n}} (w_r - 1) + \cdots + P_0 = 0 . \tag{64}$$

From this equation it follows that

$$\frac{1}{(\xi_r K_{(r)1} \ldots \xi_r K_{(r)n})^2} = \frac{(-1)^n P_0}{1 - w_r P_{2n}} = \frac{(\mu_{(r)1} \cdots \mu_{(r)n})^2}{1 - w_r}$$

and

$$\mu_{(r)1} \cdots \mu_{(r)n} K_{(r)1} \xi_r \ldots K_{(r)n} \xi_r = (1 - w_r)^{1/2} . \tag{65}$$

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SOLUTION OF THE EQUATION OF TRANSFER FOR COHERENT SCATTERING IN AN EXPONENTIAL ATMOSPHERE BY EDDINGTON'S METHOD

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Abstract. An approximate solution of the transfer equation for coherent scattering in stellar atmospheres with Planck's function as a nonlinear function of optical depth, viz.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$$

is obtained by Eddington's method.

1. Introduction

Chandrasekhar (1960) applied the method of discrete ordinates to solve the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz., $B_\nu(T) = b_0 + b_1\tau$. The equation of transfer for coherent scattering has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient, is constant) and by Strömgren's method (when η_ν has small but arbitrary variation with optical depth (see Woolley and Stibbs, 1953). Dasgupta (1977b) applied the method of Laplace transform and Wiener–Hopf technique to find an exact solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions

$$B_\nu(T) = b_0 + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau),$$

by use of a new representation of the H-function obtained by Dasgupta (1977a).

In the present paper, we have obtained an approximate solution of the equation of transfer for coherent isotropic scattering by the method used by Eddington (Woolley and Stibbs, 1953) in an exponential atmosphere (Degl'Innocenti, 1979; Karanjai and Karanjai, 1985; Deb *et al.*, 1990),

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau},$$

where β, b_0, b_1 are positive constants.

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2. Equation of Transfer

The equation of transfer for coherent scattering can be written (cf. Woolley and Stibbs, 1953) in the form

$$\cos\theta\, \mathrm{d}I_\nu(\theta)/\rho\, \mathrm{d}x = -(k + l_\nu)I_\nu(\theta) + (1 - \varepsilon)l_\nu J_\nu + (k + \varepsilon l_\nu)B_\nu(T)\,. \quad (1)$$

To find an approximate solution of Equation (1), we proceed as follows: let

$$J_\nu = (1/4\pi)\int I_\nu(\theta)\,\mathrm{d}w\,, \quad (2a)$$

$$H_\nu = (1/4\pi)\int I_\nu(\theta)\cos\theta\,\mathrm{d}w\,, \quad (2b)$$

$$K_\nu = (1/4\pi)\int I_\nu(\theta)\cos^2\theta\,\mathrm{d}w\,, \quad (2c)$$

in which the integration is made over all directions.

By multiplying Equation (1) by $(\mathrm{d}w/4\pi)$ and $(\mathrm{d}w\cos\theta/4\pi)$ and integrating we obtain

$$\mathrm{d}H_\nu/\rho\,\mathrm{d}x = -(k + \varepsilon l_\nu)(J_\nu - B_\nu)\,, \quad (3)$$

$$\mathrm{d}K_\nu/\rho\,\mathrm{d}x = -(k + l_\nu)H_\nu\,, \quad (4)$$

where $B_\nu(T) = B_\nu$. If we measure the optical depth in the continuous spectrum outside the line so that $\mathrm{d}\tau = -k\rho\,\mathrm{d}x$ and set $l_\nu/k = \eta_\nu$, then (3) and (4) becomes

$$\mathrm{d}H_\nu/\mathrm{d}\tau = (1 + \varepsilon\eta_\nu)(J_\nu - B_\nu)\,, \quad (5)$$

$$\mathrm{d}K_\nu/\mathrm{d}\tau = (1 + \eta_\nu)H_\nu\,. \quad (6)$$

If, moreover, we assume that η_ν is independent of τ, the equation can be readily integrated. Introducing Eddington's approximation

$$K_\nu = (1/3)J_\nu\,,$$

Equations (5) and (6) can be combined to give

$$\mathrm{d}^2J_\nu/\mathrm{d}\tau^2 = q_\nu^2(J_\nu - B_\nu)\,, \quad (7)$$

where

$$q_\nu^2 = 3(1 + \varepsilon\eta_\nu)(1 + \eta_\nu)\,, \quad (8)$$

Equation (7) is to be solved subject to the boundary conditions: (A) $J_\nu = 2H_\nu$ at $\tau = 0$ and (B) the requirement that $(J_\nu - B_\nu)$ shall not increase exponentially as $\tau \to \infty$.

3. Solution of Equation (7)

Let

$$B_\nu = b_0 + b_1 e^{-\beta\tau}\,. \quad (9)$$

Then Equation (7) can be written in the form

$$d^2J_\nu/d\tau^2 = q_\nu^2 J_\nu - b_0 q_\nu^2 [1 + (b_1/b_0) e^{-\beta\tau}], \tag{10}$$

which is a second-order differential equation.

If we solve Equation (10) and use the boundary condition (B) we get

$$J_\nu = b_0 + b_1 e^{-\beta\tau} + b_2 e^{-q_\nu \tau} + [b_1\beta^2/(q_\nu^2 - \beta^2)] e^{-\beta\tau}, \tag{11}$$

where b_2 is a constant to be determined from the boundary condition (A), where $\beta \neq q_\nu$.

From Equation (11) we get

$$(dJ_\nu/d\tau)_{\tau=0} = -[\beta b_1 + b_2 q_\nu + b_1\beta^3/(q_\nu^2 - \beta^2)]. \tag{12}$$

From Equation (6) with $K_\nu = (1/3)J_\nu$ we find that

$$H_\nu = [1/3(1 + \eta_\nu)][(dJ_\nu/d\tau)]. \tag{13}$$

Hence,

$$b_2 = -\frac{\left[(1 + \eta_\nu)(b_0 + b_1) + \frac{2}{3}\beta b_1 + (1 + \eta_\nu + \frac{2}{3}\beta)\dfrac{b_1\beta^2}{q_\nu^2 - \beta^2}\right]}{1 + \eta_\nu + \frac{2}{3}q_\nu}. \tag{14}$$

Finally we get

$$J_\nu = b_0 + b_1 e^{-\beta\tau} + \left[\frac{b_1\beta^2}{q_\nu^2 - \beta^2}\right] e^{-\beta\tau} - \frac{\left[(1 + \eta_\nu)(b_0 + b_1) + \frac{2}{3}b_1\beta + (1 + \eta_\nu + \frac{2}{3}\beta)\dfrac{b_1\beta^2}{q_\nu^2 - \beta^2}\right] e^{-q_\nu\tau}}{(1 + \eta_\nu + \frac{2}{3}q_\nu)}. \tag{15}$$

Now, J_ν (the average intensity) enables us to find the intensity within the absorption line at any optical depth and in any direction by solving the fundamental equation of line formation,

$$\cos\theta\, dI_\nu(\theta)/d\tau = (1 + \eta_\nu)I_\nu(\theta) - (1 - \varepsilon)\eta_\nu J_\nu - (1 + \varepsilon\eta_\nu)B_\nu; \tag{16}$$

J_ν and B_ν being known function of τ.

The solution for $I_\nu(\theta)$ can be writen down immediately since Equation (16) is a linear differential equation with constant coefficients.

4. Residual Intensity

The residual intensity in the mean contours is given (cf. Woolley and Stibbs, 1953) by

$$r_\nu = (H_\nu/H)_{\tau=0}, \tag{17}$$

where the omission of the suffix ν means *outside the line*. By virtue of the boundary condition $J_\nu = 2H_\nu$ at $\tau = 0$ we have

$$r_\nu = (J_\nu/J)_{\tau=0}\,. \tag{18}$$

Also, outside the line $\eta_\nu = 0$ and $q_\nu = \sqrt{3}$, Equation (15) with $\tau = 0$ gives

$$J_\nu(0) = b_0 + b_1 + \frac{b_1\beta^2}{q_\nu^2 - \beta^2} - \frac{(1+\eta_\nu)(b_0+b_1) + (1+\eta_\nu+\frac{2}{3}\beta)\dfrac{b_1\beta^2}{q_\nu^2-\beta^2} + \frac{2}{3}\beta b_1}{1+\eta_\nu+\frac{2}{3}q_\nu}\,. \tag{20}$$

Hence, by Equations (18), (19), and (20) we have

$$r_\nu = \frac{\frac{2}{3}q_\nu(\beta^2 - q_\nu^2)b_0 + \frac{2}{3}q_\nu^2(\beta - q_\nu)b_1}{2\sqrt{3}(\beta^2 - 3)b_0 + 6(\beta - \sqrt{3})b_1} \times \frac{(\beta^2 - 3)(3 + 2\sqrt{3})}{(\beta^2 - q_\nu)(1 + \eta_\nu + \frac{2}{3}q_\nu)}\,. \tag{21}$$

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AN EXACT SOLUTION OF THE EQUATION OF TRANSFER WITH THREE-TERM SCATTERING INDICATRIX IN AN EXPONENTIAL ATMOSPHERE

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Abstract. The general equation for radiative transfer in the Milne–Eddington model is considered here. The scattering function is assumed to be quadratically anisotropic in the cosine of the scattering angle and Planck's intensity function is assumed for thermal emission. Here we have taken Planck's function as a nonlinear function of optical depth, viz., $B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$. The exact solution for emergent intensity from the bounding face is obtained by the method of the Laplace transform in combination with the Wiener–Hopf technique.

1. Introduction

Chandrasekhar (1960) has considered the problem of radiative transfer with general anisotropic scattering in the Milne–Eddington model to obtain the exact form of emergent intensity from the bounding face and nth approximate intensity at any optical depth by discrete ordinates procedure assuming Planck's function to be linear in the optical depth. Das (1979b) obtained an exact solution of this problem using the Laplace transform and the Wiener–Hopf technique. Wilson and Sen (1964) solved the same problem by a modified spherical-harmonic method. In this paper we considered the equation of transfer with anisotropic scattering in the M–E model with Planck's function as a nonlinear function of optical depth viz.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$$

(Degl'Innocenti, 1979), where b_0, b_1, and β are three positive constants.

2. Basic Equation and Boundary Conditions

The equation of transfer in a stellar atmosphere can be written (cf. Chandrasekhar, 1960; Das, 1979b) as

$$\mu\, \mathrm{d}I_\nu(x, \mu)/\rho\, \mathrm{d}x = (k_\nu + \sigma_\nu)I_\nu(x, \mu) - (1/2)\sigma_\nu(1 - \varepsilon_\nu) \times \times \int_{-1}^{+1} P(\mu, \mu')I_\nu(x, \mu')\, \mathrm{d}\mu' - (k_\nu + \varepsilon_\nu\sigma_\nu)B_\nu(T)\,, \tag{1}$$

where

$$P(\mu, \mu') = \sum_{l=0}^{2} W_l P_l(\mu) P_1(\mu') \tag{2}$$

is the phase function for non-conservative scattering with a three-term indicatrix; $I_\nu(x, \mu)$, the specific intensity in the direction arc cos μ at a depth x; k_ν, the absorption coefficient; arc cos μ is being measured from outward drawn normal to the face $x = 0$; σ_ν, the scattering coefficient; ρ, the density of the atmosphere; $B_\nu(T)$, Planck's function; T, the local temperature at a depth x; ε_ν, the collision constant; and ν, the frequency. We define the optical depth t_ν in terms of the scattering and absorption coefficient and the optical depth τ_ν in terms of the absorption coefficient;

$$t_\nu = \int_x^\infty (k_\nu + \sigma_\nu)\rho \, \mathrm{d}x \,, \tag{3}$$

$$\tau_\nu = \int_x^\infty k_\nu \rho \, \mathrm{d}x \,; \tag{4}$$

with

$$\mathrm{d}t_\nu = -(k_\nu + \sigma_\nu)\rho \, \mathrm{d}x \,, \tag{5}$$

$$\mathrm{d}\tau_\nu = -k_\nu \rho \, \mathrm{d}x \,. \tag{6}$$

If we follow Degl'Innocenti (1979) and Karanjai and Karanjai (1985) we adopt

$$B_\nu(\tau_\nu) = B_\nu^{(0)} + B_\nu^{(1)} e^{-\alpha\tau_\nu} \,, \tag{7}$$

where $B_\nu^{(0)}$, $B_\nu^{(1)}$, and α are three positive constants.

Hence, Equation (7) with Equations (5) and (6) becomes

$$B_\nu(t_\nu) = b_0 + b_1 e^{-\beta t_\nu} \,, \tag{8}$$

where

$$b_0 = B_\nu^{(0)} \,, \qquad b_1 = B_\nu^{(1)} \quad \text{and} \quad \beta = \alpha k_\nu/(k_\nu + \sigma_\nu) \,. \tag{9}$$

In this model we shall assume that

$$\eta_\nu = (k_\nu + \sigma_\nu)^{-1} \tag{10}$$

is constant with optical depth. Equation (1) with Equations (3) and (8) becomes

$$\mu \, \mathrm{d}I(t, \mu)/\mathrm{d}t = I(t, \mu) - (1 - c_0/w_0)B(t) - $$
$$-(1/2) \int_{-1}^{+1} (c_0 + c_1\mu\mu' + \tfrac{1}{4}c_2(3\mu^2 - 1)(3\mu'^2 - 1) I(t, \mu') \, \mathrm{d}\mu' \,, \tag{11}$$

where c_0, c_1, and c_2 are given by

$$c_0/w_0 = c_1/w_1 = c_2/w_2 = \sigma(1 - \varepsilon)/(k + \sigma)\,; \tag{12}$$

and for convenience, we have omitted the subscript ν. For the solution of Equation (11) we have the boundary conditions

$$I(0, -\mu) = 0\,, \qquad 0 < \mu \leq 1 \tag{13a}$$

and

$$I(t, \mu)\exp(-t/\mu) \to 0 \quad \text{when} \quad t \to \infty\,, \quad |\mu| \leq 1\,. \tag{13b}$$

3. Solution for Emergent Intensity

The Laplace transform of $F(t)$ is denoted by $F^*(s)$, where $F^*(s)$ is defined by

$$F^*(s) = s\int_0^\infty \exp(-st)F(t)\,\mathrm{d}t\,, \quad \mathrm{Re}\,s > 0\,; \tag{14}$$

and we set

$$I_m(t) = (1/2)\int_{-1}^{+1} \mu^m I_m^*(s, \mu)\,\mathrm{d}\mu\,, \quad m = 0, 1, 2\,, \tag{15}$$

which implies that

$$I_m^*(s) = (1/2)\int_{-1}^{+1} \mu^m I_m^*(s, \mu)\,\mathrm{d}\mu\,, \quad m = 0, 1, 2\,. \tag{15}$$

Equation (11) with Equation (15), takes the form

$$\mu\,\mathrm{d}I(t, \mu)/\mathrm{d}t = I(t, \mu) - [c_0 I_0(t) + c_1\mu I_1(t) + \\ + \tfrac{1}{4}c_2(3\mu^2 - 1)(3I_2(t) - I_0(t))] - (1 - c_0/w_0)B(t)\,. \tag{17}$$

Now subjecting Equation (17) to the Laplace transform as defined in Equation (14) we have, using the boundary conditions,

$$(\mu s - 1)I^*(s, \mu) = \mu s I(0, \mu) - (1 - c_0/w_0)B^*(s) - (c_0 I_0^*(s) + \\ + c_1\mu I_1^*(s) + \tfrac{1}{4}c_2((3\mu^2 - 1)(3I_2^*(s) - I_0^*(s)))\,. \tag{18}$$

Equation (18) gives

$$I(0, \mu) = (c_0 I_0^*(1/\mu) + c_1\mu I_1^*(1/\mu) + \tfrac{1}{4}c_2(3\mu^2 - 1) + \\ + (3I_2^*(1/\mu) - I_0^*(1/\mu)) + (1 - c_0/w_0)B^*(1/\mu)\,. \tag{19}$$

Equation (19) with $\mu = s^{-1}$, s is complex, takes the form

$$I(0, s^{-1}) = (c_0 - \tfrac{1}{4}c_2(3s^{-2} - 1)I_0^*(s) + c_1 s^{-1} I_1^*(s) + \\ + \tfrac{3}{4}c_2(3s^{-2} - 1)I_2^*(s) + (1 - c_0/w_0)B^*(s)\,, \tag{20}$$

we shall apply the operator

$$(1/2)\int_{-1}^{+1} \cdots \mu \,\mathrm{d}\mu \tag{21}$$

on both sides of Equation (18) to get

$$-(1 - c_0)s^{-1}I_0^*(s) + I_1^*(s) = (1/2)\int_0^1 \mu I(0, \mu)\,\mathrm{d}\mu - \\ - (1 - c_0/w_0)s^{-1}B^*(s) \tag{22}$$

and

$$-(1 - \tfrac{1}{3}c_1)s^{-1}I_1^*(s) + I_2^*(s) = (1/2)\int_0^1 \mu^2 I(0, \mu)\,\mathrm{d}\mu\,, \tag{23}$$

we shall also apply the operator

$$(1/2)\int_{-1}^{+1} \cdots \mathrm{d}\mu/(\mu s - 1) \tag{24}$$

on both sides of Equation (18) to get

$$as^{-1} - (1 - c_0/w_0)B^*(s)t_0 s^{-1} = [1 + c_0 t_0 s^{-1} - \tfrac{1}{4}c_2(3t_2 s^{-1} - \\ - t_0 s^{-1})]I_0^*(s) + c_1 t_1 s^{-1} I_1^*(s) + \tfrac{3}{4}c_2[3t_2 s^{-1} - t_0 s^{-1}]I_2^*(s)\,, \tag{25}$$

where

$$as^{-1} = (1/2)\int_0^1 \mu s(\mu s - 1)^{-1} I(0, \mu)\,\mathrm{d}\mu \tag{26}$$

and

$$t_m s^{-1} = (1/2)\int_{-1}^{+1} (\mu s - 1)^{-1}\mu^m\,\mathrm{d}\mu\,, \quad m = 0, 1, 2\,. \tag{27}$$

If we follow the usual procedure for elimination of $I_0^*(s)$, $I_1^*(s)$, and $I_2^*(s)$ among Equations (26), (22), (23), and (25), after some lengthy calculations setting $s = z^{-1}$, we have

$$T(z)I(0, z) = (1/2)\int_0^1 x(x - z)^{-1}L(x, z)I(0, x)\,dx + + (1 - c_0/w_0)B^*z^{-1}, \tag{28}$$

where

$$T(z) = 1 - 2z^2\int_0^1 \psi(x)\,dx(z^2 - x^2)^{-1}, \tag{29}$$

$$\psi(x) = (1/2)(A + Bx^2 + Cx^4), \tag{30}$$

$$L(x, z) = A - \tfrac{3}{4}c^2x^2 + (B + C + \tfrac{3}{4}c_2)xz - (1/3)Cz^2 + Cx^2z^2, \tag{31}$$

$$B^*z^{-1} = b_0 + b_1/(1 + \beta z) = (d_0 + d_1z)/(1 + \beta z), \tag{32}$$

where

$$d_0 = b_0 + b_1, \qquad d_1 = b_0\beta, \tag{33}$$

$$A = c_0 + \tfrac{1}{4}c_2, \qquad B = c_1(1 - c_0) - \tfrac{3}{4}c_2 - \tfrac{3}{4}c_2(1 - c_0)(1 - c_1/3), \tag{34}$$

$$C = \tfrac{9}{4}c_2(1 - c_0)(1 - c_1/3), \tag{35}$$

where we shall assume that

$$\psi(x) = \tfrac{1}{2}(A + Bx^2 + Cx^4) > 0 \tag{36}$$

and

$$\psi_0 = \int_0^1 \psi(x)\,dx < \tfrac{1}{2}. \tag{37}$$

But for

$$y = k(k + \sigma) < 1, \tag{38}$$

B^*z^{-1} is analytic in $(-y^{-1}, 0)^c$, bounded at the origin and $0 < y < 1$. According to Busbridge (1960), the equation for $T(z)$ possesses the following properties: $T(z)$ is analytic in z for $(-1, 1)^c$, bounded at the origin, has a pair of zeros at $z = \pm K$ ($K > 1$), K is real and can be expressed as

$$T(z) = [H(z)H(-z)]^{-1}, \tag{39}$$

where $H(z)$ and $H(-z)$ have the following properties: $H(z)$ is analytic for $z \in (-1, 0)^c$, bounded at the origin, has a pole at $z = -K$. $H(-z)$ is analytic for $z \in (0, 1)^c$, bounded at the origin, has a pole at $z = K$.

If we follow Busbridge (1960), Das (1979a) and Dasgupta (1977) we have for $\psi_0 < \frac{1}{2}$,

$$H(z) = 1 + zH(z) \int_0^1 \psi(x)H(x)\,(x + z)^{-1}\,dx \tag{40}$$

or

$$H(z) = (A_0 + H_0 z)/(z + K) - M(z)\,, \tag{41}$$

where

$$M(z) = \int_0^1 P(x)\,dx/(x + z)\,, \tag{42}$$

$$P(x) = \phi(x)/H(x)\,, \tag{43}$$

$$\phi(x) = \pi^{-1} Y_0(x)/[T_0^2(x) + Y_0^2(x)]\,, \tag{44}$$

$$T_0(x) = 1 - 2x^2 \int_0^1 (\psi(t) - \psi(x))\,dt/(x^2 - t^2) - \psi(x)x\log(1 + x)/(1 - x)\,, \tag{45}$$

$$Y_0(x) = \pi x \psi(x)\,, \tag{46}$$

$$A_0 = (1 + P_{-1})K\,, \tag{47}$$

$$P_{-1} = \int_0^1 x^{-1}P(x)\,dx\,, \tag{48}$$

$$H_0 = (1 - 2\psi_0)^{-1/2}\,. \tag{49}$$

Equation (28) with Equation (39) takes the form

$$I(0, z)/H(z) = H(-z)G(z) + (1 - c_0/w_0)H(-z)B^*z^{-1}\,, \tag{50}$$

where

$$G(z) = (1/2) \int_0^1 x(x - z)^{-1}L(x, z)I(0, x)\,dx\,, \tag{51}$$

we shall assume that

$$I(0, z) \quad \text{is regular for} \quad \operatorname{Re} z > 0\,, \tag{52}$$

bounded at the origin. Equation (51) with the above assumption on $I(0, z)$ gives the following properties of $G(z)$: $G(z)$ is regular on $(0, 1)^c$, bounded at the origin $O(z)$ when $z \to \infty$.

Equation (50) with Equations (32) and (51) gives

$$I(0,z)/H(z) = H(-z)\left[(1/2)\int_0^1 x(x-z) + L(x,z)I(0,x)\,dx + \right.$$
$$\left. + (1 - c_0/w_0)(d_0 + d_1 z)/(1+\beta z)\right]. \tag{53}$$

Equation (53) can be put in the form

$$I(0,z)/H(z) = H(-z)\left[(1/2)\int_0^1 x(x-z)^{-1}L(x,z)I(0,x)\,dx + \right.$$
$$\left. + (1 - c_0/w_0)(d_0/z + d_1)/(z^{-1} + \beta)\right]. \tag{54}$$

Therefore, the left-hand side of Equation (54) is regular for $\mathrm{Re}\,z > 0$ and bounded at the origin and the right-hand side of Equation (54) is regular for z on $(0, 1)^c$ and bounded at the origin and tends to a linear polynomial in z, say $(x_0 + x_1 z)$ when $z \to \infty$. Hence, by a modified form of Liouville's theorem we have

$$I(0,z) = [x_0 + x_1 z]H(z) \tag{55}$$

and

$$(1/2)\int_0^1 xL(x,z)I(0,x)\,dx/(x-z) +$$
$$+ (1 - c_0/w_0)(d_0 + d_1 z)/(1+\beta z) = (x_0 + x_1 z)/H(-z). \tag{56}$$

Equation (55) will give emergent intensity from the bounding face if x_0 and x_1 are determined. We shall now determine the constants x_0 and x_1. If we set $z = 0$ in Equation (56), we have

$$(1/2)\int_0^1 L(x,0)I(0,x)\,dx + d_0(1 - c_0/w_0) = x_0. \tag{57}$$

Equation (57) with Equation (55) gives

$$x_0 y_1 + x_1 y_2 + z_1 = 0, \tag{58}$$

where

$$y_1 = (1/2)\int_0^1 L(x,0)H(x)\,dx - 1, \tag{59}$$

$$y_2 = (1/2)\int_0^1 xL(x,0)H(x)\,dx, \tag{60}$$

$$z_1 = (1 - c_0/w_0)d_0. \tag{61}$$

As $T(z)$ has a zero at $z = K$, Equation (28) gives

$$(1/2) \int_0^1 xL(x, K)I(0, x)\,\mathrm{d}x/(x - K) + \\ + (1 - c_0/w_0)(d_0 + d_1K)/(1 + \beta K) = 0\,, \tag{62}$$

Equation (62) with Equation (55) gives

$$x_0 y_3 + x_1 y_4 + z_2 = 0\,, \tag{63}$$

where

$$y_3 = (1/2) \int_0^1 xL(x, K)H(x)\,\mathrm{d}x/(x - K)\,, \tag{64}$$

$$y_4 = (1/2) \int_0^1 x^2L(x, K)H(x)\,\mathrm{d}x/(x - K)\,, \tag{65}$$

$$z_2 = (1 - c_0/w_0)(d_0 + d_1K)/(1 + \beta K)\,, \tag{66}$$

Equations (58) and (68) give

$$x_0 = (y_2 z_2 - z_1 y_4)/(y_1 y_4 - y_3 y_2)\,, \tag{67}$$

$$x_1 = (z_1 y_3 - y_1 z_2)/(y_1 y_4 - y_3 y_2)\,, \tag{68}$$

where

$$(y_1 y_4 - y_3 y_2) \neq 0.$$

Hence, Equation (55) with Equations (67) and (68) gives the emergent intensity from the bounding face of the atmosphere.

