APPENDIX III

6.3 The relation (4.38) of chapter 4

To establish the relation (4.38) , of Chapter 4 , I consider

$$D_m(x) = \sum_{r=1}^{k} \eta_r \omega_r \sum_i \frac{a_i \mu_{(r)i}^{m}}{1 + \mu_{(r)i}\, x} =$$

$$= (-1)^m \sum_{r=1}^{k} \eta_r \omega_r \sum_i \frac{a_i \mu_{(r)i}^{m}}{1 - \mu_{(r)i} x} \qquad (6.16)$$

I can derive a single recursion formula for $D_m(x)$ .Then

$$D_m(x) = \frac{1}{x}\left[\sum_{r=1}^{k} \eta_r \omega_r \sum_i a_i \mu_{(r)}^{m-1} \left(1 - \frac{1}{1 + \mu_{(r)}^{x}} \right) \right] =$$

$$= \frac{1}{x}\left[\psi_{m-1} - D_{m-1} \right] \qquad (6.17)$$

where $\psi_m = \sum_r \eta_r \omega_r \sum_i a_i \mu_{(r)i}^{m}$ (6.18)

From this formula I have

$$D_m(x) = \frac{\psi_{m-1}}{x} - \frac{\psi_{m-2}}{x^2} + \ldots\ldots + (-1)^{m-2} \frac{\psi_1}{x^{m-1}} +$$

$$+ \frac{(-1)^{m-1}}{x^m} \left[\psi_0 - D_0(x) \right] \qquad (m = 0,1,\ldots 4n) \qquad (6.19)$$

and

$$\psi_0 = 2 \sum_{r=1}^{k} \eta_r \omega_r \qquad (6.20)$$

Let $P_{2n}(\mu) = \sum_{j=0}^{n} P_{2n}\mu^{2n}$ i.e. p_{2j} be the coefficient of μ^{2j}

in the Legendre polynomial $P_{2n}(\mu)$,

then

$$\sum_{j=0}^{n} P_{2j} \; D_{2j}\left(\zeta_r K_{(r)\alpha} \right) = \sum_{r=1}^{k} \eta_r \omega_r \sum_i \frac{a_i}{1 + \mu_{(r)} K_{(r)\alpha}} \times$$

$$\times \left[\sum_{j=0}^{n} P_{2j} \mu^{2j} \right] \qquad (6.21)$$

Since $\mu_{(r)i}$'s are the zeros of $P_{2n}(\mu)$, Equation (6.21) reduces to

$$\sum_{j=0}^{n} P_{2j} D_{2j} \left(\zeta_r K_{(r)\alpha} \right) = 0 \qquad (6.22)$$

Substituting for $D_{2j}\left(\zeta_r K_{(r)\alpha} \right)$ from Equation (6.20) into Equation (6.22) I get required form of the characteristic equation as

$$\frac{P_{2n} \left(1 - M/N \right)}{\zeta_r^{2n} \; K_{(r)\alpha}^{2n}} + \dots\dots\dots\dots + P_o = 0 \qquad (6.23)$$

where M and N are given by the equation (4.39).

From this equation it follows that

$$\frac{1}{\left(\zeta_r K_{(r)1} \; \dots \; \zeta_r k_{(r)n} \right)^2} = \frac{(-1)^n \; P_o}{(1 - M/N) \; P_{2n}} =$$

$$= \frac{(\mu_{(r)1} \cdots\cdots \mu_{(r)n})^2}{(1 - M/N)} \qquad (6.24)$$

and $\mu_{(r)1} \cdot \mu_{(r)2} \cdots\cdot \mu_{(r)n} \cdot \zeta_r K_{(r)1} \cdots \zeta_r K_{(r)n} =$

$$= (1 - M/N)^{1/2} \qquad (6.25)$$

which is the required relation.

SOLUTION OF THE EQUATION OF TRANSFER FOR INTERLOCKED MULTIPLETS BY THE METHOD OF DISCRETE ORDINATES WITH THE PLANCK FUNCTION AS A NONLINEAR FUNCTION OF OPTICAL DEPTH

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Abstract. The equation of transfer for interlocked multiplets has been solved by the method of discrete ordinates, originally due to Chandrasekhar, considering nonlinear form of the Planck function to be

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}.$$

1. Introduction

Woolley and Stibbs (1953) applied the theory of formation of absorption lines by coherent scattering to the case of interlocking without redistribution and deduced the equation of transfer in the Milne–Eddington model. They have also obtained a solution for the case of triplets by Eddington's approximate method. Busbridge and Stibbs (1954) applied the principle of invariance governing the law of diffuse reflection with a slight modification to solve exactly the equation of transfer in the M–E model. Dasgupta and Karanjai (1972) applied Sobolev's probabilistic method to solve the transfer equation for the case of interlocking without redistribution. Another exact solution of the equation of transfer has been given by Dasgupta (1956) by his form of the Wiener–Hopf technique. Karanjai and Barman (1981) applied the extension of the method of discrete ordinates to find an exact solution of the problem of line formation by interlocking in the M–E model. Dasgupta (1978) obtained an exact solution of the transfer equation for non-coherent scattering arising from interlocking of principal lines without redistribution by Laplace transformation and the Wiener–Hopf technique using a new representation of the H-function obtained by Dasgupta (1977). While solving the transfer equation, Dasgupta considered the Planck function to be linear in t (optical depth), i.e., $B_\nu(T) = B(t) = b_0 + b_1\tau$. Karanjai and Karanjai (1985) solved the equation of transfer for interlocked multiplets with the Planck function as a nonlinear function of optical depth following the method used by Dasgupta (1978). They considered two nonlinear forms of $B_\nu(T)$, viz.:

(1) an exponential atmosphere (Delg'Innocenti, 1979) in which

$$B_\nu(T) = B(t) = b_0 + b_1 e^{-\beta t};$$

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(2) an atmosphere (Busbridge, 1955) in which

$$B_\nu(T) = B(t) = b_0 + b_1 t + E_2(t) .$$

In this paper, we have obtained the solution of the equation of transfer for interlocked multiplets by discrete ordinate method in an exponential atmosphere in which

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau} ,$$

where τ is the optical depth.

2. The Equation of Transfer

The equation of transfer considered here is of the form (Woolley and Stibbs, 1953)

$$\mu \frac{dI_r(\tau, \mu)}{d\tau} = (1 + \eta_r)I_r(\tau, \mu) - (1 + \varepsilon\eta_r)B_\nu(T) - \\ - \tfrac{1}{2}(1 - \varepsilon)\alpha_r \sum_{p=1}^{n} \eta_p \int_{-1}^{+1} I_p(\tau, \mu')\, d\mu' , \tag{1}$$

where τ denotes the optical depth and $\eta_r = k_r/k$ denoting the line absorption coefficient for the rth line and k the continuous absorption coefficient which is assumed to be constant for each line. In the present case we consider that the collision constant ε and Planck's function remain constant for each line. We also consider an exponential atmosphere for which Planck's function, i.e., the thermal source function is given (Degl'Innocenti, 1979) by

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau} , \tag{2}$$

where b_0, b_1, and β are three positive constants.

Now, if we use Equation (2) in Equation (1) we have the transfer equation for the rth interlocked line in the form

$$\mu \frac{dI_r(\tau, \mu)}{d\tau} = (1 + \eta_r)I_r(\tau, \mu) - (1 + \varepsilon\eta_r)(b_0 + b_1 e^{-\beta\tau}) - \\ - \tfrac{1}{2}(1 - \varepsilon)\alpha_r \sum_{p=1}^{n} \int_{-1}^{+1} I_p(\tau, \mu')\, d\mu' , \tag{3}$$

where

$$\alpha_r = \eta_r/(\eta_1 + \eta_2 + \cdots + \eta_k) , \quad r = 1, 2, \ldots, k ; \tag{4}$$

so that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1 . \tag{5}$$

Equation (3) is to be solved subject to the boundary conditions

$$I_r(0, -\mu') = 0, \quad (0 < \mu \leq 1) \tag{6}$$

and

$$I_r(\tau, \mu) e^{-\tau/\mu} \to 0 \quad \text{as} \quad \tau \to \infty . \tag{7}$$

3. Solution of Equation (3)

Following Busbridge (1953) and Stibbs (1953), let

$$I_r(\tau, \mu) = b_0 + \frac{b_1 T_r}{1 + \beta\mu\xi_r} e^{-\beta\tau} + I_r^*(\tau, \mu) \tag{8}$$

represent the solution of Equation (3), where

$$T_r = \frac{\xi_r(1 + \varepsilon\eta_r)}{1 - \frac{1}{2\beta}(1 - \varepsilon)\eta_r \log \frac{1 + \eta_r + \beta}{1 + \eta_r - \beta}} \tag{9}$$

and

$$\xi_r = \frac{1}{1 + \eta_r} . \tag{10}$$

This consists of two parts. The first part consists of the solution for a bounded atmosphere as τ tends to infinity. The second part: viz., $I_r^*(\tau, \mu)$ represents the departure of the asymptotic solution from the value $I_r(\tau, \mu)$ as we approach the boundary.

Now if we insert $I_r(\tau, \mu)$ from Equation (8) in Equation (3) and taking

$$w_r = \frac{(1 - \varepsilon)\eta_r}{1 + \eta_r}, \tag{11}$$

we have the equation

$$\xi_r\mu \frac{dI_r^*(\tau, \mu)}{d\tau} = I_r^*(\tau, \mu) - \frac{w_r}{2} \frac{1}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \int_{-1}^{+1} I_p^*(\tau, \mu')\, d\mu' \right] \tag{12}$$

together with the boundary conditions

$$I_r^*(0, -\mu') = -b_0 - \frac{b_1 T_r}{1 - \beta\xi_r\mu'} \tag{13}$$

and

$$I_r^*(\tau, \mu) e^{-\tau/\mu} \to 0 \quad \text{as} \quad \tau \to \infty . \tag{14}$$

For the sake of convenience, Equation (12) can be rewritten in the form

$$\xi_r \mu \frac{\mathrm{d}I^*_{(r)}(\tau,\mu)}{\mathrm{d}\tau} = I^*_{(r)}(\tau,\mu) - \frac{W_r/2}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \int_{-1}^{+1} I^*_{(p)}(\tau,\mu')\,\mathrm{d}\mu' \right], \tag{15}$$

together with the boundary conditions

$$I^*_{(r)}(0,-\mu) = -\frac{b_1 T_r}{1-\xi_r \beta\mu} - b_0 \tag{16}$$

and

$$I^*_{(r)}(\tau,\mu)\, e^{-\tau/\mu} \to 0 \quad \text{as} \quad \tau \to \infty\,. \tag{17}$$

Equation (15) can be replaced by the system of $2n$ linear equations

$$\xi_r \mu_{(r)i} \frac{\mathrm{d}I^*_{(r)i}}{\mathrm{d}\tau} = I^*_{(r)i} - \frac{w_r}{2} \frac{1}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \sum_j a_j I^*_{(p)_j} \right], \tag{18}$$

$$(i = \pm 1, \pm 2, \ldots, \pm n),$$

where the $\mu_{(r)i}$'s ($i = \pm 1, \ldots, \pm n$ and $\mu_{(r)_{-i}} = -\mu_{(r)_i}$) are the zeros of the Legendre polynomials $P_{2n}(\mu)$ which are dependent on the lines of interlocking and a_j's ($j = \pm 1, \ldots, \pm n$) and ($a_{-j} = a_j$) are corresponding Gaussian weights. However, it is to be noted that there is no term with $j = 0$. For simplicity, we write

$$I^*_{(r)i} \quad \text{for} \quad I^*_{(r)i}(\tau, \mu_{(r)i}) \tag{19}$$

in Equation (18).

The system of Equations (18) admits of integrals of the form

$$I^*_{(r)i} = g_{(r)i}\, e^{-K\tau}, \quad (i = \pm 1, \ldots, \pm n), \tag{20}$$

where $g_{(r)i}$'s and K are constants.

Now if we insert this form for $I^*_{(r)i}$ in Equation (18) we have

$$g_{(r)i}[1 + \xi_r \mu_{(r)i} K] = \frac{w_r}{2} \frac{1}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p \sum_j a_j g_{(p)_j} \right], \tag{21}$$

$$\therefore\ g_{(r)i} = w_r \frac{\text{constant}}{1 + \xi_r \mu_{(r)i} K}\,. \tag{22}$$

If we insert for $g_{(r)i}$ from Equation (22) back into Equation (21) we obtain the characteristic equation in the form

$$1 = \frac{1}{2} \frac{1}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p w_p \sum_{j} \frac{a_j}{1 + \xi_p \mu_{(p)_j} K} \right], \tag{23}$$

in which $a_j = a_{-j}$ and $\mu_{(r)_{-j}} = -\mu_{(r)_j}$.

We can rewrite the characteristic equation in the form

$$1 = \frac{1}{\sum_{p=1}^{k} \eta_p} \left[\sum_{p=1}^{k} \eta_p w_p \sum_{j=1}^{n} \frac{a_j}{1 - \xi_p^2 \mu_{(p)_j}^2 K^2} \right]. \tag{24}$$

This is the characteristic equation which gives the values of $K_{(r)}$. If $w_r < 1$ ($r = 1, 2, \dots, k$), the characteristic equation (24) gives distinct non-zero roots which occur in pairs as $\pm K_{(r)\alpha}$ ($\alpha = 1, 2, \dots, n$).

Therefore, Equation (18) admits the $2n$ independent integrals of the form

$$I^*_{(r)i} = w_r \frac{\text{constant}}{1 \pm \xi_r \mu_{(r)i} K_{(r)\alpha}} e^{\mp K_{(r)\alpha} \tau}. \tag{25}$$

According to Chandrasekhar (1960), the solutions (20) satisfying our requirements of the boundedness of the solutions are

$$I^*_{(r)i} = w_r b_1 \sum_{\alpha=1}^{n} \frac{L_{(r)\alpha} e^{-K_{(r)\alpha}\tau}}{1 + \xi_r K_{(r)\alpha} \mu_{(r)i}}, \tag{26}$$

together with the boundary condition

$$I^*_{(r)-i} = -b_0 - \frac{b_1 T_r}{1 - \xi_r \beta \mu_{(r)}} \quad \text{at} \quad \tau = 0. \tag{27}$$

4. The Elimination of the Constant and the Expression of the Law of Diffuse Reflection in Closed Form

The boundary conditions and the emergent intensity can be expressed in the form

$$S_r(\mu_{(r)i}) = 0, \quad (i = 1, 2, \dots, n) \tag{28}$$

and

$$I^*_{(r)}(0, \mu) = w_r b_1 S_r(-\mu) - \frac{T_r/w_r}{1 + \xi_r \beta \mu} - \frac{b_0}{w_r b_1}, \tag{29}$$

where

$$S_r(\mu) = \sum_{\alpha=1}^{n} \frac{L_{(r)\alpha}}{1 - \xi_r K_{(r)}\mu} + \frac{T_r/w_r}{1 - \xi_r \beta \mu} + \frac{b_0}{w_r b_1}. \tag{30}$$

Next we observe that the function

$$(1 - \xi_r \beta\mu) \prod_{\alpha=1}^{n} (1 - \xi_r K_{(r)\alpha}\mu) S_r(\mu)$$

in a polynomial of degree $(n + 1)$ in μ which vanishes for $\mu = \mu_i$, $i = 1, 2, \ldots, n$. There must accordingly exist a relation of the form

$$(1 - \xi_r \beta\mu) \prod_{\alpha=1}^{n} (1 - \xi_r K_{(r)\alpha}\mu) S_r(\mu) \sim (\mu - C_r) \prod_{i=1}^{n} (\mu - \mu_i), \tag{31}$$

where C_r is a constant.

The constant of proportionality can be found by comparing the coefficient of the highest power of μ (namely, μ^{n+1}).

So we have, from Equation (31)

$$S_r(\mu) = \frac{(-1)^{n+1}}{b_1 w_r} \xi_r K_{(r)1} \cdots \xi_r K_{(r)n} \xi_r \beta \frac{P_r(\mu)(\mu - C_r)}{R_r(\mu)(1 - \beta\xi_r \mu)}, \tag{32}$$

where

$$P_r(\mu) = \prod_{i=1}^{n} (\mu - \mu_i), \quad (i = 1, 2, \ldots, n) \tag{33}$$

and

$$R_r(\mu) = \prod_{\alpha=1}^{n} (1 - \xi_r K_{(r)\alpha}\mu), \quad (\alpha = 1, 2, \ldots, n). \tag{34}$$

Moreover, if we combine Equations (32) and (33), we obtain

$$L_{(r)\alpha} = (-1)^n \frac{b_0}{w_r b_1} \xi_r K_{(r)1} \cdots \xi_r K_{(r)n} \xi_r \beta \frac{P_r(1/\xi_r K_{(r)\alpha})}{R_{(r)\alpha}(1/\xi_r K_{(r)\alpha})} \times$$

$$\times \frac{\left(\dfrac{1}{\xi K_{(r)\alpha}} - C_r\right)}{\left(1 - \dfrac{\beta\xi_r}{K_{(r)\alpha}\xi_r}\right)}, \tag{35}$$

where

$$R_{(r)\alpha}(x) = \prod_{\gamma \neq \alpha} (1 - \xi_r K_{(r)\gamma} x) \tag{36}$$

and

$$\beta \neq K_{(r)\alpha}. \tag{37}$$

The roots of the characteristic equation (17) can be written in the form

$$\xi_r K_{(r)1} \cdots \xi_r K_{(r)n} \mu_{(r)1} \cdots \mu_{(r)n} = (1 - w_r)^{1/2}. \tag{38}$$

Now by use of Equation (38), Equation (32) becomes

$$S_r(\mu) = -\frac{b_0 \xi_r \beta (1 - w_r)^{1/2} H_r(-\mu)\,[\mu - C_r]}{w_r b_1 (1 - \beta \xi_r \mu)}, \tag{39}$$

where

$$H_r(\mu) = \frac{1}{\mu_{(r)1} \cdots \mu_{(r)n}} \frac{\prod_{i=1}^{n} (\mu + \mu_{(r)i})}{\prod_{\alpha=1}^{n} (1 + \xi_r K_{(r)\alpha} \mu)}; \tag{40}$$

and the characteristic roots are evaluated from Equation (24). If we put $\mu = 0$ in Equations (30) and (40) we have

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} + \frac{T_r}{w_r} + \frac{b_0}{w_r b_1} = \frac{b_0}{w_r b_1} \xi_r \beta (1 - w_r)^{1/2} C_r, \tag{41}$$

and we can next evaluate $\Sigma_{\alpha=1}^{n} L_{(r)\alpha}$ from Equation (35). Then

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} = (-1)^{n+1} \frac{b_0}{b_1 w_r} [\xi_r K_{(r)1} \ldots \xi_r K_{(r)n} \xi_r \beta f_r(0)], \tag{42}$$

where

$$f_r(x) = \sum_{\alpha=1}^{n} \frac{P_r(1/\xi_r K_{(r)\alpha}) \left[\frac{1}{\xi_r K_{(r)\alpha}} - C_r\right]}{R_{(r)\alpha}(1/\xi_r K_{(r)\alpha}) \left(1 - \frac{\xi_r \beta}{K_{(r)\alpha} \xi_r}\right)} R_{(r)\alpha}(x). \tag{43}$$

Now $f_r(x)$ defined in this manner is a polynomial of degree $n - 1$ in x, which takes the values

$$\frac{P_r(1/\xi_r K_{(r)\alpha}) \left[\frac{1}{\xi_r K_{(r)\alpha}} - C_r\right]}{\left(1 - \frac{\xi_r \beta}{K_{(r)\alpha} \xi_r}\right)},$$

for

$$x = 1/\xi_r K_{(r)\alpha}, \quad (\alpha = 1, 2, \ldots, n).$$

In other words,

$$(1 - \xi_r \beta x) f_r(x) - P_r(x) \qquad (x - C_r) = 0. \tag{44}$$

Therefore, we must accordingly have a relation of the form

$$(1 - \xi_r \beta x) f_r(x) - P_r(x)(x - C_r) = R_r(x)(A_r x + B_r), \tag{45}$$

where A_r and B_r are certain constants to be determined.

The constant A_r follows from the comparison of the coefficients of x^{n+1}. Thus

$$A_r = \frac{(-1)^{n+1}}{\xi_r K_{(r)1} \cdots \xi_r K_{(r)n}}. \tag{46}$$

Next, if we put $x = (\xi_r \beta)^{-1}$ in Equation (46) (cf. Chandrasekhar, 1960) we have

$$B_r = \frac{(-1)^n}{\xi_r \beta \xi_r K_{(r)1} \cdots \xi_r K_{(r)n}} + \frac{\left(C_r - \frac{1}{\beta \xi_r}\right) P_r\left(\frac{1}{\xi_r \beta}\right)}{R_r\left(\frac{1}{\xi_r \beta}\right)}, \tag{47}$$

i.e.,

$$B_r = \frac{(-1)^n}{\xi_r \beta \xi_r K_{(r)1} \cdots \xi_r K_{(r)n}} + (-1)^n \mu_{(r)1} \cdots \mu_{(r)n} \times$$

$$\times H_r\left(-\frac{1}{\beta \xi_r}\right)\left(C_r - \frac{1}{\xi_r \beta}\right). \tag{48}$$

Now if we use the relations (48), (47), and (46) we get

$$f_r(0) = -C_r P_r(0) + B_r R_r(0),$$

i.e.,

$$f_r(0) = -C_r(-1)^n \mu_{(r)1} \cdots \mu_{(r)n} + \frac{(-1)^n}{\xi_r \beta \xi_r K_{(r)1} \cdots \xi_r K_{(r)n}} +$$

$$+ (-1)^n \mu_{(r)1} \cdots \mu_{(r)n} H_r\left(-\frac{1}{\beta \xi_r}\right)\left(C_r - \frac{1}{\beta \xi_r}\right). \tag{49}$$

From Equation (43) using Equation (49) we have

$$\sum_{\alpha=1}^{n} L_{(r)\alpha} = \frac{b_0}{w_r b_1} C_r (1 - w_r)^{1/2} \xi_r \beta - \frac{b_0}{w_r b_1} +$$

$$+ \frac{b_0}{w_r b_1} \xi_r \beta (1 - w_r)^{1/2} H\left(-\frac{1}{\beta \xi_r}\right)\left(\frac{1}{\xi_r \beta} - C_r\right). \tag{50}$$

Now if we use Equation (50) in Equation (42) we get

$$C_r = \frac{1}{\xi_r \beta} + \frac{T_r b_1}{b_0 \xi_r \beta (1 - w_r)^{1/2} H_r\left(-\frac{1}{\beta \xi_r}\right)} ; \tag{51}$$

and if we combine Equation (40), the diffusely reflected intensity $I^*_{(r)}(0, \mu)$ in Equation (29) takes the form

$$I^*_r(0, \mu) = \frac{b_0 \xi_r \beta (1 - w_r)^{1/2} H_r(\mu)\,[\mu + C_r]}{1 + \beta \xi_r \mu} - \frac{T_r b_1}{1 + \beta \xi_r \mu} - b_0 . \tag{52}$$

This is the required solution in a closed form. If we combine Equation (8) at $\tau = 0$ and Equation (52) we have

$$I_r(0, \mu) = \frac{b_0 \xi_r \beta (1 - w_r)^{1/2} H_r(\mu)\,[\mu + C_r]}{1 + \beta \xi_r \mu} , \tag{53}$$

which is the required solution of Equation (3) in the nth approximation by the discrete-ordinates method.

If we put C_r from Equation (51), we get the solution in the form

$$I_r(0, \mu) = b_0 (1 - w_r)^{1/2} H_r(\mu) + \frac{b_1 T_r}{H_r(-1/\xi_r \beta)} \frac{H_r(\mu)}{(1 + \beta \xi_r \mu)} . \tag{54}$$

Chandrasekhar's (1960) equation for $I_r(0, \mu)$ in the case of coherent scattering is given by ($B_\nu(T) = b_0 + b_1 \tau$) (see also Karanjai and Barman, 1981), and

$$I_r(0, \mu) = b_1 \xi_r (1 - w_r)^{1/2} \mu H_r(\mu) + b_0 (1 - w_r)^{1/2} H_r(\mu) + \\ + b_1 (1 - w_r)^{1/2} \xi_r \left[\sum_{\alpha = 1}^{n} \frac{1}{\xi_r K_{(r)\alpha}} - \sum_{j = 1}^{n} \mu_{(r)_j} \right] . \tag{55}$$

If we compare Equations (54) and (55) we see that if we put $b_1 = 0$, we have the same solution for both cases. Moreover, for large values of β, i.e., $\beta \to \infty$. The solution (54) takes the form

$$I_r(0, \mu) = b_0 (1 - w_r)^{1/2} H_r(\mu) , \tag{56}$$

i.e., $B_\nu(T)$ then behaves like a constant or independent of τ. This fact can also be explained from the point of view that

$$B_\nu(T) = b_0 + b_1 e^{-\beta \tau} \to b_0 \quad \text{as} \quad \beta \to \infty .$$

Appendix

To establish the relation (38) we consider

$$D_m(x) = \sum_{r=1}^{k} \eta_r w_r \sum_i \frac{a_i \mu_{(r)i}^m}{1 + \mu_{(r)i} x} =$$

$$= (-1)^m \sum_{r=1}^{k} \eta_r w_r \sum_i \frac{a_i \mu_{(r)i}^m}{1 - \mu_{(r)i} x}, \tag{57}$$

we can derive a single recursion formula for $D_m x$. Then

$$D_m(x) = \frac{1}{x}\left[\sum_{r=1}^{k} \eta_r w_r \sum_i a_i \mu_{(r)i}^{m-1}\left(1 - \frac{1}{1 + \mu_{(r)i} x}\right)\right] =$$

$$= \frac{1}{x}\left[\psi_{m-1} - D_{m-1}\right], \tag{58}$$

where

$$\psi_m = \sum_r \eta_r w_r - \sum_i a_i \mu_{(r)i}^m . \tag{59}$$

From this formula we have

$$D_m(x) = \frac{\psi_{m-1}}{x} - \frac{\psi_{m-2}}{x^2} + \cdots + (-1)^{m-2}\frac{\psi_1}{x^{m-1}} +$$

$$+ \frac{(-1)^{m-1}}{x^m}\left[\psi_0 - D_0(x)\right], \quad (m = 0, 1, \ldots, 4n) \tag{60}$$

and

$$\psi_0 = 2\sum_{r=1}^{k} \eta_r w_r . \tag{61}$$

Moreover, let P_{2j} be the coefficient of μ^{2j} in the Legendre polynomial $P_{2n}(\mu)$.

Then

$$\sum_{j=0}^{n} P_{2j} D_{2j}(\xi_r K_{(r)\alpha}) =$$

$$= \sum_{r=1}^{k} \eta_r w_r \sum_i \frac{a_i}{1 + \mu_{(r)i}\xi_r K_{(r)\alpha}} \sum_{j=0}^{k} P_{z_j} \mu_{(r)i}^{z_j} . \tag{62}$$

Since the $\mu_{(r)i}$'s are the zeros $P_{2n}(\mu)$. Equation (62) reduces to

$$\sum_{j=0}^{n} P_{2j} D_{z_j}(\xi_r K_{(r)\alpha}) = 0 . \tag{63}$$

If we substitute for $D_{2j}(\xi_r K_{(r)\alpha})$ from Equation (61) into Equation (63) we get the required form of the characteristic equation as

$$\frac{P_{2n}}{\xi_r^{2n} K_{(r)\alpha}^{2n}}(w_r - 1) + \cdots + P_0 = 0 . \tag{64}$$

From this equation it follows that

$$\frac{1}{(\xi_r K_{(r)1} \ldots \xi_r K_{(r)n})^2} = \frac{(-1)^n P_0}{1 - w_r P_{2n}} = \frac{(\mu_{(r)1} \cdots \mu_{(r)n})^2}{1 - w_r}$$

and

$$\mu_{(r)1} \cdots \mu_{(r)n} K_{(r)1} \xi_r \ldots K_{(r)n} \xi_r = (1 - w_r)^{1/2} . \tag{65}$$

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SOLUTION OF THE EQUATION OF TRANSFER FOR COHERENT SCATTERING IN AN EXPONENTIAL ATMOSPHERE BY EDDINGTON'S METHOD

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Abstract. An approximate solution of the transfer equation for coherent scattering in stellar atmospheres with Planck's function as a nonlinear function of optical depth, viz.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$$

is obtained by Eddington's method.

1. Introduction

Chandrasekhar (1960) applied the method of discrete ordinates to solve the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz., $B_\nu(T) = b_0 + b_1\tau$. The equation of transfer for coherent scattering has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient, is constant) and by Strömgren's method (when η_ν has small but arbitrary variation with optical depth (see Woolley and Stibbs, 1953). Dasgupta (1977b) applied the method of Laplace transform and Wiener–Hopf technique to find an exact solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions

$$B_\nu(T) = b_0 + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau),$$

by use of a new representation of the H-function obtained by Dasgupta (1977a).

In the present paper, we have obtained an approximate solution of the equation of transfer for coherent isotropic scattering by the method used by Eddington (Woolley and Stibbs, 1953) in an exponential atmosphere (Degl'Innocenti, 1979; Karanjai and Karanjai, 1985; Deb *et al.*, 1990),

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau},$$

where β, b_0, b_1 are positive constants.

