Chapter 8

Conclusions:

Although a quest for higher order observational effects of gravity and particularly of general relativity effectively commenced during the eighties of the last century, at that time experimental verification of the results was completely beyond the then technical capability. At present technology has advanced to the point that experimental substantiation of few such effects with the onboard or future gravitational experiments appears feasible. Considering the situation and realizing the importance of the said investigations, in the present work a study has been made on few higher order effects of gravity. Here two different aspects of higher order effects of gravity are explored: On the one hand possibility of testing of general relativity and other viable theories of gravitation through study of gravitational deflection of massive particles has been discussed, on the other hand possible role of higher order gravitational deflection of light to discriminate general relativity from viable alternative theories of gravitation or in distinguishing two versions of Brans-Dicke scalar tensor theory from each other has been investigated. Besides wormhole feature in the Brans-Dicke theory is also studied. Below we provide chapter-wise summary of the work

In chapter 2 after briefly discussing the tests of gravitation conducted till date we reviewed the theoretical predictions of gravitational theories in terms of higher order observational effects and discussed about the prospects of experimental verifications of such effects. A special emphasize has given on possibility of second order tests in solar system and on strong field deflection of light.

In chapter 3 the expression for second post-Newtonian level gravitational deflection angle of massive particles is obtained in a model independent framework. It is shown that comparison of theoretical values with the observationally constructed values of post-Newtonian parameters for massive particles offers the future possibility of testing at that level competing gravitational theories as well as the equivalence principle. Advantage of studying gravitational deflection of massive particles over that of massless particles in testing gravity is also discussed here.

Possibility of distinguishing Einstein theory and scalar tensor theory, which is regarded as the most viable alternative to GR, through study of gravitational deflection of light at higher order accuracies is discussed in chapters 4-6. In chapter 4 it is shown that among the four classes of the static spherically symmetric solution of the vacuum Brans-Dicke theory of gravity only two are really independent. Further by matching exterior and interior (due to physically reasonable spherically symmetric matter source) scalar fields it is found that only Brans class I solution with certain restriction on solution parameters may represent exterior metric for a nonsingular massive object. The physical viability of the black hole nature of the solution is investigated. It is concluded that no physical black hole solution different from the Schwarzschild black hole is available in the Brans-Dicke theory.

Strong field gravitational lensing in the Brans-Dicke scalar tensor theory has been studied in chapter 5. The deflection angle for photons passing very close to the photon sphere is estimated for the static spherically symmetric space-time of the theory and the position and magnification of the relativistic images are obtained. Modeling the super massive central object of the galaxy by the Brans-Dicke space-time, numerical values of different strong lensing observable are estimated. It is found that against the expectation there is no significant scalar field effect in the strong field observable lensing parameters.

This result raises question on the potentiality of the strong field lensing to discriminate different gravitational theories.

It is well known that, in contrast to general relativity, there are two conformally related frames, the Jordan frame and the Einstein frame, in which the Brans-Dicke theory, a prototype of generic scalar-tensor theory, can be formulated. There is a long standing debate on the physical equivalence of the formulations in these two different frames. In chapter 6 it is shown here that gravitational deflection of light to second order accuracy may observationally distinguish the two versions of the Brans-Dicke theory.

In chapter 7 it is shown that among the different classes of claimed static wormhole solutions of the vacuum Brans-Dicke theory only Brans Class I solution with coupling constant ω less than -1.5 (excluding the point ω =2) gives rise to physically viable traversable wormhole geometry. Usability of this wormhole geometry for interstellar travel has been examined.

The present investigation is expected to provide a better platform for testing and distinguishing different gravitational theories.

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BRANS–DICKE THEORY: JORDAN VERSUS EINSTEIN FRAME

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It is well known that, in contrast to general relativity, there are two conformally related frames, the Jordan frame and the Einstein frame, in which the Brans–Dicke theory, a prototype of generic scalar-tensor theory, can be formulated. There is a long standing debate on the physical equivalence of the formulations in these two different frames. It is shown here that gravitational deflection of light to second order accuracy may observationally distinguish the two versions of the Brans–Dicke theory.

Keywords: Brans–Dicke theory; gravitational deflection angle.

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1. Introduction

The Brans–Dicke (BD) theory,[1] which describes gravitation through a spacetime metric ($g_{\mu\nu}$) and a massless scalar field (ϕ), is a modification or rather generalization of General Relativity (GR). The theory has recently received widespread attention due to the fact that it arises naturally as the low energy limit of many theories of quantum gravity such as the supersymmetric string theory or the Kaluza–Klein theory and is also found to be consistent with the present cosmological observations.[2–7]

As a generic aspect of any scalar–tensor theory, two frames are available to describe the BD theory. One frame is called the Jordan frame (JF) in which the BD field equations were originally written and the BD scalar field played the role of a spin-0 component of gravity. The other is the conformally rescaled Einstein frame (EF) in which the scalar field plays the role of a source matter field. There is a long standing debate as to whether the descriptions of the BD theory in the two frames, JF and EF, should be considered physically equivalent. In order to get a flavor of this debate and the resulting confusion, we should only say that physicists are *divided roughly into six groups depending on their attitude to the question.* They can be listed as follows. Some authors: (a) neglect the issue, (b) think that the two frames are physically equivalent, (c) consider them physically nonequivalent but do not provide supporting arguments, (d) regard only JF as physical but, if necessary, use EF for mathematical convenience, (e) regard only EF as physical, (f) belong to two or more of the above categories! For a detailed account, see the review.[8]

It has been argued in the literature that the physical frame is the one in which *matter couples directly (as opposed to anomalously[9]) to it, particles have constant* mass and move on geodesics of the physical metric so that the physical stress tensor is conserved.[10] In the non-physical frame, like the EF, particles have scalar field dependent masses and do not move along the geodesics of the EF metric due to the occurrence of a scalar field dependent force. This fact is manifest in the conservation of the sum of the energy–momentum tensor in the JF, the scalar field and the cosmological term (if it is taken into account). Although, it is a matter of theoretical interpretation which frame is the "true" frame, the physical metric is still the one that defines lengths and rates of ideal clocks and it is the one that should be compared with observables.

Flanagan[11] has argued that all physical observables are conformal frame invariants. Some works in cosmology do show that it is indeed the case.[12–14] Therefore, the question arises if we can take the deflection of light as a physical observable. *We state that the deflection angle is definitely a physical observable. In fact, the* observed deflection of light of appropriate magnitude by solar gravity provided the first experimental proof of general relativity. The difference in the deflection angle in JF and EF is not an effect of choosing different physical units. See the end of Sec. 3.2 for clarifications.

To resolve the issue in question (JF versus EF) in a more conclusive manner, we feel that it is necessary to go beyond mere (mostly speculative) theoretical arguments favoring one position or the other as listed above, and refer to a tangible *observational* ground to determine if the two frames are physically equivalent. There exist only very few works in this direction.[15,16] The situation is that the distinctive observational features that emerged from these works, such as different interaction nature of gravitational wave with gravitational detectors,[15] are unlikely to be observed experimentally in the near future. Therefore, in the present work, we *consider a more pragmatic premise, namely, the deflection of light by gravity* up to second order in gravitational strength in both versions of the BD theory. The aim is to explore whether both formulations give the same results or not.

The plan of the paper is as follows: In Sec. 2, we briefly review the gravitational deflection of light in a generic static, spherically symmetric spacetime (in isotropic coordinates). Explicit expressions for the second-order light deflection in JFBD theory and EFBD theory are obtained in Sec. 3 that also includes a discussion on the matter of changing units. Finally, the results are discussed in Sec. 4.

2. Second-Order Deflection Angle

A general static, spherically symmetric spacetime in isotropic coordinates is given by (geometrized units are used, unless specifically restored: $G = 1$, $c = 1$)

$$ds^2 = B(\rho)dt^2 - A(\rho)(d\rho^2 + \rho^2 \, d\theta^2 + \rho^2 \sin^2\theta \, d\varphi^2) \,. \tag{1}$$

The equation of the orbital motion of test particles can be obtained from the geodesic equations and is given by

$$\frac{d\varphi}{d\rho} = \frac{1}{\rho\sqrt{\frac{1}{Bb^2} - \frac{E}{b^2} - \frac{1}{\rho^2 A}}}\,, \tag{2}$$

where $b = \frac{J}{E}$ is the impact parameter (the perpendicular distance between the gravitating object and the tangent to the null geodesic) at large distances, E and J are proportional to the asymptotic energy and angular momentum of the particle. Because of the spherically symmetry, the motion has been considered only in the equatorial plane ($\theta = \frac{\pi}{2}$). Following the standard treatment,[17] the expression for the deflection angle for the light rays can be written as

$$\alpha(\rho_0) = I(\rho_0) - \pi \,, \tag{3}$$

where

$$I(\rho_0) = 2\int_{\rho_0}^{\infty} \frac{d\rho}{\rho}\left[\left(\frac{\rho}{\rho_0}\right)^2 \frac{A(\rho)B(\rho_0)}{A(\rho_0)B(\rho)} - 1\right]^{-\frac{1}{2}} \tag{4}$$

ρ_0 being the distance of closest approach. The relation between the impact parameter and the distance of closest approach follows from the conservation of the angular momentum of the scattering process and is given by

$$b(\rho_0) = \rho_0\sqrt{\frac{A(\rho_0)}{B(\rho_0)}}\,. \tag{5}$$

The Parametrized Post-Newtonian (PPN) formalism is usually employed to describe the gravitational theories in the solar system and also to compare predictions of general relativity to the results predicted by an alternative metric theory of gravity. This method is actually an approximation for obtaining the dynamics of a particle (in a weak gravitational field of a slowly moving gravitating source) to one higher order in $\frac{M}{\rho}$ than given by the Newtonian mechanics. The calculation of particle dynamics typically requires knowledge of g_{00} more accurately than g_{ij}. But as noted in Ref. 16, understanding about the light propagation in curved spacetime to any given order needs knowledge of every term to that order.

Following the standard PPN expansion treatment, we assume that the metric tensor is equal to the Minkowski tensor $\eta_{\mu\nu}$ plus corrections in the form of expansions in powers of $\frac{M}{\rho}$ (M is the mass of the source object). Considering only up to the second-order corrections terms, we have

$$B(\rho) = 1 - 2\frac{M}{\rho} + 2\beta\frac{M^2}{\rho^2}\,, \tag{6}$$

$$A(\rho) = 1 + 2\gamma\frac{M}{\rho} + \frac{3}{2}\delta\frac{M^2}{\rho^2}\,, \tag{7}$$

β, γ are the PPN parameters (also known as the Eddington parameters), δ can be considered as the post-PPN parameter. Several of these parameters are different for different theories. In general relativity all of them are equal to 1 as can be readily checked by expanding the Schwarzschild metric.

The expression for the angle of light deflection up to the second order follows from Eq. (4) and is given by

$$\alpha = 2(1+\gamma)\frac{M}{\rho_0} + \left[\left(2(1+\gamma) - \beta + \frac{3}{4}\delta\right)\pi - 2(1+\gamma)^2\right]\left(\frac{M}{\rho_0}\right)^2. \tag{8}$$

It is important to note that the term representing the second order effect contains all the three parameters β, γ and δ. So, knowing these PPN and post PPN parameters, the second-order effects on deflection angle for any metric theory of gravity can be estimated readily from the above expression. For the Schwarzschild metric, the deflection angle is given by

$$\alpha = 4\frac{M}{\rho_0} + \left[\frac{15\pi}{16} - 2\right]\frac{4M^2}{\rho_0^2}. \tag{9}$$

A limitation of the expression (8) is that it depends on the coordinate variable ρ. However, it can also be expressed in terms of coordinate independent variables, such as the impact parameter. In that case, the deflection angle reduces to

$$\alpha = 2(1+\gamma)\frac{M}{b} + \left[2(1+\gamma) - \beta + \frac{3}{4}\delta\right]\frac{\pi M^2}{b^2}. \tag{10}$$

3. Deflection Angle in the BD Theory

The expressions of the Eddington parameters β and γ for the theories under investigation are already known. For the BD theory in the Jordan frame (JFBD) these two PPN parameters are $\beta = 1$, $\gamma = \frac{\omega+1}{\omega+2}$, whereas for the BD theory in the Einstein frame (EFBD) both parameters are equal to 1. So our main task is to calculate the post PPN parameter δ for these theories. The parameter δ occurs only in the metric coefficient g_{ij}. So it is enough for us to consider only the static case.

3.1. *The JFBD theory*

The scalar field in JFBD theory acts as the source of the (local) gravitational coupling with $G \sim \phi^{-1}$. As a consequence, the gravitational "constant" is in fact not a constant but is determined by the total matter in the universe through an auxiliary scalar field equation. The scalar field couples to both matter and spacetime geometry and the strength of the coupling is represented by a single dimensionless constant parameter ω. It is generally considered that under the limit $\omega \to \infty$, the vacuum (or for traceless matter field) BD theory (and its dynamic generalization) reduces to the GR but the recent finding suggests that such a convergence is not always true.[18–22]

In the Jordan conformal frame, the BD action takes the form

$$\mathcal{A} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \mathcal{L}_{\text{matter}}\right), \tag{11}$$

where $\mathcal{L}_{\text{matter}}$ is the Lagrangian density of ordinary matter and R is the Ricci scalar. As stated earlier, the theory is constrained by the solar system experiments. The recent conjunction experiment with Cassini spacecraft constrains the value of the coupling constant as $|\omega| > 5 \times 10^4$.[23]

The static spherically symmetric matter free solution of the BD theory in isotropic coordinates is given by[24,25]:

$$\begin{aligned} ds^2 = {} & +\left(\frac{1-\frac{B}{\rho}}{1+\frac{B}{\rho}}\right)^{\frac{2}{\lambda}} dt^2 \\ & - \left(1+\frac{B}{\rho}\right)^4 \left(\frac{1-\frac{B}{\rho}}{1+\frac{B}{\rho}}\right)^{\frac{2(\lambda-C-1)}{\lambda}} (d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2), \end{aligned} \tag{12}$$

$$\phi = \phi_0 \left(\frac{1-\frac{B}{\rho}}{1+\frac{B}{\rho}}\right)^{\frac{C}{\lambda}} \tag{13}$$

with

$$\lambda^2 = (C+1)^2 - C\left(1 - \frac{\omega C}{2}\right), \tag{14}$$

where B, C are constants of integration. By the weak field Newtonian approximation, we can set $\frac{4B}{\lambda} = 2GM$, where G is the gravitational constant measured by a Cavendish or a similar experiment and M is the gravitating mass. Furthermore, by matching the interior and exterior (due to physically reasonable spherically symmetric matter source) scalar fields, the constant C can be identified as $C = \frac{1}{\omega+2}$.[17] These are the standard procedures for fixing constants of a metric theory of gravity. Expanding the metric coefficients and retaining only up to the second order terms in $\frac{M}{\rho}$, we get the parameter δ as

$$\delta = 1 - \frac{15\omega + 22}{6(\omega+2)^2}. \tag{15}$$

Hence, finally, the deflection angle becomes

$$\alpha = \left(\frac{2\omega+3}{2\omega+4}\right)\frac{4M}{\rho_0} + \left[\left(\frac{2\omega+3}{2\omega+4} - \frac{15\omega+22}{8(2\omega+4)^2} - \frac{1}{16}\right)\pi - 2\left(\frac{2\omega+3}{2\omega+4}\right)^2\right]\frac{4M^2}{\rho_0^2}. \tag{16}$$

In the limit $\omega \to \infty$, the above expression reduces to the general relativity value. The deflection angle can also be readily expressed in terms of impact parameter using Eqs. (10) and (15).

3.2. *The EFBD theory*

Recent cosmological observations indicate that the universe is undergoing cosmic acceleration and is dominated by a dark energy component with negative pressure.[26–29] Cosmological constant (Λ) is a straightforward and natural candidate for such a component. However, the observational upper limit on Λ is more than 120 orders smaller than what is expected naturally from a vacuum energy originating at the Planck time. An alternative realization of dark energy is in the form of a minimally coupled scalar field ϕ with a specific potential $U(\phi)$ (the so-called "quintessence") whose slowly varying energy density would mimic an effective cosmological constant. This is very reminiscent of the mechanism producing the inflationary phase. Thus a minimally coupled scalar field is an attractive possibility in modern cosmology.

The action for the EFBD theory is

$$\mathcal{A} = \int \sqrt{-\tilde{g}}\, d^4x(\tilde{R} + \mu \tilde{g}^{\alpha\beta}\tilde{\phi}_{,\alpha}\tilde{\phi}_{,\beta}). \tag{17}$$

This action is obtained from the action (11) by conformal transformation of the metric $\tilde{g}_{\alpha\beta} = \phi g_{\alpha\beta}$ and a redefinition of the scalar $\tilde{\phi} = \left(\frac{2\omega+3}{2\mu}\right)^{1/2} \ln\phi$. The extra constant μ is introduced here to fix the sign of the kinetic term, but it does not appear in metric observations.

