INDIAN JOURNAL
OF
THEORETICAL PHYSICS

Founder Director : Late Prof. K. C. Kar, D. Sc.

Published by the
INSTITUTE OF THEORETICAL PHYSICS
BIGNAN KUTIR, CALCUTTA-700004

BOARD OF EDITORS

Dr. M. Ghosh, D. Sc. **Dr. S. D. Chatterjee, D. Sc.**
Dr. S. Sen Gupta, D. Phil. **Dr. P. P. Chattarji, D. Phil.**
Dr. A. N. Basu, D. Phil. **Dr. D. K. Sinha, D. Phil.**
Dr. (Mrs.) C. Dutta, D. Phil. *(Registrar)*
A. Bhattacharyya, M. Sc., F. Inst. T. P. *(Secretary)*
J. Biswas, M. Sc., *(Asst. Secretary)*

NOTICE TO CONTRIBUTORS

1. The manuscript of each paper for publication in this journal should be typed on one side of the paper and should be accompanied by a short abstract. Two copies of each paper should be submitted.
2. All the references to the published papers should be numbered serially and given at the end of the paper.
3. Diagram should be drawn on thick white paper in Indian ink. Tables should be typed on separate sheets.
4. The papers to be published in the journal should be original on theoretical physics or on experimental physics with theoretical background.
5. Avoid mathematical steps as far as possible. All mathematical expressions, should be typed as far as possible, with unavailable symbols inserted neatly in ink. Be sure to distinguish between capital and small letters. Avoid complicated superscripts and subscripts. Avoid repetition of a complicated expression by representing it by a symbol.
6. Letters to the Editor, Review articles and Booklets on recent topics will also be published.
7. Fifty copies of reprints of each paper, except review articles and booklets are given free of cost to the contributor.

All communications to the Secretary, Institute of Theoretical Physics, Bignan Kutir, 4/1, Mohan Bagan Lane, Calcutta-700004.

Subscription Rates (net)

Per volume Rs. 40·00 *(Inland)* *Per volume* $ 10·00 *(Foreign)*
" Rs. 80·00 *(Foreign)* " £ 4·30 "

Back Volume Rates (net)

Back volumes at Rs. 120·00 *or Equivalent Dollars or Pounds per volume*

Indian Journal of Theoretical Physics is published quarterly in March, June, September and December.

INDIAN JOURNAL OF THEORETICAL PHYSICS VOL. 23, NO. 4, 1975.

Note on the unsteady flow of viscous incompressible fluid of finite depth over a fixed rigid plane base due to an arbitrary time varying pressure-gradient

By

Dilip Kumar Das

(*Received for publication in April 1973*)

[**Abstract:** In this paper, the problem of unidirectional flow of a viscous incompressible fluid under prescribed pressure gradients is solved.]

Introduction

The problem of unidirectional flow of viscous incompressible fluid acted upon by pressure gradient has received the attention of several investigators. Mithal[1] has discussed the non-steady flow of a viscous incompressible fluid through a straight tube of circular cross-section in the presence of a time-varying pressure-gradient. The non-steady flow of viscous incompressible fluid through a straight channel with two parallel flat walls has also been studied by Satya Prakash[2] in the presence of a time-varying pressure-gradient. In the present note we consider the laminer motion set up in a viscous incompressible fluid by pressure-gradient acting in the direction of the flow. In such a case the pressure gradient is a function of time but is independent of the space co-ordinates. The general solution is obtained by using the method of Laplace Transforms. Velocity distributions for some particular types of pressure gradients namely (1) finite (2) impulsive and (3) admitting of periodic pulse are calculated in finite forms.

Formulation of the problem

A co-ordinate system is chosen in which the origin lies on the free surface of the fluid and the x-axis denotes the direction of the flow and y-axis is normal to x and normal to the boundary. Let the depth of the fluid be h. Let us choose the motion to be unidirectional, then the equation of continuity becomes

$$\frac{\partial u}{\partial x}=0 \qquad \cdots \qquad (1)$$

Therefore u is a function of y, t say $u=u(y, t)$

The stokes-Navier equation for slow motion becomes

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \qquad \cdots \qquad (2)$$

$$0=\frac{\partial p}{\partial y} \qquad \cdots \qquad (3)$$

From (1), (2) and (3) it follows that $\frac{\partial p}{\partial x}$ is function of time alone.

