ACKNOWLEDGMENT

The author would like to express his appreciation to Dr. A. Das of Jalpaiguri Government Engineering College for suggesting the problem and for his help and encouragement in carrying the solution through.

REFERENCES

1. A. E. H. Love, *Theory of Elasticity*, Cambridge University Press, 1952.
2. S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Elastic Body*, Holden Day.
3. R. Greif and S.-C Chou, The propagation of radially symmetric stress waves in anisotropic, nonhomogeneous elastic media, *J. Appl. Mech.* **38**: 51–57 (1971).
4. S. C. Roy, Radial deformations of nonhomogeneous spherically anisotropic elastic media, *J. Indian Inst. Sci.* **58**(7): 303–314 (1976).
5. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1965.
6. D. F. Orchard, *Concrete Technology*, Vol. 1, Asia Publishing House, 1963.

Received September 22, 1976; revised January 17, 1977

$$\bar{\sigma}_z = -p_1 \exp\left[\frac{k}{2}(b^{2\alpha} - r^{2\alpha})\right]\left(\frac{b}{r}\right)^{\alpha+1} \times \frac{\{(kr^{2\alpha} - 1)\alpha\lambda_{13} + \lambda_{23}\}M_{\kappa^*,p}(kr^{2\alpha}) + 2\alpha\lambda_{13}kr^{2\alpha}M'_{\kappa^*,p}(kr^{2\alpha})}{\{(kb^{2\alpha} - 1)\alpha\lambda_{11} + \lambda_{12}\}M_{\kappa^*,p}(kb^{2\alpha}) + 2\alpha\lambda_{11}kb^{2\alpha}M'_{\kappa^*,p}(kb^{2\alpha})} \tag{18}$$

since

$$\lim_{z\to 0}\frac{M_{\kappa^*,p}(z)}{M_{\kappa^*,-p}(z)} = 0$$

$$\lim_{z\to 0}\frac{zM'_{\kappa^*,p}(z)}{M_{\kappa^*,-p}(z)} = 0$$

and

$$\lim_{z\to 0}\frac{zM'_{\kappa^*,\pm p}(z)}{M_{\kappa^*,\pm p}(z)} = \frac{1}{2} \pm p \tag{18a}$$

If the shell is under the action of internal pressure only, the external surface being stress-free, the stresses for such an inhomogeneous shell are obtained from (16) by taking $p_1 = 0$.

IV. NUMERICAL RESULTS

Numerical results are obtained for a cylindrical shell structure in which the internal surface is under a uniform normal pressure p_0, while the external surface is assumed to be stress-free for System I, and subjected to half the internal pressure for System II. All numerical results have been computed for the case of $b = 1.5a$.

We choose $\alpha = 0.5$ and the elastic parameters as $\lambda_{11} = 918$, $\lambda_{12} = 459$, $\lambda_{22} = 102$, $\lambda_{13} = 275$, $\lambda_{23} = 273$, and $k = 2/a$ (numerically) for Material I, which resembles Barite-cement aggregate (see Ref. 6) used extensively as radiation shielding material. The present analysis may also be useful in studying the stresses in layered media having exponentially increasing or decreasing stiffness.

For this material, $[-\bar{\sigma}_r/p_0]$, $[\bar{\sigma}_\theta/p_0]$, and $[-(\bar{\sigma}_z/p_0)]$ are plotted against (r/a), for both loading Systems I and II in Figs. 1 and 2. Similarly, Figs. 3 and 4 show the corresponding results for Material II identified by $\alpha = 1.0$, and $\lambda_{11} = 918$, $\lambda_{22} = 408$, $\lambda_{12} = 918$, $\lambda_{13} = 275$, $\lambda_{23} = 273$, with $k = 2/a^2$.

where

$$\alpha_{\pm p}(r) = \left[\alpha\left(kr^{2\alpha-1} - \frac{1}{r}\right)\lambda_{11} + \frac{\lambda_{12}}{r}\right]M_{\kappa^*,\pm p}(kr^{2\alpha})$$
$$+ 2\alpha\lambda_{11}kr^{2\alpha-1}M'_{k^*,\pm p}(kr^{2\alpha})$$

$$\beta_{\pm p}(r) = \left[\alpha\left(kr^{2\alpha-1} - \frac{1}{r}\right)\lambda_{12} + \frac{\lambda_{22}}{r}\right]M_{\kappa^*,\pm p}(kr^{2\alpha})$$
$$+ 2\alpha\lambda_{12}kr^{2\alpha-1}M'_{\kappa^*,\pm p}(kr^{2\alpha})$$

$$\nu_{\pm p}(r) = \left[\alpha\left(kr^{2\alpha-1} - \frac{1}{r}\right)\lambda_{13} + \frac{\lambda_{23}}{r}\right]M_{\kappa^*,\pm p}(kr^{2\alpha})$$
$$+ 2\alpha\lambda_{13}kr^{2\alpha-1}M'_{\kappa^*,\pm p}(kr^{2\alpha})$$

and

$$M = \alpha_p(a)\alpha_{-p}(b) - \alpha_{-p}(a)\alpha_p(b) \tag{17}$$

The prime indicates the derivative of the function with respect to its argument.

Stresses in a cylindrical shell ($a \leq r \leq b$) made of homogeneous cylindrically anisotropic material, under the same boundary conditions (15), may be found from the second, third, and fourth equations of (16) on letting $k \to 0$, and these agree with the results obtained by St. Venant (quoted in Ref. 2). For an isotropic body $\lambda_{11} = \lambda_{22} = \lambda + 2\mu$, $\lambda_{12} = \lambda_{13} = \lambda_{23} = \lambda$. Equation (11), with the application of these relations, gives $2\alpha p = 1$. When these are used in the second, third, and fourth equations of (16) along with the limit $k \to 0$, one gets Lamé's results given in Ref. 1.

A solid cylindrical body ($0 \leq r \leq b$) of nonhomogeneous cylindrically aeolotropic material undergoes compression by a uniformly distributed external pressure p_1. The stresses in such a shell are obtained from the last three equations of (16) by setting $a = 0$:

$$\bar{\sigma}_r = -p_1 \exp\left[\frac{k}{2}(b^{2\alpha} - r^{2\alpha})\right]\left(\frac{b}{r}\right)^{\alpha+1}$$
$$\times \frac{\{(kr^{2\alpha} - 1)\alpha\lambda_{11} + \lambda_{12}\}M_{\kappa^*,p}(kr^{2\alpha}) + 2\alpha\lambda_{11}kr^{2\alpha}M'_{\kappa^*,p}(kr^{2\alpha})}{\{(kb^{2\alpha} - 1)\alpha\lambda_{11} + \lambda_{12}\}M_{\kappa^*,p}(kb^{2\alpha}) + 2\alpha\lambda_{11}kb^{2\alpha}M'_{\kappa^*,p}(kb^{2\alpha})}$$

$$\bar{\sigma}_\theta = -p_1 \exp\left[\frac{k}{2}(b^{2\alpha} - r^{2\alpha})\right]\left(\frac{b}{r}\right)^{\alpha+1}$$
$$\times \frac{\{(kr^{2\alpha} - 1)\alpha\lambda_{12} + \lambda_{22}\}M_{\kappa^*,p}(kr^{2\alpha}) + 2\alpha\lambda_{12}kr^{2\alpha}M'_{\kappa^*,p}(kr^{2\alpha})}{\{(kb^{2\alpha} - 1)\alpha\lambda_{11} + \lambda_{12}\}M_{\kappa^*,p}(kb^{2\alpha}) + 2\alpha\lambda_{11}kb^{2\alpha}M'_{\kappa^*,p}(kb^{2\alpha})}$$

boundary conditions are

$$\begin{aligned} \bar{\sigma}_r &= -p_0, \qquad (r = a) \\ \bar{\sigma}_r &= -p_1, \qquad (r = b) \end{aligned} \tag{15}$$

On application of these boundary conditions in the first equation of (4) along with Eq. (14), one obtains two simultaneous equations involving the two unknowns A and B. Solving for A and B and inserting their values in (14) and (4), one obtains the complete solution for the radial displacement and stresses as

$$\begin{aligned}
\bar{u} = \frac{\exp(kr^{2\alpha}/2)r^{-\alpha}}{M}\Bigg[&\left\{p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_{-p}(a)\right. \\
&\left. - p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_{-p}(b)\right\} M_{\kappa^*,p}(kr^{2\alpha}) \\
&+ \left\{p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_p(b) - p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_p(a)\right\} M_{\kappa^*,-p}(kr^{2\alpha})\Bigg] \\
\bar{\sigma}_r = \frac{\exp(-kr^{2\alpha}/2)r^{-\alpha}}{M}\Bigg[&\left\{p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_{-p}(a)\right. \\
&\left. - p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_{-p}(b)\right\}\alpha_p(r) \\
&+ \left\{p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_p(b) - p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_p(a)\right\}\alpha_{-p}(r)\Bigg] \\
\bar{\sigma}_\theta = \frac{\exp(-kr^{2\alpha}/2)r^{-\alpha}}{M}\Bigg[&\left\{p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_{-p}(a)\right. \\
&\left. - p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_{-p}(b)\right\}\beta_p(r) \\
&+ \left\{p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_p(b) - p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_p(a)\right\}\beta_{-p}(r)\Bigg] \\
\bar{\sigma}_z = \frac{\exp(-kr^{2\alpha}/2)r^{-\alpha}}{M}\Bigg[&\left\{p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_{-p}(a)\right. \\
&\left. - p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_{-p}(b)\right\}\nu_p(r) \\
&+ \left\{p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_p(b) - p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_p(a)\right\}\nu_{-p}(r)\Bigg]
\end{aligned} \tag{16}$$

Equation (7) reduces to

$$x^2 \frac{d^2U}{dx^2} + \left\{\frac{1}{4}\left(1 - \frac{\lambda_{22}}{\alpha^2\lambda_{11}}\right) + \left(\frac{1}{2} - \frac{\lambda_{12}}{2\alpha\lambda_{11}}\right)x - \frac{x^2}{4}\right\}U = 0 \tag{9}$$

The solution of the above differential equation is (see Ref. 5)

$$U = AM_{\kappa^*,p}(x) + BM_{\kappa^*,-p}(x) \tag{10}$$

where $M_{\kappa^*,-p}(x)$ are Whittaker functions in which

$$2p = \left(\frac{\lambda_{22}}{\alpha^2\lambda_{11}}\right)^{1/2} \qquad \text{(a positive noninteger)} \tag{11}$$

and

$$\kappa^* = \frac{1}{2}\left(1 - \frac{\lambda_{12}}{\alpha\lambda_{11}}\right) \tag{12}$$

A and B being arbitrary constants.

If $2p$ is an integer or zero, the solution of Eq. (9) may be written as

$$U = CW_{\kappa^*,p}(x) + DW_{-\kappa^*,p}(-x) \tag{13}$$

where

$$W_{\kappa^*,p}(x) = \frac{\Gamma(c-1)}{\Gamma(d-c+1)} M_{\kappa^*,p}(x) + \frac{\Gamma(1-c)}{\Gamma(d)} M_{\kappa^*,-p}(x)$$

in which $c = 1 \pm 2p$ and $d = \frac{1}{2} - \kappa^* \pm p$.

Finally, the radial displacement $\bar{u}(r)$ satisfying the equilibrium Eq. (5) is obtained with the help of Eqs. (6), (8), and (10) as

$$\bar{u} = \frac{\exp(kr^{2\alpha}/2)}{k^{1/2}r^{\alpha}}[AM_{\kappa^*,p}(kr^{2\alpha}) + BM_{\kappa^*,-p}(kr^{2\alpha})] \tag{14}$$

This expression for $\bar{u}$ may be used in Eqs. (4) to get the general expressions for the stresses in terms of A and B.

We now consider a cylindrical shell $a \leq r \leq b$. The structure is made of nonhomogeneous cylindrically anisotropic material. The shell is under the influence of uniformly distributed internal and external pressures. The

For the axisymmetric case the nontrivial stress equation of equilibrium, in the absense of body forces, takes the form

$$\frac{\partial}{\partial r}\bar{\sigma}_r + \frac{1}{r}(\bar{\sigma}_r - \bar{\sigma}_\theta) = 0 \tag{3}$$

Nonzero stresses in the normal, circumferential, and longitudinal directions are

$$\begin{aligned}
\bar{\sigma}_r &= \left(\lambda_{11}\frac{d\bar{u}}{dr} + \lambda_{12}\frac{\bar{u}}{r}\right)\exp(-kr^{2\alpha}) \\
\bar{\sigma}_\theta &= \left(\lambda_{12}\frac{d\bar{u}}{dr} + \lambda_{22}\frac{\bar{u}}{r}\right)\exp(-kr^{2\alpha}) \\
\bar{\sigma}_z &= \left(\lambda_{13}\frac{d\bar{u}}{dr} + \lambda_{23}\frac{\bar{u}}{r}\right)\exp(-kr^{2\alpha})
\end{aligned} \tag{4}$$

respectively, $\bar{u}$ being the radial displacement.

III. METHOD OF SOLUTION

The equation of equilibrium (3), with the help of Eqs. (4), becomes

$$r^2\frac{d^2\bar{u}}{dr^2} + (1 - 2\alpha kr^{2\alpha})r\frac{d\bar{u}}{dr} - \left(\frac{\lambda_{22} + 2\alpha kr^{2\alpha}\lambda_{12}}{\lambda_{11}}\right)\bar{u} = 0 \tag{5}$$

Now on using the transformations

$$x = kr^{2\alpha} \qquad \text{and} \qquad \bar{u} = V\exp\left(\frac{x}{2}\right) \tag{6}$$

Eq. (5) changes to

$$x^2\frac{d^2V}{dx^2} + x\frac{dV}{dx} + \left\{-\frac{\lambda_{22}}{4\alpha^2\lambda_{11}} + \left(\frac{1}{2} - \frac{\lambda_{12}}{2\alpha\lambda_{11}}\right)x - \frac{x^2}{4}\right\}V = 0 \tag{7}$$

with

$$V = x^{-1/2}U \tag{8}$$

I. INTRODUCTION

The elastic behavior of a homogeneous cylindrically aeolotropic material was first studied by St. Venant; see, e.g., Ref. 1 or 2. Problems involving nonhomogeneous media in which the properties vary continuously with spatial position have been studied by various authors. Greif and Chou [3] have adopted a numerical integration method and used the computer in solving the vibration problem of a cylindrically anisotropic nonhomogeneous cylindrical shell (plane strain).

A plane-strain assumption is also used here to find the analytical solution for the radial deformation and corresponding stresses in a cylindrical shell made of cylindrically aeolotropic heterogeneous material under the influence of normal pressures on both boundaries. The results obtained by St. Venant [2] for the homogeneous anisotropic case and those found by Lamé [1] for the homogeneous isotropic case can be deduced from the general results. The corresponding expressions for a solid cylinder of nonhomogeneous anisotropic medium are derived here. The nonhomogeneity of the material is characterized by the elastic parameters C_{ij} (see Refs. 3 and 4) as

$$c_{ij} = \lambda_{ij} \exp(-kr^{2\alpha}) \qquad (i, j = 1, 2, 3) \tag{1}$$

where λ_{ij}, k, and α are the prescribed parameters of the material concerned.

II. FUNDAMENTAL EQUATIONS

The basic system of field equations in linear isothermal static elasticity theory is (a) the generalized Hooke's law, (b) the linearized strain displacement equations, and (c) the stress equations of equilibrium. Here the axis of anisotropy is taken to be the z-axis of the r, θ, z cylindrical coordinate system, and the Young's moduli are of the form

$$\begin{aligned} E_r &= E_1 \exp(-kr^{2\alpha}) \\ E_\theta &= E_2 \exp(-kr^{2\alpha}), \text{ etc.} \end{aligned} \tag{2}$$

For plane-strain assumption, λ_{ij} of Eq. (1) is then expressible in terms of these E_1, E_2, and the Poisson's ratios, see Ref. 3.

J. STRUCT. MECH., 5(3), 333–342 (1977)

Note on the Radial Deformation and Stresses in Anisotropic Nonhomogeneous Elastic Media

S. C. Roy

JALPAIGURI GOVERNMENT ENGINEERING COLLEGE
JALPAIGURI, WEST BENGAL, INDIA

ABSTRACT

In this paper we study the elasticity problem of a cylindrically anisotropic, elastic medium bounded by two axisymmetric cylindrical surfaces subjected to normal pressures (plane strain). The material of the structure is orthotropic with cylindrical anisotropy and, in addition, is continuously inhomogeneous with mechanical properties varying along the radius. General solutions in terms of Whittaker functions are presented. The results obtained by St. Venant for a homogeneous cylindrically anisotropic medium can be deduced from the general solutions. The problem of a solid cylinder of the same medium under the external pressure is also solved as a particular case of the above problem. Problems of the type covered in this paper are encountered in nuclear reactor design.

A C K N O W L E D G E M E N T

I express my gratitude to Dr. A. Das of Jalpaiguri Government Engineering College, Jalpaiguri for his kind help and guidance at every stage of the work on this thesis.

T-2134

R-11

Large Deflection Analysis of Annular Plate of Nonhomogeneous Material Subjected to Variable Normal Pressure and Heating

SUBHAS CHANDRA ROY

Jalpaiguri Government Engineering College, Jalpaiguri

Received 26 *November* 1975; *accepted* 30 *September* 1976

ABSTRACT

Large deflection analysis has been carried out to determine the deflection of an annular circular plate of non-homogeneous material under normal pressure subjected to two dimensional temperature distribution. Both clamped and simply supported edge conditions have been considered for two types of non-homogeneity.

5-5

IN solving large deflection problems related to plates one encounters nonlinear differential equations which cannot be exactly solved. Berger[1] neglects the strain energy due to second strain invariant and has shown the validity of his technique for different engineering problems. This technique has been used by Nash and Modeer[2], Nowinski and Ismail[3] and Das[4,5].

Strain energy method and Berger's technique are used here to tackle the large deflection problem of an annular circular plate of non-homogeneous material where the plate is under variable normal pressure and subjected to two dimensional temperature distribution, i.e. the temperature varies along the radius and the width of the plate. Young's modulus is supposed to vary as any power of the radius and it characterises the non-homogeneity of the plate. Calculations for the specific cases of both clamped and simply supported plates subjected to linearly varying pressure and a general type temperature distribution for two different types of non-homogeneity are presented.

Theory

The undeflected middle plane of the plate is chosen to be the plane of reference and its centre is taken as the origin. The Z-axis is perpendicular to the reference plane in the down-ward direction. The potential energy of deformation may be written as

$$V = U_b + U_m - U_q - W_T \qquad \ldots(1)$$

in which

$$U_b = \pi \int_b^a D\left[\nabla^2 w - 2(1-\nu)\frac{1}{\eta}\frac{dw}{d\eta}\frac{d^2w}{d\eta^2}\right]\eta\, d\eta \qquad \ldots(2)$$

$r/, dr/, dr^2/, r dr/$

$$U_m = \pi \int_b^a \frac{12d}{h^2}\left[e^2 - 2(1-\nu)e_2\right]\eta\, d\eta \qquad \ldots(3)$$

$r\,dr/$

$$U_q = 2\pi \int_b^a q.w.r.dr \qquad \ldots(4)$$

and lastly the expression for W_T is given by[6,7]

$$W_T = \int_0^{2\pi}\int_b^a\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\gamma T(\gamma,\zeta)}{1-2\nu}\left[e\gamma\gamma + e\theta\theta + e\varepsilon\varepsilon\right]\gamma\, d\theta\, d\gamma\, d\varepsilon \qquad \ldots(5)$$

$\theta\theta, zz$, $r\,d\theta\,dr\,dz/$

Now the plain-stress assumption of an isotropic non-homogenous thin plate leads to the relation[7].

$$\sigma\varepsilon = \lambda^1(e\gamma\gamma + e\theta\theta + e\varepsilon\varepsilon) + 2\mu^1 e\varepsilon\varepsilon = 0$$

or,

$$e\gamma\gamma = -\left(\frac{\lambda^1}{\lambda^1 + 2\mu^1}\right)(e\gamma\gamma + e\theta\theta) \qquad \ldots(6)$$

$\bar{e}_{zz}$, $\bar{e}_{rr} + \bar{e}_{\theta\theta}/$

Hence

$$e\gamma\gamma + e\theta\theta + e\varepsilon\varepsilon = \frac{1-2\gamma}{1-\gamma}(e - z\nabla^2 w) \qquad \ldots(7)$$

From Eqs (5) and (7) W_T becomes

$$W_T = 2\pi \int_b^a \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\alpha T}{1-\gamma}\left[e - Z\nabla^2 w\right]\gamma\, d\gamma\, d\varepsilon \qquad \ldots(8)$$

$r\,dr\,dz/$

The temperature $T(r, z)$ is assumed to take the following form

$$T(\gamma,\varepsilon) = T_0(\gamma) + g(\varepsilon)\,T_1(\gamma) \qquad \ldots(9)$$

$r/, z/, r$

and we suppose that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon g(\varepsilon)\, d\varepsilon = f(h) \qquad \ldots(10)$$

$z\,g(z)\,dz$

and

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} g(\varepsilon)\, d\varepsilon = F(h) \qquad \ldots(11)$$

$g(z)\,dz$

Eq[r] (2-4, 8-11)

with the aid of Eqs (2-4,, 8-11) Eq. (1) becomes, on neglecting the strain energy due to second strain invariant,

$$V = \pi\int_b^a \left[D\left(\left(\frac{d^2w}{d\gamma^2}\right)^2 + \frac{1}{\gamma^2}\left(\frac{dw}{d\gamma}\right)^2 + \frac{12e^2}{h^2} + \frac{2\nu}{\gamma}\,\frac{dw}{d\gamma},\,\frac{d^2w}{d\gamma^2} \right] \right.$$

$$\left. -2qw - \frac{2E\alpha}{1-\nu}\left[e\,(T_oh+T_1F) - f(h)\,T_1(\gamma)\,\nabla^2 w \right) \right]\gamma d\gamma \qquad \ldots(12)$$