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EXACT SOLUTION OF THE EQUATION OF TRANSFER IN A FINITE EXPONENTIAL ATMOSPHERE BY THE METHOD OF LAPLACE TRANSFORM AND LINEAR SINGULAR OPERATORS

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Abstract. The equation which commonly appears in radiative transfer problem in a finite atmosphere having ground reflection according to Lambert's law is considered in this paper. The Planck's function $B_\nu(T)$ is taken in the form,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}.$$

The exact solution of this equation is obtained for surface quantities in terms of the $X - Y$ equations of Chandrasekhar by the method of Laplace transform and linear singular operators.

1. Introduction

Das (1978, 1980) has solved various problems of radiative transfer in finite and semi-infinite atmosphere using a method involving Laplace transform and linear singular operators.

In this paper we have considered the one-sided Laplace transform together with the theory of linear singular operators to solve the transport equation which arises in the problem of a finite atmosphere having ground reflection according to Lambert's law taking the Planck's function as a nonlinear function of optical depth: viz.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau},$$

where b_0, b_1, and β are positive constants (Delg'Innocenti, 1979; Karanjai and Karanjai, 1985; Deb *et al.*, 1990).

2. Basic Equation and Boundary Conditions

The integro-differential equation for the intensity of radiation $I(\tau, \mu)$, at any optical depth τ for the problem of diffuse reflection and transmission in a finite atmosphere can be written in the form (Das, 1980) as

$$\mu \frac{dI_\nu(\tau, \mu)}{d\tau} = I_\nu(\tau, \mu) - \int_{-1}^{+1} \psi(\mu') I_\nu(\tau, \mu')\, d\mu' - B_\nu(T), \tag{1}$$

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where $I_\nu(\tau, \mu)$ is the intensity in the direction $\cos^{-1}\mu$ at a depth τ, the angle $\cos^{-1}\mu$ is measured from outside drawn normal to the face $\tau = 0$, $\psi(\mu)$ is the characteristic function for non-conservative scattering which satisfies the condition

$$\psi_0 = \int_0^1 \psi(\mu')\,\mathrm{d}\mu' < \tfrac{1}{2}; \quad \psi(\mu') \text{ is even}, \tag{2}$$

ν is the frequency and $B_\nu(T)$ is the Planck's source function at any optical depth. We have taken

$$B_\nu(T) = b_0 + b_1\, e^{-\beta\tau}.$$

Then Equation (1) becomes

$$\mu\,\frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} = I(\tau,\mu) - \int_{-1}^{+1} \psi(\mu')I(\tau,\mu')\,\mathrm{d}\mu' - (b_0 + b_1\, e^{-\beta\tau}), \tag{3}$$

where for convenience we have omitted the subscript ν.

The boundary conditions associated with Equation (3) are

$$I(0, -\mu) = 0, \qquad 0 < \mu \leq 1, \tag{4a}$$

$$I(\tau_0, \mu) = I_g, \qquad 0 < \mu \leq 1, \qquad \tau_0 > 0, \tag{4b}$$

where τ_0 is the thickness of the finite atmosphere and the bounding face $\tau = \tau_0$ is having ground reflection according to Lambert's law, I_g is a constant.

3. Integral Equations for Surface Quantities

Let us define $f^*(s, \mu)$ as the Laplace transform of $f(\tau, \mu)$ by

$$f^*(s,\mu) = s\int_0^{\tau_0} f(\tau,\mu)\, e^{-s\tau}\,\mathrm{d}\tau, \quad \mathrm{Re}\, s > 0; \tag{5a}$$

$$f(\tau, \mu) = 0, \quad \text{when} \quad \tau > \tau_0. \tag{5b}$$

Let us now apply the Laplace transform defined in Equation (5a) to Equation (3) to obtain the equation satisfying the boundary condition as

$$(\mu s - 1)I^*(s,\mu) = \mu s I(0,\mu) - \mu s\, e^{-\tau_0 s} I(\tau_0, \mu) - S^*(s), \tag{6}$$

where

$$S(\tau) = \int_{-1}^{+1} \psi(\mu')I(\tau,\mu')\,\mathrm{d}\mu' + b_0 + b_1\, e^{-\beta\tau} \Rightarrow \tag{7}$$

$$\Rightarrow S^*(s) = \int_{-1}^{+1} \psi(\mu')I^*(\tau, \mu')\,d\mu' + b_0(1 - e^{-s\tau_0}) + \frac{sb_1}{s+\beta}\,(1 - e^{-(s+\beta)\tau_0}). \tag{8}$$

Let us apply the operator

$$\int_{-1}^{+1} \ldots \psi(\mu)\,d\mu/(\mu s - 1), \tag{9}$$

on both sides of Equation (6) and we obtain, with Equation (8),

$$\begin{aligned} T(1/s)S^*(s) = & \int_{-1}^{+1} d\mu\,\mu s\,\psi(\mu)I(0, \mu)/(\mu s - 1) - \\ & - e^{-\tau_0 s} \int_{-1}^{+1} \mu s\,\psi(\mu)I(\tau_0, \mu)\,d\mu/(\mu s - 1) + \\ & + b_0(1 - e^{-s\tau_0}) + \frac{sb_1}{s+\beta}\,(1 - e^{-(s+\beta)\tau_0}), \end{aligned} \tag{10}$$

where

$$T(1/s) = 1 + \int_{-1}^{+1} d\mu\,\psi(\mu)/(\mu s - 1). \tag{11}$$

Equation (6) gives

$$I(0, \mu) - e^{-\tau_0/\mu} I(\tau_0, \mu) = S^*(1/\mu) \Rightarrow \tag{12}$$

$$\Rightarrow I(0, 1/s) - e^{-\tau_0 s}\, I(\tau_0, 1/s) = S^*(s). \tag{13}$$

Equation (10), together with Equation (12), gives for complex z, where $z = s^{-1}$,

$$\begin{aligned} & [I(0, z) - e^{-\tau_0/z} I(\tau_0, z)]\,T(z) = \\ & = \int_{-1}^{+1} \mu\psi(\mu)I(0, \mu)\,d\mu/(\mu - z) - e^{-\tau_0/z} \int_{-1}^{+1} \mu\psi(\mu)I(\tau_0, \mu)\,d\mu(\mu - z) + \\ & + b_0(1 - e^{-\tau_0/z}) + \frac{b_1}{1 + \beta z}\,(1 - e^{-\beta\tau_0}\, e^{-\tau_0/z}). \end{aligned} \tag{14}$$

Let us put $\alpha_0 = \beta^{-1}$, then Equation (14) becomes

$$[I(0, z) - e^{-\tau_0/z} I(\tau_0, z)]\,T(z) =$$

$$= \int_{-1}^{+1} \mu\psi(\mu)I(0,\mu)\,\mathrm{d}\mu/(\mu - z) - e^{-\tau_0/z}\int_{-1}^{+1} \mu\psi(\mu)I(\tau_0,\mu)\,\mathrm{d}\mu/(\mu - z) +$$

$$+ b_0(1 - e^{-\tau_0/z}) + \frac{b_1\alpha_0}{z+\alpha_0}\,(1 - e^{-\tau_0/\alpha_0}e^{-\tau_0/z}). \tag{15}$$

Let us set $z = -z$ in Equation (15) and multiply the resulting equation by $e^{-\tau_0/z}$ on both sides to obtain, for complex z,

$$[I(\tau_0, -z) - e^{-\tau_0/z}I(0, -z)]T(z) =$$

$$= \int_{-1}^{+1} \mu\psi(\mu)I(\tau_0,\mu)\,\mathrm{d}\mu/(\mu + z) - e^{\tau_0/z}\times$$

$$\times \int_{-1}^{+1} \mu\psi(\mu)I(0,\mu)\,\mathrm{d}\mu/(\mu + z) + b_0(1 - e^{-\tau_0/z}) -$$

$$- \frac{b_1\alpha_0}{\alpha_0 - z}\,(e^{-\tau_0/z} - e^{-\tau_0/\alpha_0}), \tag{16}$$

Equations (15) and (16) are the linear integral equations for the surface quantities under consideration.

4. Linear Singular Integral Equations

Equations (15) and (16) are the equations defined for complex z, where z does not lie between -1 and 1. When z lies between -1 and 1, Equations (15) and (16) will give the linear singular integral equations by the applications of Plemelj's formulae (cf. Mushkelishvili, 1946) with the boundary condition (4) as

$$[I(0,z) - e^{-\tau_0/z}I_g]T_0(z) = P\int_0^1 \mu\psi(\mu)I(0,\mu)\,\mathrm{d}\mu/(\mu - z) -$$

$$- e^{-\tau_0/z}\int_0^1 \mu\psi(\mu)I(\tau_0, -\mu)\,\mathrm{d}\mu/(\mu + z) -$$

$$- e^{-\tau_0/z}P\int_0^1 \mu\psi(\mu)I_g\,\mathrm{d}\mu/(\mu - z) +$$

$$+ b_0(1 - e^{-\tau_0/z}) + \frac{b_1\alpha_0}{z+\alpha_0}\,(1 - e^{-\tau_0(1/z + 1/\alpha_0)}) \tag{17}$$

and

$$I(\tau_0, -z)T_0(z) = P\int_0^1 \mu\psi(\mu)I(\tau_0, -\mu)\,\mathrm{d}\mu/(\mu - z) -$$

$$- e^{-\tau_0/z}\int_0^1 \mu\psi(\mu)I(0, \mu)\,\mathrm{d}\mu/(\mu + z) + \int_0^1 \mu\psi(\mu)I_g\,\mathrm{d}\mu/(\mu + z) +$$

$$+ b_0(1 - e^{-\tau_0/z}) - \frac{b_1\alpha_0}{\alpha_0 - z}(e^{-\tau_0/z} - e^{-\tau_0/\alpha_0}), \tag{18}$$

where

$$T_0(z) = 1 - 2z^2\int_0^1 \mathrm{d}\mu[\psi(\mu) - \psi(z)]/(z^2 - \mu^2) - 2z^2\psi(z)P\int_0^1 \mathrm{d}\mu/(z^2 - \mu^2), \tag{19}$$

in which P dneotes the Cauchy principal value of the integral.

Equations (17) and (18) are the linear singular integral equations from which we shall determine the surface quantities $I(0, z)$ and $I(\tau_0, -z)$ by the application of the theory of linear singular operators.

5. Theory of Linear Singular Operators

If we follow Das (1978, 1980), we can write the following theorems.

THEOREM 1

The linear integral equations for $z \in (0, 1)$,

$$L_+[R(z, -x_0)] = l(z, -x_0), \tag{20a}$$

$$I_-(Q(z, -x_0)] = m(z, -x_0), \tag{20b}$$

where

$$L_+[f(z, -x_0)] = f(z, -x_0)T_0(z) - P\int_0^1 \mu\psi(\mu)f(\mu, -x_0)\,\mathrm{d}\mu/(\mu - z) +$$

$$+ e^{-\tau_0/z}\int_0^1 \mu\psi(\mu)f(\mu, -x_0)\,\mathrm{d}\mu/(\mu + z), \tag{21a}$$

$$L_-[f(z, -x_0)] = f(z, -x_0)T_0(z) - P\int_0^1 \mu\psi(\mu)f(\mu, -x_0)\,\mathrm{d}\mu/(\mu - z) -$$

$$- e^{-\tau_0/z} \int_0^1 \mu\psi(\mu) f(\mu, -x_0)\, d\mu/(\mu + z), \tag{21b}$$

$$l(z, -x_0) = \frac{x_0}{z + x_0} [1 - e^{-\tau_0(1/z + 1/x_0)}] + \frac{x_0}{z - x_0} [e^{-\tau_0/z} - e^{-\tau_0/x_0}], \tag{22a}$$

$$m(z, -x_0) = \frac{x_0}{z + x_0} [1 - e^{-\tau_0(1/z + 1/x_0)}] - \frac{x_0}{z - x_0} [e^{-\tau_0/z} - e^{-\tau_0/x_0}], \tag{22b}$$

admit of solutions of the form

$$R(z, -x_0) = S(z, -x_0) + T(z, -x_0), \tag{23a}$$

$$Q(z, -x_0) = S(z, -x_0) - T(z, -x_0), \tag{23b}$$

where

$$S(z, -x_0) = x_0[X(z)X(x_0) - Y(z)Y(x_0)]/(z + x_0) \tag{24}$$

and

$$T(z, -x_0) = x_0[X(z)Y(x_0) - Y(z)X(x_0)]/(x_0 - z). \tag{25}$$

With constraints on $X(z)$ and $Y(z)$ as

(i) when $\psi_0 < \frac{1}{2}$

$$1 = K \int_0^1 X(\mu)\psi(\mu)\, d\mu/(K - \mu) + e^{-\tau_0/K} K \int_0^1 Y(\mu)\psi(\mu)\, d\mu/(K + \mu), \tag{26a}$$

$$e^{-\tau_0/K} = K \int_0^1 Y(\mu)\psi(\mu)\, d\mu/(K - \mu) + K e^{-\tau_0/K} \int_0^1 X(\mu)\psi(\mu)\, d\mu/(K + \mu); \tag{26b}$$

(ii) when $\psi_0 = \frac{1}{2}$

$$1 = \int_0^1 \psi(\mu) [X(\mu) + Y(\mu)]\, d\mu, \tag{27a}$$

$$\tau_0 \int_0^1 \psi(\mu) Y(\mu)\, d\mu = \int_0^1 \mu\psi(\mu) [X(\mu) - Y(\mu)]\, d\mu \tag{27b}$$

and K is the positive root of the function $T(z)$, when $\psi_0 < \frac{1}{2}$, defined by

$$T(z) = 1 + \int_{-1}^{+1} z\psi(\mu)\,\mathrm{d}\mu/(\mu - z) \tag{28}$$

and where $[X(z) - Y(z)]$ and $[X(z) + Y(z)]$ are the respective solutions of

$$L_+[f(z)] = (1 - e^{-\tau_0/z})\left(1 - \int_0^1 \psi(\mu)f(\mu)\,\mathrm{d}\mu\right) \tag{29}$$

and

$$L_-[f(z)] = (1 + e^{-\tau_0/z})\left(1 - \int_0^1 \psi(\mu)f(\mu)\,\mathrm{d}\mu\right). \tag{30}$$

THEOREM 2

As the operators L_+ and L_- are linear for $z \in (0, 1)$, then for any constant C, we have

$$L_\pm(CF(z, -x_0)) = CL_\pm(F(z, -x_0)) \tag{31}$$

and

$$L_\pm(zf(z)) = zL_\mp(f(z) - (1 \mp e^{-\tau_0/z})\int_0^1 \mu\psi(\mu)f(\mu)\,\mathrm{d}\mu\,. \tag{32}$$

THEOREM 3

If $R(z, -x_0)$ and $Q(z, -x_0)$ are the solutions of

$$L_+[R(z, -x_0)] = l(z, -x_0)\,, \tag{33a}$$

$$L_-[Q(z, -x_0)] = m(z, -x_0)\,, \tag{33b}$$

then

$$L_+(M(z)) = \int_0^1 \psi(-x_0)l(z, -x_0)\,\mathrm{d}x_0\,, \tag{34}$$

$$L_-(N(z)) = \int_0^1 \psi(-x_0)m(z, -x_0)\,\mathrm{d}x_0\,, \tag{35}$$

admit of a solution of

$$M(z) = \int_0^1 \psi(-x_0)R(z, -x_0)\,\mathrm{d}x_0\,, \tag{36}$$

$$N(z) = \int_0^1 \psi(-x_0)Q(z, -x_0)\,\mathrm{d}x_0\,. \tag{37}$$

6. Solution for Surface Quantities

Linear singular integral equations (17) and (18) are the required integral equations from which we will have to determine $I(0, z)$ and $I(\tau_0, -z)$, the quantities under consideration, by the application of the theory of linear singular operators indicated in Section 5.

Equations (17) and (18) on addition and after some rearrangement give

$$L_+[I(0, z) + I(\tau_0, -z) - e^{-\tau_0/z}I_g] =$$

$$= 2b_0(1 - e^{-\tau_0/z}) + b_1 l(z, -\alpha_0) + I_g \int_0^1 \psi(\mu)l(z, -\mu)\,\mathrm{d}\mu\,. \tag{38}$$

Equations (17) and (18) on subtraction and after some manipulation give

$$L_-[I(0, z) - I(\tau_0, -z) - e^{-\tau_0/z}I_g] =$$

$$= b_1 m(z, -\alpha_0) - I_g \int_0^1 \psi(\mu)m(z, -\mu)\,\mathrm{d}\mu\,, \tag{39}$$

where $l(z, -\mu)$ and $m(z, -\mu)$ are given by Equations (22a) and (22b). Equations (38) and (39), with Theorems 1, 2, and 3 of Section 5, will give us the desired quantities $I(0, z)$ and $I(\tau_0, -z)$. The solution of Equation (38) is given by

$$[I(0, z) + I(\tau_0, -z) - I_g e^{-\tau_0/z}] =$$

$$= \frac{2b_0}{1 - G_0}(X(z) - Y(z)) + b_1 R(z, -\alpha_0) + I_g \int_0^1 R(z, -\mu)\psi(\mu)\,\mathrm{d}\mu\,, \tag{40}$$

where

$$G_0 = \int_0^1 \psi(\mu)\,[X(\mu) - Y(\mu)]\,\mathrm{d}\mu\,. \tag{41}$$

The solution of Equation (39) is given by

$$[I(0, z) - I(\tau_0, -z) - I_g e^{-\tau_0/z}] =$$

$$= b_1 Q(z, -\alpha_0) - I_g \int_0^1 \psi(\mu)Q(z, -\mu)\,\mathrm{d}\mu\,. \tag{42}$$

Equations (40) and (42) on addition give $I(0, z)$ and Equations (38) and (42) on subtraction give $I(\tau_0, -z)$ as

$$I(0, z) = I_g e^{-\tau_0/z} + I_g \int_0^1 \psi(\mu) T(z, -\mu)\, d\mu + \frac{b_0}{1 - G_0} [X(z) - Y(z)] + b_1 S(z, -\mu) \tag{43}$$

and

$$I(\tau_0, -z) = \frac{b_0}{1 - G_0} [X(z) - Y(z)] + b_1 T(z, -\alpha_0) + I_g \int_0^1 \psi(\mu) S(z, -\mu)\, d\mu , \tag{44}$$

where $S(z, -\mu)$ and $T(z, -\mu)$ are given by Equations (24) and (25).

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SOLUTION OF THE EQUATION OF TRANSFER FOR INTERLOCKED MULTIPLETS WITH PLANCK FUNCTION AS A NONLINEAR FUNCTION OF OPTICAL DEPTH

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Abstract. The equation of transfer for interlocked multiplets has been solved exactly by the method used by Busbridge and Stibbs (1954) for exponential form of the Planck function $B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$.

1. Introduction

The equation of transfer in the Milne–Eddington model for interlocking without redistribution have been discussed by Woolley and Stibbs (1953), where a clear statement of the problem will be found. Taking the Planck function to be linear, they have obtained a solution by means of Eddington's approximation and calculated the residual intensities and the total absorption in the emergent flux for doublet and triplet lines. Busbridge and Stibbs (1954) applied the principle of invariance governing the law of diffuse reflection with a slight modification to solve exactly the equation of transfer in the M–E model. Dasgupta and Karanjai (1972) applied Sobolev's probabilistic method to solve the same problem. Karanjai and Barman (1981) applied the extension of the method of discrete ordinates to solve the problem. Dasgupta (1978) obtained an exact solution of the problem by Laplace transform and Wiener–Hopf technique using a new representation of the H-function obtained by Dasgupta (1977). The same problem has also been solved by Karanjai and Karanjai (1985) by the method used by Dasgupta (1978) and by Deb *et al.* (1991) by discrete ordinate method using the Planck function as an exponential function of optical depth.

In this paper we have solved the same problem by the method used by Busbridge and Stibbs (1954), using the Planck function $B_\nu(T)$ as an exponential function of optical depth (Degl'Innocenti, 1979)

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}.$$

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2. Equation of Transfer

Woolley and Stibbs (1953) made certain assumptions, viz., (i) that the lines are so close together that variations of the continuous absorption coefficient k and of the Planck function $B_\nu(T)$ with wavelength may be neglected. This also means that the lower states are nearly equal in excitation potential and that they have the same classical damping constant. Then the values of $\eta_1, \eta_2, \ldots, \eta_k$ (the ratios of the line absorption coefficients to k) are proportional to the transition probabilities for spontaneous emission from the upper state to the respective lower states; (ii) that $\eta_1, \eta_2, \ldots, \eta_k$ are independent of depth; (iii) that the coefficient ε, which is introduced to allow for thermal emission associated with the absorption is independent of both frequency and depth.

In the present paper, we have further assumed that (iv)

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}, \tag{1}$$

where β is a constant and $\tau = \int_0^x k\rho \, dx$, x being the depth below the surface of the atmosphere. By (i) b_0, b_1, and τ are independent of ν.

Then the equation of transfer for interlocked multiplets can be written as

$$\mu \frac{dI_r(\tau, \mu)}{d\tau} = (1 + \eta_r) I_r(\tau, \mu) - (1 + \varepsilon\eta_r)(b_0 + b_1 e^{-\beta\tau}) - \\ - (1 - \varepsilon)\alpha_r \sum_{p=1}^{k} \frac{1}{2} \eta_p \int_{-1}^{+1} I_p(\tau, \mu') \, d\mu', \quad (r = 1, 2, \ldots, k), \tag{2}$$

where

$$\alpha_r = \eta_r / (\eta_1 + \eta_2 + \cdots, \eta_k), \tag{3}$$

and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1. \tag{4}$$

In Equation (2) the subscript r denotes the quantity corresponding to the line of frequency ν_r. The Equation (2) have to be solved subject to the boundary conditions,

$$I_r(0, -\mu') = 0, \quad (0 < \mu' \leq 1, \quad r = 1, 2, \ldots, k) \tag{5}$$

together with a condition limiting $I_r(\tau, \mu)$ for large τ. We shall assume that $I_r(\tau, \mu)$ is at most linear in τ for large τ. Formal solutions of Equation (2) are easily found, but they do not satisfy Equation (5).

These are

$$I_r(\tau, \mu) = b_0 + \frac{b_1 T_r}{1 + \xi_r \beta\mu} e^{-\beta\tau}, \quad (r = 1, 2, \ldots, k), \tag{5a}$$

write

$$I_r(\tau, \mu) = b_0 + \frac{b_1 T_r}{1 + \xi_r \beta\mu} e^{-\beta\tau} + I_r^*(\tau, \mu), \quad (r = 1, 2, \ldots, k), \tag{6}$$

where

$$T_r = \frac{\lambda_r}{1 - \dfrac{1}{2\beta}(1 - \varepsilon)\eta_r \log \dfrac{1 + \beta\xi_r}{1 - \beta\xi_r}} \tag{7}$$

and

$$\lambda_r = (1 + \varepsilon\eta_r)/(1 + \eta_r), \tag{8}$$

$$\xi_r = 1/(1 + \eta_r). \tag{9}$$

Then $I_r^*(\tau, \mu)$ satisfies the equation

$$\mu \frac{dI_r^*(\tau, \mu)}{d\tau} = (1 + \eta_r)I_r^*(\tau, \mu) - (1 - \varepsilon)\alpha_r \times$$

$$\times \sum_{p=1}^{k} \frac{1}{2} \eta_p \int_{-1}^{+1} I_p^*(\tau, \mu')\, d\mu', \quad (r = 1, 2, \ldots, k) \tag{10}$$

together with the boundary condition

$$I_r^*(0, -\mu) = \frac{b_1 T_r}{\xi_r \beta\mu' - 1} - b_0, \quad (0 < \mu' \leq 1, \quad r = 1, 2, \ldots, k). \tag{11}$$

Moreover, $I_r(\tau, \mu)$ must be at most linear in τ as $\tau \to \infty$.

Now we have the problem of a scattering atmosphere (exponential) subject to external radiation whose intensity is given by Equation (11). We want to find the emergent intensity $I_r^*(0, \mu)$ of frequency ν_r. This will be the intensity of the diffusely reflected radiation and can be calculated when the appropriate scattering function is known.