A static spherically symmetric vacuum solution to the EFBD theory (with the cosmological constant $\Lambda = 0$) is the well-known Buchdahl solution[30] which is also variously referred (as demonstrated in Ref. 31) to as JNW[32] or Wyman solution.[33] Similar to its counterpart (Schwarzschild solution) in GR, this solution also correctly explains all the post-Newtonian tests of GR. However, in contrast to the Schwarzschild solution, Buchdahl solution does not represent a black hole spacetime but possesses a strong globally naked singularity, respecting the "scalar no hair theorem"[34] which purports to exclude the availability of any knowledge of a scalar field from the exterior of a spherically symmetric black hole. Whether a naked singularity occurs generically in a physically realistic collapse is a subject of considerable debate.[35]

The Buchdahl solution,[30] in isotropic coordinates, is given by

$$\begin{aligned} ds^2 &= \left(\frac{1-\frac{m}{2\rho}}{1+\frac{m}{2\rho}}\right)^{2\xi} dt^2 \\ &\quad - \left(1-\frac{m}{2\rho}\right)^{2(1-\xi)}\left(1+\frac{m}{2\rho}\right)^{2(1+\xi)}[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \end{aligned} \tag{18}$$

and the expression for the scalar field is given by

$$\phi(\rho) = \sqrt{\frac{2(1-\xi^2)}{\mu}} \ln\left(\frac{1-\frac{m}{2\rho}}{1+\frac{m}{2\rho}}\right), \qquad \rho > m/2. \tag{19}$$

The Arnowitt–Deser–Misner (ADM) mass of the source corresponding to the above solution is given by $M = \xi m$. The effect of scalar field is usually described in terms of a scalar charge defined as $q = m\sqrt{2(1-\xi^2)/\mu}$. Expanding the metric coefficients in $\frac{M}{\rho}$ and comparing with Eqs. (6) and (7), we get

$$\delta = 1 - \frac{1}{3}(1-\xi^2)\,. \tag{20}$$

Thus the final expression for the second-order deflection angle becomes

$$\alpha = 4\frac{M}{\rho_0} + \left[\frac{1}{16}(14+\xi^2)\pi - 2\right]\frac{4M^2}{\rho_0^2}\,. \tag{21}$$

One can obtain the general relativity result by taking $\xi = 1$.

A conformal transformation is usually regarded as a change in physical units. Hence, a natural question is whether the difference between the deflection angles in JF and EF, as revealed from Eqs. (16) and (21) respectively, is an effect of selecting different conventions of physical units. To see that this is not the case here, we get from (12), with $\frac{4B}{\lambda} = 2GM$, to first order, $g_{tt}^{JF} = 1 - \frac{2GM}{\rho}$ and $g_{\rho\rho}^{JF} = 1 + \frac{2(C+2)GM}{\rho}$. Now, via the conformal transformation $g_{\alpha\beta}^{EF} = \phi g_{\alpha\beta}^{JF}$, we get, to first order, $g_{tt}^{EF} = 1 - \frac{2B(C+2)}{\lambda\rho}$ and $g_{\rho\rho}^{EF} = 1 + \frac{2B(C+2)}{\lambda\rho}$. If we think that EF is the physical frame, then, again by standard Newtonian identification, $\frac{2B(C+2)}{\lambda} = 2GM$, we get $g_{tt}^{EF} = 1 - \frac{2GM}{\rho}$, $g_{\rho\rho}^{EF} = 1 + \frac{2GM}{\rho}$. [Using the relation $\xi = \frac{1}{\lambda}\left(\frac{C+2}{2}\right)$ and $B = \frac{mG}{2}$, we do get $M = \xi m$.] Clearly, just by changing units, the components of the metric tensors in JF and EF cannot be reconciled even in the first order. One of the underlying reasons could be that the numerical values of scalar invariants, like the Ricci scalar, change under conformal transformations. Another reason could be that the conformal transformation from JF to EF and its inverse do not preserve the exact specific form of either action.

4. Discussion

Our main observations are as follows:

(a) The JFBD theory contains an adjustable coupling parameter ω. As ω increases, the post-Newtonian expansions of the BD theory increasingly approach the corresponding GR expressions. As a result, observations cannot rule out the JFBD theory in favor of GR, but can only place limits on the coupling parameter ω. Using the present lower bound on ω as obtained from the recent conjunction experiment with Cassini spacecraft, we found that the second-order deflection angles of light in the GR and in the JFBD theory are the *same* up to an accuracy of 100 *pico arc seconds*. Hence the proposed experiments for measuring the deflection of light to second order accuracy, such as the LATOR experiment[36,37] which is expected to achieve an accuracy of nearly 10 *nano arc second* in angular measurement, would not impose any further constraint on the coupling parameter ω. However, it should be noted that, from the accurate observation of the first-order deflection of light, the LATOR mission will measure the PPN parameter γ very precisely which in turn will provide better information on the value of ω. Similar conclusions will also hold for the generalized scalar–tensor theories for which $\omega = \omega(\phi)$.

(b) The EFBD theory contains a scalar field that couples minimally to gravity. Then, at the first PPN order, there is no effect of the scalar field in the deflection angle. The difference from GR occurs only in the magnitudes of the second and higher orders. We observe that the second-order deflection angle depends on scalar charge in addition to the ADM mass of the source object and the bending is reduced under the effect of the scalar charge. If the scalar charge is just 10% of the total ADM mass of the sun, the *difference* between the deflection angle (up to second order) of light in the Schwarzschild and in the Buchdahl spacetime is around 7 *nano arc sec.* So the LATOR mission, for the first time, might detect signatures of the minimally coupled scalar field. The difference is significant compared to the much lesser (in principle zero, as ω is increased without limit) difference between the Schwarzschild and JFBD theory and its measurement could observationally distinguish between JF and EF. This is what we wanted to argue in this paper.

(c) We have not touched upon the cosmological issues. It might be interesting to know if tests at the solar system level be used to set the boundary conditions for a cosmological problem. Even if the results from LATOR favor one of the two frames or theories (GR or scalar tensor), there still remains the problem as to how to explain the cosmological differences of the two frames. This is a task for the future.

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Testing gravity at the second post-Newtonian level through gravitational deflection of massive particles

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Expression for second post-Newtonian level gravitational deflection angle of massive particles is obtained in a model independent framework. Comparison of theoretical values with the observationally constructed values of post-Newtonian parameters for massive particles offers the future possibility of testing at that level competing gravitational theories as well as the equivalence principle. Advantage of studying gravitational deflection of massive particles over that of massless particles in testing gravity is discussed.

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I. INTRODUCTION

A consequence of general relativity (GR) is that light rays are deflected by gravity. Historically, observations of this aspect of gravity provided one of the early proofs in favor of GR (the explanation of perihelion advance was the earliest.) The effect is nowadays routinely used as a tool to study various features of the Universe, such as viewing fainter or distant sources, estimating the masses of galaxies etc. [1].

Like photons, particles having masses are also deflected by gravity. The general relativistic corrections in the equation of motion of a massive test particle moving in bound orbits have been studied with great accuracy in the literature [2]. These studies have great relevance in comparing theory with observations of gravitational waves from compact binary systems. However, study of orbits of *unbound* massive particles in gravitational field has not received sufficient attention as of now. The main reason could be that the observational aspect of unbounded massive particles was not very practical: there was no known astrophysical source of free point particles that can be detected easily with good angular precision. The situation seems to have improved somewhat. Recent theoretical studies [3] favor the existence of local astrophysical sources of relativistic neutral particles like neutrons and neutrinos with observable fluxes. Besides, high energy neutrons are produced during solar flares [4]. Moreover, with the advent of new technology new experiments have been proposed [5], primarily to study gravitational deflection of light with high precision, in which laser interferometry will be employed between two spacecrafts/space stations whose line of sight pass close to the sun. Hence there might be a possibility that, in the future, neutron or some other neutral particle may be used in a similar experiment instead of photon, thus providing an opportunity for studying gravitational deflection of massive particles. Henceforth, we use the abbreviation for post-Newtonian as PN such that first-PN effect is of the order of $(1/\rho)$, second-PN effect is of the order of $(1/\rho^2)$ and so on.

The expected angular precision of the planned astrometric missions using optical interferometry is at the level of microarcseconds (μ arcsec) and hence these experiments would measure the effects of gravity on light at the second-PN order (c^{-4}). Though measurements with massive particles at the level of microarcsecond accuracy is way beyond the present technical capability, it can still be cautiously hoped that astrometric missions in the distant future using massive particle interferometry would have angular precisions close to that to be obtained using laser (optical) interferometry. Whatever be the technical scenario, a study of theoretical aspects of gravitational deflection of massive particles at the second-PN approximation is useful in its own right. (To our knowledge, the deflection angle for massive particles has been theoretically estimated in the literature with an accuracy of only first-PN order [6] so far.)

In the present article, we shall formulate the second-PN contribution to the gravitational deflection of massive particles in a *model independent* way but with a special emphasis on the sun as gravitating object. The corresponding PN parameters for light deflection then follow as a corollary. The key idea here is to exploit an advantage offered by the kinematics of massive particles over that of massless ones: The velocity of the probing massive particle can be altered. The investigation (i) helps us circumvent some difficulties related with photon deflection in the second-PN order, (ii) allows us to "construct" the coordinate solar radius from the particle deflection data itself and moreover, (iii) offers a possible further test of the equivalence principle. These issues are discussed at the end.

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II. GRAVITATIONAL DEFLECTION OF MASSIVE PARTICLES AT SECOND POST-NEWTONIAN ORDER

We consider the general static and spherically symmetric spacetime in isotropic coordinate which is given by (we use geometrized units i.e. $G = 1$, $c = 1$)

$$ds^2 = -B(\rho)dt^2 + A(\rho)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 sin^2\theta d\phi^2). \tag{1}$$

Restricting to orbits in the equatorial plane ($\theta = \pi/2$), the expression for the deflection angle for particles moving with a velocity V as measured by an asymptotic rest observer can be written as [7]

$$\alpha(\rho_o) = I(\rho_o) - \pi \tag{2}$$

with (see Appendix)

$$I(\rho_o) = 2\int_{\rho_o}^{\infty} \frac{d\rho}{\rho^2} A^{-(1/2)}(\rho)\left[\frac{1}{J^2}\{B^{-1}(\rho) - E\} - \frac{1}{\rho^2 A}\right]^{-(1/2)}, \tag{3}$$

where ρ_o being the distance of the closest approach,

$$J = \rho_o[A(\rho_o)\{B^{-1}(\rho_o) - E\}]^{1/2} \tag{4}$$

and

$$E = 1 - V^2. \tag{5}$$

The PN formalism in some orders [7,8] is usually employed to describe the gravitational theories in the solar system and also to compare predictions of GR with the results predicted by an alternative metric theory of gravity. This method actually is an approximation for obtaining the dynamics of a particle (in a weak gravitational field under the influence of a slowly moving gravitational source) to one higher order in $\frac{M}{\rho}$ (M is the mass of the static gravitating object) than given by the Newtonian mechanics. Following the PN expansion method, we assume the metric tensor is equal to the Minkowski tensor $\eta_{\mu\nu}$ plus corrections in the form of expansions in powers of $\frac{M}{\rho}$ and considering up to the second-PN correction terms, we have

$$B(\rho) = 1 - 2\frac{M}{\rho} + 2\beta_i \frac{M^2}{\rho^2} - \frac{3}{2}\epsilon_i \frac{M^3}{\rho^3}, \tag{6}$$

$$A(\rho) = 1 + 2\gamma_i \frac{M}{\rho} + \frac{3}{2}\delta_i \frac{M^2}{\rho^2}. \tag{7}$$

β_i, γ_i are the PN parameters (also known as the Eddington parameters), δ_i and ϵ_i can be considered as the second-PN parameters, i stands for either γ or m denoting photons or massive particles, respectively. Several of these parameters are different for different theories. In GR, all of them are equal to 1 as can be readily checked by expanding the Schwarzschild metric.

We should note that the metric coefficients above are independent of any specific model; they result solely from the assumption of central symmetry. Starting with the expansion (6) and (7) *per se*, the expression for the angle of deflection for unbound particles up to the second-PN order follows from Eq. (3) and when $\frac{2M}{\rho} << V^2$, and it works out to

$$\alpha_m = a_m \frac{M}{\rho_o} + b_m\left(\frac{M}{\rho_o}\right)^2, \tag{8}$$

where

$$a_m = 2\left(\gamma_m + \frac{1}{V^2}\right), \tag{9}$$

$$b_m = \frac{3\delta_m \pi}{4} + (2 + 2\gamma_m - \beta_m)\frac{\pi}{V^2} - 2\left(\gamma_m + \frac{1}{V^2}\right)^2. \tag{10}$$

The above expressions are also valid for massless particles (α_γ, a_γ, b_γ) as may be seen under the substitution $V = 1$. Clearly the deflection angle would be larger for particles in comparison to that of photons. Our calculation shows that the term representing the second order effect (b_m) contains only the three parameters β_m, γ_m, δ_m and does not contain ϵ_m, a cubic order contribution. This implies that calculation of the deflection of *unbound* particle orbits (including photons) by gravity to any given order needs only the knowledge of every term to that order in the expansions. In other words, to second-PN order, one needs to consider both in g_{oo} and g_{ij} terms only up to $\frac{M^2}{\rho^2}$. Similarly, to third-PN order, which is not our interest here, we would need expansions of both the metric components up to order $\frac{M^3}{\rho^3}$ and so on. This is in contrast to the case of planetary dynamics (*bound* orbits) where the calculation typically requires knowledge of g_{oo} more accurately than g_{ij} (For instance, to calculate the planetary precession to the order of M, one expands g_{oo} up to $\frac{2\beta_m M^2}{\rho^2}$ while g_{ij} is expanded up to only $\frac{2\gamma_m M}{\rho}$; for next order accuracy, one would need to consider the complete expansion as given in Eqs. (6) and (7) above so that the parameters δ_m, ϵ_m become important in this case.) The deflection angle α_m for the Schwarzschild spacetime can be obtained by taking $\beta_m = \gamma_m = \delta_m = 1$.

The deflection angle also can be expressed in terms of coordinate independent variables, such as the impact parameter b which is the perpendicular distance from the center of the gravitating object to the tangent to the geodesic at the closest approach. In that case, ρ has to be replaced by b in Eq. (8), Eq. (9) would remain unaltered but Eq. (10) would change to

$$b_m = \frac{3\delta_m \pi}{4} + (2 + 2\gamma_m - \beta_m)\frac{\pi}{V^2} + 2\left(\gamma_m + \frac{1}{V^2}\right)\left(1 - \frac{1}{V^2}\right). \tag{11}$$

Since impact parameter is the ratio of the angular momentum and energy of the particle as measured by an observer at rest far from the gravitating object, it is a formally measurable quantity but is not very suitable for practical measurements [9].

III. OTHER SIGNIFICANT EFFECTS

In the present work, the mass distribution of the gravitating object is assumed to be mainly spherically symmetric; any deviation from such symmetry would produce their effects. The effect of quadrupole moment of the mass distribution on the deflection angle is proportional to $\frac{J_Q M R^2}{\rho_o^3}$, where R is the average radius of sun. Thus, even a small quadrupole moment parameter J_Q could produce significant contribution to deflection. However, the effect is limited largely to the first-PN order (~ 0.1 μ arcsec) while in the second-PN order the effect is too small ($\sim 10^{-7}$–10^{-8} μ arcsec). If the gravitating object also has angular momentum, its effect on the deflection angle contributes to the second-PN order but it can be separated out. All these are discussed below.

A. Effect of quadrupole moment of the mass distribution

Theoretical value of solar quadrupole moment J_Q, though it depends strongly on solar model used, is very small, of the order of 10^{-7} [5,8]. Since our study is aimed at sun as the gravitating object we have ignored higher order terms involving J_Q. Thus due to the quadrupole moment of the mass distribution the effective mass parameter becomes $M_{\mathrm{eff}} = M\left[1 + \frac{J_Q R^2}{2\rho^2}(3\cos^2\theta - 1)\right]$ which leads to the following corrections in the components of the metric tensors [5,8,10]:

$$\delta g_{oo}(\rho) = J_Q \frac{MR^2}{\rho^3}(3cos^2\theta - 1) \tag{12}$$

and

$$\delta g_{jk}(\rho) = -\delta_{jk}\gamma_i J_Q \frac{MR^2}{\rho^3}(3cos^2\theta - 1), \tag{13}$$

where R is the average radius of the mass distribution and θ is the angle between radius vector and the z-axis and hence in the equatorial plane $\theta = \pi/2$. In the equatorial plane, the deflection caused by the quadrupole moment calculates to

$$\alpha_{\mathrm{QM}} = \frac{2J_Q MR^2}{\rho_o^3}\left(\frac{\gamma_m}{3} + \frac{1}{V^2}\right). \tag{14}$$

Assuming $R \sim \rho_o$ at the closest approach to the sun and taking $V = 0.75$, $\beta_m = \gamma_m = \delta_m \sim 1$, for sun $\frac{M_\odot}{R_\odot} = 2.12 \times 10^{-6}$, this first-PN quadrupole term $\alpha_{\mathrm{QM}} \sim 0.1$ μarcsec. It is roughly 7 orders of magnitude less than the first-PN deflection $a_m \frac{M}{\rho_o} \sim 0.8$ sec and is more than 1 order of magnitude less than the second-PN contribution $b_m(\frac{M}{\rho_o})^2 \sim 3.4$ μarcsec. We have not displayed the next higher order quadrupole terms involving J_Q^2 and second order terms in $1/\rho^2$ containing J_Q here because they have magnitude in the range 10^{-7} to 10^{-8} μarcsec, too small to be of any practical significance. We can justifiably ignore these second-PN quadrupole contributions. The quadrupole contribution to the deflection of light is given in Ref. [11].