For minimum of V Euler's variational equations are

$$\frac{\partial v}{\partial u} - \frac{\partial}{\partial\gamma}\,\frac{\partial v}{\partial u\gamma} = 0 \qquad \ldots(13)$$

$$\frac{\partial\nu}{\partial w} - \frac{\partial}{\partial\eta}\,\frac{\partial\nu}{\partial w\eta} + \frac{\partial^2}{\partial\eta^2}\,\frac{\partial\nu}{\partial w\eta\eta} = 0 \qquad \ldots(14)$$

Eqs. (12) and (13) lead to

$$\frac{\partial}{\partial\eta}\left[\frac{12De}{h^2} - \frac{E\alpha}{1-\nu}\,(Toh+T_1F)\right] = 0 \qquad \ldots(15)$$

from which

$$\frac{12De}{h^2} - \frac{F\alpha}{1-\nu}\left[To(\eta)\,h + T_1(\eta).\,F(h)\right] = \beta^2 D_0 \qquad \ldots(16)$$

where B is a normalized constant of integration. Eqs. (12) and (14) yield with the aid of Eq. (16)

$$-\frac{d}{dr}\left[\frac{d}{dr}\left(D\,r\,\frac{d^2w}{dr^2}\right) - \frac{D}{r}\,\frac{dw}{dr} + \nu\,\frac{dD}{dr}\,\frac{dw}{dr} - D_o\beta^2\left(\frac{rdw}{dr}\right)\right] = r\left[g(r) - \frac{\alpha f(h)}{1-\nu}\,\nabla^2 \left(E(\gamma)T_1(r)\right)\right] \qquad \ldots(17)$$

$$E(\gamma) = E_o\,\gamma^{2m} \qquad \ldots(18)$$

in which E is a constant and m may take up any value and for

$$g(\gamma) - \frac{\alpha f(h)}{1-\nu}.\,E_o.\nabla^2\left(\gamma^{2m}T_1(r)\right) = \gamma^{n-1}\,(\Sigma\,a_p\,\gamma^p) \qquad \ldots(19)$$

n being any number greater than -1 and ($p=0, 1, 2\ldots.$) Eq. (17) stands as

$$\frac{d}{d\gamma}\left[\gamma^{2m+1}\left(\frac{d^3w}{d\gamma^3} + \frac{2m+1}{\gamma}\,\frac{d^2w}{d\gamma^2} + \left(\frac{2m\nu-1}{\gamma^2} - \frac{\beta^2}{\gamma^{2m}}\right)\frac{dw}{d\gamma}\right)\right] = \frac{1}{D_o}\sum_{p=0} a_p\gamma^{n+p} \qquad \ldots(20)$$

The general expression of the lateral displacement W may be found from Eq. (20) to be

$$W = \frac{A_1}{2\,(i\beta)^2\,D_o(1-m)}\,\frac{(-1)^s\,\rho^{2(1+s)}}{(1+s)[(1^2-\mu^2)\cdots[(2s+1)^2-\mu^2)]}$$

$$+\frac{\rho}{i\beta}\left[A_2\left((\mu-1)J_\mu(\rho)\,S_{-1,\,\mu-1}(\rho) - J_{\mu-1}(\rho)S_{o,\,\mu}(\rho)\right)\right.$$

$$\left. + A_3\left((\mu-1)\,Y_\mu(\rho)S_{-1,\,\mu-1}(\rho) - Y_{\mu-1}(\rho)\,S_{o,\mu}(\rho)\right)\right]$$

$$+\frac{1}{D_o\,(1-m)^3}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\frac{a_p}{\eta+p+1}\left(\frac{1-m}{i\beta}\right)^{\eta+2}$$

$$\frac{(-1)^s\,\rho^{\eta+2+2s}}{(\eta+2+2s)\left[\left\{(\eta+1)^2-\mu^2\right\}\cdots\left\{\eta+2s+1)^2-\mu^2\right\}\right]} \qquad \ldots(21)$$

for $m \neq 1$, where

$$\mu^2 = 1 + \frac{2m(1-\nu)}{(1-m)^2}$$

$$\rho = \frac{i\beta}{1-m}\,\gamma^{1-m}$$

$$\eta = \frac{n+p+1}{1-m} \quad \text{and} \qquad \ldots(22)$$

G-144

Here $J\mu(\rho), Y\mu(\rho)$ are the Bessel functions of the first and the second kind, $S_{\eta,\mu}(\rho)$ is the Lommel's function and A_1, A_2, A_3 and A_4 are integration constants. On assuming μ to be zero on both the boundaries, Eqs. (16) and (21) lead to the result;

$$\frac{\beta^2h^2}{24(1-m)}\left[a^{2(1-m)}-b^{2(1-m)}\right]+\frac{\alpha(1+\gamma)}{h}\left[\xi(a)-\xi(b)\right]$$

$$=\frac{1-m}{2(i\beta)^2}\Bigg[\frac{A_2^2}{2}\left[\rho^2\left\{J\mu^2(\rho)-J_{\mu+1}(\rho)\,J_{\mu+1}(\rho)\right\}\right]_{\rho_b}^{\rho_a}$$

$$+\frac{A_3^2}{2}\left[\rho^2\left\{Y\mu^2(\rho)-Y_{\mu-1}(\rho)\,Y_{\mu+1}(\rho)\right\}\right]_{\rho_b}^{\rho_a}$$

$$+\frac{A_2A_3}{2}\left[\rho^2\left\{2J\mu(\rho)Y\mu(\rho)-J_{\mu-1}(\rho)\,Y_{\mu+1}(\rho)-J_{\mu+1}(\rho)\,Y_{\mu+1}(\rho)\right\}\right]_{\rho_b}^{\rho_a}$$

$$+\frac{2A_1A_2}{D_o\,(i\beta)\,(1-m)}\sum_{s=0}^{\infty}(-1)^s$$

$$\frac{\left[(2s+\mu+1)\,\rho J\mu(\rho)\,S_{2s+1,\mu-1}(\rho)-\rho J_{\mu-1}(\rho)\,S_{2+2s,\mu}(\rho)\right]_{\rho_b}^{\rho_a}}{(1^2-\mu^2)\,\ldots\ldots\left\{(2s+1)^2-\mu^2\right\}}$$

$$+\frac{2A_1A_3}{D_o(i\beta)\,(1-m)}\sum_{s=0}^{\infty}(-1)^s$$

$$\frac{\left[\rho\left\{(2s+\mu+1)\,Y\mu(\rho)\,S_{2s+1,\mu-1}(\rho)-Y_{\mu-1}(\rho)\,S_{2+2s,\mu}(\rho)\right\}\right]_{\rho_b}^{\rho_a}}{(1^2-\mu^2)\,\ldots\ldots\left\{(2s+1)^2-\mu^2\right\}}$$

$$+\frac{2i\beta A_2}{D_o(1-m)^3}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\frac{a_p}{n+p+1}\left[\frac{1-m}{i\beta}\right]^{\eta+2}(-1)^s$$

$$\frac{\left[\rho\left\{(\eta+1+2s+\mu)\,J\mu(\rho)\,S_{\eta+1+2s,\mu-1}(\rho)-J_{\mu-1}(\rho)\,S_{\eta+2+2s,\mu}(\rho)\right\}\right]_{\rho_b}^{\rho_a}}{\left[(\eta+1)^2-\mu^2\right]\ldots\left[(\eta+2s+1)^2-\mu^2\right]}$$

$$+\frac{2i\beta A_3}{D_o(1-m)^3}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\frac{a_p}{n+p+1}\left(\frac{1-m}{i\beta}\right)^{\eta+2}(-1)^s$$

$$\frac{\left[\rho(\eta+1+2s+\mu)\,Y\mu(\rho)\,S_{\eta+2s+1,\mu-1}(\rho)-Y_{\mu-1}(\rho)\,S_{\eta+1+2s,\mu}(\rho)\right]_{\rho_b}^{\rho_a}}{\left[(\eta+1)^2-\mu^2\right]\ldots\left[(\eta+2s+1)^2-\mu^2\right]}$$

$$+\frac{A_1^2}{2D_o^2\,(1-m)^2(i\beta)^2}\sum_{s=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s(-1)^s$$

$$\frac{\left[\rho_a^{\,2+2s+s_1}-\rho_b^{\,2+2s+s_1}\right]}{(2+s+s_1)\left[(1^2-\mu^2)\ldots\left\{(1+2s)^2-\mu^2\right\}\right]\left[(1^2-\mu^2)\left\{(1+2s_1)^2-\mu^2\right\}\right]}$$

$$\frac{2A_1}{D_o^2\,(1-m)^4}\sum_{p=0}^{\infty}\frac{a_p}{(n+p+1)}\left[\frac{1-m}{i\beta}\right]^{\eta+2}\sum_{s=0}^{\infty}\sum_{s_1=0}^{\infty}$$

$$\frac{(-1)^s\,(-1)^{s_1}\left[\rho_a^{\,4+\eta+2s+2s_1}-\rho_b^{\,4+\eta+2s+2s_1}\right]}{(4+\eta+2s_1)\left[(1^2-\mu^2)\ldots\left\{(1+2s)^2-\mu^2\right\}\right]\left[\left\{(\eta+1)^2-\mu^2\right\}\ldots\left\{(\eta+1+2s_1)^2-\mu^2\right\}\right]}$$

$$+\frac{(i\beta)^2}{D_o^2(1-m)^6}\sum_{p=0}^{\infty}\sum_{p_1=0}^{\infty}\frac{a_p\,a_{p1}}{(\eta+p+1)(\eta+p_1+1)}$$

$$\left(\frac{1-m}{i\beta}\right)^{\eta+\eta_1+4}\sum_{s=0}^{\infty}\sum_{s_1=0}^{\infty}$$

$$\frac{(-1)^s(-1)^{s_1}\left[\rho_a^{\,4+\eta+\eta_1+2s+2s_1}-\rho_b^{\,4+\eta+\eta_1+2s+2s_1}\right]}{(4+\eta+\eta_1+2s+2s_1)\left[\left\{(\eta+1)^2-\mu^2\right\}\ldots\left\{(\eta+2s+1)^2-\mu^2\right\}\right]\left[\left\{(\eta_1+1)^2-\mu^2\right\}\ldots\left\{(\eta_1+2s_1+1)^2-\mu^2\right\}\right]}\Bigg] \quad \ldots(23)$$

in which we have taken

G-112

$r/, r/$ $\cancel{\gamma}\left(h\, To\,(\cancel{\gamma}) + F\,(h)\, T_1(\cancel{\gamma})\right) = \frac{d\cancel{\gamma}}{d\cancel{\gamma}}$ $r/, \frac{dr}{dr}/$

$$\frac{i\beta}{1\text{-}m}\, a^{1-m} = p_a$$

$$\frac{i\beta}{1\text{-}m}\, b^{1-\cancel{m}} = p_{\cancel{b}}$$

$$\frac{n+p_1+1}{1\text{-}m} = \cancel{\eta_1} \quad \ldots(24)$$

Relevant Equations for $m=1$

The solution of Eq. (20) for $m=1$ gives the l ral displacement ~~W~~ in the following form

lc

$r/, r^{k}/$

$$w = B_1 \log \cancel{\gamma} + \frac{B_2}{k}\cancel{\gamma^{k}} - \frac{B_3}{k}\cancel{\gamma^{-k}} + B_4 + \frac{1}{D_o}$$

$r^{-k}/$

lc

$$\sum_{p=0}^{\infty} \frac{a_p \cancel{\gamma^{n+p+1}}}{(n+p+1)^2[(n+p+1)^2-k^2]} \quad \ldots(25)$$

$r^{n+p+1}/$

in which $k^2 = 2\,(1\text{-}\nu) + \beta^2$ and $k \pm (n+p+1)$ ...(26)
Even if $k = (n+p+1)$ or $-(n+p+1)$ or both the last term in Eq. (25) would be slightly changed through indefinite integrations like $r \log r\, dr$ or $\int r\,(\log)$ dr as the case arises.

The equation corresponding to the Eq. (23) happens to be

$r/, r/$

$$\frac{B^2h^2}{12}\log\left(\frac{a}{b}\right) + \frac{\alpha(1+\nu)}{h}\left[\cancel{\gamma}(a) - \cancel{\gamma}(b)\right]$$

$$= \frac{1}{2}\Bigg[(B_1^2 + 2B_2B_3)\left(\log\frac{a}{b}\right)$$

$$+ \frac{B_2^2}{2k}\left(a^{2k}\text{-}b^{2k}\right) - \frac{B_3^2}{2k}\left(a^{-2k}\text{-}b^{-2k}\right)$$

$$+ \frac{2B_1B_2}{k}\left(a^{k}\text{-}b^{k}\right) - \frac{2B_1B_3}{k}\left(a^{-k}\text{-}b^{-k}\right)$$

$(n+p+1)^2/$

$$+ \frac{2B_1}{D_o}\sum_{p=0}^{\infty}\frac{a_p\,(a^{n+p+1} - b^{n+p+1})}{\cancel{(n+r+1)}[(n+p+1)^2\text{-}k^2]}$$

$$+ \frac{2B_2}{D_o}\sum_{p=0}^{\infty}\frac{a_p(a^{n+p+1+k} - b^{n+p+1+k})}{(n+p+1)\,(n+p+1+k)^2\,(n+p+1\text{-}k)}$$

$$+ \frac{2B_3}{D_o}\sum_{p=0}^{\infty}\frac{a_p(a^{n+n+1-k} - b^{n+p+1-k})}{(n+p+1)\,(n+p+1-k)^2(n+p+1+k)}$$

$$+ \frac{1}{D_o^2}\sum_{p=0}^{\infty}\sum_{p=0}^{\infty}$$

$$\frac{a_p a_{p_1}(a^{2n+p+p_1+2} - b^{2n+p+p_1+2})}{(n+p+1)(n+p_1+1)[(n+p+1)^2 - k^2](2n+p+p_1+2)}\Bigg] \quad \ldots(27)$$

Numerical Results

Clamped boundaries — On a clamped edge we shall have

$$u = w = \frac{dw}{dr} \quad \ldots(28)$$

Using the boundary conditions (28) for the edges $r=a$ and $r=b$ we get the constants A' s from Eq. (21) in a matrix form $A\ A\ A\ A$ from its augmented matrix

$\{A_1, A_2, A_3, A_4\}$

R-S-Smal,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & b_{15} \\ a_{21} & a_{22} & a_{23} & 1 & b_{25} \\ \dfrac{S_{o,\mu}(\rho_a)}{i\beta D_o(1-m)} & J\mu(\rho_a) & Y\mu(\rho_a) & 0 & b_{35} \\ \dfrac{S_{o,\mu}(\rho_b)}{i\beta D_o(1-m)} & J\mu(\rho_o) & Y\mu(\rho_o) & 0 & b_{45} \end{bmatrix} \qquad \ldots\ldots (29)$$

$$a_{11} = \frac{1}{2(\beta^2 D_o(1-m)} \sum_{s=0}^{\infty} \frac{(-1)^s \rho_a^{2(1+s)}}{[\{(1^2-\mu^2)(3^2-\mu^2)\ldots\ldots \{(2s+1)^2-\mu^2\}](1+s)}$$

iβ/ ∧⊃∞, b 3, s

$$a_{12} = \frac{\rho_a}{i\beta}\left\{(\mu-1) J\mu(\rho_c) S_{-1,\mu-1}(\rho_a) - J_{\mu-1}(\rho_a) S_{o,\mu}(\rho_a)\right\}$$

$$a_{13} = \frac{\rho}{i\beta}\left\{(\mu-1) Y\mu(\rho_a) S_{-1,\mu-1}(\rho_a) - Y(\rho_a) S_{o,\mu}(\rho_a)\right\}$$

ρa/ ∧⊃2, a_p/

$$b^{15} = \frac{1}{(m-1)^3 D_o} \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{1-m}{i\beta}\right)^{\eta+} \frac{ap}{(\eta+p+1)} \cdot \frac{(-1)^s \rho_a^{\eta+2+2s}}{[(\eta+1)^2-\mu^2]\ldots[(\eta+2s+1)^2-\mu^2](\eta+2+2s)}$$

$\rho_a^{\eta+2+2s}$/ s/, ρa/ ∧⊃ρo

$$b^{35} = \frac{i\beta}{(m-1)^3} \sum_{p=0}^{\infty} \left(\frac{1-m}{i\beta}\right)^{\eta+2} \frac{ap}{n+p+1} J_{\eta\,\mu}(\rho a) \qquad \ldots (30)$$

Replacing ρ_a by ρ_o in the expressions of a_{11}, a_{12}, a_{13}; b_{15}, b_{35} we get $a_{21}, a_{22}, a_{23}; b_{25}, b_{45}$ respectively

For a Particular case in which

r/, r/ r/,

$$T_o(\gamma) = T_o^{(1)}/\gamma,\ T_1(\gamma) = T_1^{(1)}$$

r^{-2m}/, r/ 1/9 r^3/,

$$\left\{\gamma^{-4m}[redacted]1+\log\gamma + \frac{\gamma^3}{}[redacted]\right\}$$

ϑo/ ϑ, α/

$$\nu_o = \frac{aT_o^{(1)}}{h^2} = 0.01 \text{ and } \nu_1 = \frac{\nu T_1^{(1)} F(h)}{h^3} = 0.20 \qquad \ldots (31)$$

and where
$a=1$, $b=0.01$, $p=1=n$, $a_o=0$, $\nu=0.25$, $m=0.5$ ... (32)
add for an assumed value of $i\beta = 0.5$, the augmented matrix (29) gives

∧⊃- #

$$\{A_1\ A_2\ A_3\ A_4\} = \frac{a_1}{D_o}$$

$\{0.029930 D_o\ 0.535250\ \text{-}\ 0.000026\ 0.0010137\}$ (33)

Now Eqs (31–33) and $i\beta=0{\cdot}5$ help us determine the appropriate load factor from Eq (23) to be

$$\frac{a_1}{D_o h} = 8.6631 \qquad \ldots (34)$$

Eqs (33) and (34) clearly give the value of the arbitrary constants A_1, A_2, A_3 and A_4 of Eq (51) as
$\{A_1, A_2, A_3, A_4\} = \text{-}8.6631\ h\{0.029930\ D_o\ 0.535250\ 0.000026\ 0.0010137\}$ 35

#

For such a case we have plotted $\dfrac{10^3 w}{8.6731 h}$ against r in Fig. 1.

Simply supported boundaries for the simply supported edges the required boundary conditions are

∂/

$$[u]_{r=a,b} = 0,\ [w]_{r=a,b} = 0 \text{ and} \left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right]_{r=a,b} = 0 \qquad \ldots [36]$$

dr^2/

R-5 Somule

...(36)

The corresponding argmented matrix for the evalution of A in this case is

$$\begin{matrix} a_{11} & a_{12} & c_{3} & 1 & c_{15} \\ a_{21} & a_{22} & a_{23} & 1 & b_{25} \\ a_{31} & a_{32} & a_{33} & 0 & M_1 \\ a_{41} & a_{42} & a_{43} & 0 & M_2 \end{matrix} \quad \ldots(37)$$

where

2s+1

$$a_{31} = \frac{1}{(i\beta)D_o}\left[\frac{\nu-m}{1-m} S_{o},_{\mu}(\rho_a) + \sum_{s=0}^{\infty} \frac{(-1)^s \;(\ldots+2s)\; \rho_a^{2s\times 1}}{(1^2-\mu^2)\;(3^2-\mu^2)\ldots\ldots[(2s+1)^2-\mu^2]}\right]$$

$$a_{32} = [\nu-m)\,J\mu(\rho_a) + \frac{1-m}{2}\rho_a\{J_{\mu-1}(\rho_a)-J_{\mu+1}(\rho_a)\}]$$

$$a_{33} = [(\nu-m)Y\mu'(\rho_a) + \frac{1-m}{2}\rho_a\{Y_{\mu-1}(\rho_a)-Y_{\mu\times 1}(\rho_a)\}]$$

$\mu+1$

$$M_1 = \frac{i\beta}{(m-1)^3}\sum_{p=0}^{\infty}\left[\frac{1-m}{i\beta}\right]^{\eta+2}\frac{ap}{n+p+1}\left[(\nu-m)\,J_{\eta\cdot\mu}(pa) + (1-m)\sum_{s=0}^{\infty}\frac{(-1)^s(\eta+1+2s)\rho_a\eta^{+1+2s}}{\{(\eta+1)^2-\mu^2\}\ldots\ldots\{(\eta+2s+1)-\mu^2\}}\right.$$

...(38)

Replacing ρ_a by ρ_a in the expressions of a_{31}, a_{32} a_{33}, M_1, we get a_{41}, a_{42}, a_{43}; M_2 respectively.

For the particular case mentioned in Eq. (31) and (32) and for the same assumed value of $i\beta = 0{\cdot}5$ the augmented matrix (37) gives

$$A_1\;A_2\;A_3\;A_4 = \frac{a_1}{D_o}\{0{\cdot}078773\;D_o\;\;1{\cdot}2803\;\;0{\cdot}000008\;\;0{\cdot}000729\} \quad \ldots(39)$$

The appropriate load factor is obtained from Eq. (23) with the help of Eqs. (31), (32, (33) and (39) in the form

$$\frac{a_1}{D_o h} = 10{\cdot}426 \quad \ldots(40)$$

The arbitrary constants A_1, A_2, A_3, A_4 of Eq. (21) may now be obtained from Eqs. (39) and (40) in completely known terms as

#

$$\{A_1\;A_2\;A_3\;A_4\} = 10{\cdot}426\,h\;\{0{\cdot}078773D\;\;1{\cdot}2803\;\;0{\cdot}000008\;\;0{\cdot}000729\} \quad \ldots(41)$$

10^3 w.