In the present problem the scattering function splits up into k^2 functions

$$S_{rs}(\mu, \mu') \quad (r = 1, 2, \ldots, k; s = 1, 2, \ldots, k)$$

but it is convenient to reunite them temporarily in the function

$$P(\nu, \nu')S(\nu, \nu'; \mu, \mu'),$$

where ν is any one of $\nu_1, \nu_2, \ldots, \nu_k$.

$$P(\nu, \nu') = \alpha_\nu \sum_{p=1}^{k} \delta(\nu_p - \nu') \tag{12}$$

δ denoting Dirac's δ-function, and

$$S(\nu_r, \nu_s; \mu, \mu') = S_{rs}(\mu, \mu') . \tag{13}$$

Then the law of diffuse reflection for the atmosphere can be written as (Stibbs, 1953; Busbridge, 1953),

$$I_r^{\text{ref}}(0, \mu) = \frac{1}{2\mu} \int_0^\infty P(\nu, \nu')\, \mathrm{d}\nu' \int_0^1 S(\nu, \nu'; \mu, \mu') I_{\nu'}^{\text{inc}}(0, -\mu')\, \mathrm{d}\mu' , \tag{14}$$

The equivalent form in terms of the functions $S_{rs}(\mu, \mu')$ is

$$I_r^{\text{ref}}(0, \mu) = \alpha_r \sum_{p=1}^{k} \frac{1}{2\mu} \int_0^1 S_{rp}(\mu, \mu') I_p^{\text{inc}}(0, -\mu')\, \mathrm{d}\mu' . \tag{15}$$

3. Scattering Function

If we follow Busbridge and Stibbs (1954) we have the scattering function from frequency ν_s and direction $-\mu'$ into frequency ν_r and direction μ, in the form

$$S_{rs}(\mu, \mu') = \eta_r(1 - \lambda_s) \frac{\mu\mu'}{\xi_r \mu + \xi_s \mu'} H(\xi_r \mu) H(\xi_s \mu') , \tag{16}$$

where

$$H(\xi_r \mu) = 1 + \tfrac{1}{2}\xi_r \mu H(\xi_r \mu) \sum_{p=1}^{k} \alpha_p(1 - \lambda_p) \int_0^1 \frac{H(\xi_p \mu')\, \mathrm{d}\mu'}{\xi_r \mu + \xi_p \mu'} . \tag{17}$$

4. *H*-function

Following Busbridge and Stibbs (1954), Equation (17) can be written as

$$1/H(\xi_r \mu) = \left(\sum_{p=1}^{k} \alpha_p \lambda_p \right)^{1/2} + \frac{1}{2} \sum_{p=1}^{k} \alpha_p(1 - \lambda_p) \int_0^1 \frac{\xi_p \mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'}\, \mathrm{d}\mu' , \tag{18}$$

5. Emergent Intensity

From Equations (11), (15), and (9) we have

$$I_r^*(0, \mu) = \frac{\alpha_r}{2\mu} \sum_{p=1}^{k} \int_0^1 S_{rp}(\mu, \mu') \left(\frac{b_1 T_p}{\xi_p \beta \mu' - 1} - b_0 \right) . \tag{19}$$

If we substitute from Equation (16) we get

$$I_r^*(0,\mu) = \tfrac{1}{2}\alpha_r H(\xi_p\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) \int_0^1 \frac{\mu'}{\xi_r\mu + \xi_p\mu'} \times$$

$$\times \left(\frac{b_1 T_p}{\xi_p \beta\mu' - 1} - b_0\right) H(\xi_p\mu')\,\mathrm{d}\mu' =$$

$$= \tfrac{1}{2}\alpha_r b_1 H(\xi_r\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) T_p \int_0^1 \frac{\mu' H(\xi_p\mu')\,\mathrm{d}\mu'}{(\xi_r\mu + \xi_p\mu')(\xi_p\beta\mu' - 1)} -$$

$$- \tfrac{1}{2}\alpha_p b_0 H(\xi_p\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) \int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu'\,. \tag{20}$$

If we use the relations

$$\frac{1}{(\xi_p\beta\mu' - 1)(\xi_p\mu + \xi_p\mu')} = \frac{1}{(\xi_r\beta\mu + 1)}\left[\frac{\beta}{\xi_p\beta\mu - 1} - \frac{1}{\xi_r\mu + \xi_p\mu'}\right], \tag{21}$$

we get from Equation (20)

$$I_p^*(0,\mu) = \tfrac{1}{2}\alpha_r b_1 H(\xi_r\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) T_p \int_0^1 \frac{\mu'}{\xi_r\beta\mu + 1} \times$$

$$\times \left[\frac{\beta}{\xi_p\beta\mu' - 1} - \frac{1}{\xi_r\mu + \xi_r\mu'}\right] H(\xi_p\mu')\,\mathrm{d}\mu' - \tfrac{1}{2}\alpha_r b_0 H(\xi_r\mu) \times$$

$$\times \sum_{p=1}^{k} \xi_r(1-\lambda_p) \int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu' =$$

$$= \tfrac{1}{2}\alpha_r b_1 H(\xi_r\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) T_p \left(\frac{\beta}{\xi_r\beta\mu + 1}\right) \times$$

$$\times \int_0^1 \frac{\mu' H(\xi_p\mu')\,\mathrm{d}\mu'}{\xi_p\beta\mu' - 1} - \tfrac{1}{2}\alpha_r b_1 H(\xi_r\mu) \sum_{p=1}^{k} \xi_r \times$$

$$\times (1-\lambda_p) T_p \left(\frac{1}{\xi_p\beta\mu + 1}\right) \int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu' - \tfrac{1}{2}\alpha_r b_0 \times$$

$$\times H(\xi_r\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) \int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu'\,. \tag{22}$$

From Equation (6),

$$I_r(0, \mu) = b_0 + \frac{b_1 T_r}{1 + \xi_r \beta\mu} + I_r^*(0, \mu) . \tag{23}$$

If we use Equations (18), (22), (23) we get

$$\begin{aligned} I_r(0, \mu) = {} & \left(b_0 + \frac{b_1 T_r}{1 + \xi_r \beta\mu}\right)\left\{\left(\sum_{p=1}^{k} \alpha_p \lambda_p\right)^{1/2} + \right. \\ & \left. + \frac{1}{2} \sum_{p=1}^{k} \alpha_p (1 - \lambda_p) \int_0^1 \frac{\xi_p \mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'} \,\mathrm{d}\mu' \right\} + \tfrac{1}{2}\alpha_r b_1 \times \\ & \times H(\xi_r \mu) \sum_{p=1}^{k} \xi_r (1 - \lambda_p) \left(\frac{T_p \beta}{\xi_r \beta\mu + 1}\right) \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_p \beta\mu' - 1} \,\mathrm{d}\mu' - \\ & - \tfrac{1}{2}\alpha_r b_1 H(\xi_r \mu) \sum_{p=1}^{k} \xi_r (1 - \lambda_p) \left(\frac{T_p}{\xi_r \beta\mu + 1}\right) \times \\ & \times \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'} \,\mathrm{d}\mu' - \tfrac{1}{2}\alpha_r b_0 H(\xi_r \mu) \sum_{p=1}^{k} \xi_r \times \\ & \times (1 - \lambda_p) \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'} \,\mathrm{d}\mu' ; \end{aligned} \tag{24}$$

and thus

$$\begin{aligned} I_r(0, \mu) = {} & b_0 H(\xi_r \mu) \left\{\left(\sum_{p=1}^{k} \alpha_p \lambda_p\right)^{1/2} + \right. \\ & \left. + \sum_{p=1}^{k} (\alpha_p \xi_p - \alpha_r \xi_r)(1 - \lambda_p) \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'} \,\mathrm{d}\mu' \right\} + \\ & + b_1 \frac{H(\xi_r \mu)}{(1 + \xi_r \beta\mu)} \left\{ T_r \left(\sum_{p=1}^{k} \alpha_p \lambda_p\right)^{1/2} + \right. \\ & \left. + \frac{1}{2} \sum_{p=1}^{k} (\alpha_p \xi_p T_r - \alpha_r \xi_r T_p)(1 - \lambda_p) \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'} \,\mathrm{d}\mu' \right\} + \\ & + \tfrac{1}{2} b_1 \alpha_r \xi_r \beta \frac{H(\xi_r \mu)}{(1 + \xi_r \beta\mu)} \sum_{p=1}^{k} (1 - \lambda_p) T_p \times \\ & \times \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_p \beta\mu' - 1} \,\mathrm{d}\mu' , \end{aligned} \tag{25}$$

which is the final form of the emergent intensity in the rth line.

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TIME-DEPENDENT SCATTERING AND TRANSMISSION FUNCTION IN AN ANISOTROPIC TWO-LAYERED ATMOSPHERE

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Abstract. In this paper we consider the time-dependent diffuse reflection and transmission problems for a homogeneous anisotropically-scattering atmosphere of finite optical depth and solve it by the principle of invariance. Also we consider the time-dependent diffuse reflection and transmission of parallel rays by a slab consisting of two anisotropic homogeneous layers, whose scattering and transmission properties are known. It is shown how to express the time-dependent reflected and transmitted intensities in terms of their components. In a manner similar to that given by Tsujita (1968), we assumed that the upward-directed intensities of radiation at the boundary of the two layers are expressed by the sum of products of some auxiliary functions depending on only one argument. Then, after some analytical manipulations, three groups of systems of simultaneous integral equations governing the auxiliary functions are obtained.

1. Introduction

Sobolev (1956) dealt with the one-dimensional problem of time-dependent diffuse reflection and transmission by a probabilistic method. Diffuse reflection of time-dependent parallel rays by a semi-infinite atmosphere was treated by Ueno (1962) on the basis of the principle of invariance. Bellman *et al.* (1962) obtained an integral equation governing diffuse reflection of time-dependent parallel rays from the lower boundary of a finite inhomogeneous atmosphere. Ueno (1965) also obtained this equation by probabilistic method. Matsumoto (1967a) derived functional equations in the integral radiation allowing for the time-dependence given byDirac's δ-function and Heaviside unit step-function. Matsumoto (1967b) also derived a complete set of functional equations for the scattering (S) and transmission (T) functions which govern the laws of diffuse reflection and transmission of time-dependent parallel rays by a finite, inhomogeneous, plane-parallel, non-emitting, and isotropically-scattering atmosphere, where the dependence of the time of the incident radiation is given by Dirac's δ-function and Heaviside's unit step-function. A formulation of the time-dependent H-function was accomplished by means of the Laplace transform in the time-domain. Numerical evaluation of the H-function based on numerical inversion of the Laplace transform presented by Bellman *et al.* (1966) was made.

Recently, Karanjai and Biswas (1988) derived the time-dependent X- and Y-functions

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for homogeneous, plane-parallel, non-emitting, and isotropic atmosphere of finite optical thickness using the integral equation method devleoped by Rybicki (1971), Biswas and Karanjai (1990a) have derived the time-dependent H-, X-, and Y-function in a homogeneous atmosphere scattering anisotropically with Dirac's δ-function and Heaviside unit step-function type time-dependent incidence. Biswas and Karanjai (1990b) have also derived the solution of diffuse reflection and transmission problem for homogeneous isotropic atmosphere of finite optical depth. In this paper we derived the nonlinear integral equations for X- and Y-functions (Chandrasekhar, 1960) for anisotropically-scattering atmosphere. The anisotropy is represented by means of a phase function which can be expressed in terms of finite-order Legendre polynomials. The principal of invariance is applied to derive the functional equations for time-dependent scattering and transmission functions. Next we considered the time-dependent diffuse reflection and transmission of plane-parallel rays by a slab consisting of two homogeneous anisotropically-scattering layers, whose scattering and transmission functions are known. The problem of the time-independent scattering and transmission of radiation in plane-parallel atmosphere of two layers was treated first by Van de Hulst (1963; also see Tsujita, 1968). Hawking (1961) dealt with the problem analytically starting with Milne's integral equation. Later on, Hansen (see Tsujita, 1968) formulated the scattering and transmission functions in a medium consisting of two optically thin layers by the invariant imbedding partical-counting method. Gutshabash (1957) formulated the problem as solutions of simultaneous integral equations. So far as his equations are solvable, the scattering and transmission functions required are given exactly for two layers of different albedos and different large optical thickness. We have extended the same problem (Tsujita, 1968) for the time-dependent transfer of radiation.

2. Derivation of Fundamental Equations

2.1. Formulation of the Problem

In an anisotropically-scattering medium, the intensity of radiation $I(\tau, \mu, \phi, t)$ at any time t, any optical depth τ, in the direction $\cos^{-1}\mu$, satisfies the equation of transfer

$$\frac{1}{c}\,\frac{\partial I(\tau, \mu, \phi, t)}{\partial t} + \mu\,\frac{\partial I(\tau, \mu, \phi, t)}{\partial \tau} + I(\tau, \mu, \phi, t) = J(\tau, \mu, \phi, t)\,, \tag{1}$$

in which the source function $J((\tau, \mu, \phi, t)$ is given by

$$J(\tau, \mu, \phi, t) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} P(\mu, \phi; \mu', \phi')I(\tau, \mu', \phi', t)\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,, \tag{2}$$

where $P(\mu, \phi; \mu', \phi')$, the general phase function and c represents the velocity of light. In the above, μ and ϕ represent, respectively, the cosine of the zenith distance and the azimuthal angle. We decompose the intensity of radiation field into two components for two directions, viz., intensity directed towards the lower surface of the atmosphere ($I^+(\tau, \mu, \phi, t)$) and intensity directed towards the upper surface of the atmosphere ($I^-(\tau, \mu, \phi, t)$).

We consider the initial boundary conditions

$$I(\tau, \mu, \phi, 0) = 0\,, \tag{3}$$

$$I^+(0, \mu, \phi, t) = I_{\text{inc}}(\mu, \phi, t)\,, \tag{4}$$

$$I^-(\tau_1, \mu, \phi, t) = I^*_{\text{inc}}(\mu, \phi, t)\,. \tag{5}$$

Equations (4) and (5) asserts that the lower and the upper surfaces are illuminated. However, we shall restrict ourselves for the time being to the case of illumination on the upper surface ($\tau = 0$) by means of an instantaneously collimated beam of light at time $t = 0$. The other surface will be free from any incident radiation. We now distinguish between the reduced incident intensity which is incident on boundary surface and penetrates to the depth τ without suffering any collision and diffuse radiation which arises due to different processes (Chandrasekhar, 1960). For the total radiation field we have

$$I^+(\tau, \mu, \phi, t) = I_d^+(\tau, \mu, \phi, t) + I_{\text{inc}}\left(\mu, \phi, t - \frac{\tau}{c\mu}\right) \exp\left(-\frac{\tau}{\mu}\right), \tag{6}$$

$$I^-(\tau, \mu, \phi, t) = I_d^-(\tau, \mu, \phi, t) + I^+_{\text{inc}}\left(\mu, \phi, t - \frac{\tau_1 - \tau}{c\mu}\right) \exp\left(-\frac{\tau_1 - \tau}{\mu}\right), \tag{7}$$

where the subscript 'd' represent diffuse fields. If we substitute these expression for $I^+(\tau, \mu, \phi, t)$ and $I^-(\tau, \mu, \phi, t)$ in Equation (1) we get two separate equations of transfer for two components

$$\left(c^{-1}\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} + 1\right) I_d^+(\tau, \mu, \phi, t) = J(\tau, \mu, \phi, t)\,, \tag{8}$$

$$\left(c^{-1}\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} + 1\right) I_d^-(\tau, \mu, \phi, t) = J(\tau, \mu, \phi, t)\,, \tag{9}$$

where

$$J(\tau, \mu, \phi, t) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} I_d(\tau, \mu', \phi', t)\, \times$$

$$\times\, P(\mu, \phi; \mu', \phi')\mu'\, \mathrm{d}\phi' + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{1} P(\mu, \phi; \mu', \phi') \times$$

$$\times I_{\text{inc}}\left(\mu', \phi', t - \frac{\tau}{c\mu}\right) \exp\left(-\frac{\tau}{\mu}\right) d\mu' \, d\phi' + \frac{1}{4\pi} \times$$

$$\times \int_0^{2\pi} \int_0^1 I^*_{\text{inc}}\left(\mu', \phi', t - \frac{\tau_1 - \tau}{c\mu}\right) \exp\left(-\frac{\tau_1 - \tau}{\mu}\right) \times$$

$$\times P(\mu, \phi; \mu', \phi') \, d\mu' \, d\phi' \,. \tag{10}$$

Let us now put in Equation (10)

$$I_{\text{inc}}(\mu, \phi, t) = F\delta(t)\delta(\mu - \mu_0)\delta(\phi - \phi_0)\,, \tag{11}$$

$$I^*_{\text{inc}}(\mu, \phi, t) = 0\,; \tag{12}$$

where F is a constant.

Hence, we get

$$J(\tau, \mu, \phi, t) = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} I_d(\tau_1, \mu', \phi', t) P(\mu, \phi; \mu', \phi') \, d\mu' \, d\phi' +$$

$$+ \tfrac{1}{4} F P(\mu, \phi; \mu_0, \phi_0) \exp\left(-\frac{\tau}{\mu_0}\right) \delta\left(t - \frac{\tau}{c\mu_0}\right). \tag{13}$$

The new set of boundary conditions are given by

$$I_d^+(\tau, \mu, \phi, t) = 0\,, \tag{14a}$$

$$I_d^-(\tau, \mu, \phi, t) = 0\,. \tag{14b}$$

This simplification of boundary conditions are the characteristic of such formulation. Let us now define the scattering and transmission function (cf. Matsumoto, 1967a) as

$$S(\tau, \mu, \phi; \mu_0, \phi_0, t) = I_d^-(0, \mu, \phi, t)\,, \tag{15}$$

$$I(\tau, \mu, \phi; \mu_0, \phi_0, t) = I_d^+(\tau_1, \mu, \phi, t)\,. \tag{16}$$

2.2. Principle of Invariance

We shall now derive the functional equations for these two functions. The four principles of invariance (Matsumoto, 1969) for this problem take the following forms:

(A) The intensity $I_d^-(\tau, \mu, \phi, t)$ in the upward direction at time t and at depth τ is given by

$$I_d^-(\tau, \mu, \phi, t) = F\mu^{-1} S\left(\tau_1 - \tau; \mu, \phi: \mu_0, \phi_0, t - \frac{\tau}{c\mu_0}\right) \exp\left(-\frac{\tau}{\mu_0}\right) +$$

$$+ \frac{1}{4\pi\mu} \int_0^t dt' \int_0^1 \int_0^{2\pi} S(\tau_1 - \tau; \mu, \phi; \mu', \phi', t - t') I_d^+ \times$$

$$\times (\tau, \mu', \phi', t') \, d\mu' \, d\phi' \,. \tag{17}$$

(B) The intensity $I_d^+(\tau, \mu, \phi, t)$ in the downward direction at time t and at a depth τ is given by

$$I_d^+(\tau, \mu, \phi, t) = F\mu^{-1}T(\tau; \mu, \phi; \mu_0, \phi_0, t) + \frac{1}{4\pi\mu}\int_0^t dt' \times \\ \times \int_0^1 \int_0^{2\pi} S(\tau; \mu, \phi; \mu', \phi', t - t')I_d^-(\tau, \mu', \phi', t')\, d\mu'\, d\phi' . \quad (18)$$

(C) The diffuse reflection of the incident radiation by the entire atmosphere is given by

$$F\mu^{-1}S(\tau_1; \mu; \phi; \mu_0, \phi_0, t) = F\mu^{-1}(\tau; \mu, \phi, \mu', \phi', t) + \\ + I_d^-\left(\tau, \mu, \phi, t - \frac{\tau}{c\mu}\right)\exp\left(-\frac{\tau}{\mu}\right) + \frac{1}{4\pi\mu}\int_0^t dt' \times \\ \times \int_0^1 \int_0^{2\pi} T(\tau; \mu, \phi; \mu', \phi', t - t')I_d^-(\tau, \mu', \phi', t')\, d\mu'\, d\phi' . \quad (19)$$

(D) The diffuse transmission of the incident radiation by the entire atmosphere is given by

$$F\mu^{-1}T(\tau_1; \mu, \phi; \mu_0, \phi_0, t) = F\mu^{-1}T\left(\tau_1 - \tau; \mu, \phi; \mu_0, \phi_0, t - \frac{\tau}{c\mu_0}\right) \times \\ \times \exp\left(-\frac{\tau}{c\mu_0}\right) + I_d^+\left(\tau, \mu, \phi, t - \frac{\tau_1 - \tau}{c\mu}\right)\exp\left(-\frac{\tau_1 - \tau}{\mu}\right) + \\ + \frac{1}{4\pi\mu}\int_0^t dt' \int_0^1 \int_0^{2\pi} T(\tau_1 - \tau; \mu, \phi, \mu_0, \phi_0, t - t') \times \\ \times I_d^+(\tau, \mu', \phi', t')\, d\mu'\, d\phi' . \quad (20)$$

A derivation of these four equations is based on classical intuitive physical arguments (Ambartsumian, 1943; Chandrasekhar, 1960; Presendorfer, 1958). Although these equations do not provide a complete knowledge of radiation intensity at any depth (or neutron distribution in a given medium) but only the reflected and transmitted intensities, it has some real advantages for numerical computations.

2.3. Integral equations for the scattering and transmission function

We differentiate Equation (17) with respect to τ and take the limit as $\tau \to 0$

$$\lim_{\tau \to 0} \frac{dI_d^-(\tau, \mu, \phi, t)}{d\tau} = -F\mu^{-1}\left[(c\mu_0)^{-1}\frac{\partial}{\partial t} + (\mu_0)^{-1} + \frac{\partial}{\partial \tau_1}\right] \times$$

$$\times S(\tau_1, \mu, \phi; \mu_0, \phi_0, t) + \frac{1}{4\pi\mu}\int_0^t dt' \times$$

$$\times \int_0^{2\pi}\int_0^1 S(\tau_1; \mu, \phi; \mu', \phi', t - t')\left[\frac{dI_d^+(\tau, \mu', \phi', t')}{d\tau}\, d\mu'\, d\phi'\right]_{\tau=0}. \quad (21)$$

From Equation (8), we get by use of Equation (14)

$$\lim_{\tau \to 0} \frac{dI_d^+(\tau, \mu', \phi', t')}{d\tau} = \frac{J(0, \mu', \phi', t')}{\mu'}, \quad (22)$$

where

$$J(0, \mu', \phi', t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \frac{F}{\mu''} S(\tau_1, \mu'', \phi'', t)\, d\mu''\, d\phi'' +$$

$$+ \tfrac{1}{4}F\delta(t')P(\mu, \phi; \mu_0, \phi_0). \quad (23)$$

In deriving Equation (23) we have used the expression for $J(\tau, \mu, \phi, t)$, Equation (9) now yields, after use of Equations (14) and (15)

$$\lim_{\tau \to 0} \frac{dI_1^-(\tau, \mu, \phi, t)}{d\tau} = -\frac{J(0, \mu, \phi, t)}{\mu} +$$

$$+\left(c^{-1}\frac{\partial}{\partial t} + 1\right)\mu^{-1}F\mu^{-1}S(\tau_1, \mu, \phi; \mu_0, \phi_0, t). \quad (24)$$

If we substitute Equations (22) and (24) in Equation (17), after cancellation and rearrangements of terms, we get

$$\frac{\partial S(\tau_1; \mu, \phi, \mu_0, \phi_0, t)}{\partial \tau_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + 1\right) \times$$

$$\times S(\tau_1; \mu, \phi, \mu_0, \phi_0, t) = P(\mu, \phi; \mu_0, \phi_0)\delta(t) +$$

$$+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(\mu, \phi; \mu'', \phi'')S(\tau_1; \mu'', \phi''; \mu_0, \phi_0, t)\frac{d\mu''}{\mu''}\, d\phi'' +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(-\mu', \phi'' - \mu_0, \phi_0)\,\frac{\mathrm{d}\mu'}{\mu'}\,\mathrm{d}\phi' +$$

$$+\frac{1}{16\pi^2}\int_0^t \mathrm{d}t'\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu_0',\phi_0',t-t')\times$$

$$\times P(-\mu',\phi';\mu'',\phi'')S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,t)\,\frac{\mathrm{d}\mu'}{\mu'}\,\mathrm{d}\phi'\,\frac{\mathrm{d}\mu''}{\mu''}\,\mathrm{d}\phi''\,. \quad (25)$$

Equation (25) is the required functional equation of the time-dependent S-function. Again, if we differentiate Equations (18), (19), and (20) with respect to τ and taking the limit as $\tau \to \tau_1$ and $\tau \to 0$, respectively, and following the same procedure we get

$$\frac{\partial T(\tau_1;\mu,\phi;\mu_0,\phi_0,t)}{\partial\tau_1} + \mu^{-1}\left(1+\frac{1}{c}\frac{\partial}{\partial t}\right)I(\tau_1;\mu,\phi;\mu_0,\phi_0,t) =$$

$$= \exp\left(-\frac{\tau_1}{\mu_0}\right)\delta\left(t-\frac{\tau}{c\mu_0}\right)P(-\mu,\phi;\ -\mu_0,\phi_0) +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(-\mu,\phi;\ -\mu'',\phi'')T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,t)\,\frac{\mathrm{d}\mu''}{\mu''} +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1,\mu,\phi;\mu',\phi',t-t')\delta\left(t-\frac{\tau}{c\mu_0}\right)\exp\left(-\frac{\tau_1}{\mu_0}\right)\times$$

$$\times P(\mu,\phi;\ -\mu_0,\phi_0)\,\frac{\mathrm{d}\mu'}{\mu'}\,\mathrm{d}\phi' + \frac{1}{16\pi^2}\int_0^t \mathrm{d}t'\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}\times$$

$$\times S(\tau_1;\mu,\phi;\mu',\phi',t-t')T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,t')\times$$

$$\times P(\mu',\phi';\ -\mu'',\phi'')\,\frac{\mathrm{d}\mu'}{\mu'}\,\mathrm{d}\phi'\,\frac{\mathrm{d}\mu''}{\mu''}\,\mathrm{d}\phi''\,, \quad (26)$$

$$\frac{\partial S(\tau_1;\mu,\phi;\mu_0,\phi_0,t)}{\partial\tau_1} = P(\mu,\phi;\ -\mu_0,\phi_0)\times$$

$$\times\exp\left(-\tau_1\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\right)\delta\left(t-\frac{\tau_1}{c\mu}-\frac{\tau_1}{c\mu_0}\right) + \exp\left(-\frac{\tau_1}{\mu}\right)\times$$

$$\times \frac{1}{4\pi} \int_0^t dt' \int_0^1 \int_0^{2\pi} T(\tau_1; \mu'', \phi'', \mu_0, \phi_0, t - t') P(\mu, \phi; -\mu'', \phi'') \times$$

$$\times \delta\left(t' - \frac{\tau_1}{c\mu}\right) \frac{d\mu''}{\mu''} \, d\phi'' + \frac{1}{4\pi} \int_0^t dt' \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi', t - t') \times$$

$$\times \delta\left(t' - \frac{\tau_1}{c\mu}\right) \exp\left(-\frac{\tau_1}{\mu_0}\right) P(\mu', \phi'; -\mu_0, \phi_0) \frac{d\mu'}{\mu'} \, d\phi' + \frac{1}{16\pi^2} \int_0^t dt' \times$$

$$\times \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi', t - t') T(\tau_1; \mu'', \phi''; \mu_0, \phi_0, t') \times$$

$$\times P(\mu', \phi'; -\mu'', \phi'') \frac{d\mu'}{\mu'} \, d\phi' \, \frac{d\mu''}{\mu''} \, d\phi'' , \tag{27}$$

$$\frac{\partial T(\tau_1; \mu, \phi, \mu_0, \phi_0, t)}{\partial \tau_1} + \frac{1}{\mu_0}\left(\frac{1}{c} \frac{\partial}{\partial t} + 1\right) T(\tau_1, \mu, \phi; \mu_0, \phi_0, t) =$$

$$= P(-\mu, \phi; -\mu_0, \phi_0) \exp\left(-\frac{\tau_1}{\mu}\right) \delta\left(t - \frac{\tau_1}{c\mu}\right) + \frac{1}{4\pi} \exp\left(-\frac{\tau_1}{\mu}\right) \times$$

$$\times \int_0^1 \int_0^{2\pi} P(-\mu, \phi; \mu'', \phi'') S\left(\tau_1; \mu'', \phi'', \mu_0, \phi_0, t - \frac{\tau_1}{c\mu}\right) \times$$

$$\times \frac{d\mu''}{\mu''} \, d\phi'' + \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \mu', \phi', t) P(-\mu, \phi; -\mu_0, \phi_0) \times$$

$$\times \frac{d\mu'}{\mu'} \, d\phi' + \frac{1}{16\pi^2} \int_0^t dt' \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', t - t') \times$$

$$\times S(\tau_1; \mu'', \phi''; \mu_0, \phi_0, t) P(-\mu', \phi'; \mu'', \phi'') \frac{d\mu'}{\mu'} \, d\phi' \, \frac{d\mu''}{\mu''} \, d\phi'' . \tag{28}$$

