CHAPTER - 5

SOLUTION OF RADIATIVE TRANSFER PROBLEMS IN A FINITE ATMOSPHERE

5.1. Introduction.

Das (1978, 1980) has solved various problems of radiative transfer in finite and semi-infinite atmosphere using a method involving Laplace transform and linear singular operators.

In the present work , the one sided Laplace transform together with the theory of linear singular operators has been applied to solve the transport equation which arises in the problem of a finite atmosphere having ground reflection according to Lambert's Law taking the Planck's function as an exponential function of optical depth (Sec-5.2).

In the theory of radiative transfer for homogeneous plane-parallel stratified finite atmosphere the X- and Y-functions of Chandrasekhar (1960) play a central role . The equations satisfy a system of coupled nonlinear integral equations. Busbridge (1960) has demonstrated the existence of the solution of these coupled non-linear integral equations in terms of a particular solution of an auxiliary equation. Busbridge (1960) has obtained two coupled linear integral equations for $X(z)$ and $Y(z)$ which defined the meromorfic extensions to the complex domain $|Z|$ of the real valued solution of the coupled non-linear equations of X-and Y- functions.

Busbridge (1960) concluded that all solutions of non-linear coupled integral equations for X- and Y- functions are the solutions of the coupled linear integral equations to the extended complex plane but all solutions of the coupled linear integral equations are not solutions of the coupled non-linear integral equations. Mullikin (1964) has proved that all solution of coupled non-linear integral equations are solutions of the coupled linear integral equations but there exists a unique solution of the coupled linear integral equations with some linear constraints. Finally he has obtained the Fredholm equations of X- and Y- functions which are easy for iterative computations. Das (1979) has obtained a pair of Fredholm equations with the Wiener-Hopf technique from the coupled linear integral equations with coupled linear constraints.

In the present work, the time-dependent X- and Y- functions (Biswas and Karanjai, 1990) which gives rise to a pair of the Fredholm equations with the application of the Wiener-Hopf technique has been obtained (Sec-5.3.). These Fredholm equations define time-dependent X-functions in terms of time-dependent Y-functions and vice-versa. These representations are unique with respect to the coupled linear constraints defined by Mullikin (1964).

In the study of time-dependent radiative transfer problems in finite homogeneous plane-parallel atmospheres it is convenient to introduce X- and Y- functions (vide, Chandrasekhar , 1960). These functions satisfy non-linear coupled integral equations. Due to their important role in solving transport problems, it is advantageous to simplify the equations satisfied by them. Lahoz (1989) did this and obtained exact linear and decoupled integral equations satisfied by the time-independent X- and Y- functions.

In the present work, the same method has extended to the time-dependent radiative transfer problem (Sec-5.4). However, the equations obtained , although linear, are singular and not solvable by the standard methods applicable to Fredholm equations instead they have to be solvable by the theory of singular integral equations (vide, Muskhelishvili, 1946).

5.2. Exact Solution of the Equation of Transfer in a Finite Exponential Atmosphere by the Method of Laplace Transform and Linear Singular Operator.

5.21. Basic Equation and Boundary Conditions.

The integro-differential equation for the intensity of radiation $I(\tau,\mu)$, at an optical depth τ for the problem of diffuse reflection and transmission in a finite atmosphere can be written in the form (vide, Das, 1980) as

$$\mu \frac{dI_\nu(\tau,\mu)}{d\tau} = I_\nu(\tau,\mu) - \int_{-1}^{+1} \psi(\mu')I_\nu(\tau,\mu')d\mu' - B_\nu(T) \qquad (5.1)$$

where $I_\nu(\tau,\mu)$ is the intensity in the direction $\cos^{-1}\mu$ at a depth τ, the angle $\cos^{-1}\mu$ is measured from outside drawn normal to the face $\tau = 0$, $\psi(\mu)$ is the characteristic function for non-conservative scattering which satisfies the condition

$$\psi_0 = \int_0^1 \psi(\mu')\,d\mu' \;;\quad \psi(\mu') \text{ is even }, \qquad (5.2)$$

ν is the frequency and $B_\nu(T)$ is the Planck function at any depth (form is same as in equation (1.11)). Then equation (5.1) becomes

$$\mu \frac{dI_\nu(\tau,\mu)}{d\tau} = I_\nu(\tau,\mu) - \int_{-1}^{+1} \psi(\mu') I_\nu(\tau,\mu') d\mu' -$$

$$- \quad (b_o + b_1 e^{-\beta\tau}) \tag{5.3}$$

where for convenience I have omitted the subscript ν.

The boundary conditions associated with the equation (5.3) are

$$I(0, -\mu) = 0, \quad 0 < \mu \leq 1 \tag{5.4}$$

$$I(\tau_o, \mu) = I_g, \quad 0 < \mu \leq 1 \quad , \tau_o > 0 \tag{5.5}$$

where τ_o is the thickness of the finite atmosphere and the bounding face $\tau = \tau_o$ is having ground reflection according to Lambert's law is a constant.