A plot of $\dfrac{10}{10.426h}$ against r is given for this case in Fig. 9

Numerical Results for $m=1$

Clamped boundaries — Using the boundary conditions (28) for the edges $r=a$ and $r=b$ we get the constants Bs from Eq. (25) in a matrix form $B_1\ B_2\ B_3\ B_4$ from its augmented matrix

$$\begin{bmatrix} \log a & \dfrac{a^k}{k} & \dfrac{a^{-k}}{(-k)} & 1 & C_{15} \\ \log b & \dfrac{b^k}{k} & \dfrac{b^{-k}}{(-k)} & 1 & C_{25} \\ 1 & a^k & a^{-k} & 0 & C_{35} \\ 1 & b^k & b^{-k} & 0 & C_{45} \end{bmatrix} \quad \ldots(42)$$

where

$$C_{15} = \frac{1}{D_o} \sum_{p=0}^{\infty} a_p \frac{a^{n+p+1}}{(n+p+1)^2 \{k^2-(n+p+1)^2\}}$$

$$C_{35} = \frac{1}{D_o} \sum_{p=0}^{\infty} \frac{a_p\ a^{n+k+1}}{(d+p+1)\ \{k^2-(n+p+1)^2\}} \quad \ldots(43)$$

C_{25}. C_{45} can also be written from the expressions of C_{15} and C_{35} respectively on replacing a^{n+q+1} by b^{n+p+1} in each case.

For the particular case in which

$v=10, l=1, p=1=n, a_o=0, \gamma=0{\cdot}25, m=1$ ...(44)

and for an assumed value of $\beta^2=2{\cdot}5$, the augmented matrix (42) gives

$$B_1\ B_2\ B_3\ P_4 = \frac{a_1}{D_o}$$

$$(-7{\cdot}4^{\cdot}020 \quad {\cdot}728718 \quad 6{\cdot}83055 \quad 3{\cdot}07341) \quad \ldots(45)$$

NowoEqs. (3), (44), (45) and $\beta^2=2{\cdot}5$ are used to get appropriate load factor from Eq. (27) in the form

$$\frac{a_1}{D_o h} = 0{\cdot}15863 \quad \ldots(46)$$

Eq. (45) and (46) give clearly the value of the arbitrary constant of Eq. (25) as

$B_1\ B_2\ B_3\ B_4 = -0{\cdot}15863\ h$

$\{-7{\cdot}40020 \quad 0{\cdot}728718 \quad 6{\cdot}83065 \quad 3{\cdot}07341\}$ ...(47)

For such a case we have plotted $\dfrac{W}{0{\cdot}15863\ h}$ against

against r Fig. 2.

Simply supported boundaries — Using the boundary conditions (36) we get the constant Bs from Eq. (25) in a matrix form $(B_1\ B_2\ B_3\ B_4$ from its augmented matrix

$$\begin{bmatrix} \log a & \dfrac{a^k}{k} & \dfrac{a^{-k}}{(-k)} & 1 & C_{15} \\ \log b & \dfrac{b^k}{k} & \dfrac{b^{-k}}{(-k)} & 1 & C_{25} \\ (1-v) & [1-v-k]a^k & [1-v+k]a^{-k} & 0 & N_1 \\ [1-v] & [1-v-k]b^k & [1-v+k]b & 0 & N_2 \end{bmatrix} \quad \ldots(48)$$

in which $N_1 = \dfrac{1}{D_o} \displaystyle\sum_{p=0}^{\infty} \frac{a_p\ [\gamma+n+p]a^{n+p+1}}{[n+p+1]\ \{[n+p+1]^2-k^2\}}$

and N_2 may be obtained from the expression of N_2 just by replacing a^{n+p+1} by b^{n+p+1}

For the particular case, Eqs. (31) and (44) and for the same assumed value of $\beta^2=2{\cdot}5$, the augmented matrix (48), gives $B_1\ B_2\ B_3\ B_4 =$

$$\frac{a_1}{D_o} \left\{ -15{\cdot}2107 \quad 1{\cdot}10972 \quad 4{\cdot}64860 \quad 1{\cdot}79170 \right\} \quad \ldots(49)$$

The appropriate load factor is obtained from Eq. (27) with the help of Eqs. (31), (44), (49) and $\beta^2=2{\cdot}5$ in the form

$$\frac{a_1}{D_o h} = 0{\cdot}28673 \quad \ldots(50)$$

The arbitrary constants B_1, B_2, B_3, B_4 of Eq. (25) may be obtained from Eqs. (49) and (50) as

$\{B_1\ B_2\ B_3\ B_4\} = -0{\cdot}28673\ h$

$\{-15{\cdot}2107 \quad 1{\cdot}10972 \quad 4{\cdot}64860 \quad 1{\cdot}79170\}$ ...(51)

For this case also, a plot of $\dfrac{W}{0{\cdot}28673\ h}$ against r is given in Fig. 2.

The author his gratitude to Dr A. Das of Jalpaiguri Government Engineering College, for his valuable

References

1. BERGER, H.M., *J. appl. Mech.*, **22** (1955), 465.
2. NASH, W.A. & MODEER, J.R. (1960), *Proc. Symp. theory of thin Elastic shells,* (North-Holland), Amsterdam), 1959, 331.
3. NOWINSKI, J.L. & ISMAIL, A.I., *ZAMP*, **15** (1964), 449.
4. DAS, A., **20** (2) (1972).
5. DAS, A., *Int. J. Mech. Math.*, 1973.
4. DAS, A., *Int. J. theor. Phys.*, (1972) **20** (2) (1972).
6. HEMP, W.S., *Aircraft Engng*, (1954), 126.
7. WILLIAMS, M.L., *J. Appl. Mech.*, **22** (1955), 458; **25** (2) (1958), 251.

R-2

Ind/

Ind/

Capt

Fig. 1 — Relation between W/h and r in clamped and simply supported boundaries for $m=0\cdot5$

Fig. 2 — Relation between W/h and r in dampled and simply supported boundaries for $m=1\cdot0$

Nomenclature

a,b = outer and inner radii of the plate

$$D = \frac{Eh^3}{12[1-\nu^2]} = \frac{E_o h^3}{12[1-\nu^2]} = \text{the flexural rigidity of the plate}$$

$r^{2m} = D_o r^{2m}$

$\wedge \sim r^{2m}$

$\frac{E_o \lambda^3}{12(1-\nu^2)} r^{2m} = D_o r^{2m}$

$E = E_o r^{2m}$ = Young's modulus of the material of the plate at a distance r from the centre

$e = e + _{rr}e_{qq}$ = first strain invariant

$e_{rr} + e_{\theta\theta}$

$$e_{rr} = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{da}\right)^2 = \text{three dimensional thermal strains}$$

$$\dot{e}_{\theta\theta} = \frac{u}{r}$$

$\wedge \sim$ Radial and tangential strains

$$\varepsilon_{rr} = e_{rr} - \frac{d^2w}{dr^2} \quad = \text{radial and tangential strains}$$

$\hat{\varepsilon}_{rr}, \wedge \sim z$

$$e_{\theta\theta} = c_{\theta\theta} - \frac{z}{r}\frac{dw}{dr} e_{rr}$$

∇, e_θ

$\delta, \wedge \bar{e}_{zz}$ three dimensional thermal strains

h = thickness of the plate

$q=q(r)$ = normal load intensity

$T=T(r,z)$ = temperature distribution in the plat

$\wedge \sim \alpha$

= coefficient of linear expansion of the material of the plate

U_m, U_b, U_q = membrane strain energy, bending strain energy, and energy contribution from pressure loading respectively, in the absence of heating or cooling

U,W = radial and lateral displacements

V = potential energy of deformation

W_T = energy contribution from heating

λ, μ = lame 's constants

σ_z = lateral stress

ν = Poisson's ratio of the material of the plate.

$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$

$= \frac{d}{dr} + \frac{1}{r}\frac{d}{dr}$

TIME-HARDENING AND TIME-SOFTENING ANISOTROPIC NON-HOMOGENEOUS SPHERICAL SHELL

Subhas Chandra Roy
JALPAIGURI GOVERNMENT ENGINEERING COLLEGE,
JALPAIGURI, W. B., INDIA

ABSTRACT : *The transien tdisplacements and stresses in radially symmetric motion of a thick spherical shell are obtained where the loads applied to both surfaces of the shell are special type functions of time. The material of the structure is spherically isotropic and, in addition, is continuously inhomogeneous with mechanical properties varying along the radius and depending also on time.*

1. Introduction

A generalization of the elastic properties has been made by Paria [1]. The Young's modulus has been assumed to be a function of time but independent of co-ordinates. According to him, the time-hardening material is that in which the Young's modulus increases with time and time-softening material is that in which Young's modulus decreases with time. The physical justification for such assumption may be found in concrete, for example, in which the elastic properties vary during its maturing periods i. e. during the formation of gel. The chemical action is termed as a process of hydration and it is known to be exothermic. The elastic properties of materials may vary due to variations in temperature, moistnre contents and similar other varying factors. Das [2] has introduced this new idea to obtain transient displacements and stresses that exist in thick elastic shells under internally and externally applied time-dependent loads in radial motion of an infinite long circular cylindrical shell and in radially symmetric motion of a spherical shell.

The physical justification given above is still valid in the cases in which the elastic co-efficients vary from point to point and time to time. Das [3] and Roy [4] have successfully introduced this idea to a thin circular plate and an infinitely long thick cylindrical shell, respectively.

This paper is intended to solve in a general manner, the problem of radial vibration of a spherical shell, subjected to dynamic loads, the elastic co-efficients being dependent on time as well as on the co-ordinates of the point under consideration.

2. Basic Equations

The radial motion of a thick spherical shell made of spherically istropic and continuously nonhomogeneous material may be represented by the followig equations,

$$\frac{\partial \sigma_r}{\partial r}+\frac{2}{r}(\sigma_r-\sigma_\phi)=\rho\,\frac{\partial^2 u}{\partial t} \quad \ldots \quad (1)$$

$$\sigma_r = c_{11}\,\frac{\partial u}{\partial r}+2c_{12}\,\frac{u}{r} \quad \ldots \quad (2)$$

$$\sigma_\theta=\sigma_\phi=c_{12}\frac{\partial u}{\partial r}+[c_{22}+c_{23}]\frac{u}{r} \quad \ldots \quad (3)$$

where c_{ij} are functions of elastic moduli, Poisson's ratios, radius (for a nonhomogeneous material [5]) and time. For the present paper it is chosen as

$$c_{ij}(r,t)=\mu_{ij}\,.f(r)\left[1+\alpha\exp\left(-\frac{t}{t_0}\right)\right] \quad \ldots \quad (4)$$

where $t_0>0$, is the reference value of time, α is the prescribed parameter and μ_{ij}, the elastic constant. From (4) c_{ij} runs from $\mu_{ij}\,.f(r)\,[1+\alpha]$ at $t=0$ $t_0\,\mu_{ij}\,.f(r)$ as t approaches infinity. If α is positive, say α_1^2, the material is time-softening and if α is negative, say $(-\alpha_2^2)$, it is time-hardening. It is known that $\alpha_2<1$ in case of concrete maturing.

Poisson's ratio, may be assumed to be constant in time and space, since, its changes in time are not essential and the changes with the co-ordinates of the point considered are negligible in relation to those of the modulus, a tleast, in concrete. The time dependency of the density of the material is not at all appreciable but it may vary from point to point. Here the density is assumed to be

$$\rho=\frac{\mu_{11}\,f(r)}{C_0^2} \quad \ldots \quad (5)$$

C_0^2 is a constant. Using (2), (3), (4) and (5) on (1) we get,

$$\frac{\partial^2 u}{\partial r^2}+\left(2+\frac{r}{f}\frac{\partial f}{\partial r}\right)\frac{1}{r}\frac{\partial u}{\partial r}+\frac{2}{\mu_{11}}\left\{\mu_{12}\left(1+\frac{r}{f}\frac{\partial f}{\partial r}\right)-(\mu_{22}+\mu_{23})\right\}\frac{u}{r^2}$$

$$=\frac{1}{C_0^2\{1+\alpha\exp(-t/t_0)\}}\,\frac{\partial^2 u}{\partial t^2} \qquad t>0,\ a\leqslant r\leqslant b \quad \ldots \quad (6)$$

The initial conditions are

$$u=0 \quad \ldots \quad (7)$$

and

$$\partial u/(\partial t)=0, \qquad t=0,\ a\leqslant r\leqslant b \quad \ldots \quad (8)$$

Boundary conditions are

$$\left[\sigma_r (r, t) \right]_{r=a} = \left[\mu_{11} \frac{\partial u}{\partial r} + 2\mu_{12} \frac{u}{r} \right] f(r) [1+\alpha \exp(-t/(t_0)]$$
$$= A (\exp -t/(2t_0)), \; r=a, \; t>0 \qquad \cdots \qquad (9)$$

$$\left[\sigma_r (r, t) \right]_{r=b} = \left[\mu_{11} \frac{\partial u}{\partial r} + 2\mu_{12} \frac{u}{r} \right] f(r) [1+\alpha \exp(-t/(t_0))]$$
$$= B (\exp - t/(2t_0)), \; r=b, \; t>0 \qquad \cdots \qquad (10)$$

3. The Solution of the Problem

For

$$f(r) = 1/r \qquad \cdots \qquad (11)$$

the equation (6) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{m^2}{r^2} u = \frac{1}{C_0^2[1+\alpha \exp(-t/(t_0))]} \frac{\partial^2 u}{\partial t^2} \qquad \cdots \qquad (12)$$

where

$$m^2 = (\mu_{22}+\mu_{23})/\mu_{11} \qquad \cdots \qquad (13)$$

From Cinelli [6], the displacement $u(r, t)$, which is the solntion of (12), satisfying boundary conditions (9) and (10) is found to be

$$u(r, t) = \pi^2/2 \sum_{\xi_j} \xi_j^2 [\xi_j J_m'(\xi_j b)+k J_m]^2 \frac{\bar{u}(\xi_j, t)}{F_m(\xi_j)} C_m(r, \xi_j) \qquad \cdots \qquad (14)$$

where $C_m(r, \xi_j)$, and ξ_j are given in the equations (2), (4) and (5) of Cinelli [6], with constants,

$$h=(2\mu_{12}/a_2)/(\mu_{11}/a) \text{ and } k=(2\mu_{12}/b^2)/(\mu_{11}/b) \qquad \cdots \qquad (15)$$

For elastic constants $\mu_{22}= -\mu_{23}$ we find from (13) $m=0$, and for $\mu_{11}=2(\mu_{22}+\mu_{23})$ $m=1$, again for $9\mu_{11}=8(\mu_{22}+\mu_{23})$, the value of m becomes (3/2). For this last case i. e., where $m=(3_2/2)$, we can use

$$J_{\frac{3}{2}}(\xi_j r) = \left(\frac{2}{\pi\xi_j r} \right)^{\frac{1}{2}} \left(\frac{\sin \xi_j r}{\xi_j r} - \cos \xi_j r \right) \qquad \cdots \qquad (16)$$

$$Y_{\frac{3}{2}}(\xi_j r) = -\left(\frac{2}{\pi\xi_j r} \right)^{\frac{1}{2}} \left(\frac{\cos \xi_j r}{\xi_j r} + \sin \xi_j r \right) \qquad \cdots \qquad (17)$$

and

$$C_{3/2}(r, \xi_j) = \left(\frac{2}{\pi\xi_j r} \right)^{1/2} \left\{ \left(\frac{\sin \xi_j r}{\xi_j r} - \cos \xi_j r \right) \times [\xi_j Y'_{3/2}(\xi_j a) \right.$$
$$\left. + hY_{3/2}(\xi_j a)] + \left(\frac{\cos \xi_j r}{\xi_j r} + \sin \xi_j r \right) [\xi_j J'_{3/2}(\xi_j a) + hJ_{3/2}(\xi_j a)] \right\} \qquad (18)$$

so that $u(r,t)$ of (14) may be expressed in terms of Bessel functions of non-integral order and hence in terms of circular functions. Now $\bar{u}(\xi_j,t)$ which is a generalized Hankel transform of $u(r. t)$, (vide [7]), and used in equation

(24) is a solution of the following differential equation

$$\frac{\partial^2 \bar{u}}{\partial t^2} + C_0^2\, \xi_j^2\left[1+\alpha \exp\left(-\frac{t}{t_0}\right)\right]\bar{u} = \frac{2}{\pi \mu_{11}}\, C_0^2 \left[P_{j,m} b.B\left(\exp \frac{t}{2t_0}\right) - a.A\left(\exp - \frac{t}{2t_0}\right)\right] \quad \ldots \quad (19)$$

in which

$$P_{j,m} = \frac{\xi_j\, J'_m(\xi_j\, a) + hJ_m(\xi_j\, a)}{\xi_j\, J'_m(\xi_j b) + KJ_m(\xi_j\, a)} = \text{a constant.} \quad \ldots \quad (20)$$

The corresponding initial conditions [from (7) and (8)] are

$$\bar{u} = \frac{\partial \bar{u}}{\partial t} = 0, \quad t=0, \quad a \leqslant r \leqslant b \quad \ldots \quad (21)$$

The solution of the equation (19) satisfying the initial conditions (21) is given by, (vide [2]),

$$\begin{aligned}\bar{u}(\xi_j, t) = \sum_{n=0}^{\infty} \Bigg[\frac{i\pi x_0}{2 \sinh \pi k_j} \Big\{ & R_{j,m,n}(x_0)\Big(J'_{ikj}(x_0)\, J_{-ikj}(x) \\ & - J'_{-ikj}(x_0)\, J_{ikj}(x)\Big) + R'_{j,m,n}(x_0)\Big(J_{-ikj}(x_0)\, J_{ikj}(x) \\ & - J_{ikj}(x_0)\, J_{-ikj}(x)\Big)\Big\} + R_{j,m,n}(x)\Bigg] \end{aligned} \quad \ldots \quad (22)$$

and

$$\begin{aligned}\bar{u}(\xi_j, t) = \sum_{n=0}^{\infty} \Bigg[\frac{-\pi y_0}{2 \sinh \pi k_j} \Big\{ & Q_{j,m,n}(iy_0)\Big(J'_{ikj}(iy_0)\, J_{-ikj}(iy) \\ & - J'_{-ikj}(iy_0)\, J_{ikj}(iy)\Big) + Q'_{j,m,n}(iy_0)\Big(J_{ikj}(iy_0)\, J_{ikj}(iy) \\ & - J_{ikj}(iy_0)\, J_{-ikj}(iy)\Big)\Big\} + Q_{j,m,n}(iy)\Bigg] \end{aligned} \quad \ldots \quad (23)$$

respectively, where $x=\alpha_1 k_j \tau$, $y=\alpha_2 k_j \tau$ and $x_0=[x]_{t=0}=\alpha_1 k_j$, $y_0=[y]_{t=0}=\alpha_2 k_j$. The functions $R_{j,m,n}(x)$ and $Q_{j,m,u}(iy)$ used in (22) and (23) are defined as follows

$$R_{j,m,n}(x) = \frac{8t_0^2 c_0^2}{\pi \mu_{11}}\left[P_{j,m} \frac{b.b_n S_{n+n_1, ik_j}(x)}{(\alpha_1 k_j)^{n+n_1+1}} - \frac{a.a_n S_{n+n_2, ik_j}(x)}{(\alpha_1 k_j)^{n+n_2+1}}\right] \ldots \quad (24)$$

$$Q_{j,m,n}(iy) = \frac{8t_0^2 c_0^2}{\pi \mu_{11}}\left[P_{j,m} \frac{b.b_n S_{n+n_1, ik_j}(iy)}{(i\alpha_2 k_j)^{n+n_1+1}} - \frac{a.a_n S_{n+n_2, ik_j}(iy)}{(i\alpha_2 k_j)^{n+n_2+1}}\right] \ldots \quad (25)$$

in which $S_{\mu\nu}(z)$ is a Lommel function and

$$t=2t_0 \log(1/\tau), \quad 2\xi_j\, t_0\, c_0 = k_j \text{ (chosen as non-integer here).} \ldots \quad (26)$$

The preseribed functions $A(\tau)$ and $B(\tau)$ are supposed to be in the form

$$A(\tau) = \sum_{n=0}^{\infty} a_n\, \tau^{n+n_2+1}, \quad B(\tau) = \sum_{n=3}^{\infty} B_n\, \tau^{n+n_1+1} \ldots \quad (27)$$

in which n_1, n_2 may take up any value.

The solution of (19) for time-softening material is given by (22) where as the equation (23) denotes the solution of (19) if the material concerned is time-hardening.

Now the expression for the stresses σ_r, σ_θ and σ_ϕ as in equation (2) and (3) can easily be obtained by inserting $u\ (r,t)$ from the equation (14) where $\bar{u}\ (\xi_j,t)$ would be replaced by (22) or (23) as the case demands.

In a particular case, when the outer surface is stress-free and the inner surface is under an exponentially decaying load, then

$$b_n=0, \text{ and } A\ (\tau)=a_0\tau \qquad \dots \qquad (28)$$

Therefore

$$R_{j,m,n}{}^{(x)}=R_{j,m}{}^{(x)}=-\frac{8t_0{}^2c_0{}^2}{\pi\mu_{11}}\times\frac{a.a_0\ S_{0,i}{}^{(x)}{}_{k_j}}{\alpha_1 k_j} \qquad \dots \qquad (29)$$

and

$$Q_{j,m,n}{}^{(iv)}=Q_{j,m}{}^{(iv)}=-\frac{8t_0^2c_0^2}{\pi\mu_{11}}\times\frac{a.a_0 S_{0,l}^{(iv)}k_j}{i\alpha_2 k_j} \qquad \dots \qquad (30)$$

so that the equation (22) or (23) reduces to one term only.