Equations (25), (26), (27), and (28) are the required functional equations for 'S' and 'T' functions. Let us now introduce the Laplace transform with respect to the time-variable which enables us to eliminate (at least formally) the time-variable,

$$\frac{\partial S(\tau_1;\mu,\phi;\mu_0,\phi_0,s)}{\partial\tau_1}+\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\left(1+\frac{s}{c}\right)S(\tau_1;\mu,\phi;\mu_0,\phi_0,s)=$$

$$=P(\mu,\phi;-\mu_0,\phi_0)+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}P(\mu,\phi;\mu'',\phi'')\times$$

$$\times S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\frac{d\mu''}{\mu''}\,d\phi''+$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi;\mu',\phi',s)P(-\mu',\phi';-\mu_0,\phi_0)\frac{d\mu'}{\mu'}\,d\phi'+$$

$$+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi,\mu',\phi',s)S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\times$$

$$\times P(-\mu',\phi';\mu'',\phi'')\frac{d\mu'}{\mu'}\,d\phi'\frac{d\mu''}{\mu''}\,d\phi''\,, \qquad (29)$$

$$\frac{\partial T(\tau_1;\mu,\phi;\mu_0,\phi_0,s)}{\partial\tau_1}+\left(1+\frac{s}{c}\right)T(\tau_1;\mu,\phi;\mu_0,\phi_0,s)\mu^{-1}=$$

$$=P(-\mu,\phi;\mu_0,\phi_0)\exp\left(-\frac{\tau_1 s}{c\mu_0}\right)+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\times$$

$$\times P(-\mu,\phi;-\mu'',\phi'')\frac{d\mu''}{\mu''}\,d\phi''+\frac{1}{4\pi}\exp\left(-\frac{\tau_1}{\mu_0}\right)\exp\left(-\frac{\tau_1 s}{c\mu_0}\right)\times$$

$$\times\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi,\mu',\phi',s)P(\mu,\mu';-\mu_0,\phi_0)\frac{d\mu'}{\mu'}\,d\phi'+$$

$$+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}S(\tau_1;\mu,\phi;\mu',\phi',s)P(\mu',\phi';-\mu'',\phi'')\times$$

$$+T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\frac{d\mu'}{\mu'}\,d\phi'\frac{d\mu''}{\mu''}\,d\phi''\,, \qquad (30)$$

$$\frac{\partial S(\tau_1;\mu,\phi;\mu_0,\phi_0,s)}{\partial\tau_1}=\exp\left(-\tau_1\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right)\times$$

$$\times\exp\left(-\frac{\tau_1 s}{c}\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right)P(\mu,\phi;-\mu_0,\phi_0)+\frac{1}{4\pi}\exp\left(-\frac{\tau_1}{\mu_0}\right)\times$$

$$\times\exp\left(-\frac{\tau_1 s}{c\mu}\right)\int_0^1\int_0^{2\pi}T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)P\times$$

$$\times(\mu,\phi,-\mu'',\phi'')\frac{d\mu''}{\mu''}d\phi''+\frac{1}{4\pi}\exp\left(-\frac{\tau_1}{\mu_0}\right)\exp\left(-\frac{\tau_1 s}{c\mu_0}\right)\times$$

$$\times\int_0^1\int_0^{2\pi}I(\tau_1;\mu,\phi,\mu',\phi,s)P(\mu',\phi';-\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi'+$$

$$+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\phi;\mu',\phi',s)P(\mu',\phi';-\mu'',\phi'')\times$$

$$\times T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\frac{d\mu''}{\mu''}d\phi''\frac{d\mu'}{\mu'}d\phi',\qquad(31)$$

$$\frac{\partial T(\tau_1;\mu,\phi,\mu_0,\phi_0,s)}{\partial\tau_1}+\frac{1}{\mu_0}\left(1+\frac{s}{c}\right)T(\tau_1;\mu,\phi;\mu_0,\phi_0,s)=$$

$$=\exp\left(-\frac{\tau_1}{\mu}\right)\exp\left(-\frac{\tau_1 s}{\mu}\right)\mathrm{P}(-\mu,\phi;-\mu_0,\phi_0)+$$

$$+\frac{1}{4\pi}\exp\left(-\frac{\tau_1}{\mu}\right)\exp\left(-\frac{\tau_1 s}{c\mu}\right)\int_0^1\int_0^{2\pi}P(-\mu,\phi;\mu'',\phi'')\times$$

$$\times S(\tau_1;\mu'',\phi'',\mu_0,\phi_0,s)\frac{d\mu''}{\mu''}d\phi''+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\phi;\mu',\phi',s)\times$$

$$\times P(-\mu',\phi',-\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi'+\frac{1}{16\pi^2}$$

$$\times\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\phi,\mu',\phi',s)P(-\mu',\phi',\mu'',\phi'')\times$$

$$\times S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\frac{d\mu'}{\mu'}d\phi'\frac{d\mu''}{\mu''}d\phi''.\qquad(32)$$

2.4. The reduction of the integral equations

We have

$$P(\mu, \phi; \mu', \phi') = \sum_{m=0}^{N} (2 - \delta_{0,m}) \left[\sum_{l=m}^{N} w_l^m P_l^m(\mu) P_l^m(\mu') \right] \cos m(\phi' - \phi). \tag{33}$$

If we follow Chandrasekhar (1960), we obtain

$$S(\tau_1; \mu, \phi; \mu_0, \phi_0, s) = \sum_{m=0}^{N} S^{(m)}(\tau_1; \mu, \mu_0; s) \cos m(\phi_0 - \phi) \tag{34}$$

$$T(\tau_1; \mu, \phi; \mu_0, \phi_0, s) = \sum_{m=0}^{N} T^{(m)}(\tau_1; \mu, \mu_0, s) \cos m(\phi_0 - \phi). \tag{35}$$

If we substitute these expansions of S and T in Equations (29)–(32) and after some rearrangements we get

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\left(1 + \frac{s}{c}\right) S^{(m)}(\tau_1; \mu, \mu_0, s) + \frac{S^{(m)}(\tau_1; \mu; \mu_0; s)}{\partial \tau_1} =$$

$$= (2 - \delta_{0,m}) \sum_{l=m}^{N} (-1)^{m+l} w_l^m \left[P_l^m(\mu) + \frac{(-l)^{l+m}}{2(2 - \delta_{0,m})} \times \right.$$

$$\left. \times \int_0^1 S^m(\tau; \mu, \mu', s) P_l^m(\mu') \frac{d\mu'}{\mu'} \right] \left[P_l^m(\mu_0) + \frac{(-1)^{l+m}}{2(2 - \delta_{0,m})} \times \right.$$

$$\times \int_0^1 P_l^m(\mu'') S^{(m)}(\tau_1; \mu'', \mu_0, s) \frac{d\mu''}{\mu''}, \tag{36}$$

$$\frac{1}{\mu}\left(1 + \frac{s}{c}\right) T^{(m)}(\tau_1; \mu, \mu_0, s) + \frac{\partial T^{(m)}(\tau_1; \mu, \mu_0, s)}{\partial \tau_1} =$$

$$= (2 - \delta_{0,m}) \sum_{l=m}^{N} w_l^m \left[P_l^m(\mu) + \frac{(-l)^{l+m}}{2(2 - \delta_{0,m})} \times \right.$$

$$\left. \times \int_0^1 S^{(m)}(\tau_1; \mu, \mu', s) P_l^m(\mu) \frac{d\mu'}{\mu'} \right] \times$$

$$\times \left[\exp\left[-\frac{\tau_1}{\mu_0}\left(1 + \frac{s}{c}\right) \right] P_l^m(\mu_0) + \right.$$

$$\left. + \frac{1}{2(2 - \delta_{0,m})} \int_0^1 T^{(m)}(\tau_1; \mu'', \mu_0, s) P_l^m(\mu'') \frac{d\mu''}{\mu''} \right], \tag{37}$$

$$\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1} = (2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}w_l^m \times$$

$$\times P_l^m(\mu)\exp\left(-\frac{\tau_1}{\mu}\left(1+\frac{s}{c}\right)\right)+\frac{1}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 T^{(m)}(\tau_1;\mu,\mu',s)P_l^m(\mu')\,\frac{d\mu'}{\mu'}\Bigg]\times$$

$$\times\left[P_l^m(\mu_0)\exp\left[-\frac{\tau_1}{\mu_0}\left(l+\frac{s}{c}\right)\right]+\frac{1}{2(2-\delta_{0,m})}\times\right.$$

$$\times\int_0^1 P_l^m(\mu'')T^{(m)}(\tau_1;\mu'',\mu_0,s)\,\frac{d\mu''}{\mu''}\Bigg], \tag{38}$$

$$\frac{1}{\mu_0}\left(1+\frac{s}{c}\right)T^{(m)}(\tau_1;\mu;\mu_0;s)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1}=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}w_l^m\left[P_l^m(\mu)\exp\left(-\frac{\tau_1}{\mu}\left(1+\frac{s}{c}\right)\right)+\frac{1}{2(2-\delta_{0,m})}\times\right.$$

$$\times\int_0^1 T^{(m)}(\tau_1;\mu,\mu_0,s)P_l^m(\mu')\,\frac{d\mu'}{\mu'}\Bigg]\times$$

$$\times\left[P_l^m(\mu_0)+\frac{(-l)^{l+m}}{2(2-\delta_{0,m})}\int_0^1 P_l^m(\mu'')S^{(m)}(\tau_1;\mu'',\mu_{0,s})\,\frac{d\mu''}{\mu''}\right]. \tag{39}$$

If we now let

$$\psi_l^m(\tau_1;\mu,s)=P_l^m(\mu)+\frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\int_0^l S^{(m)}(\tau_1;\mu,\mu',s)\,P_l^m(\mu')\,\frac{d\mu'}{\mu'} \tag{40}$$

and

$$\phi_l^m(\tau_1;\mu,s)=\exp\left(-\frac{\tau_1}{\mu}\left(1+\frac{s}{c}\right)\right)P_l^m(\mu)+\frac{1}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 T^{(m)}(\tau_1;\mu,\mu',s)P_l^m(\mu)\,\frac{d\mu'}{\mu'}, \tag{41}$$

then, in view of principle of reciprocity (Chandrasekhar, 1960) we can rewrite Equations (36)–(39) in the form

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\left(1+\frac{s}{c}\right)S^{(m)}(\tau_1;\mu,\mu_0,s)+\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1}=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}w_l^m\psi_l^m(\tau_1;\mu,s)\psi_l^m(\tau_1;\mu_0,s)\,, \tag{42}$$

$$\frac{1}{\mu}\left(1+\frac{s}{c}\right)T^{(m)}(\tau_1;\mu,\mu_0,s)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1}=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}w_l^m\psi_l^m(\tau_1;\mu,s)\phi_l^m(\tau_1;\mu_0,s) \tag{43}$$

and

$$\frac{\partial S^{(m)}(\tau_1;\hat{\mu};\mu_0,s)}{\partial\tau_1}=(2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}w_l^m$$

$$\times\,\phi_l^m(\tau_1;\mu,s)\phi_l^m(\tau_1;\mu_0,s) \tag{44}$$

and

$$\frac{1}{\mu_0}\left(1+\frac{s}{c}\right)T^{(m)}(\tau_1;\mu,\mu_0,s)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1}=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}w_l^m\phi_l^m(\tau_1;\mu,s)\psi_l^m(\tau_1;\mu,s)\,. \tag{45}$$

Now by use of Equations (42) and (44) we get

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\left(1+\frac{s}{c}\right)S^{(m)}(\tau_1;\mu,\mu_0,s)=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}w_l^m\,[\psi_l^m(\tau_1;\mu,s)\psi_l^m(\tau_1;\mu_0,s)-$$

$$-\,\phi_l^m(\tau_1;\mu,s)\phi_1^m(\tau_1;\mu_0,s)]\,; \tag{46}$$

and by use of Equations (43) and (45)

$$\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)\left(1+\frac{s}{c}\right)T^{1m2}(\tau_1,\mu,\mu_0,s)=(2-\delta_{0,m})\sum_{l=m}^{N}\times$$

$$\times\, w_l^m[\phi_l^m(\tau_1;\mu,s)\psi_l^m(\tau_l;\mu_0,s)-\psi_l^m(\tau_1;\mu,s)\phi_l^m(\tau_1;\mu_0,s)]\,. \qquad (47)$$

Equations (46) and (47) are the two fundamental equations of our problem.

3. Solution

3.1. Legendre expansion of the phase function and the principle of invariance

Let us now consider that the atmosphere consists of two different layers. Denoting the quantities in the upper layer by subscript '1' and the quantities in the lower by subscript '2' and if we use Equations (46) and (47) we have

$$S_i^{(m)}(\tau_i;\mu,\mu_0,s)=\frac{\mu\mu_0}{\mu+\mu_0}(2-\delta_{0,m})\sum_{l=m}^{N}(-l)^{l+m}\frac{w_{i,l}^{(m)}}{Q}\times$$

$$\times\,\psi_l^m(\tau_i;\mu,s)\psi_l^m(\tau_i;\mu_0,s)-\phi_l^m(\tau_i;\mu,s)-\phi_l^m(\tau_i;\mu,s)\phi_l^m(\tau_i;\mu_0,s)\,, \qquad (48)$$

$$T_i^{(m)}(\tau_i;\mu,\mu_0,s)=\frac{\mu\mu_0}{\mu-\mu_0}(2-\delta_{0,m})\sum_{l=m}^{N}\frac{w_{i,l}^{(m)}}{Q}\times$$

$$\times\,[\phi_l^m(\tau_i,\mu,s)\psi_l^m(\tau_i;\mu,s)-\psi_l^m(\tau_i;\mu,s)\phi_l^m(\tau_i;\mu_0,s)]\,, \quad (49)$$

$$\psi_l^m(\tau_i;\mu,s)=P_l^m(\mu)+\frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\int_0^1 S_i^{(m)}(\tau_i;\mu,\mu',s)P_l^m(\mu')\,\frac{d\mu'}{\mu'} \qquad (50)$$

and

$$\phi_l^m(\tau_i;\mu,s)=P_l^m(\mu)\exp\left(-\frac{\tau_i Q}{\mu}\right)+\frac{1}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 T_i^{(m)}(\tau_i;\mu,\mu',s)P_l^{(m)}(\mu')\,\frac{d\mu'}{\mu'}\,; \qquad (51)$$

where

$$Q=1+\frac{s}{c}\quad\text{and}\quad i=1,2\,. \qquad (52)$$

If we use the above representations and again if we use Equations (34) and (35) we can write the scattering and transmission function in each layer as

$$S_i(\tau_i; \mu, \phi; \mu_0, \phi_0, s) = \sum_{m=0}^{N} S_i^{(m)}(\tau_i; \mu, \mu_0, s) \cos m(\phi_0 - \phi); \tag{53}$$

$$T_i(\tau_i; \mu, \phi; \mu_0, \phi_0, s) = \sum_{m=0}^{N} T_i^{(m)}(\tau_i; \mu, \mu_0, s) \cos m(\phi_0 - \phi)$$

$$(i = 1, 2). \tag{54}$$

In what follows we inquire into how represent the scattering and transmission functions in the whole atmosphere. If we follow Tsujita, we introduce diffuse radiation intensities $I_1(\tau_i; \mu, \phi; \mu_0, \phi_0, s)$ and $I_2(\tau_i, \mu, \phi; \mu_0, \phi_0; s)$ which leave the upper and lower layers in the direction (μ, ϕ) with respect to the boundary between the two layes, where (μ_0, ϕ_0) denotes the direction of the incident radiation at the upper surface $\tau = 0$

$$I_1(\tau_1; \mu, \phi; \mu_0, \phi_0, s) \quad \text{and} \quad I_2(\tau_1; \mu, \phi; \mu_0, \phi_0, s)$$

must satisfy the conditions

$$I_1(\tau_1, \mu, \phi; \mu_0, \phi_0, s) = 0 \quad \text{for} \quad 0 < \mu < 1, \tag{55}$$

$$I_2(\tau_1; \mu, \phi; \mu_0, \phi_0, s) = 0 \quad \text{for} \quad -1 < \mu < 0. \tag{56}$$

Then from the principle of invariances (A)–(B) we have after the Laplace transform with respect to time variable

$$I_2^{(m)}(\tau_1; \mu, \mu_0, s) = F\mu^{-1} S_2^{(m)}(\tau_2; \mu, \mu_0, s) \exp\left(-\frac{Q\tau_1}{\mu_0}\right) +$$

$$+ \frac{1}{2(2\delta - \delta_{0,m})\mu} \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s) I_1^{(m)}(\tau_1; \mu', \mu_0, s)\, d\mu'\, d\phi', \tag{57}$$

$$I_1^{(m)}(\tau_1; \mu, \mu_0, s) = F\mu^{-1} T_1^{(m)}(\tau_1; \mu, \mu_0, s) + \frac{1}{2(2 - \delta_{0,m})\mu} \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1, \mu, \mu', s) I_2^{(m)}(\tau_1, \mu', \mu_0, s)\, d\mu'\, d\phi'. \tag{58}$$

From (C)–(D),

$$F\mu^{-1} S(\tau_0; \mu, \phi; \mu_0, \phi_0, s) = F\mu^{-1} S_1(\tau_1; \mu, \phi; \mu_0, \phi_0, s) +$$

$$+ I_2(\tau; \mu, \phi; \mu_0, \phi_0, s) \exp\left(-\frac{\tau_1 \phi}{\mu}\right) + \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T_1 \times$$

$$\times (\tau_1; \mu, \phi; \mu', \phi, s) I_2(\tau_1; \mu', \phi'; \mu_0, \phi_0, s)\, d\mu'\, d\phi' \tag{59}$$

and

$$F\mu^{-1}T(\tau_0;\mu,\phi;\mu_0,\phi_0,s) = F\mu^{-1}T_2(\tau_2;\mu,\phi;\mu_0,\phi_0,s)\times$$

$$\times\exp\left(-\frac{\tau_1 Q}{\mu_0}\right) + I_1(\tau_1;\mu,\phi;\mu_0,\phi_0,s)\exp\left(-\frac{\tau_2 Q}{\mu}\right)+$$

$$+\frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T_2(\tau_2;\mu,\phi;\mu',\phi',s)\times$$

$$\times I_1(\tau_1;\mu',\phi';\mu_0,\phi_0,s)\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,; \tag{60}$$

where τ_0, τ_1, and τ_2 are the optical thickness of the whole atmosphere, the upper and the lower layer, respectively. Furthermore, we assume that $I_i(\tau_1,\mu,\phi,\mu',\phi',s)$ can be expanded in the form

$$I_i(\tau_1;\mu,\phi;\mu',\phi',s) = \sum_{m=0}^{N} I_i^{(m)}(\tau_1;\mu,\mu',s)\cos m(\phi'-\phi)\,,$$

$$(i=1,2)\,. \tag{61}$$

If we substitute this expansion in Equations (58) and (57) and taking account of Equations (53) and (54) and allowing for

$$\int_0^{2\pi}\cos m(\phi''-\phi)\cos n(\phi'-\phi'')\,\mathrm{d}\phi'' = \delta_{m,n}\pi\cos m(\phi'-\phi)\,(m\neq 0, n\neq 0) =$$

$$= 2\pi \qquad (m=n=0)\,, \tag{62}$$

we obtain

$$I_1^{(m)}(\tau_1;\mu,\mu_0,s) = F\mu^{-1}T_1^{(m)}(\tau_1;\mu,\mu_0,s)+$$

$$+\frac{1}{2(2-\delta_{0,m})\mu}\int_0^1 S_1^{(m)}(\tau_1;\mu,\mu',s)I_2^{(m)}(\tau_1;\mu',\mu_0,s)\,\mathrm{d}\mu\,, \tag{63}$$

$$I_2^{(m)}(\tau_1;\mu,\mu_0,s) = F\mu^{-1}S_2^{(m)}(\tau_1;\mu;\mu_0,s)\exp\left(-\frac{\tau_1 Q}{\mu_0}\right)+$$

$$+\frac{1}{2(2-\delta_{0,m})}\int_0^1 S_2^{(m)}(\tau_2;\mu,\mu',s)I_1^{(m)}(\tau;\mu',\mu_0,s)\,\mathrm{d}\mu'\,. \tag{64}$$

3.2. Auxiliary Functions and Their Functional Relations

Let us now consider some auxiliary functions in terms of which $I_1(\tau_1;\mu,\phi;\mu_0,\phi_0,s)$ and $I_2(\tau_1;\mu,\phi;\mu_0,\phi_0,s)$ are formed. If we assume that they depend on only one argument, we seek functional relations satisfied by them and then solve the system of equations. For convenience, we put

$$I_1^{(m)}(\tau_1, \mu, \mu_0, s) = F \frac{\mu_0}{\mu - \mu_0} \sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu, \mu_0, s), \tag{65}$$

$$I_2^{(m)}(\tau_1 ; \mu, \mu_0, s) = F \frac{\mu_0}{\mu + \mu_0} \sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu, \mu_0, s). \tag{66}$$

If we insert Equations (65), (66), (48), and (49) into Equations (63) and (64) and rearrange them approximately, we have

$$\begin{aligned}\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu, \mu_0, s) &= (2 - \delta_{0,m}) \sum_{l=m}^{N} \frac{w_{1,l}^{(m)}}{Q} \phi_l^{(m)}(\tau_1, \mu, s) \times \\ &\times \psi_l^m(\tau_1, \mu_0, s) - \psi_l^m(\tau_1, \mu, s)\phi_l^m(\tau_1, \mu_0, s) + \\ &+ \frac{1}{2} \int_0^1 \left\{ \sum_{l=m}^{N} (-1)^{l+m} \frac{w_{1,l}^{(m)}}{Q} [\psi_1^m(\tau_1, \mu, s)\psi_1^m(\tau_1, \mu', s) - \right. \\ &\left. - \phi_l^m(\tau_1, \mu, s)\phi_1^m(\tau_1 ; \mu', s)] \right\} \left[\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s) \right] \times \\ &\times \left[\frac{\mu}{\mu + \mu'} - \frac{\mu_0}{\mu' + \mu_0} \right] d\mu',\end{aligned} \tag{67}$$

$$\begin{aligned}\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu, \mu_0, s) &= (2 - \delta_{0,m}) \sum_{l=m}^{N} (-1)^{l+m} \frac{w_{2,l}^{(m)}}{Q} \times \\ &\times [\psi_l^m(\tau_2, \mu, s)\psi_l^m(\tau_2, \mu_0, s) - \phi_l^m(\tau_2, \mu, s)\phi_l^m(\tau_2, \mu_0, s)] \times \\ &\times \exp\left(- \frac{\tau_1 Q}{\mu_0} \right) + \frac{1}{2} \int_0^1 \left\{ \sum_{l=m}^{N} (-1)^{l+m} \frac{w_{2,l}^{(m)}}{Q} \times \right. \\ &\left. \times [\psi_l^m(\tau_2, \mu, s)\psi_l^m(\tau_2, \mu', s) - \phi_l^m(\tau_2, \mu, s)\phi_l^m(\tau_2, \mu', s)] \right\} \times \\ &\times \left[\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu, \mu_0, s) \right] \left[\frac{\mu}{\mu + \mu'} + \frac{\mu}{\mu' - \mu_0} \right] d\mu',\end{aligned} \tag{68}$$

we rewrite Equation (67) as

$$\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu, \mu_0, s) = \sum_{l=m}^{N} \frac{w_{1,l}^{(m)}}{Q} \phi_l^m(\tau_1, \mu, s) \times$$

$$\times \left[(2 - \delta_{0,m})\psi_l^m(\tau_1, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \int_0^1 \phi_l^m(\tau_1, \mu', s) \times \right.$$

$$\times \frac{\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s)}{\mu' + \mu_0} \, d\mu' \Bigg] - \sum_{l=m}^{N} \frac{w_l^{(m)}}{Q} \psi_l^m(\tau_1, \mu, s) \times$$

$$\times \Bigg[(2 - \delta_{0,m}) \phi_l^m(\tau, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \int_0^1 \psi_l^m(\tau_1, \mu', s) \times$$

$$\times \frac{\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s)}{\mu' + \mu_0} \, d\mu' \Bigg] + \frac{\mu}{2} \sum_{l=m}^{N} (-1)^{l+m} \frac{w_{1,l}^{(m)}}{Q} \times$$

$$\times \left[\int_0^1 \frac{\psi_l^m(\tau_1, \mu, s) \psi_l^m(\tau_1, \mu', s) - \phi_l^m(\tau_1, \mu, s) \phi_l^m(\tau_1, \mu', s)}{\mu + \mu'} \right] \times$$

$$\times \left[\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s) \, d\mu' \right]. \tag{69}$$

