✱ *CHAPTER IV*

EXACT RESULTS FOR HYPERFINE INTERACTION

✱ A major part of the contents of this chapter has already been reported in Refs.(91) and (92).

IV.1. Introduction :

As pointed out in the earlier chapter, the spin-dependent potentials are not yet known accurately. The recently discovered hyperfine splitting of P-states cannot be accomodated comfortably within the framework of the standard Breit-Fermi form. The uncertainty regarding the long-range confining part of the potential is one of the factors contributing to the complexity of the problem. However, the S-states of the quarkonia have only hyperfine splitting and therefore, a careful study of η_b and η_c states can provide some useful information about the spin-dependent potential $V_4(r)$ and hence about $V_2(r)$, if Breit-Fermi form [Eqs.(3.1,3.3-3.6)] is accepted. It may be mentioned that the standard approach leads to highly singular terms, like $1/r^3$ and $\delta^3(r)$, for the hyperfine interactions. As has been pointed out by Bhaduri *et al.*,[93] a treatment of such singular potentials is not reliable as the hamiltonian becomes unbounded. The conventional approach has been to ignore this difficulty and treat the singular terms perturbatively. Some attempts have, of course, been made to recast the spin-dependent potentials in less singular form. Introducing an additional parameter, some authors[94] used a cut-off to reduce the singularity. Ono and Schöberl[95] replaced $\delta^3(r)$ by a short-ranged function and Gupta[96] obtained a form of the $Q\bar{Q}$ potential which is not more singular than $1/r^2$. Obviously, the problem needs further investigation. Apart from the hyperfine splitting, the decay widths of the $^3S_1$ and $^1S_0$ states may provide some essential information. Some experimental results on these decay widths are already available. Model independent or exact results or bounds on the wave-functions at the origin, even if weak, will be very useful in this context. We present in this chapter some general results which are valid for a large class of $Q\bar{Q}$ potentials with a generally expected radial dependence.

The presentation in this chapter is as follows. In section IV.2, some inequalities for the wave-function at the origin, valid for a general class of $Q\bar{Q}$ potential have been obtained. In section IV.3, we make use of the $Q\bar{Q}$ potential obtained by Gupta which exhibits an explicit mass dependence and obtain some exact results for the hyperfine splittings of the S-states. These are used to obtain limits on the decay width of the η_b states. The final section gives our conclusions.

IV.2. Some inequalities for S-state wave-functions :

For a class of $Q\bar{Q}$ potentials, it is possible to prove the inequality,

$$\psi_s(0) > \psi_t(0) \quad , \tag{4.1}$$

where $\psi_t(r)$ and $\psi_s(r)$ denote the triplet and singlet radial wave-functions. We write $\psi_t(r) = u_t(r)/r$ and $\psi_s(r) = u_s(r)/r$ where $u_t(r)$ and $u_s(r)$, chosen real and positive, satisfy the

Schrödinger equations

$$\frac{d^2u_t}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_t - V_o(r) - \frac{1}{4}V_\sigma(r)\right]u_t = 0 \qquad (4.2)$$

and

$$\frac{d^2u_s}{dr^2} + \frac{2\mu}{\hbar^2}\left[E_s - V_o(r) + \frac{3}{4}V_\sigma(r)\right]u_s = 0 \quad . \qquad (4.3)$$

We assume that the spin-independent potential $V_o(r)$ is funnel-shaped with $dV_o/dr > 0$ and $d^2V_o/dr^2 < 0$ and that the hyperfine potential $V_\sigma(r)$ decreases monotonically, $dV_\sigma/dr < 0$. While the first assumption is generally accepted, the second is reasonable or at least there is no known objection against it. Note that $V_\sigma(r)$ is short-ranged and in the Breit-Fermi form, $V_\sigma = \nabla^2 V_v$, V_v being the vector exchange potential. We, however, ignore any $\delta^3(r)$ type term and assume that $V_\sigma(r)$ can be given in terms of a short-ranged continuous function.