From (4) it is evident that the elastic co-efficients would be independent of time if we make $\alpha \to 0$. In such a case, we use the following relations, Watson [7],

$$J_\nu'(z)=\frac{1}{2}\left[J_{\nu-1}{}^{(z)}-J_{\nu+1}{}^{(z)}\right]$$

$$J_\mu(az)\ J_\nu(bz)=\frac{\left(\frac{az}{2}\right)^\mu\left(\frac{bz}{2}\right)^\nu}{\Gamma(\nu+1)}\ \sum_{l=0}^{\infty}\left\{\frac{(-1)^l\left(\frac{az}{2}\right)^{2l}}{l\,!\ \Gamma\,(\mu+l+1}\ 2F^1(-l,-\mu-l\ ;\ \nu+1\ ;\ \frac{b^2}{a^2})\right\} \qquad \dots \qquad (31)$$

and put the Lommel functions in series form and collect the terms independent of α, i.e., α_1 or α_2 from the expression of $\bar{u}\ (\xi_j,t)$ in (22) or (23) to find that

$$\bar{u}\ (\xi_j,t)=\frac{4c_0t_0}{\pi\mu_{11}\xi_j}\ \sum_{n=0}^{\infty}\Bigg[P_{j,m}\frac{b.b_n}{[(n+n_1+1)^2+4c_0^2t_0^2\xi_j^2]}\Bigg\{(n+n_1+1)\sin\ (c_0\xi_jt)$$

$$-(2c_0\ t_0\ \xi_j)\left[\cos c_0\ \xi_jt-\exp\left(-\frac{n+n_1+1}{2t_0}.t\right)\right]\Bigg\}-\frac{a.a_n}{[(n+n_2+1)^2+4c_0^2\ t_0^2\ \xi_j^2}$$

$$\times\left\{(n+n_2+1)\sin c_0\ \xi_jt-2\ c_0t_0\xi_i\left[\cos c_0\ \xi_j\ t-\exp\left(-\frac{n+n_2+1}{2t_0}.\ t\right)\right]\right\}\Bigg]$$

And for the above particular case defined in equation (28) $\bar{u}\ (\xi_j,t)$ reduces to

$$\bar{u}(\xi_j,t)=-\frac{4c_0t_0}{\pi\mu_{11}\xi_j}\left[\frac{a.a_0}{1+4c_0^2\ t_0^3\ \xi_j^2}\right]\Bigg\{\sin c_0\xi_jt-2c_0t_0\xi_jt\Bigg[\cos c_0\xi_jt$$

$$-\exp\left(-\frac{t}{2t_0}\right)\Bigg]\Bigg\} \qquad \dots \qquad (33)$$

It is interesting to note that $\bar{u}(\xi_j, t)$ statisfying the initial conditions (21) can also be directly obtained from (19) when the elastic coefficients are independent of time i. e., where $\alpha \to 0$, by using Laplace transform and convolution integral as

$$\bar{u}(\xi_j, t) = \frac{2c_0}{\pi\mu_{11}\xi_j}\int_0^t \left\{ P_{s,m}\, b.B\left(\exp - \frac{T}{2t_0}\right) - a.A\left(\exp - \frac{T}{2t_0}\right)\right\} \sin c_0\, \xi_j\, (t-\tau)\, dT \quad \cdots \quad (34)$$

in which A $(\exp - t/2t_0)$ and B $(\exp - t/2t_0)$, the radial stresses on the inner and outer surfaces, may be any functions of time. In particular, when they are expressible in series form as in (27), the equation (34) gives the desired result obtained in (32).

I am indebted to Dr. A. Das for his kind guidance and valuable suggestion during the preparation of this paper.

References

1. Paria, G. (1966), *I. J. M. M.* 4, 2, 117.
2. Das, A. (1972), *I. J. M. M.* 10, 1 and 2, 26-32.
3. Das, A. (1973), *Bull. Cal. Math. Soc.*, 65, 115-122.
4. Roy, S. C. "*Time-hardening and Time-softening anisotropic non-homogenous Cylindrical Shell*,, Communicated to Indian J. *Theo. Phys.*
5. Grief. G. and Chou, S. C. (1971). *J. Appl. Mech.*, 38, 1, 51-57,
6. Cinelli, G. (1966), *J. Appl. Mech.*, 33, 4, 825-830.
7. Watson, G. (1952), *Theory of Bessel Function*. Cambridge University Press, 43, 45, 47 and 148.

Received, 29 January 1975 ; Revised, 19 February 1977

Indian Journal of Technology
Vol. 14, October 1976, pp. 478-480

Torsional Vibration of a Non-homogeneous Cone with Spherical Caps

SUBHAS CHANDRA ROY
Jalpaiguri Government Engineering College, Jalpaiguri

Received 10 *June* 1975; *accepted* 15 *October* 1975

The torsional vibration of a truncated cone with spherical caps is discussed. Equations of motion are solved under two sets of boundary conditions: (i) When both the spherical end caps are fixed, and (ii) when one end cap is fixed and the other is free. The frequencies of vibration have been calculated.

TORSIONAL vibration of a cone under different edge conditions is a well-known mathematical problem. The shell under consideration here is a truncated cone with spherical caps, made of such a material that the rigidity varies as a power of the distance from the vertex of the cone and the density is proportional to some other power of the distance from the vertex of the cone. The problem with such a wide variation of parameters is inevitably complicated; however, it is mathematically tractable for at least two types of boundary conditions. The conditions are evaluated and the roots of the frequency equations are presented.

Problem and Its Solution

Let us use the spherical polar co-ordinates (r, θ, ϕ) with the vertex of the cone as the origin and the axis of the cone coinciding with the coordinate system.

The equations of motion according to Love[1] are

$$\frac{\partial\tau_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\varphi}+\frac{1}{r}[2\tau_{rr}-\tau_{\theta\theta}-\tau_{\varphi\varphi}+\tau_{r\theta}\cot\theta] = \frac{\rho\partial^2_{u_r}}{\partial t^2} \quad ...(1)$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}[(\tau_{\theta\theta}-\tau_{\varphi\varphi})\cot\theta+3\tau_r\phi] = \frac{\rho\partial^2_{u_\theta}}{\partial t^2} \quad ...(2)$$

$$\frac{\partial\tau_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\varphi\varphi}}{\partial\varphi}+\frac{1}{r}[3\tau_{r\varphi}+2\tau_{\theta\varphi}\cot\theta] = \frac{\rho\partial^2_{u_\phi}}{\partial t^2} \quad ...(3)$$

We assume here

$$u_r = u_\theta = 0 \quad ...(4)$$

$$u_\phi = f(r)\sin\theta\, e^{ipt},\ (i=\sqrt{-1}) \quad ...(5)$$

The expressions of the rigidity and the density are chosen to be

$$\mu = \mu_0\, r^{2m} \quad ..(6)$$

and

$$\rho = \rho_0\, r^{2n} \quad ...(7)$$

respectively.

μ_0, ρ_0 are constants, and m and n may take any value.

All the stress components except $\tau_{r\varphi}$ are zero. This stress component is given by

$$\tau_{r\varphi} = \mu_0\, r^{2m}\left(\frac{df}{dr}-\frac{f}{r}\right)\sin\theta\ e^{ipt} \quad ...(8)$$

Since

$$\tau_{\theta\varphi} = \mu_0\, r^{2m-1}\left(\frac{\partial u_\phi}{\partial\theta}-u_\varphi\cot\theta\right)=0 \quad ...(9)$$

the conical surface is free from stress.

Eqs. (1) and (2) are identically satisfied, while Eq. (3) reduces to

$$\frac{\partial}{\partial r}\left[\mu_0 r^{2m}\left(\frac{df}{dr}-\frac{f}{r}\right)\right]+3\mu_0\, r^{2m-1}\left(\frac{df}{dr}-\frac{f}{r}\right) = -\rho_0 r^{2n}p^2 f \quad ...(10)$$

Eq. (10) may be rewritten as

$$r^2\frac{d^2f}{dr^2}+2(m+1)r\frac{df}{dr}+\left[-2(m+1)+\frac{\rho_0}{\mu_0}p^2r^{2(n-m+1)}\right] f=0 \quad ...(11)$$

The solution of Eq. (11) according to Forsyth is

$$f(r)=r^{-\frac{1}{2}(2m+1)}\left[AJ_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{r^{n-m+1}}{n-m+1}\right)+BY_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{r^{n-m+1}}{n-m+1}\right)\right]\ \text{for } m\neq n+1 \quad ...(12)$$

where

$$\mu=\frac{2m+3}{2(n-m+1)} \quad ...(13)$$

and A and B are arbitrary constants.

Eqs. (5) and (12) clearly show that

$$u_\phi=r^{-\frac{1}{2}(2m+1)}\left[AJ_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{r^{n-m+1}}{n-m+1}\right)+BY_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{r^{n-m+1}}{n-m+1}\right)\right]\sin\theta e^{ipt} \quad ...(14)$$

Case 1 — We take end caps to be fixed so that the boundary conditions are

$$\left.\begin{aligned}u_\phi &= 0,\ \text{on } r=a\\ &= 0,\ \text{on } r=b\end{aligned}\right\} \quad ...(15)$$

where a and b are the radii of the spherical caps $(b>a)$.

Therefore, from Eqs. (14) and (15) we get

$$AJ_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{a^{n-m+1}}{n-m+1}\right)+BY_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{a^{n-m+1}}{n-m+1}\right)=0 \quad \dots(16)$$

$$AJ_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{b^{n-m+1}}{n-m+1}\right)+BY_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{b^{n-m+1}}{n-m+1}\right)=0 \quad \dots(17)$$

Assuming

$$p\sqrt{\frac{\rho_0}{\mu_0}}\frac{a^{n-m+1}}{n-m+1}=x \quad \dots(18)$$

$$p\sqrt{\frac{\rho_0}{\mu_0}}\frac{b^{n-m+1}}{n-m+1}=\alpha x,\ (\alpha>1) \quad \dots(19)$$

and eleminating A and B from Eqs. (16) and (17), we find the frequency equation in the form

$$J_\mu(x)Y_\mu(\alpha x)-Y_\mu(x)J_\mu(\alpha x)=0 \quad \dots(20)$$

The sth root of Eq. (20), in order of magnitude, according to Gray and Mathews[3] is

$$x_\mu^{(s)}=\delta+\frac{p'}{\delta}+\frac{q-p'^2}{\delta^3}+\frac{r-4p'q+2p'^3}{\delta^5}+\dots\dots$$

where

$$\delta=\frac{s\pi}{\alpha-1},\ p'=\frac{4\mu^2-1}{8\alpha},\ q=\frac{4(4\mu^2-1)(4\mu^2-25)(\alpha^3-1)}{3(8\alpha)^3(\alpha-1)}$$

$$r=\frac{32(4\mu^2-1)(16\mu^4-456\mu^2+1073)(\alpha^5-1)}{5(8\alpha)^5(\alpha-1)} \quad \dots(21)$$

In a particular case, where $n=2m+0{\cdot}5$, Eq. (13) gives $\mu=1$ and for $\alpha=2$ the lowest root of Eq. (20) may be found from Eq. (21) as

$$x_1^{(1)}=3{\cdot}1971$$

Using the above root in Eq. (18) one gets,

$$p=3{\cdot}1971\frac{2m+3}{2a^{m+1\cdot5}}\sqrt{\frac{\mu_0}{\rho_0}} \quad \dots(22)$$

Again, for $m=-3/2$ Eq. (13) gives $\mu=0$ and the first three roots of Eq. (20) for different values of α are obtained from Eq. (21) as given in Table 1. Frequency p follows from Eq. (18) as

$$p=\frac{x_0^{(s)}\ (2n+5)}{2.a^{n+2\cdot5}}\sqrt{\frac{\mu_0}{\rho_0}} \quad \dots(23)$$

Case 2 — Let us assume that one end cap is fixed at $r=a$ and the other end at $r=b$ is free, so that the boundary conditions are

$$\left.\begin{aligned} u_\varphi &= 0 \text{ on } r=a\\ \text{and } \tau_{r\varphi} &= 0 \text{ on } r=b\end{aligned}\right\} \quad \dots(24)$$

when μ is a non-integer the solution of Eq. (11) may be found from Eq. (12) only on replacing Y_μ by $J_{-\mu}$ and hence Eq. (14) for this case stands as

$$u_\phi=r^{-\frac{1}{2}(2m+1)}\left[AJ_\mu\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{r^{n-m+1}}{n-m+1}\right)+BJ_{-\mu}\left(p\sqrt{\frac{\rho_0}{\mu_0}}\frac{r^{n-m+1}}{n-m+1}\right)\right]\sin\theta\ e^{ipt}$$

Also

$$\tau_{r\phi}=\frac{\mu_0 r^{2m}}{r^{m+3/2}}\Bigg[-(m+3/2)\left\{AJ_\mu\left(\frac{p}{n-m+1}\sqrt{\frac{\rho_0}{\mu_0}}r^{n-m+1}\right)+BJ_{-\mu}\left(\frac{p}{n-m+1}\sqrt{\frac{\rho_0}{\mu_0}}r^{n-m+1}\right)\right\}$$
$$+p\sqrt{\frac{\rho_0}{\mu_0}}r^{n-m+1}\left\{AJ'_\mu\left(\frac{p}{n-m+1}\sqrt{\frac{\rho_0}{\mu_0}}r^{n-m+1}\right)+BJ'_{-\mu}\left(\frac{p}{n-m+1}\sqrt{\frac{\rho_0}{\mu_0}}r^{n-m+1}\right)\right\}\Bigg] \quad \dots(25)$$

Applying the boundary conditions stated in Eq. (24) and using Eqs. (18) and (19) we get,

$$AJ_\mu(x)+BJ_{-\mu}(x)=0 \quad \dots(26)$$

and

$$A\left[-\left(m+\frac{3}{2}\right)J_\mu(\alpha x)+p\sqrt{\frac{\rho_0}{\mu_0}}b^{n-m+1}J'_\mu(\alpha x)\right]+B\left[-\left(m+\frac{3}{2}\right)J_{-\mu}(\alpha x)+p\sqrt{\frac{\rho_0}{\mu_0}}b^{n-m+1}J'_{-\mu}(\alpha x)\right]=0 \quad \dots(27)$$

Eliminating A and B from Eqs. (26) and (27), we get the frequency equation as

$$\left(m+\frac{3}{2}\right)[J_\mu(\alpha x)J_{-\mu}(x)-J_\mu(x)J_{-\mu}(\alpha x)]$$
$$=p\sqrt{\frac{\rho_0}{\mu_0}}b^{n-m+1}[J'_{-\mu}(\alpha x)J_\mu(x)-J'_{-\mu}(\alpha x)J_\mu(x)] \quad \dots(28)$$

In a particular case, when $n=3m+2$, Eq. (13) shows that $\mu=\frac{1}{2}$ and, therefore, Eq. (28) reduces to

$$(\alpha-1)x\cot(\alpha-1)x+\left(\frac{1}{\alpha}-1\right)=0 \quad \dots(29)$$

For different values of α, the first three roots of Eq. (29) are tabulated in Table 2, vide Carslaw and Jaeger[4].

Therefore, from Eqs. (18) and (19)

$$p=\frac{(2m+3)\{(\alpha-1)x_{\frac{1}{2}}^{(s)}\}}{b^{2m+3}-a^{2m+3}}\sqrt{\frac{\mu_0}{\rho_0}} \quad \dots(30)$$

TABLE 1 — FIRST THREE ROOTS OF EQ. 20 FOR DIFFERENT VALUES OF α AS OBTAINED FROM EQ. 21

α	$x_0^{(1)}$	$x_0^{(2)}$	$x_0^{(3)}$
1·2	15·7014	31·4126	47·1217
1·5	6·2702	12·5598	18·8451
2·0	3·1230	6·2734	9·4182
3·0	1·5485	3·1291	4·7030
4·0	1·0244	2·0809	3·1322

TABLE 2 — FIRST THREE ROOTS OF EQ. 29 FOR DIFFERENT VALUES OF x

α	$(\alpha-1)\alpha_{\frac{1}{2}}^{(1)}$	$(\alpha-1)x_{\frac{1}{2}}^{(2)}$	$(\alpha-1)x_{\frac{1}{2}}^{(3)}$
1·25	1·4320	4·6696	7·8284
2·5	1·0528	4·5822	7·7770
5·0	0·7593	4·5379	7·7511
10·0	0·5423	4·5157	7·7382
20·0	0·3854	4·5045	7·7317

Relevant case $(m = n+1)$: When $m = n+1$, the differential equation (Eq. (11) reduces to

$$r^2\frac{d^2f}{dr^2}+2(m+1)r\frac{df}{dr}+\left[-2(m+1)+\frac{\rho_0}{\mu_0}p^2\right]f=0$$

The solution of the above equation is

$$f(r) = r^{-(m+\frac{1}{2})}\left[A\cos\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log r\right)\right.$$
$$\left.+B\sin\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log r\right)\right] \quad ...(31)$$

Therefore,

$$u_\phi = r^{-(m+\frac{1}{2})}\left[A\cos\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log r\right)\right.$$
$$\left.+B\sin\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log r\right)\right]\sin\theta e^{ipt}$$

$$\tau_{r\phi} = \frac{\mu_0 r^{2m}}{r^{m+3/2}}$$
$$\left[A\left\{-(m+3/2)\cos\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log r\right)\right.\right.$$
$$\left.-\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\sin\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log r\right)\right\}$$
$$+B\left\{-(m+3/2)\sin\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log r\right)\right.$$
$$\left.\left.+\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\cos\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log r\right)\right\}\right]$$
$$\sin\theta e^{ipt} \quad ...(32)$$

Case (i): Using the boundary conditions given in Eqs. (15) on (32) one finds

$$A\cos\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log a\right)$$
$$+B\sin\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log a\right)=0 \quad ...(33)$$

$$A\cos\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log b\right)$$
$$+B\sin\left(\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log b\right)=0 \quad ...(34)$$

Eliminating A and B from Eqs. (33) and (34) the required frequency equation becomes

$$\sin\left\{\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\,.\log\frac{b}{a}\right\}=0 \quad ...(35)$$

From Eq. (35) one obtains

$$p=\left[\frac{\mu_0}{\rho_0}\left\{\frac{\bar{n}^2\pi^2}{(\log b/a)^2}+\left(m+\frac{3}{2}\right)^2\right\}\right]^{\frac{1}{2}} \quad ...(36)$$

where $\bar{n}$ is zero or any integer.

Case (ii): Taking

$$\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log a = y \quad ...(37)$$

and

$$\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\log b = \beta y \quad ...(38)$$

TABLE 3 — FIRST THREE ROOTS OF EQ. (41) FOR DIFFERENT VALUES OF m, WHEN $b = 2{\cdot}7183\ a$

m	$\frac{y_1}{\log a}$	$\frac{y_2}{\log a}$	$\frac{y_3}{\log a}$
−0·5	0	4·4934	7·7253
−1·5	1·5708	4·7124	7·8540
−2·5	2·0288	4·9132	7·9787

TABLE 4 — FIRST THREE ROOTS OF EQ. 41 FOR DIFFERENT VALUES OF m, WHEN $b = 1{\cdot}6488\ a$

m	$\frac{y_1}{2\log a}$	$\frac{y_2}{2\log a}$	$\frac{y_3}{2\log a}$
−0·5	1·1656	4·6042	7·7899
−1·5	1·5708	4·7124	7·8540

and applying the boundary conditions, Eqs. (24) on (32), one finds

$$A\cos y+B\sin y = 0 \quad ...(39)$$

and

$$A\left[-\left(m+\frac{3}{2}\right)\cos\beta y-\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\sin\beta y\right]$$
$$+B\left[-\left(m+\frac{3}{2}\right)\sin\beta y+\sqrt{p^2\frac{\rho_0}{\mu_0}-(m+3/2)^2}\cos\beta y\right]$$
$$=0 \quad ...(40)$$

Eliminating A and B from Eqs. (39) and (40), we get the frequency equation as

$$\left\{\left(\frac{\log b}{\log a}-1\right)y\right\}\cot\left\{\left(\frac{\log b}{\log a}-1\right)y\right\}-\left(m+\frac{3}{2}\right)\log\frac{b}{a}=0 \quad ...(41)$$

If $b = 2{\cdot}7183a$, then for different values of m the first three roots of Eq. (41) are given in Table 3, vide Carslaw and Jaeger[4].

When $b = 1{\cdot}6488a$, for different values of m, the first three roots of Eq. (41) are given in Table 4, vide Carslaw and Jaeger[4].

When the roots are known from Tables 3 and 4, the frequency may be found from Eq. (37) as

$$p=\sqrt{\left\{\left(\frac{y}{\log a}\right)^2+\left(m+\frac{3}{2}\right)^2\right\}\frac{\mu_0}{\rho_0}} \quad ...(42)$$

Acknowledgement

The author conveys his grateful thanks to Dr A. Das, Jalpaiguri Government Engineering College, for his valuable suggestions during the preparation of this paper.

References

1. LOVE, A. E. H., *A treatise on the mathematical theory of elasticity* 4th edn (Dover Publications, New York) 1944.
2. FORSYTH, A. R., *A treatise on differential equations*, 6th edn, (Macmillan & Co., London), 1961.
3. GRAY, A. & MATHEWS, G. B., *A treatise on Bessel functions and their applications to physics*, 2nd edn (Dover Publications, New York), 1966.
4. CARSLAW, H. S. & JAEGER, J. C., *Conduction of heat in solids*, 2nd edn (Oxford University Press), 1959.