If we take account of Equation (48), we write the third term of the right-hand side of the above equation as

$$\frac{1}{2(2 - \delta_{0,m})} \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s) \left[\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s) \right] \frac{d\mu'}{\mu'} . \tag{70}$$

Then we put

$$\alpha_{1,l}^{(m)}(\mu_0, s) = (2 - \delta_{0,m}) \psi_l^m(\tau_1, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \times$$

$$\times \int_0^1 \phi_l^m(\tau_1, \mu', s) \frac{\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^m(\mu', \mu_0, s)}{\mu + \mu'} \, d\mu' , \tag{71}$$

$$\alpha_{2,l}^{(m)}(\mu_0, s) = (2 - \delta_{0,m}) \phi_l^m(\tau_1, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \times$$

$$\times \int_0^1 \psi_l^m(\tau_1, \mu', s) \frac{\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^m(\mu', \mu_0, s)}{\mu' + \mu_0} \, d\mu' . \tag{72}$$

If we make use of Equations (70), (71), (72) and rewrite Equation (69) once more, we have

$$A_l^{(m)}(\mu,\mu_0,s) = \alpha_{1,l}^{(m)}(\mu_0,s)\phi_l^m(\tau_1,\mu,s) - \alpha_{2,l}^{(m)}(\mu_0,s)\times$$

$$\times\ \psi_l^m(\tau_1,\mu,s) + \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{2,l}^{(m)}}{w_{1,l}^{(m)}}\,Q\right)\times$$

$$\times\int_0^1 S_1^{(m)}(\tau_1,\mu,\mu',s)B_l^{(m)}(\mu',\mu_0,s)\,\frac{d\mu'}{\mu'}\,. \tag{73}$$

On the other hand, by rewriting Equation (68), we have

$$\sum_{l=m}^{N} w_{2,l}^{(m)}B_l^{(m)}(\mu,\mu_0,s) = \sum_{l=m}^{N}\frac{w_{2,l}^{(m)}}{Q}(-1)^{l+m}\psi_l^m(\tau_2,\mu,s)\times$$

$$\times\left[(2-\delta_{0,m})\psi_l^m(\tau_2,\mu_0,s)\exp\left(-\frac{\tau_1 Q}{\mu_0}\right)+\frac{\mu_0}{2}\times\right.$$

$$\left.\times\int_0^1 \psi_1^m(\tau_2,\mu',s)\,\frac{\sum_{l=m}^{N} w_{1,l}^{(m)}A_l^{(m)}(\mu',\mu_0,s)}{\mu'-\mu_0}\,d\mu'\right]-$$

$$-\sum_{l=m}^{N}\frac{w_{2,l}^{(m)}}{Q}(-1)^{l+m}\phi_l^m(\tau_2,\mu,s)\left[(2-\delta_{0,m})\phi_l^m(\tau_2,\mu,s)\times\right.$$

$$\times\exp\left(-\frac{\tau_1 Q}{\mu_0}\right)+\frac{\mu_0}{2}\int_0^1 \phi_1^m(\tau_2,\mu',s)\times$$

$$\left.\times\,\frac{\sum_{l=m}^{N} w_{1,l}^{(m)}A_l^{(m)}(\mu',\mu_0,s)}{\mu'-\mu_0}\,d\mu'\right]+\frac{1}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 S_2^{(m)}(\tau_2;\mu,\mu',s)\left[\sum_{l=m}^{N} w_{1,l}^{(m)}B_1^{(m)}(\mu',\mu_0,s)\right]\frac{d\mu'}{\mu'}\,. \tag{74}$$

Then we write $\alpha_{3,l}^{(m)}(\mu_0, s)$ and $\alpha_{4,l}^{(m)}(\mu_0, s)$ as

$$\alpha_{3,l}^{(m)}(\mu_0, s) = (2 - \delta_{0,m})\psi_l^m(\tau_2; \mu_0, s) \exp\left(-\frac{\tau_1 Q}{\mu_0}\right) + \\ + \frac{\mu_0}{2}\int_0^1 \psi_l^m(\tau_2, \mu', s)\,\frac{\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu', \mu_0, s)}{\mu' - \mu_0}\,\mathrm{d}\mu' , \tag{75}$$

$$\alpha_{4,l}^{(m)}(\mu_0, s) = (2 - \delta_{0,m})\phi_l^m(\tau_2, \mu_0, s) \exp\left(-\frac{\tau_1 Q}{\mu_0}\right) + \\ + \frac{\mu_0}{2}\int_0^1 \phi_l^m(\tau_2, \mu', s)\,\frac{\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu', \mu_0, s)}{\mu' - \mu_0}\,\mathrm{d}\mu' , \tag{76}$$

If we make use of Equations (75) and (76) and rewrite Equation (74) once more, we have

$$B_l^{(m)}(\mu, \mu_0, s) = \alpha_{3,l}^{(m)}(\mu_0, s)\psi_l^m(\tau_2, \mu, s) - \alpha_{4,l}^{(m)} \times \\ \times (\mu_0, s)\phi_l^m(\tau_2, \mu, s) + \frac{1}{2(2 - \delta_{0,m})}\left(\frac{w_{1,l}^{(m)}}{w_{2,l}^{(m)}}\right) Q \times \\ \times \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s) A_l^m(\mu', \mu_0, s)\,\frac{\mathrm{d}\mu'}{\mu'} . \tag{77}$$

From Equations (73) and (77) we get

$$A_l^{(m)}(\mu, \mu_0, s) = \alpha_{1,l}^{(m)}(\mu_0, s)\phi_l^m(\tau_1, \mu, s) - \alpha_{2,l}^{(m)}(\mu_0, s) \times \\ \times \psi_l^m(\tau_1, \mu, s) + \alpha_{3,l}^{(m)}(\mu_0, s)\,\frac{1}{2(2 - \delta_{0,m})}\left(\frac{w_{2,l}^{(m)}}{w_{1,l}^{(m)}}\right) Q \times \\ \times \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\phi_l^m(\tau_2, \mu', s) - \alpha_{4,l}^{(m)}(\mu_0, s)\,\frac{1}{2(2 - \delta_{0,m})} \times \\ \times \left(\frac{w_{2,l}^{(m)}}{w_{1,l}^{(m)}}\right) Q \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu_0, s)\phi_1^m(\tau_2, \mu', s)\,\frac{\mathrm{d}\mu'}{\mu'} + \\ + \frac{1}{4(2 - \delta_{0,m})^2}\int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\int_0^1 S_2^{(m)}(\tau_1; \mu, \mu'', s) A_l^{(m)} \times \\ \times (\mu'', \mu_0, s)\,\frac{\mathrm{d}\mu''}{\mu''}\,\frac{\mathrm{d}\mu'}{\mu'} , \tag{78}$$

and

$$B_l^{(m)}(\mu, \mu_0, s) = \alpha_{1,l}^{(m)}(\mu_0, s)\,\frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{1,l}^{(m)}}{w_{2,l}^{(m)}}\right) Q \times$$

$$\times \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\phi_l^m(\tau_1, \mu', s)\,\frac{\mathrm{d}\mu'}{\mu'} - \alpha_{2,l}^{(m)}(\mu_0, s) \times$$

$$\times \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{1,l}^{(m)}}{w_{2,l}^{(m)}}\right) Q \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\psi_l^m(\tau_1, \mu', s) \times$$

$$\times \frac{\mathrm{d}\mu'}{\mu'}\;\alpha_{3,l}^{(m)}(\mu_0, s)\psi_l^m(\tau_2, \mu, s) - \alpha_{4,l}^{(m)}(\mu_0, s) \times$$

$$\times \phi_l^m(\tau_2, \mu', s) + \frac{1}{4(2-\delta_{0,m})^2}\int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s) \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu'', s) B_l^{(m)}(\mu'', \mu_0, s)\,\frac{\mathrm{d}\mu''}{\mu''}\,\frac{\mathrm{d}\mu'}{\mu'}\,. \tag{79}$$

Again, from Equations (78) and (79), if we use Equations (73) and (77) we get

$$A_l^{(m)}(\mu, \mu_0, s) = \alpha_{1,l}^{(m)}(\mu_0, s)\beta_{1,l}^{(m)}(\mu, s) - \alpha_{2,l}^{(m)}(\mu_0, s) \times$$

$$\times \beta_{2,l}^{(m)}(\mu, s) + \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{2,l}^{(m)}}{w_{1,l}^{(m)}}\right) Q\alpha_{3,l}^{(m)}(\mu_0, s) \times$$

$$\times \beta_{3,l}^{(m)}(\mu, s) - \alpha_{4,l}^{(m)}(\mu_0, s)\beta_{4,l}^{(m)}(\mu, s)\,, \tag{80}$$

$$B_l^{(m)}(\mu, \mu_0, s) = \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{1,l}^{(m)}}{w_{2,l}^{(m)}}\right) Q\alpha_{1,l}^{(m)}(\mu_0, s) \times$$

$$\times \gamma_{1,l}^{(m)}(\mu, s) - \alpha_{2,l}^{(m)}(\mu_0, s)\gamma_{2,l}^{(m)}(\mu, s) + \alpha_{3,l}^{(m)}(\mu_0, s) \times$$

$$\times \gamma_{3,l}^{(m)}(\mu, s) - \alpha_{4,l}^{(m)}(\mu_0, s)\gamma_{4,l}^{(m)}(\mu, s)\,, \tag{81}$$

$$B_{1,l}^{(m)}(\mu, s) = \phi_l^m(\tau_1, \mu, s) + \frac{1}{4(2-\delta_{0,m})^2} \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\gamma_{1,l}^{(m)}(\mu', s)\,\frac{\mathrm{d}\mu'}{\mu'}\,, \tag{82}$$

$$\beta_{2,l}^{(m)}(\mu, s) = \psi_l^{(m)}(\tau_1, \mu, s) + \frac{1}{4(2-\delta_{0,m})^2} \times$$
$$\times \int_0^1 S_1^{(m)}(\tau_1, \mu, \mu', s)\gamma_{2,l}^{(m)}(\mu', s)\frac{d\mu'}{\mu'}, \tag{83}$$

$$\beta_{3,l}^{(m)}(\mu, s) = \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\gamma_{3,l}^{(m)}(\mu', s)\frac{d\mu'}{\mu'}, \tag{84}$$

$$\beta_{4,l}^{(m)}(\mu, s) = \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\gamma_{4,1}^{(m)}(\mu', s)\frac{d\mu'}{\mu'}, \tag{85}$$

$$\gamma_{1,l}^{(m)}(\mu, s) = \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\beta_{1,l}^{(m)}(\mu', s)\frac{d\mu'}{\mu'}, \tag{86}$$

$$\gamma_{2,l}^{(m)}(\mu, s) = \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\beta_{2,l}^{(m)}(\mu', s)\frac{d\mu'}{\mu'}, \tag{87}$$

$$\gamma_{3,l}^{(m)}(\mu, s) = \psi_l^{(m)}(\tau_2, \mu, s) + \frac{1}{4(2-\delta_{0,m})^2} \times$$
$$\times \int_0^1 S_2^{(m)}(\tau_2, \mu, \mu', s)\beta_{3,l}^{(m)}(\mu', s)\frac{d\mu'}{\mu'}, \tag{88}$$

$$\gamma_{4,l}^{(m)}(\mu, s) = \phi_l^{m}(\tau_2; \mu, s) + \frac{1}{4(2-\delta_{0,m})^2} \times$$
$$\times \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\beta_{4,l}^{(m)}(\mu', s)\frac{d\mu'}{\mu'}. \tag{89}$$

If we combine Equation (82) with Equation (86), Equation (83) with Equation (87), Equation (84) with Equation (88), and Equation (85) with Equation (89). We can determine $\beta_{i,l}^{(m)}(\mu, s)$ and $\gamma_{i,l}^{(m)}(\mu, s)$ $(i = 1, 2, 3, 4)$ numerically. From Equations (71), (72), (75), (76), (80), and (81) $\alpha_{i,l}^{(m)}(\mu_0, s)$, $A_l^{(m)}(\mu, \mu_0, s)$, and $B_l^{(m)}(\mu, \mu_0, s)$ can be calculated and then from Equations (65) and (66), $I_1^{(m)}(\tau_1, \mu, \mu_0, s)$ and $I_2^{(m)}(\tau_2, \mu, \mu_0, s)$ are determined. Thus we obtained $S(\tau_0, \mu, \phi, \mu_0, \phi_0, s)$ and $T(\tau_0, \mu, \phi; \mu_0, \phi_0, s)$ from Equations (59) and (60).

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AN EXACT SOLUTION OF THE EQUATION OF TRANSFER FOR COHERENT SCATTERING IN AN EXPONENTIAL ATMOSPHERE

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Abstract. An exact solution of the transfer equation for coherent scattering in stellar atmospheres with Planck's function as a nonlinear function of optical depth, of the form

$$B_\nu(T) = b_0 + b_1\, e^{-\beta\tau},$$

is obtained by the method of the Laplace transform and Wiener–Hopf technique.

1. Introduction

Chandrasekhar (1960) applied the method of discrete ordinates to solve the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz., $B_\nu(T) = b_0 + b_1\tau$. The equation of transfer for coherent scattering has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient, is constant) and by Strömgren's method (when η_ν has small but arbitrary variation with optical depth) (see Woolley and Stibbs, 1953). Dasgupta (1977b) applied the method of the Laplace transform and Wiener–Hopf technique to find an exact solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions, viz.,

$$B_\nu(T) = b_0 + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau),$$

by use of a new representation of the H-function obtained by Dasgupta (1977a). Recently, Karanjai and Deb (1990) solved the equation of transfer for coherent isotropic scattering in an exponential atmosphere by Eddington's method.

In this paper, we have obtained an exact solution of the equation of transfer for coherent isotropic scattering by the method of the Laplace transform and Wiener–Hopf technique in an exponential atmosphere (Degl'Innocenti, 1979; Karanjai and Karanjai, 1985; and Karanjai and Deb, 1990), where

$$B_\nu(T) = b_0 + b_1\, e^{-\beta\tau},$$

where b_0, b_1, and β are positive constants.

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2. Equation of Transfer

The equation of transfer considered here is of the form

$$dI_\nu(\tau, \mu)/d\tau = I_\nu(\tau, \mu) - wJ_\nu(\tau) - (1 - w)B_\nu(T), \tag{1}$$

where we have taken Planck's function $B_\nu(T)$ as

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}, \tag{2}$$

$$0 < (1 - \varepsilon_\nu)/(1 + \eta_\nu) = w < 1, \tag{2a}$$

$$l_\nu/k = \eta_\nu, \qquad 0 < \varepsilon_\nu < 1; \tag{2b}$$

l_ν, k being the line and continuous absorption coefficient; τ, the optical depth in the total absorption coefficient; ε_ν, the collision constant; and $I_\nu(\tau, \mu)$ is the intensity in the frequency, in the direction $\cos^{-1}\mu$, $J_\nu(\tau)$ is the average intensity

$$J_\nu(\tau) = (1/2) \int_{-1}^{+1} I_\nu(\tau, \mu)\, d\mu. \tag{2c}$$

For the solution of Equation (1) we have the boundary conditions

(i) $I_\nu(0, -\mu) = 0, \quad 0 < \mu < 1,$

(*ii*) $I_\nu(\tau, \mu)\, e^{-\tau/\mu} \to 0 \quad \text{as} \quad \tau \to \infty.$

3. Solution for Emergent Intensity

The Laplace transform of $F(\tau)$ is denoted by $F^*(s)$, where $F^*(s)$ is defined by

$$F^*(s) = s \int_0^\infty \exp(-s\tau)F(\tau)\, d\tau, \qquad Rl\, s > 0. \tag{3}$$

The formal solution of Equation (1) (Dasgupta, 1977b) is

$$I_\nu(0, \mu) = wJ_\nu^*(1/\mu) + (1 - w)B_\nu^*(1/\mu). \tag{4}$$

The Laplace transformation of Equation (1) with necessary re-arangement (Dasgupta, 1977b) yields

$$T(z)I_\nu(0, z) = wG_\nu(z) + (1 - w)B_\nu^*(1/z), \tag{5}$$

where

$$T(z) = 1 - (w/2)z \log[(z + 1)/(z - 1)], \tag{6}$$

and

$$G_\nu(z) = (1/2) \int_0^1 xI_\nu(0, x)\, dx/(x - z). \tag{7}$$

$T(z)$ has its roots $\pm k$, real for $0 < w \leq 1$

$$k(>1) \to \infty \quad \text{as} \quad w \to 1 .$$

According to Dasgupta (1974) we have

$$H(z) \to H_0 + H_{-1}/z + \cdots \quad \text{as} \quad z \to \infty , \tag{8}$$

where

$$H_0 = (1 - w)^{-1/2} \tag{9}$$

and

$$H_{-1} = -(wH_0^2/2) \int_0^1 xH(x)\,\mathrm{d}x . \tag{10}$$

By the well-known relation (Busbridge, 1960)

$$1/T(z) = H(z)H(-z) \quad \text{on} \quad [-1, 1]^c , \tag{11}$$

we rewrite Equation (5) as

$$I_\nu(0, z)/H(z) = H(-z)\,[wG_\nu(z) + (1 - w)B_\nu^*(1/z)] . \tag{12}$$

If we use the Laplace transformation of Equation (2) by Equation (3) we have

$$B_\nu^*(s) = b_0 + sb_1/(s + \beta) . \tag{13}$$

For $s = z^{-1}$

$$B_\nu^*(1/z) = b_0 + b_1/(1 + \beta z) = (d_0 + d_1 z)/(1 + \beta z) \quad \text{(say)} , \tag{14}$$

where

$$d_1 = b_0\beta \quad \text{and} \quad d_0 = b_0 + b_1 .$$

If we insert Equation (14) in Equation (12) we have

$$I_\nu(0, z)/H(z) = H(-z)\,[wG_\nu(z) + (1 - w)\,(d_0 + d_1 z)/(1 + \beta z)] \tag{15}$$

which can be rewriten as

$$I_\nu(0, z)/H(z) = H(-z)\,[wG(z) + (1 - w)\,(d_0/z + d_1)/(1/z + \beta)] . \tag{16}$$

Now as $z \to \infty$, $G_\nu(z) \to 0(1/z)$, since we seek solution $I_\nu(0, z)$ regular for $\operatorname{Re} z > 0$ and continuous on $[0, 1]^c$ and since $H(z)$ is regular on $[-1, 0]^c/[-k]$, $-k$ is a simple pole of $H(z)$, $1/H(z)$ being regular on $[-1, 0]^c$.

We see that the left-hand side of Equation (16) is regular at least for $\operatorname{Re} z > 0$ except perhaps at ∞, and the right-hand side of Equation (16) is regular at on $[0, 1]^c$ except at ∞, both sides being bounded at the origin.

The right-hand side of Equation (16) is

$$C_0 \quad \text{as} \quad z \to \infty , \tag{17}$$

where

$$C_0 = H_0(1 - w)d_1/\beta . \tag{18}$$

Hence, by a modified Liouville's theorem both sides of Equation (16) can be equated to C_0, so that the left-hand side of (16) is

$$C_0 \quad \text{as} \quad z \to \infty , \tag{19}$$

the right-hand side of (16) is

$$C_0 \quad \text{as} \quad z \to \infty . \tag{20}$$

Equation (16) can be put in the form

$$I(0, z)/H(z) = C_0 = H_0(1 - w)d_1\beta . \tag{21}$$

If we use the relationship $d_1 = b_0\beta$ in (21) we get when z

$$I(0, z) = H(z)\,(1 - w)H_0b_0 . \tag{22}$$

Since we have $H_0 = (1 - w)^{-1/2}$.

Hence, from Equation (22) we get

$$I(0, z) = H(z)\,(1 - w)^{1/2}b_0 , \tag{23}$$

which is the same as deducted by Karanjai and Karanjai (1985).

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SOLUTION OF THE EQUATION OF TRANSFER FOR COHERENT SCATTERING IN AN EXPONENTIAL ATMOSPHERE BY BUSBRIDGE'S METHOD

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Abstract. A solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a nonlinear function of optical depth, viz.

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$$

is obtained by the method developed by Busbridge (1953).

1. Introduction

Chandrasekhar (1960) applied the method of discrete ordinates to solve the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz.,

$$B_\nu(T) = b_0 + b_1 \tau .$$

The equation of transfer for coherent scattering has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient is constant) and by Strömgren's method (when η_ν has small but arbitrary variation with optical depth; see Woolley and Stibbs, 1953). Busbridge (1953) solved the same problem by a new method using Chandrasekhar's ideas. Dasgupta (1977b) applied the method of Laplace transform and Wiener–Hopf technique to find an exact solution of the transfer equation for coherent scattering in the stellar atmosphere with Planck's function as a sum of elementary functions, viz.,

$$B_\nu(T) = b_0 + b_1 \tau + \sum_{r=2}^{n} b_r E_r(\tau) ,$$

using a new representation of the H-function obtained by Dasgupta (1977a). Recently, Karanjai and Deb (1991a, b) solved the equation of transfer for coherent isotropic scattering in an exponential atmosphere by Eddington's method and the method of Laplace transform and Wiener–Hopf technique. In this paper, we have obtained a solution of the equation of transfer for coherent scattering in an exponential atmosphere,

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i.e.,

$$B_\nu(T) = b_0 + b_1\, e^{-\beta\tau};$$

where b_0, b_1, and β are three positive constants, by the method used by Busbridge (1953).

2. Equation of Transfer

With the usual notation of transfer for the Milne–Eddington model can be written (Busbridge, 1953; Chandrasekhar, 1960) as

$$\mu \frac{\mathrm{d}I_\nu}{\rho\,\mathrm{d}z} = (k_\nu + \sigma_\nu)I_\nu - \tfrac{1}{2}\sigma_\nu \int_{-1}^{+1} I_\nu\,\mathrm{d}\mu' - k_\nu B_\nu(T)\,, \tag{1}$$

where z is the depth below the surface; k_ν, the continuous absorption coefficient; and σ_ν is the line-scattering coefficient. We assume that k_ν and σ_ν are independent of depth and we write

$$t = \int_0^z \rho(k_\nu + \sigma_\nu)\,\mathrm{d}z\,, \tag{2a}$$

$$\tau = \int_0^z \rho k_\nu\,\mathrm{d}z\,, \tag{2b}$$

$$\eta_\nu = \frac{\sigma_\nu}{k_\nu}, \qquad \lambda_\nu = \frac{1}{1+\eta_\nu} = \frac{k_\nu}{k_\nu + \sigma_\nu}. \tag{3}$$

Then

$$\tau = \lambda_\nu t$$

and

$$B_\nu(T) = b_0 + b_1\, e^{-\beta\tau} = b_0 + b_1\, e^{-\beta\lambda_\nu t}\,, \tag{4}$$

where $B_\nu(T)$ is the Planck's function.

Substituting into Equation (1), we get

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu(t,\mu) - \tfrac{1}{2}(1-\lambda_\nu) \int_{-1}^{+1} I_\nu(t,\mu')\,\mathrm{d}\mu' - \lambda_\nu(b_0 + b_1\, e^{-\beta\lambda_\nu t})\,. \tag{5}$$

Equation (5) has to be solved subject to the boundary conditions

$$I_\nu(0, -\mu') = 0\,, \quad (0 < \mu' < 1) \tag{6a}$$

and

$$I_\nu(t, \mu')\, e^{-t/\mu} \to 0 \quad \text{as} \quad t \to \infty\,. \tag{6b}$$

3. Solution for Emergent Intensity

For convenience we suppress the subscript ν to the various quantities and consider a particular solution of Equation (5), which does not satisfy Equation (6a) in the form (Busbridge, 1953)

$$I(t, \mu) = b_0 + \frac{T_1 b_1}{1 + \beta\lambda\mu}\, e^{-\beta\lambda t}\,, \tag{7}$$

where

$$T_1 = \frac{\lambda}{1 - \dfrac{1}{2\lambda\beta}(1 - \lambda)\log\dfrac{1 + \lambda\beta}{1 - \lambda\beta}} \tag{8}$$

as readily verified by substitution. We, therefore, write (cf. Busbridge, 1953)

$$I(t, \mu) = b_0 + \frac{T_1 b_1}{1 + \beta\lambda\mu}\, e^{-\beta\lambda t} + I^*(t, \mu)\,. \tag{9}$$

Then $I^*(t, \mu)$ satisfied the integro-differential equation

$$\mu\frac{\mathrm{d}I^*(t, \mu)}{\mathrm{d}t} = I^*(t, \mu) - \tfrac{1}{2}(1 - \lambda)\int_{-1}^{+1} I^*(t, \mu')\,\mathrm{d}\mu'\,, \tag{10}$$

together with the boundary conditions

$$I^*(0, -\mu') = -\frac{T_1 b_1}{1 - \beta\lambda\mu} - b_0 \quad (0 < \mu' < 1) \tag{11a}$$

and

$$I^*(t, \mu)\, e^{-t/\mu} \to 0 \quad \text{as} \quad t \to \infty\,, \tag{11b}$$

we require the emergent intensity $I^*(0, \mu)$. This is the sum of $I_1^*(0, \mu)$, where $I_1^*(t, \mu)$ is the solution of Equation (10).