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2. Equation of Transfer

The equation of transfer for coherent scattering can be written (cf. Woolley and Stibbs, 1953) in the form

$$\cos\theta\, \mathrm{d}I_\nu(\theta)/\rho\, \mathrm{d}x = -(k + l_\nu)I_\nu(\theta) + (1-\varepsilon)l_\nu J_\nu + (k + \varepsilon l_\nu)B_\nu(T)\,. \qquad (1)$$

To find an approximate solution of Equation (1), we proceed as follows: let

$$J_\nu = (1/4\pi)\int I_\nu(\theta)\,\mathrm{d}w\,, \qquad (2a)$$

$$H_\nu = (1/4\pi)\int I_\nu(\theta)\cos\theta\,\mathrm{d}w\,, \qquad (2b)$$

$$K_\nu = (1/4\pi)\int I_\nu(\theta)\cos^2\theta\,\mathrm{d}w\,, \qquad (2c)$$

in which the integration is made over all directions.

By multiplying Equation (1) by $(\mathrm{d}w/4\pi)$ and $(\mathrm{d}w\cos\theta/4\pi)$ and integrating we obtain

$$\mathrm{d}H_\nu/\rho\,\mathrm{d}x = -(k + \varepsilon l_\nu)(J_\nu - B_\nu)\,, \qquad (3)$$

$$\mathrm{d}K_\nu/\rho\,\mathrm{d}x = -(k + l_\nu)H_\nu\,, \qquad (4)$$

where $B_\nu(T) = B_\nu$. If we measure the optical depth in the continuous spectrum outside the line so that $\mathrm{d}\tau = -k\rho\,\mathrm{d}x$ and set $l_\nu/k = \eta_\nu$, then (3) and (4) becomes

$$\mathrm{d}H_\nu/\mathrm{d}\tau = (1 + \varepsilon\eta_\nu)(J_\nu - B_\nu)\,, \qquad (5)$$

$$\mathrm{d}K_\nu/\mathrm{d}\tau = (1 + \eta_\nu)H_\nu\,. \qquad (6)$$

If, moreover, we assume that η_ν is independent of τ, the equation can be readily integrated. Introducing Eddington's approximation

$$K_\nu = (1/3)J_\nu\,,$$

Equations (5) and (6) can be combined to give

$$\mathrm{d}^2J_\nu/\mathrm{d}\tau^2 = q_\nu^2(J_\nu - B_\nu)\,, \qquad (7)$$

where

$$q_\nu^2 = 3(1 + \varepsilon\eta_\nu)(1 + \eta_\nu)\,, \qquad (8)$$

Equation (7) is to be solved subject to the boundary conditions: (A) $J_\nu = 2H_\nu$ at $\tau = 0$ and (B) the requirement that $(J_\nu - B_\nu)$ shall not increase exponentially as $\tau \to \infty$.

3. Solution of Equation (7)

Let

$$B_\nu = b_0 + b_1 e^{-\beta\tau}\,. \qquad (9)$$

Then Equation (7) can be written in the form

$$d^2J_\nu/d\tau^2 = q_\nu^2 J_\nu - b_0 q_\nu^2 [1 + (b_1/b_0)\, e^{-\beta\tau}]\,, \tag{10}$$

which is a second-order differential equation.

If we solve Equation (10) and use the boundary condition (B) we get

$$J_\nu = b_0 + b_1 e^{-\beta\tau} + b_2 e^{-q_\nu\tau} + [b_1\beta^2/(q_\nu^2 - \beta^2)]\, e^{-\beta\tau}\,, \tag{11}$$

where b_2 is a constant to be determined from the boundary condition (A), where $\beta \neq q_\nu$.

From Equation (11) we get

$$(dJ_\nu/d\tau)_{\tau=0} = -[\beta b_1 + b_2 q_\nu + b_1\beta^3/(q_\nu^2 - \beta^2)]\,. \tag{12}$$

From Equation (6) with $K_\nu = (1/3)J_\nu$ we find that

$$H_\nu = [1/3(1 + \eta_\nu)]\,[(dJ_\nu/d\tau)]\,. \tag{13}$$

Hence,

$$b_2 = -\frac{\left[(1 + \eta_\nu)(b_0 + b_1) + \frac{2}{3}\beta b_1 + (1 + \eta_\nu + \frac{2}{3}\beta)\,\dfrac{b_1\beta^2}{q_\nu^2 - \beta^2}\right]}{1 + \eta_\nu + \frac{2}{3}q_\nu}\,. \tag{14}$$

Finally we get

$$J_\nu = b_0 + b_1 e^{-\beta\tau} + \left[\frac{b_1\beta^2}{q_\nu^2 - \beta^2}\right] e^{-\beta\tau} - $$

$$- \frac{\left[(1 + \eta_\nu)(b_0 + b_1) + \frac{2}{3}b_1\beta + (1 + \eta_\nu + \frac{2}{3}\beta)\,\dfrac{b_1\beta^2}{q_\nu^2 - \beta^2}\right] e^{-q_\nu\tau}}{(1 + \eta_\nu + \frac{2}{3}q_\nu)}\,. \tag{15}$$

Now, J_ν (the average intensity) enables us to find the intensity within the absorption line at any optical depth and in any direction by solving the fundamental equation of line formation,

$$\cos\theta\, dI_\nu(\theta)/d\tau = (1 + \eta_\nu)I_\nu(\theta) - (1 - \varepsilon)\eta_\nu J_\nu -$$

$$- (1 + \varepsilon\eta_\nu)B_\nu\,; \tag{16}$$

J_ν and B_ν being known function of τ.

The solution for $I_\nu(\theta)$ can be writen down immediately since Equation (16) is a linear differential equation with constant coefficients.

4. Residual Intensity

The residual intensity in the mean contours is given (cf. Woolley and Stibbs, 1953) by

$$r_\nu = (H_\nu/H)_{\tau=0}\,, \tag{17}$$

where the omission of the suffix ν means *outside the line*. By virtue of the boundary condition $J_\nu = 2H_\nu$ at $\tau = 0$ we have

$$r_\nu = (J_\nu/J)_{\tau=0}\,. \tag{18}$$

Also, outside the line $\eta_\nu = 0$ and $q_\nu = \sqrt{3}$, Equation (15) with $\tau = 0$ gives

$$J_\nu(0) = b_0 + b_1 + \frac{b_1\beta^2}{q_\nu^2 - \beta^2} -$$

$$- \frac{(1 + \eta_\nu)(b_0 + b_1) + (1 + \eta_\nu + \frac{2}{3}\beta)\dfrac{b_1\beta^2}{q_\nu^2 - \beta^2} + \frac{2}{3}\beta b_1}{1 + \eta_\nu + \frac{2}{3}q_\nu}\,. \tag{20}$$

Hence, by Equations (18), (19), and (20) we have

$$r_\nu = \frac{\frac{2}{3}q_\nu(\beta^2 - q_\nu^2)b_0 + \frac{2}{3}q_\nu^2(\beta - q_\nu)b_1}{2\sqrt{3}(\beta^2 - 3)b_0 + 6(\beta - \sqrt{3})b_1} \times$$

$$\times \frac{(\beta^2 - 3)(3 + 2\sqrt{3})}{(\beta^2 - q_\nu)(1 + \eta_\nu + \frac{2}{3}q_\nu)}\,. \tag{21}$$

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AN EXACT SOLUTION OF THE EQUATION OF TRANSFER WITH THREE-TERM SCATTERING INDICATRIX IN AN EXPONENTIAL ATMOSPHERE

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Abstract. The general equation for radiative transfer in the Milne–Eddington model is considered here. The scattering function is assumed to be quadratically anisotropic in the cosine of the scattering angle and Planck's intensity function is assumed for thermal emission. Here we have taken Planck's function as a nonlinear function of optical depth, viz., $B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$. The exact solution for emergent intensity from the bounding face is obtained by the method of the Laplace transform in combination with the Wiener–Hopf technique.

1. Introduction

Chandrasekhar (1960) has considered the problem of radiative transfer with general anisotropic scattering in the Milne–Eddington model to obtain the exact form of emergent intensity from the bounding face and nth approximate intensity at any optical depth by discrete ordinates procedure assuming Planck's function to be linear in the optical depth. Das (1979b) obtained an exact solution of this problem using the Laplace transform and the Wiener–Hopf technique. Wilson and Sen (1964) solved the same problem by a modified spherical-harmonic method. In this paper we considered the equation of transfer with anisotropic scattering in the M–E model with Planck's function as a nonlinear function of optical depth viz.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$$

(Degl'Innocenti, 1979), where b_0, b_1, and β are three positive constants.

2. Basic Equation and Boundary Conditions

The equation of transfer in a stellar atmosphere can be written (cf. Chandrasekhar, 1960; Das, 1979b) as

$$\mu \,\mathrm{d}I_\nu(x, \mu)/\rho \,\mathrm{d}x = (k_\nu + \sigma_\nu)I_\nu(x, \mu) - (1/2)\sigma_\nu(1 - \varepsilon_\nu) \times \times \int_{-1}^{+1} P(\mu, \mu')I_\nu(x, \mu')\,\mathrm{d}\mu' - (k_\nu + \varepsilon_\nu\sigma_\nu)B_\nu(T), \qquad (1)$$

where

$$P(\mu, \mu') = \sum_{l=0}^{2} W_l P_l(\mu) P_1(\mu') \tag{2}$$

is the phase function for non-conservative scattering with a three-term indicatrix; $I_\nu(x, \mu)$, the specific intensity in the direction arc cos μ at a depth x; k_ν, the absorption coefficient; arc cos μ is being measured from outward drawn normal to the face $x = 0$; σ_ν, the scattering coefficient; ρ, the density of the atmosphere; $B_\nu(T)$, Planck's function; T, the local temperature at a depth x; ε_ν, the collision constant; and ν, the frequency. We define the optical depth t_ν in terms of the scattering and absorption coefficient and the optical depth τ_ν in terms of the absorption coefficient;

$$t_\nu = \int_x^\infty (k_\nu + \sigma_\nu)\rho \, dx , \tag{3}$$

$$\tau_\nu = \int_x^\infty k_\nu \rho \, dx ; \tag{4}$$

with

$$dt_\nu = -(k_\nu + \sigma_\nu)\rho \, dx , \tag{5}$$

$$d\tau_\nu = -k_\nu \rho \, dx . \tag{6}$$

If we follow Degl'Innocenti (1979) and Karanjai and Karanjai (1985) we adopt

$$B_\nu(\tau_\nu) = B_\nu^{(0)} + B_\nu^{(1)} e^{-\alpha\tau_\nu} , \tag{7}$$

where $B_\nu^{(0)}$, $B_\nu^{(1)}$, and α are three positive constants.

Hence, Equation (7) with Equations (5) and (6) becomes

$$B_\nu(t_\nu) = b_0 + b_1 e^{-\beta t_\nu} , \tag{8}$$

where

$$b_0 = B_\nu^{(0)} , \qquad b_1 = B_\nu^{(1)} \quad \text{and} \quad \beta = \alpha k_\nu/(k_\nu + \sigma_\nu) . \tag{9}$$

In this model we shall assume that

$$\eta_\nu = (k_\nu + \sigma_\nu)^{-1} \tag{10}$$

is constant with optical depth. Equation (1) with Equations (3) and (8) becomes

$$\mu \, dI(t, \mu)/dt = I(t, \mu) - (1 - c_0/w_0)B(t) - $$

$$-(1/2) \int_{-1}^{+1} (c_0 + c_1\mu\mu' + \tfrac{1}{4}c_2(3\mu^2 - 1)(3\mu'^2 - 1)I(t, \mu') \, d\mu' , \tag{11}$$

where c_0, c_1, and c_2 are given by

$$c_0/w_0 = c_1/w_1 = c_2/w_2 = \sigma(1 - \varepsilon)/(k + \sigma)\,; \tag{12}$$

and for convenience, we have omitted the subscript ν. For the solution of Equation (11) we have the boundary conditions

$$I(0, -\mu) = 0\,, \qquad 0 < \mu \leq 1 \tag{13a}$$

and

$$I(t, \mu)\exp(-t/\mu) \to 0 \quad \text{when} \quad t \to \infty\,, \quad |\mu| \leq 1\,. \tag{13b}$$

3. Solution for Emergent Intensity

The Laplace transform of $F(t)$ is denoted by $F^*(s)$, where $F^*(s)$ is defined by

$$F^*(s) = s\int_0^\infty \exp(-st)F(t)\,\mathrm{d}t\,, \quad \mathrm{Re}\,s > 0\,; \tag{14}$$

and we set

$$I_m(t) = (1/2)\int_{-1}^{+1} \mu^m I_m^*(s, \mu)\,\mathrm{d}\mu\,, \quad m = 0, 1, 2\,, \tag{15}$$

which implies that

$$I_m^*(s) = (1/2)\int_{-1}^{+1} \mu^m I_m^*(s, \mu)\,\mathrm{d}\mu\,, \quad m = 0, 1, 2\,. \tag{15}$$

Equation (11) with Equation (15), takes the form

$$\mu\,\mathrm{d}I(t, \mu)/\mathrm{d}t = I(t, \mu) - [c_0 I_0(t) + c_1\mu I_1(t) + \\ + \tfrac{1}{4}c_2(3\mu^2 - 1)(3I_2(t) - I_0(t))] - (1 - c_0/w_0)B(t)\,. \tag{17}$$

Now subjecting Equation (17) to the Laplace transform as defined in Equation (14) we have, using the boundary conditions,

$$(\mu s - 1)I^*(s, \mu) = \mu s I(0, \mu) - (1 - c_0/w_0)B^*(s) - (c_0 I_0^*(s) + \\ + c_1\mu I_1^*(s) + \tfrac{1}{4}c_2((3\mu^2 - 1)(3I_2^*(s) - I_0^*(s)))\,. \tag{18}$$

Equation (18) gives

$$I(0, \mu) = (c_0 I_0^*(1/\mu) + c_1\mu I_1^*(1/\mu) + \tfrac{1}{4}c_2(3\mu^2 - 1) + \\ + (3I_2^*(1/\mu) - I_0^*(1/\mu)) + (1 - c_0/w_0)B^*(1/\mu)\,. \tag{19}$$

Equation (19) with $\mu = s^{-1}$, s is complex, takes the form

$$I(0, s^{-1}) = (c_0 - \tfrac{1}{4}c_2(3s^{-2} - 1)I_0^*(s) + c_1 s^{-1} I_1^*(s) + \\ + \tfrac{3}{4}c_2(3s^{-2} - 1)I_2^*(s) + (1 - c_0/w_0)B^*(s), \quad (20)$$

we shall apply the operator

$$(1/2) \int_{-1}^{+1} \cdots \mu \, d\mu \quad (21)$$

on both sides of Equation (18) to get

$$-(1 - c_0)s^{-1} I_0^*(s) + I_1^*(s) = (1/2) \int_0^1 \mu I(0, \mu) \, d\mu - \\ - (1 - c_0/w_0)s^{-1} B^*(s) \quad (22)$$

and

$$-(1 - \tfrac{1}{3}c_1)s^{-1} I_1^*(s) + I_2^*(s) = (1/2) \int_0^1 \mu^2 I(0, \mu) \, d\mu, \quad (23)$$

we shall also apply the operator

$$(1/2) \int_{-1}^{+1} \cdots d\mu/(\mu s - 1) \quad (24)$$

on both sides of Equation (18) to get

$$as^{-1} - (1 - c_0/w_0)B^*(s)t_0 s^{-1} = [1 + c_0 t_0 s^{-1} - \tfrac{1}{4}c_2(3t_2 s^{-1} - \\ - t_0 s^{-1})]I_0^*(s) + c_1 t_1 s^{-1} I_1^*(s) + \tfrac{3}{4}c_2[3t_2 s^{-1} - t_0 s^{-1}]I_2^*(s), \quad (25)$$

where

$$as^{-1} = (1/2) \int_0^1 \mu s(\mu s - 1)^{-1} I(0, \mu) \, d\mu \quad (26)$$

and

$$t_m s^{-1} = (1/2) \int_{-1}^{+1} (\mu s - 1)^{-1} \mu^m \, d\mu, \quad m = 0, 1, 2. \quad (27)$$

If we follow the usual procedure for elimination of $I_0^*(s)$, $I_1^*(s)$, and $I_2^*(s)$ among Equations (26), (22), (23), and (25), after some lengthy calculations setting $s = z^{-1}$, we have

$$T(z)I(0, z) = (1/2)\int_0^1 x(x-z)^{-1}L(x, z)I(0, x)\,dx + (1 - c_0/w_0)B^*z^{-1}, \quad (28)$$

where

$$T(z) = 1 - 2z^2\int_0^1 \psi(x)\,dx(z^2 - x^2)^{-1}, \quad (29)$$

$$\psi(x) = (1/2)(A + Bx^2 + Cx^4), \quad (30)$$

$$L(x, z) = A - \tfrac{3}{4}c^2x^2 + (B + C + \tfrac{3}{4}c_2)xz - (1/3)Cz^2 + Cx^2z^2, \quad (31)$$

$$B^*z^{-1} = b_0 + b_1/(1 + \beta z) = (d_0 + d_1 z)/(1 + \beta z), \quad (32)$$

where

$$d_0 = b_0 + b_1, \qquad d_1 = b_0\beta, \quad (33)$$

$$A = c_0 + \tfrac{1}{4}c_2, \qquad B = c_1(1 - c_0) - \tfrac{3}{4}c_2 - \tfrac{3}{4}c_2(1 - c_0)(1 - c_1/3), \quad (34)$$

$$C = \tfrac{9}{4}c_2(1 - c_0)(1 - c_1/3), \quad (35)$$

where we shall assume that

$$\psi(x) = \tfrac{1}{2}(A + Bx^2 + Cx^4) > 0 \quad (36)$$

and

$$\psi_0 = \int_0^1 \psi(x)\,dx < \tfrac{1}{2}. \quad (37)$$

But for

$$y = k(k + \sigma) < 1, \quad (38)$$

B^*z^{-1} is analytic in $(-y^{-1}, 0)^c$, bounded at the origin and $0 < y < 1$. According to Busbridge (1960), the equation for $T(z)$ possesses the following properties: $T(z)$ is analytic in z for $(-1, 1)^c$, bounded at the origin, has a pair of zeros at $z = \pm K$ $(K > 1)$, K is real and can be expressed as

$$T(z) = [H(z)H(-z)]^{-1}, \quad (39)$$

where $H(z)$ and $H(-z)$ have the following properties: $H(z)$ is analytic for $z \in (-1, 0)^c$, bounded at the origin, has a pole at $z = -K$. $H(-z)$ is analytic for $z \in (0, 1)^c$, bounded at the origin, has a pole at $z = K$.

If we follow Busbridge (1960), Das (1979a) and Dasgupta (1977) we have for $\psi_0 < \frac{1}{2}$,

$$H(z) = 1 + zH(z) \int_0^1 \psi(x)H(x)\,(x+z)^{-1}\,\mathrm{d}x \tag{40}$$

or

$$H(z) = (A_0 + H_0 z)/(z + K) - M(z)\,, \tag{41}$$

where

$$M(z) = \int_0^1 P(x)\,\mathrm{d}x/(x+z)\,, \tag{42}$$

$$P(x) = \phi(x)/H(x)\,, \tag{43}$$

$$\phi(x) = \pi^{-1} Y_0(x)/[T_0^2(x) + Y_0^2(x)]\,, \tag{44}$$

$$T_0(x) = 1 - 2x^2 \int_0^1 (\psi(t) - \psi(x))\,\mathrm{d}t/(x^2 - t^2) - $$

$$- \psi(x)x \log(1+x)/(1-x)\,, \tag{45}$$

$$Y_0(x) = \pi x \psi(x)\,, \tag{46}$$

$$A_0 = (1 + P_{-1})K\,, \tag{47}$$

$$P_{-1} = \int_0^1 x^{-1} P(x)\,\mathrm{d}x\,, \tag{48}$$

$$H_0 = (1 - 2\psi_0)^{-1/2}\,. \tag{49}$$

Equation (28) with Equation (39) takes the form

$$I(0,z)/H(z) = H(-z)G(z) + (1 - c_0/w_0)H(-z)B^* z^{-1}\,, \tag{50}$$

where

$$G(z) = (1/2) \int_0^1 x(x-z)^{-1} L(x,z) I(0,x)\,\mathrm{d}x\,, \tag{51}$$

we shall assume that

$$I(0,z) \quad \text{is regular for} \quad \mathrm{Re}\,z > 0\,, \tag{52}$$

bounded at the origin. Equation (51) with the above assumption on $I(0, z)$ gives the following properties of $G(z)$: $G(z)$ is regular on $(0, 1)^c$, bounded at the origin $O(z)$ when $z \to \infty$.

Equation (50) with Equations (32) and (51) gives

$$I(0, z)/H(z) = H(-z)\left[(1/2)\int_0^1 x(x - z) + L(x, z)I(0, x)\,dx + \right.$$
$$\left. + (1 - c_0/w_0)(d_0 + d_1 z)/(1 + \beta z)\right]. \quad (53)$$

Equation (53) can be put in the form

$$I(0, z)/H(z) = H(-z)\left[(1/2)\int_0^1 x(x - z)^{-1} L(x, z)I(0, x)\,dx + \right.$$
$$\left. + (1 - c_0/w_0)(d_0/z + d_1)/(z^{-1} + \beta)\right]. \quad (54)$$

Therefore, the left-hand side of Equation (54) is regular for $\operatorname{Re} z > 0$ and bounded at the origin and the right-hand side of Equation (54) is regular for z on $(0, 1)^c$ and bounded at the origin and tends to a linear polynomial in z, say $(x_0 + x_1 z)$ when $z \to \infty$. Hence, by a modified form of Liouville's theorem we have

$$I(0, z) = [x_0 + x_1 z]H(z) \quad (55)$$

and

$$(1/2)\int_0^1 xL(x, z)I(0, x)\,dx/(x - z) +$$
$$+ (1 - c_0/w_0)(d_0 + d_1 z)/(1 + \beta z) = (x_0 + x_1 z)/H(-z). \quad (56)$$

Equation (55) will give emergent intensity from the bounding face if x_0 and x_1 are determined. We shall now determine the constants x_0 and x_1. If we set $z = 0$ in Equation (56), we have

$$(1/2)\int_0^1 L(x, 0)I(0, x)\,dx + d_0(1 - c_0/w_0) = x_0. \quad (57)$$

Equation (57) with Equation (55) gives

$$x_0 y_1 + x_1 y_2 + z_1 = 0, \quad (58)$$

where

$$y_1 = (1/2)\int_0^1 L(x, 0)H(x)\,dx - 1, \quad (59)$$

$$y_2 = (1/2)\int_0^1 xL(x, 0)H(x)\,dx, \quad (60)$$

$$z_1 = (1 - c_0/w_0)d_0. \quad (61)$$

As $T(z)$ has a zero at $z = K$, Equation (28) gives

$$(1/2)\int_0^1 xL(x, K)I(0, x)\,\mathrm{d}x/(x - K) + $$
$$+ (1 - c_0/w_0)(d_0 + d_1K)/(1 + \beta K) = 0\,, \tag{62}$$

Equation (62) with Equation (55) gives

$$x_0 y_3 + x_1 y_4 + z_2 = 0\,, \tag{63}$$

where

$$y_3 = (1/2)\int_0^1 xL(x, K)H(x)\,\mathrm{d}x/(x - K)\,, \tag{64}$$

$$y_4 = (1/2)\int_0^1 x^2L(x, K)H(x)\,\mathrm{d}x/(x - K)\,, \tag{65}$$

$$z_2 = (1 - c_0/w_0)(d_0 + d_1K)/(1 + \beta K)\,, \tag{66}$$

Equations (58) and (68) give

$$x_0 = (y_2z_2 - z_1y_4)/(y_1y_4 - y_3y_2)\,, \tag{67}$$

$$x_1 = (z_1y_3 - y_1z_2)/(y_1y_4 - y_3y_2)\,, \tag{68}$$

where

$$(y_1y_4 - y_3y_2) \neq 0.$$

Hence, Equation (55) with Equations (67) and (68) gives the emergent intensity from the bounding face of the atmosphere.

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EXACT SOLUTION OF THE EQUATION OF TRANSFER IN A FINITE EXPONENTIAL ATMOSPHERE BY THE METHOD OF LAPLACE TRANSFORM AND LINEAR SINGULAR OPERATORS

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Abstract. The equation which commonly appears in radiative transfer problem in a finite atmosphere having ground reflection according to Lambert's law is considered in this paper. The Planck's function $B_\nu(T)$ is taken in the form,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau} .$$

The exact solution of this equation is obtained for surface quantities in terms of the $X - Y$ equations of Chandrasekhar by the method of Laplace transform and linear singular operators.

1. Introduction

Das (1978, 1980) has solved various problems of radiative transfer in finite and semi-infinite atmosphere using a method involving Laplace transform and linear singular operators.

In this paper we have considered the one-sided Laplace transform together with the theory of linear singular operators to solve the transport equation which arises in the problem of a finite atmosphere having ground reflection according to Lambert's law taking the Planck's function as a nonlinear function of optical depth: viz.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau} ,$$

where b_0, b_1, and β are positive constants (Delg'Innocenti, 1979; Karanjai and Karanjai, 1985; Deb *et al.*, 1990).

2. Basic Equation and Boundary Conditions

The integro-differential equation for the intensity of radiation $I(\tau, \mu)$, at any optical depth τ for the problem of diffuse reflection and transmission in a finite atmosphere can be written in the form (Das, 1980) as

$$\mu \frac{\mathrm{d}I_\nu(\tau, \mu)}{\mathrm{d}\tau} = I_\nu(\tau, \mu) - \int_{-1}^{+1} \psi(\mu') I_\nu(\tau, \mu')\, \mathrm{d}\mu' - B_\nu(T) , \tag{1}$$

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where $I_\nu(\tau, \mu)$ is the intensity in the direction $\cos^{-1}\mu$ at a depth τ, the angle $\cos^{-1}\mu$ is measured from outside drawn normal to the face $\tau = 0$, $\psi(\mu)$ is the characteristic function for non-conservative scattering which satisfies the condition

$$\psi_0 = \int_0^1 \psi(\mu')\,\mathrm{d}\mu' < \tfrac{1}{2}; \quad \psi(\mu') \text{ is even}, \tag{2}$$

ν is the frequency and $B_\nu(T)$ is the Planck's source function at any optical depth. We have taken

$$B_\nu(T) = b_0 + b_1\,e^{-\beta\tau}.$$

Then Equation (1) becomes

$$\mu\,\frac{\mathrm{d}I(\tau, \mu)}{\mathrm{d}\tau} = I(\tau, \mu) - \int_{-1}^{+1} \psi(\mu')I(\tau, \mu')\,\mathrm{d}\mu' - (b_0 + b_1\,e^{-\beta\tau}), \tag{3}$$

where for convenience we have omitted the subscript ν.

The boundary conditions associated with Equation (3) are

$$I(0, -\mu) = 0, \qquad 0 < \mu \leq 1, \tag{4a}$$

$$I(\tau_0, \mu) = I_g, \qquad 0 < \mu \leq 1, \qquad \tau_0 > 0, \tag{4b}$$

where τ_0 is the thickness of the finite atmosphere and the bounding face $\tau = \tau_0$ is having ground reflection according to Lambert's law, I_g is a constant.

3. Integral Equations for Surface Quantities

Let us define $f^*(s, \mu)$ as the Laplace transform of $f(\tau, \mu)$ by

$$f^*(s, \mu) = s \int_0^{\tau_0} f(\tau, \mu)\,e^{-s\tau}\,\mathrm{d}\tau, \quad \mathrm{Re}\,s > 0; \tag{5a}$$

$$f(\tau, \mu) = 0, \quad \text{when} \quad \tau > \tau_0. \tag{5b}$$

Let us now apply the Laplace transform defined in Equation (5a) to Equation (3) to obtain the equation satisfying the boundary condition as

$$(\mu s - 1)I^*(s, \mu) = \mu s I(0, \mu) - \mu s\,e^{-\tau_0 s} I(\tau_0, \mu) - S^*(s), \tag{6}$$

where

$$S(\tau) = \int_{-1}^{+1} \psi(\mu')I(\tau, \mu')\,\mathrm{d}\mu' + b_0 + b_1\,e^{-\beta\tau} \Rightarrow \tag{7}$$

$$\Rightarrow S^*(s) = \int_{-1}^{+1} \psi(\mu')I^*(\tau, \mu')\,\mathrm{d}\mu' + b_0(1 - e^{-s\tau_0}) + \frac{sb_1}{s+\beta}(1 - e^{-(s+\beta)\tau_0}). \tag{8}$$

Let us apply the operator

$$\int_{-1}^{+1} \ldots \psi(\mu)\,\mathrm{d}\mu/(\mu s - 1), \tag{9}$$

on both sides of Equation (6) and we obtain, with Equation (8),

$$\begin{aligned} T(1/s)S^*(s) = &\int_{-1}^{+1} \mathrm{d}\mu\, \mu s\, \psi(\mu)I(0, \mu)/(\mu s - 1) - \\ &- e^{-\tau_0 s}\int_{-1}^{+1} \mu s\, \psi(\mu)I(\tau_0, \mu)\,\mathrm{d}\mu/(\mu s - 1) + \\ &+ b_0(1 - e^{-s\tau_0}) + \frac{sb_1}{s+\beta}(1 - e^{-(s+\beta)\tau_0}), \end{aligned} \tag{10}$$

where

$$T(1/s) = 1 + \int_{-1}^{+1} \mathrm{d}\mu\, \psi(\mu)/(\mu s - 1). \tag{11}$$

Equation (6) gives

$$I(0, \mu) - e^{-\tau_0/\mu} I(\tau_0, \mu) = S^*(1/\mu) \Rightarrow \tag{12}$$

$$\Rightarrow I(0, 1/s) - e^{-\tau_0 s} I(\tau_0, 1/s) = S^*(s). \tag{13}$$

Equation (10), together with Equation (12), gives for complex z, where $z = s^{-1}$,

$$\begin{aligned} &[I(0, z) - e^{-\tau_0/z} I(\tau_0, z)]T(z) = \\ &= \int_{-1}^{+1} \mu\psi(\mu)I(0, \mu)\,\mathrm{d}\mu/(\mu - z) - e^{-\tau_0/z}\int_{-1}^{+1} \mu\psi(\mu)I(\tau_0, \mu)\,\mathrm{d}\mu(\mu - z) + \\ &+ b_0(1 - e^{-\tau_0/z}) + \frac{b_1}{1+\beta z}(1 - e^{-\beta\tau_0} e^{-\tau_0/z}). \end{aligned} \tag{14}$$

Let us put $\alpha_0 = \beta^{-1}$, then Equation (14) becomes

$$[I(0, z) - e^{-\tau_0/z} I(\tau_0, z)]T(z) =$$

$$= \int_{-1}^{+1} \mu\psi(\mu)I(0,\mu)\,\mathrm{d}\mu/(\mu - z) - e^{-\tau_0/z} \int_{-1}^{+1} \mu\psi(\mu)I(\tau_0,\mu)\,\mathrm{d}\mu/(\mu - z) +$$

$$+ b_0(1 - e^{-\tau_0/z}) + \frac{b_1\alpha_0}{z + \alpha_0}(1 - e^{-\tau_0/\alpha_0} e^{-\tau_0/z}). \tag{15}$$

Let us set $z = -z$ in Equation (15) and multiply the resulting equation by $e^{-\tau_0/z}$ on both sides to obtain, for complex z,

$$[I(\tau_0, -z) - e^{-\tau_0/z} I(0, -z)]T(z) =$$

$$= \int_{-1}^{+1} \mu\psi(\mu)I(\tau_0,\mu)\,\mathrm{d}\mu/(\mu + z) - e^{\tau_0/z} \times$$

$$\times \int_{-1}^{+1} \mu\psi(\mu)I(0,\mu)\,\mathrm{d}\mu/(\mu + z) + b_0(1 - e^{-\tau_0/z}) -$$

$$- \frac{b_1\alpha_0}{\alpha_0 - z}(e^{-\tau_0/z} - e^{-\tau_0/\alpha_0}), \tag{16}$$

Equations (15) and (16) are the linear integral equations for the surface quantities under consideration.