B. Effect of rotation

The angular momentum of the gravitating object is assumed small as in the case of sun. The resulting leading term of the relevant metric tensor is

$$g_{oi} = \frac{4Ma}{\rho}, \tag{15}$$

where a is the angular momentum per unit mass of the object. The contribution of the rotation to the deflection angle is then given by [6]

$$\alpha_{\mathrm{rot}} = \frac{4MaV}{\rho_o^2}. \tag{16}$$

The value of a can be positive or negative depending on the direction of rotation. When the angular momentum of the gravitating object is antiparallel with the direction of the incoming particle, a is positive and hence rotation causes larger deflection whereas for parallel angular momentum, a is negative and the deflection angle will be less. Thus the rotational effect can be easily separated out from other contributions by studying the deflection of particles at two opposite sides of the gravitating object.

The gravitational deflection angle of light with an accuracy up to second-PN order readily follows from Eqs. (8)–(10), (14), and (16) using $V = 1$.

IV. EXTRACTING POST-PN PARAMETERS FROM MEASUREMENTS

To extract post-PN parameters from gravitational deflection of massive particles, one first has to measure deflection angles α_m for different values of V of the probing massive particle grazing the sun. Then, a least square fitting of the recorded deflection angle data with Eq. (8) through Eqs. (9) and (10) will result in the PN values β_m, γ_m, δ_m *together* with the solar radius ρ_o in isotropic coordinate. If GR is a correct theory to second-PN order, then the best fit will give $\beta_m = \gamma_m = \delta_m = 1$, and if the weak equivalence principle holds then $\beta_m = \gamma_m = \delta_m = \beta_\gamma = \gamma_\gamma = \delta_\gamma = 1$. In the case of light, there is only one probe velocity available, namely, $V = c = 1$, and there is no option to fit β_γ, γ_γ, δ_γ separately. One just proceeds to check whether the measurement is consistent with GR prediction obtained by simply assumimg all the PN parameters as unity. But the difficulty with this procedure is that, for light grazing the sun, one needs the other parameter, the solar radius, whose value to the required level of accuracy is not available, nor can it be consistently obtained, together with other PN parameters, from the observed data itself. These points are illustrated later, in Sec. V.

It should be mentioned that the entire calculation of deflection angle could be performed in any other coordinates, standard or harmonic and so on. Expression for deflection angle in any other coordinates also can be obtained directly from Eqs. (8) to (10) just by applying appropriate transformations. The observable deflection angle α_m is of course coordinate choice independent. Thus change to another coordinate system will result merely in the corresponding functional changes in the expressions for the coefficients a_m, b_m, respectively, and in the value of the coordinate radius of the sun (say, to r_o if we change to standard system) [12]. Note that once used in a certain coordinate system, the PN values β_m, γ_m, δ_m are to remain fixed for a given theory of gravitation (GR or Brans-Dicke theory, etc.) in any other coordinate system.

V. DISCUSSION

The study of gravitational deflection of massive particles is important for several reasons which are discussed below.

First of all, observations of gravitational deflection of massive particles with μ arcsec precision could probe the gravitational theories at the second-PN level without any difficulty as explained at the end of Sec. II. The second order prediction for gravitational deflection of light as evolved from different studies is plagued by the following factors: When the light ray just grazes the limb of the sun, one needs a consistent value of coordinate solar radius to be put into the expression for deflection α_γ calculated in different coordinate systems (with fixed GR values of unity for PN parameters). Now there is a long known value for the solar radius $R_\odot$, ($R_\odot = 6.961 \times 10^8$ km [13]) measured under *Euclidean* approximation! But, even in the expression for α_γ in the Schwarzschild isotropic system, the radial coordinate ρ_o is erroneously identified with the same $R_\odot$. Then it gives a second-PN contribution of $\sim$3.5 μ arcsec to deflection angle in GR [14]. If one uses the standard Schwarzschild system instead, the second-PN contribution to the deflection angle of light in GR following from Eqs. (8)–(10) would be $[\frac{15\pi}{16} - 1]\frac{4M^2}{r_o^2}$ which is numerically about 7 μarcsec, provided the standard coordinate distance r_o of closest approach is identified again with the same $R_\odot$. The deflection angle can also be expressed in terms of coordinate independent variables, such as the impact parameter b. In that case, the second-PN contribution to deflection angle in GR becomes $\frac{15\pi}{16}\frac{4M^2}{b^2}$, and when at closest approach b is identified with $R_\odot$, the magnitude of second-PN deflection angle is $\sim$11 μarcsec [15]. Thus there exists difficulties about the interpretation of the prediction of GR (or in fact of any viable gravitational theory) at the second-PN order. The single fixed value for $R_\odot$ is used in all calculations because there cannot be any way to get consistent values for the solar radius from the higher order light deflection data due to its unique trajectory grazing the sun.

There is a more fundamental reason for these anomalies. It is that the measurements of solar radius usually employ Euclidean geometry as an approximation [6] whereas the angle of gravitational deflection or other GR effects are principally based on the consideration of curved spacetime. But, comparing points in two different geometries, i.e., in curved spacetime and flat spacetime, is totally meaningless [16]. The Euclidean approximation works tolerably well only up to the first order, that is, in weak field gravity caused by a source like the sun. The magnitude of the second order contribution is, however, of the same order as the error that arises due to such an approximation. Hence the numerical value of gravitational deflection angle of light cannot be unambiguously predicted at the level of second-PN order within the theoretical scheme currently in practice. Since the deflection angle for massive particles depends also on the *velocity* of the particle which gives us an extra freedom, the stated ambiguity can be easily avoided by measuring deflection angles for two or more velocities of the probing massive particle.

In GR, a coordinate length like ρ_o is not directly measurable, it can only can be indirectly constructed from the values of actual measurements. The PN parameters and also the otherwise unknown coordinate solar radius ρ_o (or equivalently, r_o in standard Schwarzschild coordinates) can be constructed through least square fitting with the measured deflection angles α_m and probing velocities V using the Eqs. (8)–(10). The idea is that the values of coordinates, ρ_o and r_o, which refer to the same radial point, should be treated more like other PN parameters (β_m, γ_m, δ_m) due to the fact that the "flat geometry" spacetime points cannot be algebraically identified in a curved spacetime [16]. Technically, however, the flat radial distances can be constructed by using metric gravity itself (Eddington expansion) in terms of a large set of unknown PN parameters (β_m, γ_m, δ_m) including ρ_o by fitting them with the observed data [7]. This method has been adopted, for example, by Shapiro and his group in the radar echo delay observations [7,17]. The resulting parameter values can then be compared with the theoretical predictions of deflection in GR as well as in other competing theories (like Brans-Dicke theory) in the second-PN order involving both massive and massless particles.

The study of gravitational deflection of massive particles is also important in the context of testing the weak equivalence principle which is one of the fundamental postulates of general relativity. The principle states that the trajectory of a freely falling object is independent of its internal structure and composition. In other words all particles are coupled with spacetime geometry universally. The principle has been tested with great accuracy through different experiments, notable among them are the Eötvös type experiments [10] where comparison of gravitational and inertial masses of objects are made by measuring their acceleration in a known gravitational field. For massless particles like photons, however, such measurements obviously cannot be performed. Instead, in such situation the principle is tested by examining whether the gravitational (second-PN) coupling parameter γ is universal for all particles, massive or massless. On the basis of supernova 1987 neutrino and optical data [18], a limit of $|\gamma_\gamma - \gamma_m| \leq 3.4 \times 10^{-3}$ has actually been found [19]. However, the mass of a neutrino m_{ν_e} is very small (if not zero), the present upper limit being $m_{\nu_e} \leq 3$ eV. Hence, a more conclusive experiment would be to examine whether the gravitational couplings for photon and massive particles (other than neutrinos) are the same or not. The observational value of $|\gamma_\gamma - \gamma_m|$ should provide a direct answer as to the degree of validity of the principle in question.

VI. REMARKS ON PARTICLE DEFLECTION EXPERIMENT

The main concern, which is still far from resolved, is whether realistic experiments for observing gravitational deflection of massive particles can be devised or not. Here, we only speculate on some possibilities. The most important requisite in this context is to generate a beam of suitable test particles. Charged particles like protons or electrons have to be excluded as test particles because they suffer electromagnetic interactions by the interplanetary magnetic field. Among neutral particles, neutrinos are unlikely to serve the purpose as their speeds are almost, if not exactly, the same as the speed of light. Thus, neutrons seem to be the only feasible candidate. They are known to be produced during solar flares but they can at best be used to study the gravitational deflection by an intermediate planet. If astrophysical sources of neutrons other than the sun are detected in future experiments, the problem of searching the test particle beam would be resolved automatically. Otherwise, one might hope to generate the beam only artificially. However, since neutrons are unstable with a mean lifetime of 886 sec, only neutrons with a minimum speed of 0.75 c can be used as test particles so that they do not decay during the travel from one micro-spacecraft to another. Though in (man-made) accelerator experiments (at earth) neutrons can be accelerated to such speeds, it seems improbable at the present stage of technology that neutrons can be accelerated to such high energies from a micro-spacecraft. This is a challenge for the future.

Alternatively, stable massive objects, such as a bullet, can also be used as test particles but they must have a minimum speed of $\sim 6 \times 10^7$ cm sec^{-1} so that its total energy remains positive throughout the path (from one micro-spacecraft/earth to another spacecraft) and would not be captured by solar gravity. The fastest man-made object (Helios 2 solar probe) has a speed of about 7×10^6 cm sec^{-1}.

Any meaningful information on second order effects can be extracted from measurements of gravitational deflection of massive particles at the level of second order accuracy only when the uncertainty of the first-PN contribution [Eq. (9)] is smaller than the second-PN contributions. Since radius is considered as free parameter in the proposed scheme, the uncertainty in the first-PN contribution of deflection angle is entirely due to uncertainty in the knowledge of the speed of the massive test particles. For solar gravity the ratio of second order to first order contributions of deflection angle is around 10^{-6}. Thus for a meaningful second order measurement of particle deflection, the relative uncertainty of first order deflection angle $\frac{\Delta\alpha_1}{\alpha_1}$ must be less than 10^{-6} which in turn requires $\frac{\Delta v}{v} < 10^{-6}$. This should not be a major problem as a comparable level of accuracy in measurement of particle velocity has already been achieved in different experiments [20].

Particle detectors with directional resolution at the level of μ arcsec accuracy is certainly beyond the present technical capability. The maximum directional accuracy of operating particle telescopes is limited to around 100 mili-arcsec [21]. Configuring neutron interferometer instrument $S18$ as a Bonse-Hart small angle scattering camera, an angular resolution of few($\sim$10)mili $-$ arcsec has been achieved [22]. Maximum directional accuracy achieved so far using electromagnetic radiation based telescopes is also of the same order. However, currently planned astrometric missions employing optical interferometry have set their goal to achieve a directional accuracy at the level of μ arcsec. Thus, hopefully, achieving the required level of μ arcsec accuracy in particle detection might not be too far away.

VII. CONCLUSION

The subtlety of the observational meaning of coordinate distance is not unknown to the physics community [7]. One could live with the ambiguous predictions in higher order light deflection had the stakes been not high. It is imperative to test at a higher order level which theory of gravitation, GR or other theories, fits better. The far-reaching implications of the answer do not require any elaboration. What we analyzed above is a possible theoretical scheme for testing gravity at the second order.

For the scheme to work in practice, the level of experimental accuracy seems extremely demanding. But one should recall that when the second order deflection of light was first calculated theoretically in the early eighties [12], experimental verification of the result was completely beyond the then technical capability. Now, after 25 years, technology has been developed to the stage that measuring deflection angle due to solar gravity at the second-PN order appears feasible. However, as discussed in Sec. V, the numerical value of gravitational deflection angle of light cannot be uniquely predicted at the level of second-PN order within the existing theoretical scheme. As a result the proposed experiments with light are unlikely to provide any fruitful test of GR at that order. This is due to a question of principle related to the lack of a consistent parameter fitting procedure with light and not a question of attainable accuracy in experimental measurements. It has been shown that such a situation can be circumvented by using a kinematical freedom available with a massive test particle, viz., its velocity that can be altered at will unlike in the case with light. One can then measure the deflection angles for two or more velocities of the probing massive particle. However, it is understood that such measurements are completely beyond the present technical feasibility.

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APPENDIX

The standard equations for a geodesic, namely,

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{A1}$$

for the general metric (1) become

$$\frac{d^2\rho}{ds^2} + \frac{A'}{2A}\left(\frac{d\rho}{ds}\right)^2 - \rho\left(1 + \frac{\rho A'}{2A}\right) = 0, \tag{A2}$$

$$\frac{d^2\theta}{ds^2} + \left(\frac{2}{\rho} + \frac{A'}{A}\right)\frac{d\rho}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0, \tag{A3}$$

$$\frac{d^2\phi}{ds^2} + \left(\frac{2}{\rho} + \frac{A'}{A}\right)\frac{d\rho}{ds}\frac{d\theta}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0, \tag{A4}$$

$$\frac{d^2t}{ds^2} + \frac{B'}{B}\frac{d\rho}{ds}\frac{dt}{ds} = 0 \tag{A5}$$

(primes denoting differentiation with respect to ρ). If we choose $\theta = \pi/2$ and $d\theta/ds = 0$ initially, Eq. (A3) warrants that they would remain the same always. Thus normalizing time coordinate suitably, one obtains for orbits in the equatorial plane from Eq. (A5)

$$\frac{dt}{ds} = B^{-1}. \tag{A6}$$

Integrating Eq. (A4)

$$\rho^2\frac{d\phi}{ds} = A^{-1}J^2, \tag{A7}$$

where J is a constant of integration. From Eqs. (A2), (A6), and (A7), one finally obtains

$$\frac{1}{A\rho^4}\left(\frac{d\rho}{d\phi}\right)^2 + \frac{1}{A\rho^2} - \frac{1}{J^2}\left(\frac{1}{B} - E\right) = 0, \tag{A8}$$

which leads to Eq. (3). J can be conveniently expressed in terms of distance at closest approach. At the point of closest approach, $d\rho/d\phi$ vanishes. Using this in Eq. (A8), one recovers Eq. (4).

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[12] Recall that the mass of a black hole can always be independently measured either by strong field gravitational lensing or by other methods. Once we know the mass M, we can *deduce* its coordinate horizon radius, e.g., $\rho_{\rm hor} = M/2$ or $r_{\rm hor} = 2M$. On the other hand, the invariant proper horizon radius $l_{\rm hor}$ of a Schwarzschild black hole of mass M can be obtained by integrating the proper radial distance in the interior metric of a homogeneous star, viz., $l = \int_0^R (1 - \frac{2Mr^2}{R^3})^{-(1/2)} dr$ where R is the coordinate radius of a star. If we formally put the extreme value for the radius $R = 2M$ and integrate, we get $l_{\rm hor} = \pi M$. But respecting the equilibrium condition for a star, one should take $R > 2.25M$. For example, with $R = 2.26M$, the proper radius l becomes $2.94M$. There is no direct way to measure this proper radius, neither does it appear as a

PN parameter. In practice, for an uncollapsed object like the sun, the radius is identified, *strictly technically*, with the fitted values of the observed parameters ρ_o or r_o depending on the employed coordinate system.

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Strong field gravitational lensing in scalar–tensor theories

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Abstract
Strong field gravitational lensing in the Brans–Dicke scalar–tensor theory has been studied. The deflection angle for photons passing very close to the photon sphere is estimated for the static spherically symmetric spacetime of the theory and the position and magnification of the relativistic images are obtained. Modelling the super massive central object of the galaxy by the Brans–Dicke spacetime, numerical values of different strong lensing observables are estimated. It is found that against the expectation there is no significant scalar field effect on the strong field observable lensing parameters. This result raises question on the potentiality of strong field lensing to discriminate different gravitational theories.