5.22. Integral Equations for Surface Quantities.

Let us define

$$f^*(s,\mu) = s\int_0^{\tau_o} f(\tau,\mu)\ e^{-s\tau}\ d\tau \ , \quad \text{Re } s>0 \tag{5.6}$$

$$f(\tau,\mu) = 0 \ , \text{ when } \quad \tau > \tau_o \tag{5.7}$$

Let us now apply the Laplace transform defined in equation (5.6) to equation (5.7) to obtain the equation satisfying the boundary condition as

$$(\mu s - 1)\ I^{*}(s,\mu) = \mu s I(0,\mu) - \mu s e^{-\tau_{o}s} - S^{*}(s) \qquad (5.8)$$

where

$$S(\tau) = \int_{-1}^{+1} \psi(\mu')I(\tau,\mu')d\mu' + (b_{o} + b_{1}e^{-\beta\tau}) \qquad (5.9)$$

i.e.,

$$S^{*}(s) = \int_{-1}^{+1} \psi(\mu')I^{*}(\tau,\mu')d\mu' + b_{o}(1 - e^{-s\tau o}) +$$

$$+ \frac{sb_{1}}{s+\beta}(1 - e^{-(s+\beta)\tau_{o}}) \qquad (5.10)$$

Let us apply the operator

$$\int_{-1}^{+1} \psi(\mu)d\mu/(\mu s - 1)\ , \qquad (5.11)$$

on both sides of equation (5.8) and I obtain , with equation (5.10)

$$T(1/s)\ S^{*}(s) = \int_{-1}^{+1} \psi(\mu)I(0,\mu)d\mu/(\mu s - 1) -$$

$$- e^{\tau_{o}s} \int_{-1}^{+1} \mu s\psi(\mu)I(0,\mu)d\mu/(\mu s - 1) + b_{o}(1 - e^{-s\tau o}) +$$

$$+ \frac{sb_{1}}{s+\beta}(1 - e^{-(s+\beta)\tau_{o}}) \qquad (5.12)$$

where $$T(1/s) = 1 + \int_{-1}^{+1} \psi(\mu)d\mu/(\mu s - 1) \;, \tag{5.13}$$

Equation (5.8) gives

$$I(0,\mu) - e^{-\tau_o/\mu} I(\tau_o,\mu) = S^*(1/\mu) \;\Rightarrow \tag{5.14}$$

$$\Rightarrow\; I(0,1/s) - e^{-\tau_o s} I(\tau_o,1/s) = S^*(s) \tag{5.15}$$

Equation (5.12) , together with equation (5.14) , gives for complex z , where z = 1/s,

$$[I(0,z) - e^{-\tau_o/z} I(\tau_o,z)]\; T(z) = \int_{-1}^{+1} \mu\psi(\mu)I(0,\mu)d\mu/(\mu - z) -$$

$$- e^{-\tau_o/z} \int_{-1}^{+1} \mu\psi(\mu)I(\tau_o,\mu)d\mu/(\mu - z) + b_o(1 - e^{-\tau_o/z}) +$$

$$+ \frac{b_1}{1 + \beta z}\,(1 - e^{-(1/z+\beta)\tau_o}) \tag{5.16}$$

Let us put $\alpha_o = \beta^{-1}$, then equation (16) becomes

$$[I(0,z) - e^{-\tau_o/z} I(\tau_o,z)]\; T(z) = \int_{-1}^{+1} \mu\psi(\mu)I(0,\mu)d\mu/(\mu - z) -$$

$$- e^{-\tau_o/z} \int_{-1}^{+1} \mu\psi(\mu) I(\tau_o,\mu) d\mu/(\mu - z) + b_o(1 - e^{-\tau_o/z}) +$$

$$+ \frac{b_1\alpha_o}{z + \alpha_o} (1 - e^{-(1/z + 1/\alpha_o)\tau_o}) \qquad (5.17)$$

Let us put $z = -z$ in equation (5.17) and multiply the resulting equation by $e^{-\tau_o/z}$ on both sides to obtain , for complex z,

$$[I(\tau_o,-z) - e^{-\tau_o/z} I(0,-z)]T(z) = \int_{-1}^{+1} \mu\psi(\mu) I(\tau_o,\mu) d\mu/(\mu + z) -$$

$$- e^{-\tau_o/z} \int_{-1}^{+1} \mu\psi(\mu) I(0,\mu) d\mu/(\mu + z) + b_o(1 - e^{-\tau_o/z}) -$$

$$- \frac{b_1\alpha_o}{\alpha_o - z} (e^{-\tau_o/z} - e^{-\tau_o/\alpha_o}) \qquad (5.18)$$

Equations (5.17) and (5.18) are the linear integral equations for the surface quantities under consideration.

