Chapter 3

TESTING GRAVITY AT THE SECOND POST-NEWTONIAN LEVEL THROUGH GRAVITATIONAL DEFLECTION OF MASSIVE PARTICLES

3.1 Introduction

A consequence of general relativity (GR) is that light rays are deflected by gravity. Historically, observations of this aspect of gravity provided the first proof in favor of GR. The effect is nowadays routinely used as a tool to study various features of the universe, such as viewing fainter or distant sources, estimating the masses of galaxies etc.[1].

Like photons, particles having masses are also deflected by gravity. The general relativistic corrections in the equation of motion of a massive test particle moving in bound orbits have been studied with great accuracy in the literature [2]. These studies have great relevance in comparing theory with observations of gravitational waves from compact binary systems. However, for unbound orbits of massive particle, the question of accuracy beyond first order has not received sufficient attention as of now. The main reason could be that the observational aspect of unbounded massive particles was not very practicle: there was no known astrophysical source of free point particles that can be detected easily with good angular precision. The situation seems to have improved somewhat. Recent theoretical studies [3] favor the existence of local astrophysical sources of relativistic neutral particles like neutrons and neutrinos with observable fluxes. Besides, high energy neutrons are produced during solar flares [4]. Moreover, with the advent of new technology new experiments have been proposed [5], primarily to study gravitational deflection of light with high precision, in which laser interferometry will be employed between two micro-spacecrafts whose line of sight pass close to the sun. Hence there might be a possibility that in the future, neutron or some other neutral particle may be used in a similar experiment instead of photon thus providing an opportunity of studying gravitational deflection of massive particles. Henceforth, we use the abbreviation for post-Newtonian as PN such that first-PN effect is of the order of $(1/\rho)$, second-PN effect is of the order of $(1/\rho^2)$ and so on.

The expected angular precision of the planned astrometric missions using optical interferometry is at the level of microarcseconds (μ arcsec) and hence these experiments would measure the effects of gravity on light at the second-PN order (c^{-4}). Though measurements with massive particles at the level of microarcsecond accuracy is way beyond the present technical capability, it can still be cautiously hoped that astrometric missions in the distant future using massive particle interferometry would have angular precisions close to that to be obtained using laser (optical) interferometry. Whatever be the technical scenario, a study of theoretical aspects of gravitational deflection angle for massive particles at the second-PN approximation is useful in its own right. (To our knowledge, the deflection angle for massive particles has been theoretically estimated in the literature with an accuracy of only first-PN order [6] so far).

Accordingly, in this chapter, the second- PN contribution to the gravitational deflection of massive particles by a gravitating object will be estimated in a model independent way but with a special emphasis on the sun as gravitating object. The corresponding PN parameters for light deflection then follow as a corollary. The importance of this investigation is manifold: It allows us to obviate the ambiguities related with photon deflection in that order; it allows us to construct the true coordinate solar radius from measurements and moreover, it constitute a possible futher test of the equivalence principle. These are discussed at the end.

3.2 Gravitational deflection of massive particles at second post-Newtonian order

We consider the general static and spherically symmetric spacetime in isotropic coordinate which is given by (we use the geometrized units i.e. G=1, c=1)

$$ds^2 = -B(\rho)dt^2 + A(\rho)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\phi^2) \tag{3.1}$$

The standard equations for a geodesic, namely

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{3.2}$$

for the general metric (3.1) become

$$\frac{d^2\rho}{ds^2} + \frac{A'}{2A}\left(\frac{d\rho}{ds}\right)^2 - \rho\left(1 + \frac{\rho A'}{2A}\right) = 0 \tag{3.3}$$

$$\frac{d^2\theta}{ds^2} + \left(\frac{2}{\rho} + \frac{A'}{A}\right)\frac{d\rho}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \tag{3.4}$$

$$\frac{d^2\phi}{ds^2} + \left(\frac{2}{\rho} + \frac{A'}{A}\right)\frac{d\rho}{ds}\frac{d\theta}{ds} - 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \tag{3.5}$$

$$\frac{d^2t}{ds^2} + \frac{B'}{B}\frac{d\rho}{ds}\frac{dt}{ds} = 0 \tag{3.6}$$

(primes denoting differentiation with respect to ρ). If we choose $\theta = \pi/2$ and $\frac{d\theta}{ds} = 0$ initially, Eq. (3.4) warrants that they would remain the same always. Thus normalizing time coordinate suitably, one obtains for orbits in the equatorial plane from Eq.(3.6)

