CHAPTER - 4

ELASTODYNAMIC GREEN'S FUNCTION

[6]ELASTODYNAMIC GREEN'S FUNCTION FOR TIME-HARMONIC RING SOURCE IN AN INFINITE ANISOTROPIC MEDIUM

1. INTRODUCTION

The study of the scattering of the elastic waves by cracks or inclusions in an elastic medium is of considerable importance in view of its application in the quantitative nondestructive evaluation of materials. When there are only axially symmetric scatterers like circular cracks or inclusions, the incidence of torsional waves results in torsional waves only where as the incidence of logitudinal waves gives rise longitudinal wave consisting of axial and radial components of the displacement which are coupled. The torsional wave modes are useful in detecting longitudinal flaws in cylindrical materials and longitudinal wave modes are useful for detecting transverse flaws and corrosion defects.

Keeping this fact in view different investigators studied diffraction of elastic waves by cirdular cracks and inclusions in different times. Work of Kundu and Bostrom [1991], Kundu [1990], Krenk and Schmidt [1982], Schmidt and Krenk [1982] may be mentioned in this connection.

While studing elastodynamic scattering due to axisymmetric scatter, calculation can be greatly simplified if an elastodynamic Green's function for ring source is available and used.

In our paper the elastodynamic Green's function for a torsional ring source as well as for a axial and a radial ring source in a transversely isotropic solid has been obtained. The axis of material

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symmetric and the axis of the ring source have been assumed to coincide. The Green's functions are derived using Fourier and Hankel transforms in the integral form. Using stationary phase method, the far field displacement has been derived and depicted by means of graphs in the case of Graphite-epoxy composite. Recently, the elastodynamic Green's function for a circular ring source in homogeneous isotropic elastic medium has been derived by Lu [1998, 1999].

2. RING SOURCE OF FORCE IN RADIAL DIRECTION

Consider a cylindrical polar co-ordinate system r, θ, z with the cetre of the ring source on the z-axis so that the ring source of radius $r = r'$ is located at $z = z'$. The time dependence of all the field quantities, assumed to be of the form exp(-iwt), will be suppressed in the subsequent development.

The material is assumed to be a unidirectionally fiber-reinforced composite solid whose fiber diameter is small compared to the wave length so that one can consider the material as a transversely isotropic solid. The fiber direction is parallel to the z-axis.

Since the problem is axisymmetric for radial ring source of force, so the only non-vanishing displacement components are $u_{rr}(r,z)$ and $u_{zr}(r,z)$ where the first subscript denotes the direction of the displacement and the second one the direction of the ring source of force. The equations of motion for the probleme are

$$c_1\left(\frac{\partial^2 u_{rr}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{rr}}{\partial r}-\frac{u_{rr}}{r^2}\right)+c_3\frac{\partial^2 u_{zr}}{\partial r\partial z}+c_5\left(\frac{\partial^2 u_{zr}}{\partial r\partial z}+\frac{\partial^2 u_{rr}}{\partial z^2}\right)+\rho\omega^2 u_{rr}=-\frac{\delta(r-r')\,\delta(z-z')}{r} \tag{2.1}$$

and

$$(c_3 + c_5)\left(\frac{\partial^2 u_{rr}}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_{rr}}{\partial r}\right) + c_5\left(\frac{\partial^2 u_{zr}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{zr}}{\partial r}\right) + c_4\frac{\partial^2 u_{zr}}{\partial z^2} + \rho\omega^2 u_{zr} = 0 \qquad (2.2)$$

where ρ is the material density and the term $\dfrac{\delta(r-r')\,\delta(z-z')}{r}$ represent the radial ring source of force.

We apply the Fourier transform to the axial co-ordinate, z, and the Hankel transform to the radial co-ordinate, r. The Fourier transform is defined as

$$U^*(r,\eta;r',z') = \int_{-\infty}^{\infty} U(r,z;r',z')\, e^{-i\eta z}\, dz \qquad (2.3a)$$

$$U(r,z;r',z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(r,\eta;r',z')\, e^{i\eta z}\, d\eta \qquad (2.3b)$$

while the Hankel transform is defined as

$$\bar{U}^*_{rr}(\xi,\eta;r',z') = \int_0^{\infty} U^*_{rr}(r,\eta;r',z')\, J_1(\xi r)\, r\, dr \qquad (2.4a)$$

$$U^*_{rr}(r,\eta;r',z') = \int_0^{\infty} \bar{U}^*_{rr}(\xi,\eta;r',z')\, J_1(\xi r)\, \xi\, d\xi \qquad (2.4b)$$

$$\bar{U}^*_{zr}(\xi,\eta;r',z') = \int_0^{\infty} U^*_{zr}(r,\eta;r',z')\, J_0(\xi r)\, r\, dr \qquad (2.5a)$$

$$U^*_{zr}(r,\eta;r',z') = \int_0^{\infty} \bar{U}^*_{zr}(\xi,\eta;r',z')\, J_0(\xi r)\, \xi\, d\xi \qquad (2.5b)$$

where η and ξ are transform parameters and $J_0(z)$ and $J_1(z)$ are the Bessel functions of the first kind of order zero and one respectively.

Using Fourier transform with respect to z to equations (2.1) and (2.2) one obtains,

$$c_1\left(\frac{\partial^2 u^*_{rr}}{\partial r^2} + \frac{1}{r}\frac{\partial u^*_{rr}}{\partial r} - \frac{u^*_{rr}}{r^2}\right) + i\eta(c_3 + c_5)\frac{\partial u^*_{zr}}{\partial r} - c_5\eta^2 u^*_{rr} + \rho\omega^2 u^*_{rr} = -\frac{\delta(r-r')\, e^{-i\eta z'}}{r} \qquad (2.6a)$$

$$i\eta(c_3+c_5)\left(\frac{\partial u_{rr}^*}{\partial r}+\frac{u_{rr}^*}{r}\right) + c_5\frac{1}{r}\left[\frac{\partial}{\partial r}(r\frac{\partial u_{zr}^*}{\partial r})\right] - c_4\eta^2 u_{zr}^* + \rho\omega^2 u_{zr}^* = 0 \qquad (2.6b)$$

Use of equation (2.5b) and (2.5a) to equations (2.6a) and (2.6b) respectively yield

$$(\rho\omega^2 - c_1\xi^2 - c_5\eta^2)\bar{u}_{rr}^* - i\xi\eta(c_3+c_5)\bar{u}_{zr}^* = -J_1(\xi r')e^{-i\eta z'} \qquad (2.7a)$$

and

$$i\xi\eta(c_3+c_5)\bar{u}_{rr}^* + (\rho\omega^2 - c_5\xi^2 - c_4\eta^2)\bar{u}_{zr}^* = 0. \qquad (2.7b)$$

Solving equations (2.7a) and (2.7b) one obtains

$$\bar{u}_{rr}^* = -\frac{(\rho\omega^2 - c_4\eta^2 - c_5\xi^2)}{c_4c_5(\eta^2-\eta_1^2)(\eta^2-\eta_2^2)}J_1(\xi r')e^{-i\eta z'} \qquad (2.8a)$$

$$\bar{u}_{zr}^* = \frac{i\xi\eta(c_3+c_5)}{c_4c_5(\eta^2-\eta_1^2)(\eta^2-\eta_2^2)}J_1(\xi r')e^{-i\eta z'} \qquad (2.8b)$$

where

$$\eta_j^2 = \frac{1}{2c_4c_5}\left\{b_2 - b_1\xi^2 + (-1)^j(B_1\xi^4 + B_2\xi^2 + B_3)^{1/2}\right\} \qquad (2.9a)$$