5.23. Linear Singular Integral Equations.

Equation (5.17) and (5.18) are the equations defined for complex z , where does not lie between -1 and 1 . When z lies between -1 and 1 , equation (5.17) and (5.18) will give the linear singular integral equations by the application of Plemelj's formulae (vide, Mushkelishvili,1946) with boundary conditions (4.4) and (5.5) as

$$[I(0,z) - e^{-\tau_o/z} I_g]\, T_o(z) = P\int_o^1 \mu\psi(\mu) I(0,\mu)d\mu/(\mu - z) -$$

$$- e^{-\tau_o/z} \int_o^1 \mu\psi(\mu) I(\tau_o,-\mu)d\mu/(\mu + z) -$$

$$- e^{-\tau_o/z} P\int_o^1 \mu\psi(\mu) I_g d\mu/(\mu - z) + + b_o(1 - e^{-\tau_o/z}) +$$

$$+ \frac{b_1\alpha_o}{z + \alpha_o}(1 - e^{-(1/z + 1/\alpha_o)\tau_o}) \qquad (5.19)$$

and $$I(\tau_o,-z)\, T_o(z) = P\int_o^1 \mu\psi(\mu) I(\tau_o,-\mu)d\mu/(\mu - z) -$$

$$- e^{-\tau_o/z} \int_o^1 \mu\psi(\mu) I(0,\mu)d\mu/(\mu + z) +$$

$$+ e^{-\tau_o/z}\int_o^1 \mu\psi(\mu) I_g d\mu/(\mu + z) + b_o(1 - e^{-\tau_o/z}) -$$

$$- \frac{b_1\alpha_o}{\alpha_o - z}(e^{\tau_o/z} - e^{-\tau_o/\alpha_o}) \qquad (5.20)$$

where

$$T_o(z) = 1 - 2z^2 \int_0^1 d\mu\,[\psi(\mu) - \psi(z)] / (z^2 - \mu^2) -$$

$$- 2z^2\psi(z)\, P\int_0^1 d\mu/(z^2 - \mu^2) \qquad (5.21)$$

in which P denotes the Cauchy principal value of the integral.

Equations (5.19) and (5.20) are the linear singular integral equations from which I shall determine the surface quantities $I(0,z)$ and $I(\tau_o, -z)$ by the application of the theory of linear singular operators.

5.24. Theory of Linear Singular Operators.

Following Das [1978,1980] I can write the following theorems.

THEOREM 1.

The linear integral equations for $z \in (0,1)$,

$$L_+[R(z,-x_o)] = l(z,-x_o) \;, \qquad (5.22)$$

$$L_-[R(z,-x_o)] = m(z,-x_o) \;, \qquad (5.23)$$

where

$$L_+ \ [f(z,-x_o) = f(\mu,-x_o)T_o(z) - P\int_0^1 \mu\psi(\mu)f(\mu,-x_o)d\mu/(\mu - z) +$$

$$+ e^{-\tau_o/z}\int_0^1 \mu\psi(\mu)f(\mu,-x_o)d\mu/(\mu + z) \qquad (5.24)$$

$$L_- \ [f(z,-x_o) = f(\mu,-x_o)T_o(z) - P\int_0^1 \mu\psi(\mu)f(\mu,-x_o)d\mu/(\mu - z) -$$

$$+ e^{-\tau_o/z}\int_0^1 \mu\psi(\mu)f(\mu,-x_o)d\mu/(\mu + z) \qquad (5.25)$$

where

$$l(z,-x_o) = \frac{x_o}{z + x_o}[1 - e^{-(1/z + 1/x_o)\tau_o}] +$$

$$+ \frac{x_o}{z - x_o}[e^{-\tau_o/z} - e^{-\tau_o/x_o}] \qquad (5.26)$$

$$m(z,-x_o) = \frac{x_o}{z + x_o}[1 - e^{-(1/z + 1/x_o)\tau_o}] +$$

$$- \frac{x_o}{z - x_o}[e^{-\tau_o/z} - e^{-\tau_o/x_o}] \qquad (5.27)$$

admit of solutions of the form

$$R(z,-x_o) = S(z,-x_o) + T(z,-x_o) \qquad (5.28)$$

$$Q(z,-x_o) = S(z,-x_o) + T(z,-x_o) \qquad (5.29)$$

where

$$S(z,-x_o) = x_o[X(z)X(x_o) - Y(z)\ Y(x_o)]/(z + x_o) \qquad (5.30)$$

$$T(z,-x_o) = x_o[X(z)Y(x_o) - Y(z)\ X(x_o)]/(x_o - z) \qquad (5.31)$$

With constraints on X(z) and Y(z) as

(i) when ψ_o <1/2

$$1 = K\int_0^1 X(\mu)\psi(\mu)d\mu/(K - \mu) +$$

$$+ e^{-\tau_o/K} K\int_0^1 Y(\mu)\psi(\mu)d\mu/(K + \mu) \qquad (5.32)$$

$$e^{-\tau_o/K} = K\int_0^1 Y(\mu)\psi(\mu)d\mu/(K - \mu) +$$

$$+ e^{-\tau_o/K} K\int_0^1 X(\mu)\psi(\mu)d\mu/(K + \mu) \qquad (5.33)$$

(ii) when ψ_o = 1/2

$$1 = \int_0^1 [X(\mu) + Y(\mu)]\psi(\mu)d\mu \qquad (5.34)$$

$$+ \tau_o\int_0^1 Y(\mu)\psi(\mu)d\mu = \int_0^1 [X(\mu) - Y(\mu)]\mu\psi(\mu)d\mu \qquad (5.35)$$