Subject to the boundary condition

$$I_1^*(0, -\mu') \equiv 0\,, \quad (0 < \mu' < 1) \tag{12}$$

and $I_2^*(0, \mu)$ which is the diffusely reflected intensity corresponding to the incident intensity given by Equation (11). It can be shown that unless $\lambda_\nu = 0$ (which is not so),

$$I_1^*(t, \mu) = 0\,. \tag{13}$$

Hence,

$$I^*(0, \mu) = I_2^*(0, \mu) = \frac{1}{2\mu} \int_0^1 S(\mu, \mu') \left(\frac{T_1 b_1}{\beta\lambda\mu - 1} - b_0 \right) d\mu' , \tag{14}$$

where (cf. Chandrasekhar, 1960)

$$S(\mu, \mu') = (1 - \lambda) \frac{\mu\mu'}{\mu + \mu'} H(\mu)H(\mu') \tag{15}$$

and $H(\mu)$ is the solution of

$$H(\mu) = 1 + \tfrac{1}{2}(1 - \lambda)\mu H(\mu) \int_0^1 \frac{H(\mu')}{\mu + \mu'} d\mu' . \tag{16}$$

From Equations (14) and (15), we have

$$I(0, \mu) = \tfrac{1}{2}(1 - \lambda)H(\mu) \int_0^1 \left(\frac{T_1 b_1}{\beta\lambda\mu - 1} - b_0 \right) \frac{\mu'\mu}{\mu' + \mu} H(\mu') d\mu' =$$

$$= \tfrac{1}{2}(1 - \lambda)H(\mu)T_1 b_1 \int_0^1 \frac{\mu' H(\mu') d\mu'}{(\mu' + \mu)(\beta\lambda\mu' - 1)} -$$

$$- \tfrac{1}{2}(1 - \lambda)H(\mu)b_0 \int_0^1 \frac{\mu'}{\mu' + \mu} H(\mu') d\mu' =$$

$$= \tfrac{1}{2}(1 - \lambda)H(\mu) \frac{T_1 b_1}{\beta\lambda} \int_0^1 \frac{H(\mu')}{\mu + \mu'} d\mu' +$$

$$+ \tfrac{1}{2}(1 - \lambda)H(\mu) \frac{T_1 b_1}{\beta\lambda} \int_0^1 \frac{H(\mu') d\mu'}{(\mu' + \mu)(\beta\lambda\mu - 1)} -$$

$$- \tfrac{1}{2}(1 - \lambda)H(\mu)b_0 \int_0^1 \left(1 - \frac{\mu}{\mu + \mu'} \right) H(\mu') d\mu' . \tag{17}$$

After some rearrangement and with Equation (16), this gives

$$I^*(0, \mu) = \frac{H(\mu)T_1 b_1}{1 + \beta\lambda\mu} \frac{1}{H(-1/\beta\lambda)} - \frac{T_1 b_1}{1 + \beta\lambda\mu} + \\ + (H(\mu) - 1)b_0 - \tfrac{1}{2}(1 - \lambda)H(\mu)b_0\alpha_0 \tag{18}$$

where

$$\alpha_n = \int_0^1 H(\mu)\mu^n \, d\mu . \tag{19}$$

Following Chandrasekhar (1960)

$$1 - \tfrac{1}{2}(1 - \lambda)\alpha_0 = \lambda^{1/2} , \tag{20}$$

we have from Equations (9) and (18)

$$I(0, \mu) = H(\mu)\lambda^{1/2}b_0 + \frac{H(\mu)T_1 b_1}{1 + \beta\lambda\mu} \frac{1}{H(-1/\beta\lambda)} , \tag{21}$$

which represents our solution.

Appendix

We have to show that

$$I_1^*(t, \mu) = 0 . \tag{A.1}$$

For this, with the usual notation (cf. Chandrasekhar, 1960), we have

$$I_1^*(t, \mu) \simeq \frac{1}{2}(1 - \lambda) \sum_{\alpha=1}^{n} \{L_\alpha e^{-k_\alpha t}/(1 + \mu k_\alpha)\} , \tag{A.2}$$

where the constants L_α are determined by the equations

$$\sum_{\alpha=1}^{n} L_\alpha/(1 - \mu_i k_\alpha) = 0 , \quad (i = 1, 2, 3, \ldots, n) . \tag{A.3}$$

Since

$$\prod_{\alpha=1}^{n} (1 - \mu k_\alpha) \sum_{\alpha=1}^{n} L_\alpha/(1 - \mu k_\alpha)$$

is a polynomial in μ of degree $(n - 1)$ with n distinct zero, it is identically zero.

Hence, every $L_\alpha = 0$, and in the limit, as $n \to \infty$

$$I_1^*(t, \mu) = 0 .$$

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SOLUTION OF THE EQUATION OF TRANSFER FOR COHERENT SCATTERING IN AN EXPONENTIAL ATMOSPHERE BY THE METHOD OF DISCRETE ORDINATES

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Abstract. A solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a nonlinear function of optical depth, viz.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$$

is obtained by the method of discrete ordinates originally due to Chandrasekhar.

1. Introduction

Busbridge (1953) solved the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz. $B_\nu(T) = b_0 + b_1\tau$ by a modified principle of invariance method. Chandrasekhar (1960) solved the same problem by the method of discrete ordinates. The same problem has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient is constant) and by Strömgren's method (when η_ν, has small but arbitrary variation with optical depth) (see Woolley and Stibbs, 1953).

Dasgupta (1977b) applied the method of Laplace transform and Wiener–Hopf technique to find an exact solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions, viz.,

$$B_\nu(T) = b_0 + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau) ,$$

using a new representation of the H-function obtained by Dasgupta (1977a). Recently, Karanjai and Deb (1991, 1992a) solved the equation of transfer for coherent isotropic scattering in an exponential atmosphere by Eddington's method and by the method of Laplace transform and Wiener–Hopf technique.

By use of a method developed by Busbridge (1953), Karanjai and Deb (1992b) solved the same problem.

In this paper, we have obtained a solution of the equation of transfer for coherent isotropic scattering in an exponential atmosphere by the method of discrete ordinates, where $B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$ and b_0, b_1 and β are three positive constants.

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2. Equation of Transfer

The equation of transfer considered here is of the form

$$\mu \frac{dI_\nu}{\rho dz} = (k_\nu + \sigma_\nu) I_\nu - \tfrac{1}{2}\sigma_\nu \int_{-1}^{+1} I_\nu \, d\mu' - k_\nu B_\nu(T) \tag{1}$$

(Busbridge, 1953; and Chandrasekhar, 1960) where z is the depth below the surface; k_ν, the continuous absorption coefficient; and σ_ν, the line-scattering coefficient. We assume that k_ν and σ_ν are independent of depth and we write

$$t = \int_0^z \rho(k_\nu + \sigma_\nu)\, dz \ , \tag{2a}$$

$$\tau = \int_0^z \rho k_\nu \, dz \ , \tag{2b}$$

$$\eta_\nu = \sigma_\nu / k_\nu , \ \lambda_\nu = 1/(1 + \eta_\nu) = \frac{k_\nu}{k_\nu + \sigma_\nu} \ . \tag{3}$$

Then $\tau = \lambda_\nu t$ and

$$B_\nu(T) = b_0 + b_1 \, e^{-\beta\tau} \ , \tag{4a}$$

i.e.,

$$B_\nu(T) = b_0 + b_1 \, e^{-\beta\lambda_\nu t} \ . \tag{4b}$$

If we substitute in Equation (1) we get

$$\mu \ \frac{dI_\nu(t, \mu)}{dt} = I_\nu(t, \mu) - \tfrac{1}{2}(1 - \lambda_\nu) \int_{-1}^{+1} I_\nu(t, \mu') \, d\mu' - \lambda_\nu (b_0 + b_1 \, e^{-\beta\lambda_\nu t}) \ . \tag{5}$$

Equation (5) has to be solved subject to the boundary conditions

$$I_\nu(0, -\mu) = 0, \quad (0 < \mu \leq 1) \tag{6a}$$

and

$$I_\nu(t, \mu) \, e^{-t/\mu} \to 0 \quad \text{as} \quad t \to \infty, \quad |\mu| \leq 1 \ . \tag{6b}$$

Now a particular solution of Equation (5), which does not satisfy Equation (6a) is

$$I_\nu(t, \mu) = b_0 + \frac{T_\nu b_1}{1 + \alpha_\nu \mu} \, e^{-\alpha_\nu t} \ , \tag{7}$$

where

$$T_\nu = \frac{\lambda_\nu}{1 - \frac{1}{2}(1 - \lambda_\nu)\log\dfrac{1 + \alpha_\nu}{1 - \alpha_\nu}} \tag{8a}$$

and

$$\alpha_\nu = \beta\lambda_\nu \tag{8b}$$

as readily verified by substitution.

If we follow Busbridge (1953) we write

$$I_\nu(t, \mu) = b_0 + b_1 \frac{T_\nu}{1 + \alpha_\nu\mu} e^{-\alpha_\nu t} + I_\nu^*(t, \mu) \ . \tag{9}$$

Then $I_\nu^*(t, \mu)$ satisfies the integro-differential equation

$$\mu \frac{dI_\nu^*(t, \mu)}{dt} = I_\nu^*(t, \mu) - \frac{1}{2}(1 - \lambda_\nu) \int_{-1}^{+1} I_\nu^*(t, \mu')\, d\mu' \ , \tag{10}$$

together with the boundary conditions

$$I_\nu^*(0, -\mu) = -b_1 \frac{T_\nu}{1 - \alpha_\nu\mu} - b_0 \tag{11a}$$

and

$$I_\nu^*(t, \mu)\, e^{-t/\mu} \to 0 \quad \text{as} \quad t \to \infty, \quad |\mu| \le 1 \ . \tag{11b}$$

3. Solution for Emergent Intensity

For convenience we suppress the subscript ν to the various quantities and in the nth approximation, we replace Equation (10) by the system of $2n$ linear equations

$$\mu_i \frac{dI_i^*}{dt} = I_i^* - \frac{1}{2}(1 - \lambda) \sum_j a_j I_j^* \ , \quad i = \pm 1, \pm 2, \ldots, \pm n \ ; \tag{12}$$

where the μ_i's ($i = \pm 1, \pm 2, \ldots, \pm n$ and $\mu_{-i} = -\mu_i$) are the zeros of the Legendre polynomial $P_{2n}(\mu)$. a_j's ($j = \pm 1, \ldots, \pm n$ and $a_{-j} = a_j$) are corresponding Gaussian weights. However, it is to be noted that there is no term with $j = 0$. For simplicity, in Equation (12) we write

$$I_i^* \quad \text{for} \quad I_i^*(t, \mu_i) \ . \tag{13}$$

The system of Equations (12) admits of integral of the form

$$I_i^* = g_i\, e^{-kt} \quad (i = \pm 1, \ldots, \pm n) \ , \tag{14}$$

where the g_i's and k are constants.

Now if we insert this form for I_i^* in Equation (12) we have

$$g_i |1 + \mu_i k| = \tfrac{1}{2}(1 - \lambda) \sum_j a_j g_j \ , \qquad (15)$$

$$\therefore g_i = (1 - \lambda) \frac{\text{constant}}{1 + \mu_i k} \ . \qquad (16)$$

If we insert for g_i from Equation (16) back into Equation (15) we obtain the characteristic equation in the form

$$1 = \tfrac{1}{2}(1 - \lambda) \sum_j \frac{a_j}{1 + \mu_j k} \ . \qquad (17)$$

If we remember that $a_j = a_{-j}$ and $\mu_{-j} = -\mu_j$ we can rewrite the characteristic equation in the form

$$1 = (1 - \lambda) \sum_{j=1}^{n} \frac{a_j}{1 - \mu_i^2 k^2} \ . \qquad (18)$$

This is the characteristic equation which gives the values of k. If $\lambda > 0$, the characteristic Equation (18) gives distinct non-zero roots which occur in pairs as $\pm k_r$ $(r = 1, 2, \ldots, n)$.

Therefore, Equation (12) admits the $2n$ independent integrals of the form

$$I_i^* = (1 - \lambda) \frac{\text{constant}}{1 \pm \mu_i k_r} e^{\pm k_r t} \ . \qquad (19)$$

According to Chandrasekhar (1960), the solutions (14) satisfying our requirements that the solutions are bounded by

$$I_i^* = (1 - \lambda)\, b_1 \sum_{r=1}^{n} \frac{L_r e^{-k_r t}}{1 + k_r \mu_i} \ , \qquad (20)$$

together with the boundary condition

$$I_{-i}^* = -\frac{b_1 T}{1 - \alpha \mu_{-i}} - b_0 \quad \text{at} \quad t = 0 \ . \qquad (21)$$

4. The Elemination of the Constants and the Expression of the Law of Diffuse Reflection in Closed Form

The boundary condition and the emergent intensity can be expressed in the form

$$S(\mu_i) = 0 \quad (i = 1, 2, \ldots, n) \qquad (22)$$

and

$$I^*(0, \mu) = (1 - \lambda) b_1 \left[S(-\mu) - \frac{T/(1 - \lambda)}{1 + \alpha\mu} - \frac{b_0}{(1 - \lambda) b_1} \right] , \qquad (23)$$

where

$$S(\mu) = \sum_{r=1}^{n} \frac{L_r}{1 - k_r\mu} + \frac{T/(1-\lambda)}{1 - \alpha\mu} + \frac{b_0}{(1-\lambda)b_1} . \tag{24}$$

Next we observe that the function

$$(1 - \alpha\mu) \prod_{r=1}^{n} (1 - k_r\mu) S(\mu)$$

is a polynomial of degree $n + 1$ in μ which vanishes for $\mu = \mu_i, i = 1, 2, \ldots, n$. There must accordingly exist a relation of the form

$$(1 - \alpha\mu) \prod_{r=1}^{n} (1 - k_r\mu) S(\mu) \propto (\mu - C) \prod_{i=1}^{n} (\mu - \mu_i) , \tag{25}$$

where C is a constant.

The constant of proportionality can be found by comparing the coefficients of the highest power of μ (viz. μ^{n+1}).

Thus, from Equation (25) we have

$$S(\mu) = \frac{(-1)^{n+1} b_0}{b_1(1-\lambda)} k_1 \ldots k_n \alpha \frac{P(\mu)(\mu - C)}{R(\mu)(1 - \alpha\mu)} , \tag{26}$$

where

$$P(\mu) = \prod_{i=1}^{n} (\mu - \mu_i) \quad i = 1, i, \ldots, n , \tag{27}$$

and

$$R(\mu) = \prod_{r=1}^{n} (1 - k_r\mu) \quad r = 1, r, \ldots, n . \tag{28}$$

Moreover, combining Equations (26) and (27) we obtain

$$L_r = (-1)^n \frac{b_1}{b_0(1-\lambda)} k_1 \ldots k_n \alpha \times \frac{P(1/k_r)(1/k_r - C)}{R_r(1/k_r)(1 - \alpha/k_r)} , \tag{29}$$

where

$$R_r(x) = \prod_{h \neq r} (1 - k_h x) \tag{30}$$

and

$$\alpha \neq k_r . \tag{31}$$

The roots of the characteristic equation (18) can be written in the form

$$k_1 k_2 \ldots k_n \mu_1 \mu_2 \ldots \mu_n = \lambda^{1/2} . \tag{32}$$

Now by use of Equation (32), Equation (26) becomes

$$S(\mu) = -\frac{b_0 \alpha \lambda^{1/2} H(-\mu)(\mu - C)}{(1-\lambda) b_1 (1 - \alpha\mu)} , \tag{33}$$

where

$$H(\mu) = \frac{1}{\mu_1\mu_2 \ldots \mu_n} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{r=1}^{n} (1 + k_r\mu)} ; \tag{34}$$

and the characteristic roots are evaluated from Equation (24).

If we put $\mu = 0$ in Equations (24) and (34) we have

$$\sum_{r=1}^{n} L_r + \frac{T}{1-\lambda} + \frac{b_0}{(1-\lambda)b_1} = \frac{b_0\lambda^{1/2}C\alpha}{(1-\lambda)b_1} . \tag{35}$$

We can next evaluate $\sum_{r=1}^{n} L_r$ from Equation (29). Then

$$\sum_{r=1}^{n} L_r = (-1)^{n+1} \frac{b_0}{b_1(1-\lambda)} k_1k_2 \ldots k_n \alpha f(0) , \tag{36}$$

where

$$f(x) = \sum_{r=1}^{n} \frac{P(1/k_r)(1/k_r - C)}{R_r(1/k_r)(1 - \alpha/k_r)} . \tag{37}$$

Now $f(x)$ defined in this manner is a polynomial of degree $(n-1)$ in x which takes the values

$$\frac{P(1/k_r)(1/k_r - C)}{(1 - \alpha/k_r)} .$$

for

$$x = 1/k_r \quad (r = 1, 2, \ldots, n) .$$

In other words,

$$(1 - \alpha x)f(x) - P(x)(x - C) = 0 . \tag{38}$$

Therefore, we must accordingly have a relation of the form

$$(1 - \alpha x)f(x) - P(x)(x - C) = R(x)(Ax + B) , \tag{39}$$

where A and B are certain constants to be determined. The constant A follows from the comparison of the coefficient of x^{n+1}. Thus

$$A = \frac{(-1)^{n+1}}{k_1k_2 \ldots k_n} . \tag{40}$$

Next, if we put $x = \alpha^{-1}$ in Equation (40) we have

$$B = \frac{(-1)^n}{\alpha k_1k_2 \ldots k_n} + \frac{(C - 1/\alpha)P(\alpha^{-1})}{R(\alpha^{-1})} , \tag{41}$$

i.e.,

$$B = \frac{(-1)^n}{\alpha k_1 k_2 \ldots k_n} + (-1)^n \mu_1 \ldots \mu_n H(-1/\alpha)(C - \alpha^{-1}) \ . \tag{42}$$

Now by use of the relations (42), (41), and (40) we get

$$f(0) = -CP(0) + BR(0) = -C(-1)^n \mu_1 \mu_2 \ldots \mu_n + $$

$$+ \frac{(-1)^n}{\alpha k_1 k_2 \ldots k_n} + (-1)^n \mu_1 \mu_2 \ldots \mu_n H(-\alpha^{-1})(C - \alpha^{-1}) \ . \tag{43}$$

From the Equation (37) using Equation (43) we have

$$\sum_{r=1}^{n} L_r = \frac{b_0}{(1-\lambda)b_1} C\lambda^{1/2}\alpha - \frac{b_0}{(1-\lambda)b_1} + $$

$$+ \frac{b_0 \alpha \lambda^{1/2} H(-\alpha^{-1})(\alpha^{-1} - C)}{(1-\lambda)b_1} \ . \tag{44}$$

By use of Equation (44) in Equation (38) we get

$$C = \frac{1}{\alpha} + \frac{Tb_1}{b_0 \alpha \lambda^{1/2} H(-\alpha^{-1})} \ . \tag{45}$$

If, moreover, we combine Equation (44), the diffusely reflected intensity $I^*(0, \mu)$ in Equation (23) takes the form

$$I^*(0, \mu) = \frac{b_0 \alpha \lambda^{1/2} H(\mu)[\mu + C]}{1 + \alpha\mu} - \frac{Tb_0}{1 + \alpha\mu} - b_0 \ . \tag{46}$$

This is the required solution in closed form. If we combine Equation (9) at $t = 0$ and Equation (46) we have

$$I(0, \mu) = \frac{b_0 \alpha \lambda^{1/2} H(\mu)[\mu + C]}{1 + \alpha\mu} \ , \tag{47}$$

which is the required solution of Equation (5) in the nth approximation by the discrete ordinate method.

On putting C from Equation (45) we get the solution in the form

$$I(0, \mu) = b_0 \lambda^{1/2} H(\mu) + \frac{b_1 T H(\mu)}{1 + \alpha\mu} \frac{1}{H(-\alpha^{-1})} \ . \tag{48}$$

Chandrasekhar's (1960) solution for $I(0, \mu)$ in the case of coherent scattering is given by (for $B_\nu(T) = b_0 + b_1\tau$)

$$I(0, \mu) = b_0 \lambda^{1/2} H(\mu) + b_1 \lambda^{3/2} H(\mu)\mu + \tfrac{1}{2} b_1 \lambda(1 - \lambda) H(\mu)\alpha_1 \ , \tag{49}$$

where

$$\alpha_n = \int_0^1 H(\mu)\mu^n \, d\mu \ . \tag{50}$$

If we compare Equations (48) and (49) we see that by putting $b_1 = 0$ we have the same solution for both the cases. Moreover for large values of β (i.e., $\beta \to \infty$) the solutions (48) takes the form

$$I(0, \mu) = b_0 \lambda^{1/2} H(\mu) \ ; \tag{51}$$

i.e., β then behaves like a constant or independent of τ. This fact can also be explained from the point of view that

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau_\nu} \to b_0 \quad \text{as} \quad \beta \to \infty \ .$$

Also the result obtained by Karanjai and Deb (1992b) is the same as obtained here.

Appendix

To establish the relation (32) we consider

$$D_m(x) = (1 - \lambda) \sum_i \frac{a_i \mu_i^m}{1 + \mu_i x} = (-1)^m (1 - \lambda) \sum_i \frac{a_i \mu_i^m}{1 - \mu_i x} \ , \tag{52}$$

$$(m = 0, 1, \ldots, 4n) \ .$$

We can derive a single recursion formula for $D_m(x)$. Then

$$D_m(x) = \frac{1}{x} \left[(1 - \lambda) \sum_i a_i \mu_i^{m-1} \left(1 - \frac{1}{1 + \mu_i x} \right) \right] =$$

$$= \frac{1}{x} [\psi_{m-1} - D_{m-1}] \ , \tag{53}$$

where

$$\psi_m = (1 - \lambda) - \sum_i a_i \mu_i^m \ . \tag{54}$$

From this formula we have

$$D_m(x) = \frac{\psi_{m-1}}{x} - \frac{\psi_{m-2}}{x^2} + \ldots + (-1)^{m-2} \frac{\psi_1}{x^{m-1}} + \frac{(-1)^{m-1}}{x^m} \times$$

$$\times [\psi_0 - D_0(x)] \ (m = 0, 1, \ldots, 4n) \tag{55}$$

and

$$\psi_0 = 2(1 - \lambda) \ . \tag{56}$$

Moreover, let P_{2j} be the coefficient of μ^{2j} in the Legendre polynomial $P_{3n}(\mu)$, then

$$\sum_{j=0}^{n} P_{2j} D_{2j}(K_r) = (1 - \lambda) \sum_{j} \frac{a_j}{1 + \mu_i k_r} \times \sum_{j=0} P_{2j} \mu_i \ . \tag{57}$$

Since the μ_i's are the zeros of $P_{2n}(\mu)$. Equation (57) reduces to

$$\sum_{j=0}^{n} P_{2j} D_{2j}(k_z) = 0 \ . \tag{58}$$

If we substitute for $D_{2j}(k_r)$ from Equation (56) into Equation (58) we get the required form of the characteristic equation as

$$-\frac{P_{2n}\lambda}{k_r^{2n}} + \cdots + P_0 = 0 \ . \tag{59}$$

From this equation it follows that

$$\frac{1}{(k_1 k_2 \ldots k_n)^2} = \frac{(-1)^n P_0}{\lambda P_{2n}} = \frac{(\mu_1 \mu_2 \cdots \mu_n)^2}{\lambda} \tag{60}$$

i.e.,

$$\mu_1 \mu_2 \ldots \mu_n \, k_1 k_2 \ldots k_n = \lambda^{1/2} \ . \tag{61}$$

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THE TIME-DEPENDENT X- AND Y-FUNCTIONS

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Abstract. The application of the Wiener–Hopf technique to the coupled linear integral equation of time-dependent X- and Y-functions gives rise to the Fredholm equations with simpler kernels. The time-dependent X-function is expressed in terms of time-dependent Y-function and *vice versa*. These are unique in representation with respect to coupled linear constraints.

1. Introduction

In the theory of radiative transfer for homogeneous plane-parallel stratified finite atmosphere the X- and Y-functions of Chandrasekhar (1960) play a central role. These equations satisfy a system of coupled nonlinear integral equations. Busbridge (1960) has demonstrated the existence of the solutions of these coupled nonlinear integral equations in terms of a particular solution of an auxilliary equation. Busbridge (1960) has obtained two coupled linear integral equations for $X(z)$ and $Y(z)$ which defined the meromorphic extension to the complex domain $|Z|$ of the real valued solution of the coupled nonlinear integral equations of X- and Y-functions. Busbridge (1960) concludes that all solutions of nonlinear coupled integral equations for X- and Y-functions are the solutions of the coupled linear integral equations to the extended complex plane but all solutions of the coupled linear integral equations are not solutions of the coupled nonlinear integral equations. Mullikin (1964) has proved that all solutions of coupled nonlinear integral equations are solutions of the coupled linear integral equations but there exist a unique solution of the coupled linear integral equations with some linear constraints. Finally he has obtained the Fredholm equation of X- and Y-functions which are easy for iterative computations. Das (1979) has obtained a pair of the Fredholm equations with the Wiener–Hopf technique from the coupled linear integral equations with coupled linear constraints.

In this paper we have considered the time-dependent X- and Y-functions (Biswas and Karanjai, 1990) which give rise to a pair of the Fredholm equations with the application of the Wiener–Hopf technique. These Fredholm equations define time-dependent X-functions in terms of time-dependent Y-functions and *vice versa*. These representations are unique with respect to the coupled linear constraints defined by Mullikin (1964).

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2. Basic Equation

The coupled nonlinear integral equations satisfied by the time-dependent X- and Y-functions (Biswas and Karanjai, 1990) are of the form

$$X(\tau_1, \mu, s) = 1 + \frac{w}{2Q}\,\mu \int_0^1 \frac{X(\tau_1, \mu, s)X(\tau_1, x, s) - Y(\tau_1, \mu, s)Y(\tau_1, x, s)}{\mu + x}\,\mathrm{d}x\,,$$

$$0 \leq \mu < 1\,, \qquad (1)$$

$$Y(\tau_1, \mu, s) = \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{w}{2Q}\,\mu \int_0^1 \frac{Y(\tau_1, \mu, s)X(\tau_1, x, s)X(\tau_1, \mu, s)Y(\tau_1, x, s)}{\mu - x}\,\mathrm{d}x\,,$$

$$0 \leq \mu < 1\,, \qquad (2)$$

where

$$Q = 1 + \frac{s}{c}\,, \qquad (3)$$

τ_1 is the thickness of the atmosphere; c, the velocity of light; and s, Laplace transform parameter.