Reprinted from "Journal of the Indian Institute of Science", Vol. 58, No. 7, July 1976

RADIAL DEFORMATIONS OF NONHOMOGENEOUS SPHERICALLY ANISOTROPIC ELASTIC MEDIA

SUBHAS CHANDRA ROY

(Department of Mathematics, Jalpaiguri Government Engineering College, Jalpaiguri, West Bengal, India)

Received on December 22, 1975 and in revised form on May 10, 1976

ABSTRACT

This paper deals with the elasticity problem of a spherically anisotropic elastic medium bounded by two concenrtric spherical surfaces subjected to normal pressures. The material of the structure is spherically anisotropic and, in addition, is continuously inhomogeneous with mechanical properties varying exponentially along the radius. An exact solution of the problem in terms of Whittaker functions is presented. The St. Venant's solution in the case of homogeneous material and Lamè's solution in the case of homogeneous isotropic material are derived here from the general solution. The problem of a solid sphere of the same medium under the external pressure is also solved as a particular case of the above problem. Lastly, the displacements and stresses of a composite sphere consisting of a solid spherical body made of homogeneous material and a nonhomogeneous concentric spherical shell covering the inclusion, both of them being spherically anisotropic, are obtained when the sphere is under uniform compression,

Keywords: Radial deformation, Stresses, Spherical shell, Nonhomogeneity, Inclusion.

1. INTRODUCTION

The elastic behavior of a spherically aeolotropic material was first studied by St. Vernant in 1865. He considered the problem of a spherical shell under uniform internal and external pressures and applied his results to some piezometer experiments. A description of his analysis may be found in the treatise by Love [1] or in the book of anisotropic elasticity by Lekhnitskii [2].

Increasing use of composite materials in aerospace applications calls for the study of problems of nonhomogeneous anisotrpic elastic media. Grief and Chou [3] have treated a dynamic problem of a non-homogeneous cylindrically aeolotropic shell. Sengupta and Basu Mallick [4] have investiagted the radical deformation of a nonhomogeneous spherically anisotropic elastic shell under uniform internal and external pressures. All of them consider the elastic prameters to be proportional to n-th power of the radius.

An attempt has been made here to find the analytical solution for the radial deformation and corresponding stresses in a spherical shell made of spherically aeolotropic heterogeneous material under the influence of normal pressures on both boundaries. The corresponding results for homogeneous spherically aeolotropic material are derived here as a particular case and these were obtained by St. Venant, as quoted in Lekhnitskii [2]. The expressions calculated by Lamè and given in Love [1] for homogeneous isotropic bodies are found from the general results. The results for a solid sphere of nonhomogeneous spherically anisotropic medium under the external pressure are derived from the general expressions when the radius of the inner surface approaches zero. At the end, radial displacements and stresses for both the portions of nonhomogeneous spherically anisotropic shell having concentric homogeneous spherically anisotropic inclusion are presented here, when the outer surface is loaded with a uniform normal pressure. In all the cases, the nonhomogeneity of the material is characterized by the elastic parameters C_{ij}, *vide* Grief and Chou [3], Sengupta and Basu Mallick [4] as

$$C_{ij} = \lambda_{ij} \exp(-kr), \ (i, j = 1, 2, 3) \tag{1}$$

a new variation, where λ_{ij} and k are the prescribed parameters of the material concerned.

2. Fundamental Equations

The basic system of field equations in linear isothermal static elasticity theory are:

(*a*) the generalized Hooke's law, (*b*) the linearized strain displacement equations, and (*c*) the stress equations of equilibrium. Here the centre of a spherical shell or sphere is taken as origin and spherical polar co-ordinates (r, θ, ϕ) are used.

For a spherically anisotropic body, the generalized Hooke's law may be written as, *vide* Lekhnitskii]2]

$$\begin{aligned}
\bar{\sigma}_r &= c_{11}\,\bar{e}_{rr} + c_{12}\,\bar{e}_{\theta\theta} + c_{12}\,\bar{e}_{\phi\phi} \\
\bar{\sigma}_\theta &= c_{12}\,\bar{e}_{rr} + c_{22}\,\bar{e}_{\theta\theta} + c_{23}\,\bar{e}_{\phi\phi} \\
\bar{\sigma}_\phi &= c_{12}\,\bar{e}_{rr} + c_{23}\,\bar{e}_{\theta\theta} + c_{22}\,\bar{e}_{\phi\phi} \\
\bar{\tau}_{\theta\phi} &= \tfrac{1}{2}(c_{22} - c_{23})\,\bar{e}_{\theta\phi} \\
\bar{\tau}_{r\theta} &= c_{44}\,\bar{e}_{r\theta} \\
\bar{\tau}_{r\phi} &= c_{44}\bar{e}_{r\phi}
\end{aligned} \tag{2}$$

where C_{ij} are functions of the elastic moduli, Poison's ratios, and (for a non-homogeneous material) also functions of the spatial position. For the present problem C_{ij} are already mentioned in equation (1).

Now for a purely radial deformation of the body, the displacement components $(\bar{u}, \bar{v}, \bar{w})$ must be of the type $\bar{u} = \bar{u}(r)$, $\bar{v} = 0$ and

$$\bar{w} = 0. \tag{3}$$

Due to this assumption the strain components, in terms of displacements, are

$$\bar{e}_{rr} = \frac{d\bar{u}}{dr}, \quad \bar{e}_{\theta\theta} = \frac{\bar{u}}{r} = \bar{e}_{\phi\phi}$$

$$\bar{e}_{\theta\phi} = \bar{e}_{r\theta} = \bar{e}_{r\phi} = 0. \tag{4}$$

The non-zero stress components in equations (2), in terms of displacement and λ_{ij}, may now be written as

$$\bar{\sigma}_r = \exp(-kr)\left[\lambda_{11}\frac{d\bar{u}}{dr} + 2\lambda_{12}\frac{\bar{u}}{r}\right]$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \exp(-kr)\left[\lambda_{12}\frac{d\bar{u}}{dr} + (\lambda_{22} + \lambda_{23})\frac{\bar{u}}{r}\right]. \tag{5}$$

In the absence of body forces, two equations of equilibrium are identically satisfied and the non-trivial equation of equilibrium becomes

$$\frac{d}{dr}(\bar{\sigma}_r) + \frac{2}{r}(\bar{\sigma}_r - \bar{\sigma}_\theta) = 0.$$

This equation of equilibrium, with the help of equations [5], stands as

$$r^2\frac{d^2\bar{u}}{dr^2} + (2 - kr)\,r\frac{d\bar{u}}{dr} - 2\left\{\frac{\lambda_{22} + \lambda_{23} + (kr - 1)\lambda_{12}}{\lambda_{11}}\right\}\bar{u} = 0. \tag{6}$$

3. Method of Solution

We use transformations

$$x = kr \quad \text{and} \quad \bar{u} = V\exp\frac{x}{2} \tag{7}$$

in the equation (6) and rewrite it accordingly,

$$x^2\frac{d^2 v}{dx^2} + 2x\frac{dv}{dx} + \left\{\frac{-2(\lambda_{22} + \lambda_{23} - \lambda_{12})}{\lambda_{11}} + \left(1 - \frac{2\lambda_{12}}{\lambda_{11}}\right)x - \frac{x^2}{4}\right\}V = 0. \tag{8}$$

Again for

$$V = x^{-1} U, \tag{9}$$

the equation (8) reduces to

$$x^2 \frac{d^2 U}{dx^2} + \left\{\frac{-2(\lambda_{22} + \lambda_{23} - \lambda_{12})}{\lambda_{11}} + \left(1 - \frac{2\lambda_{12}}{\lambda_{11}}\right) x - \frac{x^2}{4}\right\} U = 0 \tag{10}$$

Following Whittaker and Watson [5] one can write down the solution of the equation (10) in the form

$$U = AM_{\overset{*}{k}, p}(x) + BM_{\overset{*}{k}, -p}(x) \tag{11}$$

where $M_{\overset{*}{k}, \pm p}(x)$ are Whittaker functions in which

$$2p = \left\{1 + \frac{8}{\lambda_{11}}(\lambda_{22} + \lambda_{23} - \lambda_{12})\right\}^{\frac{1}{2}} > 0, \qquad \text{(noninteger)} \tag{12}$$

and

$$\overset{*}{k} = 1 - \frac{2\lambda_{12}}{\lambda_{11}}. \tag{13}$$

A and B are arbitrary constants.

If $2p$ be an integer or zero, the solution of the equation (10) may be written as

$$U = AW_{\overset{*}{k}, p}(x) + BW_{-\overset{*}{k}, p}(-x) \tag{14}$$

where

$$W_{\overset{*}{k}, p}(x) = \frac{\Gamma(c-1)}{\Gamma(d-c+1)} M_{\overset{*}{k}, p}(x) + \frac{\Gamma(1-c)}{\Gamma(d)} M_{\overset{*}{k}, -p}(x) \tag{15}$$

in which $C = 1 \pm 2p$, $d = \frac{1}{2} - \overset{*}{k} \pm p$ and $\Gamma(t)$ is a gamma function of t.

Finally the radial displacement $\bar{u}(r)$ satisfying the equilibrium equation (6) is obtained with the help of equations (7), (9) and (11) as

$$\bar{u} = \frac{\exp(kr/2)}{kr}[AM_{\overset{*}{k}, p}(kr) + BM_{\overset{*}{k}, -p}(kr)]. \tag{16}$$

Substituting this expression for u in equations (5), the nonvanishing stresses may now be obtained in general form

$$\left.\begin{aligned} \bar{\sigma}_r &= \frac{\exp(-kr/2)}{r}[A\alpha_p(r) + B\alpha_{\ p}(r)] \\ \bar{\sigma}_\theta = \bar{\sigma}_\phi &= \frac{\exp(-kr/2)}{r}[A\beta_p(r) + B\beta_{-p}(r) \end{aligned}\right\} \tag{17}$$

where

$$\left.\begin{aligned} \alpha_{\pm p}(r) &= \left[\frac{kr-2}{2kr}\lambda_{11} + \frac{2\lambda_{12}}{kr}\right] M^{*}_{k,\pm p}(kr) + \lambda_{11} M'^{*}_{k,\pm p}(kr) \\ \beta_{\pm p}(r) &= \left[\frac{kr-2}{2kr}\lambda_{12} + \frac{\lambda_{22}+\lambda_{23}}{kr}\right] M^{*}_{k,\pm p}(kr) + \lambda_{12} M'^{*}_{k,\pm p}(kr) \end{aligned}\right\} \tag{18}$$

The prime indicates the derivative of the function with respect to its argument.

4. The Problem of a Non-homogeneous Spherical Shell

We consider here a spherical shell $a \leq r \leq b$. The structure is made of nonhomogeneous spherically anisotropic material. The shell is under the influence of uniformly distributed internal and external pressures. The boundary conditions are as follows:

$$\left.\begin{aligned} \bar{\sigma}_r &= -p_0, \text{ on the surface } r = a \\ \bar{\sigma}_r &= -p_1, \text{ on the surface } r = b \end{aligned}\right\} \tag{19}$$

On application of these boundary conditions in the first equation of (17), we get

$$A\alpha_p(a) + B\alpha_{-p}(a) + p_0\, a \exp(ka/2) = 0$$

$$A\alpha_p(b) + B\alpha_{-p}(b) + p_1\, b \exp(kb/2) = 0.$$

Solving the above equations for A and B and inserting their values in equations (16) and (17), one obtains the complete solution for radial displacement and stresses as

$$\bar{u} = \frac{\exp(kr/2)}{Mkr}\left[\{p_1 b\alpha_{-p}(a)\exp(kb/2) - p_0 a\alpha_{-p}(b)\exp(ka/2)\} M_{k^*,p}(kr)\right.$$
$$\left. + \{p_0 a\alpha_p(b)\exp(ka/2) - p_1 b\alpha_p(a)\exp(kb/2)\} M_{k^*,-p}(kr)\right],$$

$$\bar{\sigma}_r = \frac{\exp(-kr/2)}{Mr}\left[\{p_1 b\alpha_{-p}(a)\exp(kb/2) - p_0 a\alpha_{-p}(b)\right.$$
$$\times \exp(ka/2)\}\,\alpha_p(r) + \{p_0 a\alpha_p(b)\exp(ka/2) - p_1 b\alpha_p(a)$$
$$\times \exp(kb/2)\}\,\alpha_{-p}(r),$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \frac{\exp(-kr/2)}{Mr}\left[\{p_1 b\alpha_{-p}(a)\exp(kb/2) - p_0 a\alpha_{-p}(b)\right.$$
$$\times \exp(ka/2)\}\,\beta_p(r) + \{p_0 a\alpha_p(b)\exp(ka/2) - p_1 b\alpha_p(a)$$
$$\left.\times \exp(kb/2)\}\,\beta_{-p}(r)\right],$$

where

$$M = \alpha_p(a)\,\alpha_{-p}(b) - \alpha_{-p}(a)\,\alpha_p(b). \tag{20}$$

4.1 *Dilation of the Cavity*

The volume of the cavity of the shell changes under the action of internal and external pressures; the relative change of this volume V_0 say is found to be

$$v = \frac{\nabla V_0}{V_0} = \left(\frac{3\bar{u}}{r}\right)_{r=a} = \frac{3\exp(ka/2)}{Mka^2}\,[\{p_1 b\alpha_{-p}(a)\exp(kb/2)$$
$$- p_0 a\alpha_{-p}(b)\exp(ka/2)\}\,M_{k^*,\,p}(ka) + \{p_0 a\alpha_p(b)\exp(ka/2)$$
$$- p_1 b\alpha_{-p}(a)\exp(kb/2)\}\,M_{k^*,\,-p}(ka).] \tag{21}$$

4.2 *Stress Concentration in the Neighbourhood of the Cavity*

It is of interest to find the stress distribution in the vicinity of a spherical cavity. Generally the stress reaches its maximum value in the area which passes through the radius vector near the inner surface. Therefore we have on the surface of the cavity $r = a$

$$[\bar{\sigma}_\theta]_{\max} = [\bar{\sigma}_\phi]_{\max} = \frac{\exp(-ka/2)}{Ma}\,[\{p_1 b\alpha_{-p}(a)\exp(kb/2)$$
$$- p_0 a\alpha_{-p}(b)\exp(ka/2)\}\,\beta_p(a) + \{p_0 a\alpha_p(b)\exp(ka/2)$$
$$- p_1 b\alpha_p(a)\exp(kb/2)\}\,\beta_{-p}(a)]. \tag{22}$$

4.3 *Stresses in a Compressed Shell Due to Internal Pressure*

If the spherical shell undergoes compression due to the action of internal pressure only, the external surface being stress-free, the stresses of such a compressed shell are obtained from the equations (20) to be

$$\bar{\sigma}_r = \frac{p_0 a\exp[k(a-r)/2]}{Mr}\,[\alpha_p(b)\,\alpha_{-p}(r) - \alpha_{-p}(b)\,\alpha_p(r)],$$
$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \frac{p_0 a\exp[k(a-r)/2]}{Mr}\,[\alpha_p(b)\,\beta_{-p}(r) - \alpha_{-p}(b)\,\beta_p(r)]. \tag{23}$$

4.4 *St. Venant's Solution*

A spherical shell ($a \leq r \leq b$), made of homogeneous spherically anisotropic material, is considered under the same boundary conditions (19).

Stresses of such a shell may be found from the second and third equations of (20) on making $k \mapsto 0$ and they are

$$\sigma_r = \underset{k \to 0}{\mathrm{lt}} (\bar{\sigma}_r) = \frac{r^{p-3/2}}{\left(\frac{a}{b}\right)^p - \left(\frac{b}{a}\right)^p} \left[\left(\frac{p_1 b^{3/2}}{a^p} - \frac{p_0 a^{3/2}}{b^p}\right) + \frac{1}{r^{2p}} \left(p_0 a^{3/2} b^p - p_1 b^{3/2} a^p\right)\right],$$

Similarly,

$$\sigma_\theta = \sigma_\phi = \left[\frac{r^{p-3/2}}{\left(\frac{a}{b}\right)^p - \left(\frac{b}{a}\right)^p}\right] \left[\left(\frac{p_1 b^{3/2}}{a^p} - \frac{p_0 a^{3/2}}{b^p}\right) \times \frac{2(\lambda_{22} + \lambda_{23}) + (2p-1)\lambda_{12}}{4\lambda_{12} + (2p-1)\lambda_{11}} + \frac{1}{r^{2p}} \left(p_0 a^{3/2} b^p - p_1 b^{3/2} a^p\right) \times \frac{2(\lambda_{22} + \lambda_{23}) - (2p+1)\lambda_{12}}{4\lambda_{12} - (2p+1)\lambda_{11}}\right] \tag{24}$$

respectively.

It is to be noted that the following limits are used to compute the stresses in (24)

$$\underset{k \to 0}{\mathrm{Lt}} \frac{M^*_{k, \pm p}(k\xi)}{M^*_{k, \pm p}(kr)} = \left(\frac{\xi}{r}\right)^{\frac{1}{2} \pm p}, \quad \underset{k \to 0}{\mathrm{Lt}} \frac{k\xi M'^*_{k, \pm p}(k\xi)}{M^*_{k, \pm p}(kr)} = \left(\frac{\xi}{r}\right)^{\frac{1}{2} \pm p}, \quad \left(\tfrac{1}{2} \pm p\right) \tag{24 a}$$

Same sign of p is to be retained for the above limits.

The expressions of stresses in equations (24) for the above shell problem are the same found by St. Venant and given in Lekhnitskii [2].

4.5 *Isotropic Body and Love's Results*

The elastic property of a spherically anisotropic material is describes by five elastic parameters and they reduce to two independent parameter for isotropic material. It is treated as a special case of spherical anisotropy. For thepresent nonhomogeneous problem the relations are

$$C_{11} = C_{22} = (\lambda_0 + 2\mu_0) \exp(-kr), \; C_{12} = C_{23} = \lambda_0 \exp(-kr)$$

$$C_{44} = \mu_0 \exp(-kr) \tag{25}$$

where λ_0 and μ_0 are Lamè constants. Following equation (1) and equation (25) the above relations in terms of λ_{ij} are

$$\lambda_{11} = \lambda_{22} = \lambda_0 + 2\mu_0, \qquad \lambda_{12} = \lambda_{23} = \lambda_0, \qquad \lambda_{44} = \mu_0. \tag{26}$$

The results obtained in equations (20)–(23) for spherically anisotropic nonhomogeneous bodies can also be used as the results for isotropic nonhomogeneous bodies if we replace there λ_{ij} by λ_0 and μ_0 as given in equations (26). We further note that the application of relations (26) in equation (12) follows

$$2p = 3. \tag{27}$$

As a test case we make use of equations (26) and (27) in the last two equations of (20) and take the limit as $k \to 0$ (for homogeneous material) and arrive at the following results:

$$\sigma_r = \frac{p_0 a^3 - p_1 b^3}{b^3 - a^3} + \frac{1}{r^3}\frac{(p_1 - p_0)\, a^3\, b^3}{b^3 - a^3},$$

$$\sigma_\theta = \sigma_\phi = \frac{p_0 a^3 - p_1 b^3}{b^3 - a^3} - \frac{1}{2}\cdot\frac{1}{r^3}\frac{(p_1 - p_0)\, a^3\, b^3}{b^3 - a^3}. \tag{28}$$

These are the stresses in a spherical shell ($a \leq r \leq b$) of homogeneous isotropic elastic material, subjected to internal and external pressures on the boundaries as in equations (19). These expressions were calculated by Lamè and are presented in Love [1].