η_1^2 and η_2^2 are the roots of the quadratic equation in η^2 given by

$$c_4c_5\eta^4 + \left\{(c_1c_4 - c_3^2 - 2c_3c_5)\xi^2 - \rho\omega^2(c_4+c_5)\right\}\eta^2 + \left\{c_1c_5\xi^4 - \rho\omega^2(c_1+c_5)\xi^2 + \rho^2\omega^4\right\} = 0 \quad (2.9b)$$

and

$$\begin{aligned} b_1 &= c_1c_4 - c_3^2 - 2c_3c_5 \\ b_2 &= \rho\omega^2(c_4+c_5) \\ B_1 &= b_1^2 - 4c_1c_4c_5^2 \\ B_2 &= 4\rho\omega^2c_4c_5(c_1+c_5) - 2b_1b_2 \\ B_3 &= \rho^2\omega^4(c_4-c_5)^2. \end{aligned} \qquad (2.10)$$

From equation (2.9a) it can be shown that η_j (j=1,2) is equal to zero at $\xi=k_j$ where

$$k_1 = \omega\sqrt{\rho/c_1}\,, \qquad k_2 = \omega\sqrt{\rho/c_5} \qquad (2.11)$$

For $\xi < k_j$; η_j^2 is positive and η_j is taken to be a positive real number whereas for $\xi > k_j$; η_j^2 is negative and η_j is taken to be a positive imaginary number.

Now we apply equation (2.3b) to equations (2.8a) and (2.8b) to obtain

$$\bar{u}_{rr}(\xi,z,;r',z') = \frac{i}{2c_4c_5}\left[-\frac{(\rho\omega^2-c_5\xi^2-c_4\eta_1^2)}{\eta_1(\eta_1^2-\eta_2^2)}e^{i\eta_1|z-z'|}+\frac{(\rho\omega^2-c_5\xi^2-c_4\eta_2^2)}{\eta_2(\eta_1^2-\eta_2^2)}e^{i\eta_2|z-z'|}\right]J_1(\xi r') \tag{2.12a}$$

and

$$\bar{u}_{zr}(\xi,z,;r',z') = -\frac{\mathrm{sgn}(z-z')(c_3+c_5)}{2c_4c_5(\eta_1^2-\eta_2^2)}\left[e^{i\eta_1|z-z'|}-e^{i\eta_2|z-z'|}\right]\xi J_1(\xi r'). \tag{2.12b}$$

Taking Hankel inversion one obtains

$$u_{rr}(r,z;r',z') = \frac{i}{2c_4c_5}\sum_{j=1}^{2}(-1)^j\int_0^\infty \sqrt{\xi}F_j(\xi)e^{i\eta_j|z-z'|}J_1(\xi r)\,d\xi \tag{2.13a}$$

where

$$F_j(\xi) = \frac{\sqrt{\xi}(\rho\omega^2-c_5\xi^2-c_4\eta_j^2)}{\eta_j(\eta_1^2-\eta_2^2)}J_1(\xi r') \tag{2.13b}$$

and

$$u_{zr}(r,z;r',z') = \frac{\mathrm{sgn}(z-z')(c_3+c_5)}{2c_4c_5}\sum_{j=1}^{2}(-1)^j\int_0^\infty \sqrt{\xi}E(\xi)e^{i\eta_j|z-z'|}J_0(\xi r)\,d\xi \tag{2.14a}$$

where

$$E(\xi) = \frac{\xi^{3/2}J_1(\xi r')}{\eta_1^2-\eta_2^2}. \tag{2.14b}$$

3. FAR FIELD CALCULATION FOR RING SOURCE OF FORCE IN RADIAL DIRECTION

In the near field (small z and r), the integrals in equation (2.13a) and in equation (2.14a) can be evaluated numerically. However, in the far field, the numerical evaluation of these integrals is awkward because of the rapid oscillation introduced in the integral by the Bessel functions and trigonometric functions.

Assume the ring source to be situated at z=0-plane so that z'=0. Next for large ξr, taking the first term in the asymptotic expansion of the Bessel function

$$J_m(\xi r) \approx \frac{1}{\sqrt{2\pi\xi r}}\left[e^{i\xi r-i(2m+1)\pi/4}+e^{-i\xi r+i(2m+1)\pi/4}\right]. \tag{3.1}$$

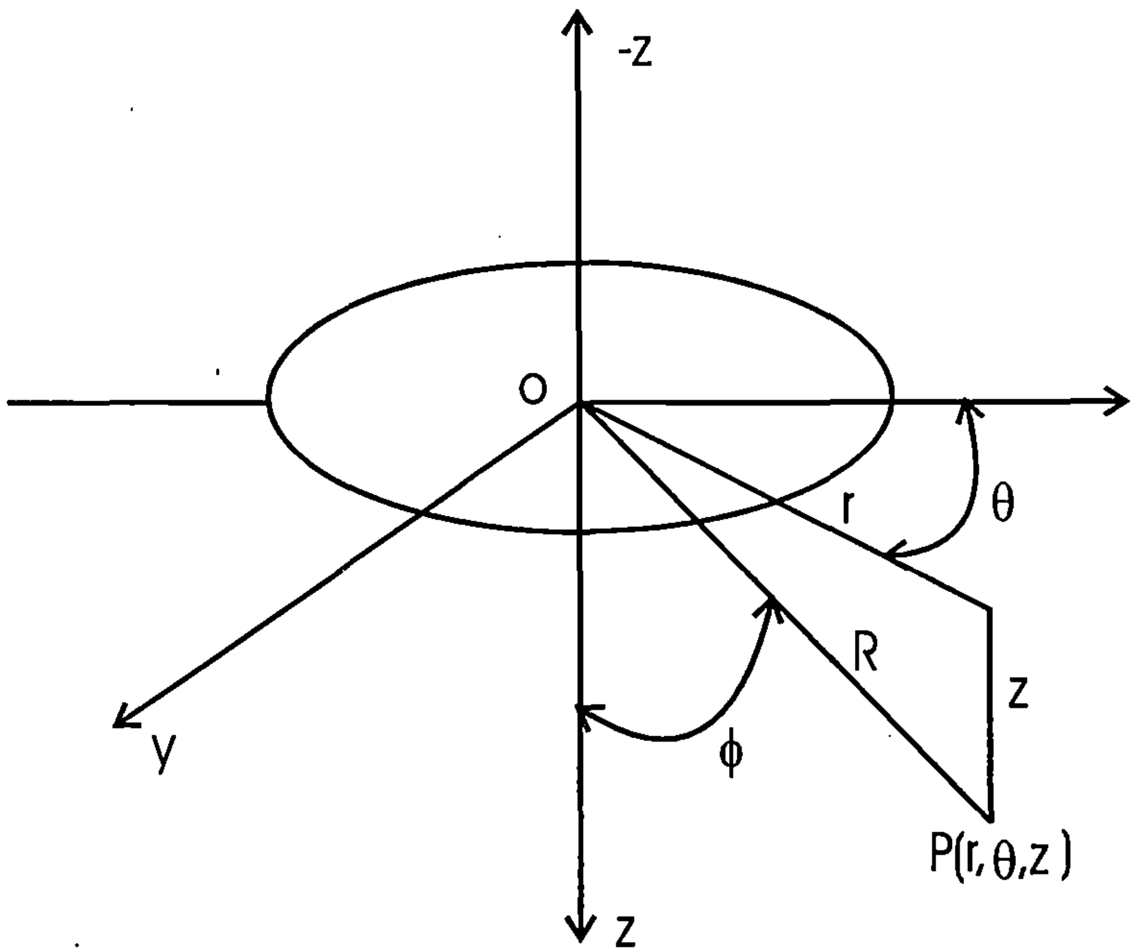

Fig.1. Schematic diagram of the problem geometry.