4. Linear Singular Integral Equations

Equations (15) and (16) are the equations defined for complex z, where z does not lie between -1 and 1. When z lies between -1 and 1, Equations (15) and (16) will give the linear singular integral equations by the applications of Plemelj's formulae (cf. Mushkelishvili, 1946) with the boundary condition (4) as

$$[I(0,z) - e^{-\tau_0/z} I_g]T_0(z) = P \int_0^1 \mu\psi(\mu)I(0,\mu)\,\mathrm{d}\mu/(\mu - z) -$$

$$- e^{-\tau_0/z} \int_0^1 \mu\psi(\mu)I(\tau_0, -\mu)\,\mathrm{d}\mu/(\mu + z) -$$

$$- e^{-\tau_0/z} P \int_0^1 \mu\psi(\mu)I_g\,\mathrm{d}\mu/(\mu - z) +$$

$$+ b_0(1 - e^{-\tau_0/z}) + \frac{b_1\alpha_0}{z + \alpha_0}(1 - e^{-\tau_0(1/z + 1/\alpha_0)} \tag{17}$$

and

$$I(\tau_0, -z)T_0(z) = P\int_0^1 \mu\psi(\mu)I(\tau_0, -\mu)\,d\mu/(\mu - z) -$$

$$- e^{-\tau_0/z}\int_0^1 \mu\psi(\mu)I(0, \mu)\,d\mu/(\mu + z) + \int_0^1 \mu\psi(\mu)I_g\,d\mu/(\mu + z) +$$

$$+ b_0(1 - e^{-\tau_0/z}) - \frac{b_1\alpha_0}{\alpha_0 - z}(e^{-\tau_0/z} - e^{-\tau_0/\alpha_0}), \tag{18}$$

where

$$T_0(z) = 1 - 2z^2\int_0^1 d\mu[\psi(\mu) - \psi(z)]/(z^2 - \mu^2) - 2z^2\psi(z)P\int_0^1 d\mu/(z^2 - \mu^2), \tag{19}$$

in which P dneotes the Cauchy principal value of the integral.

Equations (17) and (18) are the linear singular integral equations from which we shall determine the surface quantities $I(0, z)$ and $I(\tau_0, -z)$ by the application of the theory of linear singular operators.

5. Theory of Linear Singular Operators

If we follow Das (1978, 1980), we can write the following theorems.

THEOREM 1

The linear integral equations for $z \in (0, 1)$,

$$L_+[R(z, -x_0)] = l(z, -x_0), \tag{20a}$$

$$I_-(Q(z, -x_0)] = m(z, -x_0), \tag{20b}$$

where

$$L_+[f(z, -x_0)] = f(z, -x_0)T_0(z) - P\int_0^1 \mu\psi(\mu)f(\mu, -x_0)\,d\mu/(\mu - z) +$$

$$+ e^{-\tau_0/z}\int_0^1 \mu\psi(\mu)f(\mu, -x_0)\,d\mu/(\mu + z), \tag{21a}$$

$$L_-[f(z, -x_0)] = f(z, -x_0)T_0(z) - P\int_0^1 \mu\psi(\mu)f(\mu, -x_0)\,d\mu/(\mu - z) -$$

$$- e^{-\tau_0/z} \int_0^1 \mu \psi(\mu) f(\mu, -x_0)\, d\mu/(\mu + z)\,, \tag{21b}$$

$$l(z, -x_0) = \frac{x_0}{z + x_0} [1 - e^{-\tau_0(1/z + 1/x_0)}] + \frac{x_0}{z - x_0} [e^{-\tau_0/z} - e^{-\tau_0/x_0}]\,, \tag{22a}$$

$$m(z, -x_0) = \frac{x_0}{z + x_0} [1 - e^{-\tau_0(1/z + 1/x_0)}] - \frac{x_0}{z - x_0} [e^{-\tau_0/z} - e^{-\tau_0/x_0}]\,, \tag{22b}$$

admit of solutions of the form

$$R(z, -x_0) = S(z, -x_0) + T(z, -x_0)\,, \tag{23a}$$

$$Q(z, -x_0) = S(z, -x_0) - T(z, -x_0)\,, \tag{23b}$$

where

$$S(z, -x_0) = x_0[X(z)X(x_0) - Y(z)Y(x_0)]/(z + x_0) \tag{24}$$

and

$$T(z, -x_0) = x_0[X(z)Y(x_0) - Y(z)X(x_0)]/(x_0 - z)\,. \tag{25}$$

With constraints on $X(z)$ and $Y(z)$ as

(i) when $\psi_0 < \frac{1}{2}$

$$1 = K \int_0^1 X(\mu)\psi(\mu)\, d\mu/(K - \mu) + e^{-\tau_0/K} K \int_0^1 Y(\mu)\psi(\mu)\, d\mu/(K + \mu)\,, \tag{26a}$$

$$e^{-\tau_0/K} = K \int_0^1 Y(\mu)\psi(\mu)\, d\mu/(K - \mu) + K\, e^{-\tau_0/K} \int_0^1 X(\mu)\psi(\mu)\, d\mu/(K + \mu)\,; \tag{26b}$$

(ii) when $\psi_0 = \frac{1}{2}$

$$1 = \int_0^1 \psi(\mu)\, [X(\mu) + Y(\mu)]\, d\mu\,, \tag{27a}$$

$$\tau_0 \int_0^1 \psi(\mu) Y(\mu)\, d\mu = \int_0^1 \mu\psi(\mu)\, [X(\mu) - Y(\mu)]\, d\mu \tag{27b}$$

and K is the positive root of the function $T(z)$, when $\psi_0 < \frac{1}{2}$, defined by

$$T(z) = 1 + \int_{-1}^{+1} z\psi(\mu)\,\mathrm{d}\mu/(\mu - z) \tag{28}$$

and where $[X(z) - Y(z)]$ and $[X(z) + Y(z)]$ are the respective solutions of

$$L_+[f(z)] = (1 - e^{-\tau_0/z})\left(1 - \int_0^1 \psi(\mu)f(\mu)\,\mathrm{d}\mu\right) \tag{29}$$

and

$$L_-[f(z)] = (1 + e^{-\tau_0/z})\left(1 - \int_0^1 \psi(\mu)f(\mu)\,\mathrm{d}\mu\right). \tag{30}$$

THEOREM 2

As the operators L_+ and L_- are linear for $z \in (0, 1)$, then for any constant C, we have

$$L_\pm(CF(z, -x_0)) = CL_\pm(F(z, -x_0)) \tag{31}$$

and

$$L_\pm(zf(z)) = zL_\mp(f(z) - (1 \mp e^{-\tau_0/z})\int_0^1 \mu\psi(\mu)f(\mu)\,\mathrm{d}\mu\,. \tag{32}$$

THEOREM 3

If $R(z, -x_0)$ and $Q(z, -x_0)$ are the solutions of

$$L_+[R(z, -x_0)] = l(z, -x_0)\,, \tag{33a}$$

$$L_-[Q(z, -x_0)] = m(z, -x_0)\,, \tag{33b}$$

then

$$L_+(M(z)) = \int_0^1 \psi(-x_0)l(z, -x_0)\,\mathrm{d}x_0\,, \tag{34}$$

$$L_-(N(z)) = \int_0^1 \psi(-x_0)m(z, -x_0)\,\mathrm{d}x_0\,, \tag{35}$$

admit of a solution of

$$M(z) = \int_0^1 \psi(-x_0)R(z, -x_0)\,\mathrm{d}x_0\,, \tag{36}$$

$$N(z) = \int_0^1 \psi(-x_0) Q(z, -x_0)\, dx_0 . \tag{37}$$

6. Solution for Surface Quantities

Linear singular integral equations (17) and (18) are the required integral equations from which we will have to determine $I(0, z)$ and $I(\tau_0, -z)$, the quantities under consideration, by the application of the theory of linear singular operators indicated in Section 5.

Equations (17) and (18) on addition and after some rearrangement give

$$L_+[I(0, z) + I(\tau_0, -z) - e^{-\tau_0/z} I_g] =$$
$$= 2b_0(1 - e^{-\tau_0/z}) + b_1 l(z, -\alpha_0) + I_g \int_0^1 \psi(\mu) l(z, -\mu)\, d\mu . \tag{38}$$

Equations (17) and (18) on subtraction and after some manipulation give

$$L_-[I(0, z) - I(\tau_0, -z) - e^{-\tau_0/z} I_g] =$$
$$= b_1 m(z, -\alpha_0) - I_g \int_0^1 \psi(\mu) m(z, -\mu)\, d\mu , \tag{39}$$

where $l(z, -\mu)$ and $m(z, -\mu)$ are given by Equations (22a) and (22b). Equations (38) and (39), with Theorems 1, 2, and 3 of Section 5, will give us the desired quantities $I(0, z)$ and $I(\tau_0, -z)$. The solution of Equation (38) is given by

$$[I(0, z) + I(\tau_0, -z) - I_g e^{-\tau_0/z}] =$$
$$= \frac{2b_0}{1 - G_0} (X(z) - Y(z)) + b_1 R(z, -\alpha_0) + I_g \int_0^1 R(z, -\mu) \psi(\mu)\, d\mu , \tag{40}$$

where

$$G_0 = \int_0^1 \psi(\mu) [X(\mu) - Y(\mu)]\, d\mu . \tag{41}$$

The solution of Equation (39) is given by

$$[I(0, z) - I(\tau_0, -z) - I_g e^{-\tau_0/z}] =$$
$$= b_1 Q(z, -\alpha_0) - I_g \int_0^1 \psi(\mu) Q(z, -\mu)\, d\mu . \tag{42}$$

Equations (40) and (42) on addition give $I(0, z)$ and Equations (38) and (42) on subtraction give $I(\tau_0, -z)$ as

$$I(0, z) = I_g e^{-\tau_0/z} + I_g \int_0^1 \psi(\mu) T(z, -\mu) \, d\mu + + \frac{b_0}{1 - G_0} [X(z) - Y(z)] + b_1 S(z, -\mu) \tag{43}$$

and

$$I(\tau_0, -z) = \frac{b_0}{1 - G_0} [X(z) - Y(z)] + + b_1 T(z, -\alpha_0) + I_g \int_0^1 \psi(\mu) S(z, -\mu) \, d\mu , \tag{44}$$

where $S(z, -\mu)$ and $T(z, -\mu)$ are given by Equations (24) and (25).

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SOLUTION OF THE EQUATION OF TRANSFER FOR INTERLOCKED MULTIPLETS WITH PLANCK FUNCTION AS A NONLINEAR FUNCTION OF OPTICAL DEPTH

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Abstract. The equation of transfer for interlocked multiplets has been solved exactly by the method used by Busbridge and Stibbs (1954) for exponential form of the Planck function $B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$.

1. Introduction

The equation of transfer in the Milne–Eddington model for interlocking without redistribution have been discussed by Woolley and Stibbs (1953), where a clear statement of the problem will be found. Taking the Planck function to be linear, they have obtained a solution by means of Eddington's approximation and calculated the residual intensities and the total absorption in the emergent flux for doublet and triplet lines. Busbridge and Stibbs (1954) applied the principle of invariance governing the law of diffuse reflection with a slight modification to solve exactly the equation of transfer in the M–E model. Dasgupta and Karanjai (1972) applied Sobolev's probabilistic method to solve the same problem. Karanjai and Barman (1981) applied the extension of the method of discrete ordinates to solve the problem. Dasgupta (1978) obtained an exact solution of the problem by Laplace transform and Wiener–Hopf technique using a new representation of the H-function obtained by Dasgupta (1977). The same problem has also been solved by Karanjai and Karanjai (1985) by the method used by Dasgupta (1978) and by Deb *et al.* (1991) by discrete ordinate method using the Planck function as an exponential function of optical depth.

In this paper we have solved the same problem by the method used by Busbridge and Stibbs (1954), using the Planck function $B_\nu(T)$ as an exponential function of optical depth (Degl'Innocenti, 1979)

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}.$$

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2. Equation of Transfer

Woolley and Stibbs (1953) made certain assumptions, viz., (i) that the lines are so close together that variations of the continuous absorption coefficient k and of the Planck function $B_\nu(T)$ with wavelength may be neglected. This also means that the lower states are nearly equal in excitation potential and that they have the same classical damping constant. Then the values of $\eta_1, \eta_2, \ldots, \eta_k$ (the ratios of the line absorption coefficients to k) are proportional to the transition probabilities for spontaneous emission from the upper state to the respective lower states; (ii) that $\eta_1, \eta_2, \ldots, \eta_k$ are independent of depth; (iii) that the coefficient ε, which is introduced to allow for thermal emission associated with the absorption is independent of both frequency and depth.

In the present paper, we have further assumed that (iv)

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}, \tag{1}$$

where β is a constant and $\tau = \int_0^x k\rho\,\mathrm{d}x$, x being the depth below the surface of the atmosphere. By (i) b_0, b_1, and τ are independent of ν.

Then the equation of transfer for interlocked multiplets can be written as

$$\mu \frac{\mathrm{d}I_r(\tau,\mu)}{\mathrm{d}\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\varepsilon\eta_r)(b_0 + b_1 e^{-\beta\tau}) - (1-\varepsilon)\alpha_r \sum_{p=1}^{k} \frac{1}{2}\eta_p \int_{-1}^{+1} I_p(\tau,\mu')\,\mathrm{d}\mu', \quad (r = 1, 2, \ldots, k), \tag{2}$$

where

$$\alpha_r = \eta_r/(\eta_1 + \eta_2 + \cdots, \eta_k), \tag{3}$$

and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1. \tag{4}$$

In Equation (2) the subscript r denotes the quantity corresponding to the line of frequency ν_r. The Equation (2) have to be solved subject to the boundary conditions,

$$I_r(0, -\mu') = 0, \quad (0 < \mu' \leq 1, \quad r = 1, 2, \ldots, k) \tag{5}$$

together with a condition limiting $I_r(\tau,\mu)$ for large τ. We shall assume that $I_r(\tau,\mu)$ is at most linear in τ for large τ. Formal solutions of Equation (2) are easily found, but they do not satisfy Equation (5).

These are

$$I_r(\tau,\mu) = b_0 + \frac{b_1 T_r}{1+\xi_r\beta\mu} e^{-\beta\tau}, \quad (r = 1, 2, \ldots, k), \tag{5a}$$

write

$$I_r(\tau, \mu) = b_0 + \frac{b_1 T_r}{1 + \xi_r \beta \mu} e^{-\beta\tau} + I_r^*(\tau, \mu), \quad (r = 1, 2, \ldots, k), \tag{6}$$

where

$$T_r = \frac{\lambda_r}{1 - \dfrac{1}{2\beta}(1 - \varepsilon)\eta_r \log \dfrac{1 + \beta\xi_r}{1 - \beta\xi_r}} \tag{7}$$

and

$$\lambda_r = (1 + \varepsilon\eta_r)/(1 + \eta_r), \tag{8}$$

$$\xi_r = 1/(1 + \eta_r). \tag{9}$$

Then $I_r^*(\tau, \mu)$ satisfies the equation

$$\mu \frac{dI_r^*(\tau, \mu)}{d\tau} = (1 + \eta_r) I_r^*(\tau, \mu) - (1 - \varepsilon)\alpha_r \times$$

$$\times \sum_{p=1}^{k} \frac{1}{2} \eta_p \int_{-1}^{+1} I_p^*(\tau, \mu')\, d\mu', \quad (r = 1, 2, \ldots, k) \tag{10}$$

together with the boundary condition

$$I_r^*(0, -\mu) = \frac{b_1 T_r}{\xi_r \beta \mu' - 1} - b_0, \quad (0 < \mu' \leq 1, \quad r = 1, 2, \ldots, k). \tag{11}$$

Moreover, $I_r(\tau, \mu)$ must be at most linear in τ as $\tau \to \infty$.

Now we have the problem of a scattering atmosphere (exponential) subject to external radiation whose intensity is given by Equation (11). We want to find the emergent intensity $I_r^*(0, \mu)$ of frequency ν_r. This will be the intensity of the diffusely reflected radiation and can be calculated when the appropriate scattering function is known.

In the present problem the scattering function splits up into k^2 functions

$$S_{rs}(\mu, \mu') \quad (r = 1, 2, \ldots, k; s = 1, 2, \ldots, k)$$

but it is convenient to reunite them temporarily in the function

$$P(\nu, \nu') S(\nu, \nu'; \mu, \mu'),$$

where ν is any one of $\nu_1, \nu_2, \ldots, \nu_k$.

$$P(\nu, \nu') = \alpha_\nu \sum_{p=1}^{k} \delta(\nu_p - \nu') \tag{12}$$

δ denoting Dirac's δ-function, and

$$S(\nu_r, \nu_s; \mu, \mu') = S_{rs}(\mu, \mu') . \tag{13}$$

Then the law of diffuse reflection for the atmosphere can be written as (Stibbs, 1953; Busbridge, 1953),

$$I_r^{\mathrm{ref}}(0, \mu) = \frac{1}{2\mu} \int_0^\infty P(\nu, \nu')\,\mathrm{d}\nu' \int_0^1 S(\nu, \nu'; \mu, \mu') I_{\nu'}^{\mathrm{inc}}(0, -\mu')\,\mathrm{d}\mu' , \tag{14}$$

The equivalent form in terms of the functions $S_{rs}(\mu, \mu')$ is

$$I_r^{\mathrm{ref}}(0, \mu) = \alpha_r \sum_{p=1}^{k} \frac{1}{2\mu} \int_0^1 S_{rp}(\mu, \mu') I_p^{\mathrm{inc}}(0, -\mu')\,\mathrm{d}\mu' . \tag{15}$$

3. Scattering Function

If we follow Busbridge and Stibbs (1954) we have the scattering function from frequency ν_s and direction $-\mu'$ into frequency ν_r and direction μ, in the form

$$S_{rs}(\mu, \mu') = \eta_r(1 - \lambda_s) \frac{\mu\mu'}{\xi_r\mu + \xi_s\mu'} H(\xi_r\mu) H(\xi_s\mu') , \tag{16}$$

where

$$H(\xi_r\mu) = 1 + \tfrac{1}{2}\xi_r\mu H(\xi_r\mu) \sum_{p=1}^{k} \alpha_p(1 - \lambda_p) \int_0^1 \frac{H(\xi_p\mu')\,\mathrm{d}\mu'}{\xi_r\mu + \xi_p\mu'} . \tag{17}$$

4. H-function

Following Busbridge and Stibbs (1954), Equation (17) can be written as

$$1/H(\xi_r\mu) = \left(\sum_{p=1}^{k} \alpha_p\lambda_p \right)^{1/2} + \frac{1}{2} \sum_{p=1}^{k} \alpha_p(1 - \lambda_p) \int_0^1 \frac{\xi_p\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu' , \tag{18}$$

5. Emergent Intensity

From Equations (11), (15), and (9) we have

$$I_r^*(0, \mu) = \frac{\alpha_r}{2\mu} \sum_{p=1}^{k} \int_0^1 S_{rp}(\mu, \mu') \left(\frac{b_1 T_p}{\xi_p\beta\mu' - 1} - b_0 \right) . \tag{19}$$

If we substitute from Equation (16) we get

$$I_r^*(0,\mu) = \tfrac{1}{2}\alpha_r H(\xi_p\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) \int_0^1 \frac{\mu'}{\xi_r\mu + \xi_p\mu'} \times$$

$$\times \left(\frac{b_1 T_p}{\xi_p\beta\mu' - 1} - b_0\right) H(\xi_p\mu')\,\mathrm{d}\mu' =$$

$$= \tfrac{1}{2}\alpha_r b_1 H(\xi_r\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) T_p \int_0^1 \frac{\mu' H(\xi_p\mu')\,\mathrm{d}\mu'}{(\xi_r\mu + \xi_p\mu')(\xi_p\beta\mu' - 1)} -$$

$$- \tfrac{1}{2}\alpha_p b_0 H(\xi_p\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) \int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu'\,. \tag{20}$$

If we use the relations

$$\frac{1}{(\xi_p\beta\mu' - 1)(\xi_p\mu + \xi_p\mu')} = \frac{1}{(\xi_r\beta\mu + 1)}\left[\frac{\beta}{\xi_p\beta\mu - 1} - \frac{1}{\xi_r\mu + \xi_p\mu'}\right], \tag{21}$$

we get from Equation (20)

$$I_p^*(0,\mu) = \tfrac{1}{2}\alpha_r b_1 H(\xi_r\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) T_p \int_0^1 \frac{\mu'}{\xi_r\beta\mu + 1} \times$$

$$\times \left[\frac{\beta}{\xi_p\beta\mu' - 1} - \frac{1}{\xi_r\mu + \xi_r\mu'}\right] H(\xi_p\mu')\,\mathrm{d}\mu' - \tfrac{1}{2}\alpha_r b_0 H(\xi_r\mu) \times$$

$$\times \sum_{p=1}^{k} \xi_r(1-\lambda_p) \int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu' =$$

$$= \tfrac{1}{2}\alpha_r b_1 H(\xi_r\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) T_p \left(\frac{\beta}{\xi_r\beta\mu + 1}\right) \times$$

$$\times \int_0^1 \frac{\mu' H(\xi_p\mu')\,\mathrm{d}\mu'}{\xi_p\beta\mu' - 1} - \tfrac{1}{2}\alpha_r b_1 H(\xi_r\mu) \sum_{p=1}^{k} \xi_r \times$$

$$\times (1-\lambda_p) T_p \left(\frac{1}{\xi_p\beta\mu + 1}\right) \int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu' - \tfrac{1}{2}\alpha_r b_0 \times$$

$$\times H(\xi_r\mu) \sum_{p=1}^{k} \xi_r(1-\lambda_p) \int_0^1 \frac{\mu' H(\xi_p\mu')}{\xi_r\mu + \xi_p\mu'}\,\mathrm{d}\mu'\,. \tag{22}$$

From Equation (6),

$$I_r(0, \mu) = b_0 + \frac{b_1 T_r}{1 + \xi_r \beta\mu} + I_r^*(0, \mu)\,. \tag{23}$$

If we use Equations (18), (22), (23) we get

$$\begin{aligned} I_r(0, \mu) = {} & \left(b_0 + \frac{b_1 T_r}{1 + \xi_r \beta\mu}\right)\left\{\left(\sum_{p=1}^{k} \alpha_p \lambda_p\right)^{1/2} + \right. \\ & \left. + \frac{1}{2} \sum_{p=1}^{k} \alpha_p (1 - \lambda_p) \int_0^1 \frac{\xi_p \mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'}\, d\mu' \right\} + \tfrac{1}{2}\alpha_r b_1 \times \\ & \times H(\xi_r \mu) \sum_{p=1}^{k} \xi_r (1 - \lambda_p)\left(\frac{T_p \beta}{\xi_r \beta\mu + 1}\right) \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_p \beta\mu' - 1}\, d\mu' - \\ & - \tfrac{1}{2}\alpha_r b_1 H(\xi_r \mu) \sum_{p=1}^{k} \xi_r (1 - \lambda_p)\left(\frac{T_p}{\xi_r \beta\mu + 1}\right) \times \\ & \times \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'}\, d\mu' - \tfrac{1}{2}\alpha_r b_0 H(\xi_r \mu) \sum_{p=1}^{k} \xi_r \times \\ & \times (1 - \lambda_p) \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'}\, d\mu'\,; \end{aligned} \tag{24}$$

and thus

$$\begin{aligned} I_r(0, \mu) = {} & b_0 H(\xi_r \mu)\left\{\left(\sum_{p=1}^{k} \alpha_p \lambda_p\right)^{1/2} + \right. \\ & \left. + \sum_{p=1}^{k} (\alpha_p \xi_p - \alpha_r \xi_r)(1 - \lambda_p) \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'}\, d\mu' \right\} + \\ & + b_1 \frac{H(\xi_r \mu)}{(1 + \xi_r \beta\mu)} \left\{ T_r \left(\sum_{p=1}^{k} \alpha_p \lambda_p\right)^{1/2} + \right. \\ & \left. + \frac{1}{2} \sum_{p=1}^{k} (\alpha_p \xi_p T_r - \alpha_r \xi_r T_p)(1 - \lambda_p) \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_r \mu + \xi_p \mu'}\, d\mu' \right\} + \\ & + \tfrac{1}{2} b_1 \alpha_r \xi_r \beta \frac{H(\xi_r \mu)}{(1 + \xi_r \beta\mu)} \sum_{p=1}^{k} (1 - \lambda_p) T_p \times \\ & \times \int_0^1 \frac{\mu' H(\xi_p \mu')}{\xi_p \beta\mu' - 1}\, d\mu'\,, \end{aligned} \tag{25}$$

which is the final form of the emergent intensity in the rth line.

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TIME-DEPENDENT SCATTERING AND TRANSMISSION FUNCTION IN AN ANISOTROPIC TWO-LAYERED ATMOSPHERE

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Abstract. In this paper we consider the time-dependent diffuse reflection and transmission problems for a homogeneous anisotropically-scattering atmosphere of finite optical depth and solve it by the principle of invariance. Also we consider the time-dependent diffuse reflection and transmission of parallel rays by a slab consisting of two anisotropic homogeneous layers, whose scattering and transmission properties are known. It is shown how to express the time-dependent reflected and transmitted intensities in terms of their components. In a manner similar to that given by Tsujita (1968), we assumed that the upward-directed intensities of radiation at the boundary of the two layers are expressed by the sum of products of some auxiliary functions depending on only one argument. Then, after some analytical manipulations, three groups of systems of simultaneous integral equations governing the auxiliary functions are obtained.

1. Introduction

Sobolev (1956) dealt with the one-dimensional problem of time-dependent diffuse reflection and transmission by a probabilistic method. Diffuse reflection of time-dependent parallel rays by a semi-infinite atmosphere was treated by Ueno (1962) on the basis of the principle of invariance. Bellman *et al.* (1962) obtained an integral equation governing diffuse reflection of time-dependent parallel rays from the lower boundary of a finite inhomogeneous atmosphere. Ueno (1965) also obtained this equation by probabilistic method. Matsumoto (1967a) derived functional equations in the integral radiation allowing for the time-dependence given byDirac's δ-function and Heaviside unit step-function. Matsumoto (1967b) also derived a complete set of functional equations for the scattering (S) and transmission (T) functions which govern the laws of diffuse reflection and transmission of time-dependent parallel rays by a finite, inhomogeneous, plane-parallel, non-emitting, and isotropically-scattering atmosphere, where the dependence of the time of the incident radiation is given by Dirac's δ-function and Heaviside's unit step-function. A formulation of the time-dependent H-function was accomplished by means of the Laplace transform in the time-domain. Numerical evaluation of the H-function based on numerical inversion of the Laplace transform presented by Bellman *et al.* (1966) was made.