4.6 *Solid Spherical Body*

A solid spherical body ($0 \leq r \leq b$) of nonhomogeneous spherically aeolotropic material undergoes compression by an uniformly distributed external pressure p_1. The stresses of such a sphere are obtained from the last two equations of (20) when the cavity of radius 'a' diminishes to zero, *i.e.*, as $a \to 0$:

$$\bar{\sigma}_r = -\,p_1 (b/r)^2 \exp[k(b-r)/2] \times \frac{\{(kr-2)\,\lambda_{11} + 4\lambda_{12}\}\, M^*_{k,p}(kr) + 2kr\,\lambda_{11}\, M'^*_{k,p}(kr)}{\{(kb-2)\,\lambda_{11} + 4\lambda_{12}\}\, M^*_{k,p}(kb) + 2kb\,\lambda_{11} M'^*_{k,p}(kb)},$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = -\,p_1 (b/r)^2 \exp[k(b-r)/2] \times \frac{\{(kr-2)\,\lambda_{12} + 2(\lambda_{22} + \lambda_{23})\}\, M^*_{k,p}(kr) + 2kr\,\lambda_{12}\, M'^*_{k,p}(kr)}{\{(kb-2)\,\lambda_{11} + 4\lambda_{12}\}\, M^*_{k,p}(kb) + 2kb\lambda_{11}\, M'^*_{k,p}(kb)}, \tag{29}$$

since

$$\underset{z\to 0}{\mathrm{Lt}} \frac{M^*_{k,p}(z)}{M^*_{k,-p}(z)} = 0, \quad \underset{z\to 0}{\mathrm{Lt}} \frac{zM'^*_{k,p}(z)}{M^*_{k,-p}(z)} = 0$$

and

$$\underset{z\to 0}{\mathrm{Lt}} \frac{zM'^*_{k,\pm p}(z)}{M^*_{k,\pm p}(z)} = \tfrac{1}{2} \pm p. \tag{29 a}$$

5. The Problem of a Composite Sphere

We now consider a homogeneous solid sphere $(0 \leq r \leq a)$ of spherically anisotropic matrial surrounded by a nonhomogeneous concentric spherical shell $(a \leq r \leq b)$ of spherically anisotropic medium and the whole body is acted upon by a uniform radial pressure on the external bounding surface $r = b$. At the surface of separation $r = a$ the materials are sufficiently rough to ensure the continuity of radial stresses and displacements. The relevant boundary conditions are then

$$\bar{\sigma}_r = -p_1 \text{ on the surface } r = b$$

and

$$u = \bar{u}, \quad \sigma_r = \bar{\sigma}_r \text{ on the surface } r = a \tag{30}$$

In the case of a homogeneous solid sphere $(0 \leq r \leq a)$ $C_{ij} = \lambda_{ij}$ for $k = 0$ in equation (1) and the stress equation of equilibrium corresponding to the equation (6) turns out to be

$$\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) - \frac{2\,(C_{22} + C_{23} - C_{12})}{C_{11}}\, u = 0.$$

The general solution of the above equation may be put as

$$u = \mathrm{C}r^{m-1/2} + \mathrm{D}r^{m-1/2}$$

where

$$m = \{\tfrac{1}{4} + 2\,(C_{22} + C_{23} - C_{12})/C_{11}\}^{1/2} \tag{31}$$

To ensure the finiteness of the stress at every point of the solid sphere, including the neighbourhood of the origin, we are to take the displacement and stresses as (supposing $m > 3/2$)

$$\begin{aligned} u &= Cr^{(m-1)/2} \\ \sigma_r &= Ca_m r^{(m-3)/2} \\ \sigma_\theta &= \sigma_\phi = Cb_m r^{(m-3)/2} \end{aligned} \tag{32}$$

where

$$a_m = 2C_{12} + C_{11}(m - 1/2), \quad b_m = (C_{22} + C_{23}) + C_{12}(m - 1/2). \tag{33}$$

And for the spherical shell ($a \le r \le b$) the displacements and stresses are given in equations (16) and (17). Boundary conditions (30) are applied to equations (16), (17) and (32) to get

$$\begin{aligned} &A\alpha_p(b) + B\alpha_{-p}(b) + p_1 b \exp(kb/2) = 0 \\ &\frac{\exp(ka/2)}{ka}[AM^*_{k,p}(ka) + BM^*_{k,-p}(ka)] = Ca^{(m-1/2)} \\ &\frac{\exp(-ka/2)}{a}[A\alpha_p(a) + B\alpha_{-p}(a)] = Ca_m a^{(m-3/2)}. \end{aligned} \tag{34}$$

Solving the above equations for A, B and C and inserting their values in equations (32) and in equations (16), (17) we get the following sets of results:

Displacement and stresses in the sphere ($0 \le r < a$)

$$\begin{aligned} u &= \frac{p_1 b \exp[k(a+b)/2]}{Nka^{(m+1/2)}}[\alpha_p(a) M^*_{k,-p}(ka) - \alpha_{-p}(a) M^*_{k,p}(ka)] \\ &\quad \times r^{(m-1/2)}, \\ \sigma_r &= \frac{p_1 b \exp[k(a+b)/2]}{Nka^{m+1/2}}[\alpha_p(a) M^*_{k,-p}(ka) - \alpha_{-p}(a) M^*_{k,p}(ka)] \\ &\quad \times a_m r^{m-3/2}, \\ \sigma_\theta &= \sigma_\phi = \frac{p_1 b \exp[k(a+b)/2]}{Nka^{m+1/2}}[\alpha_p(a) M^*_{k,-p}(ka) - \alpha_{-p}(a) M^*_{k,p}(ka)] \\ &\quad \times b_m r^{m-3/2}, \end{aligned} \tag{35}$$

Displacement and stresses in the spherical shell ($a \le r \le b$)

$$\begin{aligned} \bar{u} &= \frac{p_1 b \exp[k(b+r)/2]}{Nkr}\left[\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,-p}(ka) - \alpha_{-p}(a)\right\}\right. \\ &\quad \left.\times M^*_{k,p}(kr) - \left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - \alpha_p(a)\right\} M^*_{k,-p}(kr)\right] \\ \bar{\sigma}_\theta &= \bar{\sigma}_\phi = \frac{p_1 b \exp[k(b-r)/2]}{Nr}\left[\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,-p}(ka) - \alpha_{-p}(a)\right\}\right. \\ &\quad \left.\times \beta_p(r) - \left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - \alpha_p(a)\right\} \beta_{-p}(r)\right], \end{aligned}$$

$$\bar{\sigma}_r = \frac{p_1 b \exp[k(b-r)/2]}{Nr}\left[\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - \alpha_{-p}(a)\right\}\alpha_p(r)\right.$$

$$\left. - \left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - \alpha_p(a)\right\}\alpha_{-p}(r)\right],$$

where

$$N = \alpha_{-p}(b)\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,p}(ka) - \alpha_p(a)\right\}$$

$$- \alpha_p(b)\left\{\frac{a_m \exp(ka)}{ka} M^*_{k,-p}(ka) - \alpha_{-p}(a)\right\}. \tag{36}$$

6. Numerical Results

All the results are for structures with finite outer radius b that is twice the inner radius a. It should be noted that the results previously derived are quite general. The problem investigated involve inhomogeneous materials with properties varying exponemtially with the radius according to the equation (1).

We choose the elastic constants $\lambda_{11} = 26\cdot 92$, $\lambda_{12} = 13\cdot 46$, $\lambda_{22} = 8\cdot 47$, $\lambda_{23} = 3\cdot 12$, $\lambda_{44} = 6\cdot 53$ in terms of a unit 10^{11} dynes per square centimeter and $k = 2/a$ (numerically) for Material I. The present analysis may be useful in studying the stresses for layered media having exponentially increasing or decreasing stiffness. We make use of the values of λ_{ij} in equations (12) and (13) to obtain $p = 1/3$ and $\overset{*}{k} = 0$ and from equations (18) we get.

r/a	$\alpha_p(r)/\Gamma(4/3)$	$\alpha_{-p}(r)/\Gamma(2/3)$	$\beta_p(r)/\Gamma(4/3)$	$\beta_{-p}(r)/\Gamma(2/3)$
1	54·03	24·89	29·96	12·87
2	161·40	64·33	89·03	35·30

For the first problem the internal surface of the shell structure is always under a uniform normal pressure p_0, whereas its surface is supposed to be stress-free for system I and is subjected to a pressure which is half of the internal pressure for the system II. Our main interest lies in computing the stress concentration near the vicinity of the caivity. The above obtained values for Material I are applied to the last equation of (20) and it yields $M = -540\cdot 9\ \Gamma(2/3)\ \Gamma(4/3)$. Ultimately the equation (22) shows $[\bar{\sigma}_\theta]_{\max.} = (-.2776\, p_0)$ and $(-.5297\, p_0)$ for the leading systems I and II respectively. Also the third equation of (20) leads to $[\bar{\sigma}_\theta] = (\cdot 0105\, p_0)$ and $(-.2748\, p_0)$ for the loading systems I and II respectively.

In the second problem with inclusion we choose $C_{11} = 6 \cdot 17$, $C_{12} = 2 \cdot 17$, $C_{22} = 5 \cdot 97$, $C_{23} = 2 \cdot 62$ $C_{44} = 1 \cdot 64$ (with the same unit mentioned previously) for Material II for the homogeneous portion $0 \leq r \leq a$. Using these values of C_{ij} in equations (31) and (33) the values of m, a_m, b_m are found to be $m = 1 \cdot 527$, $a = 10 \cdot 67$, $b_m = 10 \cdot 82$. The other portion $a \leq r \leq b$ is filled up with Material I. We calculate the value of the constant N of equation (36) for Material I and find $N = -2054\,\Gamma(2/3)\,\Gamma(4/3)$. Stresses of equations (35) and (36) may now be had from the table below:

TABLE I

Nonhomogeneous			Homogeneous	
r/a	$-\bar{\sigma}_r/p_1$	$-\bar{\sigma}_\theta/p_1$	$-\sigma_r/p_1$	$-\sigma_\theta/p_1$
1·0	1·635	·960	1·635	1·657
1·4	1·435	·793		
1·6	1·252	·693		
1·8	1·113	·615		
2·0	1	·554		

7. ACKNOWLEDGEMENT

The author would like to express his appreciation to Dr. A. Das of Jalpaiguri Government Engineering College for suggesting the problem and for his help and encouragement in carrying it through. The author is also grateful to the referees for their valuable comments.

REFERENCES

[1] Love, A. E. H. .. *Theory of Elasticity*, Cambridge University Press, U.K., 1952, pp. 142.

[2] Lekhnitskii, S. G. .. *Theory of Elasticity of an Anisotropic Elastic Body*, Holden Day, 1963.

[3] Grief, R. and Chou, S. C. .. The propagation of radially symmetric stress waves in anisotropic nonhomogeneous elastic media, *Jour. of Appl. Mech.*, 1971, **38** (1), 51–57.

[4] Sengupta, P. R. and Basu Mallick, A. N. Radial deformation of a nonhomogeneous spherically aniso. tropic elastic sphere with a concentric spherical inclusion, *Ind. Jour. of Mech. and Math.* 1970, **8** (2), 1–9.

[5] Whittaker, E. T. and Watson, G. N. *A Course of Modern Analysis*, Cambridge University Press, 1965, p. 337.

RADIAL DEFORMATION OF A NONHOMOGENEOUS SPHERICALLY ISOTROPIC ELASTIC SPHERE WITH A CONCENTRIC SPHERICAL INCLUSION

By

Subhas Chandra Roy

Reprinted from "The Proceedings of the Indian Academy of Sciences". Vol. LXXXV, Sec. 'A', No. 5, 1977.

Proc. Indian Acad. Sci., Vol. 85 A, No. 5, 1977, pp. 338–350.

Radial deformation of a nonhomogeneous spherically isotropic elastic sphere with a concentric spherical inclusion

SUBHAS CHANDRA ROY

Jalpaiguri Government Engineering College, Jalpaiguri, West Bengal

MS received 25 September 1975; in revised form 16 December 1976

ABSTRACT

The elasticity of a spherically isotropic medium bounded by two concentric spherical surfaces subjected to normal pressures is discussed. The material of the structure is spherically isotropic and, in addition, is continuously inhomogeneous with mechanical properties varying exponentially as the square of the radius. An exact solution of the problem in terms of Whittaker functions is presented. The St. Venant's solution in the case of homogeneous material and Lamé's solution in the case of homogeneous isotropic material are derived from the general solution. The problem of a solid sphere of the same medium under the external pressure is also solved as a particular case of the above problem. Finally, the displacements and stresses of a composite sphere consisting of a solid spherical body made of homogeneous material and a nonhomogeneous concentric spherical shell covering the inclusion, both of them being spherically isotropic, are obtained when the sphere is under uniform compression.

1. INTRODUCTION

THE elastic behaviour of a spherically isotropic material was first studied by St. Venant in 1865. He considered the problem of a spherical shell under uniform internal and external pressures and applied his results to some piezometer experiments.[1,2]

The increasing use of composite materials in aerospace applications requires the study of nonhomogeneous anisotropic elastic media. Greif and Chou[3] treated a dynamic problem of a non-homogeneous cylindrically aeolotropic shell. Sengupta and Basu Mallick[4] investigated the radial deformation of a nonhomogeneous spherically isotropic elastic shell under uniform internal and external pressures. All of them considered the elastic parameters to be proportional to nth power of the radius.

An attempt has been made in this paper to get an analytical solution for the radial deformation and corresponding stresses in a spherical shell made of spherically isotropic heterogeneous material under the influence of normal pressures on both boundaries. The corresponding results for homogeneous spherically isotropic material are derived as a particular case obtained by St. Venant.[2] The expressions calculated by Lamé[1] for homogeneous isotropic bodies, are found from the general results. The results for a solid sphere of nonhomogeneous spherically isotropic medium under the external pressure are derived from the general expressions when the radius of the inner surface approaches zero. At the end, radial displacements and stresses for both the portions of nonhomogeneous spherically isotropic shell having concentric homogeneous spherically isotropic inclusion are presented here, when the outer surface is loaded with a uniform normal pressure. In all the cases, the nonhomogeneity of the material is charactrerized by the elastic parameters C_{ij} (*vide* Greif and Chou,[3] Sengupta and Basu Mallick[4] and Bhaduri[5]) as

$$C_{ij} = \lambda_{ij} \exp(-\kappa r^2) \; (i.j = 1, 2, 3) \tag{1}$$

where λ_{ij} and κ are the prescribed parameters of the material concerned.

2. FUNDAMENTAL EQUATIONS

The basic system of field equations in linear isothermal static elasticity theory are: (*a*) the generalized Hooke's law, (*b*) the linearized strain displacement equations, and (*c*) the stress equations of equilibrium. Here the centre of a spherical shell or sphere is taken as origin and spherical polar co-ordinates (r, θ, ϕ) are used.

For a spherically isotropic body the generalized Hooke's law may be written as[2],

$$\begin{aligned}
\bar{\sigma}_r &= C_{11}\,\bar{e}_{rr} + C_{12}\,\bar{e}_{\theta\theta} + C_{12}\,\bar{e}_{\phi\phi} \\
\bar{\sigma}_\theta &= C_{12}\,\bar{e}_{rr} + C_{22}\,\bar{e}_{\theta\theta} + C_{23}\bar{e}_{\phi\phi} \\
\bar{\sigma}_\phi &= C_{12}\,\bar{e}_{rr} + C_{23}\,\bar{e}_{\theta\theta} + C_{22}\,\bar{e}_{\phi\phi} \\
\bar{\tau}_{\theta\phi} &= \tfrac{1}{2}(C_{22} - C_{23})\,\bar{e}_{\theta\phi} \\
\bar{\tau}_{r\phi} &= C_{44}\,\bar{e}_{r\phi} \\
\bar{\tau}_{r\theta} &= C_{44}\,\bar{e}_{r\theta}
\end{aligned} \tag{2}$$

where C_{ij} are functions of the elastic moduli, Poisson's ratios, and (for a non-homogeneous material) also functions of the spatial position. For the present problem C_{ij} are already mentioned in eq. (1).

Now for a purely radial deformation of the body, the displacement components $(\bar{u}, \bar{v}, \bar{w})$ must be of the type

$$\bar{u} = \bar{u}(r),\ \bar{v} = 0 \text{ and } \bar{w} = 0 \tag{3}$$

Due to this assumption the strain components, in terms of displacements, are

$$\bar{e}_{rr} = \frac{d\bar{u}}{dr},\ \bar{e}_{\theta\theta} = \frac{\bar{u}}{r} = \bar{e}_{\phi\phi}$$

$$\bar{e}_{\theta\phi} = \bar{e}_{r\theta} = \bar{e}_{r\phi} = 0. \tag{4}$$

The non-zero stress components in eq. (2), in terms of displacement and λ_{ij}, may now be written as

$$\bar{\sigma}_r = \exp(-\kappa r^2)\left[\lambda_{11}\frac{d\bar{u}}{dr} + 2\lambda_{12}\frac{\bar{u}}{r}\right],$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi \exp(-\kappa r^2)\left[\lambda_{12}\frac{d\bar{u}}{d_r} + (\lambda_{22} + \lambda_{23})\frac{\bar{u}}{r}\right]. \tag{5}$$

In the absence of body forces, two equations of equilibrium are identically satisfied and the non-trivial equation of equilibrium becomes

$$\frac{d}{dr}(\bar{\sigma} r) + \frac{2}{r}(\bar{\bar{\sigma}}_r - \bar{\sigma}_\theta) = 0.$$

This equation of equilibrium, with the help of eq. (5), stands as

$$r^2\frac{d^2\bar{u}}{dr^2} + 2(1 - \kappa r^2)\, r\frac{d\bar{u}}{dr} - 2$$

$$\times\left\{\frac{(\lambda_{22} + \lambda_{23} + (2\kappa r^2 - 1)\lambda_{12}}{\lambda_{11}}\right\}\ \bar{u} = 0. \tag{6}$$

3. Method of Solution

We use transformations

$$x = \kappa r^2 \text{ and } \bar{u} = V\exp\frac{x}{2} \tag{7}$$

in eq. (6) and rewrite it accordingly

$$x^2\frac{d^2V}{dx^2} + \frac{3}{2}x\frac{dV}{dx} + \left\{-\frac{(\lambda_{22} + \lambda_{23} - \lambda_{12})}{2\lambda_{11}}\right.$$

$$\left. + \left(\frac{3}{4} - \frac{\lambda_{12}}{\lambda_{11}}\right)x - \frac{x^2}{4}\right\}V = 0. \tag{8}$$

Again for $V = x^{-3/4}\,U$ (9)

eq. (8) reduces to

$$4x^2 \frac{d^2U}{dx^2} + \left[\frac{3}{4} - \frac{2(\lambda_{22} + \lambda_{23} - \lambda_{12})}{\lambda_{11}} + \left(3 - \frac{4\lambda_{12}}{\lambda_{11}}\right) x - x^2\right] U = 0 \; . \tag{10}$$

Following Whittaker and Watson[6] one can write down the solution of eq. (10) in the form

$$U = AM_{k^*,p}(x) + BM_{k^*,-p}(x) \tag{11}$$

where $M_{k^*\pm p}(x)$ are Whittaker functions in which

$$2p = \left\{\frac{\lambda_{11} + 8(\lambda_{22} + \lambda_{23} - \lambda_{12})}{4\lambda_{11}}\right\}^{1/2} > 0, \tag{12}$$

(noninteger)

and

$$k^* = \frac{3}{4} - \frac{\lambda_{12}}{\lambda_{11}} \tag{13}$$

A and B are arbitrary constants.

If $2p$ be an integer or zero, the solution of eq. (10) may be written as

$$U = AW_{k^*,p}(x) + BW_{-k^*,p}(-x) \tag{14}$$

where

$$W_{k^*,p}(x) = \frac{\Gamma(c-1)}{\Gamma(d-c+1)} M_{k^*,p}(x) + \frac{\Gamma(1-c)}{\Gamma(d)} M_{k^*,-p}(x) \tag{15}$$

in which $c = 1 \pm 2p$, $d = \frac{1}{2} - k^* \pm p$ and $\Gamma(t)$ is a gamma function of t.

Finally the radial displacement $\bar{u}(r)$ satisfying the equilibrium eq. (6) is obtained with the help of eqs (7), (9) and (11) as

$$\bar{u} = \exp\left(\frac{\kappa r^2}{2}\right)(\kappa r^2)^{-3/4}\left[AM_{k^*,p}(\kappa r^2) + BM_{k^*,-p}(\kappa r^2)\right] \tag{16}$$

Substituting this expression for $\bar{u}$ in eq. (5), the nonvanishing stresses may now be obtained in general form

$$\bar{\sigma}_r = \exp\left(-\frac{\kappa r^2}{2}\right)(\kappa r^2)^{-3/4}\left[A\alpha_p(r) + B\alpha_{-p}(r)\right] \tag{17}$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \exp\left(-\frac{\kappa r^2}{2}\right)(\kappa r^2)^{-3/4}\left[A\,\beta_p(r) + B\,\beta_{-p}(r)\right]$$

where

$$\alpha_{\pm p}(r) = \left[\left(\kappa r - \frac{3}{2r}\right)\lambda_{11} + \frac{2\lambda_{12}}{r}\right] M_{k^*, \pm p}(\kappa r^2) + 2\lambda_{11}\,\kappa r$$

$$\times\; M'_{k^*, \pm p}(\kappa r^2)$$

$$\beta_{\pm p}(r) = \left[\left(\kappa r - \frac{3}{2r}\right)\lambda_{12} + \frac{\lambda_{22}+\lambda_{23}}{r}\right] M_{k^*, \pm p}(\kappa r^2)$$

$$+2\lambda_{12}\,\kappa r M'_{k^*, \pm p}(\kappa r^2). \tag{18}$$

The prime indicates the derivative of the function with respect to its argument.