If we follow Chandrasekhar (1960) Equations (1) and (2) can be written as

$$X(\tau_1, \mu, s) = 1 + \frac{\mu}{Q} \int_0^1 \frac{\psi(x)}{x + \mu}\,[X(\tau_1, \mu, s)X(\tau_1, x, s) -$$

$$- Y(\tau_1, \mu, s)Y(\tau_1, x, s)]\,\mathrm{d}x\,, \quad 0 \leq \mu < 1\,, \qquad (4)$$

$$Y(\tau_1, \mu, s) = \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{\mu}{Q} \int_0^1 \frac{\psi(x)}{x - \mu}\,[Y(\tau_1, x, s)X(\tau_1, \mu, s) -$$

$$- X(\tau_1, x, s)Y(\tau_1, \mu, s)]\,\mathrm{d}x\,, \quad 0 \leq \mu < 1\,; \qquad (5)$$

where $\psi(x)$, the characteristic function satisfying the Hölder condition on $0 \leq x \leq 1$, is non-negative and satisfies the condition

$$\psi_0 = \int_0^1 \psi(x)\,\mathrm{d}x \leq \tfrac{1}{2}\,. \qquad (6)$$

The atmosphere is said to be conservative when $\psi_0 = \frac{1}{2}$ and non-conservative otherwise.

The dispersion function $T(z, s)$, $z \in (-1, 1)^c$ can be defined by

$$T(z, s) = 1 - \frac{2z^2}{Q} \int_0^1 \psi(x)\, dx T(z^2 - x^2) \tag{6a}$$

and

$$T(z, s) = (H(z, s)H(-z, s))^{-1}, \tag{6b}$$

where

$$H(z, s) = 1 + zH(z, s) \int_0^1 \frac{\psi(x)H(x, s)\, dx}{x + z}. \tag{7}$$

According to Busbridge (1960), the only zeros of $T(z, s)$ are at $z = \pm K, K > 1$, when $\psi_0 < \frac{1}{2}$ and $K \to \infty$ when $\psi_0 = \frac{1}{2}$.

Following Busbridge (1960), Dasgupta (1977), and Das (1978) $H(z, s)$ is meromorphic on $(-1, 0)^c$ having a simple pole at $z = -K$ and tend to 1 as $z \to 0_+$. It can be represented by

$$H(z, s) = \frac{A_0 + H_0 z}{K + z} - \int_0^1 \frac{P(x, s)\, dx}{x + z}, \quad K > 1, \psi_0 < \tfrac{1}{2}, \tag{8}$$

$$H(z, s) = h_1 z + h_0 - \int_0^1 \frac{P(x, s)\, dx}{x + z}, \quad K \to \infty, \psi_0 = \tfrac{1}{2}; \tag{9}$$

where

$$\begin{aligned}
A_0 &= (1 + P_{-1})K, \qquad P_{-1} = \int_0^1 P(x, s)\, dx/x, \\
H_0 &= \left(1 - 2\int_0^1 \psi(x)\, dx\right)^{-1/2}, \\
h_1 &= \left(2\int_0^1 x^2 \psi(x)\, dx\right)^{-1/2}, \\
h_0 &= (1 + P_{-1}), \\
P(x, s) &= \phi(x, s)/H(x, s), \\
\phi(x, s) &= x\psi(x)/(T_0^2(x, s) + \pi^2 x^2 \psi^2(x)),
\end{aligned} \tag{10}$$

$$T_0(x, s) = 1 - \frac{2x^2}{Q} \int_0^1 (\psi(t) - \psi(x))\, dt/(x^2 - t^2) -$$

$$- \frac{x\psi(x)}{Q} \log((1 + x)/(1 - x)),$$

where $\phi(x, s)$ is non-negative and continuous on (0, 1), tends to $\psi(0)x$ as $x \to 0_+$, tends to $0((\log(1 - x)^{-2}))$ when $x \to 1_-$, and $1/H(z, s)$ is regular on $(-1, 0)^c$.

If we follow Busbridge (1960) and Mullikin (1964) we find that the coupled linear equations satisfied by $X(z, s)$ and $Y(z, s)$ for $z \in (-1, 1)^c$ are of the form

$$X(z, s)T(z, s) = 1 + zU(X)(z, s) - z\exp(-(\tau_1/z)Q)V(Y)(z, s), \tag{11}$$

$$Y(z, s)T(z, s) = (\exp(-(\tau_1/z)Q) + zU(Y)(z, s)) -$$
$$- z\exp(-(\tau_1/z)Q)V(X)(z, s), \tag{12}$$

with constraints for $\psi_0 < \frac{1}{2}$,

$$0 = 1 + KU(X)(K, s) - K\exp(-(\tau_1/K)Q)V(Y)(K, s), \tag{13a}$$

$$0 = (\exp(-(\tau_1/K)Q) + KU(Y)(K, s)) - K\exp(-(\tau_1/K)V(X)(K, s), \tag{13b}$$

for $\psi_0 = \frac{1}{2}$,

$$1 = \int_0^1 \psi(x)(X(x, s) + Y(x, s))\, dx, \tag{14a}$$

$$\tau_1 \int_0^1 Y(x, s)\psi(x)\, dx = \int_0^1 x\psi(x)(X(x, s) - Y(x, s))\, dx. \tag{14b}$$

The other conditions for which $X(z, s)$ and $Y(z, s)$ hold are

$$X(z, s) \to H(z, s) \quad \text{when} \quad \tau_1 \to \infty, \tag{15a}$$

$$Y(z, s) \to \hat{u} \quad \text{when} \quad \tau_1 \to \infty, \tag{15b}$$

where for $M = X$ or Y

$$V(M)(z, s) = \int_0^1 \psi(x)M(x, s)\, dx/(x + z) \tag{16}$$

is analytic for $z \in (-1, 0)^c$ bounded at the origin $O(z^{-1})$ when $z \to \infty$ and

$$U(M)(z, s) = \int_0^1 \psi(x) M(x, s)\, \mathrm{d}x/(x - z) \tag{17}$$

is analytic for $z \in (0, 1)^c$, bounded at the origin $O(z^{-1})$ when $z \to \infty$.

3. Fredholm Equations

Equations (11) and (12) with Equations (6b) can be written in the form

$$X(z, s)/H(z, s) = H(-z, s)(1 + zU(X)(z, s)) - z\exp(-(\tau_1/z)Q) \times \times H(-z, s)V(Y)(z, s), \tag{18}$$

$$Y(z, s)/H(z, s) = H(-z, s)((\exp(-\tau_1/z)Q) + zU(Y)(z, s) - - z\exp(-(\tau_1/z)Q)H(-z, s)V(X)(z, s). \tag{19}$$

We shall assume that $X(z, s)$ and $Y(z, s)$ are regular for $\mathrm{Re}\, z > 0$ and bounded at the origin. Equation (8) gives

$$H(-z, s) = \frac{A_0 - H_0 z}{(K - z)} - \int_0^1 \frac{P(x, s)}{x - z}\, \mathrm{d}x \quad \text{for} \quad \psi_0 < \tfrac{1}{2}. \tag{20}$$

Hence

$$V(M)(z, s) \int_0^1 \frac{P(x, s)}{x - z}\, \mathrm{d}x = D(M, P_0)(z, s) + D(P, M_0)(z, s), \tag{21}$$

where

$$D(M, P_0)(z, s) = \int_0^1 \frac{\psi(x) M(x, s) P_0(x, s)\, \mathrm{d}x}{x + z} \tag{22}$$

and

$$D(P, M_0)(z, s) = \int_0^1 \frac{\psi(x) P(x, s) M_0(x, s)}{x - z}\, \mathrm{d}x, \tag{23}$$

where

$$P_0(z, s) = \int_0^1 \frac{P(x, s)\, \mathrm{d}x}{x + z} \tag{24}$$

is regular on $(-1, 0)^c$, bounded at the origin $O(z^{-1})$ when $z \to \infty$,

$$M_0(z, s) = \int_0^1 \frac{\psi(x) M(x, s)\, dx}{x + z}, \tag{25}$$

is regular on $(-1, 0)^c$, bounded at the origin $O(z^{-1})$ when $z \to \infty$ and $D(M, P_0)(z, s)$ is regular for z on $(-1, 0)^c$, bounded at the origin and $O(z^{-1})$ when $z \to \infty$ and $D(P, M_0)(z, s)$ is regular for z, on $(0, 1)^c$ bounded at the origin, and $O(z^{-1})$ when $z \to \infty$. Hence, Equations (18) and (19) can for $\psi_0 < \frac{1}{2}$ be written in the form

$$X(z, s)/H(z, s) + z \exp(-(\tau_1/z)Q) \times \\ \times \left(\frac{A_0 - H_0 z}{K - z} V(Y)(z, s) - D(Y, P_0)(z, s)\right) = \\ = H(-z, s)(1 + zU(X)(z, s) + z \exp(-(\tau_1/z)Q) D(P, Y_0)(z, s)), \tag{26}$$

$$Y(z, s)/H(z, s) + z \exp(-(\tau_1/z)Q) \times \\ \times \left(\frac{A_0 - H_0 z}{K - z} V(X)(z, s) - D(X, P_0)(z, s)\right) = \\ = H(-z, s)(\exp(-(\tau_1/z)z) + zU(Y)(z, s)) + \\ + z \exp(-(\tau_1/z)Q) D(P, X_0)(z, s). \tag{27}$$

The left-hand side of Equations (26) and (27) are regular for $\operatorname{Re} z > 0$ and bounded at the origin; the right-hand side of Equations (26) and (27) are regular for z, on $(0, 1)^c$, bounded at the origin and tends to constants, say, A and B, respectively, when $z \to \infty$.

Hence, by a modified form of Liouville's theorem we have

$$X(z, s) = H(z, s)\left[z \exp(-(\tau_1/z)Q)\left(D(Y, P_0)(z, s) - \\ - \frac{A_0 - H_0 z}{K - z} V(Y)(z, s)\right) + A\right], \tag{28}$$

$$Y(z, s) = H(z, s)\left[z \exp(-(\tau_1/z)Q)\left(D(X, P_0)(z, s) - \\ - \frac{A_0 - H_0 z}{K - z} V(X)(z, s)\right) + B\right], \tag{29}$$

Equations (28) and (29) together with Equations (15a) and (15b) gives

$$A = 1, \qquad B = 0. \tag{30}$$

Hence for $\psi_0 = \frac{1}{2}$, the expression of $X(z, s)$ and $Y(z, s)$ are

$$X(z, s) = H(z, s)\,[1 + z\exp(-(\tau_1/z)Q)\,(D(Y, P_0)\,(z, s) - \\ - (-h_1 z + h_0)V(Y)\,(z, s))]\,, \tag{31}$$

$$Y(z, s) = H(z, s) z \exp(-(\tau_1/z)Q)\,(D(X, P_0)\,(z, s)) - \\ - (-h_1 z + h_0)V(X)\,(z, s))\,. \tag{32}$$

Hence, following Mullikin (1964) Equations (28) and (29) together with Equations (13a) and (13b) give unique representations of time-dependent X- and Y-functions for $\psi_0 < \frac{1}{2}$ and Equations (31) and (32) together with Equations (14a) and (14b) give unique representations of X- and Y-functions for $\psi_0 = \frac{1}{2}$.

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EXACT SOLUTION OF THE EQUATION OF TRANSFER WITH PLANETARY PHASE FUNCTION

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Abstract. We have considered the transport equation for radiative transfer to a problem in semi-infinite atmosphere with no incident radiation and scattering according to planetary phase function $w(1 + x\cos\theta)$. Using Laplace transform and the Wiener–Hopf technique, we have determined the emergent intensity and the intensity at any optical depth. The emergent intensity is in agreement with that of Chandrasekhar (1960).

1. Introduction

The transport equation for the intensity of radiation in a semi-infinite atmosphere with no incident radiation and scattering according to the phase function $w(1 + x\cos\theta)$ has been considered. This equation has been solved by Chandrasekhar (1960) using his principle of invariance to get the emergent radiation. The singular eigen function approach of Case (1960) is also applied to get the intensity of radiation at any optical depth. Boffi (1970) has also applied the two sided Laplace transform to get the emergent intensity and the intensity at any optical depth. Das (1979) solved exactly the equation of transfer for scattering albedo $w < 1$ using the Laplace transform and the Wiener–Hopf technique and also deduced the intensity at any optical depth by inversion.

In this paper we have solved the above problem exactly by a method based on the use of the Laplace transform and the Wiener–Hopf technique. The intensity at any optical depth is also derived by inversion.

2. Basic Equation and its Solution

The equation of transfer appropriate to the problem (Chandrasekhar, 1960) is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} w \int_{-1}^{+1} I(\tau, \mu')(1 + x\mu\mu')\, d\mu' , \tag{1}$$

where the symbols have their usual meaning.

We shall have the following boundary conditions

$$I(0,-\mu)=0, \quad 0<\mu<1\,; \tag{2a}$$

$$I(\tau,\mu)\to L_0\exp(k\tau)\,\frac{1+x(1-w)(\mu/k)}{1-k\mu}, \quad \text{as } \tau\to\infty\,; \tag{2b}$$

where L_0 is a constant and k is the positive root, less than 1, of the transcendental equation.

$$1=\frac{w}{2k}\left[1+\frac{x(1-w)}{k^2}\right]\log\left(\frac{1+k}{1-k}\right)- -\frac{1}{k^2}xw(1-w)\,. \tag{3}$$

Let us define

$$f^*(s)=s\int_0^\infty \exp(-s\tau)f(\tau)\,\mathrm{d}\tau, \quad \text{R}1/s>0\,. \tag{4}$$

Let us set

$$I_m(\tau)=\frac{1}{2}\int_{-1}^{+1} I(\tau,\mu')\mu'^m\,\mathrm{d}\mu', \quad \text{where } m=0,1\,, \tag{5}$$

which gives

$$I_0^*(s)=\frac{1}{2}\int_{-1}^{+1} I^*(s,\mu')\,\mathrm{d}\mu' \tag{6}$$

and

$$I_1^*(s)=\frac{1}{2}\int_{-1}^{+1} I^*(s,\mu')\mu'\,\mathrm{d}\mu'\,, \tag{7}$$

Equation (1) with Equation (5) takes the form

$$\frac{\mu\,\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau}=I(\tau,\mu)-wI_0(\tau)-wx\mu I_1(\tau)\,. \tag{8}$$

Now, subjecting Equation (8) to the Laplace transform as define in Equation (4), we have, using the boundary conditions,

$$(\mu s-1)I^*(s,\mu)=\mu sI(0,\mu)-wI_0^*(s) -xw\mu I_1^*(s)\,. \tag{9}$$

Equation (9) gives (on putting $s=1/\mu$)

$$I(0,\mu)=wI_0^*(1/\mu)+xw\mu I_1^*(1/\mu)\,. \tag{10}$$

Equation (10) with $\mu = 1/s$, s is complex, takes the form

$$I(0, 1/s) = wI_0^*(s) + xws^{-1}I_1^*(s)\,, \tag{11}$$

we apply the operator

$$\frac{1}{2}\int_{-1}^{+1} \ldots \mathrm{d}\mu \tag{12}$$

on both sides of Equation (9) to get

$$I_1^*(s) - (1-w)s^{-1}I_0^*(s) = \frac{1}{2}\int_0^1 \mu I(0, \mu)\,\mathrm{d}\mu \tag{13}$$

we apply the operator

$$\frac{1}{2}\int_{-1}^{+1} \ldots \mathrm{d}\mu/(\mu s - 1)\,, \tag{14}$$

$$a(1/s) = 1 + wt_0(1/s) + xwt_1(1/s)I_1^*(s)\,, \tag{15}$$

where

$$a(1/s) = \frac{1}{2}\int_0^1 \mu s(\mu s - 1)^{-1} I(0, \mu)\,\mathrm{d}\mu \tag{16}$$

and

$$t_m(1/s) = \frac{1}{2}\int_{-1}^{+1} (\mu s - 1)^{-1}\mu^m\,\mathrm{d}\mu, \quad m = 0, 1\,. \tag{17}$$

Eliminating $I_0^*(s)$, $I_1^*(s)$ among Equations (11), (13) and (15) and setting $s = 1/z$, we have

$$T(z)I(0, z) = \frac{w}{2}\int_0^1 \frac{\mu}{\mu - z} \times [1 + \mu x(1-w)z]I(0, \mu)\,\mathrm{d}\mu\,, \tag{18}$$

where

$$T(z) = 1 + wx(1-w)z^2 + w[1 + x(1-w)z^2]t_0(z)\,, \tag{19}$$

where

$$t_0(z) = \frac{z}{2}\int_{-1}^{+1} \frac{\mathrm{d}\mu}{\mu - z}\,. \tag{20}$$

Following Chandrasekhar (1960) and considering Equation (3), we see that $T(z)$ has a pair of roots at $z = \pm k^{-1}$ and

$$T(z) = \frac{1}{H(z)H(-z)}, \quad z \in (-1, 1)^c, \tag{21}$$

where $H(z)$ is Chandrasekhar's H-function for planetary scattering. Equation (18) with Equation (21) takes the form

$$\frac{I(0, z)}{H(z)} = H(-z)\frac{w}{2}\int_0^1 \frac{\mu}{\mu - z} \times$$
$$\times [1 + \mu x(1 - w)z]I(0, \mu)\, d\mu, \tag{22}$$

Equation (22) can be written as

$$\frac{I(0, z)}{H(z)} = H(-z)wG(z),$$

where

$$G(z) = \frac{1}{2}\int_0^1 \frac{\mu}{\mu - z}[1 + \mu x(1 - w)z]I(0, \mu)\, d\mu. \tag{23}$$

Let us seek solution $I(0, z)$ of Equation (22) by Wiener–Hopf technique on the assumption that $I(0, z)$ is regular for $\mathrm{Rl}\, z > 0$ and bounded at the origin.

Equation (23) with the above assumption on $I(0, z)$ gives the following properties of $G(z)$: $G(z)$ is regular on $(0, 1)^c$, bounded at the origin and a constant as $z \to \infty$. Equation (23) then gives

$$\frac{(1 - kz)I(0, z)}{H(z)} = w(1 - kz)H(-z)G(z), \tag{24}$$

where $H(-z)$, $H(z)$, $1/H(z)$ has the following properties: $H(z)$ is regular for $z \in (-1, 0)^c$, uniformly bounded at the origin has a simple pole at $z = -(1/k)$, $k < 1$; k is real on the negative real axis and bounded at infinity and tends to $H_0 + H_{-1}z^{-1} + H_{-2}z^{-2} + \cdots$ when $z \to \infty$.

Hence, $1/H(z)$ is regular for z in $(-1, 0)^c$ and bounded at the origin. Similarly, $H(-z)$ is regular for $z \in (0, 1)^c$ uniformly bounded at the origin has a simple pole at $z = 1/k$, $k < 1$; k is real, on the positive side of the real axis and bounded at infinity and tends to $H_0 - H_{-1}z^{-1} + H_{-2}z^{-2} - \cdots$ when $z \to \infty$.

Following the properties of $H(z)$, $1/H(z)$, $H(-z)$ (Busbridge, 1960) the left hand side of Equation (24) is regular for $\mathrm{Rl}\, z > 0$, bounded at the origin and the right hand side of Equation (24) is regular for $z \in (0, 1)^c$ and bounded at the origin and tends to a polynomial say $A + Bz$, as $z \to \infty$.

Hence by a modified form of Liouville's theorem

$$\frac{(1-kz)I(0,z)}{H(z)} = A + Bz, \quad \text{when } z \in (-1, 0)^c \tag{25}$$

and

$$A + Bz = w(1-kz)H(-z)G(z), \quad \text{when } z \in (0,1)^c \,. \tag{26}$$

Equation (25) gives the emergent radiation as

$$I(0,z) = \frac{(A+Bz)H(z)}{1-kz}, \tag{27}$$

where the constants A and B are two arbitrary constants to be determined later on.

3. Intensity at Any Optical Depth

The radiation intensity at an optical depth τ is given by the inversion integral as

$$I(\tau,\mu) = (1/2\pi i) \lim_{\delta\to\infty} \int_{c-i\delta}^{c+i\delta} \exp(s\tau) \times$$

$$\times\, I^*(s,\mu)\, ds/s, \quad c > 0 \,. \tag{28}$$

Equation (9) with Equation (11) takes the form

$$I^*(s,\mu)/s = \phi(s,\mu)/(s-1/\mu)\,, \tag{29}$$

where

$$\phi(s,\mu) = I(0,\mu) - I(0,1/s) + \frac{w(s-1/s)}{s} I_0^*(s)\,. \tag{30}$$

But

$$\lim_{s\to 1/\mu} (s-1/\mu) I^*(s,\mu)\exp(s\tau)/s \to 0\,. \tag{31}$$

Hence the integrand of Equations (28) is regular for $s \in (-\infty, -1)^c$ and has simple pole at $s = \pm k$, $k < 1$.

Hence by Cauchy's residue theorem, Equation (28) gives

$$I(\tau,\mu) = R_p + \lim_{R\to\infty} (1/2\pi i) \int_\Gamma I^*(s,\mu)e^{s\tau}\, ds/s\,, \tag{32}$$

where R_p is the sum of the residues of the poles at $s = \pm k$ and $\Gamma = \Gamma_1 \cup CD \cup \nu \cup EF \cup \Gamma_2$. [$\Gamma_1$ and Γ_2 are arcs of the circle of radius R having centre at $s = 0$ (clockwise) and ν is an arc of a small circle of radius r having centre at $s = -1$ (anticlockwise) and CD and EF are the lower edge and upper edge of

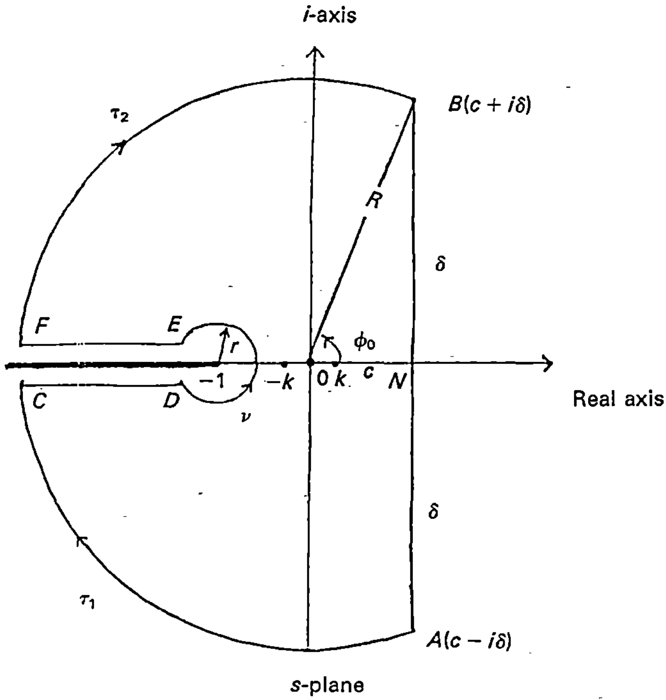

Fig. 1.

the singular line $(-R, -1)]$ (Figure 1). Hence, following Kourganoff (1963) we have

$$\int_{\Gamma_1 \cup \Gamma_2} I^*(s, \mu) \exp(s\tau)\, ds/s \to 0, \quad \text{when } R \to \infty \tag{33}$$

and

$$\int_{\nu} I^*(s, \mu) \exp(s\tau)\, ds/s \to 0, \quad \text{when } r \to 0\,. \tag{33a}$$

Hence in the limit of $R \to \infty$, $r \to 0$, Equation (32) with Equations (33) and (33a) becomes

$$I(\tau, \mu) = R_p + (1/2\pi i) \int_{CD} I^*(s, \mu) e^{s\tau}\, ds/s + $$

$$+ (1/2\pi i) \int_{EF} I^*(s, \mu) e^{s\tau}\, ds/s\,. \tag{34}$$

Here on CD and EF,

$$s = -\upsilon, \qquad \upsilon \geqslant 1 \tag{34a}$$

and on CD,

$$H(1/s) = \frac{X(1/v) + i\pi Y(1/v)}{H(1/v)Z(1/v)} \tag{35}$$

and on EF,

$$H(1/s) = \frac{X(1/v) - i\pi Y(1/v)}{H(1/v)Z(1/v)}; \tag{36}$$

where

$$X(1/v) = 1 + wx(1-w)v^{-2} - w[1 + x(1-w)v^{-2}] \times \times \frac{1}{2v}\log\left(\frac{v+1}{v-1}\right), \tag{37}$$

$$Y(1/v) = (w/2)v^{-1}; \tag{38}$$

$$Z(1/v) = (X^2(1/v) + \pi^2 Y^2(1/v, \mu)). \tag{39}$$

Therefore on CD

$$\phi(s, \mu) = V(1/v, \mu) - i\pi W(1/v, \mu) \tag{40}$$

and on EF,

$$\phi(s, \mu) = V(1/v, \mu) + i\pi W(1/v, \mu), \tag{41}$$

where

$$V(1/v, \mu) = I(0, \mu) - \left[\frac{(B - vA)(1/v)}{(v + k)H(1/k)Z(1/v)}\right] \times \times \left\{1 + \frac{v + 1/\mu}{1 + x(1-w)/v^2}\right\} + \frac{(v + 1/\mu)w\alpha_1/2}{1 + x(1-w)/v^2}, \tag{42}$$

$$W(1/v, \mu) = \left[\frac{(B - vA)Y(1/v)}{(v + k)H(1/k)Z(1/v)}\right]\left[1 + \frac{v + 1/\mu}{1 + x(1-w)/v^2}\right].$$