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1. Introduction

Scalar–tensor (ST) theories of gravitation [1], in which gravity is mediated by one or several long range scalar field(s) in addition to the usual tensor fields present in Einstein's theory, are widely considered as the most viable alternatives to Einstein's general theory of relativity (GR). The inclusion of scalar fields in the gravitational sector is justified from the fact that their presence is inevitable in most of the theoretical attempts to unify gravity with other fundamental interactions, such as the superstring theory, supergravity or modern revival of the Kaluza–Klein theory. Cosmological observations too insist on the introduction of a long-range scalar field; almost all scenarios of cosmological inflation are based on the scalar field.

The introduction of scalar fields obviously leads to corrections to general relativistic dynamics. These deviations from GR can be expressed in terms of the coupling function $\omega(\varphi)$ that characterizes a ST theory and represents the strength of the coupling between the scalar field (φ) and the curvature. Experimental observations, however, suggest that the contribution of the scalar field is not more than a very small fraction of that of the tensor field, if not zero, and as a result ST theories are severely constrained by the requirement that $|\omega(\varphi)|$ is very large. The VLBI observations of radio wave deflection demand $|\omega| > 500$ [2], whereas the recent conjunction experiment with the Cassini spacecraft [3] imposes even a harder restriction $|\omega| > 5 \times 10^4$. Such a limit on $\omega(\varphi)$ obviously raises doubt on the existence of a gravitational scalar field because in the limit $|\omega| \to \infty$ the post-Newtonian expansions of ST gravity reduce to those of GR [1] (however, see also [4]). The small contribution of the scalar field (relative to that of the tensor field) has been explained through the idea [5] that most of the ST theories are cosmologically evolved towards a state with practically no scalar admixture to gravity during the matter dominated era. This means that for a large class of ST theories $\omega(\varphi)$ cosmologically evolves towards a very large value. So the present lower bound on $\omega(\varphi)$ does not rule out ST gravity.

The theoretical speculation [5] is that at the present epoch $\omega(\varphi) \sim 1.4 \times 10^6 (\Omega_o^3/H_o)^{1/2}$ where Ω_o is the ratio of current density to closure density and H_o is the Hubble constant in units of 100 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$. So in the present time $\omega(\varphi)$ could have finite large value. For such large $\omega(\varphi)$, deviations of gravity in the solar system from the general relativistic values are extremely small. This motivates new searches [6] for small deviations at levels better than 10^{-5} or even 10^{-7} of the post-Newtonian effects but obviously it is a very difficult task. However, even for large $\omega(\varphi)$, ST theories may produce interesting departures from GR at the strong field scenario. Here the strong field is distinguished from the weak gravity through the quantity $\frac{GM}{rc^2}$ [8] (in the strong field regime higher order terms in $\frac{GM}{rc^2}$ cannot be ignored). For instance, in the case of generation of gravitational waves, ST gravity allows binary systems (consisting of two massive compact objects) to emit dipole radiation whereas GR admits only quadrupole and higher angular modes. The total gravitational energy radiated by a given source is also different in these theories. As a result, experiments like the Laser Interferometric Gravitational-Wave Observatory (LIGO) may discriminate these two theories or at least yield a stronger bound on $\omega(\varphi)$ than is achievable from the solar system measurements [7].

Gravitational lensing by a massive compact lens is considered another potential tool for studying strong fields. Theoretical investigations [9–12] suggest that while propagating close to a massive compact object (e.g. a black hole) light rays take several turns around the lens before reaching the observer, and as a result, apart from primary and secondary images, a set of infinite images on both sides of the optic axis will be produced which are termed as relativistic images. Though in such a situation the primary and secondary lensed images carry important information on various orders of post-post-Newtonian effects [13], these relativistic images are the main signature of the strong field lensing. However, unless the source is almost perfectly aligned with the lens and the observer, relativistic images will be very faint as a result of high demagnification. With the indication that the radio-source Sgr A* in the galactic centre hosts a supermassive object (black hole) of 3.6×10^6 solar masses [14] a possibility has developed of studying lensing phenomena in the strong gravity regime. It is thus imperative to investigate the effects of a scalar field in a strong field situation and to look for its possible observational signatures. This is precisely the aim of the present work.

An essential pre-requisite for studying strong field lensing is to have knowledge of exact explicit solution(s) of the theory. But no such solution is currently available for the generalized ST gravity. Hence our discussion would be restricted only to the simplest version of ST gravity that is developed by Jordan, Fierz, Brans and Dicke, and is commonly known as Brans–Dicke (BD) theory [15] (this is a standard approach, see, for example, [7]). In the BD theory, there is only one scalar field and $\omega(\varphi)$ is assumed to be a fixed constant. It is to be noted that in dealing with the scalar–tensor theories in general and the BD theory in particular, one envisages two types of frames, namely the Jordan and Einstein frames, which are conformally connected. Sometimes it is mathematically more preferable to use the Einstein representation, as the spin-2 and spin-0 fields are decoupled in the latter frame from each other and the behaviour of the fields is more readily manageable, but experimentally observed quantities are those that are written in the Jordan frame [5, 16], which is also known as a physical frame. Ordinary (normal) matter has universal coupling to (physical) metric in the Jordan frame which implies that test particles follow geodesics of the geometry and physical rods and clocks measure the Jordan frame metric. The weak equivalence principle, conservation laws and the constancy of the non-gravitational constants are only preserved in this frame. An undesirable feature of the frame is that the energy density of scalar fields is not positive definite here, though there are some ambiguities in the definition of energy density itself [17]. But more importantly it has been shown very recently [18] that Minkowski space is stable in this frame with respect to inhomogeneous scalar and tensor perturbations, at least at the linear order. Einstein frame formulation leads to a well-defined energy–momentum tensor for the scalar field but in this frame ordinary matter is non-minimally coupled with the scalar field and consequently test particles do not move on geodesics of the Einstein frame metric. Virbhadra and Ellis [19] have already studied gravitational lensing numerically in the strong field regime in the BD theory but in the Einstein frame. Bozza [20] too obtained analytical expressions of strong field lensing in the Einstein frame BD theory. For obvious reasons, here we would like to study strong field lensing for the BD theory in the physical frame. Since the two frames are conformally coupled, in many cases we will exploit the Einstein frame results [20] without going in to all the detailed calculations starting from *ab initio*.

The paper is organized as follows. The BD theory in Jordan and Einstein frames and its spherically symmetric vacuum solutions will be revisited in section 2. In section 3, after reviewing the strong field lensing technique for a general static, spherically symmetric spacetime, the deflection angle for the physical metric of the Jordan frame BD theory will be obtained. Expressing the gravitational field due to a super massive central object of the galaxy by the BD theory, an estimation of observational strong lensing parameters will be given in section 4 along with the similar estimation when the lens is represented by a Schwarzschild black hole. A discussion of the results will be made in section 5.

2. The BD theory in Jordan and Einstein frames

As mentioned in the previous section, the BD theory can be formulated in two distinguished conformal frames—the Jordan frame and the Einstein frame.

2.1. The BD theory in the Jordan frame

In the Jordan conformal frame, the BD action takes the form (we use geometrized units such that $G = c = 1$ and follow the signature $-, +, +, +$)

$$\mathcal{A} = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left(\varphi R + \frac{\omega}{\varphi}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right) + \mathcal{A}_{\mathrm{matter}}[\psi_m, g_{\mu\nu}]. \tag{1}$$

The last term is the action of the ordinary matter fields, ψ_m, which couple only to the metric $g_{\mu\nu}$ and not to the scalar field. Variation of (1) with respect to $g^{\mu\nu}$ and φ gives, respectively, the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi}{\varphi}T_{\mu\nu} - \frac{\omega}{\varphi^2}\left(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{\mu\nu}\varphi^{,\sigma}\varphi_{,\sigma}\right) - \frac{1}{\varphi}(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi), \tag{2}$$

$$\Box\varphi = \frac{8\pi T}{(2\omega+3)} \tag{3}$$

where R is the Ricci scalar, $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta A_{\text{matter}}}{\delta g^{\mu\nu}}$ is the physical frame energy–momentum tensor and $T = T^{\mu}_{\mu}$ is the trace of the matter energy momentum tensor. As evident from the above field equations that in the Jordan BD theory the scalar field acts as the source of the (local) gravitational coupling with $G \sim \varphi^{-1}$ and consequently the gravitational 'constant' is not in fact a constant.

Since Birkhoff's theorem does not hold in the presence of a scalar field, several static solutions of the BD theory seems possible even in spherically symmetric vacuum situations. Four forms of the static spherically symmetric vacuum solution of the BD theory are available in the literature, which are named after Brans [21] (in fact Brans class I solution was discovered jointly by Brans and Dicke [15] and hereafter we shall call it the BD class I solution). However, recent studies [22, 23] suggest that only two classes of solution are really independent; choice of imaginary parameters in the BD class I solution leads to the class II solution whereas under a redefinition of the radial variable class III solution maps to class IV. Further, by matching exterior and interior (due to physically reasonable spherically symmetric matter source) scalar fields it has been found that only the BD class I solution with certain restriction on solution parameters may represent exterior metric for a non-singular massive object. The BD class I solution (in isotropic coordinates) is given by

$$ds^2 = -\left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{\frac{2}{\lambda}} dt^2 + \left(1 + \frac{B}{\rho}\right)^4 \left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{\frac{2(\lambda - C - 1)}{\lambda}} (d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2) \tag{4}$$

$$\varphi = \varphi_0 \left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{\frac{C}{\lambda}} \tag{5}$$

with the constraint condition

$$\lambda^2 = (C + 1)^2 - C\left(1 - \frac{\omega C}{2}\right) \tag{6}$$

where B and C are arbitrary constants.

Matching of exterior and interior scalar fields demands

$$C = -\frac{1}{\omega + 2}, \qquad 2B/\lambda = M \quad \text{and} \quad \lambda = \sqrt{\frac{2\omega + 3}{2\omega + 4}}. \tag{7}$$

An important point to note is that though the BD class I solution is not the unique solution of the BD theory but it is the most general physically acceptable static spherically symmetric solution of the theory [22]. In the limit ω tends to ∞ this solution reduces to the Schwarzschild metric with a constant scalar field. Other claimed *new* spherically symmetric static vacuum solutions of the BD theory are found essentially limiting cases of the BD class I solution [24].

In general, the BD class I solution exhibits a naked singularity; all curvature invariants diverge at the horizon $\rho = B$ (it exhibits black hole nature only when $-2 > \omega > -(2 + \frac{1}{\sqrt{3}})$ [22], such small values of ω are already ruled out by observations). Here it is worthwhile mentioning that the naked singularity is undesirable to many physicists, but whether a naked singularity occurs generically in a physically realistic collapse is a subject of considerable debate [25]. Since no proof of cosmic censorship hypothesis is available, only observation can give a final verdict on the issue. The BD class I solution with the coupling constant ω less than -1.5 (excluding the point $\omega = 2$) also gives rise to physically viable traversable wormhole geometry, though it is not very suitable for interstellar travel [26].

Under the coordinate transformation

$$r = \rho\left(1 + \frac{B}{\rho}\right)^2 \tag{8}$$

the BD class I metric takes the form

$$ds^2 = -\left(1-\frac{4B}{r}\right)^{\frac{1}{\lambda}} dt^2 + \left(1-\frac{4B}{r}\right)^{-\frac{C+1}{\lambda}} dr^2 + \left(1-\frac{4B}{r}\right)^{1-\frac{C+1}{\lambda}} (r^2\,d\theta^2 + r^2\sin^2\theta\, d\phi^2) \tag{9}$$

which is mathematically more convenient for studying strong field lensing.

2.2. *The BD theory in the Einstein frame*

Defining a conformally related metric through what has been known as Dicke transformations

$$\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu} \tag{10}$$

and a redefinition of the scalar field

$$d\tilde{\varphi} = \left[\frac{2\omega+3}{2}\right]^{\frac{1}{2}} \frac{d\varphi}{\varphi}, \tag{11}$$

one finds from equation (1) the BD action in the Einstein frame variable $(\tilde{g}_{\mu\nu}, \tilde{\varphi})$,

$$\tilde{\mathcal{A}} = \frac{1}{16\pi}\int \sqrt{-\tilde{g}}\,d^4x(\tilde{R} + 2\tilde{g}^{\alpha\beta}\tilde{\varphi}_{,\alpha}\tilde{\varphi}_{,\beta}) + \tilde{\mathcal{A}}_{\text{matter}}[\psi_m, \tilde{g}_{\mu\nu}, \tilde{\varphi}]. \tag{12}$$

In the above equation derivatives are with respect to $\tilde{g}_{\mu\nu}$. Two important aspects of the Einstein frame action are that the metric and scalar field parts are untangled here, the dynamics of the gravity is governed solely by the Ricci scalar $\tilde{R}$ and secondly here matter fields couple to both $\tilde{g}_{\mu\nu}$ and $\tilde{\varphi}$. It is important to recognize that the Einstein frame energy–momentum tensor is not that measured in the local Lorentz frame, i.e. it is not the physical energy–momentum tensor.

The Einstein frame field equations follow by varying the action (12) with respect to $\tilde{g}^{\mu\nu}$ and $\tilde{\varphi}$:

$$\tilde{R}_{\alpha\beta} - \tfrac{1}{2}\tilde{g}_{\alpha\beta}\tilde{R} = -8\pi\tilde{T}_{\alpha\beta} - 2\left(\tilde{\varphi}_{,\alpha}\tilde{\varphi}_{,\beta} - \tfrac{1}{2}\tilde{g}_{\alpha\beta}\tilde{\varphi}_{,\sigma}\tilde{\varphi}^{,\sigma}\right) \tag{13}$$

$$\Box\tilde{\varphi} = 2\pi\frac{d\ln\varphi}{d\tilde{\varphi}}\tilde{T}. \tag{14}$$

The static spherically symmetric vacuum solution of the above field equations that is conformally related to the BD class I solution is the Buchdahl solution [27] which in the so-called standard coordinates becomes [28] the more familiar JNW [29] or Wyman solution [30] and is given by (leaving out tilde)

$$ds^2 = -\left(1-\frac{4B}{r}\right)^{\gamma} dt^2 + \left(1-\frac{4B}{r}\right)^{-\gamma} dr^2 + \left(1-\frac{4B}{r}\right)^{1-\gamma} (r^2\,d\theta^2 + r^2\sin^2\theta\, d\phi^2) \tag{15}$$

and

$$\varphi(r) = \sqrt{\frac{2(1-\gamma^2)}{16\pi}}\ln\left(1-\frac{4B}{r}\right). \tag{16}$$

The above form of the solution is conformally related to the BD class I solution in the standard coordinates (9).

3. Deflection angle in the strong field regime

Lensing theory in the strong field regime has been developed in stages by several researchers. The occurrence of relativistic images was brought forward by Darwin [9] and Atkinson [10] in their pioneering works in the field. The lens equation in the strong regime was mainly developed by Fritelli and Newman [31], Virbhadra and Ellis [12], Bozza *et al* [32] and Perlick [33]. After a detailed numerical study of strong field lensing produced by a Schwarzschild black hole, Virbhadra and Ellis [12] first explored observational consequences of the phenomena when the lens is the massive black hole of the galactic centre. Noting the possibility that detection of relativistic images may not be impossible in future and hence they could be used to test strong field gravity, extensive study of relativistic images started to take place. Bozza *et al* [32] developed an analytical technique of obtaining deflection angle in the strong field situation and showed that the deflection angle diverges logarithmically as light rays approach the photon sphere of a Schwarzschild black hole. Such a study was extended by Eiroa *et al* [34] for lensing due to the Reissner–Nordström (RN) spacetime. Later Bozza [20] extended the method of analytical lensing for a general class of static spherically symmetric metrics and demonstrated that the logarithmic divergence of the deflection angle at the photon sphere is a common feature for such spacetimes. Exploiting Bozza's method, strong field lensing has been carried out in several interesting cases, such as lensing due to the charged black hole of heterotic string theory [35], black holes from braneworlds [36], Einstein–Born–Infeld black holes [37], wormholes, monopole [33] etc. Very recently Bozza *et al* [38] have studied strong field lensing due to the Kerr black hole for equatorial observers. An interesting consequence of strong field gravitational deflection is the retro lensing [39] which occurs when the source is in between the observer and the lens or the observer is in between the source and the lens in contrast to the case of standard lensing where lens is situated in between the source and the observer. The phenomenon is almost the same as standard lensing except for the fact that relativistic images are formed, in this case for deflection angles closer to odd multiples of π rather than even multiples. Holtz and Wheeler [39] studied retro lensing due to a Schwarzschild black hole in the galactic bulge with the Sun as a source. Eiroa and Torres [40] considered the analytical retro lensing due to a general spherically symmetric static lens. Without remaining confined to the highly aligned case of source, lens and observer geometry, Bozza and Mancini [41] explored retro lensing due to the massive black hole of the galactic centre with the nearby (to lens) bright star S2 as source. The time delay between different relativistic images was estimated by Bozza and Mancini [42] and was later applied by several authors to some interesting cases [43]. In the present work we would employ Bozza's analytical method to obtain the deflection angle in the strong field regime under the framework of the Jordan BD theory.