Let us suppose that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=f(t)$$

Here we assume that there is no external force and ρ denotes the density of the fluid, ν the kinematic co-efficient of viscosity, u the component of velocity along x-direction ; t the time and p the pressure at any point whose co-ordinates are (x, y).

The initial and boundary conditions are

$$\left.\begin{array}{llll} (a) & u=0 & \text{when} & -h\leqslant y\leqslant 0, \quad t\geqslant 0 \\ (b) & u=0 & \text{when} & y=-h, \quad t>0 \\ (c) & \frac{\partial u}{\partial y}=0 & \text{when} & y=0, \quad t>0 \end{array}\right\} \qquad \cdots \qquad (4)$$

Method of solution

Let us introduce the Laplace transform

$$\bar{u}=\int_0^\infty e^{-\lambda t}\, u\, dt$$

The subsidiary equation corresponding to (2) is

$$\frac{d^2\bar{u}}{dy^2}=\frac{\lambda}{\nu}\bar{u}=-\frac{\bar{f}(\lambda)}{\nu} \qquad \cdots \qquad (5)$$

Equation (4) reduces to

$$\left.\begin{array}{lll} (a) & \bar{u}=0 & \text{when} \quad -h\leqslant y\leqslant 0 \\ (b) & \bar{u}=0 & \text{on} \quad y=-h \\ (c) & \dfrac{d\bar{u}}{dy}=0 & \text{on} \quad y=0 \end{array}\right\} \qquad \cdots \qquad (6)$$

Solution of equation (5) subject to the conditions (6) is

$$\bar{u}=\frac{\bar{f}(\lambda)}{\lambda}\left[1-\frac{\cos h\sqrt{\frac{\lambda}{\nu}}\,y}{\cos h\sqrt{\frac{\lambda}{\nu}}\,h}\right] \qquad \cdots \qquad (7)$$

Now, $\bar{f}(\lambda)$ is the Laplace transform of $f(t)$ and $\left\{\frac{1}{\lambda}-\frac{1}{\lambda}\frac{\cos h\sqrt{\frac{\lambda}{\nu}}y}{\cos h\sqrt{\frac{\lambda}{\nu}}h}\right\}$

is the Laplace transform of $4\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\frac{\pi y}{2h}}{(2n+1)\pi}$

$e^{-\frac{(2n+1)^2\pi^2\nu t}{4h^2}}=g(y,t)$ say[3].

Thus applying convolution theorem, we have (*c. f.* 4)

$$u(y,t)=\int_0^t f(\tau)\, g\,(y, t-\tau)\, d\tau \qquad \cdots \qquad (8)$$

$$=\int_0^t g(y,\tau)\, f(t-\tau)\, d\tau \qquad \cdots \quad (8a)$$

$$=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n \cos(2n+1)\frac{\pi y}{2h}}{2n+1}\int_0^t e^{-\frac{(2n+1)^2\pi^2\nu t}{4h^2}} f(t-\tau)\, d\tau \qquad \cdots \quad (8b)$$

Particular types of pressure gradients

(*a*) *Constant pressure gradient* :—

We take

$$f(t)=A\,H(t) \qquad \cdots \quad (9)$$

where $H(t)$ is Heaviside's unit function, A is a constant.

i. e., $H(t)=1$ for $t>0$

$=0$ for $t\leqslant 0$

Substituting from (9) in (8) we get

$$u(y,t)=\frac{4A}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n \cos(2n+1)\frac{\pi y}{2h}}{2n+1}\int_0^t e^{-\frac{(2n+1)^2\pi^2\nu t}{4h^2}} H(t-\tau)\, d\tau \qquad \cdots \quad (10)$$

Where $H(t-\tau)=1$ when $t>\tau$

$=0$ when $t<\tau$.