Displacement components given by equation (2.13a) and equation (2.14a) become,

$$u_{rr}(r,z;r',0) = \frac{i}{2c_4c_5\sqrt{2\pi r}}\sum_{j=1}^{2}(-1)^j\int_0^\infty F_j(\xi)\left[e^{-3\pi i/4}e^{i\eta_j|z|+i\xi r} + e^{3\pi i/4}e^{i\eta_j|z|-i\xi r}\right]d\xi \quad (3.2)$$

$$u_{zr}(r,z;r',0) = \frac{sgn(z)(c_3+c_5)}{2c_4c_5\sqrt{2\pi r}}\left[\sum_{j=1}^{2}(-1)^j\left\{e^{-i\pi/4}\int_0^\infty E(\xi)e^{i\eta_j|z|+i\xi r}d\xi + e^{i\pi/4}\int_0^\infty E(\xi)e^{i\eta_j|z|-i\xi r}\right\}\right] \quad (3.3)$$

So, oscillations in the integrand are introduced by the factors of the form $\exp(i\eta_j|z| \pm i\xi r)$. This type of oscillatory integrals can be evaluated by the method of stationary phase [1950] for large distance away from the source.

For this purpose, we substitute

$$|z| = R\cos\phi \quad \text{and} \quad r = R\sin\phi \quad (3.4)$$

where R is large; then

$$e^{i\eta_j|z|\pm i\xi r} = e^{iR(\eta_j\cos\phi \pm \xi\sin\phi)} = e^{iR\psi_j(\xi)} \text{ or } e^{iR\chi_j(\xi)} \quad (3.5)$$

where $\psi_j(\xi) = \eta_j\cos\phi + \xi\sin\phi$, $\quad\chi_j(\xi) = \eta_j\cos\phi - \xi\sin\phi$. (3.6)

With the help of equation (3.5), equation (3.2) and equation (3.3) can be expressed as

$$u_{rr}(r,z;r',0) = \frac{1}{2c_4c_5\sqrt{2\pi R\sin\phi}}\sum_{j=1}^{2}(-1)^j\left[e^{-i\pi/4}\int_0^\infty F_j(\xi)e^{iR\psi_j(\xi)}d\xi - e^{i\pi/4}\int_0^\infty F_j(\xi)e^{iR\chi_j(\xi)}d\xi\right] \quad (3.7)$$

and

$$u_{zr}(r,z;r',0) = \frac{sgn(z)(c_3+c_5)}{2c_4c_5\sqrt{2\pi R\sin\phi}}\sum_{j=1}^{2}(-1)^j\left[e^{-i\pi/4}\int_0^\infty E(\xi)e^{iR\psi_j(\xi)}d\xi + e^{i\pi/4}\int_0^\infty E(\xi)e^{iR\chi_j(\xi)}d\xi\right]. \quad (3.8)$$

The statonary points are given by $\psi_j'(\xi)=0$ and $\chi_j'(\xi)=0$ which yield,

$$\frac{2b_1\xi - (-1)^j(2B_1\xi^3 + B_2\xi)/B(\xi)}{[b_2 - b_1\xi^2 + (-1)^jB(\xi)]^{1/2}} = \pm\sqrt{8c_4c_5}\tan\phi \quad (3.9)$$

where $$B(\xi) = [B_1\xi^4 + B_2\xi^2 + B_3]^{1/2} \qquad (3.10)$$

and positive sign in the equation (3.9) on right hand side corresponds to $\psi_j'(\xi)=0$ whereas negative sign corresponds to $\chi_j'(\xi)=0$.

For an anisotropic material the roots of equation (3.9) can be solved numerically. We know that for Graphite-epoxy composite, c_1=13.92 Gpa, c_2=6.92 Gpa, c_3=6.44 Gpa, c_4=160.73 Gpa, c_5=7.07 Gpa, ρ=1578 kg/m^3, [1991]. For this type of material, it can be shown that the equation $\chi_j'(\xi)=0$ does not give any positive root of ξ. $\psi_1'(\xi)=0$ gives only one positive root whereas $\psi_2'(\xi)=0$ gives three positive roots for some values of ϕ and a single positive root for other values of ϕ. These roots are denoted by ξ_{01} and ξ_{02}^m respectively. For multiple values of ξ_{02} , the superscript m takes the values 1, 2, 3. The values of ξ_{01} and ξ_{02}^m for different values of ϕ are given in the following table :

Table-1. Roots of the equations $\psi_1'(\xi)=0$ and $\psi_2'(\xi)=0$.

ϕ	ξ_{01}	ξ_{02}^1	ξ_{02}^2	ξ_{02}^3
10°	0.5079	0.0971	0	0
15°	0.5845	0.1460	0	0
20°	0.6200	0.1952	0	0
25°	0.6402	0.2448	0	0
30°	0.6537	0.2948	0.8194	0.9172
35°	0.6636	0.3452	0.7995	0.9459
40°	0.6714	0.3958	0.7873	0.9626
45°	0.6779	0.4463	0.7780	0.9738
50°	0.6835	0.4964	0.7702	0.9815
55°	0.6885	0.5454	0.7630	0.9871
60°	0.6931	0.5927	0.7558	0.9913
65°	0.6974	0.6377	0.7478	0.9943
70°	0.7015	0.6815	0.7363	0.9965
75°	0.7054	0	0	0.9981

For far field computation we can neglect all the terms containing $e^{-i\xi r}$ in equation (3.7) and also in equation (3.8) because $\chi_j'(\xi) = 0$ does not give any positive root. Using the stationary-phase method the far field components of displacement are thus given by

$$u_{rr}(r,z;r',0) = \frac{e^{-i\pi/4}}{2c_4 c_5 R\sqrt{\sin\phi}}\left[\sum_{m=1}^{M} \frac{F_2(\xi_{02}^m)}{\sqrt{|\psi_2''(\xi_{02}^m)|}} \exp\left\{iR\psi_2(\xi_{02}^m) + \frac{i\pi}{4}\operatorname{sgn}\psi_2''(\xi_{02}^m)\right\}\right.$$

$$\left. - \frac{F_1(\xi_{01})}{\sqrt{|\psi_1''(\xi_{01})|}} \exp\left\{iR\psi_1(\xi_{01}) + \frac{i\pi}{4}\operatorname{sgn}\psi_1''(\xi_{01})\right\}\right] \tag{3.11}$$

and

$$u_{zr}(r,z;r',0) = \frac{\operatorname{sgn} z (c_3 + c_5) e^{-i\pi/4}}{2c_4 c_5 R\sqrt{\sin\phi}}\left[\sum_{m=1}^{M} \frac{E(\xi_{02}^m)}{\sqrt{|\psi_2''(\xi_{02}^m)|}} \exp\left\{iR\psi_2(\xi_{02}^m) + \frac{i\pi}{4}\operatorname{sgn}\psi_2''(\xi_{02}^m)\right\}\right.$$

$$\left. - \frac{E(\xi_{01})}{\sqrt{|\psi_1''(\xi_{01})|}} \exp\left\{iR\psi_1(\xi_{01}) + \frac{i\pi}{4}\operatorname{sgn}\psi_1''(\xi_{01})\right\}\right] \tag{3.12}$$

where

$$\psi_1'(\xi_{01}) = 0 \quad \text{and} \quad \psi_2'(\xi_{02}^m) = 0 \tag{3.13}$$

and M is the number of roots of the equation $\psi_2'(\xi) = 0$ and

$$\psi_j''(\xi) = -\frac{\cos\phi}{\sqrt{8c_4c_5}}\left[\left\{2b_1 - (-1)^j B^{-3}(\xi)\left(2B_1^2\xi^6 + 3B_1B_2\xi^4 + 6B_1B_3\xi^2 + B_2B_3\right)\right\}\right.$$

$$\times \left\{b_2 - b_1\xi^2 + (-1)^j B(\xi)\right\} + \frac{1}{2}\left.\left\{2b_1\xi - (-1)^j B^{-1}(\xi)(2B_1\xi^3 + B_2\xi)\right\}^2\right]$$

$$\times \left\{b_2 - b_1\xi^2 + (-1)^j B(\xi)\right\}^{-3/2} \tag{3.14}$$

where b_1, b_2, B_1, B_2, B_3, $B(\xi)$ have been defined previously.