4. THE PROBLEM OF A NONHOMOGENEOUS SPHERICAL SHELL

We consider here a spherical shell $a \leqslant r \leqslant b$. The structure is made of nonhomogeneous spherically isotropic material. The shell is under the influence of uniformly distributed internal and external pressures. The boundary conditions are as follows:

$$\left.\begin{array}{l} \bar{\sigma}_r = -\,p_0, \text{ on the surface } r = a \\ \bar{\sigma}_r = -\,p_1 \text{ on the surface } r = b \end{array}\right\} \tag{19}$$

On application of these boundary conditions in the first equation of (17)

$$A\alpha_p(a) + B\,\alpha_{-p}(a) + p_0 \exp\left(\frac{\kappa a^2}{2}\right).\ (\kappa a^2)^{3/4} = 0$$

$$A\,\alpha_p(b) + B\,\alpha_{-p}(b) + p_1 \exp\left(\frac{\kappa b^2}{2}\right).\ (\kappa b^2)^{3/4} = 0\ .$$

Solving the above equations for A and B and inserting their values in eqs (16) and (17), one obtains the complete solution for radial displacement and stresses as

$$\bar{u} = \frac{\exp\left(\frac{\kappa r^2}{2}\right) r^{-3/2}}{M}\left[\left\{p_1\, b^{3/2} \exp\left(\frac{\kappa b^2}{2}\right) \alpha_{-p}(a)\right.\right.$$

$$\left. - \,p_0\, a^{3/2} \exp\left(\frac{\kappa a^2}{2}\right) \alpha_{-p}(b)\right\} M_{k^*, p}(\kappa r^2) + \left\{p_0\, a^{3/2} \exp\left(\frac{\kappa a^2}{2}\right)\right.$$

$$\left.\left. \times\, \alpha_p(b) - p_1\, b^{2/3} \exp\left(\frac{\kappa b^2}{2}\right) \alpha_{+p}(a)\right\} M_{k^*, -p}(\kappa r^2)\right]$$

$$\bar{\sigma}_r = \frac{\exp\left(-\frac{\kappa r^2}{2}\right) r^{-3/2}}{M} \Big[\Big\{ p_1\, b^{3/2} \exp\left(\frac{kb^2}{2}\right) \alpha_{-p}(a)$$

$$- p_0\, a^{3/2} \exp\left(\frac{\kappa a^2}{2}\right) \alpha_{-p}(b) \Big\} \alpha_p(r)$$

$$+ \Big\{ p_0\, a^{3/2} \exp\left(\frac{\kappa a^2}{2}\right) \alpha_p(b) - p_1\, b^{3/2} \exp\left(\frac{\kappa b^2}{2}\right) \alpha_p(a) \Big\} \alpha_{-p}(r) \Big]$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \frac{\exp\left(-\frac{\kappa r^2}{2}\right) r^{-3/2}}{M} \Big[\Big\{ p_1\, b^{3/2} \exp\left(\frac{\kappa b^2}{2}\right)$$

$$\times\, \alpha_p(a) - p_0\, a^{3/2} \exp\left(\frac{\kappa a^2}{2}\right) \times \alpha_{-p}(b) \Big\} \beta_p(r)$$

$$+ \Big\{ p_0\, a^{3/2} \exp\left(\frac{\kappa a^2}{2}\right) \alpha_p(b)$$

$$- p_1 b^{3/2} \exp\left(\frac{\kappa b^2}{2}\right) \alpha_p(a) \Big\} \beta_{-p}(r) \Big]$$

where the constant $M = \alpha_p(a)\, \alpha_{-p}(b) - \alpha_{-p}(a)\, \alpha_{+p}(b)$. (20)

5. Dilatation of the Cavity

The volume of the cavity of the shell changes under the action of internal and external pressures; the relative change of this volume V_0 say is found to be

$$\vartheta = \frac{\nabla V_0}{V_0} = \left(\frac{3\bar{u}}{r}\right)_{r=a} = \frac{3 \exp\left(+\frac{\kappa a^2}{2}\right) a^{-5/2}}{M}$$

$$\times \Big[\Big\{ p_1\, b^{3/2} \exp\left(\frac{\kappa b^2}{2}\right) \alpha_{-p}(a) - p_0\, a^{3/2} \exp\left(\frac{\kappa a^2}{2}\right) \alpha_{-p}(b) \Big\}$$

$$\times\, M_{k^*,p}(\kappa a^2) + \Big\{ p_0\, a^{3/2} \exp\left(\frac{\kappa a^2}{2}\right) \alpha_p(b)$$

$$- p_1\, b^{3/2} \exp\left(\frac{\kappa b^2}{2}\right) \alpha_p(a) \Big\} M_{k^*,-p}(\kappa a^2) \Big] \quad . \tag{21}$$

6. Stresses in a Compressed Shell Due to Internal Pressure

If the spherical shell undergoes compression due to the action of internal pressure only, the external surface being stress-free, the stresses of such a compressed shell as obtained from eq. (20) are

$$\bar{\sigma}_r = \frac{p_0 \exp\left[\kappa \dfrac{(a^2 - r^2)}{2}\right]}{M} \left(\frac{a}{r}\right)^{3/2} [\alpha_p (b) \alpha_p (r) - \alpha_{-p} (b) \alpha_p (r)]$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \frac{p_0 \exp\left[\kappa \dfrac{(a^2 - r^2)}{2}\right]}{M} \left(\frac{a}{r}\right)^{3/2}$$

$$[\alpha_p (b) \beta_{-p} (r) - \alpha_{-p} (b) \beta_p (r)]. \tag{22}$$

ST. VENANT'S SOLUTION

A spherical shell ($a \leqslant r \leqslant b$), made of homogeneous spherically isotropic material, is considered under the same boundary conditions (19). Stresses of such a shell may be found from the second and third equations of (20) on making $\kappa \to 0$ and they are

$$\sigma_r = \underset{k \to 0}{Lt} (\bar{\sigma}_r) = \frac{r^{2p-3/2}}{\left(\frac{a}{b}\right)^{2p} - \left(\frac{b}{a}\right)^{2p}} \left[\left(\frac{p_1 b^{3/2}}{a^2 p} - \frac{p_0 a^{3/2}}{b^2 p}\right)\right.$$

$$\left. + \frac{1}{r^{4p}} (p_0\, a^{3/2}\, b^{2p} - p_1\, b^{3/2}\, a^{2p})\right]$$

similarly

$$\sigma_\theta = \sigma_\phi = \left[\frac{r^{2p-3/2}}{\left(\frac{a}{b}\right)^{2p} - \left(\frac{b}{a}\right)^{2p}}\right] \left[\frac{2(\lambda_{22} + \lambda_{23}) + (4p - 1)\lambda_{12}}{4\lambda_{12} + (4p - 1)\lambda_{11}}\right.$$

$$\times \left(\frac{p_1 b^{3/2}}{a^{2p}} - \frac{p_0 a^{3/2}}{b^{2p}}\right) + \frac{1}{r^{4p}} (p_0\, a^{3/2}\, b^{2p} - p_1 b^{3/2}\, a^{2p})$$

$$\left.\times \frac{2(\lambda_{22} + \lambda_{23}) - (4p + 1)\lambda_{12}}{4\lambda_{12} - (4p + 1)\lambda_{11})}\right] . \tag{23}$$

It is to be noted that the following limits are used to compute the stresses in (23)

$$\underset{k \to 0}{\mathrm{Lt}} \frac{M_{k^*, \pm p} (\kappa \xi^2)}{M_{k^*, \pm p} (k r^2)} = \left(\frac{\xi}{r}\right)^{1 \pm 2p} \;;\; \underset{k \to 0}{\mathrm{Lt}} \frac{k \xi^2 M'_{k^*, \pm p} (\kappa \xi^2)}{M_{k^*, \pm p} (k r^2)}$$

$$= \frac{1}{2} \left(\frac{\xi}{r}\right)^{1 \pm 2p} (1 \pm 2p) \tag{23 a}$$

Same sign of p is to be retained for the above limits.

The expressions of stresses in eq. (23) for the above shell problem are the same found by St. Venant and given in Lekhnitskii.[2]

ISOTROPIC BODY AND LAME'S RESULTS

The elastic property of a spherically isotropic material is described by five elastic parameters and they reduce to two independent parameters for isotropic material. It is treated as a special case of spherical isotropy. For the present nonhomogeneous problem the relations are

$$C_{11} = C_{22} = (\lambda_0 + 2\mu_0) \exp\ (-kr^2),\quad C_{12} = C_{23} = \lambda_0 \exp\ (-kr^2)$$

$$C_{44} = \mu_0 \exp\ (-kr^2) \tag{24}$$

where λ_0 and μ_0 are Lamé constants. Following eqs (1) and (24) the above relation in terms of λ_{ij} are

$$\lambda_{11} = \lambda_{22} = \lambda_0 + 2\mu_0,\ \lambda_{12} = \lambda_{23} = \lambda_0,\ \lambda_{44} = \mu_0. \tag{25}$$

The results in eqs (20)–(22) for spherically isotropic nonhomogeneous bodies can also be used as the results for isotropic nonhomogeneous bodies if we replace λ_{ij} by λ_0 and μ_0 as given in eq. (25). We further note that the application of relations (25) in eq. (12) follows

$$2p = \frac{3}{2}\ . \tag{26}$$

We make use of eqs (25) and (26) in the last two equations of (20) and take the limit as $\kappa \to 0$ (for homogeneous material) and arrive at the following results:

$$\sigma_r = \frac{p_0 a^3 - p_1 b^3}{b^3 - a^3} + \frac{1}{r^3}\frac{(p_1 - p_0)\, a^3 b^3}{b^3 - a^3},$$

$$\sigma_\theta = \sigma_\phi = \frac{p_0 a^3 - p_1 b^3}{b^3 - a^3} - \frac{1}{2}\cdot\frac{1}{r^3}\frac{(p_1 - p_0)\, a^3 b^3}{b^3 - a^3}\ . \tag{27}$$

These are the stresses in a spherical shell ($a \leqslant r \leqslant b$) of homogeneous isotropic elastic material, subjected to internal and external pressures on the boundaries as in eq. (19). These expressions were calculated by Lamé' and are presented in Love.[1]

SOLID SPHERICAL BODY

A solid spherical body ($0 \leqslant r \leqslant b$) of nonhomogeneous spherically isotropic material undergoes compression by a uniformly distributed external pressure p_1. The stresses of such a sphere are obtained from the last two equations of (20) when the cavity of radius 'a' diminishes to zero, *i.e.*, as $a \to 0$

$$\bar{\sigma}_r = -p_1 \left(\frac{b}{r}\right)^{5/2} \exp\left[\frac{\kappa}{2}(b^2 - r^2)\right]$$

$$\times \frac{\{[(2\kappa r^2 - 3)\lambda_{11} + 4\lambda_{12}] M_{k^*,p}(\kappa r^2) + 4\lambda_{11}\kappa r^2 M'_{k^*,p}(\kappa r^2)\}}{\{[(2\kappa b^2 - 3)\lambda_{11} + 4\lambda_{12}] M_{k^*,p}(\kappa b^2) + 4\lambda_{11}\kappa b^2 M'_{k^*,p}(\kappa b^2)\}}$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = -p_1 \left(\frac{b}{r}\right)^{5/2} \exp\left[\frac{\kappa}{2}(b^2 - r^2)\right]$$

$$\times \frac{\{[(2\kappa r^2 - 3)\lambda_{12} + 2(\lambda_{22} + \lambda_{23})] M_{k^*,p}(\kappa r^2) + 4\lambda_{12}\kappa r^2 M'_{k^*,p}(\kappa r^2)\}}{\{[(2\kappa b^2 - 3)\lambda_{11} + 4\lambda_{12}] M_{k^*,p}(\kappa b^2) + 4\lambda_{11}\kappa b^2 M'_{k^*,p}(\kappa b^2)\}} \tag{28}$$

since

$$\operatorname*{Lt}_{z\to 0} \frac{M_{k^*,p}(z)}{M_{k^*,-p}(z)} = 0 \quad \operatorname*{Lt}_{z\to 0} \frac{zM'_{k^*,p}(z)}{M_{k^*-p}(z)} = 0$$

and

$$\operatorname*{Lt}_{z\to 0} \frac{zM'_{k^*,-p}(z)}{M_{k^*,-p}(z)} = \frac{1}{2} - p. \tag{28 a}$$

7. THE PROBLEM OF A COMPOSITE SPHERE

We now consider a homogeneous solid sphere ($0 \leqslant r \leqslant a$) of spherically isotropic material surrounded by a nonhomogeneous concentric spherical shell ($a \leqslant r \leqslant b$) of spherically isotropic medium and the whole body is acted upon by a uniform radial pressure on the external bounding surface $r = b$. At the surface of separation $r = a$ the materials are sufficiently rough to ensure the continuity of radial stress and displacements. The relevant boundary conditions are then

$$\bar{\sigma}_r = -p_1, \quad \text{on the surface } r = b$$

and $u = \bar{u}$, $\sigma_r = \bar{\sigma}_r$, on the surface $r = a$. (29)

In the case of a homogeneous solid sphere $0 \leqslant r \leqslant a$, $C_{ij} = \lambda_{ij}$ for $\kappa = 0$ in eq. (1) and the stress equation of equilibrium corresponding to eq. (16) turns out to be

$$\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) - \frac{2(C_{22} + C_{23} - C_{12})}{C_{11}} u = 0.$$

The general solution of the above equation may be put as

$$u = Cr^{m-1/2} + Dr^{-m-1/2}$$

where

$$m = \left\{\frac{1}{4} + \frac{2(C_{22} + C_{23} - C_{12})}{C_{11}}\right\}^{1/2}. \tag{30}$$

To ensure the finiteness of the stress at every point of the solid sphere, including the neighbourhood of the origin, we are to take the displacement and stresses as (supposing $m > 3/2$)

$$u = Cr^{m-1/2}$$

$$\sigma_r = Ca_m \, r^{m-3/2}$$

$$\sigma_\theta = \sigma_\phi = Cb_m \, r^{m-3/2} \tag{31}$$

where

$$a_m = 2C_{12} + C_{11}\,(m - \tfrac{1}{2})\,, \;\; b_m = (C_{22} + C_{23}) + C_{12}\,(m - \tfrac{1}{2})$$

And for the spherical shell ($a \leqslant r \leqslant b$) the displacements and stresses are given in eqs (16) and (17). Boundary conditions (29) are applied to eq. (16), (17) and (31) to get

$$A\alpha_p\,(b) + B\alpha_{-p}\;(b) + p_1 \exp\left(\frac{\kappa b^2}{2}\right)\;(\kappa b^2)^{3/4} = 0 \tag{32}$$

$$\exp\left(\frac{\kappa a^2}{2}\right)(\kappa a^2)^{-3/4}\,[AM_{k^*,\,p}\,(\kappa a^2) + BM_{k^*,-p}\,(\kappa a^2)] = Ca^{m-1/2}$$

$$\exp\left(-\,\frac{\kappa a^2}{2}\right)(\kappa a^2)^{-3/4}\,[A\alpha_p\,(a) + B\alpha_{-p}\,(a)] = Ca_m\,.\,a^{m-3/2}. \tag{33}$$

Solving the above equations for A, B and C and inserting their values in eq. (31) and eqs (16)–(17) we get the following sets of results:

Displacement and stresses in the sphere ($0 \leqslant r \leqslant a$)

$$u = \frac{p_1 \exp\left[\frac{\kappa}{2}\,(a^2 + b^2)\right]\,.\,b^{3/2}}{Na^{m+1}}\;[\alpha_p\,(a)\,M_{k^*,\,-p}\,(\kappa a^2)$$
$$-\,\alpha_{-p}\,(a)\,M_{k^*,\,p}\;(\kappa a^2)]\;r^{m-1/2},$$

$$\sigma_r = \frac{p_1 \exp\left[\frac{\kappa}{2}\,(a^2 + b^2)\right]b^{3/2}}{Na^{m+1}}\;[\alpha_p\,(a)\,M_{k^*,-p}\,(\kappa a^2)$$
$$-\,\alpha_{-p}\,(a)\,M_{k^*,\,p}\,(\kappa a^2)]\;a_m\,r^{m-3/2},$$

$$\sigma_\theta = \sigma_\phi = \frac{p_1 \exp\left[\frac{\kappa}{2}\,(a^2 + b^2)\right]b^{3/2}}{N\,.\,a^{m+1}}\;[\alpha_p\,(a)\,M_{k^*,-p}\,(\kappa a^2)$$
$$-\,\alpha_{-p}\,(a)\,M_{k^*,\,p}\,(\kappa a^2)]\;b_m\,r^{m-3/2}. \tag{34}$$

Displacement and stresses in the spherical shell ($a \leqslant r \leqslant b$)

$$\bar{u} = \frac{p_1 \exp\left[\frac{\kappa}{2}\,(b^2 + r^2)\right]\left(\frac{b}{r}\right)^{3/2}}{N} \left[\left\{\frac{a_m}{a} \exp\,(\kappa a^2)\, M_{k^*-p}\,(\kappa a^2)\right.\right.$$

$$\left. - \alpha_{-p}(a)\right\} M_{k^*,\,p}\,(\kappa r^2) - \left\{\frac{a_m}{a} \exp\,(\kappa a^2)\, M_{k^*,p}\,(\kappa a^2)\right.$$

$$\left.\left. - \alpha_p\,(a)\right\} M_{k^*,-p}\,(\kappa r^2)\right],$$

$$\bar{\sigma}_r = \frac{p_1 \exp\left[\frac{\kappa}{2}\,(b^2 - r^2)\right]\left(\frac{b}{r}\right)^{3/2}}{N} \left[\left\{\frac{a_m}{a} \exp\,(\kappa a^2)\, M_{k^*,-p}\,(\kappa a^2)\right.\right.$$

$$\left. - \alpha_{-p}\,(a)\right\} \alpha_p\,(r) - \left\{\frac{a_m}{a} \exp\,(\kappa a^2)\, M_{k^*,\,p}\,(\kappa a^2)\right.$$

$$\left.\left. - \alpha_p\,(a)\right\} \alpha_{-p}\,(r)\right]$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = \frac{p_1 \exp\left[\frac{\kappa}{2}\,(b^2 - r^2)\right] \cdot \left(\frac{b}{r}\right)^{3/2}}{N}$$

$$\times \left[\left\{\frac{a_m}{a} \exp\,(\kappa a^2)\, M_{k^*,-p}\,(\kappa a^2) - \alpha_{-p}\,(a)\right\} \beta_p\,(r)\right.$$

$$\left. - \left\{\frac{a_m}{a} \exp\,(\kappa a^2)\, M_{k^*,\,p}\,(\kappa a^2) - \alpha_p\,(a)\right\} \beta_{-p}\,(r)\right]$$

where

$$N = \alpha_{-p}\,(b) \left\{\frac{a_m}{a} \exp\,(\kappa a^2)\, M_{k^*,\,p}\,(\kappa a^2) - \alpha_p\,(a)\right\}$$

$$- \alpha_p\,(b) \left\{\frac{a_m}{a} \exp\,(\kappa a^2)\, M_{k^*,-p}\,(\kappa a^2) - \alpha_{-p}\,(a)\right\}. \tag{35}$$

8. NUMERICAL RESULTS

All the results are for structures with finite outer radius b that is twice the inner radius a. It should be noted that the results previously derived are quite general. The problem investigated involve inhomogeneous materials with properties varying exponentially with the square of the radius according to eq. (1).

We choose the elastic constants $\lambda_{11} = 26\cdot 92$, $\lambda_{12} = 20\cdot 19$, $\lambda_{22} = 16\cdot 56$, λ_{25} $6\cdot 247$, $\lambda_{44} = 6\cdot 53$ in terms of a unit 10^{11} dynes per square centimetres and $\kappa = 2/a^2$ (numerically) for material I in the spherical shell ($a \leqslant r \leqslant b$) and for the inclusion ($0 \leqslant r \leqslant a$) we choose $C_{11} = 6\cdot 17$, $C_{12} = 2\cdot 17$, $C_{22} = 5\cdot 97$, $C_{23} = 2\cdot 62$, $C_{44} = 1\cdot 64$ (with the same unit mentioned previously) for material II. The elastic properties of material I are comparable with concrete with barite as concrete aggregate used extensively for radiation sheilding[7] and the elastic properties of material I resemble those of magnesium. The present analysis may also be useful in studying the stresses for layered media having exponentially increasing or decreasing stiffness.

Using these values of C_{ij} in eqs (30) and (32) the values of m, a_m, b_m are found to be $m = 1\cdot 527$, $a_m = 10\cdot 67$, $b_m = 10\cdot 82$. The other portion $a \leqslant r \leqslant b$ is filled up with material I. We make use of the values of 'λ_{ij}'s in eqs (12) and (13) to obtain $p = 1/3$, and $k^* = 0$. We also calculate the value of the constant N of eq. (35) for material I and find $N = -118082\cdot 0 \times \Gamma\frac{2}{3}\,\Gamma\frac{4}{3}/(a^2)$. Stresses of eqs (34) and (25) may now be had from table 1.

Table 1.

	Nonhomogeneous		Homogeneous	
r	$-\dfrac{\bar{\sigma}_r}{p_1}$	$-\dfrac{\bar{\sigma}_\theta}{p_1}$	$-\dfrac{\sigma_r}{p_1}$	$-\dfrac{\sigma_\theta}{p_1}$
$r = a$	1·79	1·17	1·79	1·81
$r = \sqrt{2}\,a$	1·25	1·03		
$r = \sqrt{3}\,a$	1·07	·78		
$r = 2a$	1	·74		

ACKNOWLEDGEMENT

The author would like to express his appreciation to Dr. A. Das of Jalpaiguri Government Engineering College for suggesting the problem and for his help and encouragement in this work. The author is also grateful to the referee for his valuable comments.

REFERENCES

1. Love, A. E. H., *Theory of Elasticity*, Cambridge University Press, U.K., p. 142 (1952).
2. Lekhnitskii, S. G., *Theory of Elasticity of an Anisotropic Elastic Body* (1963).

THE BANGALORE PRESS

J. STRUCT. MECH., 5(3), 333–342 (1977)

Note on the Radial Deformation and Stresses in Anisotropic Nonhomogeneous Elastic Media

S. C. Roy

JALPAIGURI GOVERNMENT ENGINEERING COLLEGE
JALPAIGURI, WEST BENGAL, INDIA

ABSTRACT

In this paper we study the elasticity problem of a cylindrically anisotropic, elastic medium bounded by two axisymmetric cylindrical surfaces subjected to normal pressures (plane strain). The material of the structure is orthotropic with cylindrical anisotropy and, in addition, is continuously inhomogeneous with mechanical properties varying along the radius. General solutions in terms of Whittaker functions are presented. The results obtained by St. Venant for a homogeneous cylindrically anisotropic medium can be deduced from the general solutions. The problem of a solid cylinder of the same medium under the external pressure is also solved as a particular case of the above problem. Problems of the type covered in this paper are encountered in nuclear reactor design.

I. INTRODUCTION

The elastic behavior of a homogeneous cylindrically aeolotropic material was first studied by St. Venant; see, e.g., Ref. 1 or 2. Problems involving nonhomogeneous media in which the properties vary continuously with spatial position have been studied by various authors. Greif and Chou [3] have adopted a numerical integration method and used the computer in solving the vibration problem of a cylindrically anisotropic nonhomogeneous cylindrical shell (plane strain).

A plane-strain assumption is also used here to find the analytical solution for the radial deformation and corresponding stresses in a cylindrical shell made of cylindrically aeolotropic heterogeneous material under the influence of normal pressures on both boundaries. The results obtained by St. Venant [2] for the homogeneous anisotropic case and those found by Lamé [1] for the homogeneous isotropic case can be deduced from the general results. The corresponding expressions for a solid cylinder of nonhomogeneous anisotropic medium are derived here. The nonhomogeneity of the material is characterized by the elastic parameters C_{ij} (see Refs. 3 and 4) as

$$c_{ij} = \lambda_{ij} \exp(-kr^{2\alpha}) \qquad (i, j = 1, 2, 3) \tag{1}$$

where λ_{ij}, k, and α are the prescribed parameters of the material concerned.