To derive the inequality we follow the steps given in Ref. (97) which generalise the techniques developed by Grosse and Martin.[98] The steps are outlined below :

a) We know that at a large distance, there is no spin-spin interaction, $V_\sigma(r) \to 0$ as $r \to \infty$. From the large distance behavior of the Eqs. (4.2) and (4.3), we see that

$$u_t(r) > u_s(r) \quad \text{as} \quad r \to \infty \text{ , since } E_t > E_s \text{ ,}$$

as given by experimental results for both $b\bar{b}$ and $c\bar{c}$ states.

b) We note that the Wronskian

$$I(r) = \left(u_s \frac{du_t}{dr} - u_t \frac{du_s}{dr} \right)(r)$$

$$= \frac{2\mu}{\hbar^2} \int_0^r u_t u_s \, [\, V_\sigma(r) - (E_t - E_s) \,]dr \qquad (4.4)$$

is also given by

$$I(r) = - \frac{2\mu}{\hbar^2} \int_r^\infty u_t u_s \, [\, V_\sigma(r) - (E_t - E_s) \,] \, dr \qquad (4.5)$$

as $I(\infty) = 0$. Note that for $r \sim 0$, $I(r) > 0$ and for $r \to \infty$, $I(r) > 0$.

c) We note that since V_σ is monotonic, the integrand has only one zero, say at $r = r_o$. Hence for $r < r_o$, we use Eq. (4.4) and for $r > r_o$, we use Eq. (4.5) to show that $I(r) > 0$ for all r. One can now prove that $u_t - u_s$ has only one zero, say at $r = r_1$. From (4.4), we get

$$u_s(r_1) \, [\, u_t'(r_1) - u_s'(r_1) \,] > 0 \,,$$

and since $u > 0$, we see that $u_t - u_s$ can vanish only once.

d) Since $u_t(r) > u_s(r)$ as $r \to \infty$ and $u_t - u_s$ vanishes only once, we see that $u_s > u_t$ near the origin.

e) We now consider the expression

$$|\psi_t(0)|^2 - |\psi_s(0)|^2 = \int_0^\infty (u_t^2 - u_s^2)\frac{dV_o}{dr}dr + \frac{1}{4}\int_0^\infty (3u_s^2 + u_t^2)\frac{dV_\sigma}{dr}dr$$

$$< \int_0^\infty (u_t^2 - u_s^2)\frac{dV_o}{dr}dr \ , \quad \text{since} \quad \frac{dV_o}{dr} < 0 \ .$$

Since both u_t and u_s are normalised , we have

$$|\psi_t(0)|^2 - |\psi_s(0)|^2 < \int_0^\infty (u_t^2 - u_s^2) \left[\frac{dV_o}{dr} - \frac{dV_o}{dr}\Big|_{r=r_1} \right] dr$$

$$< 0 \ , \qquad \text{since} \quad \frac{d^2V_o}{dr^2} < 0 \ . \tag{4.6}$$

Thus the singlet wave-function is larger than the triplet wave-function at the origin, although in perturbative calculations, one takes $\psi_t(r) = \psi_s(r)$.

The inequality (4.6) may be converted into useful inequalities for the decay widths of the η_b and η_c states. We first note some useful QCD relations involving wave-functions at the origin of the vector meson v :

$$\Gamma(\ v \rightarrow \mu^+\mu^-\ ) = \frac{16\pi}{M_v^2}\ e_Q^2\ \alpha^2\ |\psi_t(0)|^2 \ , \tag{4.7}$$

$$\Gamma(\ v \rightarrow 3g\ ) = \frac{160}{81M_v^2}\ \alpha_s^3\ (\pi^2 - 9)\ |\psi_t(0)|^2 \ , \tag{4.8}$$

$$\Gamma(\ v \rightarrow 2g\gamma\ ) = \frac{128}{9}\ \alpha\ \frac{e_Q^2}{M_v^2}\ \alpha_s^2\ (\pi^2 - 9)\ |\psi_t(0)|^2 \ , \tag{4.9}$$

$$\Gamma(\ \eta_Q \rightarrow 2g\ ) = \frac{32\pi}{3M^2(\eta_Q)}\ \alpha_s^2\ |\psi_s(0)|^2 \ . \tag{4.10}$$