4. RING SOURCE OF FORCE IN AXIAL DIRECTION

For an axial ring source of force, the equations of motion become

$$c_1\left(\frac{\partial^2 u_{rz}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{rz}}{\partial r}-\frac{u_{rz}}{r^2}\right)+c_3\frac{\partial^2 u_{zz}}{\partial r\partial z}+c_5\left(\frac{\partial^2 u_{zz}}{\partial r\partial z}+\frac{\partial^2 u_{rz}}{\partial z^2}\right)+\rho\omega^2 u_{rz}=0 \qquad (4.1)$$

and

$$(c_3+c_5)\left(\frac{\partial^2 u_{rz}}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_{rz}}{\partial z}\right)+c_5\left(\frac{\partial^2 u_{zz}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{zz}}{\partial r}\right)+c_4\frac{\partial^2 u_{zz}}{\partial z^2}+\rho\omega^2 u_{zz}=-\frac{\delta(r-r')\delta(z-z')}{r}. \qquad (4.2)$$

Using the similar procedure which has been applied previously, we obtain,

$$\bar{u}^*_{rz}(\xi,\eta;r',z') = -\frac{i\xi\eta(c_3+c_5)J_0(\xi r')e^{-i\eta z'}}{c_4c_5(\eta^2-\eta_1^2)(\eta^2-\eta_2^2)} \qquad (4.3)$$

$$\bar{u}^*_{zz}(\xi,\eta;r',z') = -\frac{(\rho\omega^2-c_1\xi^2-c_5\eta^2)J_0(\xi r')e^{-i\eta z'}}{c_4c_5(\eta^2-\eta_1^2)(\eta^2-\eta_2^2)}. \qquad (4.4)$$

Taking Fourier inversion and Hankel inversion we obtain finally,

$$u_{rz}(r,z;r',0) = \frac{\mathrm{sgn}(z)(c_3+c_5)}{2c_4c_5}\left[\sum_{j=1}^{2}\frac{(-1)^{j+1}}{\sqrt{2\pi r}}\left\{e^{-3\pi i/4}\int_0^\infty H(\xi)\,e^{i\eta_j|z|+i\xi r}d\xi + e^{3\pi i/4}\int_0^\infty H(\xi)e^{i\eta_j|z|-i\xi r}d\xi\right\}\right] \qquad (4.5)$$

where

$$H(\xi) = \frac{\xi^{3/2}J_0(\xi r')}{(\eta_1^2-\eta_2^2)} \qquad (4.6)$$

$$u_{zz}(r,z;r',0) = \frac{i}{2c_4c_5\sqrt{2\pi r}}\int_0^\infty\sum_{j=1}^{2}(-1)^j\,N_j(\xi)\left[e^{-\pi i/4}e^{i\eta_j|z|+i\xi r}+e^{\pi i/4}e^{i\eta_j|z|-i\xi r}\right]d\xi \qquad (4.7)$$

where

$$N_j(\xi) = \frac{\sqrt{\xi}(\rho\omega^2-c_1\xi^2-c_5\eta_j^2)}{\eta_j(\eta_1^2-\eta_2^2)}J_0(\xi r'). \qquad (4.8)$$

Using the method of stationary phase for evaluation of the integrals arising in equation (4.5) and (4.6) for large distance away from the ring source we have

$$u_{rz}(r,z;r',0) = \frac{\operatorname{sgn} z(c_3+c_5)e^{-3\pi i/4}}{2c_4c_5R\sqrt{\sin\phi}}\left[\frac{H(\xi_{01})}{\sqrt{|\psi_1''(\xi_{01})|}}\exp\left\{iR\psi_1(\xi_{01})+\frac{i\pi}{4}\operatorname{sgn}\psi_1''(\xi_{01})\right\}\right.$$

$$\left. - \sum_{m=1}^{M}\frac{H(\xi_{02}^m)}{\sqrt{|\psi_2''(\xi_{02}^m)|}}\exp\left\{iR\psi_2(\xi_{02}^m)+\frac{i\pi}{4}\operatorname{sgn}\psi_2''(\xi_{02}^m)\right\}\right] \tag{4.9}$$

and

$$u_{zz}(r,z;r',0) = \frac{e^{i\pi/4}}{2c_4c_5R\sqrt{\sin\phi}}\left[\sum_{m=1}^{M}\frac{N_2(\xi_{02}^m)}{\sqrt{|\psi_2''(\xi_{02}^m)|}}\exp\left\{iR\psi_2(\xi_{02}^m)+\frac{i\pi}{4}\operatorname{sgn}\psi_2''(\xi_{02}^m)\right\}\right.$$

$$\left. - \frac{N_1(\xi_{01})}{\sqrt{|\psi_1''(\xi_{01})|}}\exp\left\{iR\psi_1(\xi_{01})+\frac{i\pi}{4}\operatorname{sgn}\psi_1''(\xi_{01})\right\}\right]. \tag{4.10}$$

5. NUMERICAL RESULTS FOR RADIAL AND AXIAL RING SOURCE

Numerical results for normalized displacement components $|2c_4Ru_{rr}|$, $|2c_4Ru_{zr}|$ for radial ring source and $|2c_4Ru_{rz}|$, $|2c_4Ru_{zz}|$ for axial ring source at large distace away from the source for different values of k_2r' (=2,8) have been plotted graphically for different values of R/r' (=20, 40, 60) in the graphite-epoxy composite for which the material properties have been presented in the earlier section. Variation of normalized displacement components for different values of ϕ varying from 10° to 75° where $\phi = \tan^{-1}\frac{r}{|z|}$ has been depicted by means of graphs (Figs.2-9). It is interesting to naote that for smaller angles less than 25° and for angles greater than 70°, the variation of the normalized displacement components with values of R/r' is not prominent.