Recently, Karanjai and Biswas (1988) derived the time-dependent X- and Y-functions

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for homogeneous, plane-parallel, non-emitting, and isotropic atmosphere of finite optical thickness using the integral equation method devleoped by Rybicki (1971), Biswas and Karanjai (1990a) have derived the time-dependent H-, X-, and Y-function in a homogeneous atmosphere scattering anisotropically with Dirac's δ-function and Heaviside unit step-function type time-dependent incidence. Biswas and Karanjai (1990b) have also derived the solution of diffuse reflection and transmission problem for homogeneous isotropic atmosphere of finite optical depth. In this paper we derived the nonlinear integral equations for X- and Y-functions (Chandrasekhar, 1960) for anisotropically-scattering atmosphere. The anisotropy is represented by means of a phase function which can be expressed in terms of finite-order Legendre polynomials. The principal of invariance is applied to derive the functional equations for time-dependent scattering and transmission functions. Next we considered the time-dependent diffuse reflection and transmission of plane-parallel rays by a slab consisting of two homogeneous anisotropically-scattering layers, whose scattering and transmission functions are known. The problem of the time-independent scattering and transmission of radiation in plane-parallel atmosphere of two layers was treated first by Van de Hulst (1963; also see Tsujita, 1968). Hawking (1961) dealt with the problem analytically starting with Milne's integral equation. Later on, Hansen (see Tsujita, 1968) formulated the scattering and transmission functions in a medium consisting of two optically thin layers by the invariant imbedding partical-counting method. Gutshabash (1957) formulated the problem as solutions of simultaneous integral equations. So far as his equations are solvable, the scattering and transmission functions required are given exactly for two layers of different albedos and different large optical thickness. We have extended the same problem (Tsujita, 1968) for the time-dependent transfer of radiation.

2. Derivation of Fundamental Equations

2.1. Formulation of the Problem

In an anisotropically-scattering medium, the intensity of radiation $I(\tau, \mu, \phi, t)$ at any time t, any optical depth τ, in the direction $\cos^{-1}\mu$, satisfies the equation of transfer

$$\frac{1}{c}\frac{\partial I(\tau, \mu, \phi, t)}{\partial t} + \mu\frac{\partial I(\tau, \mu, \phi, t)}{\partial \tau} + I(\tau, \mu, \phi, t) = J(\tau, \mu, \phi, t)\,, \tag{1}$$

in which the source function $J((\tau, \mu, \phi, t)$ is given by

$$J(\tau, \mu, \phi, t) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} P(\mu, \phi; \mu', \phi')I(\tau, \mu', \phi', t)\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,, \tag{2}$$

where $P(\mu, \phi; \mu', \phi')$, the general phase function and c represents the velocity of light. In the above, μ and ϕ represent, respectively, the cosine of the zenith distance and the azimuthal angle. We decompose the intensity of radiation field into two components for two directions, viz., intensity directed towards the lower surface of the atmosphere ($I^+(\tau, \mu, \phi, t)$) and intensity directed towards the upper surface of the atmosphere ($I^-(\tau, \mu, \phi, t)$).

We consider the initial boundary conditions

$$I(\tau, \mu, \phi, 0) = 0\,, \tag{3}$$

$$I^+(0, \mu, \phi, t) = I_{\text{inc}}(\mu, \phi, t)\,, \tag{4}$$

$$I^-(\tau_1, \mu, \phi, t) = I^*_{\text{inc}}(\mu, \phi, t)\,. \tag{5}$$

Equations (4) and (5) asserts that the lower and the upper surfaces are illuminated. However, we shall restrict ourselves for the time being to the case of illumination on the upper surface ($\tau = 0$) by means of an instantaneously collimated beam of light at time $t = 0$. The other surface will be free from any incident radiation. We now distinguish between the reduced incident intensity which is incident on boundary surface and penetrates to the depth τ without suffering any collision and diffuse radiation which arises due to different processes (Chandrasekhar, 1960). For the total radiation field we have

$$I^+(\tau, \mu, \phi, t) = I_d^+(\tau, \mu, \phi, t) + I_{\text{inc}}\left(\mu, \phi, t - \frac{\tau}{c\mu}\right)\exp\left(-\frac{\tau}{\mu}\right), \tag{6}$$

$$I^-(\tau, \mu, \phi, t) = I_d^-(\tau, \mu, \phi, t) + I^+_{\text{inc}}\left(\mu, \phi, t - \frac{\tau_1 - \tau}{c\mu}\right)\exp\left(-\frac{\tau_1 - \tau}{\mu}\right), \tag{7}$$

where the subscript 'd' represent diffuse fields. If we substitute these expression for $I^+(\tau, \mu, \phi, t)$ and $I^-(\tau, \mu, \phi, t)$ in Equation (1) we get two separate equations of transfer for two components

$$\left(c^{-1}\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} + 1\right)I_d^+(\tau, \mu, \phi, t) = J(\tau, \mu, \phi, t)\,, \tag{8}$$

$$\left(c^{-1}\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} + 1\right)I_d^-(\tau, \mu, \phi, t) = J(\tau, \mu, \phi, t)\,, \tag{9}$$

where

$$J(\tau, \mu, \phi, t) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} I_d(\tau, \mu', \phi', t)\,\times$$

$$\times\, P(\mu, \phi; \mu', \phi')\mu'\,\mathrm{d}\phi' + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{1} P(\mu, \phi; \mu', \phi')\,\times$$

$$\times I_{\text{inc}}\left(\mu', \phi', t - \frac{\tau}{c\mu}\right) \exp\left(-\frac{\tau}{\mu}\right) d\mu' \, d\phi' + \frac{1}{4\pi} \times$$

$$\times \int_0^{2\pi} \int_0^1 I_{\text{inc}}^*\left(\mu', \phi', t - \frac{\tau_1 - \tau}{c\mu}\right) \exp\left(-\frac{\tau_1 - \tau}{\mu}\right) \times$$

$$\times P(\mu, \phi; \mu', \phi') \, d\mu' \, d\phi' . \tag{10}$$

Let us now put in Equation (10)

$$I_{\text{inc}}(\mu, \phi, t) = F\delta(t)\delta(\mu - \mu_0)\delta(\phi - \phi_0), \tag{11}$$

$$I_{\text{inc}}^*(\mu, \phi, t) = 0 ; \tag{12}$$

where F is a constant.

Hence, we get

$$J(\tau, \mu, \phi, t) = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} I_d(\tau_1, \mu', \phi', t) P(\mu, \phi; \mu', \phi') \, d\mu' \, d\phi' +$$

$$+ \tfrac{1}{4} F P(\mu, \phi; \mu_0, \phi_0) \exp\left(-\frac{\tau}{\mu_0}\right) \delta\left(t - \frac{\tau}{c\mu_0}\right). \tag{13}$$

The new set of boundary conditions are given by

$$I_d^+(\tau, \mu, \phi, t) = 0 , \tag{14a}$$

$$I_d^-(\tau, \mu, \phi, t) = 0 . \tag{14b}$$

This simplification of boundary conditions are the characteristic of such formulation. Let us now define the scattering and transmission function (cf. Matsumoto, 1967a) as

$$S(\tau, \mu, \phi; \mu_0, \phi_0, t) = I_d^-(0, \mu, \phi, t) , \tag{15}$$

$$I(\tau, \mu, \phi; \mu_0, \phi_0, t) = I_d^+(\tau_1, \mu, \phi, t) . \tag{16}$$

2.2. Principle of Invariance

We shall now derive the functional equations for these two functions. The four principles of invariance (Matsumoto, 1969) for this problem take the following forms:

(A) The intensity $I_d^-(\tau, \mu, \phi, t)$ in the upward direction at time t and at depth τ is given by

$$I_d^-(\tau, \mu, \phi, t) = F\mu^{-1} S\left(\tau_1 - \tau; \mu, \phi: \mu_0, \phi_0, t - \frac{\tau}{c\mu_0}\right) \exp\left(-\frac{\tau}{\mu_0}\right) +$$

$$+ \frac{1}{4\pi\mu} \int_0^t dt' \int_0^1 \int_0^{2\pi} S(\tau_1 - \tau; \mu, \phi; \mu', \phi', t - t') I_d^+ \times$$

$$\times (\tau, \mu', \phi', t') \, d\mu' \, d\phi' . \tag{17}$$

(B) The intensity $I_d^+(\tau, \mu, \phi, t)$ in the downward direction at time t and at a depth τ is given by

$$I_d^+(\tau, \mu, \phi, t) = F\mu^{-1}T(\tau; \mu, \phi; \mu_0, \phi_0, t) + \frac{1}{4\pi\mu}\int_0^t dt' \times$$
$$\times \int_0^1 \int_0^{2\pi} S(\tau; \mu, \phi; \mu', \phi', t-t') I_d^-(\tau, \mu', \phi', t')\, d\mu'\, d\phi' . \tag{18}$$

(C) The diffuse reflection of the incident radiation by the entire atmosphere is given by

$$F\mu^{-1}S(\tau_1; \mu; \phi; \mu_0, \phi_0, t) = F\mu^{-1}(\tau; \mu, \phi, \mu', \phi', t) +$$
$$+ I_d^-\left(\tau, \mu, \phi, t - \frac{\tau}{c\mu}\right)\exp\left(-\frac{\tau}{\mu}\right) + \frac{1}{4\pi\mu}\int_0^t dt' \times$$
$$\times \int_0^1 \int_0^{2\pi} T(\tau; \mu, \phi; \mu', \phi', t-t') I_d^-(\tau, \mu', \phi', t')\, d\mu'\, d\phi' . \tag{19}$$

(D) The diffuse transmission of the incident radiation by the entire atmosphere is given by

$$F\mu^{-1}T(\tau_1; \mu, \phi; \mu_0, \phi_0, t) = F\mu^{-1}T\left(\tau_1 - \tau; \mu, \phi; \mu_0, \phi_0, t - \frac{\tau}{c\mu_0}\right) \times$$
$$\times \exp\left(-\frac{\tau}{c\mu_0}\right) + I_d^+\left(\tau, \mu, \phi, t - \frac{\tau_1 - \tau}{c\mu}\right)\exp\left(-\frac{\tau_1 - \tau}{\mu}\right) +$$
$$+ \frac{1}{4\pi\mu}\int_0^t dt' \int_0^1 \int_0^{2\pi} T(\tau_1 - \tau; \mu, \phi, \mu_0, \phi_0, t - t') \times$$
$$\times I_d^+(\tau, \mu', \phi', t')\, d\mu'\, d\phi' . \tag{20}$$

A derivation of these four equations is based on classical intuitive physical arguments (Ambartsumian, 1943; Chandrasekhar, 1960; Presendorfer, 1958). Although these equations do not provide a complete knowledge of radiation intensity at any depth (or neutron distribution in a given medium) but only the reflected and transmitted intensities, it has some real advantages for numerical computations.

2.3. Integral Equations for the Scattering and Transmission Function

We differentiate Equation (17) with respect to τ and take the limit as $\tau \to 0$

$$\lim_{\tau \to 0} \frac{\mathrm{d}I_d^-(\tau, \mu, \phi, t)}{\mathrm{d}\tau} = -F\mu^{-1}\left[(c\mu_0)^{-1}\frac{\partial}{\partial t} + (\mu_0)^{-1} + \frac{\partial}{\partial \tau_1}\right] \times$$

$$\times S(\tau_1, \mu, \phi; \mu_0, \phi_0, t) + \frac{1}{4\pi\mu}\int_0^t \mathrm{d}t' \times$$

$$\times \int_0^{2\pi}\int_0^1 S(\tau_1; \mu, \phi; \mu', \phi', t - t')\left[\frac{\mathrm{d}I_d^+(\tau, \mu', \phi', t')}{\mathrm{d}\tau}\,\mathrm{d}\mu'\,\mathrm{d}\phi'\right]_{\tau=0}. \quad (21)$$

From Equation (8), we get by use of Equation (14)

$$\lim_{\tau \to 0} \frac{\mathrm{d}I_d^+(\tau, \mu', \phi', t')}{\mathrm{d}\tau} = \frac{J(0, \mu', \phi', t')}{\mu'}, \quad (22)$$

where

$$J(0, \mu', \phi', t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \frac{F}{\mu''} S(\tau_1, \mu'', \phi'', t)\,\mathrm{d}\mu''\,\mathrm{d}\phi'' +$$

$$+ \tfrac{1}{4}F\delta(t')P(\mu, \phi; \mu_0, \phi_0). \quad (23)$$

In deriving Equation (23) we have used the expression for $J(\tau, \mu, \phi, t)$, Equation (9) now yields, after use of Equations (14) and (15)

$$\lim_{\tau \to 0} \frac{\mathrm{d}I_1^-(\tau, \mu, \phi, t)}{\mathrm{d}\tau} = -\frac{J(0, \mu, \phi, t)}{\mu} +$$

$$+\left(c^{-1}\frac{\partial}{\partial t} + 1\right)\mu^{-1}F\mu^{-1}S(\tau_1, \mu, \phi; \mu_0, \phi_0, t). \quad (24)$$

If we substitute Equations (22) and (24) in Equation (17), after cancellation and rearrangements of terms, we get

$$\frac{\partial S(\tau_1; \mu, \phi, \mu_0, \phi_0, t)}{\partial \tau_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + 1\right) \times$$

$$\times S(\tau_1; \mu, \phi, \mu_0, \phi_0, t) = P(\mu, \phi; \mu_0, \phi_0)\delta(t) +$$

$$+ \frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(\mu, \phi; \mu'', \phi'')S(\tau_1; \mu'', \phi''; \mu_0, \phi_0, t)\,\frac{\mathrm{d}\mu''}{\mu''}\,\mathrm{d}\phi'' +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(-\mu', \phi'' - \mu_0, \phi_0)\,\frac{\mathrm{d}\mu'}{\mu'}\,\mathrm{d}\phi' +$$

$$+\frac{1}{16\pi^2}\int_0^t \mathrm{d}t' \int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} S(\tau_1\,;\mu, \phi; \mu_0', \phi_0', t-t')\times$$

$$\times P(-\mu', \phi'\,;\mu'', \phi'')S(\tau_1\,;\mu'', \phi''\,;\mu_0, \phi_0, t)\,\frac{\mathrm{d}\mu'}{\mu'}\,\mathrm{d}\phi'\,\frac{\mathrm{d}\mu''}{\mu''}\,\mathrm{d}\phi''\,. \quad (25)$$

Equation (25) is the required functional equation of the time-dependent S-function. Again, if we differentiate Equations (18), (19), and (20) with respect to τ and taking the limit as $\tau \to \tau_1$ and $\tau \to 0$, respectively, and following the same procedure we get

$$\frac{\partial T(\tau_1\,;\mu, \phi; \mu_0, \phi_0, t)}{\partial\tau_1} + \mu^{-1}\left(1 + \frac{1}{c}\frac{\partial}{\partial t}\right)I(\tau_1\,;\mu, \phi; \mu_0, \phi_0, t) =$$

$$= \exp\left(-\frac{\tau_1}{\mu_0}\right)\delta\left(t - \frac{\tau}{c\mu_0}\right)P(-\mu, \phi;\ -\mu_0, \phi_0) +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(-\mu, \phi;\ -\mu'', \phi'')T(\tau_1\,;\mu'', \phi''\,;\mu_0, \phi_0, t)\,\frac{\mathrm{d}\mu''}{\mu''} +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1, \mu, \phi; \mu', \phi', t-t')\delta\left(t - \frac{\tau}{c\mu_0}\right)\exp\left(-\frac{\tau_1}{\mu_0}\right)\times$$

$$\times P(\mu, \phi;\ -\mu_0, \phi_0)\,\frac{\mathrm{d}\mu'}{\mu'}\,\mathrm{d}\phi' + \frac{1}{16\pi^2}\int_0^t \mathrm{d}t' \int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}\times$$

$$\times S(\tau_1\,;\mu, \phi; \mu', \phi', t-t')T(\tau_1\,;\mu'', \phi''\,;\mu_0, \phi_0, t')\times$$

$$\times P(\mu', \phi'\,;\ -\mu'', \phi'')\,\frac{\mathrm{d}\mu'}{\mu'}\,\mathrm{d}\phi'\,\frac{\mathrm{d}\mu''}{\mu''}\,\mathrm{d}\phi''\,, \quad (26)$$

$$\frac{\partial S(\tau_1\,;\mu, \phi; \mu_0, \phi_0, t)}{\partial\tau_1} = P(\mu, \phi;\ -\mu_0, \phi_0)\times$$

$$\times \exp\left(-\tau_1\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)\right)\delta\left(t - \frac{\tau_1}{c\mu} - \frac{\tau_1}{c\mu_0}\right) + \exp\left(-\frac{\tau_1}{\mu}\right)\times$$

$$\times \frac{1}{4\pi} \int_0^t dt' \int_0^1 \int_0^{2\pi} T(\tau_1; \mu'', \phi'', \mu_0, \phi_0, t - t') P(\mu, \phi; -\mu'', \phi'') \times$$

$$\times \delta\left(t' - \frac{\tau_1}{c\mu}\right) \frac{\mathrm{d}\mu''}{\mu''} \mathrm{d}\phi'' + \frac{1}{4\pi} \int_0^t \mathrm{d}t' \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi', t - t') \times$$

$$\times \delta\left(t' - \frac{\tau_1}{c\mu}\right) \exp\left(-\frac{\tau_1}{\mu_0}\right) P(\mu', \phi'; -\mu_0, \phi_0) \frac{\mathrm{d}\mu'}{\mu'} \mathrm{d}\phi' + \frac{1}{16\pi^2} \int_0^t \mathrm{d}t' \times$$

$$\times \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', \phi', t - t') T(\tau_1; \mu'', \phi''; \mu_0, \phi_0, t') \times$$

$$\times P(\mu', \phi'; -\mu'', \phi'') \frac{\mathrm{d}\mu'}{\mu'} \mathrm{d}\phi' \frac{\mathrm{d}\mu''}{\mu''} \mathrm{d}\phi'' , \tag{27}$$

$$\frac{\partial T(\tau_1; \mu, \phi, \mu_0, \phi_0, t)}{\partial \tau_1} + \frac{1}{\mu_0}\left(\frac{1}{c} \frac{\partial}{\partial t} + 1\right) T(\tau_1, \mu, \phi; \mu_0, \phi_0, t) =$$

$$= P(-\mu, \phi; -\mu_0, \phi_0) \exp\left(-\frac{\tau_1}{\mu}\right) \delta\left(t - \frac{\tau_1}{c\mu}\right) + \frac{1}{4\pi} \exp\left(-\frac{\tau_1}{\mu}\right) \times$$

$$\times \int_0^1 \int_0^{2\pi} P(-\mu, \phi; \mu'', \phi'') S\left(\tau_1; \mu'', \phi'', \mu_0, \phi_0, t - \frac{\tau_1}{c\mu}\right) \times$$

$$\times \frac{\mathrm{d}\mu''}{\mu''} \mathrm{d}\phi'' + \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \mu', \phi', t) P(-\mu, \phi; -\mu_0, \phi_0) \times$$

$$\times \frac{\mathrm{d}\mu'}{\mu'} \mathrm{d}\phi' + \frac{1}{16\pi^2} \int_0^t \mathrm{d}t' \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \phi; \mu', t - t') \times$$

$$\times S(\tau_1; \mu'', \phi''; \mu_0, \phi_0, t) P(-\mu', \phi'; \mu'', \phi'') \frac{\mathrm{d}\mu'}{\mu'} \mathrm{d}\phi' \frac{\mathrm{d}\mu''}{\mu''} \mathrm{d}\phi'' . \tag{28}$$

Equations (25), (26), (27), and (28) are the required functional equations for 'S' and 'T' functions. Let us now introduce the Laplace transform with respect to the time-variable which enables us to eliminate (at least formally) the time-variable,

$$\frac{\partial S(\tau_1;\mu,\phi;\mu_0,\phi_0,s)}{\partial \tau_1} + \left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\left(1+\frac{s}{c}\right)S(\tau_1;\mu,\phi;\mu_0,\phi_0,s) =$$

$$= P(\mu,\phi;-\mu_0,\phi_0) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} P(\mu,\phi;\mu'',\phi'')\times$$

$$\times S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\frac{d\mu''}{\mu''}d\phi'' +$$

$$+\frac{1}{4\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu',\phi',s)P(-\mu',\phi';-\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi' +$$

$$+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi,\mu',\phi',s)S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\times$$

$$\times P(-\mu',\phi';\mu'',\phi'')\frac{d\mu'}{\mu'}d\phi'\frac{d\mu''}{\mu''}d\phi'', \tag{29}$$

$$\frac{\partial T(\tau_1;\mu,\phi;\mu_0,\phi_0,s)}{\partial \tau_1} + \left(1+\frac{s}{c}\right)T(\tau_1;\mu,\phi;\mu_0,\phi_0,s)\mu^{-1} =$$

$$= P(-\mu,\phi;\mu_0,\phi_0)\exp\left(-\frac{\tau_1 s}{c\mu_0}\right) + \frac{1}{4\pi}\int_0^1\int_0^{2\pi} T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\times$$

$$\times P(-\mu,\phi;-\mu'',\phi'')\frac{d\mu''}{\mu''}d\phi'' + \frac{1}{4\pi}\exp\left(-\frac{\tau_1}{\mu_0}\right)\exp\left(-\frac{\tau_1 s}{c\mu_0}\right)\times$$

$$\times\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi,\mu',\phi',s)P(\mu,\mu';-\mu_0,\phi_0)\frac{d\mu'}{\mu'}d\phi' +$$

$$+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\phi;\mu',\phi',s)P(\mu',\phi';-\mu'',\phi'')\times$$

$$+ T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\frac{d\mu'}{\mu'}d\phi'\frac{d\mu''}{\mu''}d\phi'', \tag{30}$$

$$\frac{\partial S(\tau_1;\mu,\phi;\mu_0,\phi_0,s)}{\partial\tau_1} = \exp\left(-\tau_1\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right)\times$$

$$\times\exp\left(-\frac{\tau_1 s}{c}\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right)P(\mu,\phi;-\mu_0,\phi_0)+\frac{1}{4\pi}\exp\left(-\frac{\tau_1}{\mu_0}\right)\times$$

$$\times\exp\left(-\frac{\tau_1 s}{c\mu}\right)\int_0^1\int_0^{2\pi}T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)P\times$$

$$\times(\mu,\phi,-\mu'',\phi'')\frac{d\mu''}{\mu''}\,d\phi''+\frac{1}{4\pi}\exp\left(-\frac{\tau_1}{\mu_0}\right)\exp\left(-\frac{\tau_1 s}{c\mu_0}\right)\times$$

$$\times\int_0^1\int_0^{2\pi}I(\tau_1;\mu,\phi,\mu',\phi,s)P(\mu',\phi';-\mu_0,\phi_0)\frac{d\mu'}{\mu'}\,d\phi'+$$

$$+\frac{1}{16\pi^2}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\phi;\mu',\phi',s)P(\mu',\phi';-\mu'',\phi'')\times$$

$$\times T(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\frac{d\mu''}{\mu''}\,d\phi''\frac{d\mu'}{\mu'}\,d\phi', \tag{31}$$

$$\frac{\partial T(\tau_1;\mu,\phi,\mu_0,\phi_0,s)}{\partial\tau_1}+\frac{1}{\mu_0}\left(1+\frac{s}{c}\right)T(\tau_1;\mu,\phi;\mu_0,\phi_0,s)=$$

$$=\exp\left(-\frac{\tau_1}{\mu}\right)\exp\left(-\frac{\tau_1 s}{\mu}\right)\mathrm{P}(-\mu,\phi;-\mu_0,\phi_0)+$$

$$+\frac{1}{4\pi}\exp\left(-\frac{\tau_1}{\mu}\right)\exp\left(-\frac{\tau_1 s}{c\mu}\right)\int_0^1\int_0^{2\pi}P(-\mu,\phi;\mu'',\phi'')\times$$

$$\times S(\tau_1;\mu'',\phi'',\mu_0,\phi_0,s)\frac{d\mu''}{\mu''}\,d\phi''+\frac{1}{4\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\phi;\mu',\phi',s)\times$$

$$\times P(-\mu',\phi',-\mu_0,\phi_0)\frac{d\mu'}{\mu'}\,d\phi'+\frac{1}{16\pi^2}$$

$$\times\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi}T(\tau_1;\mu,\phi,\mu',\phi',s)P(-\mu',\phi',\mu'',\phi'')\times$$

$$\times S(\tau_1;\mu'',\phi'';\mu_0,\phi_0,s)\frac{d\mu'}{\mu'}\,d\phi'\frac{d\mu''}{\mu''}\,d\phi''. \tag{32}$$

2.4. THE REDUCTION OF THE INTEGRAL EQUATIONS

We have

$$P(\mu, \phi; \mu', \phi') = \sum_{m=0}^{N} (2 - \delta_{0,m}) \left[\sum_{l=m}^{N} w_l^m P_l^m(\mu) P_l^m(\mu') \right] \cos m(\phi' - \phi) . \tag{33}$$

If we follow Chandrasekhar (1960), we obtain

$$S(\tau_1; \mu, \phi; \mu_0, \phi_0, s) = \sum_{m=0}^{N} S^{(m)}(\tau_1; \mu, \mu_0; s) \cos m(\phi_0 - \phi) \tag{34}$$

$$T(\tau_1; \mu, \phi; \mu_0, \phi_0, s) = \sum_{m=0}^{N} T^{(m)}(\tau_1; \mu, \mu_0, s) \cos m(\phi_0 - \phi) . \tag{35}$$

If we substitute these expansions of S and T in Equations (29)–(32) and after some rearrangements we get

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\left(1 + \frac{s}{c}\right) S^{(m)}(\tau_1; \mu, \mu_0, s) + \frac{S^{(m)}(\tau_1; \mu; \mu_0; s)}{\partial \tau_1} =$$

$$= (2 - \delta_{0,m}) \sum_{l=m}^{N} (-1)^{m+l} w_l^m \left[P_l^m(\mu) + \frac{(-l)^{l+m}}{2(2 - \delta_{0,m})} \times \right.$$

$$\left. \times \int_0^1 S^m(\tau; \mu, \mu', s) P_l^m(\mu') \frac{d\mu'}{\mu'} \right] \left[P_l^m(\mu_0) + \frac{(-1)^{l+m}}{2(2 - \delta_{0,m})} \times \right.$$

$$\left. \times \int_0^1 P_l^m(\mu'') S^{(m)}(\tau_1; \mu'', \mu_0, s) \frac{d\mu''}{\mu''} \right], \tag{36}$$

$$\frac{1}{\mu}\left(1 + \frac{s}{c}\right) T^{(m)}(\tau_1; \mu, \mu_0, s) + \frac{\partial T^{(m)}(\tau_1; \mu, \mu_0, s)}{\partial \tau_1} =$$

$$= (2 - \delta_{0,m}) \sum_{l=m}^{N} w_l^m \left[P_l^m(\mu) + \frac{(-l)^{l+m}}{2(2 - \delta_{0,m})} \times \right.$$

$$\left. \times \int_0^1 S^{(m)}(\tau_1; \mu, \mu', s) P_l^m(\mu) \frac{d\mu'}{\mu'} \right] \times$$

$$\times \left[\exp\left[-\frac{\tau_1}{\mu_0}\left(1 + \frac{s}{c}\right)\right] P_l^m(\mu_0) + \right.$$

$$\left. + \frac{1}{2(2 - \delta_{0,m})} \int_0^1 T^{(m)}(\tau_1; \mu'', \mu_0, s) P_l^m(\mu'') \frac{d\mu''}{\mu''} \right], \tag{37}$$

$$\frac{\partial S^{(m)}(\tau_1\,;\mu,\mu_0,s)}{\partial \tau_1} = (2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}w_l^m \times$$

$$\times P_l^m(\mu)\exp\left(-\frac{\tau_1}{\mu}\left(1+\frac{s}{c}\right)\right)+\frac{1}{2(2-\delta_{0,m})}\times$$

$$\times \int_0^1 T^{(m)}(\tau_1\,;\mu,\mu',s)P_l^m(\mu')\,\frac{d\mu'}{\mu'}\Bigg]\times$$

$$\times\left[P_l^m(\mu_0)\exp\left[-\frac{\tau_1}{\mu_0}\left(l+\frac{s}{c}\right)\right]\right]+\frac{1}{2(2-\delta_{0,m})}\times$$

$$\times \int_0^1 P_l^m(\mu'')T^{(m)}(\tau_1\,;\mu'',\mu_0,s)\,\frac{d\mu''}{\mu''}\Bigg], \tag{38}$$

$$\frac{1}{\mu_0}\left(1+\frac{s}{c}\right)T^{(m)}(\tau_1\,;\mu\,;\mu_0\,;s)+\frac{\partial T^{(m)}(\tau_1\,;\mu,\mu_0,s)}{\partial \tau_1}=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}w_l^m\left[P_l^m(\mu)\exp\left(-\frac{\tau_1}{\mu}\left(1+\frac{s}{c}\right)\right)+\frac{1}{2(2-\delta_{0,m})}\times\right.$$

$$\times \int_0^1 T^{(m)}(\tau_1\,;\mu,\mu_0,s)P_l^m(\mu')\,\frac{d\mu'}{\mu'}\Bigg]\times$$

$$\times\left[P_l^m(\mu_0)+\frac{(-l)^{l+m}}{2(2-\delta_{0,m})}\int_0^1 P_l^m(\mu'')S^{(m)}(\tau_1\,;\mu'',\mu_{0,s})\,\frac{d\mu''}{\mu''}\right]. \tag{39}$$

If we now let

$$\psi_l^m(\tau_1\,;\mu,s)=P_l^m(\mu)+\frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\int_0^l S^{(m)}(\tau_1\,;\mu,\mu',s)\,P_l^m(\mu')\,\frac{d\mu'}{\mu'} \tag{40}$$

and

$$\phi_l^m(\tau_1\,;\mu,s)=\exp\left(-\frac{\tau_1}{\mu}\left(1+\frac{s}{c}\right)\right)P_l^m(\mu)+\frac{1}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 T^{(m)}(\tau_1\,;\mu,\mu',s)P_l^m(\mu)\,\frac{d\mu'}{\mu'}\,, \tag{41}$$

then, in view of principle of reciprocity (Chandrasekhar, 1960) we can rewrite Equations (36)–(39) in the form

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\left(1+\frac{s}{c}\right) S^{(m)}(\tau_1;\mu,\mu_0,s)+\frac{\partial S^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1}=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}w_l^m\psi_l^m(\tau_1;\mu,s)\psi_l^m(\tau_1;\mu_0,s)\,, \tag{42}$$

$$\frac{1}{\mu}\left(1+\frac{s}{c}\right)T^{(m)}(\tau_1;\mu,\mu_0,s)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1}=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}w_l^m\psi_l^m(\tau_1;\mu,s)\phi_l^m(\tau_1;\mu_0,s) \tag{43}$$

and

$$\frac{\partial S^{(m)}(\tau_1;\hat{\mu};\mu_0,s)}{\partial\tau_1}=(2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}w_l^m$$

$$\times\,\phi_l^m(\tau_1;\mu,s)\phi_l^m(\tau_1;\mu_0,s) \tag{44}$$

and

$$\frac{1}{\mu_0}\left(1+\frac{s}{c}\right)T^{(m)}(\tau_1;\mu,\mu_0,s)+\frac{\partial T^{(m)}(\tau_1;\mu,\mu_0,s)}{\partial\tau_1}=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}w_l^m\phi_l^m(\tau_1;\mu,s)\psi_l^m(\tau_1;\mu,s)\,. \tag{45}$$

Now by use of Equations (42) and (44) we get

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\left(1+\frac{s}{c}\right)S^{(m)}(\tau_1;\mu,\mu_0,s)=$$

$$=(2-\delta_{0,m})\sum_{l=m}^{N}(-1)^{l+m}w_l^m\,[\psi_l^m(\tau_1;\mu,s)\psi_l^m(\tau_1;\mu_0,s)-$$

$$-\,\phi_l^m(\tau_1;\mu,s)\phi_1^m(\tau_1;\mu_0,s)]\,; \tag{46}$$

and by use of Equations (43) and (45)

$$\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)\left(1+\frac{s}{c}\right)T^{1m2}(\tau_1,\mu,\mu_0,s) = (2-\delta_{0,m})\sum_{l=m}^{N} \times$$

$$\times\ w_l^m[\phi_l^m(\tau_1;\mu,s)\psi_l^m(\tau_l;\mu_0,s) - \psi_l^m(\tau_1;\mu,s)\phi_l^m(\tau_1;\mu_0,s)]\,. \quad (47)$$

Equations (46) and (47) are the two fundamental equations of our problem.