Thus when $t<\tau$, we have

$$u(y,t)=0 \qquad \cdots \quad (10a)$$

When $t>\tau$ we have

$$u(y, t)=\frac{4A}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n \cos (2n+1)\frac{\pi y}{2h}}{2n+1}\int_0^t e^{-\frac{(2n+1^2 \pi^2 \nu\tau}{4h^2}} d\tau$$

$$=\frac{16Ah^2}{\pi^3\nu}\sum_{n=0}^{\infty}\frac{(-1)^{n+1} \cos (2n+1)\frac{\pi y}{2h}}{(2n+1)^3}\left[e^{-\frac{(2n+1)^2\pi^2\nu t}{4h^2}}-1\right] \quad \cdots \quad (11)$$

(*b*) *Impulsive pressure-gradient* :—

we take

$$f(t)=A\,\delta(t)$$

where A is constant

Therefore $\quad f(t-\tau)=A\,\delta(t-\tau) \quad \cdots \quad (12)$

Making substitution from (12) in (8*a*) we get

$$u(y, t)=A\int_0^t g\,(y, \tau)\,\delta\,(t-\tau)\,d\tau$$

$$=A\,g\,(y, t)$$

$$=\frac{4A}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n \cos (2n+1)\frac{\pi y}{2h}}{(2n+1)}e^{-\frac{(2n+1)^2\pi^2\nu\tau}{4h^2}} \quad \cdots \quad (13)$$

(*c*) *Periodic pulse of pressure gradient* :—

we take

$$f(t)=B \sin \omega t \quad \text{when} \quad 0\leqslant t\leqslant\frac{\pi}{\omega}$$

$$=0 \quad \text{when} \quad t>\frac{\pi}{\omega}$$

Now

$$\bar{f}(\lambda)=B\int_0^{\frac{\pi}{\omega}} e^{-\lambda t}\sin\omega t\,dt$$

$$=\frac{B\omega}{\lambda^2+\omega^2}\left(1+e^{-\frac{\pi\lambda}{\omega}}\right) \quad \cdots \quad (14)$$

Making substitution from (14) in Equation (7) we get

$$\bar{u}(y,\lambda)=\frac{B\omega}{\lambda(\lambda^2+\omega^2)}\left(1-\frac{\cos h\sqrt{\frac{\lambda}{\nu}}y}{\cos h\sqrt{\frac{\lambda}{\nu}}h}\right)\left(1+e^{-\frac{\pi\lambda}{\omega}}\right)$$

Now

$$L^{-1}\left[\frac{B\omega}{\lambda(\lambda^2+\omega^2)}\left\{1-\frac{\cos h\sqrt{\frac{\lambda}{\nu}}y}{\cos h\sqrt{\frac{\lambda}{\nu}}h}\right\}\right]=-\frac{B}{\omega}\cos\omega t$$

$$+\frac{B}{2\omega}\left[e^{i\omega t}\frac{\cos h\sqrt{\frac{i\omega}{\nu}}y}{\cos h\sqrt{\frac{i\omega}{\nu}}h}+e^{-i\omega t}\frac{\cos\sqrt{\frac{i\omega}{\nu}}y}{\cos\sqrt{\frac{i\omega}{\nu}}h}\right]$$

$$+\sum_{n=0}^{\infty}\frac{4B(-1)^n\omega\cos(2n+1)\frac{\pi y}{2h}}{(2n+1)\pi\left[\frac{(2n+1)^4\pi^4\nu^2}{16h^4}+\omega^2\right]}e^{-\frac{(2n+1)^2\pi^2\nu t}{4h^2}}=\phi(y,t) \text{ say}$$

Then

$$u(y,t)=\phi(y,t)+\phi\left(y,t-\frac{\pi}{\omega}\right)H\left(t-\frac{\pi}{\omega}\right) \quad \cdots \quad (15)$$

where $H\left(t-\frac{\pi}{\omega}\right)=1$ when $t>\frac{\pi}{\omega}$

$=0$ when $t\leqslant\frac{\pi}{\omega}$.

Thus when $0 \leqslant t \leqslant \frac{\pi}{\omega}$

$$u(y, t) = -\frac{B}{\omega}\cos\omega t + \frac{B}{2\omega}\left[e^{i\omega t}\frac{\cos h\sqrt{\frac{i\omega}{\nu}}y}{\cos h\sqrt{\frac{i\omega}{\nu}}h} + e^{-i\omega t}\frac{\cos\sqrt{\frac{i\omega}{\nu}}y}{\cos\sqrt{\frac{i\omega}{\nu}}h}\right]$$

$$= \frac{4B\omega}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\frac{\pi y}{2h}}{(2n+1)\left[\frac{(2n+1)^4\pi^4\nu^2}{16h^4}+\omega^2\right]}e^{-\frac{(2n+1)^2\pi^2\nu t}{4h^2}} \quad (16)$$