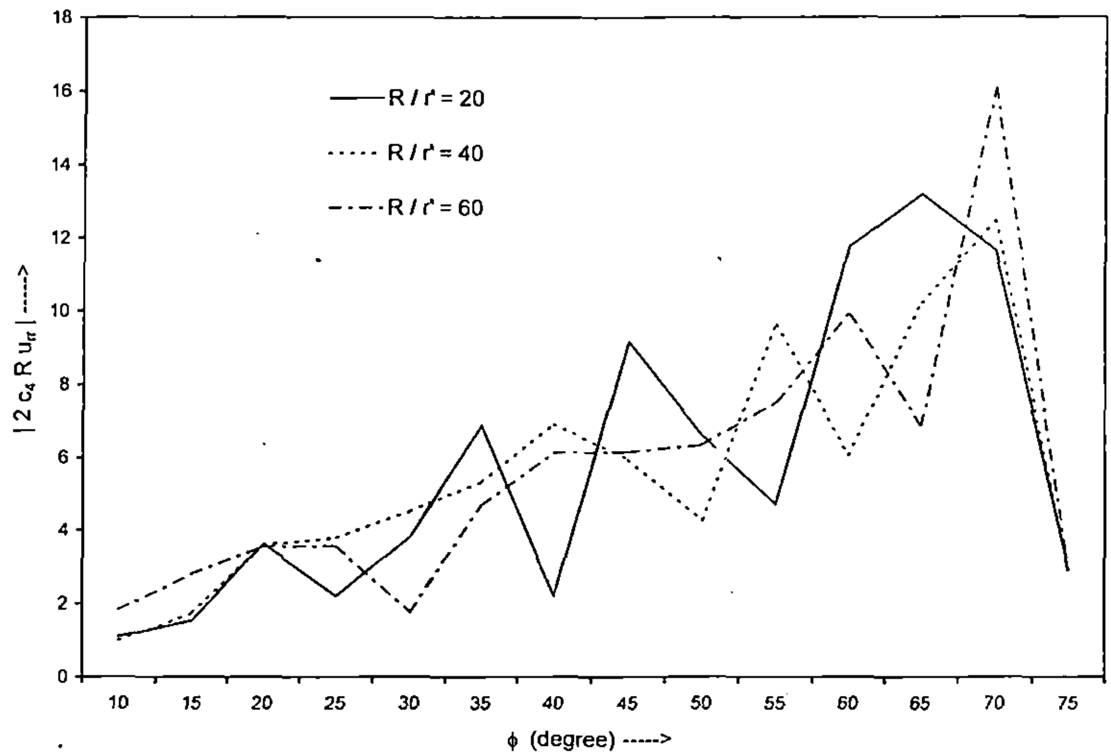

Fig.2. Normalized radial displacement due to radial ring source in graphite-epoxy composite

(for $k_2 r' = 2$).

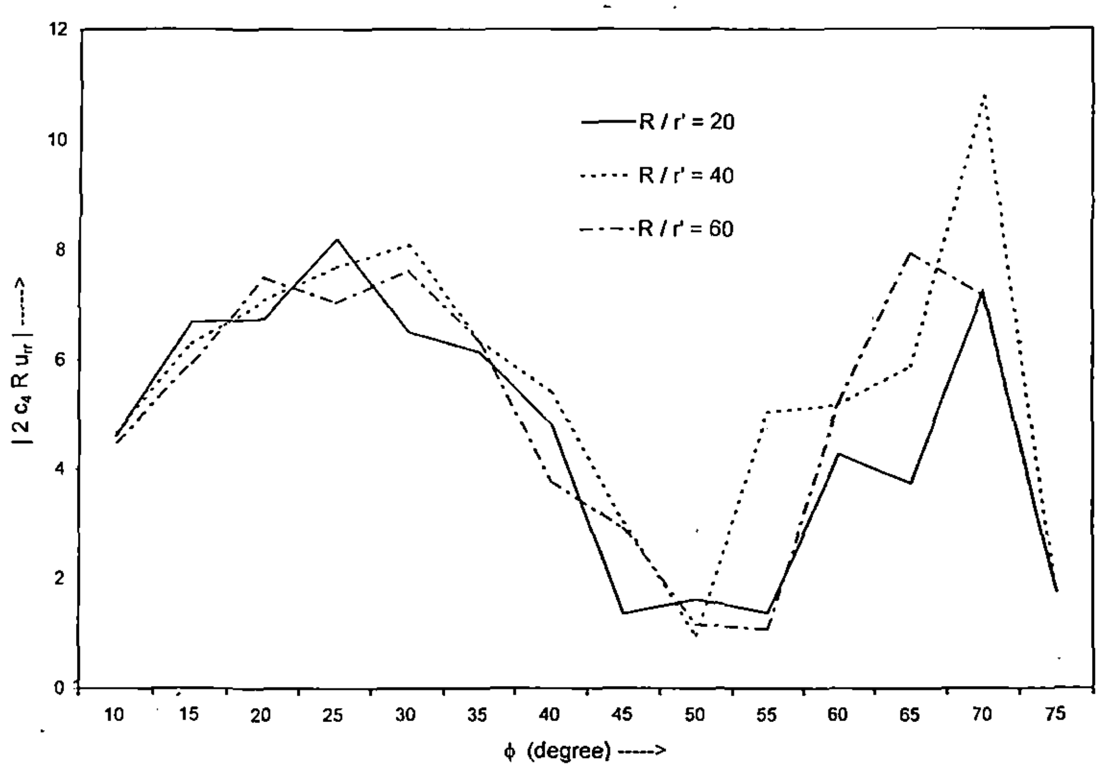

Fig.3. Normalized radial displacement due to radial ring source in graphite-epoxy composite

(for $k_2 r' = 8$).

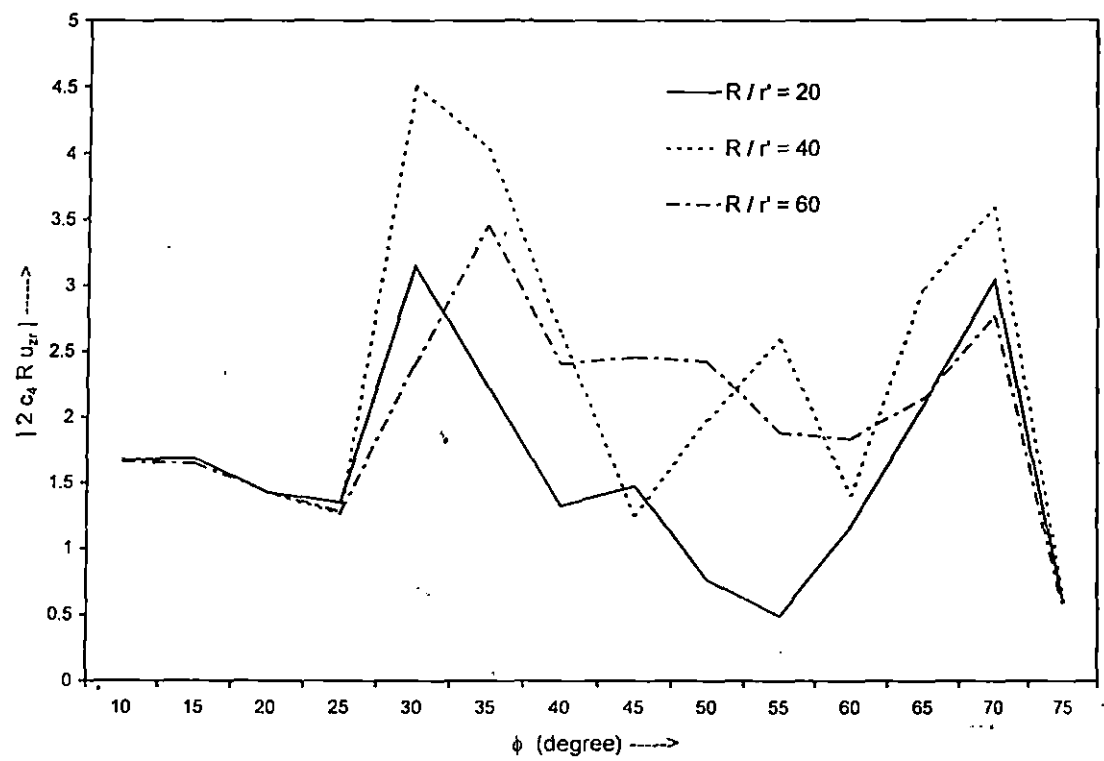

Fig.4. Normalized axial displacement due to radial ring source in graphite-epoxy composite

(for $k_2 r' = 2$).