We consider the lens geometry as follows. A light ray from a source (S) is deflected by the lens (L) of mass M and reaches an observer (O). The background spacetime is taken as asymptotically flat, both the source and the observer are placed in the flat spacetime. The line joining the lens and the observer (OL) is taken as the optic axis for this configuration. β and θ are the angular position of the source and the image with respect to the optic axis, respectively. The distances between the observer and lens, the lens and source and the observer and source are $d_{\mathrm{ol}}, d_{\mathrm{ls}}$ and d_{os} respectively (all distances are expressed in terms of the Schwarzschild radius $r_s = 2M$, M being the mass of the lens). The position of the source and of the image are related through the so-called lens equation [12]

$$\tan\theta - \tan\beta = d[\tan\theta + \tan(\alpha - \theta)] \tag{17}$$

where α is the deflection angle, $d = \frac{d_{\mathrm{ls}}}{d_{\mathrm{os}}}$ for standard lensing, i.e. the lens is between the source and the observer and $\frac{d_{\mathrm{os}}}{d_{\mathrm{ol}}}$ is for retro lensing with the source being in between the observer and the lens. We shall skip the case of observer in between the source and the lens. For positive β, the above relation only gives images on the same side ($\theta > 0$) of the source. Images on the other side can be obtained by taking negative values of β. The first and main step of getting image positions is to calculate the deflection angle.

For a general static and spherically symmetric spacetime of the form

$$ds^2 = -A(x)\,dt^2 + B(x)\,dx^2 + C(x)(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{18}$$

where $x = r/2M$, and as $x \to \infty$, $A(x) \to 1$, $B(x) \to 1$, $C(x) \to x^2$, the deflection angle as a function of the closest approach x_o ($x_o = r_o/2M$) is given by

$$\alpha(x_o) = I(x_o) - \pi \tag{19}$$

$$I(x_o) = 2\int_{x_o}^{\infty} \frac{\sqrt{B(x)}\,dx}{\sqrt{C(x)}\sqrt{\frac{C(x)A(x_o)}{C(x_o)A(x)} - 1}}. \tag{20}$$

With the decrease of the closest approach x_o the deflection angle will increase and, for a certain value of x_o, the deflection angle will become 2π so that the light ray will make a complete loop around the lens. If x_o decreases further, the light ray will wind several times around the lens before reaching the observer and finally when x_o is equal to the radius of the photon sphere (x_{ps}) the deflection angle will become unboundedly large and the incident photon will be captured by the lens object.

Bozza develops the following technique to evaluate the integral (20) close to its divergence. *The divergent integral is first splitted into two parts to separate out the divergent ($I_D(x_o)$)* and the regular parts ($I_R(x_o)$). Then both of them are expanded around $x_o = x_{ps}$ and are approximated by the leading terms. At first the integrand of equation (20) is expressed as a function of a new convenient variable z which is defined by

$$z = \frac{A(x) - A(x_o)}{1 - A(x_o)} \tag{21}$$

so that

$$I(x_o) = \int_0^1 R(z, x_o) f(z, x_o)\,dz \tag{22}$$

where

$$R(z, x_o) = \frac{2\sqrt{A(x)B(x)}}{C(x)A'(x)}(1 - A(x_o))\sqrt{C(x_o)} \tag{23}$$

$$f(z, x_o) = \frac{1}{\sqrt{A(x_o) - A(x)C(x_o)/C(x)}}. \tag{24}$$

The function $R(z, x_o)$ is regular for all values of z and x_o but $f(z, x_o)$ diverges as $z \to 0$, i.e., as one approaches to the photon sphere. The integral (22) is then split into two parts,

$$I(x_o) = I_D(x_o) + I_R(x_o), \tag{25}$$

where

$$I_D(x_o) = \int_0^1 R(0, x_{ps}) f_o(z, x_o)\,dz \tag{26}$$

includes the divergence and

$$I_R(x_o) = \int_0^1 g(z, x_o)\,dz \tag{27}$$

is regular in z and x_o. The function $f_o(z, x_o)$ is the expansion of the argument of the square root in the divergent function $f(z, x_o)$ up to the second order in z,

$$f_o(z, x_o) = \frac{1}{\sqrt{p(x_o)z + q(x_o)z^2}}, \tag{28}$$

where

$$p(x_o) = \frac{1 - A(x_o)}{C(x_o)A'(x_o)}[C'(x_o)A(x_o) - C(x_o)A'(x_o)] \tag{29}$$

$$q(x_o) = \frac{(1 - A(x_o))^2}{2C(x_oA'^3(x_o)}[2C(x_o)C'(x_o)A'^2(x_o) + (C(x_o)C''(x_o) \\ - 2C'^2(x_o))A(x_o)A'(x_o) - C(x_o)C'(x_o)A(x_o)A''(x_o)] \tag{30}$$

and the function $g(z, x_o)$ is simply the difference of the original integrand and the divergent integrand

$$g(z, x_o) = R(z, x_o)f(z, x_o) - R(0, x_{ps})f_o(z, x_o). \tag{31}$$

As $x_o \to x_{ps}$, $p(x_o) \to 0$ and hence the integral (26) diverges logarithmically. Expanding both the integral around $x_o = x_{ps}$ and approximating by the leading terms, Bozza obtained the analytical expression of the deflection angle close to the divergence in the form [20]

$$\alpha(\theta) = -u \log\left(\frac{\theta D_{OL}}{b_{ps}} - 1\right) + v + O(b - b(x_{ps})) \tag{32}$$

where

$$u = \frac{R(0, x_{ps})}{2\sqrt{q(x_{ps})}} \tag{33}$$

$$v = -\pi + v_R + u \log\frac{2q(x_{ps})}{A(x_{ps})} \tag{34}$$

$$v_R = I_R(x_{ps}), \qquad I_R(x_o) = \int_o^1 g(z, x_o)\, dz. \tag{35}$$

The coefficient v_R may not be computed analytically for all metrics but can be evaluated numerically.

3.1. Strong gravitational deflection due to the BD spacetime

Since the Brans class I metric in standard coordinate is conformally related with the JNW metric, the integral $I(x_o$ for the metric is the same as for the JNW metric with the parameter γ being replaced by $\frac{C+2}{2\lambda}$. But the implication of this change is non-trivial. This can be easily understood from the fact that at the (first) post-Newtonian level the deflection angle for the JNW metric is $\frac{4M}{R}$, where $M = 2\gamma B$ is the gravitational mass of the lensing object and R is the radius of the lensing object, which is the same as that of general relativity whereas for the Jordan BD theory the deflection angle is $\frac{2M}{R}\left(1 + \frac{2\omega+3}{2\omega+4}\right)$. As a result the solar system observations have so far not imposed any restriction on the parameter γ that represents the effect of the scalar field in the Einstein frame BD theory but, as mentioned already, the Jordan frame parameter ω is already severely constrained by the observations.

The radius of the photon sphere for the BD class I metric is

$$x_{\rm ps} = \frac{1}{2} + \frac{2\omega+3}{2\omega+4}. \tag{36}$$

For finite ω, this is smaller than the photon sphere radius of the Schwarzschild spacetime. The expression for impact parameter at photon sphere is given by

$$b(x_{\rm ps}) = \left(\frac{1}{2} + \sqrt{\frac{2\omega+3}{2\omega+4}}\right)\left(\frac{2\sqrt{2\omega+3}+\sqrt{2\omega+4}}{2\sqrt{2\omega+3}-\sqrt{2\omega+4}}\right)^{\left(-\frac{1}{2}+\sqrt{\frac{2\omega+3}{2\omega+4}}\right)}. \tag{37}$$

Exploiting the results of the strong field lensing for the JNW spacetime, the coefficients u and v of the deflection angle in the strong field regime for the BD class I metric have been obtained as follows:

$$u = 1 \tag{38}$$

$$v = -\pi + v_R + \log\left[\frac{(3\omega+4)}{(2\omega+3)}\left(1 - \left(\frac{2\sqrt{2\omega+3}+\sqrt{2\omega+4}}{2\sqrt{2\omega+3}-\sqrt{2\omega+4}}\right)^{\sqrt{(2\omega+3)/(2\omega+4)}}\right)^2\right] \tag{39}$$

where

$$v_R = 0.9496 + 0.1199\left(1 - \sqrt{\frac{2\omega+3}{2\omega+4}}\right) + \text{higher order terms in } \left(1 - \sqrt{\frac{2\omega+3}{2\omega+4}}\right). \tag{40}$$

It can be seen that, as $\omega \to \infty$, all the coefficients approach the GR value.

4. Strong field observable

Once the deflection angle is known, the position of the images can be obtained from equation (17). In the strong field regime and when the source, lens and observer are highly aligned, the lens equation becomes [32]

$$\beta \doteq \theta - d\Delta\alpha_n, \tag{41}$$

where $\Delta\alpha_n = \alpha - 2n\pi$ is the offset of the deflection angle α and n is an integer. If θ_n^0 are the image positions corresponding to $\alpha = 2n\pi$, the above equation gives

$$\theta_n^0 = \frac{b(x_{\rm ps})}{d_{\rm ol}}(1 + e_n) \tag{42}$$

where

$$e_n = {\rm e}^{(v-2n\pi)/u} \tag{43}$$

and thus the position of the nth relativistic image can be approximated as [20]

$$\theta_n = \theta_n^0 + \frac{b(x_{\rm ps})e_n}{udd_{\rm ol}}(\beta - \theta_n^0). \tag{44}$$

The magnification of the nth relativistic image is given by (approximating the position of the images by θ_n^0)

$$\mu_n = \frac{1}{(\beta/\theta)\partial\beta/\partial\theta} \simeq e_n \frac{b(x_{\rm ps})^2(1+e_n)}{u\beta dd_{\rm ol}^2}. \tag{45}$$

Table 1. Estimates of the lensing observable in the BD theory for the central massive object of our galaxy.

	Standard lensing			Retro lensing		
		BD			BD	
Observable	Schwarzschild	$\omega = 500$	$\omega = 50\,000$	Schwarzschild	$\omega = 500$	$\omega = 50000$
θ_∞ (μ arcsec)	25.0417	25.0280	25.0415	25.0417	25.0280	25.0415
s (μ arcsec)	0.031340	0.031325	0.031338	0.725217	0.724877	0.725213
r_m (magnitudes)	6.82	6.82	6.82	6.85	6.85	6.85

In the simplest situation if only the outermost image can be resolved as a single image then its angular separation from the remaining bunch of relativistic images is

$$s = \theta_1 - \theta_\infty, \tag{46}$$

where $\theta_\infty = b_{\rm ps}/d_{\rm ol}$ is the angular position of a set of relativistic images in the limit $n \to \infty$. If r denotes the ratio of the flux from the outermost relativistic image and those from the remaining relativistic images, then

$$r = \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n}. \tag{47}$$

For highly aligned source, lens and observer geometry, these observable take the simple form

$$s_{\rm SL} = \theta_\infty \, {\rm e}^{(v-2\pi)/u} \tag{48}$$

$$r_{\rm SL} \simeq {\rm e}^{2\pi/u} + {\rm e}^{v/u} - 1 \tag{49}$$

for standard lensing and

$$s_{\rm SL} = \theta_\infty \, {\rm e}^{(v-\pi)/u} \tag{50}$$

$$r_{RL} \simeq {\rm e}^{2\pi/u} + {\rm e}^{(v+\pi)/u} - 1 \tag{51}$$

for retro lensing. Since the deflection angle is already known, the strong lensing parameters, namely the position of the relativistic images, the angular separation between the outermost relativistic image and the remaining relativistic images and their flux ratio, readily follow from equations (48)–(51) for both standard and retro lensing. By measuring these parameters one should be able to identify the nature of the lensing object.

4.1. Lensing by the super massive galactic centre

To get an idea of the numerical values of the scalar field effect in a strong lens observation, we model the gravitational field of the super massive galactic centre of the Milky Way by the BD spacetime. The mass of the central object of our galaxy is estimated to be 3.6×10^6 of the solar mass and its distance is around 7.6 kpc [14]. Therefore $d_{\rm ol} \sim 2.14 \times 10^{10}$. The angular position of the relativistic images (θ_∞), the angular separation of the outermost relativistic image with the remaining bunch of relativistic images (s) and the relative magnification of the outermost relativistic image with respect to the other relativistic images (r) are estimated by taking $\omega = 500$ and 50 000 (the lower bounds obtained from two observations) for standard as well as retro lensing and are given in table 1 (magnification is converted to magnitudes: $r_m = 2.5 \, {\rm Log} r$). The same observable parameters when the lens is a Schwarzschild black hole are also given in table 1 for comparison. It is clear from table 1 that the observational predictions of the GR and the BD theories are almost the same within the given accuracy.

5. Discussion

For certain values of the coupling parameter ω, the scalar tensor theories, which are among the best motivated alternatives to GR, agree with GR in the post-Newtonian limit up to any desired accuracy, and hence weak-field observations cannot rule out the scalar–tensor theories in favour of general relativity. But in the strong field regime usually the full features of a theory come into play. As a result, strong field predictions of different theories are expected to be divergent. With this anticipation in the present work strong field gravitational lensing is studied in the framework of the BD theory which is the simplest and most studied scalar–tensor theory.

The strong field deflection angle is calculated for the BD spacetime and different strong lensing parameters such as the angular positions of the relativistic images, angular separation between the outermost relativistic image and the rest of the images and also their relative magnifications are estimated for both standard as well as retro lensing scenarios. It is found that all the parameters of strong field deflection in the BD theory reduce to GR values in the limit $\omega \rightarrow \infty$ as in the case of weak field lensing. The nature of such convergence is not identical but similar to the weak field scenario. This implies that against the hope there is no significant scalar field effect on the strong field observable lensing parameters. Here one may be tempted to say that the said observation was expected *a priori* because the radius of the photon sphere in the BD theory (equation (36)) has ω dependence similar to that of the weak field observables of the theory such as the post-Newtonian (PPN) deflection angle [44]. It is to be noted that the radius of horizon of the BD and the Schwarzschild spacetimes are exactly the same, yet the curvature components (or invariants) are very dissimilar at horizon; for Schwarzschild spacetime, they are finite whereas for the BD spacetime they diverge as the horizon approaches. Hence *a priori* it was not possible to guess the outcome of the problem, particularly in view of the fact that the Einstein frame representation of the theory gives large deviation of strong field deflection angle parameters from those for Schwarzschild spacetime.

It has been already realized that observation of relativistic images is not easy [12] though, see [20, 37, 38]. To observe relativistic images, the resolution of the detecting telescope needs to be of the order of μ-arcsec or even better (the resolution achieved so far is only of the order of m-arcsec or slightly better) (whereas weak field gravitational deflection can be detected with just arcsec observational accuracy). Proposed optical interferometer based telescopes on the International Space Station are expected to achieve angular resolution of about 0.01 μ arcsec [45]. Hence numerical values of the lensing parameters have been estimated at the level of nano-arcsec expressing the gravitational field due to a massive compact object at the centre of the galaxy by the BD spacetime. When compared with the corresponding lensing observable due to the Schwarzschild black hole, it is clear that detection of relativistic images will not give any special advantage over weak field observations to discriminate scalar–tensor theories from GR. For instance, an observational accuracy of 0.01 μ arcsec could yield only a bound of $\omega > 1000$ whereas with an observational accuracy of 0.1 nano-arcsec, the lower bound of ω could be raised up to about 1.5×10^5. In contrast, measurements of gravitational deflection of light by the solar gravity with an angular precesion of 0.01 μ arcsec could yield a bound of $\omega > 10^8$ [8, 44, 45]. However, strong field observations have their own merits; observation of relativistic images with finite ω would be a test for the ST gravity in the strong field regime.

Another interesting observation is that the strong field deflection angle in the BD theory is smaller than that of GR. Here one may recall that from weak field analysis Bekenstein and Sanders provided the theorem that *in a generic ST theory of gravity, the scalar field cannot* *enhance lensing* [46]. The present work indicates that Bekenstein–Sanders theorem may be valid also in the strong field regime.

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LETTER

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On static spherically symmetric solutions of the vacuum Brans-Dicke theory

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Abstract It is shown that among the four classes of the static spherically symmetric solutions of the vacuum Brans-Dicke theory of gravity only two are really independent. Further, by matching exterior and interior (due to physically reasonable spherically symmetric matter source) scalar fields it is found that only the Brans class I solution with a certain restriction on the solution parameters may represent an exterior metric for a nonsingular massive object. The physical viability of the black hole nature of the solution is investigated. It is concluded that no physical black hole solution different from the Schwarzschild black hole is available in the Brans-Dicke theory.