Now, Equation (33) with Equations (29), (34a), (40) and (41) gives

$$I(\tau, \mu) = R_p - \frac{1}{2\pi i}\int_1^\infty \frac{\{v(1/\nu, \mu) - i\pi W(1/v, \mu)\}}{v + 1/\mu} e^{-v\tau}\, dv + + \frac{1}{2\pi i}\int_1^\infty \frac{V(1/v, \mu) + i\pi W(1/v, \mu)}{v + 1/\mu} e^{-v\tau}\, dv. \tag{44}$$

Hence when $\mu > 0$, Equation (44) give

$$I(\tau, \mu) = R_p + \int_1^\infty W(1/v, \mu)\, e^{-v\tau}\, dv/(v + 1/\mu)\,, \tag{45}$$

where $\mu < 0$, we shall assume that $(V(1/v, \mu) \pm i\pi W(1/v, \mu)\, e^{-v\tau}$ satisfies Hölder condition on $(1, \infty)$ and we have by Plemelj's formula (Muskhelishvili, 1946)

$$\frac{1}{2\pi i}\int_1^\infty \frac{V(1/v, \mu) \pm i\pi W(1/v, \mu)}{v + 1/\mu}\, e^{-v\tau}\, dv = \pm\frac{1}{2}(V(-\mu, \mu) \pm$$

$$\pm\, i\pi W(-\mu, \mu))e^{\tau/\mu} + \frac{1}{2\pi i} \times P\int_1^\infty \frac{V(1/v, \mu) \pm i\pi W(1/v, \mu)}{v + 1/\mu} \times$$

$$\times\, e^{-v\tau}\, dv\,, \tag{46}$$

where P denotes the Cauchy principal value of the integral. Hence Equation (44) with Equation (46) for $\mu < 0$ gives

$$I(\tau, \mu) = R_p + V(-\mu, \mu)\, e^{\tau/\mu} + P\int_1^\infty \frac{W(1/v, \mu)\, e^{-v\tau}}{v + 1/\mu}\, dv\,, \tag{47}$$

where

$$R_p = R_k + R_{-k}\,, \tag{48}$$

where, $R_{\pm k}$ is the residue of the integral in Equation (32) at $s = \pm k$, and R_k is given by

$$R_k = \lim_{s\to k}(s - k)I^*(s, \mu)\, e^{s\tau}/s$$

$$= \lim_{s\to k}\frac{H(1/s)(As + B)s}{\{s^2 + x(1 - w)\}(1 - s\mu)}[1 + x(1 - w)/s]\, e^{s\tau}$$

$$= \frac{H(1/k)(Ak + B)k}{[k^2 + x(1 - w)](1 - k\mu)}[1 + x(1 - w)/k]\, e^{k\tau}\,. \tag{49}$$

Similarly, R_{-k} is given by

$$R_{-k} = \lim_{s\to(-k)}(s + k)I^*(s, \mu)\, e^{s\tau}/s$$

$$= \lim_{s\to(-k)}\frac{(s + k)H(1/s)(As + B)s}{(s - k)\{s^2 + x(1 - w)\}(1 - s\mu)} \times$$

$$\times\, [1 + x(1 - w)/s]\, e^{s\tau}$$

$$= \frac{(B - Ak)k[1 - x(1 - w)/k]\, e^{-k\tau}}{2k\{k^2 + x(1 - w)\}(1 + k\mu)}\lim_{s\to(-k)}(s + k)/T(1/s)$$

$$= \frac{(B - Ak)[1 - x(1 - w)k]e^{-k\tau}}{2\{k^2 + x(1 - w)\}(1 + k\mu)}[dT(1/s)/ds]^{-1}_{s=-k} \tag{50}$$

4. Determination of constants A and B

When $z \to 0$, from Equation (26) we get

$$A = (w/2) \int_0^1 I(0, \mu)\, d\mu . \tag{51}$$

From Equation (51) and Equation (25) we get after simplification

$$A\left[1 - \frac{w}{2} \int_0^1 \frac{H(\mu)\, d\mu}{1 - k\mu}\right] = \frac{wB}{2k}\left[-\alpha_0 + \int_0^1 \frac{H(\mu)\, d\mu}{1 - k\mu}\right] = m , \tag{52}$$

where

$$\alpha_0 = \int_0^1 H(\mu)\, d\mu, \qquad m = \text{constant} .$$

$H(z)$ has a simple pole at $z = -(1/k)$ where

$$1/H(z) = 1 - zH(z) \int_0^1 \frac{\psi(z)H(\mu)\, d\mu}{\mu + z} , \tag{53}$$

where

$$\psi(\mu) = \frac{w}{2}[1 + x(1 - w)\mu^2] . \tag{54}$$

Equation (53) has a zero at $z = -(1/k)$ and so

$$1 + \frac{1}{k} \int_0^1 \frac{\psi(\mu)H(\mu)\, d\mu}{\mu - 1/k} = 0 . \tag{55}$$

In Equation (55) putting the value of $\psi(\mu)$ and simplifying and using Equation (52) we get

$$A = \frac{2mN}{kQ} \Big/ \left(\frac{x(1 - w)}{k} - c\right), \qquad B = \frac{2mN}{Q(k + c)} ,$$
$$N = k^2 + x(1 - w), \qquad Q = 2 - w\alpha_0, \qquad c = \frac{xw(1 - w)\alpha_1}{Q} \tag{56}$$

$$A + B\mu = \frac{2mN}{QR}\left\{\left(1 + \frac{c}{k}\right) + \left(\frac{x(1 - w)}{k} - c\right)\right\} . \tag{57}$$

Putting

$$\mu = 1/k \text{ we get } kA + B = \frac{2mN^2}{QkR} \tag{58}$$

where

$$R = \left\{\frac{x(1-w)}{k} - c\right\}(k+c)\,. \tag{59}$$

If we use Equations (58) and (59) we get from Equation (27)

$$I(0,\mu) = \frac{(kA+B)k}{k^2 + x(1-w)}\left[\left(1+\frac{c}{k}\right) + \left\{\frac{x(1-w)}{k} - c\right\}\mu\right]\frac{H(\mu)}{1-k\mu}, \tag{60}$$

when $\tau \to \infty$ from Equations (47), (48) and (49) we get

$$I(\tau,\mu) \to \frac{H(1/k)(Ak+B)k}{[k^2 + x(1-w)](1-k\mu)} \times [1 + x(1-w)/k]e^{k\tau}\,. \tag{61}$$

Hence Equation (61) with Equation (2b) gives

$$\frac{(Ak+B)k}{k^2 + x(1-w)} = \frac{L_0}{H(1/k)}, \tag{62}$$

$$I(0,\mu) = \frac{L_0}{H(1/k)}\left\{1 + \frac{c}{k} + \mu\left[\frac{x(1-w)}{k} - c\right]\right\}\frac{H(\mu)}{1-k\mu}, \tag{63}$$

which is the expression obtained by Chandrasekhar (1960).

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AN EXACT LINEARIZATION AND DECOUPLING OF THE INTEGRAL EQUATIONS SATISFIED BY TIME-DEPENDENT X- AND Y-FUNCTIONS

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Abstract. We discuss a simple method of linearization and decoupling of the integral equations satisfied by time-dependent X- and Y-functions which play an important rôle in the study of non-stationary radiative transfer problems.

1. Introduction

In the study of the time-dependent radiative transfer problems in finite homogeneous plane-parallel atmospheres, it is convenient to introduce X- and Y-functions (Chandrasekhar, 1960). These functions satisfy non-linear coupled integral equations. Due to their important rôle in solving transport problems, it is advantageous to simplify the equations satisfied by them, and, if possible, do so in an exact manner. Lahoz (1989) did this and obtained exact linear and decoupled integral equations satisfied by the time- independent X- and Y-functions.

In this paper we have extended the same method to the time-dependent radiative transfer problem. However, the equations obtained, although linear, are singular and not solvable by the standard methods applicable to Fredholm equations; instead they have to be solved by the theory of singular integral equations (Muskhelishvili, 1946).

2. Analysis

The integral equations incorporating the various invariances of the time-dependent problem of diffuse refelection and transmission can be reduced to one or more pairs of integral equations of the following form (Biswas and Karanjai, 1990).

$$X(\mu, s) = 1 + \frac{W}{2}\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu' \frac{X(\mu, s)X(\mu', s) - Y(\mu, s)Y(\mu', s)}{\mu + \mu'}, \tag{1}$$

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$$Y(\mu,s) = \exp\{(-\tau_1/\mu)Q\} + \frac{W}{2Q}\int_0^1 d\mu' \frac{Y(\mu,s)X(\mu',s) - X(\mu,s)Y(\mu',s)}{\mu-\mu'}, \quad (2)$$

Following Chandrasekhar (1960), we can write the above equations in the form:

$$X(\mu,s) = 1 + \frac{\mu}{Q}\int_0^1 d\mu' \psi(\mu') \frac{X(\mu,s)X(\mu',s) - Y(\mu,s)Y(\mu',s)}{\mu+\mu'}, \quad (3)$$

$$\begin{aligned} Y(\mu,s) &= \exp\{(-\tau_1/\mu)Q\} + \frac{\mu}{Q}\int_0^1 d\mu' \times \\ &\times \psi(\mu') \frac{Y(\mu,s)X(\mu',s) - X(\mu,s)Y(\mu',s)}{\mu-\mu'}, \end{aligned} \quad (4)$$

where τ_1 is the optical thickness of the atmosphere and $Q = 1 + s/c$, where c is the velocity of light, s is the Laplace invariant of the time variable and the characteristic function $\psi(\mu)$ is an even polynomial in μ satisfying

$$\psi_0 = \int_0^1 \psi(\mu)\, d\mu \leq \frac{1}{2}, \quad (5)$$

where $\psi_0 = \frac{1}{2}$ holds, $\psi(\mu)$ is said to be conservative; and non-conservative otherwise.

Clearly, Eqs. (3) and (4) are non-linear and coupled. These equations have been linearized in an exact manner (Mullikin, 1964). The results are

$$\begin{aligned} X(\mu,s)K(\mu,s) &= 1 + \frac{\mu}{Q}\int_0^1 d\mu' \psi(\mu') \frac{X(\mu',s)}{\mu'-\mu} - \\ &- \exp\{(-\tau_1/\mu)Q\} \frac{\mu}{Q}\int_0^1 d\mu' \psi(\mu') \frac{Y(\mu',s)}{\mu'+\mu} \end{aligned} \quad (6)$$

and

$$\begin{aligned} Y(\mu,s)K(\mu,s) &= \exp\{(-\tau_1/\mu)Q\} + \frac{\mu}{Q}\int_0^1 d\mu' \, \psi(\mu') \frac{Y(\mu',s)}{\mu'-\mu} - \\ &- \exp\{(-\tau_1/\mu)Q\} \frac{\mu}{Q}\int_0^1 d\mu' \, \psi(\mu') \frac{X(\mu',s)}{\mu'+\mu}, \end{aligned} \quad (7)$$

where $K(\mu, s)$ is defined by

$$K(\mu,s) \equiv 1 - \frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\psi(\mu)\left[\frac{1}{\mu'+\mu} - \frac{1}{\mu'-\mu}\right], \tag{8}$$

We now proceed to decouple Eqs. (4) and (5) in an exact manner (Lahoz, 1989). We introduce the following singular integral equation, which is linear in $1/T(\mu, s)$:

$$\frac{1}{T(\mu,s)} = 1 - \frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\frac{\psi(\mu')}{T(\mu',s)K(\mu',s)}\,\frac{1}{\mu'-\mu}. \tag{9}$$

which, in principle, is solvable for $T(\mu, s)$ as $\psi(\mu)$ and $K(\mu, s)$ are known functions.

Next, we multiply Eq. (6) by

$$\frac{(\mu'/Q)\psi(\mu)}{T(\mu,s)K(\mu,s)(\mu'-\mu)},$$

which we assume is well defined in $\mu \in [0, 1]$ and integrate with respect to μ from 0 to 1 to obtain

$$\begin{aligned}\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\left[\frac{\Psi(\mu')X(\mu',s)}{\mu'+\mu}\right] &= 1 - T(-\mu,s)\times\\ \times\Bigg[1 - P(\mu,s)\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\frac{\psi(\mu')Y(\mu',s)}{\mu'-\mu} &+ \frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\frac{\psi(\mu')Y(\mu',s)P(\mu',s)}{\mu'-\mu}\Bigg],\end{aligned} \tag{10}$$

where we have used Eq. (9) and defined the function $P(\mu, s)$ (in principle known) by

$$P(\mu,s) \equiv \frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\frac{\psi(\mu')\exp(-\tau_1/\mu)}{T(\mu',s)K(\mu',s)}\,\frac{1}{\mu'+\mu}. \tag{11}$$

If we substitute Eq. (10) in Eq. (5) we get the decoupled equation for $Y(\mu, s)$ as follows:

$$\begin{aligned} &Y(\mu,s)K(\mu,s) =\\ &= T(-\mu,s)\exp\{(-\tau_1/\mu)Q\}+\\ &+ T(-\mu,s)P(\mu,s)[1-\exp\{(-\tau_1/\mu)Q\}]\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\frac{\psi(\mu')Y(\mu',s)}{\mu'-\mu}+\\ &+ T(-\mu,s)\exp\{(-\tau_1/\mu)Q\}\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\frac{\psi((\mu',s))Y(\mu',s)P(\mu',s)}{\mu'-\mu}.\end{aligned} \tag{12}$$

A similar analysis yields the decoupled equation for $X(\mu, s)$:

$$X(\mu, s)K(\mu, s) = [1 - T(-\mu, s)P(\mu, s)\exp\{(-\tau_1/\mu)Q\}] \times$$
$$\times \left[1 + \frac{\mu}{Q}\int_0^1 d\mu' \frac{\psi(\mu')X(\mu', s)}{\mu' - \mu}\right] + \tag{13}$$
$$+T(-\mu, s)\exp\{(-\tau_1/\mu)Q\}\frac{\mu}{Q}\int_0^1 d\mu' \frac{\psi(\mu', s)X(\mu', s)}{\mu' - \mu}.$$

Eqs. (12) and (13) are linear, singular and decoupled and, in principle, solvable by the theory of singular integral equations (Muskhelisvili, 1946).

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Solution of A Radiative Transfer Problem with A Combined Rayleigh and Isotropic Phase Matrix

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ABSTRACT :

Chandrasekhar (1960), has considered the problem, by his discrete ordinate procedure, of the basic non-conservative matrix equation of radiative transfer for diffuse reflection for a combination of Rayleigh and isotropic scattering in a semi-infinite atmosphere. Schnatz and Siewert (1970) have obtained the exact solution of the basic transport equation for non-conservative rayleigh phase matrix by the eigen function approach of Case(1960). Bond and Siewert(1971) have obtained a rigorous general solution of a non-conservative matrix equation of transfer, which appears for consideration of polarization by the eigen function approach of Case(1960). Das (1979a) solved the basic integro-differential equation for radiative transfer in diffuse reflection in a combination of Rayleigh and isotropic scattering for a semi-infinite atmosphere exactly for the emergent intensity matrix by use of the Laplace transform and Wiener-Hopf technique.

In this paper, we shall consider the Laplace transform and Wiener-Hopf technique to solve the matrix transport equation for a scattering which scatters radiation in accordance with the phase matrix obtained from a combination of Rayleigh and isotropic scattering in a semi-infinite atmosphere. The basic matrix equation is subject to the Laplace transform to obtain an integral equation for the emergent intensity matrix. On application of the Wiener-Hopf technique this matrix integral equation gives the emergent intensity matrix in terms of a singular H-matrix and an

unknown matrix. The unknown matrix has been obtained by equating the asymptotic solution of the boundary condition at infinity.

1. INTRODUCTION :

The method of Laplace Transform and Wiener-Hopf Technique has been applied to solve problems of radiative transfer by Dasgupta (1977), Das (1979b) Karanjai and Karanjai (1985) and others. Recently Karanjai and Islam (1993) solved radiative transfer problems with anisotropic scattering by the same method. We like to solve have a particular anisotropically scattering problem where the phase matrix consists of contributions from isotropic and Rayleigh scattereing.

2. BASIC MATRIX TRANSPORT EQUATION AND BOUNDARY CONDITIONS :

The basic integro-differential equation for infinity matrix I (τ,μ) can be written in the form

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\,\omega \int_{-1}^{+1} K(\mu,\mu') \quad I(\tau,\mu') \quad d\mu' \qquad (1)$$

where τ is the optical thickness of the atmosphere, μ is the direction parameter, $I(\tau,\mu)$ is a (2x1) matrix , w $(0 < \omega < 1)$ is the albedo for single scattering. According to Burniston and Siewert (1970),

$K(\mu,\mu)$, a (2x2) matrix, can be written as

$$K(\mu,\mu') = Q(\mu) \quad Q^T (\mu') \qquad (2)$$

where $Q(\mu)$, a (2x2) matrix, can be defined by

$$Q(\mu) = \frac{3(c+2)^{1/2}}{c+2} \begin{bmatrix} c\mu^2 + \frac{2}{3}(1 - c) & (2c)^{1/2}(1 - \mu^2) \\ \frac{1}{3}(c + 2) & 0 \end{bmatrix} \quad (3)$$

$Q^T(\mu)$ is the transpose of $Q(\mu)$, and c is aparameter $(0 < c < 1)$

A solution of Equation (1) is required with the following boundary conditions

$$I(0, -\mu) = 0, \qquad 0 \le \mu \le 1 \quad (4a)$$

and $I(\tau,\mu) \longrightarrow \frac{1}{2}\omega L_o \left[\frac{k}{k - \mu}\right] e^{\tau/k} \; Q(\mu)$ as $\tau \longrightarrow \infty$, (4b)

where K is a positive root greater than one and real of the equation $T(z) = \det D(z)$ (5)

$$\text{where } D(z) = E + z\int_{-1}^{+1} \psi(\mu)\frac{d\mu}{\mu - z} \quad (6)$$

$\psi(\mu)$ is a (2x1) matrix and $\psi(\mu)$ is defined by

$$\psi(\mu) = (1/2)\omega Q^T(\mu) \; Q(\mu) \quad (7)$$

and

E is a unit matrix, $D(z)$ is a (2x2) matrix and L_0 is a specified (2x1) matrix.

3. SOLUTION FOR EMERGENT INTENSITY MATRIX :

The Laplace transform of the intensity matrix is defined by

$$I^*(s,\mu) = s\int_0^\infty e^{-s\tau} \; I(\tau,\mu) \; d\mu \;, \quad \text{Re } s > 0 \quad (8)$$

Let us set $I_u(\tau)$, a (2x1) matrix as

$$I_u(\tau) = (1/2) \int_{-1}^{+1} Q^T(\mu') \; I(\tau,\mu') \, d\mu' \qquad (9)$$

$$I_u(s) = (1/2) \int_{-1}^{+1} Q^T(\mu') \; I^*(s,\mu') \, d\mu' \qquad (10)$$

we subject the Laplace transform as defined in Equation (8) to Equation (1) to get (Using Equations (4a), (9, (10))

$$(\mu s - 1) \, I^*(s, \mu) = \mu s \, I(o, \mu) - \omega \, Q(\mu) \, I_u^*(s) \qquad (11)$$

The solution for the emergent intensity matrix arrived from Equation (11)

$$I(0,\mu) = \omega \, Q(\mu) \; I_u^*(1/\mu) \qquad (12)$$

Equation (12) gives for $\mu = 1/s$, s is complex

$$I(0,1/s) = \omega \, Q(1/s) \, I_u^*(s) \qquad (13)$$

we now apply the (2x2) matrix operator

$$(1/2)\int_{-1}^{+1} \frac{Q^T(\mu) \, d\mu}{(\mu s - 1)} \; , \qquad (14)$$

to Equation (11) to get $\quad D(1/s) \, I_u^*(s) = a(1/s) \qquad (15)$

where D (1/s) is a (2x2) matrix and a(1/s) is (2x1) matrix defined by

$$D(1/s) = E + \int_{-1}^{+1} \frac{\psi(\mu) \, d\mu}{(\mu s - 1)} \; , \qquad (16)$$

and

$$a(1/s) = (1/2) \int_{0}^{1} \frac{\mu s \, Q^T(\mu) \, I(0,\mu) \, d\mu}{(\mu s - 1)} \qquad (17)$$

respectively where

$\psi(\mu)$ is given by Equation (7) , is a (2x2) unit matrix .

Eliminating $I_u^*(s)$ between Equations (13) and (15) we get a matrix integral equation as

$$D(z)\, I(0,z) = \omega\, Q(z) \quad a(z), \text{ where } s = 1/z \tag{18}$$

Following Bond a Siewert (1971) , we have

$$T(z) = \det D(z) = \frac{1}{8}\, c\, T_1\,(z)\, T_2(z) + \left[(1-c) + \frac{3}{2}\, c\,(1-\omega)\, z^2\right] T_0(z) \tag{19}$$

and

$$T_n\,(z) = (-1)^n + 3(1-z^2)\, T_0\,(z) - (-1)^n\; 3(1-\omega) z^2 \;,\quad n = 1 \text{ or } 2 \tag{20}$$

$$T_0(z) = 1 + (1/2)\,\omega z \int_{-1}^{+1} \frac{d\mu}{\mu - z} \;, \tag{21}$$

where T(z) is analytic in the complex plane cut from -1 to +1 along the real axis with two zeros at $z = \pm\, k$, k is real ($k > 1$).

We consider the (2x2) H-matrix equation (cf. Abhyankar and Fymat, 1970) as

$$H(z) = E + zH(z) \int_0^1 H^T\,(\mu)\, \psi\,(\mu)\, d\mu \,/\, (\mu + z\,) \tag{22}$$

where $\psi(\mu)$ is given by Equation (7).

We shall assume that the (2x2) H(z) matrix is analytic in the complex plane cut from -1 to 0 , bounded at the origin , has a pole at $z = -k$, k is real ($k > 1$) and similarly the H(-z) matrix is analytic in the complex plane cut from 0 to 1, bounded at the origin, has a pole at $z = k$, k is real , ($k > 1$). Hence , $H^{-1}\,(z)$, the inverse of the H-matrix, is analytic in the complex plane cut from -1 to 0 and bounded at the origin. If the (2x2) H-matrix is a symmetric matrix, it can be proved that

$$D(z) = H^{-1}(z)\, H^{-1}(-z) \quad , \; z \in (-1,1)^{o} \tag{23}$$

Now Equation (18) together with Equation (23) takes the form

$$H^{-1}(z)\, Q^{-1}(z)\, I(0,z) \left[\frac{k - z}{k}\right]$$
$$= \omega \left[\frac{k - z}{k}\right] H(-z)\, a(z) \qquad (24)$$

where the left hand side of Equation (24) is regular for Re $z > 0$, bounded at the origin and the right hand side of Equation (24) is analytic in $(0,1)^c$, bounded at the origin and tends to a constant matrix (2x1) say A, when $z \longrightarrow \infty$ subject to the assumption that $I(0,z)$ is analytic for Re $z>0$ and bounded at the origin. Hence, by a modified form of Liouville's theorem, Equation (24) gives the emergent intensity matrix $I\,(0,z)$ as

$$I\,(0,z) = \left[\frac{k}{k - z}\right] Q(z)\; H(z)\, A \qquad (25)$$

We now determine the matrix A. The inversion integral gives the intensity matrix $I(\tau,\mu)$ as

$$I(\tau,\mu) = (1/2\pi\; i) \lim_{\nu\to\infty} \int_{\alpha-i\nu}^{\alpha+i\nu} I(s,\mu)\quad e^{s\tau}\quad ds/s\;, \quad \alpha > 0\;, \qquad (26)$$

where

$I^*(s,\mu)$ can be obtained as

$$I^*(s,\mu)/s = \left[I(0,\mu) - (\mu s)^{-1} Q^{-1}(1/s)\; Q(\mu)\right.$$
$$\left. I(0,\mu)\right]/\;(s - 1/\mu) \qquad (27)$$
$$I^*(s,\mu)/s = \left[I(0,\mu)/\;(s- 1/2) - Q(\mu)\right.$$
$$\left. H(1/s)A/(s - 1/k)\mu(s-1/\mu)\right] \qquad (28)$$

The integral of Equation (26) is analytic for s in $(-\infty,-1)^c$, has poles at $s = \pm\; 1/k$, k is real $k > 1$, where $s = 1/\mu$ is not a pole as

$$\lim_{s \longrightarrow 1/\mu} \quad (s - 1/\mu)\ I^{*}\ (s,\mu)\ e^{s\tau}/s \longrightarrow 0 \tag{29}$$

The contribution fo pole at s = 1/k will give the asymptotic solution of Equation (1) as

$$I(\tau,\mu) \longrightarrow \left[\frac{k}{k-\mu} \right] Q(\mu)\ H(k)\ e^{s/k}\ A \quad \text{when } \tau \longrightarrow \infty \tag{30}$$

Equation (4b) with Equation (30) gives the matrix A as -

$$A = (1/2)\ [\ \omega\ H^{-1}(K)\ ]\ L_0 \tag{31}$$

Equation (25) with Equation (31) gives the emergent intensity in the form

$$I(O,z) = (1/2)\ \omega\ L_0\ H^{-1}(k)\ H(z)\ Q(z)\ \left[\frac{k}{k - z} \right] \tag{32}$$

4. CONCLUSIONS :

Here we allow the values c (O < c < 1) and ω (O< ω <1) to study the general mixture of Rayleigh and isotropic scattering.

a. When ω = 1 and c (O< c <1) the basic matrix transport equation yields a conservative model for a mixture of Rayleigh and isotropic scattering.

b. When ω (O< ω <1) and c=1 , we obtain the general Rayleigh scattering problem.

c. When c = 1 and w = 1 , the problem yields Chandrasekhar's (1960) Rayleigh scattering model and $Q(\mu)$ reduces to Sekera's (1963) form for factorising the Rayleigh scattering phase matrix (Das,1979c).

d. In this problem there exists some possibilities for future development such as determination of the H-matrix expression and the values of the D(z) matrix on both sides of the cut etc.

e. There exists some possibilities to determine a characteristic function which is an even function having polynomial expression but has a transcendental form.

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