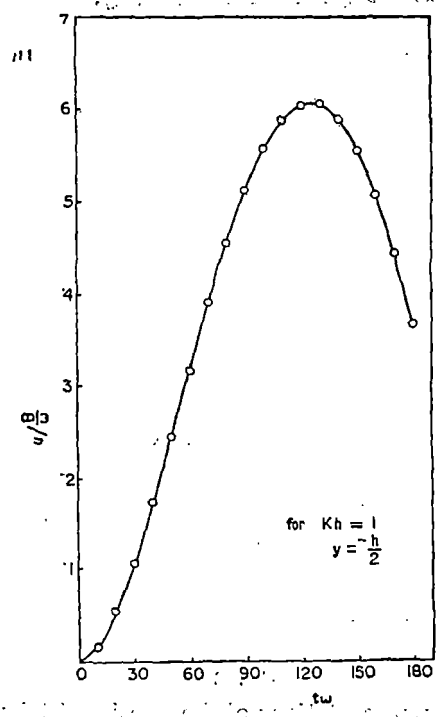

Fig. 1

When $t > \frac{\pi}{\omega}$

$$u(y, t) = \frac{4B\omega}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \cos (2n+1)\frac{\pi y}{2h}}{(2n+1)\left[\frac{(2n+1)^4 \pi^4 \nu^2}{16h^4} + \omega^2\right]}$$

$$\left[e^{-\frac{(2n+1)^2 \pi^2 \nu t}{4h^2}} + e^{-\frac{(2n+1)^2 \pi^2 \nu \left(t - \frac{\pi}{\omega}\right)}{4h^2}}\right] \quad \cdots \quad (17)$$

Equation (8*b*) gives the velocity distribution for the viscous incompressible fluid with an arbitrary pressure gradient which is a function of time. The expression (10), (13) and (15) are the velocity distribution of the same fluid with constant, impulsive and periodic pressure gradients respectively.

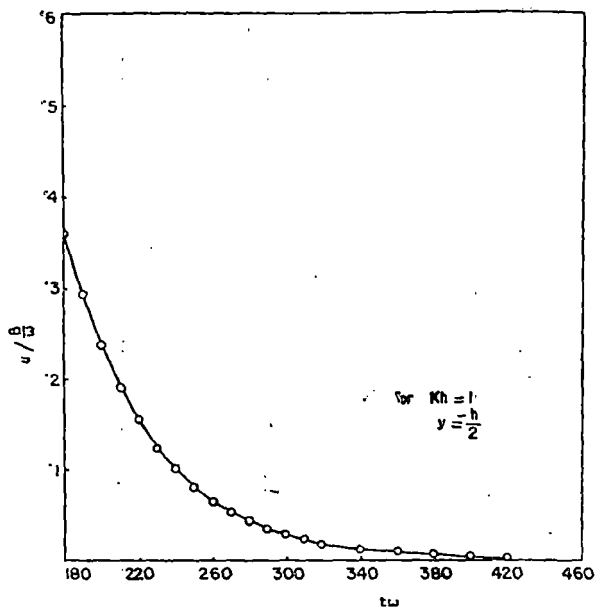

Fig. 2

A graphical representations of the velocity distribution in the range $0 < \omega t < \pi$ and $\omega t > \pi$ are depicted in Fig. 1 and Fig. 2

respectively $\left[\text{Assuming the value of } \sqrt{\frac{\omega}{2\nu}}\, h=1 \text{ and } y=-\frac{h}{2},\right.$ $\left.\sqrt{\frac{\omega}{2\nu}}=k\right]$ Fig. 1 and Fig. 2 show that velocity reaches its maximum when $\omega t \simeq 2{\cdot}18$ and after that exponentially decays

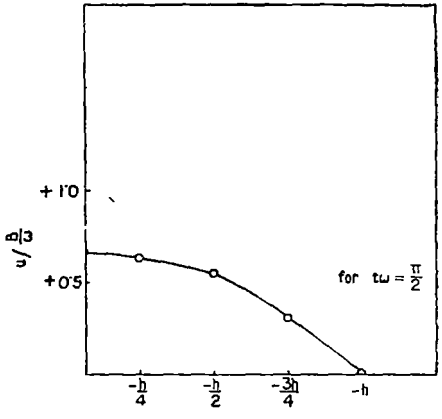

Fig. 3

as t increases. A variation in the distribution of velocity with the different height is also shown in Fig. 3 assuming the fixed value of $\omega t=\frac{\pi}{2}$

I take this opportunity of thanking Dr. P. Bhattacharyya for his kind help and guidance in preparation of this paper. I am also indebted to the University of North Bengal for the award of U. G. C. Scholarship.