Keywords Brans-Dicke theory · Static solutions · Black hole

1 Introduction

Although general relativity (GR) is one of the most beautiful physical theory and is supported by observational evidences, the sustained inability of reconciling GR with quantum mechanics and recent cosmological observations indicate that Einstein's theory needs modification. The Brans-Dicke (BD) theory [1], which describes gravitation through a spacetime metric ($g_{\mu\nu}$) and a massless scalar field (φ), is a modification of the GR. The theory has recently received interest as it arises naturally as the low energy limit of many theories of quantum gravity such as the supersymmetric string theory or the Kaluza-Klein theory and is also found consistent with present cosmological observations [2]. The theory contains

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an adjustable parameter ω that represents the strength of coupling between scalar field and the matter. For certain values of ω, the BD theory agrees with GR in post-Newtonian limit up to any desired accuracy and hence weak-field observations cannot rule out the BD theory in favor of general relativity. It is thus imperative to study strong-field cases in which these two theories could give different predictions.

Due to highly non-linear character of all viable gravitational theories, a desirable pre-requisite for studying strong field situation is to have knowledge of exact explicit solution(s) of the field equations. By Birkhoff's theorem, the static spherically symmetric vacuum solution of Einstein's theory is unique, the Schwarzschild metric. On the other hand Birkhoff's theorem does not hold in the presence of a scalar field, hence several static solutions of the BD theory seems possible even in spherically symmetric vacuum situations. Four forms of static spherically symmetric vacuum solution of the BD theory are available in the literature which are constructed by Brans himself [3]. However, it has been shown in [4] that Brans class III and class IV solutions are not different; under a mere redefinition of the radial variable one of them maps to another.

Among the all Brans solutions, class I solution is the most studied one as it is the only one which is permitted for all values of ω. The solution in general gives rise to naked singularity [5] though for some particular choices of the solution parameters it represents a black hole different from Schwarzschild one [6]. The other two classes (II and IV) of Brans solution are valid only for $\omega < -3/2$ [3] which implies non-positive contribution of matter to effective gravitational constant and thus a violation of the weak energy condition. However, this energy condition is gradually loosing its status as a kind of law as in many physical situation it could be violated [7–12]. For instance, classical systems built from scalar field non-minimally coupled to gravity violate all the energy conditions [7]. In quantum systems these violations are even more profound. The Casimir effect suggests the existence of negative energy density. So is the squeezed states of light [10]. Similarly negative energy density fields also occur in the context of Hawking evaporation of black holes [11], radiation from moving mirrors [12] and in several other situations [10]. The experimental observation of first two effects (Casimir and squeezed states of light) suggests that the idea of negative energy density have to be taken seriously. Besides Solar system observations do not impose any restriction on the sign of ω [13]. Oppenheimer-Snyder (gravitational) collapse in BD theory also has not conclusively ruled out negative ω [14]. Hence class II and IV solutions cannot be regarded as unphysical just for being negative ω solutions. These solutions or their Einstein frame variants [4, 15] have been used in literatures in different contexts such as in connection with the wormhole physics [16, 17] or to generate solutions of the string theory (in string frame) [18]. In some aspects Class IV solution, which gives rise to what is called as cold black hole [19, 20], even exhibits better behavior than class I solution. For example the tidal forces do not diverge on the horizon for this spacetime unlike the class I metric [20].

All Brans solutions, however, may not be physically relevant. There are several known exact perfect fluid interior solutions in general relativity [21]. Most of them are physically not acceptable because either the solutions have not a well defined boundary or they do not match with the Schwarzschild exterior solution at the boundary surface or perfect fluid satisfies unrealistic equations of state or due to some other valid reasons. The same could happen for the vacuum BD solutions also. In the present article we would like to examine the physical viability of the Brans solutions. The paper is organized as follows. After giving a short account of the BD theory and its static spherically symmetric solutions, it has been shown in Sect. 2 *that Brans class I and class II solutions are essentially not different.* Physical viability of Brans solutions has been examined in Sect. 3. The broad nature of the class I solution has been discussed in Sect. 4 and the physical relevance of the Brans class I black hole has been investigated in the Sect. 5. Finally the results are summarized in Sect. 6.

2 Static spherically symmetric vacuum solutions of the BD theory

In the BD theory the scalar field acts as the source of the (local) gravitational coupling with $G \sim \varphi^{-1}$ and consequently the gravitational 'constant' is not in fact a constant but is determined by the total matter in the universe through an auxiliary scalar field equation. The scalar field couples to both matter and spacetime geometry and as mentioned before the strength of the coupling is represented by the dimensionless constant ω. The theory is found consistent with the (local) observations only when ω is very large. A lower limit $|\omega| > 5 \times 10^4$ is obtained from the recent conjunction experiment with Cassini spacecraft [22]. This suggests even if scalar field exists, predictions of the BD theory are not much different from GR, particularly in the weak-field regime, because under the limit $|\omega| \to \infty$, the BD theory (and its dynamic generalization) reduces to GR [23] unless the matter field is traceless [24].

In the Jordan conformal frame, the BD action takes the form (we use geometrized units such that $G = c = 1$ and we follow the signature $-,+,+,+$)

$$\mathcal{A} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\varphi R + \frac{\omega}{\varphi} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} + \mathcal{L}_{matter} \right) \tag{1}$$

where $\mathcal{L}_{matter}$ is the Lagrangian density of ordinary matter. Variation of (1) with respect to $g^{\mu\nu}$ and φ gives, respectively, the field equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = & -\frac{8\pi}{\varphi} T_{\mu\nu} - \frac{\omega}{\varphi^2} \left(\varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} \varphi^{,\sigma} \varphi_{,\sigma} \right) \\ & - \frac{1}{\varphi} (\nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} \Box \varphi), \end{aligned} \tag{2}$$

$$\Box \varphi = \frac{8\pi T}{(2\omega + 3)} \tag{3}$$

where R is the Ricci scalar, and $T = T^\mu_\mu$ is the trace of the matter energy momentum tensor.

As stated earlier, Brans provided four classes of static spherically symmetric solutions of the above theory when $T_{\mu\nu} = 0$. The Brans class I solution (in isotropic coordinates) is given by

$$ds^2 = -e^{\alpha_o}\left(\frac{1-B/\rho}{1+B/\rho}\right)^{\frac{2}{\lambda}} dt^2 + e^{\beta_o}\left(1+\frac{B}{\rho}\right)^4\left(\frac{1-B/\rho}{1+B/\rho}\right)^{\frac{2(\lambda-C-1)}{\lambda}}$$
$$\times(d\rho^2+\rho^2 d\theta^2+\rho^2\sin^2\theta d\phi^2) \tag{4}$$

$$\varphi = \varphi_0\left(\frac{1-B/\rho}{1+B/\rho}\right)^{\frac{C}{\lambda}} \tag{5}$$

with the constraint condition

$$\lambda^2 = (C+1)^2 - C\left(1-\frac{\omega C}{2}\right) \tag{6}$$

where α_o, β_o, B, C are arbitrary constants.

The class II solution is given by

$$ds^2 = -e^{\alpha_o+\frac{4}{\Lambda}tan^{-1}(\rho'/B')}dt^2 + e^{\zeta_o-\frac{4(C+1)}{\Lambda}tan^{-1}(\rho'/B')-2ln[\rho'^2/(\rho'^2+B'^2)]}$$
$$\times(d\rho'^2+\rho'^2 d\theta^2+\rho'^2\sin^2\theta d\varphi^2) \tag{7}$$

$$\varphi = \varphi_0 e^{2C/\Lambda tan^{-1}(\rho'/B')} \tag{8}$$

with the solution parameters are related by

$$\Lambda^2 = C\left(1-\frac{\omega C}{2}\right) - (C+1)^2 \tag{9}$$

Here arbitrary constants are denoted as α_o, ζ_o, B', C.

Class III solution can be written as

$$ds^2 = -e^{\alpha_o-2\rho/B}dt^2 + e^{\beta_o-4ln(\rho/B+2(C+1)\rho/B}(d\rho^2+\rho^2 d\theta^2+\rho^2\sin^2\theta d\phi^2) \tag{10}$$

$$\varphi = \varphi_0 e^{C\rho/B} \tag{11}$$

with the condition

$$C = \frac{-1\pm\sqrt{-2\omega-3}}{\omega+2} \tag{12}$$

and finally class IV solution is

$$ds^2 = -e^{\alpha_o-2/(B\rho)}dt^2 + e^{\beta_o+2(C+1)/(B\rho)}(d\rho^2+\rho^2 d\theta^2+\rho^2\sin^2\theta d\phi^2) \tag{13}$$

$$\varphi = \varphi_0 e^{C/(B\rho)} \tag{14}$$

with

$$C = \frac{-1\pm\sqrt{-2\omega-3}}{\omega+2} \tag{15}$$

The class I and class II solutions are, however, not different. To show this we define a new radial variable $\rho = 1/\rho'$. Utilizing the identity $tan^{-1}(x) = i/2\,ln(1-ix/1+ix)$, the Eqs. (7) and (8) can be recast as

$$ds^2 = -e^{\alpha_o}\left(\frac{1-iB''/\rho}{1+iB''/\rho}\right)^{\frac{2i}{\Lambda}} dt^2 + e^{\beta_o}\left(1+\frac{B''^2}{\rho^2}\right)^2\left(\frac{1-iB''/\rho}{1+iB''/\rho}\right)^{\frac{-2i(C+1)}{\Lambda}}$$
$$\times(d\rho^2+\rho^2 d\theta^2+\rho^2\sin^2\theta d\phi^2) \tag{16}$$

$$\varphi = \varphi_0\left(\frac{1-iB''/\rho}{1+B''/\rho}\right)^{\frac{iC}{\Lambda}} \tag{17}$$

where $B'' = 1/B'$, and $\beta_o = \zeta_o - 4lnB''$. The above two equations would reduce to Eqs. (4) and (5) if we denote $\lambda \doteq -i\Lambda$ and $B = iB''$. The relation (9) will also map to relation (6) under such redefinition of constants. Therefore, class I and class II solutions are equivalent; the choice of imaginary B and λ in the class I solution leads to the class II solution. A point to be noted is that for imaginary B and λ the class I solution becomes regular at all points including the point $r = B$ and consequently the class II solution does not possess any horizon.

As mentioned before Brans class III and class IV solutions are also not different [4]; under a mere redefinition of the radial variable ($\rho \equiv 1/\rho$) one of them maps to another. Hence only two classes of solutions, class I and class IV, are found independent. These two classes can be expressed by a single form. For this purpose let us consider the transformation

$$e^{-\sigma/r} = \frac{1-B/\rho}{1+B/\rho} \tag{18}$$

($\sigma \neq 0$) under which the Eqs.(4) and (5) are reduced to the form

$$ds^2 = -e^{\alpha_o-\alpha/r}dt^2 + e^{\zeta_o+\alpha(C+1)/r}\left(\frac{\sigma/r}{\sinh(\sigma/r)}\right)^4 dr^2 + e^{\zeta_o+\alpha(C+1)/r}$$
$$\times\left(\frac{\sigma/r}{\sinh(\sigma/r)}\right)^2 r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{19}$$

and

$$\varphi = \varphi_0 e^{-\frac{\alpha C}{2r}} \tag{20}$$

$\zeta_o = \beta_o + ln(B^4/\sigma^2)$. The relation among the parameters is given by

$$4\frac{\sigma^2}{\alpha^2} = (C+1)^2 - C\left(1-\frac{\omega C}{2}\right) \tag{21}$$

The Eqs. (19)–(21) are the general form of all Brans' solutions. As is evident from the above $\sigma \neq 0$ leads to class I solution where use of some imaginary parameters results the class II solution. The choice $\sigma = 0$ gives the Brans' class IV solution and a further redefinition of the radial variable $\bar{r} = 1/r$ will immediately give the class III solution. The line element (19) is conformal to the Wyman solution [25] of the Einstein minimally coupled scalar field theory.

3 Physical viability of brans solutions

In general relativity the metric tensor is the only gravitational field variable. Hence, it is sufficient and necessary to match the interior and exterior solutions for metric tensor only. In contrast BD theory has additional scalar field which contributed to the gravitational field as well. Therefore, in this theory not only matching for metric tensor is necessary but also for the additional scalar field.

It follows from Eq. (3) that to the leading order in $1/r$ the interior (in presence of matter) scalar field satisfies the equation

$$\nabla^2 \overset{2}{\varphi} = -\frac{8\pi}{2\omega+3}\overset{o}{T} \tag{22}$$

where we have expanded scalar field as $\varphi = \varphi_o + \overset{2}{\varphi} + \overset{4}{\varphi} + \cdots$, $\overset{N}{\varphi}$ denotes the term in φ of order $1/r^{N/2}$ (here we have followed the notation of [23]) and $\overset{o}{T}$ denotes the term in T^{σ}_{σ} of order $1/r^3$ ($\overset{o}{T}^{oo}$ is the density of rest mass) [23]. Therefore to the leading order in $1/r$ (r being the radial variable) the scalar field at near the surface of the matter distribution has the following expression

$$\varphi = \varphi_o - \left(\frac{2}{2\omega+3}\right)\phi \tag{23}$$

where ϕ is the Newtonian potential defined through $\nabla^2\phi = 4\pi \overset{o}{T}$. At the surface and outside the source $\phi = -M/r$ where M is the total mass of the source as viewed by a distant observer. Utilizing the relation $G = \frac{2\omega+4}{2\omega+3}\frac{1}{\varphi_o}$ (which can be obtained from the relation $\overset{2}{g}_{oo} = -2\phi$) where G is the gravitational constant that would be measured in a real experiment, we finally get the expression for scalar field to the leading order in $1/r$ near the surface of the matter distribution (we still continue of using geometrized units)

$$\varphi = \varphi_o\left(1 + \frac{1}{\omega+2}\frac{M}{r}\right) \tag{24}$$

A physically viable external solution for scalar field must match smoothly with the above expression at the surface.

When matching to two different solutions on a common surface it is essential to choose an appropriate coordinate system. The expression (24) is written in standard coordinates (t, r, θ and ϕ). Hence the expression for scalar field in Brans class I or class IV solution needs to transform first in the standard coordinates for effective comparison. However, at the first order the standard and isotropic coordinates produce identical effects. Hence comparing the expression for scalar field of Brans class I solution with Eq. (24) and using the relation $\overset{2}{g}_{oo} = -2\phi$, we get

$$C = -\frac{1}{\omega+2}; \quad 2B/\lambda = M \tag{25}$$

The relation (6) gives $\lambda = \sqrt{2\omega+3/2\omega+4}$. Therefore, Brans class I solution may represent external gravitational field due to a reasonable matter field only when the solution parameters are given by Eq. (25). On the other hand scalar field of class IV solution can not be matched with interior solution at the surface as its 1st order term goes as $\omega^{-1/2}$ for large ω whereas boundary condition requires that it must be proportional to ω^{-1}. Thus regularity conditions at the boundary suggest that class IV solution cannot be acted as an exterior metric for a nonsingular spherical massive object.

4 Generic nature of brans class I solution

In dealing with scalar-tensor theories in general and BD theory in particular, one envisages two types of frames, viz., the Jordan and Einstein frames which are connected via the conformal relation [26]

$$\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu} \tag{26}$$

and a redefinition of the scalar field

$$d\tilde{\varphi} = \left[\frac{2\omega + 3}{2\lambda}\right]^{\frac{1}{2}} \frac{d\varphi}{\varphi} \tag{27}$$

where λ is a constant and $\tilde{g}_{\mu\nu}$ and $\tilde{\varphi}$ are the Einstein frame variables. Though experimentally observed quantities are those that are written in the Jordan frame [27] sometimes it is mathematically more preferable to use the Einstein representation as the spin-2 and spin-0 fields are decoupled in the later frame from each other and the behavior of the fields are more readily manageable. In this conformal frame the field equations are given by

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = -\lambda T_{\mu\nu}, \quad \tilde{\varphi}^{;\sigma}_{;\sigma} = 0, \tag{28}$$

and

$$T_{\mu\nu} = \tilde{\varphi}_{,\mu}\tilde{\varphi}_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\varphi}^{,\sigma}\tilde{\varphi}_{,\sigma}, \tag{29}$$

where $\tilde{R}$ is the Ricci scalar, $T_{\mu\nu}$ is the energy momentum tensor due to the massless scalar field $\tilde{\varphi}$. With the assumption of asymptotic flatness and taking scalar field to be time independent ($\dot{\varphi} = 0$), Wyman, by directly solving the field equations in a straightforward way, has shown [25] that the most general static spherically symmetric metric that satisfies the above field equations is given by

$$\begin{aligned} ds^2 = &-e^{\alpha/r}dt^2 + e^{-\alpha/r}\left(\frac{\sigma/r}{\sinh(\sigma/r)}\right)^4 dr^2 \\ &+ e^{-\alpha/r}\left(\frac{\sigma/r}{\sinh(\sigma/r)}\right)^2 r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \tag{30}$$

Later Roberts has shown [28] that the assumption of asymptotic flatness is not even required for obtaining the above general solution. Note that the well known static spherically symmetric solution of the Einstein minimally coupled scalar field theory, the Buchdahl solution [29], which is also variously referred

[15] to as Janis-Newman-Winicour solution [5], is contained in the Wyman solution as a special case.