and K is the positive root of the function T(z), when ψ_o <1/2 , defined by

$$T(z) = 1 + \int_{-1}^{+1} z\ \psi(\mu)d\mu/(\mu - z) \qquad (5.36)$$

and where $[X(\mu) - Y(\mu)]$ and $[X(\mu) + Y(\mu)]$ are the respective solutions of

$$L_+ [f(z)] = (1 - e^{-\tau_o/z}) \left\{ 1 - \int_0^1 f(\mu)\psi(\mu)d\mu \right\} \qquad (5.37)$$

$$L_- [f(z)] = (1 + e^{-\tau_o/z}) \left\{ 1 - \int_0^1 f(\mu)\psi(\mu)d\mu \right\} \qquad (5.38)$$

THEOREM 2.

As the operators L_+ and L_- are linear for $z \varepsilon (0,1)$, then for any constant C, I have

$$L_{\pm} (CF(z,-x_o)) = CL_{\pm} (F(z,-x_o) \qquad (5.39)$$

and

$$L_{\pm}(zf(z)) = zL_{\pm}(f(z) - (1 \mp e^{-\tau_o/z})\int_0^1 \mu\psi(\mu)f(\mu)d\mu \qquad (5.40)$$

THEOREM 3.

If $R(z,-x_o)$ and $Q(z,-x_o)$ are the solutions of

$$L_+[R(z,-x_o)] = l(z,-x_o) \ , \qquad (5.41)$$

$$L_-[R(z,-x_o)] = m(z,-x_o) \ , \qquad (5.42)$$

then

$$L_+[M(z)] = \int_0^1 \psi(-x_o)\ l(z,-x_o)\ dx_o\ , \qquad (5.43)$$

$$L_-[N(z)] = \int_0^1 \psi(-x_o)\ m(z,-x_o)\ dx_o\ , \qquad (5.44)$$

admit the solution of

$$M(z) = \int_0^1 \psi(-x_o)\ R(z,-x_o)\ dx_o\ , \qquad (5.45)$$

$$N(z) = \int_0^1 \psi(-x_o)\ Q(z,-x_o)\ dx_o\ , \qquad (5.46)$$

5.25. Solution for Surface Quantities.

Linear singular integral equations (5.19) and (5.20) are the required integral equations from which I will have to determine $I(0,\mu)$ and $I(\tau_o,-z)$, the quantities under consideration, by the application of the theory of linear singular operators indicated in section 5.2.4. Equations (5.19) and (5.20) on addition and after some rearrangement give

$$L_+[I(0,z) + I(\tau_o, -z) - e^{-\tau_o/z} I_g] = 2b_o(1 - e^{-\tau_o/z}) +$$

$$+ b_1 l(z , -\alpha_o) + I_g \int_o^1 \psi(\mu)\ l(z , -\mu)d\mu \qquad (5.47)$$

Equations (5.19) and (5.20) on subtraction and after manipulation give

$$L_-[I(0,z) - I(\tau_o, -z) - e^{-\tau_o/z} I_g] =$$

$$= b_1 m(z , -\alpha_o) + I_g \int_o^1 \psi(\mu)\ m(z , -\mu)d\mu \qquad (5.48)$$

where $l(z , -\mu)$ and $m(z , -\mu)$ are given by equations (5.26) and (5.27). Equations (5.47) and (5.48) with Theorems 1,2 and 3 of section 5.2.4. will give us the desired quantities $I(0,z)$ and $I(\tau_o,-z)$. The solution of equation (5.47) is given by

$$[I(0,z) + I(\tau_o , -z) - e^{-\tau_o/z} I_g] = \frac{2b_o}{1 - G_o} [X(z) - Y(z)] +$$

$$+ b_1 R(z , -\alpha_o) + I_g \int_o^1 \psi(\mu)\ R(z , -\mu)d\mu \qquad (5.49)$$

where $$G_o = \int_o^1 [X(\mu) - Y(\mu)]\psi(\mu)d\mu \qquad (5.50)$$

The solution of equation (5.48) is given by

$$[I(0,z) + I(\tau_o, -z) - e^{-\tau_o/z} I_g] =$$

$$= b_1 Q(z, -\alpha_o) - I_g \int_0^1 \psi(\mu)\, Q(z, -\mu) d\mu \qquad (5.51)$$