3. Solution

3.1. Legendre expansion of the phase function and the principle of invariance

Let us now consider that the atmosphere consists of two different layers. Denoting the quantities in the upper layer by subscript '1' and the quantities in the lower by subscript '2' and if we use Equations (46) and (47) we have

$$S_i^{(m)}(\tau_i;\mu,\mu_0,s) = \frac{\mu\mu_0}{\mu+\mu_0}(2-\delta_{0,m})\sum_{l=m}^{N}(-l)^{l+m}\frac{w_{i,l}^{(m)}}{Q}\times$$

$$\times\ \psi_l^m(\tau_i;\mu,s)\psi_l^m(\tau_i;\mu_0,s) - \phi_l^m(\tau_i;\mu,s) - \phi_l^m(\tau_i;\mu,s)\phi_l^m(\tau_i;\mu_0,s)\,, \quad (48)$$

$$T_i^{(m)}(\tau_i;\mu,\mu_0,s) = \frac{\mu\mu_0}{\mu-\mu_0}(2-\delta_{0,m})\sum_{l=m}^{N}\frac{w_{i,l}^{(m)}}{Q}\times$$

$$\times\ [\phi_l^m(\tau_i,\mu,s)\psi_l^m(\tau_i;\mu,s) - \psi_l^m(\tau_i;\mu,s)\phi_l^m(\tau_i;\mu_0,s)]\,, \quad (49)$$

$$\psi_l^m(\tau_i;\mu,s) = P_l^m(\mu) + \frac{(-1)^{l+m}}{2(2-\delta_{0,m})}\int_0^1 S_i^{(m)}(\tau_i;\mu,\mu',s)P_l^m(\mu')\,\frac{\mathrm{d}\mu'}{\mu'} \quad (50)$$

and

$$\phi_l^m(\tau_i;\mu,s) = P_l^m(\mu)\exp\left(-\frac{\tau_i Q}{\mu}\right) + \frac{1}{2(2-\delta_{0,m})}\times$$

$$\times\int_0^1 T_i^{(m)}(\tau_i;\mu,\mu',s)P_l^{(m)}(\mu')\,\frac{\mathrm{d}\mu'}{\mu'}\,; \quad (51)$$

where

$$Q = 1+\frac{s}{c} \quad \text{and} \quad i = 1,2\,. \quad (52)$$

If we use the above representations and again if we use Equations (34) and (35) we can write the scattering and transmission function in each layer as

$$S_i(\tau_i; \mu, \phi; \mu_0, \phi_0, s) = \sum_{m=0}^{N} S_i^{(m)}(\tau_i; \mu, \mu_0, s) \cos m(\phi_0 - \phi)\,; \tag{53}$$

$$T_i(\tau_i; \mu, \phi; \mu_0, \phi_0, s) = \sum_{m=0}^{N} T_i^{(m)}(\tau_i; \mu, \mu_0, s) \cos m(\phi_0 - \phi) \qquad (i = 1, 2)\,. \tag{54}$$

In what follows we inquire into how represent the scattering and transmission functions in the whole atmosphere. If we follow Tsujita, we introduce diffuse radiation intensities $I_1(\tau_i; \mu, \phi; \mu_0, \phi_0, s)$ and $I_2(\tau_i, \mu, \phi; \mu_0, \phi_0; s)$ which leave the upper and lower layers in the direction (μ, ϕ) with respect to the boundary between the two layes, where (μ_0, ϕ_0) denotes the direction of the incident radiation at the upper surface $\tau = 0$

$$I_1(\tau_1; \mu, \phi; \mu_0, \phi_0, s) \quad \text{and} \quad I_2(\tau_1; \mu, \phi; \mu_0, \phi_0, s)$$

must satisfy the conditions

$$I_1(\tau_1, \mu, \phi; \mu_0, \phi_0, s) = 0 \quad \text{for} \quad 0 < \mu < 1\,, \tag{55}$$

$$I_2(\tau_1; \mu, \phi; \mu_0, \phi_0, s) = 0 \quad \text{for} \quad -1 < \mu < 0\,. \tag{56}$$

Then from the principle of invariances (A)–(B) we have after the Laplace transform with respect to time variable

$$I_2^{(m)}(\tau_1; \mu, \mu_0, s) = F\mu^{-1} S_2^{(m)}(\tau_2; \mu, \mu_0, s) \exp\left(-\frac{Q\tau_1}{\mu_0}\right) + \frac{1}{2(2\delta - \delta_{0,m})\mu} \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s) I_1^{(m)}(\tau_1; \mu', \mu_0, s)\, d\mu'\, d\phi'\,, \tag{57}$$

$$I_1^{(m)}(\tau_1; \mu, \mu_0, s) = F\mu^{-1} T_1^{(m)}(\tau_1; \mu, \mu_0, s) + \frac{1}{2(2 - \delta_{0,m})\mu} \times \int_0^1 S_1^{(m)}(\tau_1, \mu, \mu', s) I_2^{(m)}(\tau_1, \mu', \mu_0, s)\, d\mu'\, d\phi'\,. \tag{58}$$

From (C)–(D),

$$F\mu^{-1} S(\tau_0; \mu, \phi; \mu_0, \phi_0, s) = F\mu^{-1} S_1(\tau_1; \mu, \phi; \mu_0, \phi_0, s) + I_2(\tau; \mu, \phi; \mu_0, \phi_0, s) \exp\left(-\frac{\tau_1 \phi}{\mu}\right) + \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T_1 \times (\tau_1; \mu, \phi; \mu', \phi, s) I_2(\tau_1; \mu', \phi'; \mu_0, \phi_0, s)\, d\mu'\, d\phi' \tag{59}$$

and

$$F\mu^{-1}T(\tau_0; \mu, \phi; \mu_0, \phi_0, s) = F\mu^{-1}T_2(\tau_2; \mu, \phi; \mu_0, \phi_0, s) \times$$

$$\times \exp\left(-\frac{\tau_1 Q}{\mu_0}\right) + I_1(\tau_1; \mu, \phi; \mu_0, \phi_0, s) \exp\left(-\frac{\tau_2 Q}{\mu}\right) +$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T_2(\tau_2; \mu, \phi; \mu', \phi', s) \times$$

$$\times I_1(\tau_1; \mu', \phi'; \mu_0, \phi_0, s)\, d\mu'\, d\phi' ; \tag{60}$$

where τ_0, τ_1, and τ_2 are the optical thickness of the whole atmosphere, the upper and the lower layer, respectively. Furthermore, we assume that $I_i(\tau_1, \mu, \phi, \mu', \phi', s)$ can be expanded in the form

$$I_i(\tau_1; \mu, \phi; \mu', \phi', s) = \sum_{m=0}^{N} I_i^{(m)}(\tau_1; \mu, \mu', s) \cos m(\phi' - \phi),$$

$$(i = 1, 2). \tag{61}$$

If we substitute this expansion in Equations (58) and (57) and taking account of Equations (53) and (54) and allowing for

$$\int_0^{2\pi} \cos m(\phi'' - \phi) \cos n(\phi' - \phi'')\, d\phi'' = \delta_{m,n}\pi \cos m(\phi' - \phi)\, (m \neq 0, n \neq 0) =$$

$$= 2\pi \qquad (m = n = 0), \tag{62}$$

we obtain

$$I_1^{(m)}(\tau_1; \mu, \mu_0, s) = F\mu^{-1}T_1^{(m)}(\tau_1; \mu, \mu_0, s) +$$

$$+ \frac{1}{2(2 - \delta_{0,m})\mu} \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s) I_2^{(m)}(\tau_1; \mu', \mu_0, s)\, d\mu, \tag{63}$$

$$I_2^{(m)}(\tau_1; \mu, \mu_0, s) = F\mu^{-1}S_2^{(m)}(\tau_1; \mu; \mu_0, s) \exp\left(-\frac{\tau_1 Q}{\mu_0}\right) +$$

$$+ \frac{1}{2(2 - \delta_{0,m})} \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s) I_1^{(m)}(\tau; \mu', \mu_0, s)\, d\mu'. \tag{64}$$

3.2. Auxiliary Functions and Their Functional Relations

Let us now consider some auxiliary functions in terms of which $I_1(\tau_1; \mu, \phi; \mu_0, \phi_0, s)$ and $I_2(\tau_1; \mu, \phi; \mu_0, \phi_0, s)$ are formed. If we assume that they depend on only one argument, we seek functional relations satisfied by them and then solve the system of equations. For convenience, we put

$$I_1^{(m)}(\tau_1, \mu, \mu_0, s) = F \frac{\mu_0}{\mu - \mu_0} \sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu, \mu_0, s), \tag{65}$$

$$I_2^{(m)}(\tau_1; \mu, \mu_0, s) = F \frac{\mu_0}{\mu + \mu_0} \sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu, \mu_0, s). \tag{66}$$

If we insert Equations (65), (66), (48), and (49) into Equations (63) and (64) and rearrange them approximately, we have

$$\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu, \mu_0, s) = (2 - \delta_{0,m}) \sum_{l=m}^{N} \frac{w_{1,l}^{(m)}}{Q} \phi_l^{(m)}(\tau_1, \mu, s) \times$$
$$\times \psi_l^m(\tau_1, \mu_0, s) - \psi_l^m(\tau_1, \mu, s)\phi_l^m(\tau_1, \mu_0, s) +$$
$$+ \frac{1}{2} \int_0^1 \left\{ \sum_{l=m}^{N} (-1)^{l+m} \frac{w_{1,l}^{(m)}}{Q} [\psi_1^m(\tau_1, \mu, s)\psi_1^m(\tau_1, \mu', s) - \right.$$
$$\left. - \phi_l^m(\tau_1, \mu, s)\phi_1^m(\tau_1; \mu', s)] \right\} \left[\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s) \right] \times$$
$$\times \left[\frac{\mu}{\mu + \mu'} - \frac{\mu_0}{\mu' + \mu_0} \right] \mathrm{d}\mu', \tag{67}$$

$$\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu, \mu_0, s) = (2 - \delta_{0,m}) \sum_{l=m}^{N} (-1)^{l+m} \frac{w_{2,l}^{(m)}}{Q} \times$$
$$\times [\psi_l^m(\tau_2, \mu, s)\psi_l^m(\tau_2, \mu_0, s) - \phi_l^m(\tau_2, \mu, s)\phi_l^m(\tau_2, \mu_0, s)] \times$$
$$\times \exp\left(- \frac{\tau_1 Q}{\mu_0} \right) + \frac{1}{2} \int_0^1 \left\{ \sum_{l=m}^{N} (-1)^{l+m} \frac{w_{2,l}^{(m)}}{Q} \times \right.$$
$$\left. \times [\psi_l^m(\tau_2, \mu, s)\psi_l^m(\tau_2, \mu', s) - \phi_l^m(\tau_2, \mu, s)\phi_l^m(\tau_2, \mu', s)] \right\} \times$$
$$\times \left[\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu, \mu_0, s) \right] \left[\frac{\mu}{\mu + \mu'} + \frac{\mu}{\mu' - \mu_0} \right] \mathrm{d}\mu', \tag{68}$$

we rewrite Equation (67) as

$$\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu, \mu_0, s) = \sum_{l=m}^{N} \frac{w_{1,l}^{(m)}}{Q} \phi_l^m(\tau_1, \mu, s) \times$$
$$\times \left[(2 - \delta_{0,m})\psi_l^m(\tau_1, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \int_0^1 \phi_l^m(\tau_1, \mu', s) \times \right.$$

$$\times \frac{\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s)}{\mu' + \mu_0} \, d\mu' \Bigg] - \sum_{l=m}^{N} \frac{w_l^{(m)}}{Q} \psi_l^m(\tau_1, \mu, s) \times$$

$$\times \Bigg[(2 - \delta_{0,m}) \phi_l^m(\tau, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \int_0^1 \psi_l^m(\tau_1, \mu', s) \times$$

$$\times \frac{\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s)}{\mu' + \mu_0} \, d\mu' \Bigg] + \frac{\mu}{2} \sum_{l=m}^{N} (-1)^{l+m} \frac{w_{1,l}^{(m)}}{Q} \times$$

$$\times \Bigg[\int_0^1 \frac{\psi_l^m(\tau_1, \mu, s) \psi_l^m(\tau_1, \mu', s) - \phi_l^m(\tau_1, \mu, s) \phi_l^m(\tau_1, \mu', s)}{\mu + \mu'} \Bigg] \times$$

$$\times \Bigg[\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s) \, d\mu' \Bigg]. \tag{69}$$

If we take account of Equation (48), we write the third term of the right-hand side of the above equation as

$$\frac{1}{2(2 - \delta_{0,m})} \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s) \Bigg[\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu', \mu_0, s) \Bigg] \frac{d\mu'}{\mu'} . \tag{70}$$

Then we put

$$\alpha_{1,l}^{(m)}(\mu_0, s) = (2 - \delta_{0,m}) \psi_l^m(\tau_1, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \times$$

$$\times \int_0^1 \phi_l^m(\tau_1, \mu', s) \frac{\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^m(\mu', \mu_0, s)}{\mu + \mu'} \, d\mu' , \tag{71}$$

$$\alpha_{2,l}^{(m)}(\mu_0, s) = (2 - \delta_{0,m}) \phi_l^m(\tau_1, \mu_0, s) + \frac{(-1)^{l+m}}{2} \mu_0 \times$$

$$\times \int_0^1 \psi_l^m(\tau_1, \mu', s) \frac{\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^m(\mu', \mu_0, s)}{\mu' + \mu_0} \, d\mu' . \tag{72}$$

If we make use of Equations (70), (71), (72) and rewrite Equation (69) once more, we have

$$A_l^{(m)}(\mu, \mu_0, s) = \alpha_{1,l}^{(m)}(\mu_0, s)\phi_l^m(\tau_1, \mu, s) - \alpha_{2,l}^{(m)}(\mu_0, s) \times$$

$$\times \psi_l^m(\tau_1, \mu, s) + \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{2,l}^{(m)}}{w_{1,l}^{(m)}} Q\right) \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1, \mu, \mu', s) B_l^{(m)}(\mu', \mu_0, s) \frac{d\mu'}{\mu'} . \tag{73}$$

On the other hand, by rewriting Equation (68), we have

$$\sum_{l=m}^{N} w_{2,l}^{(m)} B_l^{(m)}(\mu, \mu_0, s) = \sum_{l=m}^{N} \frac{w_{2,l}^{(m)}}{Q} (-1)^{l+m} \psi_l^m(\tau_2, \mu, s) \times$$

$$\times \left[(2-\delta_{0,m})\psi_l^m(\tau_2, \mu_0, s) \exp\left(-\frac{\tau_1 Q}{\mu_0}\right) + \frac{\mu_0}{2} \times\right.$$

$$\left.\times \int_0^1 \psi_1^m(\tau_2, \mu', s) \frac{\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu', \mu_0, s)}{\mu' - \mu_0} d\mu'\right] -$$

$$- \sum_{l=m}^{N} \frac{w_{2,l}^{(m)}}{Q} (-1)^{l+m} \phi_l^m(\tau_2, \mu, s)\left[(2-\delta_{0,m})\phi_l^m(\tau_2, \mu, s) \times\right.$$

$$\times \exp\left(-\frac{\tau_1 Q}{\mu_0}\right) + \frac{\mu_0}{2} \int_0^1 \phi_1^m(\tau_2, \mu', s) \times$$

$$\left.\times \frac{\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu', \mu_0, s)}{\mu' - \mu_0} d\mu'\right] + \frac{1}{2(2-\delta_{0,m})} \times$$

$$\times \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\left[\sum_{l=m}^{N} w_{1,l}^{(m)} B_1^{(m)}(\mu', \mu_0, s)\right] \frac{d\mu'}{\mu'} . \tag{74}$$

Then we write $\alpha_{3,l}^{(m)}(\mu_0, s)$ and $\alpha_{4,l}^{(m)}(\mu_0, s)$ as

$$\alpha_{3,l}^{(m)}(\mu_0, s) = (2 - \delta_{0,m})\psi_l^m(\tau_2; \mu_0, s)\exp\left(-\frac{\tau_1 Q}{\mu_0}\right) + \\ + \frac{\mu_0}{2}\int_0^1 \psi_l^m(\tau_2, \mu', s)\frac{\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu', \mu_0, s)}{\mu' - \mu_0}\,d\mu', \tag{75}$$

$$\alpha_{4,l}^{(m)}(\mu_0, s) = (2 - \delta_{0,m})\phi_l^m(\tau_2, \mu_0, s)\exp\left(-\frac{\tau_1 Q}{\mu_0}\right) + \\ + \frac{\mu_0}{2}\int_0^1 \phi_l^m(\tau_2, \mu', s)\frac{\sum_{l=m}^{N} w_{1,l}^{(m)} A_l^{(m)}(\mu', \mu_0, s)}{\mu' - \mu_0}\,d\mu', \tag{76}$$

If we make use of Equations (75) and (76) and rewrite Equation (74) once more, we have

$$B_l^{(m)}(\mu, \mu_0, s) = \alpha_{3,l}^{(m)}(\mu_0, s)\psi_l^m(\tau_2, \mu, s) - \alpha_{4,l}^{(m)} \times \\ \times (\mu_0, s)\phi_l^m(\tau_2, \mu, s) + \frac{1}{2(2 - \delta_{0,m})}\left(\frac{w_{1,l}^{(m)}}{w_{2,l}^{(m)}}\right) Q \times \\ \times \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s) A_l^m(\mu', \mu_0, s)\,\frac{d\mu'}{\mu'}\,. \tag{77}$$

From Equations (73) and (77) we get

$$A_l^{(m)}(\mu, \mu_0, s) = \alpha_{1,l}^{(m)}(\mu_0, s)\phi_l^m(\tau_1, \mu, s) - \alpha_{2,l}^{(m)}(\mu_0, s) \times \\ \times \psi_l^m(\tau_1, \mu, s) + \alpha_{3,l}^{(m)}(\mu_0, s)\,\frac{1}{2(2 - \delta_{0,m})}\left(\frac{w_{2,l}^{(m)}}{w_{1,l}^{(m)}}\right) Q \times \\ \times \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\phi_l^m(\tau_2, \mu', s) - \alpha_{4,l}^{(m)}(\mu_0, s)\,\frac{1}{2(2 - \delta_{0,m})} \times \\ \times \left(\frac{w_{2,l}^{(m)}}{w_{1,l}^{(m)}}\right) Q \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu_0, s)\phi_1^m(\tau_2, \mu', s)\,\frac{d\mu'}{\mu'} + \\ + \frac{1}{4(2 - \delta_{0,m})^2}\int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\int_0^1 S_2^{(m)}(\tau_1; \mu, \mu'', s) A_l^{(m)} \times \\ \times (\mu'', \mu_0, s)\,\frac{d\mu''}{\mu''}\,\frac{d\mu'}{\mu'}, \tag{78}$$

and

$$B_l^{(m)}(\mu, \mu_0, s) = \alpha_{1,l}^{(m)}(\mu_0, s) \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{1,l}^{(m)}}{w_{2,l}^{(m)}}\right) Q \times$$

$$\times \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\phi_l^m(\tau_1, \mu', s) \frac{d\mu'}{\mu'} - \alpha_{2,l}^{(m)}(\mu_0, s) \times$$

$$\times \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{1,l}^{(m)}}{w_{2,l}^{(m)}}\right) Q \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\psi_l^m(\tau_1, \mu', s) \times$$

$$\times \frac{d\mu'}{\mu'} \alpha_{3,l}^{(m)}(\mu_0, s)\psi_l^m(\tau_2, \mu, s) - \alpha_{4,l}^{(m)}(\mu_0, s) \times$$

$$\times \phi_l^m(\tau_2, \mu', s) + \frac{1}{4(2-\delta_{0,m})^2} \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s) \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu'', s) B_l^{(m)}(\mu'', \mu_0, s) \frac{d\mu''}{\mu''} \frac{d\mu'}{\mu'} . \tag{79}$$

Again, from Equations (78) and (79), if we use Equations (73) and (77) we get

$$A_l^{(m)}(\mu, \mu_0, s) = \alpha_{1,l}^{(m)}(\mu_0, s)\beta_{1,l}^{(m)}(\mu, s) - \alpha_{2,l}^{(m)}(\mu_0, s) \times$$
$$\times \beta_{2,l}^{(m)}(\mu, s) + \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{2,l}^{(m)}}{w_{1,l}^{(m)}}\right) Q\alpha_{3,l}^{(m)}(\mu_0, s) \times$$
$$\times \beta_{3,l}^{(m)}(\mu, s) - \alpha_{4,l}^{(m)}(\mu_0, s)\beta_{4,l}^{(m)}(\mu, s) , \tag{80}$$

$$B_l^{(m)}(\mu, \mu_0, s) = \frac{1}{2(2-\delta_{0,m})}\left(\frac{w_{1,l}^{(m)}}{w_{2,l}^{(m)}}\right) Q\alpha_{1,l}^{(m)}(\mu_0, s) \times$$
$$\times \gamma_{1,l}^{(m)}(\mu, s) - \alpha_{2,l}^{(m)}(\mu_0, s)\gamma_{2,l}^{(m)}(\mu, s) + \alpha_{3,l}^{(m)}(\mu_0, s) \times$$
$$\times \gamma_{3,l}^{(m)}(\mu, s) - \alpha_{4,l}^{(m)}(\mu_0, s)\gamma_{4,l}^{(m)}(\mu, s) , \tag{81}$$

$$B_{1,l}^{(m)}(\mu, s) = \phi_l^m(\tau_1, \mu, s) + \frac{1}{4(2-\delta_{0,m})^2} \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\gamma_{1,l}^{(m)}(\mu', s) \frac{d\mu'}{\mu'} , \tag{82}$$

$$\beta_{2,l}^{(m)}(\mu, s) = \psi_l^{(m)}(\tau_1, \mu, s) + \frac{1}{4(2-\delta_{0,m})^2} \times$$

$$\times \int_0^1 S_1^{(m)}(\tau_1, \mu, \mu', s)\gamma_{2,l}^{(m)}(\mu', s)\, \frac{d\mu'}{\mu'}\,, \tag{83}$$

$$\beta_{3,l}^{(m)}(\mu, s) = \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\gamma_{3,l}^{(m)}(\mu', s)\, \frac{d\mu'}{\mu'}\,, \tag{84}$$

$$\beta_{4,l}^{(m)}(\mu, s) = \int_0^1 S_1^{(m)}(\tau_1; \mu, \mu', s)\gamma_{4,1}^{(m)}(\mu', s)\, \frac{d\mu'}{\mu'}\,, \tag{85}$$

$$\gamma_{1,l}^{(m)}(\mu, s) = \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\beta_{1,l}^{(m)}(\mu', s)\, \frac{d\mu'}{\mu'}\,, \tag{86}$$

$$\gamma_{2,l}^{(m)}(\mu, s) = \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\beta_{2,l}^{(m)}(\mu', s)\, \frac{d\mu'}{\mu'}\,, \tag{87}$$

$$\gamma_{3,l}^{(m)}(\mu, s) = \psi_l^{(m)}(\tau_2, \mu, s) + \frac{1}{4(2-\delta_{0,m})^2} \times$$

$$\times \int_0^1 S_2^{(m)}(\tau_2, \mu, \mu', s)\beta_{3,l}^{(m)}(\mu', s)\, \frac{d\mu'}{\mu'}\,, \tag{88}$$

$$\gamma_{4,l}^{(m)}(\mu, s) = \phi_l^{m}(\tau_2; \mu, s) + \frac{1}{4(2-\delta_{0,m})^2} \times$$

$$\times \int_0^1 S_2^{(m)}(\tau_2; \mu, \mu', s)\beta_{4,l}^{(m)}(\mu', s)\, \frac{d\mu'}{\mu'}\,. \tag{89}$$

If we combine Equation (82) with Equation (86), Equation (83) with Equation (87), Equation (84) with Equation (88), and Equation (85) with Equation (89). We can determine $\beta_{i,l}^{(m)}(\mu, s)$ and $\gamma_{i,l}^{(m)}(\mu, s)$ ($i = 1, 2, 3, 4$) numerically. From Equations (71), (72), (75), (76), (80), and (81) $\alpha_{i,l}^{(m)}(\mu_0, s)$, $A_l^{(m)}(\mu, \mu_0, s)$, and $B_l^{(m)}(\mu, \mu_0, s)$ can be calculated and then from Equations (65) and (66), $I_1^{(m)}(\tau_1, \mu, \mu_0, s)$ and $I_2^{(m)}(\tau_2, \mu, \mu_0, s)$ are determined. Thus we obtained $S(\tau_0, \mu, \phi, \mu_0, \phi_0, s)$ and $T(\tau_0, \mu, \phi; \mu_0, \phi_0, s)$ from Equations (59) and (60).

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AN EXACT SOLUTION OF THE EQUATION OF TRANSFER FOR COHERENT SCATTERING IN AN EXPONENTIAL ATMOSPHERE

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Abstract. An exact solution of the transfer equation for coherent scattering in stellar atmospheres with Planck's function as a nonlinear function of optical depth, of the form

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau},$$

is obtained by the method of the Laplace transform and Wiener–Hopf technique.

1. Introduction

Chandrasekhar (1960) applied the method of discrete ordinates to solve the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz., $B_\nu(T) = b_0 + b_1\tau$. The equation of transfer for coherent scattering has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient, is constant) and by Strömgren's method (when η_ν has small but arbitrary variation with optical depth) (see Woolley and Stibbs, 1953). Dasgupta (1977b) applied the method of the Laplace transform and Wiener–Hopf technique to find an exact solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions, viz.,

$$B_\nu(T) = b_0 + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau),$$

by use of a new representation of the H-function obtained by Dasgupta (1977a). Recently, Karanjai and Deb (1990) solved the equation of transfer for coherent isotropic scattering in an exponential atmosphere by Eddington's method.

In this paper, we have obtained an exact solution of the equation of transfer for coherent isotropic scattering by the method of the Laplace transform and Wiener–Hopf technique in an exponential atmosphere (Degl'Innocenti, 1979; Karanjai and Karanjai, 1985; and Karanjai and Deb, 1990), where

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau},$$

where b_0, b_1, and β are positive constants.