$$\frac{dt}{ds} = B^{-1} \tag{3.7}$$

Integrating Eq.(3.5)

$$\rho^2\frac{d\phi}{ds} = A^{-1}J^2 \tag{3.8}$$

where J is a constant of integration. From Eqs.(3.3), (3.7) and (3.8), one finally obtains

$$\frac{1}{A\rho^4}\left(\frac{d\rho}{d\phi}\right)^2+\frac{1}{A\rho^2}-\frac{1}{J^2}\left(\frac{1}{B}-E\right)=0 \qquad (3.9)$$

J can be conveniently expressed in terms of distance at closest approach. At the point of closest approach, $\frac{d\rho}{d\phi}$ vanishes.

Restricting to orbits in the equatorial plane $\left(\theta=\frac{\pi}{2}\right)$, the expression for the deflection angle for particles moving with a velocity V as measured by an asymptotic rest observer can be written as [7]

$$\alpha(\rho_0)=I(\rho_0)-\pi \qquad (3.10)$$

with

$$I(\rho_0)=2\int_{\rho_0}^{\infty}\frac{d\rho}{\rho^2}A^{-1/2}(\rho)\left[\frac{1}{J^2}(B^{-1}(\rho)-E)-\frac{1}{\rho^2 A}\right]^{-1/2} \qquad (3.11)$$

where ρ_0 being the distance of the closest approach,

$$J=\rho_0\left[A(\rho_0)(B^{-1}(\rho_0)-E)\right]^{1/2} \qquad (3.12)$$

and

$$E=1-V^2 \qquad (3.13)$$

The Eqs.(3.11) and (3.12) follow from the Eq.(3.9).

The PN formalism in some orders [7,8] is usually employed to describe the gravitational theories in the solar system and also to compare predictions of GR with the results predicted by an alternative metric theory of gravity. This method actually is an approximation for obtaining the dynamics of a particle (in a weak gravitational field under the influence of a slowly moving gravitational source) to one higher order in $\frac{M}{\rho}$ (M is the mass of the static gravitating object) than given by the Newtonian mechanics. Following the PN expansion method, we assume the metric tensor is equal to the Minkowski tensor $\eta_{\mu\nu}$ plus corrections in the form of expansions in powers of $\frac{M}{\rho}$ and considering up to the second-PN correction terms, we have

$$B(\rho) = 1 - 2\frac{M}{\rho} + 2\beta_i \frac{M^2}{\rho^2} - \frac{3}{2}\varepsilon_i \frac{M^3}{\rho^3} \tag{3.14}$$

$$A(\rho) = 1 + 2\gamma_i \frac{M}{\rho} + \frac{3}{2}\delta_i \frac{M^2}{\rho^2} \tag{3.15}$$

β_i, γ_i are the PN parameters (also known as the Eddington parameters), δ_i, ε_i can be considered as the second-PN parameters, i stands for either γ or m denoting photons or massive particles respectively . Several of these parameters are different for different theories. In GR, all of them are equal to 1 as can be readily checked by expanding the Schwarzschild metric.

We should note that the metric coefficients above are independent of any specific model; they result solely from the assumption of central symmetry. Starting with the expansion (3.14) and (3.15) *per se*, the expression for the angle of deflection for unbound particles up to the second–PN order follows from Eq. (3.11) and when $\frac{2M}{\rho} << V^2$, it works out to

$$\alpha_m = a_m \frac{M}{\rho_0} + b_m \left(\frac{M}{\rho_0}\right)^2 \tag{3.16}$$

where

$$a_m = 2\left(\gamma_m + \frac{1}{V^2}\right) \tag{3.17}$$

$$b_m = \frac{3\delta_m\pi}{4} + (2+2\gamma_m - \beta_m)\frac{\pi}{V^2} - 2\left(\gamma_m + \frac{1}{V^2}\right)^2 \qquad (3.18)$$