As mentioned already, the Wyman solution is conformal to the general form of Brans' solutions (Eqs. (19)–(21)). Since the mapping between the two conformal frames, the Jordan and Einstein, is one-to-one, Eqs. (19)–(21) thus should be the general static spherically symmetric solution of the BD theory in the Jordan frame. Here it should be noted that though look very similar, the characteristics of the solutions in the Einstein frame representation are quite different from those in the Jordan frame. For instance, the Buchdahl solution, which is conformal to the Brans class I solution, always satisfies weak energy condition and exhibits strong globally naked singularity for any choice of the solution parameters unlike the Brans class I solution. As it is already shown that among the Brans solutions only class I solution with parameters constrained by the Eq. (25) represents exterior metric for a nonsingular spherical massive object, hence though the class I metric is not the unique solution of the BD theory but it is the most general physically acceptable static spherically symmetric solution of the theory. In the limit ω tends to ∞ this solution reduces to the Schwarzschild metric with constant scalar field. Recently He and Kim [30] have claimed for two *new* static vacuum solutions of the BD theory. But as already shown in [31, 32], these two classes of solutions are essentially limiting cases of the Brans class I solution.

5 Physical relevance of Brans class I black hole

In general, class I solution exhibits naked singularity; all curvature invariants diverge at the horizon $\rho = B$. But for this reason one cannot rule out the solution as whether a naked singularity occurs generically in a physical realistic collapse is still a subject of considerable debate [33]. However, as demonstrated by Campanelli and Lousto [6], class I solution also exhibits black hole nature when solution parameters obey certain constraint conditions as mentioned below. Under such restrictions the Hawking temperature for the metric becomes zero and hence the solution is recognized as cold black hole following the terminology of Bronnikov et al. [19]. It was further shown in [6] that some strong gravitational fields effects such as scattering of photons, X-ray luminosity of accretion disks, or Hawking radiation could distinguish the BD black holes from the Schwarzschild one.

Campanelli and Lousto use the following form of the solution

$$ds^2 = -\left(1-\frac{B}{r}\right)^{m+1} dt^2 - \left(1-\frac{B}{r}\right)^{n-1} dr^2 - r^2\left(1-\frac{B}{r}\right)^{n} (d\theta^2 + \sin^2\theta d\phi^2) \tag{31}$$

and

$$\varphi = \varphi_0 \left(1-\frac{B}{r}\right)^{-(m+n)/2} \tag{32}$$

where m, n are arbitrary constants. The coupling constant is related with the parameters as is given by

$$\omega = -2\frac{m^2+n^2+nm+m-n}{(m+n)^2} \tag{33}$$

The Eqs. (31)–(33) transform to the original form of the solution as given by Eqs. (4)–(6) under the radial transformation

$$r = \rho\left(1 + \frac{B}{\rho}\right)^2 \tag{34}$$

and with the identification $m = 1/\lambda - 1$, $n = 1 - \frac{C+1}{\lambda}$. Campanelli and Lousto found that for $n \leq -1$, class I solution admits black hole space time. This can be understood by studying the curvature invariants of the metric. The Ricci scalar for the class I metric (4) is given by

$$R = \frac{4\omega B^2 C^2}{\lambda^2 \rho^4 (1 + B/\rho)^8} \left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{\frac{-2(2\lambda - C - 1)}{\lambda}} \tag{35}$$

The expression for the Kretschmann scalar is quite messy. To the leading order of $1/r$ it has the expression

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \approx \frac{96B^2(2 + 2C + C^2)}{\lambda^2 \rho^6 (1 + B/\rho)^{16}} \left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{\frac{-4(2\lambda - C - 1)}{\lambda}} \tag{36}$$

and the Weyl scalar is given by

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \frac{16B^2[2B(4 + 4C + C^2 + 2\lambda^2)r - 3\lambda(C + 2)(B^2 + r^2)]^2}{3\lambda^4 \rho^{10}(1 + B/\rho)^{16}} \times \left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{\frac{-4(2\lambda - C - 1)}{\lambda}} \tag{37}$$

It can be seen easily that as $r \to B$ all curvature invariants diverge and the solution exhibits naked singularity. However, when

$$C + 1 \geq 2\lambda \tag{38}$$

curvature invariants become non-singular. If further $(C + 2 - \lambda)/\lambda > 0$ then the surface $\rho = B$ will be an outgoing null surface and hence it will act as event horizon and the solution exhibits black hole nature [6]. Note that the condition (38) implies ω to be negative and thus a violation of the weak energy condition. Now if we impose the restriction (25) which is needed for class I solution to be physically acceptable, the inequality (38) for real C and λ demands $-2 > \omega > -(2 + 1/\sqrt{3})$. Such small values of ω are already ruled out by observations.

6 Discussion

A viable theory of gravity could have several exact explicit solutions. Though many of those solutions may be useful for understanding the inherent non-linear character of gravitational theories, only physically acceptable solutions are of astrophysical interest. In this work we examine different classes of the static spherically symmetric solution of vacuum BD theory for their physical relevance.

It has been found that among the four different forms of the static spherically symmetric solution of the vacuum BD theory of gravity only two classes, Brans class I and class IV solutions, are really independent; the remaining solutions are their variant. Moreover, by matching the expressions for scalar field of the independent Brans solutions with the interior solution for the scalar field due to physically reasonable matter source, it is found that only Brans Class I solution may represent external gravitational field for nonsingular spherically symmetric matter source when the parameters of the solution have a specific dependence on coupling constant ω as given by Eq. (25). The class IV solution, though also admits all the standard weak field tests (up to the first post-Newtonian order) of gravitation, does not act as an exterior metric for any reasonable gravitating object.

Hawking theorem [34] states that the static spherically symmetric black hole solution of the BD theory is the Schwarzschild black hole. The proof relies on the assumption that the weak energy condition holds. Campanelli and Lousto demonstrated that the BD theory admits black holes different from the Schwarzschild one when the weak energy condition is not respected. As it is well known now that in many physical situation the weak energy condition could be violated, the existence of BD black holes in nature is an interesting possibility. The present investigation, however, suggests that such black holes are physically not viable; they are incompatible with the observationally imposed constraints on the solution parameters.

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WORMHOLES IN VACUUM BRANS–DICKE THEORY

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It is shown that among the different classes of claimed static wormhole solutions of the vacuum Brans–Dicke theory only Brans Class I solution with coupling constant ω less than -1.5 (excluding the point $\omega = 2$) gives rise to physically viable traversable wormhole geometry. Usability of this wormhole geometry for interstellar travel has been examined.

Keywords: Brans–Dicke theory; wormholes; traversability.

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1. Introduction

In recent years considerable interest has grown in the study of wormhole physics, either in general relativity or in alternative theories of gravitation, following the seminal work by Morris, Thorne[1] in which they introduced the concept of traversable wormholes and also obtained the properties that a spacetime must have to hold up such geometry. Though the concept of wormhole came much earlier[2] as objects connecting different regions of spacetime but such wormholes were not traversable and thus were physically uninteresting. The idea of traversable wormholes opens up several possible interesting physical applications,[3,4] for instance wormholes may be used as time machines.[3] A basic fact, however, is that for traversability it is essential to thread the wormhole throat with matter that violates the averaged null energy condition (ANEC). Most discussions of such exotic matter involve quantum field theory effects, such as the Casimir effect or Hawking evaporation. But, the quantum inequalities satisfied by the exotic matter fields tightly constrain the geometry of the wormhole by confining the exotic matter in a thin shell of size only slightly larger than the Planck length at the throat of the wormhole[5] thus essentially preventing the traversability.

The attempts to get around ANEC violation have led to increasing number of works in nonstandard gravity theories such as in the Brans–Dicke theory,[6–9] $R+R^2$ theory,[10] Einstein–Gauss–Bonnet,[11] Einstein Cartan model,[12] Kaluza–Klein[13] or in a Brane world scenario.[14] Though violation of ANEC is inevitable for traversability but some of these alternative theories allow one to use normal matter while relegating the exocity to nonstandard fields. The study of wormhole geometry in the Brans–Dicke (BD) gravity theory,[15] which describes gravitation through a spacetime metric ($g_{\mu\nu}$) and a massless scalar field (φ) that comes up naturally in most theoretical attempts at unifying gravity with other interactions or at quantizing gravity, receives special attention as the theory admits static wormholes both in vacuum[6–8] and with matter content that do not violate the ANEC by itself.[9] In this theory scalar field itself plays the role of the exotic matter and since it is a classical field, Roman–Ford restriction on the size of the traversable region is not applicable in this case. Although there are some ambiguities in the definition of energy density,[16] it is generally considered that the energy density of scalar fields is not positive-definite in the BD theory.[17] But very recently it has been shown[18] that the Minkowski space is stable in this theory with respect to inhomogeneous scalar and tensor perturbations, at least at the linear order. This means negative energy associated with the scalar–tensor gravitational waves does not cause runway solutions at the classical level in the BD theory.

The study of wormhole geometry in the BD theory has been initiated by Agnese and La Camera.[6] They have shown that the static spherically symmetric vacuum solution of the BD theory, which is often referred to as Brans class I solution, gives rise to a two-way traversable wormhole for $\omega < -2$ where ω is the characteristic coupling constant of the theory. Since Birkhoff's theorem does not hold in the presence of a scalar field, several static solutions of the BD theory is possible even in spherically symmetric vacuum situation. Brans himself provided[19] four forms of static spherically symmetric vacuum solution of the BD theory (however as far as we know no other spherically symmetric solution that describes correctly the weak field observations is available in the literature). Among all the Brans classes of solutions, class I solution receives more attraction as it is the only one which is permitted for all values of ω. The other three forms are valid only for $\omega < -3/2$ which implies non-positive contribution of matter to effective gravitational constant and thus a violation of the ANEC. Extending the work of Agnese and La Camera, Nandi *et al.* showed[7] that several other Brans classes of solutions also support wormhole geometry. They further pointed out that Brans class I solution admits wormhole geometry even when ω is positive. However, it has been shown recently[20] that only two of the Brans solutions are really independent and only class I solution represents exterior metric for a spherical gravitating object. Hence wormhole geometries corresponding to other classes of Brans solution though are mathematically viable but physically irrelevant; only wormhole geometry corresponding to Brans class I solution is physically meaningful. Recently He and Kim[8] have claimed for two *new* static vacuum wormhole solutions of the BD theory. But as already shown in Refs. 20 and 21, these two classes of solutions are essentially limiting cases of the Brans class I solution. Moreover, He–Kim classes of solutions do not satisfy all the standard weak-field observational results of gravitation. However, an important point of their analysis is that they have also considered the usability criteria[8] for effective traversability. Such a study has not yet been considered in literatures in the context of the general and physical viable class of solution (Brans class I solution) of the BD theory.

For a wormhole to be traversable not just in principle but in practice it has to satisfy several usability criteria. For convenient travel tidal gravitational forces a traveler feels must be bearably small, acceleration that the traveler experiences should not exceed much that of earth gravity as well as the time of journey to cross through the wormhole must also be finite and reasonable. In this paper we would like to study wormhole geometry in vacuum BD theory considering both traversability and usability conditions as prescribed in Ref. 1. We shall also impose the basic conditions of physical viability of the solution by demanding that the solution should represent external gravitational field for nonsingular spherical massive object and it must be consistent with the observational results. The present paper is organized as follows. After giving a short account of the BD theory and its static spherically symmetric solutions, physical viability of all Brans solutions will be discussed in Sec. 2. In Sec. 3 wormhole nature of the Brans solutions will be discussed by imposing traversability conditions. In Sec. 4 we examine the usability of the wormhole under study for traveling to distant parts of the universe. Finally the results are discussed in Sec. 5.

2. Physically Viable Spherical Symmetric Vacuum Solutions of the BD Theory

In the BD theory, which accommodates both Mach's principle and Dirac's large number hypothesis, the scalar field acts as the source of the (local) gravitational coupling with $G \sim \varphi^{-1}$ and consequently the gravitational "constant" is not in fact a constant but is determined by the total matter in the universe through an auxiliary scalar field equation. The scalar field couples to both matter and spacetime geometry and the strength of the coupling is represented by a single dimensionless constant ω. The theory is consistent with the (local) observations only when ω is very large. A lower limit $|\omega| > 5 \times 10^4$ is imposed from the recent conjunction experiment with Cassini spacecraft.[22] Here it should be mentioned that in the limit $|\omega| \to \infty$, the BD theory (and its dynamic generalization) reduces to GR unless the energy–momentum tensor is traceless.[23]

In the Jordan conformal frame, the BD action takes the form (we use geometrized units such that $G = c = 1$ and we follow the signature $-, +, +, +$)

$$\mathcal{A} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\varphi R + \frac{\omega}{\varphi} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} + \mathcal{L}_{\text{matter}} \right), \tag{1}$$

where $\mathcal{L}_{\rm matter}$ is the Lagrangian density of ordinary matter. Variation of (1) with respect to $g^{\mu\nu}$ and φ gives, respectively, the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi}{\varphi}T_{\mu\nu} - \frac{\omega}{\varphi^2}\left(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{\mu\nu}\varphi^{,\sigma}\varphi_{,\sigma}\right)$$

$$- \frac{1}{\varphi}(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi)\,, \tag{2}$$

$$\Box\varphi = \frac{8\pi T}{(2\omega+3)}\,, \tag{3}$$

where R is the Ricci scalar, and $T = T^\mu_\mu$ is the trace of the matter energy momentum tensor.

As stated earlier, though there are four Brans classes of static spherically symmetric solutions of the above theory when $T_{\mu\nu} = 0$, only two of them are actually independent. The Brans class I solution (in isotropic coordinates) is given by

$$ds^2 = \left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{2/\lambda} dt^2$$

$$- \left(1 + \frac{B}{\rho}\right)^4 \left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{\frac{2(\lambda - C - 1)}{\lambda}} (d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2)\,, \tag{4}$$

$$\varphi = \varphi_0\left(\frac{1 - B/\rho}{1 + B/\rho}\right)^{C/\lambda}, \tag{5}$$

where B, C, λ are arbitrary constants and the parameters are connected through the constraint

$$\lambda^2 = (C+1)^2 - C\left(1 - \frac{\omega C}{2}\right). \tag{6}$$

On the other hand, the class IV solution reads

$$ds^2 = e^{-2/(B\rho)}\, dt^2 - e^{2(C+1)/(B\rho)}(d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2)\,, \tag{7}$$

$$\varphi = \varphi_0 e^{C/(B\rho)}\,, \tag{8}$$

with

$$C = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2}\,. \tag{9}$$

Choice of imaginary B and λ in class I solution leads to the Brans class II solution.[20] A point to be noted is that under these choices the solution becomes regular at all points including the point $r = B$ and consequently the (class II) solution does not possess any horizon. Brans class III and class IV solutions are also not different[20,24]; under a mere redefinition of the radial variable ($\rho \equiv 1/\rho$) one of them maps to another. Both class I and class IV solutions are compatible with all the standard (up to the first post-Newtonian order) experimental tests of gravity conducted till now. The class I solution, which is the best known spherical symmetric solution of the BD theory, (in the Einstein conformal frame the corresponding solution is the well-known Buchdahl solution[25] which is also variously referred[26] to as JNW[27] or Wyman solution[28]) in general gives rise to naked singularity whereas class IV solution is supposed to give rise to the so-called cold black hole.[29]

In the quest for the physical viability of the solution we shall examine whether they match with the interior solution of the theory due to any reasonable spherical distribution of matter.[20] In general relativity the metric tensor is the only gravitational field variable. Hence, it is sufficient and necessary to match the interior and exterior solutions for metric tensor only. In contrast BD theory has additional scalar field which contributes to the gravitational field as well. Therefore, in this theory not only matching for metric tensor is necessary but also for the additional scalar field.[30] This is because the scalar field in Brans–Dicke theory (and also for its dynamic generalization) represents strength of gravitational field and locally measurable value of gravitational constant G is a function of background scalar field φ. Since at the boundary there can be only one measured value of G, the interior and exterior scalar field has to be the same there.

Moreover, metric tensor depends on the scalar field. For example the relation $(\ln\varphi)_{,i} = K(\ln g_{oo}^{1/2})_{,i}$,[31] where K is a constant, holds for static spherically symmetric solutions of the Brans–Dicke theory. This relation was derived by taking energy–momentum tensor of matter to that of a perfect fluid and the relation is unique provided the spacetime is asymptotically flat and $\frac{\varphi}{g_{oo}}$ tends uniformly to a limit at infinity and its second derivative exists everywhere.[31] So mismatching of background field at the boundary surface results mismatching of metric and hence of geometry.