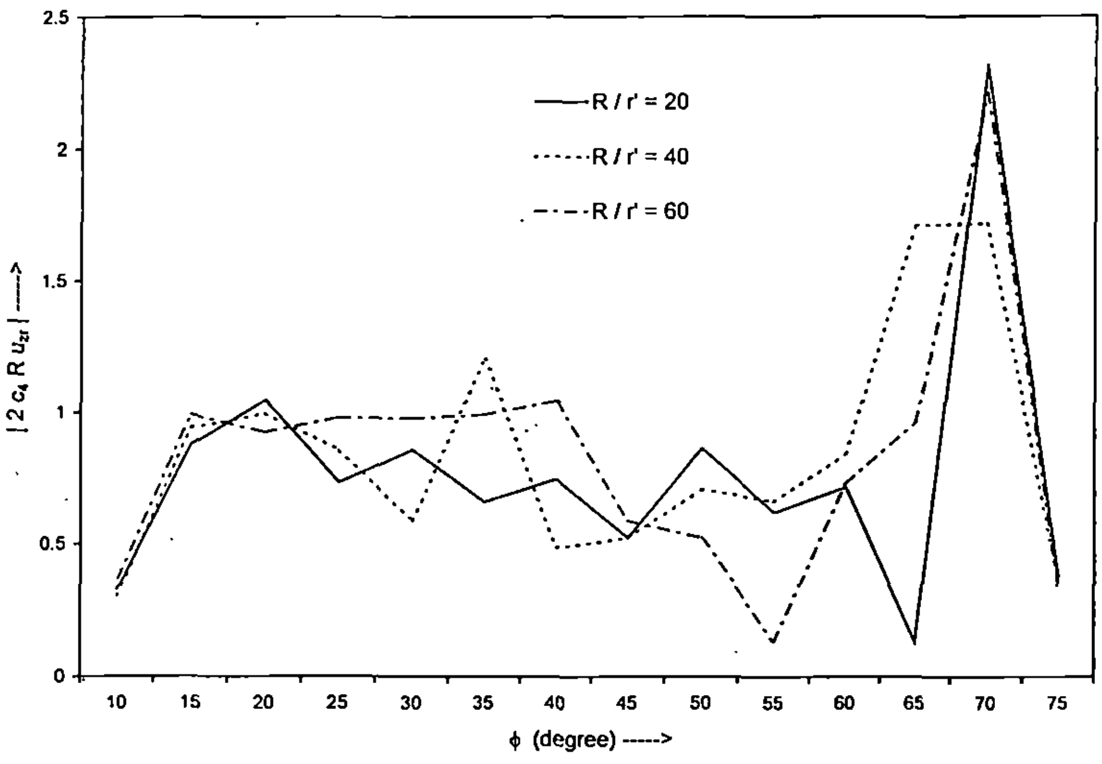

Fig.5. Normalized axial displacement due to radial ring source in graphite-epoxy composite

(for $k_2 r' = 8$).

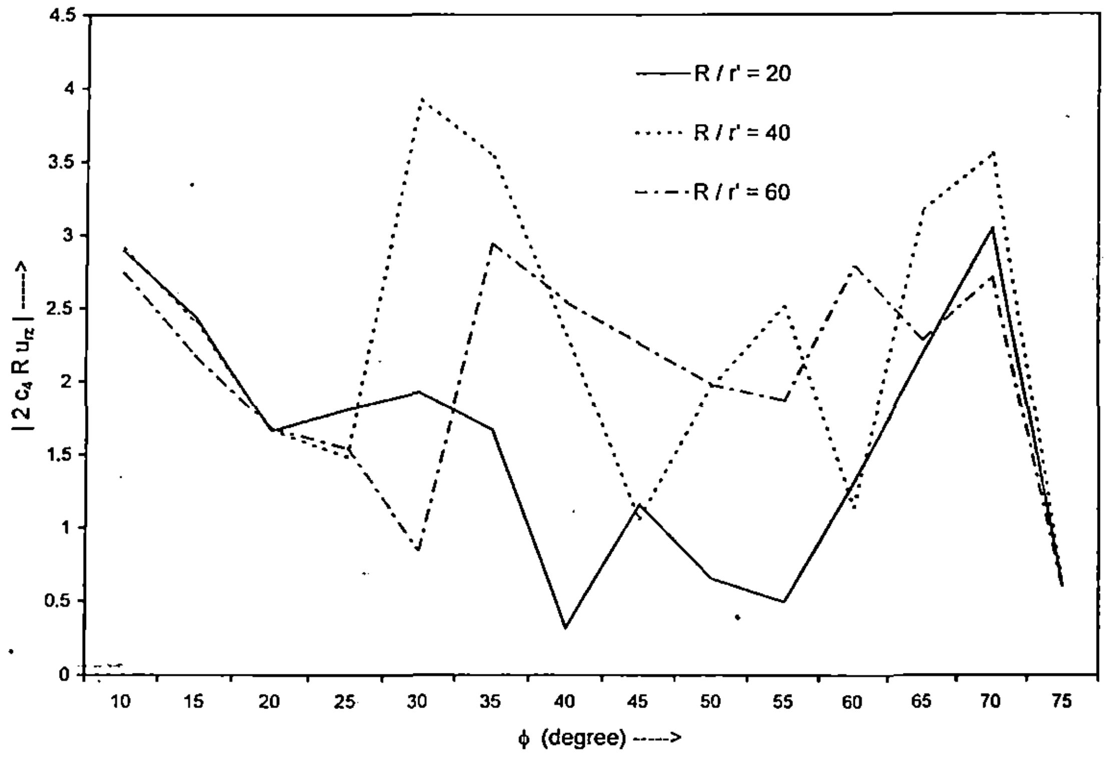

Fig.6. Normalized radial displacement due to axial ring source in graphite-epoxy composite

(for $k_2 r' = 2$).

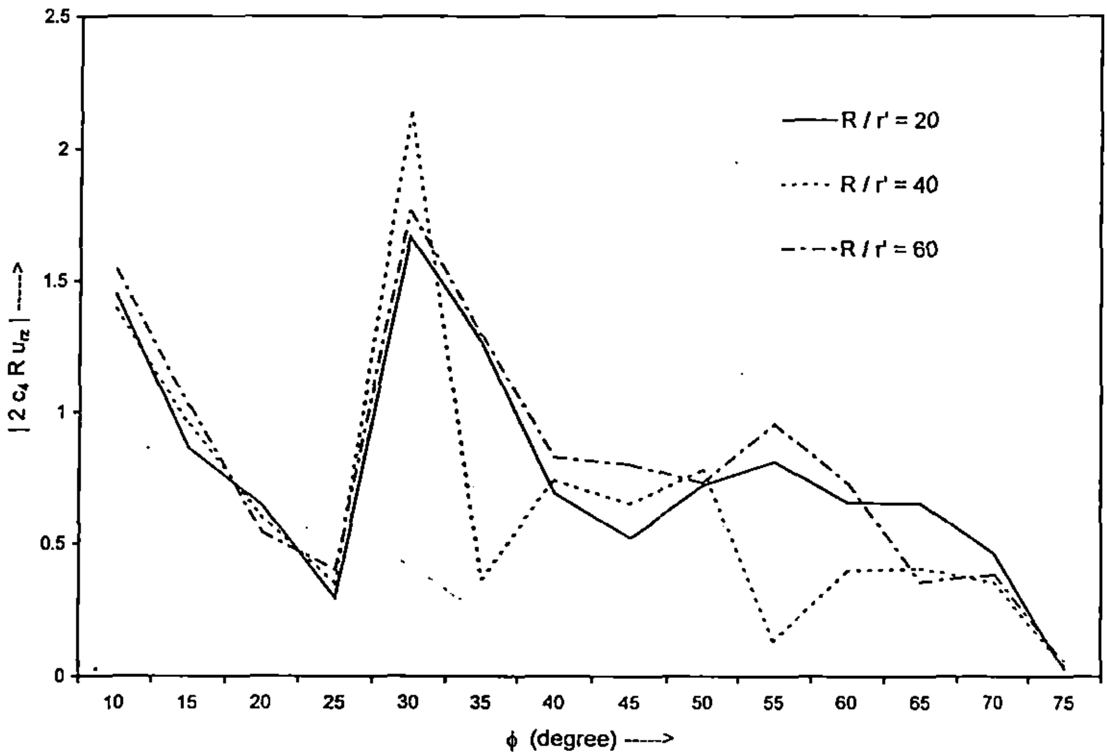

Fig.7. Normalized radial displacement due to axial ring source in graphite-epoxy composite (for $k_2 r' = 8$).