Equation (5.50) and (5.51) on addition give $I(0,z)$ and equations (5.47) and (5.51) on subtraction give $I(\tau_o, -z)$ as

$$I(0,z) = I_g\, e^{-\tau_o/z} + I_g \int_0^1 \psi(\mu)\, T(z, -\mu) d\mu +$$

$$+ \frac{b_o}{1 - G_o} [X(z) - Y(z)) + b_1 S(z, -\mu) \qquad (5.52)$$

and

$$I(\tau_o, -z) = I_g \int_0^1 \psi(\mu)\, S(z, -\mu) d\mu +$$

$$+ \frac{b_o}{1 - G_o} [X(z) - Y(z)] + b_1 T(z, -\alpha_o) \qquad (5.53)$$

where $S(z, -\mu)$ and $T(z, -\mu)$ are given by equations (5.30) and (5.31).

5.3. The Time-Dependent X- and Y- Functions.

5.31. Basic Equation.

The coupled nonlinear integral equations satisfied by the time-dependent X- and Y- function (vide, Biswas and Karanjai, 1990) are of the form

$$X(\tau_1,\mu,s) = 1 + \frac{\omega}{2Q}\mu \times$$

$$\times \int_0^1 \frac{X(\tau_1,\mu,s)X(\tau_1,x,s)-Y(\tau_1,\mu,s)Y(\tau_1,x,s)}{\mu + x}\, dx \qquad (5.54)$$

$$0 \le \mu \le 1 \quad .$$

$$Y(\tau_1,\mu,s) = \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{\omega}{2Q}\mu \times$$

$$\times \int_0^1 \frac{Y(\tau_1,\mu,s)X(\tau_1,x,s)-X(\tau_1,\mu,s)Y(\tau_1,x,s)}{\mu - x}\, dx \qquad (5.55)$$

$$0 \le \mu \le 1 \quad .$$

where $$Q = 1 + s/c \qquad (5.56)$$

τ_1 is the thickness of the atmosphere ; c, the velocity of light ; and s, Laplace transform parameter.

Following Chandrasekhar (1960) equations (5.54) and (5.55) can be written as

$$X(\tau_1,\mu,s) = 1 + \frac{\mu}{Q} \times$$

$$\times \int_0^1 \frac{X(\tau_1,\mu,s)X(\tau_1,x,s)-Y(\tau_1,\mu,s)Y(\tau_1,x,s)}{\mu + x}\, \psi(x)dx \qquad (5.57)$$

$$0 \le \mu \le 1 \quad .$$

$$Y(\tau_1,\mu,s) = \exp\left(-\frac{\tau_1 Q}{\mu}\right) + \frac{\mu}{Q} \times$$

$$\times \int_0^1 \frac{Y(\tau_1,\mu,s)X(\tau_1,x,s)-X(\tau_1,\mu,s)Y(\tau_1,x,s)}{\mu - x}\, \psi(\mu)dx \qquad (5.58)$$

$$0 \le \mu \le 1 \quad .$$

where $\psi(x)$, the characteristic function satisfying the Hölder condition on $0 \le x \le 1$, is non-negative and satisfies the condition

$$\psi_o = \int_0^1 \psi(x)\, dx \le 1/2 \qquad (5.59)$$

The atmosphere is said to be conservative when $\psi_o = 1/2$ and non-negative otherwise .

The dispersion function $T(z,s)$, $z \in (-1,1)^c$ can be defined by

$$T(z,s) = 1 - \frac{2z^2}{Q} \int_0^1 \frac{\psi(x)dx}{z^2 - x^2} \qquad (5.60)$$

and

$$T(z,s) = (H(z,s)H(-z,s))^{-1} \qquad (5.61)$$

where

$$H(z,s) = 1 + zH(z,s) \int_0^1 \frac{\psi(x)H(x,s)dx}{x + z} \qquad (5.62)$$

According to Busbridge (1960) , the only zeros of T(z,s) are

at $z = \pm K$, $K>1$, when $\psi_o < 1/2$ and when $\psi_o = 1/2$.

Following Busbridge (1960), Dasgupta (1977), and Das (1978) H(z,s) is meromorfic on $(-1,0)^c$ having a simple pole at $z = -K$ and tend to 1 as $z \longrightarrow 0_+$. It can be represented by

$$H(z,s) = \frac{A_o + H_o z}{K + z} - \int_0^1 \frac{P(x,s)dx}{x + z} \qquad (5.63)$$