Department of Mathematics
North Bengal University
Darjeeling

References

1. Mithal, K. G.—Bull. Cal. Math. Soc. Vol. 52, p. 147 (1960).
2. Satya Prakash—Bull. Cal. Math. Soc. Vol. 59, p. 55 (1967).
3. Harry Bateman,—Tables of Integral Transforms, Vol. 1.
4. Carslaw and Jaeger,—Operational Methods in Applied Mathematics.

INTRODUCTION TO THEORETICAL PHYSICS

by Prof. K. C. Kar, D. Sc.

Vol. I. Mechanics of particle (including relativity), rigid body, deformable body, fluid, statistical mechanics and thermodynamics.

Useful to students of higher physics. Special emphasis is given to the physical side of each problem. An important feature of the book is that certain examples in particle and rigid dynamics which are important for the study of higher physics are worked out by different important methods.

Price : $ 7·50 ● Rs. 40·00

Vol. II. Acoustics

Theories of different problems of acoustics are discussed emphasising throughout their physical aspects. Many new problems on forced vibration and the vibration of string are discussed for the first time. The book is useful to students of higher acoustics and is a standard treatise on the subject.

Price : $ 7·50 ● Rs. 40·00

STATISTICAL MECHANICS

VOLS. I & II

Principles and Applications

by Prof. K. C. Kar, D. Sc.

The different statistical methods are the first time classified under five heads. The book is more exhaustive than the existing treatises on the subject and is useful to advanced students of theoretical physics (including physical chemistry).

Second edition in the press

WAVE STATISTICS

Its Principles and Applications

by Prof. K. C. Kar, D. Sc.

The book has been written to present the theories developed by the author and his collaborators, for the first time in a connected form. High Velocity problems *e.g.*, X-ray scattering in the Klein Nishina region, radiative scattering and pair production are discussed in detail. Special emphasis is given to the physical sides of the different problems discussed. A special feature of the book is that certain questions which are never discussed in books on quantum mechanics, are for the first time clearly explained by the author. The questions are :—(1) Why Schrödinger equation gives correct eigen values ? (2) Why $\chi\chi^*$ gives the probability ? (3) What is the nature of Heisenberg exchange force ? What is the basis of the selection rule ? and so on.

It is hoped the book will be useful to students of Quantum Mechanics.

The author is the first and the only Indian Physicist to write a comprehensive book on his extensive theory.

Price : $ 10·00 ● Rs. 60·00

A NEW APPROACH TO THE THEORY OF RELATIVITY

by Prof. K. C. Kar, D. Sc.

In this book the relativistic problems have been discussed from new standpoints. Author's new theory of gravitation has been presented. In this theory it is explained why the gravitational field is always attractive and is never repulsive. On the basis of this theory Einstein's formulæ for perihelion motion of planets and the bending of light have been deduced with the help of simple theory of relativity. Einstein's complicated general theory, or his assumption of space-time curvature has not been used at any stage. Thus there is nothing as special or general theory of relativity. There is only one theory which is the simple theory of relativity. The phenomenon of red-shift which is an outcome of general theory of relativity, has been discussed as a purely non-relativistic effect. It is due to action of the gravitational field on the photon mass $h\nu/c^2$.

It is hoped the book will be useful to the Post graduate and advanced students who want to have a clear grasp of the fundamentals of relativity.

Price : $ 6·00 ● Rs. 30·00

Institute of Theoretical Physics
"Bignan Kutir"
4/1, Mohan Bagan Lane, Calcutta-700004

All rights reserved. Copyright by the Institute of Theoretical Physics
printed by A. K. Das at I. T. P. Press, Bignan Kutir, 4/1, Mohan Bagan Lane, Calcutta-700004.