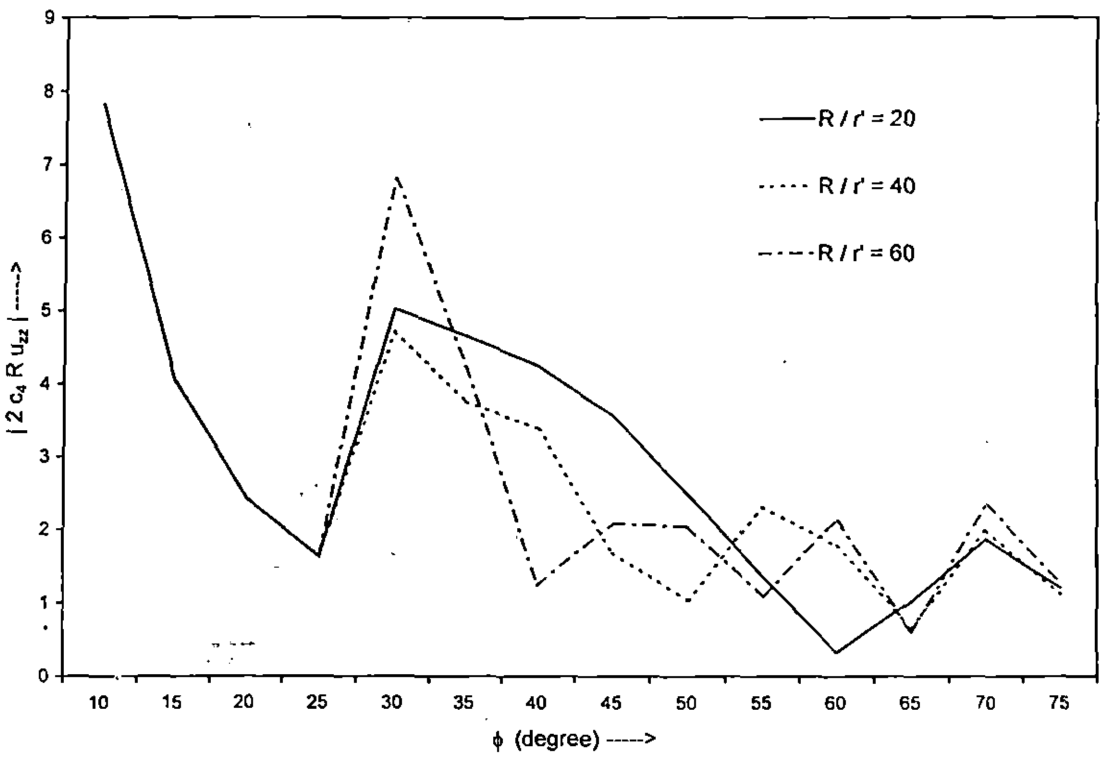

Fig.8. Normalized axial displacement due to axial ring source in graphite-epoxy composite

(for $k_2 r' = 2$).

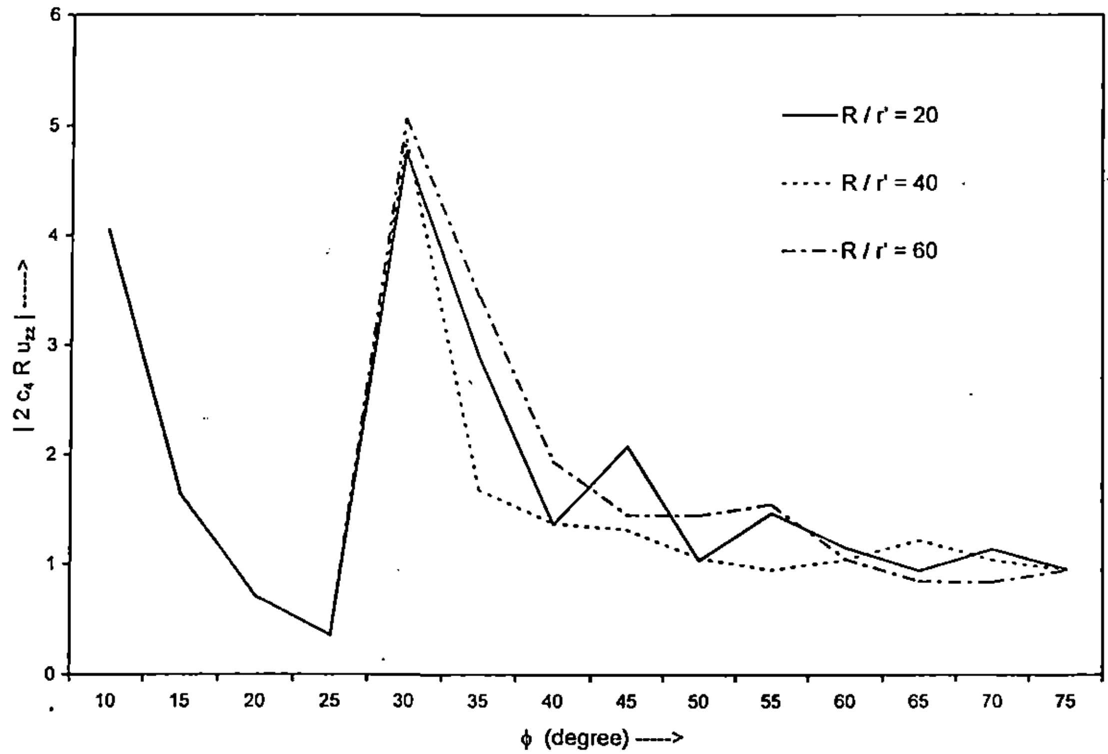

Fig.9. Normalized axial displacement due to axial ring source in graphite-epoxy composite

(for $k_2 r' = 8$).

6. TORSIONAL RING SOURCE OF FORCE

For torsional ring source of force , the equation of motion for Green's function $u_\theta(r,z;r',z')$, can be written as

$$G\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\right)+\mu\frac{\partial^2 u_\theta}{\partial z^2}+\rho\omega^2 u_\theta = -\frac{\delta(r-r')\,\delta(z-z')}{r} \tag{6.1}$$

where u_θ is the tangential component of displacement and μ and G are shear moduli in the θz and $r\theta$ directions respectively. $\frac{\delta(r-r')\,\delta(z-z')}{r}$ represents the torsional ring source, ρ is the material density. The time dependence of all the field quantities, assumed to be of the form exp(-iwt), will be suppressed in the subsequent development.

We use the Fourier transform for the axial co-ordinate, z, and the Hankel transform for the radial co-ordinate, r. The Fourier transform is defined as

$$U_\theta^*(r,\eta;r',z') = \int_{-\infty}^{\infty} U_\theta(r,z;r',z')\, e^{-i\eta z}\, dz \tag{6.2a}$$

$$U_\theta(r,z;r',z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_\theta^*(r,\eta;r',z')\, e^{i\eta z}\, d\eta \tag{6.2b}$$

The Hankel transform is defined as

$$\bar{U}_\theta^*(\xi,\eta;r',z') = \int_0^{\infty} U_\theta^*(r,\eta;r',z')\, J_1(\xi r)\, r\, dr \tag{6.3a}$$

$$U_\theta^*(r,\eta;r',z') = \int_0^{\infty} \bar{U}_\theta^*(\xi,\eta;r',z')\, J_1(\xi r)\, \xi\, d\xi \tag{6.3b}$$

where η and ξ are transform parameters.