The above expression are also valid for massless particles $(\alpha_{\gamma,} a_{\gamma,} b_\gamma)$ as may be seen under the substitution V=1. Clearly the deflection angle would be larger for particles in comparison to that of photons. Our calculation shows that the term representing the second order effect (b_m) contains only the three parameters $\beta_m, \gamma_m, \delta_m$ and does not contain ε_m, a cubic order contribution . This implies that calculation of the deflection of unbound particle orbits (including photons) by gravity to any given order needs only the knowledge of every term to that order in the expansions. In other words, to second-PN order, one needs to consider both in g_{00} and g_{ij} terms only up to $\frac{M^2}{\rho^2}$. Similarly, to third-PN order, which is not our interest here, we would need expansions of both the metric components up to order $\frac{M^3}{\rho^3}$ and so on. This is in contrast to the case of planetary dynamics (bound orbits) where the calculation typically requires knowledge of g_{00} more accurately than g_{ij} (For instance, to calculate the planetary precession to the order of M, one expands g_{00} up to $\frac{2\beta_m M^2}{\rho^2}$ while g_{ij} is expanded up to only $\frac{2\gamma_m \mathrm{M}}{\rho}$; for next order accuracy, one would need to consider the complete expansion as given in Eqs. (3.14), (3.15) above so that the parameters δ_m, ε_m become important in this case). The deflection angle α_m for the Schwarzschild spacetime can be obtained by taking $\beta_m = \gamma_m = \delta_m = 1$.

The deflection angle also can be expressed in terms of co-ordinate idependent variable, such as the impact parameter b which is the perpendicular distance from the centre of the gravitating object to the tangent to the geodesic at the closest appraoch. In that case, ρ has to be replaced by b in Eq.(3.16), the Eq. (3.17) would remain unaltered but Eq. (3.18) would change to

$$b_m = \frac{3\delta_m \pi}{4} + (2 + 2\gamma_m - \beta_m)\frac{\pi}{V^2} + 2\left(\gamma_m + \frac{1}{V^2}\right)\left(1 - \frac{1}{V^2}\right) \tag{3.19}$$

Since impact parameter is the ratio of the angular momentum and energy of the particle as measured by an observer at rest far from the gravitating object, it is a formally measurable quantity but is not very suitable for practical measurements [9].

3.3 OTHER SIGNFICANT EFFECTS

In the present work, the mass distribution of the gravitating object is assumed to be mainly spherically symmetric; any deviation from such symmetry would produce their effects. The effect of quadrupole moment of the mass distribution on the deflection angle is proportional to $\frac{J_Q MR^2}{\rho_0^{\,3}}$, where R is the average radius of sun. Thus, even a small quadrupole moment parameter J_Q could produce significant contribution to deflection. However, the effect is limited largely to the first-PN order (~ 0.1 μ arcsec) while in the second-PN order effect is too small (~ 10^{-7} -10^{-8} μ arcsec). If the gravitating object also has angular momentum, its effect on the deflection angle contributes to the second-PN order but it can be seperated out. All these are discussed below.

3.3.1 Effect of quadrupole moment of the mass distribution

Therotical value of solar quadrupole moment J_Q, though it depends strongly on solar model used, is very small, of the order of 10^{-7} [5, 8]. Since our study is aimed at sun as the gravitating object we have ignored higher order terms involving J_Q. Thus due to the quadrupole moment of the mass distribution the effective mass parameter becomes

$$M_{eff} = M\left[1+\frac{J_Q R^2}{2\rho^2}\left(3\cos^2\theta-1\right)\right]$$

which leads to the following corrections in the components of the metric tensors [5,8,10]:

$$\delta\, g_{00}(\rho) = J_Q \frac{MR^2}{\rho^3}\left(3\cos^2\theta-1\right) \tag{3.20}$$

and

$$\delta\, g_{jk}(\rho) = -\delta_{jk}\gamma_i J_Q \frac{MR^2}{\rho^3}\left(3\cos^2\theta-1\right) \tag{3.21}$$

where R is the average radius of the mass distribution and θ is the angle between radius vector and the z-axis and hence in the equatorial plane $\theta = \frac{\pi}{2}$. In the equatorial plane, the deflection caused by the quadrupole moment calculates to

$$\alpha_{QM} = \frac{2J_Q MR^2}{\rho_0^{\,3}}\left(\frac{\gamma_m}{3}+\frac{1}{V^2}\right) \tag{3.22}$$

Assuming R~ρ0 at the closest approach to the sun and taking V=0.75, $\beta_m = \gamma_m = \delta_m \sim 1$, for sun $\frac{M_\oplus}{R_\oplus} = 2.12\times10^{-6}$, the first-PN quadrupole term $\alpha_{QM} \sim 0.1$ μ arcsec. It is roughly 7 orders of magnitude less than the first-PN deflection