$$K > 1, \quad \psi_o < 1/2$$

$$H(z,s) = h_1 z + h_o - \int_0^1 \frac{P(x,s)dx}{x + z} \qquad (5.64)$$

$$, K \longrightarrow \alpha \text{ , } \psi_o = 1/2$$

where

$$A_o = (1 + P_{-1})K, \qquad P_{-1} = \int_0^1 P(x,s)\ dx/x, \qquad (5.65)$$

$$H_o = \left\{ 1 - 2 \int_0^1 \psi(x)\ dx \right\}^{-1/2} \qquad (5.66)$$

$$h_1 = \left\{ 2 \int_0^1 x^2 \psi(x)\ dx \right\}^{-1/2} \qquad (5.67)$$

$$h_o = (1 + P_{-1}) \qquad (5.68)$$

$$P(x,s) = \phi(x,s)/H(x,s) \tag{5.69}$$

$$\phi(x,s) = x\psi(x)/(T_o^2(x,s) + \pi^2 x^2 \psi^2(x)) \tag{5.70}$$

$$T_o(x,s) = 1 - \frac{2x^2}{Q}\int_0^1 \frac{\psi(t) - \psi(x)}{x^2 - t^2} -$$

$$- \frac{x\psi(x)}{Q}\log((1 + x)/(1 - x)) \tag{5.71}$$

where $\phi(x,s)$ is non-negative and continuous on (0,1), tends to $\psi(0)$, as $x \longrightarrow 0_+$, tends to $O((\log(1 - x)^{-2})$ when $x \longrightarrow 1_-$, and $1/H(z,s)$ is regular on $(-1,0)^c$.

Following Busbridge (1960) and Mullikin (1964) I find that the coupled linear equations satisfied by X(z,s) and Y(z,s) for $z \varepsilon (-1,1)^c$ are of the form

$$X(z,s)T(z,s) = 1 + zU(X)(z,s) -$$

$$- \exp(-(\tau_1/z)Q)V(Y)(z,s) \tag{5.72}$$

$$Y(z,s)T(z,s) = \exp(-(\tau_1/z)Q) + zU(Y)(z,s) -$$

$$- z\exp(-(\tau_1/z)Q)V(Y)(z,s) \tag{5.73}$$

with constraints for $\psi_o < 1/2$,

$$0 = 1 + KU(X)(K,s) - K\exp(-(\tau_1/K)QV(Y)(K,s) \tag{5.74}$$

$$0 = (\exp(-(\tau_1/K)Q) + KU(Y)(K,s)) -$$

$$- K\exp(-(\tau_1/K)V(X)(K,s) \tag{5.75}$$

for $\psi_o = 1/2$

$$1 = \int_0^1 \psi(x)(X(x,s) + Y(x,s)\ dx \qquad (5.76)$$

$$\tau_1 \int_0^1 Y(x,s)\psi(x)\ dx = \int_0^1 X\psi(x)(X(x,s) - Y(x,s))dx \qquad (5.77)$$

The other conditions for which $X(z,s)$ and $Y(x,s)$ hold are

$$X(s,s) \longrightarrow H(z,s) \quad \text{when} \quad \tau_1 \longrightarrow \alpha \qquad (5.78)$$

$$Y(z,s) \longrightarrow 0 \text{ when } \quad \tau_1 \longrightarrow \alpha \qquad (5.79)$$

where for $M = X$ or Y

$$V(M)(z,s) = \int_0^1 \psi(x)M(x,s)dx/(x + z) \qquad (5.80)$$

is analytic for $z \in (-1,1)$ bounded at the origin $O(z^{-1})$ when $z \longrightarrow \alpha$ and

$$U(M)(z,s) = \int_0^1 \psi(x)M(x,s)dx/(x - z) \qquad (5.81)$$

is analytic for $z \in (0,1)^c$, bounded at the origin $O(z^{-1})$ when $z \longrightarrow \alpha$.

5.32. Fredholm equations.

Equations (5.72)and (5.73) with equation (5.61) can be written in the form

$$X(z,s)/H(z,s) = H(-z,s)(1 + zU(X)(z,s) -$$

$$- \exp(-(\tau_1/z)Q)H(-z,s)V(Y)(z,s) \qquad (5.82)$$

$$Y(z,s)/H(z,s) = H(-z,s)(\exp(-(\tau_1/z)Q) + zU(Y)(z,s) -$$

$$- z\exp(-(\tau_1/z)Q)H(-z,s)V(Y)(z,s) \qquad (5.83)$$

I shall assume that $X(z,s)$ and $Y(z,s)$ are regular for Re $z>0$ and bounded at the origin. Equation (5.63) gives

$$H(-z,s) = \frac{A_o - H_o z}{K - z} - \int_o^1 \frac{P(x,s)dx}{x - z} \qquad (5.84)$$

for $\psi_o < 1/2$

Hence

$$V(M)(z,s)\int_o^1 \frac{P(x,s)}{x - z}dx = D(M,P_o)(z,s) + D(P,M_o)(z,s) \qquad (5.85)$$

where $$D(M,P_o)(z,s) = \int_o^1 \frac{\psi(x)M(x,s)P_o(x,s)dx}{x + z} \qquad (5.86)$$

and

$$D(P,M_o)(z,s) = \int_o^1 \frac{\psi(x)P(x,s)M_o(x,s)dx}{x - z} \qquad (5.87)$$

where $$P_o(z,s) = \int_o^1 \frac{P(x,s)}{x + z}dx \qquad (5.88)$$

is regular on $(-1,0)^c$, bounded at the origin and $O(z^{-1})$ when $z \longrightarrow \alpha$ and $D(M,P_o)(z,s)$ is regular for z on $(-1,0)^c$, bounded at the origin and $O(z^{-1})$ when $z \longrightarrow \alpha$. and $D(P,M_o)(z,s)$ is regular for z, on $(0,1)^c$ bounded at the origin , and $O(z^{-1})$ when $z \longrightarrow \alpha$.