II. FUNDAMENTAL EQUATIONS

The basic system of field equations in linear isothermal static elasticity theory is (a) the generalized Hooke's law, (b) the linearized strain displacement equations, and (c) the stress equations of equilibrium. Here the axis of anisotropy is taken to be the z-axis of the r, θ, z cylindrical coordinate system, and the Young's moduli are of the form

$$\begin{aligned} E_r &= E_1 \exp(-kr^{2\alpha}) \\ E_\theta &= E_2 \exp(-kr^{2\alpha}), \text{ etc.} \end{aligned} \tag{2}$$

For plane-strain assumption, λ_{ij} of Eq. (1) is then expressible in terms of these E_1, E_2, and the Poisson's ratios, see Ref. 3.

For the axisymmetric case the nontrivial stress equation of equilibrium, in the absense of body forces, takes the form

$$\frac{\partial}{\partial r}\bar{\sigma}_r + \frac{1}{r}(\bar{\sigma}_r - \bar{\sigma}_\theta) = 0 \tag{3}$$

Nonzero stresses in the normal, circumferential, and longitudinal directions are

$$\begin{aligned}
\bar{\sigma}_r &= \left(\lambda_{11}\frac{d\bar{u}}{dr} + \lambda_{12}\frac{\bar{u}}{r}\right)\exp\left(-kr^{2\alpha}\right) \\
\bar{\sigma}_\theta &= \left(\lambda_{12}\frac{d\bar{u}}{dr} + \lambda_{22}\frac{\bar{u}}{r}\right)\exp\left(-kr^{2\alpha}\right) \\
\bar{\sigma}_z &= \left(\lambda_{13}\frac{d\bar{u}}{dr} + \lambda_{23}\frac{\bar{u}}{r}\right)\exp\left(-kr^{2\alpha}\right)
\end{aligned} \tag{4}$$

respectively, $\bar{u}$ being the radial displacement.

III. METHOD OF SOLUTION

The equation of equilibrium (3), with the help of Eqs. (4), becomes

$$r^2\frac{d^2\bar{u}}{dr^2} + (1 - 2\alpha kr^{2\alpha})r\frac{d\bar{u}}{dr} - \left(\frac{\lambda_{22} + 2\alpha kr^{2\alpha}\lambda_{12}}{\lambda_{11}}\right)\bar{u} = 0 \tag{5}$$

Now on using the transformations

$$x = kr^{2\alpha} \qquad \text{and} \qquad \bar{u} = V\exp\left(\frac{x}{2}\right) \tag{6}$$

Eq. (5) changes to

$$x^2\frac{d^2V}{dx^2} + x\frac{dV}{dx} + \left\{-\frac{\lambda_{22}}{4\alpha^2\lambda_{11}} + \left(\frac{1}{2} - \frac{\lambda_{12}}{2\alpha\lambda_{11}}\right)x - \frac{x^2}{4}\right\}V = 0 \tag{7}$$

with

$$V = x^{-1/2}U \tag{8}$$

Equation (7) reduces to

$$x^2 \frac{d^2U}{dx^2} + \left\{\frac{1}{4}\left(1 - \frac{\lambda_{22}}{\alpha^2\lambda_{11}}\right) + \left(\frac{1}{2} - \frac{\lambda_{12}}{2\alpha\lambda_{11}}\right)x - \frac{x^2}{4}\right\} U = 0 \tag{9}$$

The solution of the above differential equation is (see Ref. 5)

$$U = AM_{\kappa^*,p}(x) + BM_{\kappa^*,-p}(x) \tag{10}$$

where $M_{\kappa^*,-p}(x)$ are Whittaker functions in which

$$2p = \left(\frac{\lambda_{22}}{\alpha^2\lambda_{11}}\right)^{1/2} \qquad \text{(a positive noninteger)} \tag{11}$$

and

$$\kappa^* = \frac{1}{2}\left(1 - \frac{\lambda_{12}}{\alpha\lambda_{11}}\right) \tag{12}$$

A and B being arbitrary constants.

If $2p$ is an integer or zero, the solution of Eq. (9) may be written as

$$U = CW_{\kappa^*,p}(x) + DW_{-\kappa^*,p}(-x) \tag{13}$$

where

$$W_{\kappa^*,p}(x) = \frac{\Gamma(c-1)}{\Gamma(d-c+1)} M_{\kappa^*,p}(x) + \frac{\Gamma(1-c)}{\Gamma(d)} M_{\kappa^*,-p}(x)$$

in which $c = 1 \pm 2p$ and $d = \frac{1}{2} - \kappa^* \pm p$.

Finally, the radial displacement $\bar{u}(r)$ satisfying the equilibrium Eq. (5) is obtained with the help of Eqs. (6), (8), and (10) as

$$\bar{u} = \frac{\exp(kr^{2\alpha}/2)}{k^{1/2}r^{\alpha}} [AM_{\kappa^*,p}(kr^{2\alpha}) + BM_{\kappa^*,-p}(kr^{2\alpha})] \tag{14}$$

This expression for $\bar{u}$ may be used in Eqs. (4) to get the general expressions for the stresses in terms of A and B.

We now consider a cylindrical shell $a \leq r \leq b$. The structure is made of nonhomogeneous cylindrically anisotropic material. The shell is under the influence of uniformly distributed internal and external pressures. The

boundary conditions are

$$\begin{aligned} \bar{\sigma}_r &= -p_0, \qquad (r = a) \\ \bar{\sigma}_r &= -p_1, \qquad (r = b) \end{aligned} \tag{15}$$

On application of these boundary conditions in the first equation of (4) along with Eq. (14), one obtains two simultaneous equations involving the two unknowns A and B. Solving for A and B and inserting their values in (14) and (4), one obtains the complete solution for the radial displacement and stresses as

$$\begin{aligned}
\bar{u} = \frac{\exp(kr^{2\alpha}/2)r^{-\alpha}}{M}\Bigg[&\left\{p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_{-p}(a)\right.\\
&\left.- p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_{-p}(b)\right\} M_{\kappa^*,p}(kr^{2\alpha})\\
&+ \left\{p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_p(b) - p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_p(a)\right\} M_{\kappa^*,-p}(kr^{2\alpha})\Bigg]\\
\bar{\sigma}_r = \frac{\exp(-kr^{2\alpha}/2)r^{-\alpha}}{M}\Bigg[&\left\{p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_{-p}(a)\right.\\
&\left.- p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_{-p}(b)\right\}\alpha_p(r)\\
&+ \left\{p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_p(b) - p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_p(a)\right\}\alpha_{-p}(r)\Bigg]\\
\bar{\sigma}_\theta = \frac{\exp(-kr^{2\alpha}/2)r^{-\alpha}}{M}\Bigg[&\left\{p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_{-p}(a)\right.\\
&\left.- p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_{-p}(b)\right\}\beta_p(r)\\
&+ \left\{p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_p(b) - p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_p(a)\right\}\beta_{-p}(r)\Bigg]\\
\bar{\sigma}_z = \frac{\exp(-kr^{2\alpha}/2)r^{-\alpha}}{M}\Bigg[&\left\{p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_{-p}(a)\right.\\
&\left.- p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_{-p}(b)\right\}\nu_p(r)\\
&+ \left\{p_0 a^\alpha \exp\left(\frac{ka^{2\alpha}}{2}\right)\alpha_p(b) - p_1 b^\alpha \exp\left(\frac{kb^{2\alpha}}{2}\right)\alpha_p(a)\right\}\nu_{-p}(r)\Bigg]
\end{aligned} \tag{16}$$

where

$$\alpha_{\pm p}(r) = \left[\alpha\left(kr^{2\alpha-1} - \frac{1}{r}\right)\lambda_{11} + \frac{\lambda_{12}}{r}\right]M_{\kappa^*,\pm p}(kr^{2\alpha}) + 2\alpha\lambda_{11}kr^{2\alpha-1}M'_{\kappa^*,\pm p}(kr^{2\alpha})$$

$$\beta_{\pm p}(r) = \left[\alpha\left(kr^{2\alpha-1} - \frac{1}{r}\right)\lambda_{12} + \frac{\lambda_{22}}{r}\right]M_{\kappa^*,\pm p}(kr^{2\alpha}) + 2\alpha\lambda_{12}kr^{2\alpha-1}M'_{\kappa^*,\pm p}(kr^{2\alpha})$$

$$\nu_{\pm p}(r) = \left[\alpha\left(kr^{2\alpha-1} - \frac{1}{r}\right)\lambda_{13} + \frac{\lambda_{23}}{r}\right]M_{\kappa^*,\pm p}(kr^{2\alpha}) + 2\alpha\lambda_{13}kr^{2\alpha-1}M'_{\kappa^*,\pm p}(kr^{2\alpha})$$

and

$$M = \alpha_p(a)\alpha_{-p}(b) - \alpha_{-p}(a)\alpha_p(b) \tag{17}$$

The prime indicates the derivative of the function with respect to its argument.

Stresses in a cylindrical shell ($a \leq r \leq b$) made of homogeneous cylindrically anisotropic material, under the same boundary conditions (15), may be found from the second, third, and fourth equations of (16) on letting $k \to 0$, and these agree with the results obtained by St. Venant (quoted in Ref. 2). For an isotropic body $\lambda_{11} = \lambda_{22} = \lambda + 2\mu$, $\lambda_{12} = \lambda_{13} = \lambda_{23} = \lambda$. Equation (11), with the application of these relations, gives $2\alpha p = 1$. When these are used in the second, third, and fourth equations of (16) along with the limit $k \to 0$, one gets Lamé's results given in Ref. 1.

A solid cylindrical body ($0 \leq r \leq b$) of nonhomogeneous cylindrically aeolotropic material undergoes compression by a uniformly distributed external pressure p_1. The stresses in such a shell are obtained from the last three equations of (16) by setting $a = 0$:

$$\bar{\sigma}_r = -p_1 \exp\left[\frac{k}{2}(b^{2\alpha} - r^{2\alpha})\right]\left(\frac{b}{r}\right)^{\alpha+1} \times \frac{\{(kr^{2\alpha} - 1)\alpha\lambda_{11} + \lambda_{12}\}M_{\kappa^*,p}(kr^{2\alpha}) + 2\alpha\lambda_{11}kr^{2\alpha}M'_{\kappa^*,p}(kr^{2\alpha})}{\{(kb^{2\alpha} - 1)\alpha\lambda_{11} + \lambda_{12}\}M_{\kappa^*,p}(kb^{2\alpha}) + 2\alpha\lambda_{11}kb^{2\alpha}M'_{\kappa^*,p}(kb^{2\alpha})}$$

$$\bar{\sigma}_\theta = -p_1 \exp\left[\frac{k}{2}(b^{2\alpha} - r^{2\alpha})\right]\left(\frac{b}{r}\right)^{\alpha+1} \times \frac{\{(kr^{2\alpha} - 1)\alpha\lambda_{12} + \lambda_{22}\}M_{\kappa^*,p}(kr^{2\alpha}) + 2\alpha\lambda_{12}kr^{2\alpha}M'_{\kappa^*,p}(kr^{2\alpha})}{\{(kb^{2\alpha} - 1)\alpha\lambda_{11} + \lambda_{12}\}M_{\kappa^*,p}(kb^{2\alpha}) + 2\alpha\lambda_{11}kb^{2\alpha}M'_{\kappa^*,p}(kb^{2\alpha})}$$

$$\bar{\sigma}_z = -p_1 \exp\left[\frac{k}{2}(b^{2\alpha} - r^{2\alpha})\right]\left(\frac{b}{r}\right)^{\alpha+1}$$
$$\times \frac{\{(kr^{2\alpha} - 1)\alpha\lambda_{13} + \lambda_{23}\}M_{\kappa^*,p}(kr^{2\alpha}) + 2\alpha\lambda_{13}kr^{2\alpha}M'_{\kappa^*,p}(kr^{2\alpha})}{\{(kb^{2\alpha} - 1)\alpha\lambda_{11} + \lambda_{12}\}M_{\kappa^*,p}(kb^{2\alpha}) + 2\alpha\lambda_{11}kb^{2\alpha}M'_{\kappa^*,p}(kb^{2\alpha})} \tag{18}$$

since

$$\lim_{z\to 0} \frac{M_{\kappa^*,p}(z)}{M_{\kappa^*,-p}(z)} = 0$$

$$\lim_{z\to 0} \frac{zM'_{\kappa^*,p}(z)}{M_{\kappa^*,-p}(z)} = 0$$

and

$$\lim_{z\to 0} \frac{zM'_{\kappa^*,\pm p}(z)}{M_{\kappa^*,\pm p}(z)} = \frac{1}{2} \pm p \tag{18a}$$

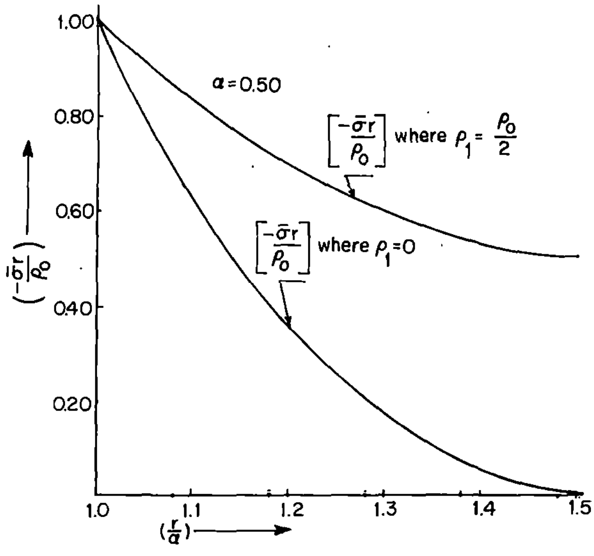
If the shell is under the action of internal pressure only, the external surface being stress-free, the stresses for such an inhomogeneous shell are obtained from (16) by taking $p_1 = 0$.

IV. NUMERICAL RESULTS

Numerical results are obtained for a cylindrical shell structure in which the internal surface is under a uniform normal pressure p_0, while the external surface is assumed to be stress-free for System I, and subjected to half the internal pressure for System II. All numerical results have been computed for the case of $b = 1.5a$.

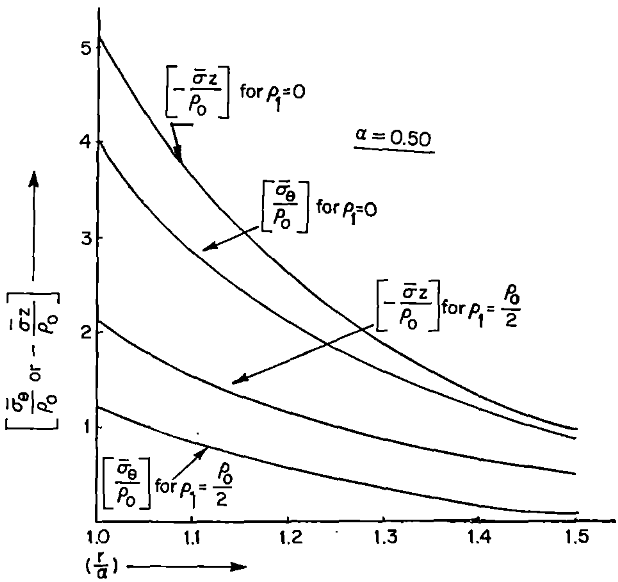
We choose $\alpha = 0.5$ and the elastic parameters as $\lambda_{11} = 918$, $\lambda_{12} = 459$, $\lambda_{22} = 102$, $\lambda_{13} = 275$, $\lambda_{23} = 273$, and $k = 2/a$ (numerically) for Material I, which resembles Barite-cement aggregate (see Ref. 6) used extensively as radiation shielding material. The present analysis may also be useful in studying the stresses in layered media having exponentially increasing or decreasing stiffness.

For this material, $[-\bar{\sigma}_r/p_0]$, $[\bar{\sigma}_\theta/p_0]$, and $[-(\bar{\sigma}_z/p_0)]$ are plotted against (r/a), for both loading Systems I and II in Figs. 1 and 2. Similarly, Figs. 3 and 4 show the corresponding results for Material II identified by $\alpha = 1.0$, and $\lambda_{11} = 918$, $\lambda_{22} = 408$, $\lambda_{12} = 918$, $\lambda_{13} = 275$, $\lambda_{23} = 273$, with $k = 2/a^2$.



a = 0.50
$\left[-\frac{\bar{\sigma}r}{P_0}\right]$ where $P_1 = \frac{P_0}{2}$
$\left[-\frac{\bar{\sigma}r}{P_0}\right]$ where $P_1 = 0$
$\left(-\frac{\bar{\sigma}r}{P_0}\right)$
1.00
0.80
0.60
0.40
0.20
1.0
1.1
1.2
1.3
1.4
1.5
$(\frac{r}{a})$

Fig. 1



$\left[-\frac{\bar{\sigma}z}{P_0}\right]$ for $P_1 = 0$
a = 0.50
$\left[\frac{\bar{\sigma}_\theta}{P_0}\right]$ for $P_1 = 0$
$\left[-\frac{\bar{\sigma}z}{P_0}\right]$ for $P_1 = \frac{P_0}{2}$
$\left[\frac{\bar{\sigma}_\theta}{P_0}\right]$ for $P_1 = \frac{P_0}{2}$
$\left[\frac{\bar{\sigma}_\theta}{P_0} \text{ or } -\frac{\bar{\sigma}z}{P_0}\right]$
5
4
3
2
1
1.0
1.1
1.2
1.3
1.4
1.5
$(\frac{r}{a})$

Fig. 2

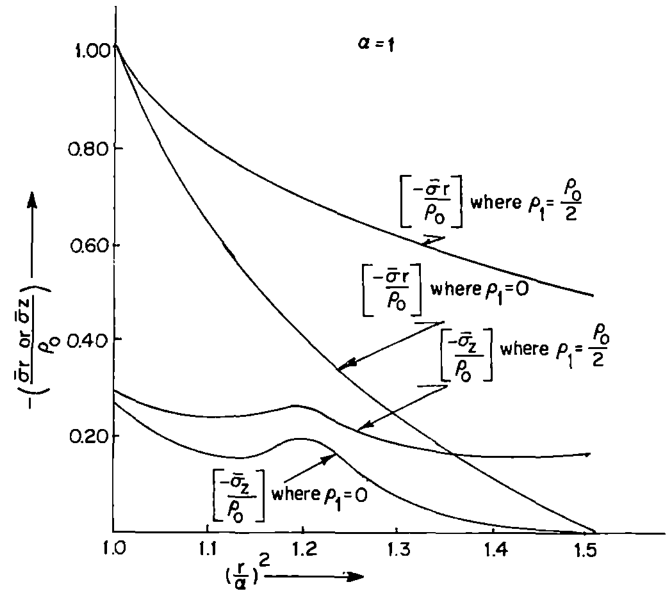

α = 1
1.00
0.80
0.60
0.40
0.20
$-\left(\frac{\bar{\sigma}r \text{ or } \bar{\sigma}z}{P_0}\right)$
$\left[\frac{-\bar{\sigma}r}{P_0}\right]$ where $P_1 = \frac{P_0}{2}$
$\left[\frac{-\bar{\sigma}r}{P_0}\right]$ where $P_1 = 0$
$\left[\frac{-\bar{\sigma}_z}{P_0}\right]$ where $P_1 = \frac{P_0}{2}$
$\left[\frac{-\bar{\sigma}_z}{P_0}\right]$ where $P_1 = 0$
1.0
1.1
1.2
1.3
1.4
1.5
$(\frac{r}{a})^2$

Fig. 3

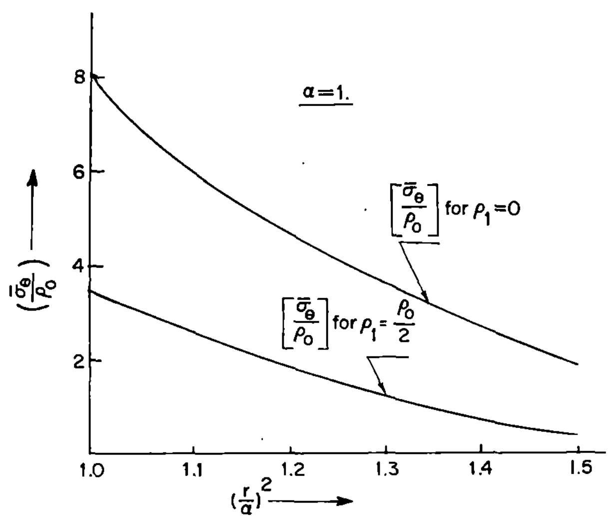

α = 1.
8
6
4
2
$\left(\frac{\bar{\sigma}_\theta}{P_0}\right)$
$\left[\frac{\bar{\sigma}_\theta}{P_0}\right]$ for $P_1 = 0$
$\left[\frac{\bar{\sigma}_\theta}{P_0}\right]$ for $P_1 = \frac{P_0}{2}$
1.0
1.1
1.2
1.3
1.4
1.5
$(\frac{r}{a})^2$

Fig. 4

ACKNOWLEDGMENT

The author would like to express his appreciation to Dr. A. Das of Jalpaiguri Government Engineering College for suggesting the problem and for his help and encouragement in carrying the solution through.

REFERENCES

1. A. E. H. Love, *Theory of Elasticity*, Cambridge University Press, 1952.
2. S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Elastic Body*, Holden Day.
3. R. Greif and S.-C Chou, The propagation of radially symmetric stress waves in anisotropic, nonhomogeneous elastic media, *J. Appl. Mech.* **38**: 51–57 (1971).
4. S. C. Roy, Radial deformations of nonhomogeneous spherically anisotropic elastic media, *J. Indian Inst. Sci.* **58**(7): 303–314 (1976).
5. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1965.
6. D. F. Orchard, *Concrete Technology*, Vol. 1, Asia Publishing House, 1963.

Received September 22, 1976; revised January 17, 1977