$a_m \dfrac{M}{\rho_0}$~ 0.8sec and is more than one order of magnitude less than the second- PN contribution $b_m\left(\dfrac{M}{\rho_0}\right)^2$~3.4 μ arcsec. We have not displayed the next higher order quadrupole terms involving $J^2{}_Q$ and second order terms in $\dfrac{1}{\rho^2}$ containing J_Q here because they have magnitude in the range 10^{-7} to 10^{-8} μ arcsec, too small to be of any practical significance . We can justifiably ignore these second-PN quadrupole contribution. The qhuadrupole contribution to the deflection of light is given in Ref. [11].

3.3.2 Effect of Rotation

The angular momentum of the gravitating object is assumed small as in the case of sun. The resulting leading term of the relevant metric tensor is

$$g_{oi} = \frac{4Ma}{\rho} \tag{3.23}$$

where a is the angular momentum per unit mass of the object. The contribution of the rotation to the deflection angle is given by [6]

$$\alpha_{rot} = \frac{4MaV}{\rho_0{}^2} \tag{3.24}$$

The value a can be positive or negative depending on the direction of rotation. When the angular momentum of the gravitating object is antiparallel with the direction of the incoming particle, a is positive and hence rotation causes larger deflection whereas for parallel angular momentum, a is negative and the deflection angle will be less. Thus the rotational effect can be easily seperated out from other contributions by studying the deflection of particles at two opposite sides of the gravitating object. The gravitational deflection angle of light with an accuracy up to second-PN order readily follows from Eqs(3.16)-(3.18), (3.22) and (3.24) using V=1.

3.4 Discussion

The study of gravitational deflection of massive particles is important for several reasons which are discussed below.

First of all, observations of gravitational deflection of massive particles with μ arcsec precision could probe the gravitational theories at the second post-Newtonian level without any ambiguity. The second order predictions of gravitational deflection of light as evolved from different studies are found to be ambiguous. This is because of eroneous identification of theoretical deflection variables with the observables or measured quantities. For in-stance, when light ray just grazes the limb of the sun and the isotropic radial coordinate ρ_0 is identified with the measured (under Euclidean approximation) solar radius, it gives a second-PN contribution of ~ 3.5 μ arcsec to deflection angle in GR [12]. But if one uses the standard Schwarzschild coordinates instead, the second-PN contribution to the deflection angle of light in GR following from Eqs.(3.16)-(3.18) would be $\left[\frac{15\pi}{16}-1\right]\frac{4M^2}{r_0^2}$ which is numerically about 7 μ arcsec for light ray grazing the limb of the sun, provided the distance of closest approach in such coordinates is identified with the measured radius of the sun. The deflection angle can also be expressed in terms of coordinate independent variables, such as the impact parameter b. In that case, the second-PN contribution to deflection angle in GR becomes $\frac{15\pi}{16}\frac{4M^2}{b^2}$ and when at closest approach, b is identified as the magnitude of second-PN deflection angle is ~ 11 μ arcsec [13]. Thus there exists a great confusion about the prediction of GR (or in fact of any viable gravitational theory) at the second-PN order.

The fundamental reason for such an ambiguity is that the measurements of solar radius usually employ Euclidean geometry as an approximation [6] whereas the angle of gravitational deflection or other GR effects are principally based on the consideration of curved spacetime. But comparing points in two different geometries i.e., in curved spacetime and flat spacetime is totally meaningless [14]. The Euclidean approximation works tolerably well only upto the first order, that is, in weak field gravity caused by a source like Sun. The magnitude of the second order contribution is, however, of the same order as the error that arises due to such an approximation. Hence the numerical value of gravitational deflection angle of light can not be unambiguously predicted at the level of second-PN order within the theoretical scheme in vogue. Since the deflection angle for massive particles depends also on the velocity of the particle, the stated ambiguity can be easily avoided by measuring deflection angles for two or more velocities of the probing massive particles.