Hence , equation (5.82) and (5.83) can for $\psi_o < 1/2$ be written in the form

$$X(z,s)/H(z,s) + \exp(-(\tau_1/z)Q)\left[\frac{A_o - H_o z}{K - z} V(Y)(z,s) - D(Y,P_o)(z,s)\right] =$$

$$= H(-z,s)(1 + zU(X)(z,s) + \exp(-(\tau_1/z)Q)(P,Y_o)(z,s)) \quad (5.89)$$

$$Y(z,s)/H(z,s) + z\exp(-(\tau_1/z)Q)\left[\frac{A_o - H_o z}{K - z} V(X)(z,s) - \right.$$

$$\left. - D(X,P_o)(z,s)\right] = H(-z,s)(\exp(-(\tau_1/z)z) + zU(Y)(z,s) +$$

$$+ z\exp(-(\tau_1/z)Q)D(P,X_o)(z,s)) \quad (5.90)$$

The left-hand side of equation (5.89) and (5.90) are regular for Re z>0 and bounded at the origin; the right-hand side of equations (5.89) and (5.90) are regular for z, on $(0,1)^c$, bounded at the origin and tends to constants , say A and B, respectively, when $z \longrightarrow \alpha$.

Hence, by modified form of Liouville's theorem I have

$$X(z,s) = H(z,s)\left[z \exp(-(\tau_1/z)Q\left\{D(Y,P_o(z,s) - \right.\right.$$

$$\left.\left. - \frac{A_o - H_o z}{K - z} V(X)(z,s)\right\} + A \right], \tag{5.91}$$

$$Y(z,s) = H(z,s)\left[z \exp(-(\tau_1/z)Q\left\{D(X,P_o(z,s) - \right.\right.$$

$$\left.\left. - \frac{A_o - H_o z}{K - z} V(X)(z,s)\right\} + B \right], \tag{5.92}$$

Equations (5.91) and (5.92) together with Equations (5.78) and (5.79) gives

$$A = 1, \quad B = 0 \tag{5.93}$$

Hence, for $\psi_o = 1/2$, the expression of X(z,s) and Y(z,s) are

$$X(z,s) = H(z,s)[1 + z \exp(-(\tau_1/z)Q)(D(Y,P_o)(z,s) -$$

$$- (- h_1 z + h_o) V(Y)(z,s))] \tag{5.94}$$

$$Y(z,s) = H(z,s)\ z \exp(-(\tau_1/z)Q)(D(Y,P_o)(z,s)) -$$

$$- (- h_1 z + h_o) V(Y)(z,s)) \tag{5.95}$$

Hence, following Mullikin (1964) equations (5.91) and (5.92) together with equations (5.74) and (5.75) give unique representation of time-dependent X- and Y- functions for ψ_o <1/2 and equations (5.94) and (5.95) together with equations (5.76) and (5.77) give unique representations of X- and Y- functions for $\psi_o = 1/2$.

5.4. *An* Exact *Linearization and Decoupling of the Integral Equations Satisfied by Time-Dependent X- and Y-Functions.*

5.41. Analysis.

The integral equations incorporating the various invariances of the time-dependent problem of diffuse reflection and transmission can be reduced to one or more pairs of integral equations of the following form (vide, Biswas and Karanjai, 1990)

$$X(\mu,s) = 1 + \frac{\omega}{2}\,\frac{\mu}{\theta}\int_0^1 \frac{X(\mu,s)X(\mu',s) - Y(\mu,s)Y(\mu',s)}{\mu + \mu'}\, d\mu' \qquad (5.96)$$

$$Y(\mu,s) = \exp[(-\tau_1/\mu)] + \frac{\omega}{2}\,\frac{\mu}{\theta}\int_0^1 \frac{Y(\mu,s)X(\mu',s) - X(\mu,s)Y(\mu',s)}{\mu - \mu'}\, d\mu' \qquad (5.97)$$

Following Chandrasekhar (1960) , I can write the above equations in the form

$X(\mu,s) = 1 +$

$$+ \frac{\mu}{Q}\int_0^1 \frac{X(\mu,s)X(\mu',s) - Y(\mu,s)Y(\mu',s)}{\mu + \mu'}\,\psi(\mu')\,d\mu' \qquad (5.98)$$

$$Y(\mu,s) = \exp[(-\tau_1/\mu)] +$$

$$+ \frac{\mu}{Q}\int_0^1 \frac{Y(\mu,s)X(\mu',s) - X(\mu,s)Y(\mu',s)}{\mu - \mu'}\,\psi(\mu')\,d\mu' \qquad (5.99)$$

where τ_1 is the optical thickness of the atmosphere and $Q = 1 + s/c$, where c is the velocity of light, s is the Laplace invariant of the time variable and the characteristic function $\psi(\mu)$ is an even polynomial in μ satisfying

$$\psi_o = \int_0^1 \psi(\mu)\,d\mu \le 1/2 \qquad (5.100)$$

where $\psi_o = 1/2$ holds, $\psi(\mu)$ is said to be conservative ; and non-conservative otherwise.