Using experimental values[28] for $\Gamma(\psi \to 3g) = 58.5$ KeV and $\Gamma(\eta_c \to 2g) = 10.3^{+3.8}_{-3.4}$ MeV, we get from Eqs. (4.8) and (4.10),

$$\frac{|\psi_t(0)|^2}{|\psi_s(0)|^2} = 0.541^{-0.146}_{+0.267} \qquad (4.11)$$

if $\alpha_s(c\bar{c}) = 0.2048$, which may be obtained by considering the ratio of (4.8) and (4.7) and using the experimental value for $\Gamma_\psi(e^+e^-) = 4.72 \pm 0.35$ KeV. Thus experimentally $\psi_t(0)/\psi_s(0) \approx 0.6 - 0.9$ for the $c\bar{c}$ system. In case of $b\bar{b}$ system, using the input value of $\Gamma(\Upsilon \to e^+e^-) = 1.34 \pm 0.05$ KeV, we get from Eq. (4.7), the value of $|\psi_t(0)|^2_{b\bar{b}} = 0.403\ \mathrm{GeV}^3$, which now becomes a lower bound for the wave-function at the origin of the yet unobserved $^1S_0$ $b\bar{b}$ state. The simple inequality (4.6) should hold for all S-states for all heavy quarkonia for the class of potentials considered and hence useful in predicting the order of the two-gluon widths for all η_Q states, yet to be discovered.

IV.3. Mass-dependent potential and decay width of η_b :

As pointed out earlier, most of the existing potential models involve highly singular interaction terms, like $\delta^3(r)$ or $1/r^3$ terms. The validity of a perturbative calculation with such terms is indeed questionable. Gupta[96] has proposed a new $Q\bar{Q}$ potential, which is less singular. This is obtained by considering a non-relativistic approximation of the $Q\bar{Q}$ scattering matrix element by treating p^2/p_o^2 (rather than p^2/M^2) as the small expansion parameter, where $p^2 = \frac{1}{4}K^2 + \frac{1}{4}S^2$, $K = p' - p$, $S = p' + p$, it being assumed that S^2 is very small compared to K^2. The approximation leads to the second-order perturbative potential

$$V(r) = -\frac{4\alpha_s}{3}\left[\frac{1}{r} - \frac{4e^{-2Mr}}{3r}\left(S^2 - \frac{3}{4}\right) - \frac{3f_1}{2r} L.S - \frac{f_2}{4r} S_{12}\right] + Ar, \quad (4.12)$$

where

$$f_1 = \left[1 - \left(1 + 2Mr\right)e^{-2Mr}\right] / M^2r^2, \quad (4.13)$$

$$f_2 = \left[1 - \left(1 + 2Mr + \frac{4}{3}M^2r^2\right)e^{-2Mr}\right] / M^2r^2. \quad (4.14)$$

The potential has some interesting features, the most notable being its explicit mass dependence. We note that for an S-state, the potential may be written as

$$V(r) = M\phi_1(Mr) + Ar, \quad (4.15)$$

where $\phi_1(Mr)$ is a matrix function of Mr. In a recent paper, Gupta *et al.*[88] have taken a mixture of scalar and vector exchange terms with an arbitrary mixing parameter B and have chosen different values of B for the $b\bar{b}$ and $c\bar{c}$ systems. A justification for this choice is not clear. If we assume that B is the same for all flavours and that the $Q\bar{Q}$ potential is, in fact, of the type (4.15), we may use the mass scaling properties of the corresponding Schrödinger equation to derive some interesting results.

Let M_c, M_b be the quark masses and let $\phi_s^c(r) = \frac{u(r)}{r}$ and $\phi_s^b(r) = \frac{v(r)}{r}$ be the singlet radial wave-functions satisfying

$$\frac{d^2u}{dr^2} + \frac{2M_c}{\hbar^2}[\, E - M_c\phi(M_c r) - Ar\,]u = 0 \qquad (4.16)$$

$$\frac{d^2v}{dr^2} + \frac{2M_b}{\hbar^2}[\, E' - M_b\phi(M_b r) - Ar\,]v = 0 \quad . \qquad (4.17)$$

We now consider a scaling transformation, $r' = \frac{M_b}{M_c} r$, of Eq. (4.17) and substitute $\omega(r) = \left(\frac{M_b}{M_c}\right)^{1/2} v(r)$. Arguments similar to those given before, will lead to the inequality

$$\left(\frac{M_b}{M_c}\right)^3 |\phi_s^c(0)|^2 \;>\; |\phi_s^b(0)|^2 \quad . \qquad (4.18)$$