In GR, a coordinate length like ρ_0 is not directly measurable, it can only can be indirectly "constructed" from the values of actual measurements. The PN parameters and also the otherwise unknown coordinate solar radius ρ_0 (or equivalently, r_0 in standard Schwarzschild coordinates) can be constructed through least square fitting with the measured deflection angles α_m and probing velocities V using the Eqs.(3.16)-(3.18). The idea is that the values of coordinates, ρ_0 and r_0, which refer to the same radial point, should be treated more like other PN parameters $(\beta_m, \gamma_m, \delta_m)$ due to the fact that the "flat geometry" spacetime points can not be algebraically identified in a curved spacetime [14]. Technically, however, the flat radial distances can be constructed by using metric gravity itself (Eddington expansion) in terms of a large set of unknown PN parameters $(\beta_m, \gamma_m, \delta_m)$ including ρ_0 by fitting them with the observed data [7]. This method has been adopted, for example, by Shapiro and his group in the radar echo delay observations [15,16]. The resulting parameter values can then be compared with the theoretical predictions of deflection in GR as well as in other competing theories (like Brans-Dicke theory) in the second-PN order involving both massive and massless particles.

The study of gravitational deflection of massive particles is also important in the context of testing the weak equivalence principle which is one of the fundamental postulates of general relativity. The principle states that the trajectory of a freely falling object is independent of its internal structure and composition. In other words all particles are coupled with spacetime geometry univerversally. The principle has been tested with great accuracy through different experiments, notable among them are the Eotvos type experiments [10] where comparison of gravitational and inertial masses of objects are made by measurements obviously can not be performed. Instead, in such situation the principle is tested by examining whether the gravitational (second-PN) coupling parameter γ is universal for all particles, massive or massless. On the basis of supernova 1987 neutrino and optical data [17], a limit of $|\gamma_\gamma - \gamma_m| \leq 3.4\times10^{-3}$ has actually been found [18]. However, the mass of a neutrino m_{ν_e} is very small (if not zero); the present upper limit being $m_{\nu_e} \leq 3$ eV. Hence, a more conclusive experiment would be to examine whether the gravitational coupling for photon and massive particles (other than neutrinos) are same or not. The observational value of $\left|\gamma_{\gamma}-\gamma_{m}\right|$ should provide a direct answer as to the degree of validity of the principle in question.

Remarks on particle deflection experiment

The main concern, which is still far from resolved, is whether realistic experiments for observing gravitational deflection of massive particles can be devised or not. Here, we only speculate on some possibilities. The most important requisite in this context is to generate a beam of suitable test particles. Charged particles like protons or electrons have to be excluded as test particles because they suffer electromagnetic interactions by the interplanetary magnetic field. Among neutral particles, neutrinos are unlikely to serve the purpose as their speeds are almost, if not exactly, the same as the speed of light. Thus, neutrons seem to be the only feasible candidate. They are known to be produced during solar flares but they can at best be used to study the gravitational deflection by an intermidiate planet. If astrophysical sources of neutrons other than the sun are detected in future experiments, the problem of searching the test particle beam would be resolved automatically. Otherwise, one might hope to generate the beam only artificially. However, since neutrons are unstable with a mean lifetime of 886 sec, only neutrons with a minimum speed of 0.75c can be used as test particles so that they do not decay during the travel from one micro-spacecraft to another. Though in (man-made) accelerator experiments (at earth) neutrons can be accelerated to such speeds, it seems improbable at the present stage of technology that neutrons can be accelerated to such high energies from a micro-spacecraft. This is a challenge for the future.

Alternatively , stable massive objects, such as a bullet, can also be used as test particles but they must have a minimum speed of $\sim 6\times10^{7}\, cm\, \sec^{-1}$ so that its total energy remains positive throughout the path (from one microspacecraft/earth to another spacecraft) and would not be capture by solar gravity. The fastest man made object (Helios 2 solar prob) has a speed of about $7\times10^{6}\, cm\, \sec^{-1}$. However, as already mentioned, accelerating a material object to speeds of~ $10^{8}\, cm\, \sec^{-1}$ and achieving the required level of μ arcsec accuracy in the deflection angle is completely beyond present technical feasibility.

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[16] Recall that the mass of a black hole can always be measured either by strong field gravitational lensing or other methods. Once we know the mass M, we can deduce its coordinate horizon radius, e.g., $\rho_{hor} = M/2$ or $r_{hor} = 2M$. (On the other hand, the proper horizon radius of a Schwarzschild black hole is imaginary which is observationally meaningless.) Usually, only the coordinate values of the radius of an uncollapsed object are used in the actual observable quantitites like the deflection angle or Shapiro radar echo delay observations. In those observations, the fitted values of ρ_0 or r_0 are identified, strictly technically, with the radius of a uncollapsed object like the sun.

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