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2. Equation of Transfer

The equation of transfer considered here is of the form

$$\mathrm{d}I_\nu(\tau, \mu)/\mathrm{d}\tau = I_\nu(\tau, \mu) - wJ_\nu(\tau) - (1 - w)B_\nu(T), \tag{1}$$

where we have taken Planck's function $B_\nu(T)$ as

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}, \tag{2}$$

$$0 < (1 - \varepsilon_\nu)/(1 + \eta_\nu) = w < 1, \tag{2a}$$

$$l_\nu/k = \eta_\nu, \qquad 0 < \varepsilon_\nu < 1; \tag{2b}$$

l_ν, k being the line and continuous absorption coefficient; τ, the optical depth in the total absorption coefficient; ε_ν, the collision constant; and $I_\nu(\tau, \mu)$ is the intensity in the frequency, in the direction $\cos^{-1}\mu$, $J_\nu(\tau)$ is the average intensity

$$J_\nu(\tau) = (1/2) \int_{-1}^{+1} I_\nu(\tau, \mu)\,\mathrm{d}\mu. \tag{2c}$$

For the solution of Equation (1) we have the boundary conditions

(i) $\quad I_\nu(0, -\mu) = 0, \quad 0 < \mu < 1,$

(*ii*) $\quad I_\nu(\tau, \mu)\, e^{-\tau/\mu} \to 0 \quad \text{as} \quad \tau \to \infty.$

3. Solution for Emergent Intensity

The Laplace transform of $F(\tau)$ is denoted by $F^*(s)$, where $F^*(s)$ is defined by

$$F^*(s) = s \int_0^\infty \exp(-s\tau)F(\tau)\,\mathrm{d}\tau, \qquad Rl\, s > 0. \tag{3}$$

The formal solution of Equation (1) (Dasgupta, 1977b) is

$$I_\nu(0, \mu) = wJ_\nu^*(1/\mu) + (1 - w)B_\nu^*(1/\mu). \tag{4}$$

The Laplace transformation of Equation (1) with necessary re-arangement (Dasgupta, 1977b) yields

$$T(z)I_\nu(0, z) = wG_\nu(z) + (1 - w)B_\nu^*(1/z), \tag{5}$$

where

$$T(z) = 1 - (w/2)z \log[(z + 1)/(z - 1)], \tag{6}$$

and

$$G_\nu(z) = (1/2) \int_0^1 xI_\nu(0, x)\,\mathrm{d}x/(x - z). \tag{7}$$

$T(z)$ has its roots $\pm k$, real for $0 < w \leq 1$

$$k(>1) \to \infty \quad \text{as} \quad w \to 1 .$$

According to Dasgupta (1974) we have

$$H(z) \to H_0 + H_{-1}/z + \cdots \quad \text{as} \quad z \to \infty , \tag{8}$$

where

$$H_0 = (1 - w)^{-1/2} \tag{9}$$

and

$$H_{-1} = -(wH_0^2/2) \int_0^1 xH(x)\, \mathrm{d}x . \tag{10}$$

By the well-known relation (Busbridge, 1960)

$$1/T(z) = H(z)H(-z) \quad \text{on} \quad [-1, 1]^c , \tag{11}$$

we rewrite Equation (5) as

$$I_\nu(0, z)/H(z) = H(-z)\,[wG_\nu(z) + (1 - w)B_\nu^*(1/z)] . \tag{12}$$

If we use the Laplace transformation of Equation (2) by Equation (3) we have

$$B_\nu^*(s) = b_0 + sb_1/(s + \beta) . \tag{13}$$

For $s = z^{-1}$

$$B_\nu^*(1/z) = b_0 + b_1/(1 + \beta z) = (d_0 + d_1 z)/(1 + \beta z) \quad (\text{say}) , \tag{14}$$

where

$$d_1 = b_0\beta \quad \text{and} \quad d_0 = b_0 + b_1 .$$

If we insert Equation (14) in Equation (12) we have

$$I_\nu(0, z)/H(z) = H(-z)\,[wG_\nu(z) + (1 - w)\,(d_0 + d_1 z)/(1 + \beta z)] \tag{15}$$

which can be rewriten as

$$I_\nu(0, z)/H(z) = H(-z)\,[wG(z) + (1 - w)\,(d_0/z + d_1)/(1/z + \beta)] . \tag{16}$$

Now as $z \to \infty$, $G_\nu(z) \to 0(1/z)$, since we seek solution $I_\nu(0, z)$ regular for $\mathrm{Re}\, z > 0$ and continuous on $[0, 1]^c$ and since $H(z)$ is regular on $[-1, 0]^c/[-k]$, $-k$ is a simple pole of $H(z)$, $1/H(z)$ being regular on $[-1, 0]^c$.

We see that the left-hand side of Equation (16) is regular at least for $\mathrm{Re}\, z > 0$ except perhaps at ∞, and the right-hand side of Equation (16) is regular at on $[0, 1]^c$ except at ∞, both sides being bounded at the origin.

The right-hand side of Equation (16) is

$$C_0 \quad \text{as} \quad z \to \infty , \tag{17}$$

where

$$C_0 = H_0(1 - w)d_1/\beta . \tag{18}$$

Hence, by a modified Liouville's theorem both sides of Equation (16) can be equated to C_0, so that the left-hand side of (16) is

$$C_0 \quad \text{as} \quad z \to \infty , \tag{19}$$

the right-hand side of (16) is

$$C_0 \quad \text{as} \quad z \to \infty . \tag{20}$$

Equation (16) can be put in the form

$$I(0, z)/H(z) = C_0 = H_0(1 - w)d_1 \beta . \tag{21}$$

If we use the relationship $d_1 = b_0\beta$ in (21) we get when z

$$I(0, z) = H(z)\,(1 - w)H_0 b_0 . \tag{22}$$

Since we have $H_0 = (1 - w)^{-1/2}$.

Hence, from Equation (22) we get

$$I(0, z) = H(z)\,(1 - w)^{1/2} b_0 , \tag{23}$$

which is the same as deducted by Karanjai and Karanjai (1985).

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SOLUTION OF THE EQUATION OF TRANSFER FOR COHERENT SCATTERING IN AN EXPONENTIAL ATMOSPHERE BY BUSBRIDGE'S METHOD

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Abstract. A solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a nonlinear function of optical depth, viz.

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$$

is obtained by the method developed by Busbridge (1953).

1. Introduction

Chandrasekhar (1960) applied the method of discrete ordinates to solve the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz.,

$$B_\nu(T) = b_0 + b_1\tau .$$

The equation of transfer for coherent scattering has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient is constant) and by Strömgren's method (when η_ν has small but arbitrary variation with optical depth; see Woolley and Stibbs, 1953). Busbridge (1953) solved the same problem by a new method using Chandrasekhar's ideas. Dasgupta (1977b) applied the method of Laplace transform and Wiener–Hopf technique to find an exact solution of the transfer equation for coherent scattering in the stellar atmosphere with Planck's function as a sum of elementary functions, viz.,

$$B_\nu(T) = b_0 + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau) ,$$

using a new representation of the H-function obtained by Dasgupta (1977a). Recently, Karanjai and Deb (1991a, b) solved the equation of transfer for coherent isotropic scattering in an exponential atmosphere by Eddington's method and the method of Laplace transform and Wiener–Hopf technique. In this paper, we have obtained a solution of the equation of transfer for coherent scattering in an exponential atmosphere,

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i.e.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau};$$

where b_0, b_1, and β are three positive constants, by the method used by Busbridge (1953).

2. Equation of Transfer

With the usual notation of transfer for the Milne–Eddington model can be written (Busbridge, 1953; Chandrasekhar, 1960) as

$$\mu \frac{\mathrm{d}I_\nu}{\rho\,\mathrm{d}z} = (k_\nu + \sigma_\nu)I_\nu - \tfrac{1}{2}\sigma_\nu \int_{-1}^{+1} I_\nu\,\mathrm{d}\mu' - k_\nu B_\nu(T), \tag{1}$$

where z is the depth below the surface; k_ν, the continuous absorption coefficient; and σ_ν is the line-scattering coefficient. We assume that k_ν and σ_ν are independent of depth and we write

$$t = \int_0^z \rho(k_\nu + \sigma_\nu)\,\mathrm{d}z, \tag{2a}$$

$$\tau = \int_0^z \rho k_\nu\,\mathrm{d}z, \tag{2b}$$

$$\eta_\nu = \frac{\sigma_\nu}{k_\nu}, \qquad \lambda_\nu = \frac{1}{1+\eta_\nu} = \frac{k_\nu}{k_\nu + \sigma_\nu}. \tag{3}$$

Then

$$\tau = \lambda_\nu t$$

and

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau} = b_0 + b_1 e^{-\beta\lambda_\nu t}, \tag{4}$$

where $B_\nu(T)$ is the Planck's function.

Substituting into Equation (1), we get

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu(t,\mu) - \tfrac{1}{2}(1-\lambda_\nu) \int_{-1}^{+1} I_\nu(t,\mu')\,\mathrm{d}\mu' - \lambda_\nu(b_0 + b_1 e^{-\beta\lambda_\nu t}). \tag{5}$$

Equation (5) has to be solved subject to the boundary conditions

$$I_\nu(0, -\mu') = 0, \quad (0 < \mu' < 1) \tag{6a}$$

and

$$I_\nu(t, \mu') e^{-t/\mu} \to 0 \quad \text{as} \quad t \to \infty . \tag{6b}$$

3. Solution for Emergent Intensity

For convenience we suppress the subscript ν to the various quantities and consider a particular solution of Equation (5), which does not satisfy Equation (6a) in the form (Busbridge, 1953)

$$I(t, \mu) = b_0 + \frac{T_1 b_1}{1 + \beta\lambda\mu} e^{-\beta\lambda t} , \tag{7}$$

where

$$T_1 = \frac{\lambda}{1 - \frac{1}{2\lambda\beta}(1 - \lambda) \log \frac{1 + \lambda\beta}{1 - \lambda\beta}} \tag{8}$$

as readily verified by substitution. We, therefore, write (cf. Busbridge, 1953)

$$I(t, \mu) = b_0 + \frac{T_1 b_1}{1 + \beta\lambda\mu} e^{-\beta\lambda t} + I^*(t, \mu) . \tag{9}$$

Then $I^*(t, \mu)$ satisfied the integro-differential equation

$$\mu \frac{dI^*(t, \mu)}{dt} = I^*(t, \mu) - \tfrac{1}{2}(1 - \lambda) \int_{-1}^{+1} I^*(t, \mu') \, d\mu' , \tag{10}$$

together with the boundary conditions

$$I^*(0, -\mu') = -\frac{T_1 b_1}{1 - \beta\lambda\mu} - b_0 \quad (0 < \mu' < 1) \tag{11a}$$

and

$$I^*(t, \mu) e^{-t/\mu} \to 0 \quad \text{as} \quad t \to \infty , \tag{11b}$$

we require the emergent intensity $I^*(0, \mu)$. This is the sum of $I_1^*(0, \mu)$, where $I_1^*(t, \mu)$ is the solution of Equation (10).

Subject to the boundary condition

$$I_1^*(0, -\mu') \equiv 0 , \quad (0 < \mu' < 1) \tag{12}$$

and $I_2^*(0, \mu)$ which is the diffusely reflected intensity corresponding to the incident intensity given by Equation (11). It can be shown that unless $\lambda_\nu = 0$ (which is not so),

$$I_1^*(t, \mu) = 0 . \tag{13}$$

Hence,

$$I^*(0,\mu) = I_2^*(0,\mu) = \frac{1}{2\mu}\int_0^1 S(\mu,\mu')\left(\frac{T_1 b_1}{\beta\lambda\mu - 1} - b_0\right)\mathrm{d}\mu', \tag{14}$$

where (cf. Chandrasekhar, 1960)

$$S(\mu,\mu') = (1-\lambda)\frac{\mu\mu'}{\mu+\mu'}H(\mu)H(\mu') \tag{15}$$

and $H(\mu)$ is the solution of

$$H(\mu) = 1 + \tfrac{1}{2}(1-\lambda)\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu+\mu'}\,\mathrm{d}\mu'. \tag{16}$$

From Equations (14) and (15), we have

$$I(0,\mu) = \tfrac{1}{2}(1-\lambda)H(\mu)\int_0^1\left(\frac{T_1 b_1}{\beta\lambda\mu - 1} - b_0\right)\frac{\mu'\mu}{\mu'+\mu}H(\mu')\,\mathrm{d}\mu' =$$

$$= \tfrac{1}{2}(1-\lambda)H(\mu)T_1 b_1\int_0^1 \frac{\mu' H(\mu')\,\mathrm{d}\mu'}{(\mu'+\mu)(\beta\lambda\mu'-1)} -$$

$$- \tfrac{1}{2}(1-\lambda)H(\mu)b_0\int_0^1 \frac{\mu'}{\mu'+\mu}H(\mu')\,\mathrm{d}\mu' =$$

$$= \tfrac{1}{2}(1-\lambda)H(\mu)\frac{T_1 b_1}{\beta\lambda}\int_0^1 \frac{H(\mu')}{\mu+\mu'}\,\mathrm{d}\mu' +$$

$$+ \tfrac{1}{2}(1-\lambda)H(\mu)\frac{T_1 b_1}{\beta\lambda}\int_0^1 \frac{H(\mu')\,\mathrm{d}\mu'}{(\mu'+\mu)(\beta\lambda\mu-1)} -$$

$$- \tfrac{1}{2}(1-\lambda)H(\mu)b_0\int_0^1\left(1 - \frac{\mu}{\mu+\mu'}\right)H(\mu')\,\mathrm{d}\mu'. \tag{17}$$

After some rearrangement and with Equation (16), this gives

$$I^*(0, \mu) = \frac{H(\mu) T_1 b_1}{1 + \beta\lambda\mu} \frac{1}{H(-1/\beta\lambda)} - \frac{T_1 b_1}{1 + \beta\lambda\mu} + \\ + (H(\mu) - 1) b_0 - \tfrac{1}{2}(1 - \lambda) H(\mu) b_0 \alpha_0 \tag{18}$$

where

$$\alpha_n = \int_0^1 H(\mu) \mu^n \, d\mu . \tag{19}$$

Following Chandrasekhar (1960)

$$1 - \tfrac{1}{2}(1 - \lambda)\alpha_0 = \lambda^{1/2} , \tag{20}$$

we have from Equations (9) and (18)

$$I(0, \mu) = H(\mu)\lambda^{1/2} b_0 + \frac{H(\mu) T_1 b_1}{1 + \beta\lambda\mu} \frac{1}{H(-1/\beta\lambda)} , \tag{21}$$

which represents our solution.

Appendix

We have to show that

$$I_1^*(t, \mu) = 0 . \tag{A.1}$$

For this, with the usual notation (cf. Chandrasekhar, 1960), we have

$$I_1^*(t, \mu) \simeq \frac{1}{2}(1 - \lambda) \sum_{\alpha=1}^{n} \{L_\alpha e^{-k_\alpha t}/(1 + \mu k_\alpha)\} , \tag{A.2}$$

where the constants L_α are determined by the equations

$$\sum_{\alpha=1}^{n} L_\alpha/(1 - \mu_i k_\alpha) = 0 , \quad (i = 1, 2, 3, \ldots, n) . \tag{A.3}$$

Since

$$\prod_{\alpha=1}^{n} (1 - \mu k_\alpha) \sum_{\alpha=1}^{n} L_\alpha/(1 - \mu k_\alpha)$$

is a polynomial in μ of degree $(n - 1)$ with n distinct zero, it is identically zero.

Hence, every $L_\alpha = 0$, and in the limit, as $n \to \infty$

$$I_1^*(t, \mu) = 0 .$$

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SOLUTION OF THE EQUATION OF TRANSFER FOR COHERENT SCATTERING IN AN EXPONENTIAL ATMOSPHERE BY THE METHOD OF DISCRETE ORDINATES

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Abstract. A solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a nonlinear function of optical depth, viz.,

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$$

is obtained by the method of discrete ordinates originally due to Chandrasekhar.

1. Introduction

Busbridge (1953) solved the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz. $B_\nu(T) = b_0 + b_1\tau$ by a modified principle of invariance method. Chandrasekhar (1960) solved the same problem by the method of discrete ordinates. The same problem has also been solved by Eddington's method (when η_ν, the ratio of line to the continuum absorption coefficient is constant) and by Strömgren's method (when η_ν, has small but arbitrary variation with optical depth) (see Woolley and Stibbs, 1953).

Dasgupta (1977b) applied the method of Laplace transform and Wiener–Hopf technique to find an exact solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions, viz.,

$$B_\nu(T) = b_0 + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau) ,$$

using a new representation of the H-function obtained by Dasgupta (1977a). Recently, Karanjai and Deb (1991, 1992a) solved the equation of transfer for coherent isotropic scattering in an exponential atmosphere by Eddington's method and by the method of Laplace transform and Wiener–Hopf technique.

By use of a method developed by Busbridge (1953), Karanjai and Deb (1992b) solved the same problem.

In this paper, we have obtained a solution of the equation of transfer for coherent isotropic scattering in an exponential atmosphere by the method of discrete ordinates, where $B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$ and b_0, b_1 and β are three positive constants.

2. Equation of Transfer

The equation of transfer considered here is of the form

$$\mu \frac{dI_\nu}{\rho dz} = (k_\nu + \sigma_\nu) I_\nu - \tfrac{1}{2}\sigma_\nu \int_{-1}^{+1} I_\nu \, d\mu' - k_\nu B_\nu(T) \tag{1}$$

(Busbridge, 1953; and Chandrasekhar, 1960) where z is the depth below the surface; k_ν, the continuous absorption coefficient; and σ_ν, the line-scattering coefficient. We assume that k_ν and σ_ν are independent of depth and we write

$$t = \int_0^z \rho(k_\nu + \sigma_\nu)\, dz \ , \tag{2a}$$

$$\tau = \int_0^z \rho k_\nu \, dz \ , \tag{2b}$$

$$\eta_\nu = \sigma_\nu / k_\nu \, , \ \lambda_\nu = 1/(1 + \eta_\nu) = \frac{k_\nu}{k_\nu + \sigma_\nu} \ . \tag{3}$$

Then $\tau = \lambda_\nu t$ and

$$B_\nu(T) = b_0 + b_1 \, e^{-\beta\tau} \ , \tag{4a}$$

i.e.,

$$B_\nu(T) = b_0 + b_1 \, e^{-\beta\lambda_\nu t} \ . \tag{4b}$$

If we substitute in Equation (1) we get

$$\mu \frac{dI_\nu(t, \mu)}{dt} = I_\nu(t, \mu) - \tfrac{1}{2}(1 - \lambda_\nu) \int_{-1}^{+1} I_\nu(t, \mu')\, d\mu' - \lambda_\nu (b_0 + b_1 \, e^{-\beta\lambda_\nu t}) \ . \tag{5}$$

Equation (5) has to be solved subject to the boundary conditions

$$I_\nu(0, -\mu) = 0, \quad (0 < \mu \leq 1) \tag{6a}$$

and

$$I_\nu(t, \mu)\, e^{-t/\mu} \to 0 \quad \text{as} \quad t \to \infty, \quad |\mu| \leq 1 \ . \tag{6b}$$

Now a particular solution of Equation (5), which does not satisfy Equation (6a) is

$$I_\nu(t, \mu) = b_0 + \frac{T_\nu b_1}{1 + \alpha_\nu \mu} e^{-\alpha_\nu t} \ , \tag{7}$$

where

$$T_\nu = \frac{\lambda_\nu}{1 - \frac{1}{2}(1 - \lambda_\nu)\log\dfrac{1 + \alpha_\nu}{1 - \alpha_\nu}} \tag{8a}$$

and

$$\alpha_\nu = \beta\lambda_\nu \tag{8b}$$

as readily verified by substitution.

If we follow Busbridge (1953) we write

$$I_\nu(t, \mu) = b_0 + b_1 \frac{T_\nu}{1 + \alpha_\nu\mu} e^{-\alpha_\nu t} + I_\nu^*(t, \mu) \ . \tag{9}$$

Then $I_\nu^*(t, \mu)$ satisfies the integro-differential equation

$$\mu \frac{\mathrm{d}I_\nu^*(t, \mu)}{\mathrm{d}t} = I_\nu^*(t, \mu) - \tfrac{1}{2}(1 - \lambda_\nu) \int_{-1}^{+1} I_\nu^*(t, \mu')\,\mathrm{d}\mu' \ , \tag{10}$$

together with the boundary conditions

$$I_\nu^*(0, -\mu) = -b_1 \frac{T_\nu}{1 - \alpha_\nu\mu} - b_0 \tag{11a}$$

and

$$I_\nu^*(t, \mu)\,e^{-t/\mu} \to 0 \quad \text{as} \quad t \to \infty, \quad |\mu| \leq 1 \ . \tag{11b}$$

3. Solution for Emergent Intensity

For convenience we suppress the subscript ν to the various quantities and in the nth approximation, we replace Equation (10) by the system of $2n$ linear equations

$$\mu_i \frac{\mathrm{d}I_i^*}{\mathrm{d}t} = I_i^* - \tfrac{1}{2}(1 - \lambda) \sum_j a_j I_j^* \ , \quad i = \pm 1, \pm 2, \ldots, \pm n \ ; \tag{12}$$

where the μ_i's ($i = \pm 1, \pm 2, \ldots, \pm n$ and $\mu_{-i} = -\mu_i$) are the zeros of the Legendre polynomial $P_{2n}(\mu)$. a_j's ($j = \pm 1, \ldots, \pm n$ and $a_{-j} = a_j$) are corresponding Gaussian weights. However, it is to be noted that there is no term with $j = 0$. For simplicity, in Equation (12) we write

$$I_i^* \quad \text{for} \quad I_i^*(t, \mu_i) \ . \tag{13}$$

The system of Equations (12) admits of integral of the form

$$I_i^* = g_i\,e^{-kt} \quad (i = \pm 1, \ldots, \pm n) \ , \tag{14}$$

where the g_i's and k are constants.

Now if we insert this form for I_i^* in Equation (12) we have

$$g_i|1+\mu_i k| = \tfrac{1}{2}(1-\lambda)\sum_j a_j g_j \,, \tag{15}$$

$$\therefore g_i = (1-\lambda)\frac{\text{constant}}{1+\mu_i k} \,. \tag{16}$$

If we insert for g_i from Equation (16) back into Equation (15) we obtain the characteristic equation in the form

$$1 = \tfrac{1}{2}(1-\lambda)\sum_j \frac{a_j}{1+\mu_j k} \,. \tag{17}$$

If we remember that $a_j = a_{-j}$ and $\mu_{-j} = -\mu_j$ we can rewrite the characteristic equation in the form

$$1 = (1-\lambda)\sum_{j=1}^{n} \frac{a_j}{1-\mu_i^2 k^2} \,. \tag{18}$$

This is the characteristic equation which gives the values of k. If $\lambda > 0$, the characteristic Equation (18) gives distinct non-zero roots which occur in pairs as $\pm k_r$ $(r = 1, 2, \ldots, n)$.

Therefore, Equation (12) admits the $2n$ independent integrals of the form

$$I_i^* = (1-\lambda)\frac{\text{constant}}{1 \pm \mu_i k_r}\, e^{\pm k_r t} \,. \tag{19}$$

According to Chandrasekhar (1960), the solutions (14) satisfying our requirements that the solutions are bounded by

$$I_i^* = (1-\lambda)\, b_1 \sum_{r=1}^{n} \frac{L_r e^{-k_r t}}{1+k_r \mu_i} \,, \tag{20}$$

together with the boundary condition

$$I_{-i}^* = -\frac{b_1 T}{1-\alpha\mu_{-i}} - b_0 \quad \text{at} \quad t = 0 \,. \tag{21}$$

4. The Elemination of the Constants and the Expression of the Law of Diffuse Reflection in Closed Form

The boundary condition and the emergent intensity can be expressed in the form

$$S(\mu_i) = 0 \quad (i = 1, 2, \ldots, n) \tag{22}$$

and

$$I^*(0,\mu) = (1-\lambda)b_1\left[S(-\mu) - \frac{T/(1-\lambda)}{1+\alpha\mu} - \frac{b_0}{(1-\lambda)b_1}\right], \tag{23}$$

where

$$S(\mu) = \sum_{r=1}^{n} \frac{L_r}{1 - k_r\mu} + \frac{T/(1-\lambda)}{1 - \alpha\mu} + \frac{b_0}{(1-\lambda)b_1} . \tag{24}$$

Next we observe that the function

$$(1 - \alpha\mu) \prod_{r=1}^{n} (1 - k_r\mu) S(\mu)$$

is a polynomial of degree $n + 1$ in μ which vanishes for $\mu = \mu_i, i = 1, 2, \ldots, n$. There must accordingly exist a relation of the form

$$(1 - \alpha\mu) \prod_{r=1}^{n} (1 - k_r\mu) S(\mu) \propto (\mu - C) \prod_{i=1}^{n} (\mu - \mu_i) , \tag{25}$$

where C is a constant.

The constant of proportionality can be found by comparing the coefficients of the highest power of μ (viz. μ^{n+1}).

Thus, from Equation (25) we have

$$S(\mu) = \frac{(-1)^{n+1} b_0}{b_1(1-\lambda)} k_1 \ldots k_n \alpha \frac{P(\mu)(\mu - C)}{R(\mu)(1 - \alpha\mu)} , \tag{26}$$

where

$$P(\mu) = \prod_{i=1}^{n} (\mu - \mu_i) \quad i = 1, i, \ldots, n , \tag{27}$$

and

$$R(\mu) = \prod_{r=1}^{n} (1 - k_r\mu) \quad r = 1, r, \ldots, n . \tag{28}$$

Moreover, combining Equations (26) and (27) we obtain

$$L_r = (-1)^n \frac{b_1}{b_0(1-\lambda)} k_1 \ldots k_n \alpha \times \frac{P(1/k_r)(1/k_r - C)}{R_r(1/k_r)(1 - \alpha/k_r)} , \tag{29}$$

where

$$R_r(x) = \prod_{h \neq r} (1 - k_h x) \tag{30}$$

and

$$\alpha \neq k_r . \tag{31}$$

The roots of the characteristic equation (18) can be written in the form

$$k_1 k_2 \ldots k_n \mu_1 \mu_2 \ldots \mu_n = \lambda^{1/2} . \tag{32}$$

Now by use of Equation (32), Equation (26) becomes

$$S(\mu) = -\frac{b_0 \alpha \lambda^{1/2} H(-\mu)(\mu - C)}{(1-\lambda) b_1 (1 - \alpha\mu)} , \tag{33}$$

where

$$H(\mu) = \frac{1}{\mu_1\mu_2 \dots \mu_n} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{r=1}^{n} (1 + k_r\mu)} ; \tag{34}$$

and the characteristic roots are evaluated from Equation (24).

If we put $\mu = 0$ in Equations (24) and (34) we have

$$\sum_{r=1}^{n} L_r + \frac{T}{1-\lambda} + \frac{b_0}{(1-\lambda)b_1} = \frac{b_0 \lambda^{1/2} C\alpha}{(1-\lambda)b_1} . \tag{35}$$

We can next evaluate $\sum_{r=1}^{n} L_r$ from Equation (29). Then

$$\sum_{r=1}^{n} L_r = (-1)^{n+1} \frac{b_0}{b_1(1-\lambda)} k_1 k_2 \dots k_n \alpha f(0) , \tag{36}$$

where

$$f(x) = \sum_{r=1}^{n} \frac{P(1/k_r)(1/k_r - C)}{R_r(1/k_r)(1 - \alpha/k_r)} . \tag{37}$$

Now $f(x)$ defined in this manner is a polynomial of degree $(n-1)$ in x which takes the values

$$\frac{P(1/k_r)(1/k_r - C)}{(1 - \alpha/k_r)} .$$

for

$$x = 1/k_r \quad (r = 1, 2, \dots, n) .$$

In other words,

$$(1 - \alpha x) f(x) - P(x)(x - C) = 0 . \tag{38}$$

Therefore, we must accordingly have a relation of the form

$$(1 - \alpha x) f(x) - P(x)(x - C) = R(x)(Ax + B) , \tag{39}$$

where A and B are certain constants to be determined. The constant A follows from the comparison of the coefficient of x^{n+1}. Thus

$$A = \frac{(-1)^{n+1}}{k_1 k_2 \dots k_n} . \tag{40}$$

Next, if we put $x = \alpha^{-1}$ in Equation (40) we have

$$B = \frac{(-1)^n}{\alpha k_1 k_2 \dots k_n} + \frac{(C - 1/\alpha)P(\alpha^{-1})}{R(\alpha^{-1})} , \tag{41}$$

i.e.,

$$B = \frac{(-1)^n}{\alpha k_1 k_2 \ldots k_n} + (-1)^n \mu_1 \ldots \mu_n H(-1/\alpha)(C - \alpha^{-1}) . \tag{42}$$

Now by use of the relations (42), (41), and (40) we get

$$f(0) = -CP(0) + BR(0) = -C(-1)^n \mu_1 \mu_2 \ldots \mu_n + \\ + \frac{(-1)^n}{\alpha k_1 k_2 \ldots k_n} + (-1)^n \mu_1 \mu_2 \ldots \mu_n H(-\alpha^{-1})(C - \alpha^{-1}) . \tag{43}$$

From the Equation (37) using Equation (43) we have

$$\sum_{r=1}^{n} L_r = \frac{b_0}{(1-\lambda)b_1} C\lambda^{1/2}\alpha - \frac{b_0}{(1-\lambda)b_1} + \\ + \frac{b_0 \alpha \lambda^{1/2} H(-\alpha^{-1})(\alpha^{-1} - C)}{(1-\lambda)b_1} . \tag{44}$$

By use of Equation (44) in Equation (38) we get

$$C = \frac{1}{\alpha} + \frac{Tb_1}{b_0 \alpha \lambda^{1/2} H(-\alpha^{-1})} . \tag{45}$$

If, moreover, we combine Equation (44), the diffusely reflected intensity $I^*(0, \mu)$ in Equation (23) takes the form

$$I^*(0, \mu) = \frac{b_0 \alpha \lambda^{1/2} H(\mu)[\mu + C]}{1 + \alpha\mu} - \frac{Tb_0}{1 + \alpha\mu} - b_0 . \tag{46}$$

This is the required solution in closed form. If we combine Equation (9) at $t = 0$ and Equation (46) we have

$$I(0, \mu) = \frac{b_0 \alpha \lambda^{1/2} H(\mu)[\mu + C]}{1 + \alpha\mu} , \tag{47}$$

which is the required solution of Equation (5) in the nth approximation by the discrete ordinate method.