It follows from Eq. (3) that to the leading order in $1/r$ the interior (in presence of matter) scalar field satisfies the equation

$$\nabla^2 \overset{2}{\varphi} = -\frac{8\pi}{2\omega+3}\overset{o}{T}, \tag{10}$$

where we have expanded scalar field as $\varphi = \varphi_o + \overset{2}{\varphi} + \overset{4}{\varphi} + \cdots, \overset{N}{\varphi}$ denotes the term in φ of order $\frac{1}{r^{N/2}}$ (here we have followed the notation of Ref. 32) and $\overset{o}{T}$ denotes the term in T^σ_σ of order $\frac{1}{r^3}$ ($\overset{o}{T}{}^{oo}$ is the density of rest mass).[32] Therefore to the leading order in $1/r$ (r being the radial variable) the scalar field near the surface of the matter distribution has the following expression

$$\varphi = \varphi_o - \left(\frac{2}{2\omega+3}\right)\phi, \tag{11}$$

where ϕ is the Newtonian potential defined through $\nabla^2\phi = 4\pi\,\overset{o}{T}$. At the surface and outside the source $\phi = -M/r$ where M is the total mass (a positive definite quantity) of the source as viewed by a distant observer. Utilizing the relation[32] $G = \frac{2\omega+4}{2\omega+3}\frac{1}{\varphi_o}$ where G is the gravitational constant that will be measured in a real experiment, we finally get the expression for scalar field to the leading order in $1/r$ near the surface of the matter distribution

$$\varphi = \varphi_o\left(1 + \frac{1}{\omega+2}\frac{M}{r}\right). \tag{12}$$

A physically viable external solution for scalar field must match smoothly with the above expression at the surface.

When matching to two different solutions on a common surface it is essential to choose an appropriate coordinate system. The expression (12) is written in standard coordinates (t, r, θ and ϕ). Hence the expression for scalar field in Brans class I or class IV solution needs to transform first in the standard coordinates for effective comparison. However, at the first order the standard and isotropic coordinates produce identical effects. Hence comparing the expression for scalar field of Brans class I solution with Eq. (12) we get

$$C = -\frac{1}{\omega+2}. \tag{13}$$

Here we have employed $2B/\lambda = M$ which is obtained from the relation $\overset{2}{g}_{oo} = 2\phi$. Consequently the constraint condition (6) gives

$$\lambda = \sqrt{\frac{2\omega+3}{2\omega+4}}. \tag{14}$$

Thus Brans class I solution (including class II variant) may represent external gravitational field due to a reasonable matter field only when the parameters C and λ are given respectively by Eqs. (13) and (14). On the other hand, scalar field of class IV solution can not be matched with interior solution at the surface as its first-order term goes as $\omega^{-1/2}$ for large ω whereas boundary condition requires that it must be proportional to ω^{-1}. Thus regularity conditions at the boundary suggest that class IV solution does not describe exterior gravitational field for a nonsingular spherical massive object.

3. Brans Wormholes

Agnese and La Camera[6] *already studied wormhole nature of Brans class I solution* exactly with the choice of C as given by Eq. (12). They use post-Newtonian values of the BD theory to fix the parameter C. However, they did not consider imaginary values of the solution parameters. On the other hand, Nandi *et al.*[7] studied all classes of solutions but they did not apply any restriction on the parameters except the constraint condition (6) and the requirement of obeying Newtonian result in the weak field limit such that $2B/\lambda = M$. Thus the relaxed condition on the parameter ω as obtained by them for holding up wormhole geometry is though mathematically correct but physically unacceptable. But more importantly none of these works consider the usability of the BD wormhole. Hence for completeness we re-investigate the problem below.

3.1. *Traversable BD wormhole*

To be a wormhole, the solution must have a throat that connects two asymptotically flat regions of spacetime. To examine whether a throat exists or not it is of convenience to cast the metric into the Morris–Thorne canonical form[1]

$$ds^2 = e^{2\chi(R)}\,dt^2 - \left[1 - \frac{b(R)}{R}\right]^{-1} dR^2 - R^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{15}$$

where R is the new radial coordinate, $\chi(R)$ is known as the redshift function and $b(R)$ is called as shape function.

The class I solution can be cast to the above form by defining a radial coordinate R which is related with r via the expression

$$R = r\left(1 + \frac{B}{r}\right)^{1+(C+1)/\lambda}\left(1 - \frac{B}{r}\right)^{1-(C+1)/\lambda}. \tag{16}$$

The functions $\chi(R)$ and $b(R)$ are given by[6,7]

$$\chi(R) = \frac{1}{\lambda}\ln\left[\frac{1 - B/r(R)}{1 + B/r(R)}\right], \tag{17}$$

$$b(R) = R\left[1 - \left(\frac{r^2(R) + B^2 - 2r(R)B(C+1)/\lambda}{r^2(R) - B^2}\right)^2\right]. \tag{18}$$

The axially symmetric embedded surface $z = Z(R)$ shaping the wormhole's spatial geometry is obtained from

$$\frac{dz}{dR} = \pm\left[\frac{R}{b(R)} - 1\right]^{-1/2}. \tag{19}$$

By definition of wormhole at throat its embedded surface is vertical. Hence the expression for the throat of the BD wormhole, which occurs at $R = R_o$ such that $b(R_o) = R_o$, is given in r-coordinate by

$$r_o^{\pm} = \frac{B}{\lambda}[C + 1 \pm \sqrt{(C+1)^2 - \lambda^2}]. \tag{20}$$

The choice (13) leads the above equation as

$$r_o^{\pm} = \frac{M}{2(\omega+2)}[\omega + 1 \pm \sqrt{-(3\omega+4)/2}]. \tag{21}$$

The throat radius thus becomes real when $\omega < -4/3$. However, positivity of the throat radius requires ω to be less than $-3/2$. An interesting range is $-2 < \omega < -1.5$ for which λ becomes imaginary. But as shown in Ref. 20 imaginary λ together with imaginary B lead to Brans class II solution. This range thus gives rise to viable wormhole geometry. The redshift function has a singularity at $r = B$ which corresponds to the point $R = 0$. Hence traversability requires $R_o > 0$ or equivalently $r_o > B$. This condition is satisfied when r_o^+ is the throat radius. The class I solution (including the class II variant) thus represents traversable two-way wormhole with throat radius r_o^+ when $\omega < -3/2$ excluding the value $\omega = 2$.

The expressions for wormhole geometry corresponding to the class II solution can be easily obtained from those of class I solution by replacing B and λ by iB and $-i\Lambda$ respectively. This gives

$$\chi(R) = \frac{2}{\Lambda}\tan^{-1}(B/r)\,, \tag{22}$$

$$b(R) = R\left[1 - \left(\frac{r^2(R) - B^2 + 2r(R)B(C+1)/\Lambda}{r^2(R) + B^2}\right)^2\right] \tag{23}$$

and

$$r_o^{\pm} = \frac{B}{\Lambda}[-(C+1) \pm \sqrt{(C+1)^2 + \Lambda^2}]\,. \tag{24}$$

These are the expressions derived in Ref. 7 (the expression for shape function in Ref. 7 contains a minor error which is probably due to printing mistake) starting from *ab initio.* Unlike class I solution here throat radius is always real for any choice of Λ and C. The ω dependence of C (13) leads to the following expression for throat radius

$$r_o^{\pm} = \frac{M}{2(\omega+2)}[(\omega+1) \mp \sqrt{-(3\omega+4)/2}] \tag{25}$$

(here Newtonian limit leads to the choice $2B/\lambda = -M$). As in the case of class I solution here also the throat radius becomes real when $\omega < -4/3$ and becomes also positive only when $\omega < -3/2$. However, the range $\omega < -2$ makes λ imaginary which in turns (together with imaginary B) maps the solution to class I solution and hence no new wormhole geometry is available. So class II solution only supports wormhole geometry when $-2 < \omega < -3/2$.

3.2. *He–Kim classes of solutions*

The claimed new class I solution in Ref. 8 is essentially a limiting case of the Brans class II solution that can be obtained with the choice $\Lambda \to \infty$ and $\frac{C}{\Lambda} \equiv C$ as already shown in Ref. 21 and the expressions for the shape and redshift functions and the radius of the throat thus follow from Eqs. (22)–(24). The expression for the redshift function as given by Eq. (17) of Ref. 8 contains a sign anomaly which in turns affected the expression (Eq. (18) of Ref. 8) for the radius of the throat. The correct expressions for the redshift function and radius of the throat for the He–Kim class I solution would be

$$b(R) = R\left[1 - \left(\frac{(r^2(R) - B^2) + 2r(R)BC}{r^2(R) + B^2}\right)^2\right] \tag{26}$$

and

$$r_o^{\pm} = BC\left[-1 \pm \left(1 + \frac{1}{C^2}\right)^{1/2}\right]\,. \tag{27}$$

But anyway these solutions do not represent exterior gravitational field due to any reasonable spherically symmetric matter distribution and hence not physically viable.

4. Usability

As mentioned already for practical traversability across a wormhole several conditions, as prescribed in Ref. 1, have to comply which imposes restrictions on the geometry of the wormhole. Here additional restriction comes from the fact that the theory is consistent with the (local) observations only when ω is very large.

4.1. *Feasible throat radius*

The throat radius has to be reasonably large for passage through wormhole. Insisting that R_o should be at least of the order of 1 m we get the following condition on ω from Eqs. (20) invoking Eqs. (13) and (14) in the limit of large ω

$$\omega \leq 1.3 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 . \tag{28}$$

So feasible throat radius demands ω to be bounded from the upper and the upper limit depends on the mass of the wormhole. The present observational restriction on ω thus sets a lower limit on the mass of the traversable BD wormhole.

4.2. *Tidal forces*

Major constraints on wormhole geometry come from the tidal forces a traveler feels while traveling across the wormhole. The tidal acceleration between two extreme parts (e.g. head to feet) of the traveler's body is given by[1]

$$\Delta a^i = -R^{i'}_{0'k'0'}\xi^{k'} , \tag{29}$$

where ξ is the vector separation between two extreme parts of the traveler's body and $R^{i'}_{0'k'0'}\xi^{k'}$ (latin indices represents spatial components) are the components of the Riemann curvature tensor in the traveler's frame (denoted as primed) which are connected with those of the static observer's frame as follows:

$$R^{1'}_{0'1'0'} = R^1_{010} \tag{30}$$

and

$$R^{2'}_{0'2'0'} = R^{3'}_{0'3'0'} = \gamma^2 R^2_{020} + \gamma^2 v^2 R^2_{121} . \tag{31}$$

Here $v(R)$ is the radial velocity of the traveler when he/she passes the radial point r, as measured by a static observer there, γ is the usual Lorentz factor ($\gamma^2 \equiv \frac{1}{1-v^2}$). For a convenient wormhole travel by human beings ($\xi \sim 2$ m) tidal accelerations should not exceed much that of earth gravity ($g_\oplus$), i.e.

$$|R^{1'}_{0'1'0'}| \le \frac{1}{(10^8\ \mathrm{m})^2}, \tag{32}$$

$$|R^{2'}_{0'2'0'}| \le \frac{1}{(10^8\ \mathrm{m})^2}. \tag{33}$$

The constraint (32) arising from radial tidal acceleration restricts the wormhole geometry whereas the constraint (33) which is due to lateral tidal acceleration imposes restriction on the velocity of the traveler. In the static observer's frame the relevant nonvanishing components of the Riemann tensor for the Brans class I metric are given by

$$R_{0101} = -\frac{4Br^3\left(\frac{r-B}{r+B}\right)^{\frac{2(C+1)}{\lambda}}((B^2+r^2)\lambda-(C+2)Br)}{(r-B)^4(r+B)^4\lambda^2}, \tag{34}$$

$$R_{0202} = \frac{2Br^3\left(\frac{r-B}{r+B}\right)^{\frac{2(C+1)}{\lambda}}((B^2+r^2)\lambda-2(C+1)Br)}{(r-B)^4(r+B)^4\lambda^2} \tag{35}$$

and

$$R_{1212} = \frac{2Br^3\left(\frac{r-B}{r+B}\right)^{\frac{2(C+1)}{\lambda}}((C+1)(B^2+r^2)-2Br\lambda)}{(r-B)^4(r+B)^4\lambda}. \tag{36}$$

For physical viability the parameters C and λ are to be substituted by Eqs. (13) and (14).

It may appear from the expression of $R_{1'0'1'0'}$ that due to the presence of $(r-B)^4$ in the denominator, the tidal acceleration would become very large at throat but since observations already constrain $1/\omega$ to a very small value, $R_{1'0'1'0'}$ effectively tends (excluding the exact point $r=B$) to GR expression $-\frac{4Br^3}{(r+B)^6}$ which is finite. Thus the condition (32) leads to the constraint $M > 10^4 M_\odot$. Hence mass of the wormhole must be very large for effective traversability.

On the other hand the condition, (33) gives

$$\gamma^2\left[\frac{2Br^3\left(\frac{r-B}{r+B}\right)^{\frac{2(C+1)}{\lambda}}}{(r-B)^4(r+B)^4\lambda}\right][(B^2+r^2)(\lambda-v^2(C+1))-2Br(C+1-v^2\lambda)]$$
$$\le \frac{1}{(10^8\ \mathrm{m})^2}. \tag{37}$$

As before in the limit of large ω the left-hand side of the above equation effectively reduces to $\frac{2Br^3}{(r+B)^6}$ which is finite and is interestingly independent of the velocity of the traveler. Hence essentially there is no restriction on the velocity of the traveler from tidal accelerations.

4.3. *Other dynamical constraints*

The acceleration that a traveler would sense (note that traveler is not freely falling) in passing through a gravitational field is given by[1]

$$a = g_{oo}^{-1} \frac{d(\gamma g_{oo})}{dl} . \tag{38}$$

The demand that traveler should not feel acceleration greater than about 1 Earth gravity while traveling through the wormhole leads to the following condition for the BD wormhole

$$\frac{(r^2+B^2)\lambda - 2rB(C+1)}{\lambda(r^2-B^2)} \left[\gamma \frac{2B}{\lambda(r^2-B^2)} - \gamma' \right] \leq 1.1 \times 10^{-16}\ \mathrm{m}^{-1} \tag{39}$$

(here prime denotes the derivative with respect to r). The acceleration would thus be very large particularly near the throat and for large ω the above condition demands $\gamma(M/M_{\odot})^{-1} \leq 2 \times 10^{-12}$, i.e. mass of the wormhole needs to be abnormally large.

For practical traversability the total journey time to cross through the wormhole must be finite and reasonably small as measured with the traveler as well as the persons waiting outside the wormhole. The journey time measured by people who live in the stations

$$\delta t = \int_{-l_1}^{l_2} \frac{dl}{v g_{oo}} , \tag{40}$$

where l_1 and l_2 coordinates of the space-stations. It is required that this time is of the order of 1 year. Certainly this condition cannot be met when the mass of the wormhole is abnormally large ($\sim 10^{12} M_{\odot}$). In that case space stations, where geometry of the spacetime must be nearly flat and acceleration of the gravity should be at maximum of the order of gravity, would be at large distances ($\sim 10^7$ pc) from the wormhole throat and even by moving with the speed of light the traveler would not be able to reach one station from another within a reasonable period of time.

5. Discussion

Recently a measure of quantifying exotic matter needed for traversable wormhole geometry has been advanced[33] in the context of general relativity. It appears that no such quantifying measure is really required in this case. As is evident from the foregoing analysis that the existence of traversable wormhole geometry for class I solution entirely depends on the mass and coupling constant ω. As $1/\omega$ goes to zero the BD theory converges to the general relativity (leaving aside the case of traceless matter source), class I solution tends to Schwarzschild solution and two-way traversability disappears. But existence of scalar field (which is pre-assumed in the BD theory) with very small (bounded by Eq. (40)) negative $1/\omega$ is sufficient to give rise (in principle) to traversable wormhole in the BD theory.

Traversability through wormhole requires violation of ANEC and hence exotic matter. Since all standard energy conditions can be violated easily at the classical level in a theory involving scalar field that coupled non-minimally with the spacetime[34] (in fact it has been shown recently[35] that a spontaneous violation of the energy conditions occurs for a wide class of scalar–tensor theories as a consequence of spontaneous scalarization[30] thus providing a natural way by which normal matter could be transformed into exotic, in the interior of neutron stars), BD theory admits traversable wormholes for particular choices of parameters. An important question is that whether BD vacuum wormholes are suited for interstellar travel or not. It has been shown above that the practical traversability restricts the mass of the wormhole and coupling parameter. The reasonable throat radius, which is required for passage through wormhole, demands ω to be bounded from upper. This upper limit depends on the mass of the wormhole and the present observational restriction on ω does not rule out a journey through BD wormholes. The tidal forces a traveler would feel while traveling across the wormhole constrains the mass of the wormhole to a large value, at least $10^4 \times M_\odot$. But most severe restriction on mass of the wormhole comes from the acceleration that a traveler would experience while approaching the throat of the wormhole. This results in a lower bound of the mass of the wormhole as $M > 5 \times 10^{12} M_\odot$. Consequently the time to cross through the wormhole would be very large. The present investigation, thus, shows that vacuum BD wormhole is though traversable in principle but not suitable for interstellar travel.

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