Applying Fourier and Hankel transform equation (6.1) becomes

$$\bar{U}_\theta^*(\xi,\eta;r',z') = \frac{J_1(\xi r')\, e^{-i\eta z'}}{G[a^2\eta^2+\xi^2-k^2]} \tag{6.4}$$

where

$$\frac{\mu}{G} = a^2 \quad \text{and} \frac{\rho\omega^2}{G} = k^2 . \tag{6.5}$$

Substituting equation (6.4) into equation (6.3b) and then taking Fourier inversion one obtains, using residue theorem

$$U_\theta(r,z;r',z') = \frac{1}{2\mu}\int_0^\infty \frac{e^{-\beta|z-z'|}}{\beta} J_1(\xi r') J_1(\xi r)\xi d\xi \tag{6.6}$$

where

$$\beta = \frac{1}{a}(\xi^2 - k^2)^{1/2} \qquad \text{for } \xi > k \tag{6.7a}$$

$$= -i\frac{1}{a}(k^2 - \xi^2)^{1/2} \qquad \text{for } \xi < k \tag{6.7b}$$

$$= -i\beta_1 \qquad \text{(say)}.$$

7. FAR-FIELD CALCULATIONS AND NUMERICAL RESULTS FOR TORSIONAL RING SOURCE

The integral arising in (6.6) can be evaluated numerically in the near field (small z, and r). However in the far-field the numerical evaluation of this integral is not possible because of the rapid oscillation introduced in the integrand by the Bessel function. For large ξr

$$J_1(\xi r) \approx \frac{1}{\sqrt{2\pi\xi r}}\left[e^{i\xi r - 3\pi i/4} + e^{-i\xi r + 3\pi i/4}\right]. \tag{7.1}$$

Substituting equation (7.1) into equation (6.6), one obtains

$$U_\theta(r,z;r',z') = \frac{e^{-\pi i/4}}{2\mu\sqrt{2\pi r}}\int_0^\infty L(\xi)e^{i\beta_1|z| + i\xi r}d\xi$$

$$- \frac{e^{\pi i/4}}{2\mu\sqrt{2\pi r}}\int_0^\infty L(\xi)e^{i\beta_1|z| - i\xi r}d\xi \tag{7.2}$$

where

$$L(\xi) = \frac{\sqrt{\xi}\, J_1(\xi r')}{\beta_1}. \tag{7.3}$$

So oscillations in the integrand are introduced by the factors of the form $\exp(i\beta_1|z| \pm i\xi r)$. This type of oscillatary integrals can be easily evaluated by the method of stationary phase [1950].

For this purpose we substitute $|z| = R\cos\phi$ and $r = R\sin\phi$ where R, the distance from the centre of the ring source, is taken to be large.

So, $$e^{i\beta_1|z|\pm i\xi r} = e^{iR(\beta_1\cos\phi \pm \xi\sin\phi)} = e^{iRf_1(\xi)} \quad \text{or} \quad e^{iRf_2(\xi)} \tag{7.4}$$

where $$f_j(\xi) = \beta_1\cos\phi - (-1)^j\xi\sin\phi, \quad j=1,2. \tag{7.5}$$

The stationary points can be obtained from $f_1'(\xi) = 0$ and $f_2'(\xi) = 0$. Thus one gets the stationary points ξ_0, where

$$\xi_0 = \pm\frac{k(\sqrt{\mu/G}\tan\phi)}{\sqrt{1+(\mu/G)\tan^2\phi}}. \tag{7.6}$$

If one defines

$$\tan\alpha = \sqrt{\mu/G}\tan\phi \tag{7.7}$$

then equation (7.6) simplifies to

$$\xi_0 = \pm\ k\sin\alpha. \tag{7.8}$$

One needs to consider only the positive value of ξ_0 since the negative value is not within the limit of the integral, hence

$$u_\theta(r,z;r',0) = \frac{e^{-i\pi/4}}{2\mu R\sqrt{\sin\phi}}\frac{L(\xi_0)}{\sqrt{|f_1''(\xi_0)|}}\exp\left\{iRf_1(\xi_0) + \frac{i\pi}{4}\operatorname{sgn} f_1''(\xi_0)\right\} \tag{7.9}$$

where

$$f_1''(\xi_0) = -\sqrt{\frac{G}{\mu}\frac{1}{k}}\,(1+\frac{\mu}{G}\tan^2\phi)^{3/2}\cos\phi. \tag{7.10}$$

Numerical results for normalized displacement $|\mu R u_\theta|$ for torsional ring source at large distances away from the source have been depicted by means of graph for graphite-epoxy composite material. For this type of material shear modulus μ=7.07 Gpa and shear modulus G=3.50 Gpa whereas density ρ=1578 Kg/m^3. For different values of $k_2 r'$(=5, 20), normalized displacement has been plotted against ϕ where $\phi = \tan^{-1}\frac{r}{|z|}$, shown in Fig.10. It is found that as the frequency increases the displacement becomes stronger near ϕ=0 and diminishes for larger values of ϕ.

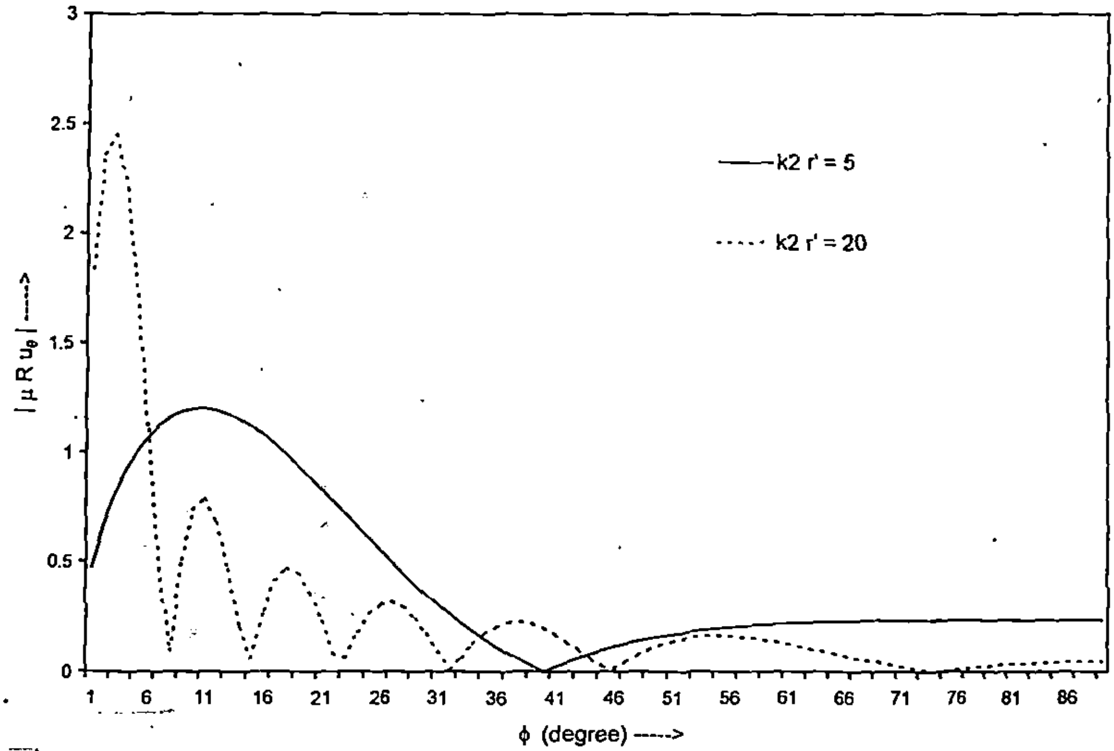

Fig.10. Normalized displacement due to torsional ring source in graphite-epoxy composite.