On putting C from Equation (45) we get the solution in the form

$$I(0, \mu) = b_0 \lambda^{1/2} H(\mu) + \frac{b_1 TH(\mu)}{1 + \alpha\mu} \frac{1}{H(-\alpha^{-1})} . \tag{48}$$

Chandrasekhar's (1960) solution for $I(0, \mu)$ in the case of coherent scattering is given by (for $B_\nu(T) = b_0 + b_1 \tau$)

$$I(0, \mu) = b_0 \lambda^{1/2} H(\mu) + b_1 \lambda^{3/2} H(\mu)\mu + \tfrac{1}{2} b_1 \lambda (1 - \lambda) H(\mu) \alpha_1 , \tag{49}$$

where

$$\alpha_n = \int_0^1 H(\mu)\mu^n \, d\mu \; . \tag{50}$$

If we compare Equations (48) and (49) we see that by putting $b_1 = 0$ we have the same solution for both the cases. Moreover for large values of β (i.e., $\beta \to \infty$) the solutions (48) takes the form

$$I(0, \mu) = b_0 \lambda^{1/2} H(\mu) \; ; \tag{51}$$

i.e., β then behaves like a constant or independent of τ. This fact can also be explained from the point of view that

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau_\nu} \to b_0 \quad \text{as} \quad \beta \to \infty \; .$$

Also the result obtained by Karanjai and Deb (1992b) is the same as obtained here.

Appendix

To establish the relation (32) we consider

$$D_m(x) = (1 - \lambda) \sum_i \frac{a_i \mu_i^m}{1 + \mu_i x} = (-1)^m (1 - \lambda) \sum_i \frac{a_i \mu_i^m}{1 - \mu_i x} \; , \tag{52}$$

$$(m = 0, 1, \ldots, 4n) \; .$$

We can derive a single recursion formula for $D_m(x)$. Then

$$D_m(x) = \frac{1}{x} \left[(1 - \lambda) \sum_i a_i \mu_i^{m-1} \left(1 - \frac{1}{1 + \mu_i x} \right) \right] =$$

$$= \frac{1}{x} [\psi_{m-1} - D_{m-1}] \; , \tag{53}$$

where

$$\psi_m = (1 - \lambda) - \sum_i a_i \mu_i^m \; . \tag{54}$$

From this formula we have

$$D_m(x) = \frac{\psi_{m-1}}{x} - \frac{\psi_{m-2}}{x^2} + \ldots + (-1)^{m-2} \frac{\psi_1}{x^{m-1}} + \frac{(-1)^{m-1}}{x^m} \times$$

$$\times [\psi_0 - D_0(x)] \; (m = 0, 1, \ldots, 4n) \tag{55}$$

and

$$\psi_0 = 2(1 - \lambda) \; . \tag{56}$$

Moreover, let P_{2j} be the coefficient of μ^{2j} in the Legendre polynomial $P_{3n}(\mu)$, then

$$\sum_{j=0}^{n} P_{2j}D_{2j}(K_r) = (1-\lambda)\sum_{j}\frac{a_j}{1+\mu_i k_r} \times \sum_{j=0} P_{2j}\mu_i \ . \tag{57}$$

Since the μ_i's are the zeros of $P_{2n}(\mu)$. Equation (57) reduces to

$$\sum_{j=0}^{n} P_{2j}D_{2j}(k_z) = 0 \ . \tag{58}$$

If we substitute for $D_{2j}(k_r)$ from Equation (56) into Equation (58) we get the required form of the characteristic equation as

$$-\frac{P_{2n}\lambda}{k_r^{2n}} + \cdots + P_0 = 0 \ . \tag{59}$$

From this equation it follows that

$$\frac{1}{(k_1k_2\ldots k_n)^2} = \frac{(-1)^n P_0}{\lambda P_{2n}} = \frac{(\mu_1\mu_2\cdots\mu_n)^2}{\lambda} \tag{60}$$

i.e.,

$$\mu_1\mu_2\cdots\mu_n\, k_1k_2\ldots k_n = \lambda^{1/2} \ . \tag{61}$$

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THE TIME-DEPENDENT X- AND Y-FUNCTIONS

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Abstract. The application of the Wiener–Hopf technique to the coupled linear integral equation of time-dependent X- and Y-functions gives rise to the Fredholm equations with simpler kernels. The time-dependent X-function is expressed in terms of time-dependent Y-function and *vice versa*. These are unique in representation with respect to coupled linear constraints.

1. Introduction

In the theory of radiative transfer for homogeneous plane-parallel stratified finite atmosphere the X- and Y-functions of Chandrasekhar (1960) play a central role. These equations satisfy a system of coupled nonlinear integral equations. Busbridge (1960) has demonstrated the existence of the solutions of these coupled nonlinear integral equations in terms of a particular solution of an auxilliary equation. Busbridge (1960) has obtained two coupled linear integral equations for $X(z)$ and $Y(z)$ which defined the meromorphic extension to the complex domain $|Z|$ of the real valued solution of the coupled nonlinear integral equations of X- and Y-functions. Busbridge (1960) concludes that all solutions of nonlinear coupled integral equations for X- and Y-functions are the solutions of the coupled linear integral equations to the extended complex plane but all solutions of the coupled linear integral equations are not solutions of the coupled nonlinear integral equations. Mullikin (1964) has proved that all solutions of coupled nonlinear integral equations are solutions of the coupled linear integral equations but there exist a unique solution of the coupled linear integral equations with some linear constraints. Finally he has obtained the Fredholm equation of X- and Y-functions which are easy for iterative computations. Das (1979) has obtained a pair of the Fredholm equations with the Wiener–Hopf technique from the coupled linear integral equations with coupled linear constraints.

In this paper we have considered the time-dependent X- and Y-functions (Biswas and Karanjai, 1990) which give rise to a pair of the Fredholm equations with the application of the Wiener–Hopf technique. These Fredholm equations define time-dependent X-functions in terms of time-dependent Y-functions and *vice versa*. These representations are unique with respect to the coupled linear constraints defined by Mullikin (1964).

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2. Basic Equation

The coupled nonlinear integral equations satisfied by the time-dependent X- and Y-functions (Biswas and Karanjai, 1990) are of the form

$$X(\tau_1, \mu, s) = 1 + \frac{w}{2Q}\,\mu \int_0^1 \frac{X(\tau_1, \mu, s)X(\tau_1, x, s) - Y(\tau_1, \mu, s)Y(\tau_1, x, s)}{\mu + x}\,\mathrm{d}x\,, \quad 0 \le \mu < 1\,, \tag{1}$$

$$Y(\tau_1, \mu, s) = \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{w}{2Q}\,\mu \int_0^1 \frac{Y(\tau_1, \mu, s)X(\tau_1, x, s)X(\tau_1, \mu, s)Y(\tau_1, x, s)}{\mu - x}\,\mathrm{d}x\,, \quad 0 \le \mu < 1\,, \tag{2}$$

where

$$Q = 1 + \frac{s}{c}\,, \tag{3}$$

τ_1 is the thickness of the atmosphere; c, the velocity of light; and s, Laplace transform parameter.

If we follow Chandrasekhar (1960) Equations (1) and (2) can be written as

$$X(\tau_1, \mu, s) = 1 + \frac{\mu}{Q}\int_0^1 \frac{\psi(x)}{x + \mu}\,[X(\tau_1, \mu, s)X(\tau_1, x, s) - Y(\tau_1, \mu, s)Y(\tau_1, x, s)]\,\mathrm{d}x\,, \quad 0 \le \mu < 1\,, \tag{4}$$

$$Y(\tau_1, \mu, s) = \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{\mu}{Q}\int_0^1 \frac{\psi(x)}{x - \mu}\,[Y(\tau_1, x, s)X(\tau_1, \mu, s) - X(\tau_1, x, s)Y(\tau_1, \mu, s)]\,\mathrm{d}x\,, \quad 0 \le \mu < 1\,; \tag{5}$$

where $\psi(x)$, the characteristic function satisfying the Hölder condition on $0 \le x \le 1$, is non-negative and satisfies the condition

$$\psi_0 = \int_0^1 \psi(x)\,\mathrm{d}x \le \tfrac{1}{2}\,. \tag{6}$$

The atmosphere is said to be conservative when $\psi_0 = \frac{1}{2}$ and non-conservative otherwise.

The dispersion function $T(z, s)$, $z \in (-1, 1)^c$ can be defined by

$$T(z, s) = 1 - \frac{2z^2}{Q} \int_0^1 \psi(x)\, \mathrm{d}x T(z^2 - x^2) \tag{6a}$$

and

$$T(z, s) = (H(z, s)H(-z, s))^{-1}, \tag{6b}$$

where

$$H(z, s) = 1 + zH(z, s) \int_0^1 \frac{\psi(x)H(x, s)\, \mathrm{d}x}{x + z}. \tag{7}$$

According to Busbridge (1960), the only zeros of $T(z, s)$ are at $z = \pm K, K > 1$, when $\psi_0 < \frac{1}{2}$ and $K \to \infty$ when $\psi_0 = \frac{1}{2}$.

Following Busbridge (1960), Dasgupta (1977), and Das (1978) $H(z, s)$ is meromorphic on $(-1, 0)^c$ having a simple pole at $z = -K$ and tend to 1 as $z \to 0_+$. It can be represented by

$$H(z, s) = \frac{A_0 + H_0 z}{K + z} - \int_0^1 \frac{P(x, s)\, \mathrm{d}x}{x + z}, \quad K > 1, \psi_0 < \tfrac{1}{2}, \tag{8}$$

$$H(z, s) = h_1 z + h_0 - \int_0^1 \frac{P(x, s)\, \mathrm{d}x}{x + z}, \quad K \to \infty, \psi_0 = \tfrac{1}{2}; \tag{9}$$

where

$$\begin{aligned}
A_0 &= (1 + P_{-1})K, \qquad P_{-1} = \int_0^1 P(x, s)\, \mathrm{d}x/x, \\
H_0 &= \left(1 - 2\int_0^1 \psi(x)\, \mathrm{d}x\right)^{-1/2}, \\
h_1 &= \left(2\int_0^1 x^2\psi(x)\, \mathrm{d}x\right)^{-1/2}, \\
h_0 &= (1 + P_{-1}), \\
P(x, s) &= \phi(x, s)/H(x, s), \\
\phi(x, s) &= x\psi(x)/(T_0^2(x, s) + \pi^2 x^2 \psi^2(x)),
\end{aligned} \tag{10}$$

$$T_0(x, s) = 1 - \frac{2x^2}{Q} \int_0^1 (\psi(t) - \psi(x))\, \mathrm{d}t/(x^2 - t^2) -$$

$$- \frac{x\psi(x)}{Q} \log((1 + x)/(1 - x)),$$

where $\phi(x, s)$ is non-negative and continuous on (0, 1), tends to $\psi(0)x$ as $x \to 0_+$, tends to $0((\log(1 - x)^{-2}))$ when $x \to 1_-$, and $1/H(z, s)$ is regular on $(-1, 0)^c$.

If we follow Busbridge (1960) and Mullikin (1964) we find that the coupled linear equations satisfied by $X(z, s)$ and $Y(z, s)$ for $z \in (-1, 1)^c$ are of the form

$$X(z, s)T(z, s) = 1 + zU(X)(z, s) - z\exp(-(\tau_1/z)Q)V(Y)(z, s), \tag{11}$$

$$Y(z, s)T(z, s) = (\exp(-(\tau_1/z)Q) + zU(Y)(z, s)) -$$

$$- z\exp(-(\tau_1/z)Q)V(X)(z, s), \tag{12}$$

with constraints for $\psi_0 < \frac{1}{2}$,

$$0 = 1 + KU(X)(K, s) - K\exp(-(\tau_1/K)Q)V(Y)(K, s), \tag{13a}$$

$$0 = (\exp(-(\tau_1/K)Q) + KU(Y)(K, s)) - K\exp(-(\tau_1/K)V(X)(K, s), \tag{13b}$$

for $\psi_0 = \frac{1}{2}$,

$$1 = \int_0^1 \psi(x)(X(x, s) + Y(x, s))\, \mathrm{d}x, \tag{14a}$$

$$\tau_1 \int_0^1 Y(x, s)\psi(x)\, \mathrm{d}x = \int_0^1 x\psi(x)(X(x, s) - Y(x, s))\, \mathrm{d}x. \tag{14b}$$

The other conditions for which $X(z, s)$ and $Y(z, s)$ hold are

$$X(z, s) \to H(z, s) \quad \text{when} \quad \tau_1 \to \infty, \tag{15a}$$

$$Y(z, s) \to \hat{u} \quad \text{when} \quad \tau_1 \to \infty, \tag{15b}$$

where for $M = X$ or Y

$$V(M)(z, s) = \int_0^1 \psi(x)M(x, s)\, \mathrm{d}x/(x + z) \tag{16}$$

is analytic for $z \in (-1, 0)^c$ bounded at the origin $O(z^{-1})$ when $z \to \infty$ and

$$U(M)(z, s) = \int_0^1 \psi(x) M(x, s)\, dx/(x - z) \tag{17}$$

is analytic for $z \in (0, 1)^c$, bounded at the origin $O(z^{-1})$ when $z \to \infty$.

3. Fredholm Equations

Equations (11) and (12) with Equations (6b) can be written in the form

$$X(z, s)/H(z, s) = H(-z, s)(1 + zU(X)(z, s)) - z \exp(-(\tau_1/z)Q) \times \times H(-z, s) V(Y)(z, s), \tag{18}$$

$$Y(z, s)/H(z, s) = H(-z, s)((\exp(-\tau_1/z)Q) + zU(Y)(z, s) - - z \exp(-(\tau_1/z)Q) H(-z, s) V(X)(z, s). \tag{19}$$

We shall assume that $X(z, s)$ and $Y(z, s)$ are regular for $\operatorname{Re} z > 0$ and bounded at the origin. Equation (8) gives

$$H(-z, s) = \frac{A_0 - H_0 z}{(K - z)} - \int_0^1 \frac{P(x, s)}{x - z}\, dx \quad \text{for} \quad \psi_0 < \tfrac{1}{2}. \tag{20}$$

Hence

$$V(M)(z, s) \int_0^1 \frac{P(x, s)}{x - z}\, dx = D(M, P_0)(z, s) + D(P, M_0)(z, s), \tag{21}$$

where

$$D(M, P_0)(z, s) = \int_0^1 \frac{\psi(x) M(x, s) P_0(x, s)\, dx}{x + z} \tag{22}$$

and

$$D(P, M_0)(z, s) = \int_0^1 \frac{\psi(x) P(x, s) M_0(x, s)}{x - z}\, dx, \tag{23}$$

where

$$P_0(z, s) = \int_0^1 \frac{P(x, s)\, dx}{x + z} \tag{24}$$

is regular on $(-1, 0)^c$, bounded at the origin $O(z^{-1})$ when $z \to \infty$,

$$M_0(z, s) = \int_0^1 \frac{\psi(x) M(x, s)\,\mathrm{d}x}{x + z}, \tag{25}$$

is regular on $(-1, 0)^c$, bounded at the origin $O(z^{-1})$ when $z \to \infty$ and $D(M, P_0)(z, s)$ is regular for z on $(-1, 0)^c$, bounded at the origin and $O(z^{-1})$ when $z \to \infty$ and $D(P, M_0)(z, s)$ is regular for z, on $(0, 1)^c$ bounded at the origin, and $O(z^{-1})$ when $z \to \infty$. Hence, Equations (18) and (19) can for $\psi_0 < \frac{1}{2}$ be written in the form

$$\begin{aligned} &X(z, s)/H(z, s) + z \exp(-(\tau_1/z)Q) \times \\ &\quad \times \left(\frac{A_0 - H_0 z}{K - z} V(Y)(z, s) - D(Y, P_0)(z, s) \right) = \\ &= H(-z, s)\,(1 + zU(X)(z, s) + z \exp(-(\tau_1/z)Q) D(P, Y_0)(z, s)), \end{aligned} \tag{26}$$

$$\begin{aligned} &Y(z, s)/H(z, s) + z \exp(-(\tau_1/z)Q) \times \\ &\quad \times \left(\frac{A_0 - H_0 z}{K - z} V(X)(z, s) - D(X, P_0)(z, s) \right) = \\ &= H(-z, s)\,(\exp(-(\tau_1/z)z) + zU(Y)(z, s)) + \\ &\quad + z \exp(-(\tau_1/z)Q) D(P, X_0)(z, s). \end{aligned} \tag{27}$$

The left-hand side of Equations (26) and (27) are regular for $\operatorname{Re} z > 0$ and bounded at the origin; the right-hand side of Equations (26) and (27) are regular for z, on $(0, 1)^c$, bounded at the origin and tends to constants, say, A and B, respectively, when $z \to \infty$.

Hence, by a modified form of Liouville's theorem we have

$$\begin{aligned} X(z, s) = H(z, s) \Bigg[& z \exp(-(\tau_1/z)Q) \Bigg(D(Y, P_0)(z, s) - \\ & - \frac{A_0 - H_0 z}{K - z} V(Y)(z, s) \Bigg) + A \Bigg], \end{aligned} \tag{28}$$

$$\begin{aligned} Y(z, s) = H(z, s) \Bigg[& z \exp(-(\tau_1/z)Q) \Bigg(D(X, P_0)(z, s) - \\ & - \frac{A_0 - H_0 z}{K - z} V(X)(z, s) \Bigg) + B \Bigg], \end{aligned} \tag{29}$$

Equations (28) and (29) together with Equations (15a) and (15b) gives

$$A = 1, \qquad B = 0. \tag{30}$$

Hence for $\psi_0 = \frac{1}{2}$, the expression of $X(z, s)$ and $Y(z, s)$ are

$$X(z, s) = H(z, s)\,[1 + z \exp(-(\tau_1/z)Q)\,(D(Y, P_0)\,(z, s) - \\ -(-h_1 z + h_0)V(Y)\,(z, s))]\,, \tag{31}$$

$$Y(z, s) = H(z, s) z \exp(-(\tau_1/z)Q)\,(D(X, P_0)\,(z, s)) - \\ -(-h_1 z + h_0)V(X)\,(z, s))\,. \tag{32}$$

Hence, following Mullikin (1964) Equations (28) and (29) together with Equations (13a) and (13b) give unique representations of time-dependent X- and Y-functions for $\psi_0 < \frac{1}{2}$ and Equations (31) and (32) together with Equations (14a) and (14b) give unique representations of X- and Y-functions for $\psi_0 = \frac{1}{2}$.

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EXACT SOLUTION OF THE EQUATION OF TRANSFER WITH PLANETARY PHASE FUNCTION

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Abstract. We have considered the transport equation for radiative transfer to a problem in semi-infinite atmosphere with no incident radiation and scattering according to planetary phase function $w(1 + x\cos\theta)$. Using Laplace transform and the Wiener–Hopf technique, we have determined the emergent intensity and the intensity at any optical depth. The emergent intensity is in agreement with that of Chandrasekhar (1960).

1. Introduction

The transport equation for the intensity of radiation in a semi-infinite atmosphere with no incident radiation and scattering according to the phase function $w(1 + x\cos\theta)$ has been considered. This equation has been solved by Chandrasekhar (1960) using his principle of invariance to get the emergent radiation. The singular eigen function approach of Case (1960) is also applied to get the intensity of radiation at any optical depth. Boffi (1970) has also applied the two sided Laplace transform to get the emergent intensity and the intensity at any optical depth. Das (1979) solved exactly the equation of transfer for scattering albedo $w < 1$ using the Laplace transform and the Wiener–Hopf technique and also deduced the intensity at any optical depth by inversion.

In this paper we have solved the above problem exactly by a method based on the use of the Laplace transform and the Wiener–Hopf technique. The intensity at any optical depth is also derived by inversion.

2. Basic Equation and its Solution

The equation of transfer appropriate to the problem (Chandrasekhar, 1960) is

$$\mu \frac{\mathrm{d}I(\tau, \mu)}{\mathrm{d}\tau} = I(\tau, \mu) - \frac{1}{2} w \int_{-1}^{+1} I(\tau, \mu')(1 + x\mu\mu')\,\mathrm{d}\mu' , \tag{1}$$

where the symbols have their usual meaning.

We shall have the following boundary conditions

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$$I(0, -\mu) = 0, \quad 0 < \mu < 1\,; \tag{2a}$$

$$I(\tau, \mu) \to L_0 \exp(k\tau)\,\frac{1 + x(1-w)(\mu/k)}{1 - k\mu}, \quad \text{as } \tau \to \infty\,; \tag{2b}$$

where L_0 is a constant and k is the positive root, less than 1, of the transcendental equation.

$$1 = \frac{w}{2k}\left[1 + \frac{x(1-w)}{k^2}\right]\log\left(\frac{1+k}{1-k}\right) - \\ - \frac{1}{k^2}xw(1-w)\,. \tag{3}$$

Let us define

$$f^*(s) = s\int_0^\infty \exp(-s\tau) f(\tau)\,\mathrm{d}\tau, \quad \mathrm{R}1/s > 0\,. \tag{4}$$

Let us set

$$I_m(\tau) = \frac{1}{2}\int_{-1}^{+1} I(\tau, \mu')\mu'^m\,\mathrm{d}\mu', \quad \text{where } m = 0, 1\,, \tag{5}$$

which gives

$$I_0^*(s) = \frac{1}{2}\int_{-1}^{+1} I^*(s, \mu')\,\mathrm{d}\mu' \tag{6}$$

and

$$I_1^*(s) = \frac{1}{2}\int_{-1}^{+1} I^*(s, \mu')\mu'\,\mathrm{d}\mu'\,, \tag{7}$$

Equation (1) with Equation (5) takes the form

$$\frac{\mu\,\mathrm{d}I(\tau, \mu)}{\mathrm{d}\tau} = I(\tau, \mu) - wI_0(\tau) - wx\mu I_1(\tau)\,. \tag{8}$$

Now, subjecting Equation (8) to the Laplace transform as define in Equation (4), we have, using the boundary conditions,

$$(\mu s - 1)I^*(s, \mu) = \mu s I(0, \mu) - wI_0^*(s) \\ -xw\mu I_1^*(s)\,. \tag{9}$$

Equation (9) gives (on putting $s = 1/\mu$)

$$I(0, \mu) = wI_0^*(1/\mu) + xw\mu I_1^*(1/\mu)\,. \tag{10}$$

Equation (10) with $\mu = 1/s$, s is complex, takes the form

$$I(0, 1/s) = wI_0^*(s) + xws^{-1}I_1^*(s)\,, \tag{11}$$

we apply the operator

$$\frac{1}{2}\int_{-1}^{+1} \ldots \mathrm{d}\mu \tag{12}$$

on both sides of Equation (9) to get

$$I_1^*(s) - (1-w)s^{-1}I_0^*(s) = \frac{1}{2}\int_0^1 \mu I(0, \mu)\,\mathrm{d}\mu \tag{13}$$

we apply the operator

$$\frac{1}{2}\int_{-1}^{+1} \ldots \mathrm{d}\mu/(\mu s - 1)\,, \tag{14}$$

$$a(1/s) = 1 + wt_0(1/s) + xwt_1(1/s)I_1^*(s)\,, \tag{15}$$

where

$$a(1/s) = \frac{1}{2}\int_0^1 \mu s(\mu s - 1)^{-1} I(0, \mu)\,\mathrm{d}\mu \tag{16}$$

and

$$t_m(1/s) = \frac{1}{2}\int_{-1}^{+1} (\mu s - 1)^{-1}\mu^m\,\mathrm{d}\mu, \quad m = 0, 1\,. \tag{17}$$

Eliminating $I_0^*(s)$, $I_1^*(s)$ among Equations (11), (13) and (15) and setting $s = 1/z$, we have

$$T(z)I(0, z) = \frac{w}{2}\int_0^1 \frac{\mu}{\mu - z} \times [1 + \mu x(1-w)z]I(0, \mu)\,\mathrm{d}\mu\,, \tag{18}$$

where

$$T(z) = 1 + wx(1-w)z^2 + w[1 + x(1-w)z^2]t_0(z)\,, \tag{19}$$

where

$$t_0(z) = \frac{z}{2}\int_{-1}^{+1} \frac{\mathrm{d}\mu}{\mu - z}\,. \tag{20}$$

Following Chandrasekhar (1960) and considering Equation (3), we see that $T(z)$ has a pair of roots at $z = \pm k^{-1}$ and

$$T(z) = \frac{1}{H(z)H(-z)}, \quad z \in (-1, 1)^c, \tag{21}$$

where $H(z)$ is Chandrasekhar's H-function for planetary scattering. Equation (18) with Equation (21) takes the form

$$\frac{I(0, z)}{H(z)} = H(-z)\frac{w}{2}\int_0^1 \frac{\mu}{\mu - z} \times$$
$$\times [1 + \mu x(1 - w)z]I(0, \mu)\, d\mu, \tag{22}$$

Equation (22) can be written as

$$\frac{I(0, z)}{H(z)} = H(-z)wG(z),$$

where

$$G(z) = \frac{1}{2}\int_0^1 \frac{\mu}{\mu - z}[1 + \mu x(1 - w)z]I(0, \mu)\, d\mu. \tag{23}$$

Let us seek solution $I(0, z)$ of Equation (22) by Wiener–Hopf technique on the assumption that $I(0, z)$ is regular for $\mathrm{Rl}\, z > 0$ and bounded at the origin.

Equation (23) with the above assumption on $I(0, z)$ gives the following properties of $G(z)$: $G(z)$ is regular on $(0, 1)^c$, bounded at the origin and a constant as $z \to \infty$. Equation (23) then gives

$$\frac{(1 - kz)I(0, z)}{H(z)} = w(1 - kz)H(-z)G(z), \tag{24}$$

where $H(-z)$, $H(z)$, $1/H(z)$ has the following properties: $H(z)$ is regular for $z \in (-1, 0)^c$, uniformly bounded at the origin has a simple pole at $z = -(1/k)$, $k < 1$; k is real on the negative real axis and bounded at infinity and tends to $H_0 + H_{-1}z^{-1} + H_{-2}z^{-2} + \cdots$ when $z \to \infty$.

Hence, $1/H(z)$ is regular for z in $(-1, 0)^c$ and bounded at the origin. Similarly, $H(-z)$ is regular for $z \in (0, 1)^c$ uniformly bounded at the origin has a simple pole at $z = 1/k$, $k < 1$; k is real, on the positive side of the real axis and bounded at infinity and tends to $H_0 - H_{-1}z^{-1} + H_{-2}z^{-2} - \cdots$ when $z \to \infty$.

Following the properties of $H(z)$, $1/H(z)$, $H(-z)$ (Busbridge, 1960) the left hand side of Equation (24) is regular for $\mathrm{Rl}\, z > 0$, bounded at the origin and the right hand side of Equation (24) is regular for $z \in (0, 1)^c$ and bounded at the origin and tends to a polynomial say $A + Bz$, as $z \to \infty$.

Hence by a modified form of Liouville's theorem

$$\frac{(1-kz)I(0,z)}{H(z)} = A + Bz, \quad \text{when } z \in (-1,0)^c \tag{25}$$

and

$$A + Bz = w(1-kz)H(-z)G(z), \quad \text{when } z \in (0,1)^c \,. \tag{26}$$

Equation (25) gives the emergent radiation as

$$I(0,z) = \frac{(A+Bz)H(z)}{1-kz}, \tag{27}$$

where the constants A and B are two arbitrary constants to be determined later on.

3. Intensity at Any Optical Depth

The radiation intensity at an optical depth τ is given by the inversion integral as

$$I(\tau,\mu) = (1/2\pi i) \lim_{\delta\to\infty} \int_{c-i\delta}^{c+i\delta} \exp(s\tau) \times$$

$$\times\, I^*(s,\mu)\, \mathrm{d}s/s, \quad c > 0\,. \tag{28}$$

Equation (9) with Equation (11) takes the form

$$I^*(s,\mu)/s = \phi(s,\mu)/(s - 1/\mu)\,, \tag{29}$$

where

$$\phi(s,\mu) = I(0,\mu) - I(0,1/s) + \frac{w(s-1/s)}{s} I_0^*(s)\,. \tag{30}$$

But

$$\lim_{s\to 1/\mu} (s-1/\mu) I^*(s,\mu) \exp(s\tau)/s \to 0\,. \tag{31}$$

Hence the integrand of Equations (28) is regular for $s \in (-\infty, -1)^c$ and has simple pole at $s = \pm k$, $k < 1$.