This behaviour is, in fact, expected even in more general cases. For a mass-independent power-law potential $V = \lambda r^{\nu}$, one gets

$$|\psi_M(0)|^2 \;\propto\; (M\lambda)^{3/(2+\nu)} \quad . \qquad (4.19)$$

Thus for a coulomb potential ($\nu = -1$), $|\psi_M(0)|^2/M^3$ is a constant, but for $\nu > -1$, $|\psi_M(0)|^2/M^3$ decreases as M increases as in the case of Gupta's[96] potential. For $-2 < \nu < -1$, the ratio increases with M. The inequality (4.18) is a weak one but can still be converted into useful results, as is shown below:

a) We note that with M_b = 4.78 GeV and M_c = 1.36 GeV, which we have used in the previous chapter to fit the fine-hyperfine spectra, the inequality gives

$$|\phi_s^c(0)|^2 > 0.023\ |\phi_s^b(0)|^2 \quad . \tag{4.20}$$

b) Using (4.10) and the experimental value of $M(\eta_c) = 2979.6^{+1.7}_{-1.6}$ MeV, we get $|\phi_s^c(0)|^2 = 0.0651$ GeV^3. Thus the upper bound of $|\phi_s^b(0)|^2$ is $\sim$ 2.83 GeV^3.

c) One may estimate $\alpha_s(b\bar{b})$ from the ratio[99]

$$\frac{\Gamma(\ \Upsilon(1S) \to 2g\gamma\ )}{\Gamma(\ \Upsilon(1S) \to 3g\ )} = \frac{36}{5}\ \frac{\alpha}{\alpha_s(b\bar{b})}\ e_Q^2 \approx \frac{3}{100} \tag{4.21}$$

giving $\alpha_s(b\bar{b}) = 0.1946$.

d) The width of η_b can now be bounded as

$$\Gamma(\ \eta_b \to 2g\ ) = \frac{32\pi}{3}\ \frac{\alpha_s^2}{M_{Q\bar{Q}}^2}\ |\phi_s^b(0)|^2$$

$$< 40 \text{ MeV} \quad . \tag{4.22}$$

e) It may be noted that the potential of Gupta for the S-states quarkonia can also be written in the form

$$V = V_o(r) + V_\sigma(\ S^2 - \frac{3}{4}\ ) \tag{4.23}$$

with $\frac{dV_o}{dr} > 0$, $\frac{d^2V_o}{dr^2} < 0$, $\frac{dV_\sigma}{dr} < 0$. Thus the first inequality $\phi_s^b(0) > \phi_t^b(0)$, should also hold.

f) The value of $\phi_t^b(0)$ is already known (from its leptonic decay width), $|\phi_t^b(0)|^2 \approx 0.403$ GeV^3. We, therefore, get a lower bound for

$$\Gamma(\ \eta_b \to 2g\ ) > 5.7 \text{ MeV} \quad , \tag{4.24}$$

g) Rosner *et al.*[100] obtained the inequality

$$\psi^2_{M'}(0) \geq \frac{M'}{M}\psi^2_M(0) \qquad (4.25)$$

which is valid for power-law potentials and also for the class of potentials discussed in this chapter. We have given elsewhere[97] a proof of this inequality for more general potentials. Since $\psi^2(0)$ for $^1S_0$ $c\bar{c}$ state is ~ 0.0651 GeV^3, we obtain the bound

$$\Gamma(\eta_b \to 2g) > 3.23 \text{ MeV} \qquad (4.26)$$

which is, however, weaker than the bound in (4.24).

IV.4. Conclusions :

We have presented in this chapter some general results on the hyperfine members of the S-states quarkonia for a general class of potentials. The results will be valid even if the Breit-Fermi form of the spin-dependent potential is not valid. We first use a flavour-independent potential and obtain a lower bound on the wave-function of the singlet S-states of $b\bar{b}$. We consider in this context Gupta's potential for the $Q\bar{Q}$ system and make use of the mass-scaling properties of the relevant Schrödinger equation to predict that the width $\Gamma(\eta_b \to 2g)$ should lie within the range 6 ~ 40 MeV. Thus even the weak inequalities considered here are useful in quarkonium spectroscopy.