Clearly , equations (5.98) and (5.99) are non-linear and coupled. These equations have been linearized in an exact manner (vide, Mullikin, 1964) . The results are

$$X(\mu,s)K(\mu,s) = 1 + \frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu - \mu'}\,\psi(\mu')\,d\mu' -$$

$$- \exp[(-\tau_1/\mu)Q\ \frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu + \mu'}\,\psi(\mu')\,d\mu' \qquad (5.101)$$

and

$$Y(\mu,s)K(\mu,s) = \exp[(-\tau_1/\mu)Q] + \frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu-\mu'}\psi(\mu')\,d\mu' -$$

$$- \exp[(-\tau_1/\mu)Q]\frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu+\mu'}\psi(\mu')\,d\mu' \qquad (5.102)$$

where $K(\mu,s)$ is defined by

$$K(\mu,s) \equiv 1 - \frac{\mu}{Q}\int_0^1 \left[\frac{1}{\mu+\mu'} - \frac{1}{\mu'-\mu}\right]\psi(\mu')\,d\mu' \qquad (5.103)$$

I now proceed to decouple equations (5.101) and (5.102) in an exact manner (vide, Lahoz, 1989). I introduce the following singular integral equation, which is linear in $1/T(\mu,s)$:

$$\frac{1}{T(\mu,s)} = 1 - \frac{\mu}{Q}\int_0^1 \left[\frac{1}{T(\mu',s)K(\mu',s)}\right]\frac{\psi(\mu')}{\mu'-\mu}\,d\mu' \qquad (5.104)$$

which in principle , is solvable for $T(\mu,s)$ as $\psi(\mu)$ and $K(\mu,s)$ are known functions.

Next, I multiply equation (5.101) by

$$\frac{(\mu'/Q)\psi(\mu)}{T(\mu,s)\;K(\mu,s)(\mu'-\mu)}, \qquad (5.105)$$

which I assume is well defined in $\mu \in [0,1]^c$ and integrate with respect to μ from 0 to 1 to obtain

$$\frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu+\mu'}\,\psi(\mu')\,d\mu' = 1 -$$

$$- T(-\mu,s)\left[1 - P(\mu,s)\,\frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu'-\mu}\,\psi(\mu')\,d\mu' +\right.$$

$$\left. + \frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu'-\mu}\,P(\mu',s)\,\psi(\mu')\,d\mu'\right] \qquad (5.106)$$

where I have used equation (5.104) and defined the function $P(\mu,s)$ (in principle known) by

$$P(\mu,s) \equiv \frac{\mu}{Q}\int_0^1 \frac{1}{\mu'+\mu}\,\frac{\exp(-\tau_1/\mu)}{T(\mu',s)K(\mu',s)}\,\psi(\mu')\,d\mu' \qquad (5.107)$$

Substituting equation (5.106) in equation (5.102) I get the decoupled equation for $Y(\mu,s)$ as follows:

$$Y(\mu,s)K(\mu,s) = T(-\mu,s)\,\exp[(-\tau_1/\mu)Q\,] + T(-\mu,s)\,P(\mu,s)$$

$$[1 - \exp[(-\tau_1/\mu)Q]]\,\frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu'-\mu}\,\psi(\mu')\,d\mu' \;+$$

$$T(-\mu,s)\,\exp[(-\tau_1/\mu)Q\,]\frac{\mu}{Q}\int_0^1 \frac{Y(\mu',s)}{\mu'-\mu}\times$$

$$\times\,\psi(\mu',s)\,\psi(\mu')\,d\mu' \qquad (5.108)$$

A similar analysis yields the decoupled equation for $X(\mu,s)$:

$$X(\mu,s)K(\mu,s) = [1 - T(-\mu,s)\,P(\mu,s)\exp[(-\tau_1/\mu)Q\,]] \times$$

$$\times \left[\, 1 + \frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu'-\mu}\,\psi(\mu')\,d\mu' \,\right] +$$

$$T(-\mu,s)\,\exp[(-\tau_1/\mu)Q\,]\frac{\mu}{Q}\int_0^1 \frac{X(\mu',s)}{\mu'-\mu} \times$$

$$\times\,\psi(\mu',s)\quad d\mu' \qquad (5.109)$$