Hence by Cauchy's residue theorem, Equation (28) gives

$$I(\tau,\mu) = R_p + \lim_{R\to\infty} (1/2\pi i) \int_\Gamma I^*(s,\mu) e^{s\tau}\, \mathrm{d}s/s\,, \tag{32}$$

where R_p is the sum of the residues of the poles at $s = \pm k$ and $\Gamma = \Gamma_1 \cup CD \cup \nu \cup EF \cup \Gamma_2$. [$\Gamma_1$ and Γ_2 are arcs of the circle of radius R having centre at $s = 0$ (clockwise) and ν is an arc of a small circle of radius r having centre at $s = -1$ (anticlockwise) and CD and EF are the lower edge and upper edge of

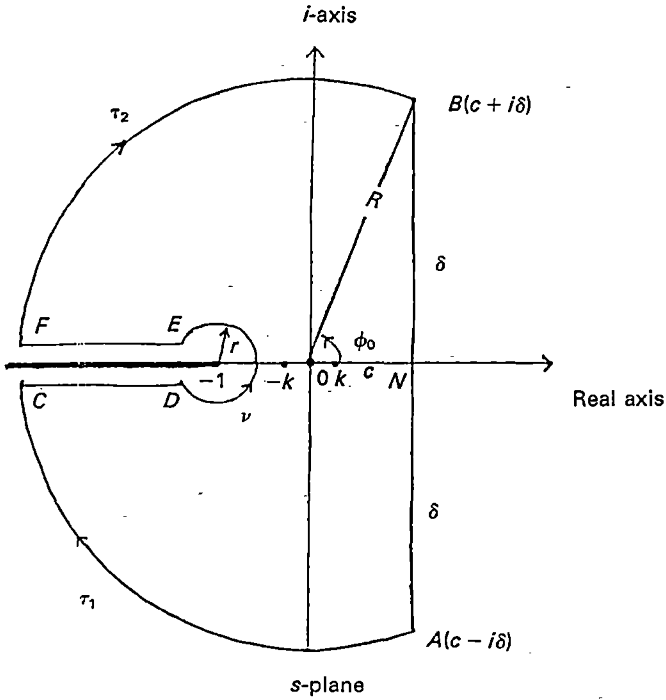

Fig. 1.

the singular line $(-R, -1)]$ (Figure 1). Hence, following Kourganoff (1963) we have

$$\int_{\Gamma_1 \cup \Gamma_2} I^*(s, \mu) \exp(s\tau)\, ds/s \rightarrow 0, \quad \text{when } R \rightarrow \infty \tag{33}$$

and

$$\int_{\nu} I^*(s, \mu) \exp(s\tau)\, ds/s \rightarrow 0, \quad \text{when } r \rightarrow 0\,. \tag{33a}$$

Hence in the limit of $R \rightarrow \infty$, $r \rightarrow 0$, Equation (32) with Equations (33) and (33a) becomes

$$I(\tau, \mu) = R_p + (1/2\pi i) \int_{CD} I^*(s, \mu) e^{s\tau}\, ds/s + + (1/2\pi i) \int_{EF} I^*(s, \mu) e^{s\tau}\, ds/s\,. \tag{34}$$

Here on CD and EF,

$$s = -\upsilon, \qquad \upsilon \geqslant 1 \tag{34a}$$

and on CD,

$$H(1/s) = \frac{X(1/v) + i\pi Y(1/v)}{H(1/v)Z(1/v)} \tag{35}$$

and on EF,

$$H(1/s) = \frac{X(1/v) - i\pi Y(1/v)}{H(1/v)Z(1/v)}; \tag{36}$$

where

$$X(1/v) = 1 + wx(1-w)v^{-2} - w[1 + x(1-w)v^{-2}] \times$$
$$\times \frac{1}{2v} \log\left(\frac{v+1}{v-1}\right), \tag{37}$$

$$Y(1/v) = (w/2)v^{-1}; \tag{38}$$

$$Z(1/v) = (X^2(1/v) + \pi^2 Y^2(1/v, \mu)). \tag{39}$$

Therefore on CD

$$\phi(s, \mu) = V(1/v, \mu) - i\pi W(1/v, \mu) \tag{40}$$

and on EF,

$$\phi(s, \mu) = V(1/v, \mu) + i\pi W(1/v, \mu), \tag{41}$$

where

$$V(1/v, \mu) = I(0, \mu) - \left[\frac{(B - vA)(1/v)}{(v + k)H(1/k)Z(1/v)}\right] \times$$
$$\times \left\{1 + \frac{v + 1/\mu}{1 + x(1-w)/v^2}\right\} + \frac{(v + 1/\mu)w\alpha_1/2}{1 + x(1-w)/v^2}, \tag{42}$$

$$W(1/v, \mu) = \left[\frac{(B - vA)Y(1/v)}{(v + k)H(1/k)Z(1/v)}\right]\left[1 + \frac{v + 1/\mu}{1 + x(1-w)/v^2}\right].$$

Now, Equation (33) with Equations (29), (34a), (40) and (41) gives

$$I(\tau, \mu) = R_p - \frac{1}{2\pi i}\int_1^\infty \frac{\{v(1/\nu, \mu) - i\pi W(1/v, \mu)\}}{v + 1/\mu} e^{-v\tau} dv +$$
$$+ \frac{1}{2\pi i}\int_1^\infty \frac{V(1/v, \mu) + i\pi W(1/v, \mu)}{v + 1/\mu} e^{-v\tau} dv. \tag{44}$$

Hence when $\mu > 0$, Equation (44) give

$$I(\tau, \mu) = R_p + \int_1^\infty W(1/\upsilon, \mu)\, e^{-\upsilon\tau}\, d\upsilon/(\upsilon + 1/\mu)\,, \tag{45}$$

where $\mu < 0$, we shall assume that $(V(1/\upsilon, \mu) \pm i\pi W(1/\upsilon, \mu)\, e^{-\nu\tau}$ satisfies Hölder condition on $(1, \infty)$ and we have by Plemelj's formula (Muskhelishvili, 1946)

$$\frac{1}{2\pi i}\int_1^\infty \frac{V(1/\upsilon, \mu) \pm i\pi W(1/\upsilon, \mu)}{\upsilon + 1/\mu}\, e^{-\upsilon\tau}\, d\upsilon = \pm\frac{1}{2}(V(-\mu, \mu) \pm$$

$$\pm\, i\pi W(-\mu, \mu))e^{\tau/\mu} + \frac{1}{2\pi i} \times P\int_1^\infty \frac{V(1/\upsilon, \mu) \pm i\pi W(1/\upsilon, \mu)}{\upsilon + 1/\mu} \times$$

$$\times\, e^{-\upsilon\tau}\, d\upsilon\,, \tag{46}$$

where P denotes the Cauchy principal value of the integral. Hence Equation (44) with Equation (46) for $\mu < 0$ gives

$$I(\tau, \mu) = R_p + V(-\mu, \mu)\, e^{\tau/\mu} + P\int_1^\infty \frac{W(1/\upsilon, \mu)\, e^{-\upsilon\tau}}{\upsilon + 1/\mu}\, d\upsilon\,, \tag{47}$$

where

$$R_p = R_k + R_{-k}\,, \tag{48}$$

where, $R_{\pm k}$ is the residue of the integral in Equation (32) at $s = \pm k$, and R_k is given by

$$\begin{aligned} R_k &= \lim_{s\to k}\,(s - k)I^*(s, \mu)\, e^{s\tau}/s \\ &= \lim_{s\to k} \frac{H(1/s)(As + B)s}{\{s^2 + x(1 - w)\}(1 - s\mu)}[1 + x(1 - w)/s]\, e^{s\tau} \\ &= \frac{H(1/k)(Ak + B)k}{[k^2 + x(1 - w)](1 - k\mu)}[1 + x(1 - w)/k]\, e^{k\tau}\,. \end{aligned} \tag{49}$$

Similarly, R_{-k} is given by

$$\begin{aligned} R_{-k} &= \lim_{s\to(-k)}(s + k)I^*(s, \mu)\, e^{s\tau}/s \\ &= \lim_{s\to(-k)} \frac{(s + k)H(1/s)(As + B)s}{(s - k)\{s^2 + x(1 - w)\}(1 - s\mu)} \times \\ &\quad \times [1 + x(1 - w)/s]\, e^{s\tau} \\ &= \frac{(B - Ak)k[1 - x(1 - w)/k]\, e^{-k\tau}}{2k\{k^2 + x(1 - w)\}(1 + k\mu)} \lim_{s\to(-k)} (s + k)/T(1/s) \\ &= \frac{(B - Ak)[1 - x(1 - w)k]e^{-k\tau}}{2\{k^2 + x(1 - w)\}(1 + k\mu)}[dT(1/s)/ds]^{-1}_{s=-k} \end{aligned} \tag{50}$$

4. Determination of constants A and B

When $z \to 0$, from Equation (26) we get

$$A = (w/2) \int_0^1 I(0, \mu) \, \mathrm{d}\mu \,. \tag{51}$$

From Equation (51) and Equation (25) we get after simplification

$$A\left[1 - \frac{w}{2}\int_0^1 \frac{H(\mu)\,\mathrm{d}\mu}{1-k\mu}\right] = \frac{wB}{2k}\left[-\alpha_0 + \int_0^1 \frac{H(\mu)\,\mathrm{d}\mu}{1-k\mu}\right] = m\,, \tag{52}$$

where

$$\alpha_0 = \int_0^1 H(\mu)\,\mathrm{d}\mu, \qquad m = \text{constant}\,.$$

$H(z)$ has a simple pole at $z = -(1/k)$ where

$$1/H(z) = 1 - zH(z)\int_0^1 \frac{\psi(z)H(\mu)\,\mathrm{d}\mu}{\mu + z}, \tag{53}$$

where

$$\psi(\mu) = \frac{w}{2}[1 + x(1-w)\mu^2]\,. \tag{54}$$

Equation (53) has a zero at $z = -(1/k)$ and so

$$1 + \frac{1}{k}\int_0^1 \frac{\psi(\mu)H(\mu)\,\mathrm{d}\mu}{\mu - 1/k} = 0\,. \tag{55}$$

In Equation (55) putting the value of $\psi(\mu)$ and simplifying and using Equation (52) we get

$$A = \frac{2mN}{kQ}\Big/\left(\frac{x(1-w)}{k} - c\right), \qquad B = \frac{2mN}{Q(k+c)},$$
$$N = k^2 + x(1-w), \qquad Q = 2 - w\alpha_0, \qquad c = \frac{xw(1-w)\alpha_1}{Q} \tag{56}$$

$$A + B\mu = \frac{2mN}{QR}\left\{\left(1 + \frac{c}{k}\right) + \left(\frac{x(1-w)}{k} - c\right)\right\}. \tag{57}$$

Putting

$$\mu = 1/k \text{ we get } kA + B = \frac{2mN^2}{QkR} \tag{58}$$

where

$$R = \left\{ \frac{x(1-w)}{k} - c \right\} (k + c) . \tag{59}$$

If we use Equations (58) and (59) we get from Equation (27)

$$I(0, \mu) = \frac{(kA + B)k}{k^2 + x(1-w)} \left[\left(1 + \frac{c}{k}\right) + \left\{ \frac{x(1-w)}{k} - c \right\} \mu \right] \frac{H(\mu)}{1 - k\mu}, \tag{60}$$

when $\tau \to \infty$ from Equations (47), (48) and (49) we get

$$I(\tau, \mu) \to \frac{H(1/k)(Ak + B)k}{[k^2 + x(1-w)](1 - k\mu)} \times [1 + x(1-w)/k] e^{k\tau} . \tag{61}$$

Hence Equation (61) with Equation (2b) gives

$$\frac{(Ak + B)k}{k^2 + x(1-w)} = \frac{L_0}{H(1/k)}, \tag{62}$$

$$I(0, \mu) = \frac{L_0}{H(1/k)} \left\{ 1 + \frac{c}{k} + \mu \left[\frac{x(1-w)}{k} - c \right] \right\} \frac{H(\mu)}{1 - k\mu}, \tag{63}$$

which is the expression obtained by Chandrasekhar (1960).

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AN EXACT LINEARIZATION AND DECOUPLING OF THE INTEGRAL EQUATIONS SATISFIED BY TIME-DEPENDENT X- AND Y-FUNCTIONS

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Abstract. We discuss a simple method of linearization and decoupling of the integral equations satisfied by time-dependent X- and Y-functions which play an important rôle in the study of non-stationary radiative transfer problems.

1. Introduction

In the study of the time-dependent radiative transfer problems in finite homogeneous plane-parallel atmospheres, it is convenient to introduce X- and Y-functions (Chandrasekhar, 1960). These functions satisfy non-linear coupled integral equations. Due to their important rôle in solving transport problems, it is advantageous to simplify the equations satisfied by them, and, if possible, do so in an exact manner. Lahoz (1989) did this and obtained exact linear and decoupled integral equations satisfied by the time- independent X- and Y-functions.

In this paper we have extended the same method to the time-dependent radiative transfer problem. However, the equations obtained, although linear, are singular and not solvable by the standard methods applicable to Fredholm equations; instead they have to be solved by the theory of singular integral equations (Muskhelishvili, 1946).

2. Analysis

The integral equations incorporating the various invariances of the time-dependent problem of diffuse refelection and transmission can be reduced to one or more pairs of integral equations of the following form (Biswas and Karanjai, 1990).

$$X(\mu, s) = 1 + \frac{W}{2}\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu' \frac{X(\mu, s)X(\mu', s) - Y(\mu, s)Y(\mu', s)}{\mu + \mu'}, \tag{1}$$

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$$Y(\mu,s)=\exp\{(-\tau_1/\mu)Q\}+\frac{W}{2Q}\int_0^1 \mathrm{d}\mu'\frac{Y(\mu,s)\,X(\mu',s)-X(\mu,s)Y(\mu',s)}{\mu-\mu'},\tag{2}$$

Following Chandrasekhar (1960), we can write the above equations in the form:

$$X(\mu,s)=1+\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\psi(\mu')\frac{X(\mu,s)\,X(\mu',s)-Y(\mu,s)\,Y(\mu',s)}{\mu+\mu'},\tag{3}$$

$$\begin{aligned} Y(\mu,s) &= \exp\{(-\tau_1/\mu)Q\}+\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\times \\ &\quad\times\psi(\mu')\frac{Y(\mu,s)X(\mu',s)-X(\mu,s)Y(\mu',s)}{\mu-\mu'}, \end{aligned}\tag{4}$$

where τ_1 is the optical thickness of the atmosphere and $Q = 1 + s/c$, where c is the velocity of light, s is the Laplace invariant of the time variable and the characteristic function $\psi(\mu)$ is an even polynomial in μ satisfying

$$\psi_0=\int_0^1 \psi(\mu)\,\mathrm{d}\mu\le\frac{1}{2},\tag{5}$$

where $\psi_0 = \frac{1}{2}$ holds, $\psi(\mu)$ is said to be conservative; and non-conservative otherwise.

Clearly, Eqs. (3) and (4) are non-linear and coupled. These equations have been linearized in an exact manner (Mullikin, 1964). The results are

$$\begin{aligned} X(\mu,s)K(\mu,s) &= 1+\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\psi(\mu')\frac{X(\mu',s)}{\mu'-\mu}- \\ &\quad-\exp\{(-\tau_1/\mu)Q\}\,\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\psi(\mu')\frac{Y(\mu',s)}{\mu'+\mu} \end{aligned}\tag{6}$$

and

$$\begin{aligned} Y(\mu,s)K(\mu,s) &= \exp\{(-\tau_1/\mu)Q\}+\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\psi(\mu')\frac{Y(\mu',s)}{\mu'-\mu}- \\ &\quad-\exp\{(-\tau_1/\mu)Q\}\,\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu'\,\psi(\mu')\frac{X(\mu',s)}{\mu'+\mu}, \end{aligned}\tag{7}$$

where $K(\mu, s)$ is defined by

$$K(\mu, s) \equiv 1 - \frac{\mu}{Q} \int_0^1 d\mu' \, \psi(\mu) \left[\frac{1}{\mu' + \mu} - \frac{1}{\mu' - \mu} \right], \tag{8}$$

We now proceed to decouple Eqs. (4) and (5) in an exact manner (Lahoz, 1989). We introduce the following singular integral equation, which is linear in $1/T(\mu, s)$:

$$\frac{1}{T(\mu, s)} = 1 - \frac{\mu}{Q} \int_0^1 d\mu' \, \frac{\psi(\mu')}{T(\mu', s) K(\mu', s)} \, \frac{1}{\mu' - \mu}. \tag{9}$$

which, in principle, is solvable for $T(\mu, s)$ as $\psi(\mu)$ and $K(\mu, s)$ are known functions.

Next, we multiply Eq. (6) by

$$\frac{(\mu'/Q)\psi(\mu)}{T(\mu, s)K(\mu, s)(\mu' - \mu)},$$

which we assume is well defined in $\mu \in [0, 1]$ and integrate with respect to μ from 0 to 1 to obtain

$$\begin{aligned} &\frac{\mu}{Q} \int_0^1 d\mu' \left[\frac{\Psi(\mu')X(\mu', s)}{\mu' + \mu} \right] = 1 - T(-\mu, s) \times \\ &\times \left[1 - P(\mu, s) \frac{\mu}{Q} \int_0^1 d\mu' \, \frac{\psi(\mu')Y(\mu', s)}{\mu' - \mu} + \frac{\mu}{Q} \int_0^1 d\mu' \, \frac{\psi(\mu')Y(\mu', s)P(\mu', s)}{\mu' - \mu} \right], \end{aligned} \tag{10}$$

where we have used Eq. (9) and defined the function $P(\mu, s)$ (in principle known) by

$$P(\mu, s) \equiv \frac{\mu}{Q} \int_0^1 d\mu' \, \frac{\psi(\mu') \exp(-\tau_1/\mu)}{T(\mu', s)K(\mu', s)} \, \frac{1}{\mu' + \mu}. \tag{11}$$

If we substitute Eq. (10) in Eq. (5) we get the decoupled equation for $Y(\mu, s)$ as follows:

$$\begin{aligned} &Y(\mu, s)K(\mu, s) = \\ &= T(-\mu, s) \exp\{(-\tau_1/\mu)Q\} + \\ &+ T(-\mu, s)P(\mu, s)[1 - \exp\{(-\tau_1/\mu)Q\}] \frac{\mu}{Q} \int_0^1 d\mu' \, \frac{\psi(\mu')Y(\mu', s)}{\mu' - \mu} + \\ &+ T(-\mu, s) \exp\{(-\tau_1/\mu)Q\} \frac{\mu}{Q} \int_0^1 d\mu' \, \frac{\psi((\mu', s))Y(\mu', s)P(\mu', s)}{\mu' - \mu}. \end{aligned} \tag{12}$$

A similar analysis yields the decoupled equation for $X(\mu, s)$:

$$X(\mu,s)K(\mu,s) = [1 - T(-\mu,s)P(\mu,s)\exp\{(-\tau_1/\mu)Q\}] \times$$
$$\times \left[1 + \frac{\mu}{Q}\int_0^1 \mathrm{d}\mu' \frac{\psi(\mu')X(\mu',s)}{\mu'-\mu}\right] + \tag{13}$$
$$+T(-\mu,s)\exp\{(-\tau_1/\mu)Q\}\frac{\mu}{Q}\int_0^1 \mathrm{d}\mu' \frac{\psi(\mu',s)X(\mu',s)}{\mu'-\mu}.$$

Eqs. (12) and (13) are linear, singular and decoupled and, in principle, solvable by the theory of singular integral equations (Muskhelisvili, 1946).

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Solution of a Radiative Transfer Problem with a Combined Rayleigh and Isotropic Phase Matrix

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ABSTRACT :

Chandrasekhar (1960), has considered the problem, by his discrete ordinate procedure, of the basic non-conservative matrix equation of radiative transfer for diffuse reflection for a combination of Rayleigh and isotropic scattering in a semi-infinite atmosphere. Schnatz and Siewert (1970) have obtained the exact solution of the basic transport equation for non-conservative rayleigh phase matrix by the eigen function approach of Case(1960). Bond and Siewert(1971) have obtained a rigorous general solution of a non-conservative matrix equation of transfer, which appears for consideration of polarization by the eigen function approach of Case(1960). Das (1979a) solved the basic integro-differential equation for radiative transfer in diffuse reflection in a combination of Rayleigh and isotropic scattering for a semi-infinite atmosphere exactly for the emergent intensity matrix by use of the Laplace transform and Wiener-Hopf technique.

In this paper, we shall consider the Laplace transform and Wiener-Hopf technique to solve the matrix transport equation for a scattering which scatters radiation in accordance with the phase matrix obtained from a combination of Rayleigh and isotropic scattering in a semi-infinite atmosphere. The basic matrix equation is subject to the Laplace transform to obtain an integral equation for the emergent intensity matrix. On application of the Wiener-Hopf technique this matrix integral equation gives the emergent intensity matrix in terms of a singular H-matrix and an

unknown matrix. The unknown matrix has been obtained by equating the asymptotic solution of the boundary condition at infinity.

1. INTRODUCTION :

The method of Laplace Transform and Wiener-Hopf Technique has been applied to solve problems of radiative transfer by Dasgupta (1977), Das (1979b) Karanjai and Karanjai (1985) and others. Recently Karanjai and Islam (1993) solved radiative transfer problems with anisotropic scattering by the same method. We like to solve have a particular anisotropically scattering problem where the phase matrix consists of contributions from isotropic and Rayleigh scattereing.

2. BASIC MATRIX TRANSPORT EQUATION AND BOUNDARY CONDITIONS :

The basic integro-differential equation for infinity matrix I (τ,μ) can be written in the form

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2} \omega \int_{-1}^{+1} K(\mu,\mu') \quad I(\tau,\mu') \quad d\mu' \qquad (1)$$

where τ is the optical thickness of the atmosphere, μ is the direction parameter, $I(\tau,\mu)$ is a (2x1) matrix , w $(0 < \omega < 1)$ is the albedo for single scattering. According to Burniston and Siewert (1970),

$K(\mu,\mu)$, a (2x2) matrix, can be written as

$$K(\mu,\mu') = Q(\mu) \quad Q^T (\mu') \qquad (2)$$

where $Q(\mu)$, a (2x2) matrix, can be defined by

$$Q(\mu) = \frac{3(c+2)^{1/2}}{c+2} \begin{bmatrix} c\mu^2 + \frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{bmatrix} \quad (3)$$

$Q^T(\mu)$ is the transpose of $Q(\mu)$, and c is aparameter $(0 < c < 1)$

A solution of Equation (1) is required with the following boundary conditions

$$I(0, -\mu) = 0, \qquad 0 \le \mu \le 1 \qquad (4a)$$

and $I(\tau,\mu) \longrightarrow \frac{1}{2}\omega L_o \left[\frac{k}{k-\mu}\right] e^{\tau/k} Q(\mu)$ as $\tau \longrightarrow \infty$, (4b)

where K is a positive root greater than one and real of the equation $T(z) = \det D(z)$ (5)

$$\text{where } D(z) = E + z\int_{-1}^{+1} \psi(\mu)\frac{d\mu}{\mu - z} \qquad (6)$$

$\psi(\mu)$ is a (2x1) matrix and $\psi(\mu)$ is defined by

$$\psi(\mu) = (1/2)\omega Q^T(\mu)\ Q(\mu) \qquad (7)$$

and

E is a unit matrix , D(z) is a (2x2) matrix and L_0 is a specified (2x1) matrix.

3. SOLUTION FOR EMERGENT INTENSITY MATRIX :

The Laplace transform of the intensity matrix is defined by

$$I^*(s,\mu) = s\int_0^\infty e^{-s\tau}\ I(\tau,\mu)\ d\mu\ , \quad \text{Re } s > 0 \qquad (8)$$

Let us set $I_u(\tau)$, a (2x1) matrix as

$$I_u\ (\tau) = (1/2) \int_{-1}^{+1} Q^T\ (\mu')\ I\ (\tau,\mu')\ d\mu' \qquad (9)$$

$$I_u\ (s) = (1/2) \int_{-1}^{+1} Q^T\ (\mu')\ I^*\ (s,\mu')\ d\mu' \qquad (10)$$

we subject the Laplace transform as defined in Equation (8) to Equation (1) to get (Using Equations (4a), (9, (10))

$$(\mu s - 1)\ I^*\ (s,\ \mu) = \mu s\ I(o,\ \mu) - \omega\ Q(\mu)\ I_u^*\ (s) \qquad (11)$$

The solution for the emergent intensity matrix arrived from Equation (11)

$$I\ (0,\mu) = \omega\ Q(\mu)\ \ I_u^*\ \ (1/\mu) \qquad (12)$$

Equation (12) gives for $\mu = 1/s$, s is complex

$$I\ (0,1/s) = \omega\ Q(1/s)\ I_u^*\ (s) \qquad (13)$$

we now apply the (2x2) matrix operator

$$(1/2)\int_{-1}^{+1} \frac{Q^T\ (\mu)\ d\mu}{(\mu s - 1)}\ , \qquad (14)$$

to Equation (11) to get $\quad D\ (1/s)\ I_u^*\ (s)\ =\ a(1/s) \qquad (15)$

where D (1/s) is a (2x2) matrix and a(1/s) is (2x1) matrix defined by

$$D(1/s) = E + \int_{-1}^{+1} \frac{\psi\ \ (\mu)\ d\mu}{(\mu s - 1)}\ , \qquad (16)$$

and

$$a(1/s) = (1/2) \int_0^1 \frac{\mu s\ Q^T(\mu)\ I(0,\mu)\ d\mu}{(\mu s - 1)} \qquad (17)$$

respectively where

$\psi\ (\mu)$ is given by Equation (7) , is a (2x2) unit matrix .

Eliminating I_u^* (s) between Equations (13) and (15) we get a matrix integral equation as

$$D(z)\, I(0,z) = \omega\, Q(z)\ \ a(z), \text{ where } s = 1/z \tag{18}$$

Following Bond a Siewert (1971) , we have

$$T(z) = \det D(z) = \frac{1}{8}\, c\, T_1(z)\, T_2(z) + \left[(1-c) + \frac{3}{2}\, c\, (1-\omega)\, z^2\right] T_0(z) \tag{19}$$

and

$$T_n(z) = (-1)^n + 3(1-z^2)\, T_0(z) - (-1)^n\ 3(1-\omega) z^2 \ , \quad n = 1 \text{ or } 2 \tag{20}$$

$$T_0(z) = 1 + (1/2)\, \omega z \int_{-1}^{+1} \frac{d\mu}{\mu - z} \ , \tag{21}$$

where $T(z)$ is analytic in the complex plane cut from -1 to $+1$ along the real axis with two zeros at $z = \pm k$, k is real ($k > 1$).

We consider the (2x2) H-matrix equation (cf. Abhyankar and Fymat, 1970) as

$$H(z) = E + zH(z) \int_0^1 H^T(\mu)\, \psi(\mu)\, d\mu \,/\, (\mu + z) \tag{22}$$

where $\psi(\mu)$ is given by Equation (7).

We shall assume that the (2x2) $H(z)$ matrix is analytic in the complex plane cut from -1 to 0 , bounded at the origin , has a pole at $z = -k$, k is real ($k > 1$) and similarly the $H(-z)$ matrix is analytic in the complex plane cut from 0 to 1, bounded at the origin, has a pole at $z = k$, k is real , ($k > 1$). Hence , $H^{-1}(z)$, the inverse of the H-matrix, is analytic in the complex plane cut from -1 to 0 and bounded at the origin. If the (2x2) H-matrix is a symmetric matrix, it can be proved that

$$D(z) = H^{-1}(z)\, H^{-1}(-z) \quad , \ z \in (-1,1)^{o} \tag{23}$$

Now Equation (18) together with Equation (23) takes the form

$$H^{-1}(z)\,Q^{-1}(z)\,I(0,z)\left[\frac{k-z}{k}\right]$$
$$= \omega\left[\frac{k-z}{k}\right] H(-z)\,a(z) \tag{24}$$

where the left hand side of Equation (24) is regular for Re $z > 0$, bounded at the origin and the right hand side of Equation (24) is analytic in $(0,1)^c$, bounded at the origin and tends to a constant matrix (2x1) say A, when $z \longrightarrow \infty$ subject to the assumption that $I(0,z)$ is analytic for Re $z>0$ and bounded at the origin. Hence, by a modified form of Liouville's theorem, Equation (24) gives the emergent intensity matrix $I(0,z)$ as

$$I(0,z) = \left[\frac{k}{k-z}\right] Q(z)\; H(z)\, A \tag{25}$$

We now determine the matrix A. The inversion integral gives the intensity matrix $I(\tau,\mu)$ as

$$I(\tau,\mu) = (1/2\pi\, i) \lim_{\nu\to\infty} \int_{\alpha-i\nu}^{\alpha+i\nu} I(s,\mu)\; e^{s\tau}\; ds/s\ , \quad \alpha > 0\ , \tag{26}$$

where

$I^*(s,\mu)$ can be obtained as

$$I^*(s,\mu)/s = [I(0,\mu) - (\mu s)^{-1} Q^{-1}(1/s)\, Q(\mu)$$
$$I(0,\mu)]/\,(s - 1/\mu) \tag{27}$$
$$I^*(s,\mu)/s = [I(0,\mu)/\,(s - 1/2) - Q(\mu)$$
$$H(1/s)A/(s - 1/k)\mu(s - 1/\mu)] \tag{28}$$

The integral of Equation (26) is analytic for s in $(-\infty,-1)^c$, has poles at $s = \pm\, 1/k$, k is real $k > 1$, where $s = 1/\mu$ is not a pole as

$$\lim_{s \longrightarrow 1/\mu} (s - 1/\mu)\ I^{*}\ (s,\mu)\ e^{s\tau}/s \longrightarrow 0 \qquad (29)$$

The contribution fo pole at s = 1/k will give the asymptotic solution of Equation (1) as

$$I(\tau,\mu) \longrightarrow [\ \frac{k}{k-\mu}\ ]\ Q(\mu)\ H(k)\ e^{s/k}\ A \quad \text{when } \tau \longrightarrow \infty \qquad (30)$$

Equation (4b) with Equation (30) gives the matrix A as -

$$A = (1/2)\ [\ \omega\ H^{-1}(K)\ ]\ L_0 \qquad (31)$$

Equation (25) with Equation (31) gives the emergent intensity in the form

$$I(O,z) = (1/2)\ \omega\ L_0\ H^{-1}(k)\ H(z)\ Q(z)\ [\ \frac{k}{k - z}\ ] \qquad (32)$$

4. CONCLUSIONS :

Here we allow the values c (O < c < 1) and ω (O< ω <1) to study the general mixture of Rayleigh and isotropic scattering.

a. When ω = 1 and c (O< c <1) the basic matrix transport equation yields a conservative model for a mixture of Rayleigh and isotropic scattering.

b. When ω (O< ω <1) and c=1 , we obtain the general Rayleigh scattering problem.

c. When c = 1 and w = 1 , the problem yields Chandrasekhar's (1960) Rayleigh scattering model and $Q(\mu)$ reduces to Sekera's (1963) form for factorising the Rayleigh scattering phase matrix (Das,1979c).

d. In this problem there exists some possibilities for future development such as determination of the H-matrix expression and the values of the D(z) matrix on both sides of the cut etc.

e. There exists some possibilities to determine a characteristic function which is an even function having polynomial expression but has a transcendental form.

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