CHAPTER - 3

SCATTERING OF WAVES

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[5]HIGH FREQUENCY SOLUTION OF ELASTODYNAMIC STRESS INTENSITY FACTORS DUE TO THE DIFFRACTION OF PLANE LONGITUDINAL WAVE BY AN EDGE CRACK IN A SEMI-INFINITE MEDIUM

1. INTRODUCTION

The problem of scattering of elastic waves by a surface breaking crack is of considerable importance in a variety of engineering applications. In fracture mechanics, the interest is in the determination of the stress field near the crack tip as a prerequisite to the study of crack propagation under dynamic loading.

Elastodynamic analysis of an edge crack has been done by Achenbach et al [1980] when the cracked half plane is subjected to time harmonic line load applied to its free surface. Stress intensity factors for three dimensional dynamic loading of a cracked half space has also been studied by Angel and Achenbach [1985]. Low frequency solution of the scattering of SH-wave by an inclined edge crack in a semi-infinite medium was studied by Dutta [1979] using the method of matched asymptotic expansion. The problem of anti-plane shear waves by an edge crack was studied by Stone et al [1980]. Scattering of body waves by an inclined surface breaking crack has also been studied by Zhang and Achenbach [1988] using Boundary Integral Equation method. Detailed discussion on the problems of fracture mechanics can be found in the books of Freund [1990], Brock [1986] and Cherepanov [1979].

In the present paper, the problem of the determination of the high frequency solution of

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elastodynamic stress intensity factor due to the incidence of a time harmonic plane longitudinal wave in the presence of a vertical edge crack in a semi-infinite medium has been studied. The solution of the diffraction problem is complicated by the presence of the free surface of the medium in addition to the crack surface and the associated sharp corners. The resulting boundary value problem for the cracked half-plane is decomposed into two problems for the quarter plane, which represent the symmetric and antisymmetric motions relative to the plane of the crack, respectively.

The plane longitudinal wave, when incident on the free surface of the semi-infinite medium, gives rise to reflected longitudinal and shear waves. These body waves are broken up into symmetric and antisymmetric parts with respect to the plane of the crack. In both the symmetric and antisymmetric motion, firstly, assuming the free surface of the semi-infinite medium to be absent, body waves are assumed to be incident on the semi-infinite crack. Using Wiener-Hopf technique, diffracted field arising from the tip of the crack consisting of both the body waves and Rayleigh surface wave moving along the surface of the crack are obtained. For high frequency, body waves after a few wave lengths are found to be insignificant so that important part of the diffracted field which reaches the corner of the 90^0 wedge formed by the plane free surface and the surface of the crack is Rayleigh wave. The Rayleigh wave on reaching the corner of the wedge is reflected back as Rayleigh wave, the reflection coefficient being given by Li et al [1992]. Diffracted body waves from the corner of the wedge being again insignificant after a few wave lengths may be neglected for high frequency solution. This reflected Rayleigh wave on reaching the crack tip again gives rise to diffracted Rayleigh wave which reaches the corner of the wedge and is again reflected back. This process of reflection of Rayleigh wave from the corner and subsequent diffraction from the edge of the crack tip continues. Using Wiener-Hopf technique, stress components just ahead of the crack tip due to the incidence of body waves and all the reflected Rayleigh waves on the crack tip have been obtained. The expression for the resulting stress intensity factors have been determined. The dependence of the stress intensity factors on the frequency and on the angle of incidence has been depicted by means of graphs.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let us consider a plane harmonic compressional (P-) wave of frequency ω to be incident on an edge crack of length h located at right angles to the free surface of an homogeneous isotropic semi-infinite medium. The x-axis is taken along the free surface and y-axis along the direction of the crack as shown in Fig.1.

In the absence of the crack, the free surface of the medium would give rise to the reflected plane (P) and inplane shear (SV) waves. Let the three waves be represented by the Stokes-Helmholtz potentials $e^{-i\omega t}\phi_I$, $e^{-i\omega t}\phi_R$ and $e^{-i\omega t}\psi_R$ respectively. Then

$$\begin{aligned} \phi_I &= A_0 \exp\left[-ik_1(x\sin\theta_1 + y\cos\theta_1)\right] \\ \phi_R &= A_R \exp\left[-ik_1(x\sin\theta_1 - y\cos\theta_1)\right] \\ \psi_R &= B_R \exp\left[-ik_2(x\sin\theta_2 - y\cos\theta_2)\right] \end{aligned} \tag{1}$$

where θ_1 is the angle of incidence, A_0 the amplitude of the incident wave, k_1, k_2 are the P and S wave numbers respectively related to their phase velocities c_1, c_2 through $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$. The remaining quantities in (1) are given by the laws of reflection of elastic waves as

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \qquad \text{(Snell's law)}$$

$$\frac{A_R}{A_0} = \frac{(\sin 2\theta_2 \sin 2\theta_1 - k^2\cos^2 2\theta_2)}{(\sin 2\theta_2 \sin 2\theta_1 + k^2\cos^2 2\theta_2)}$$

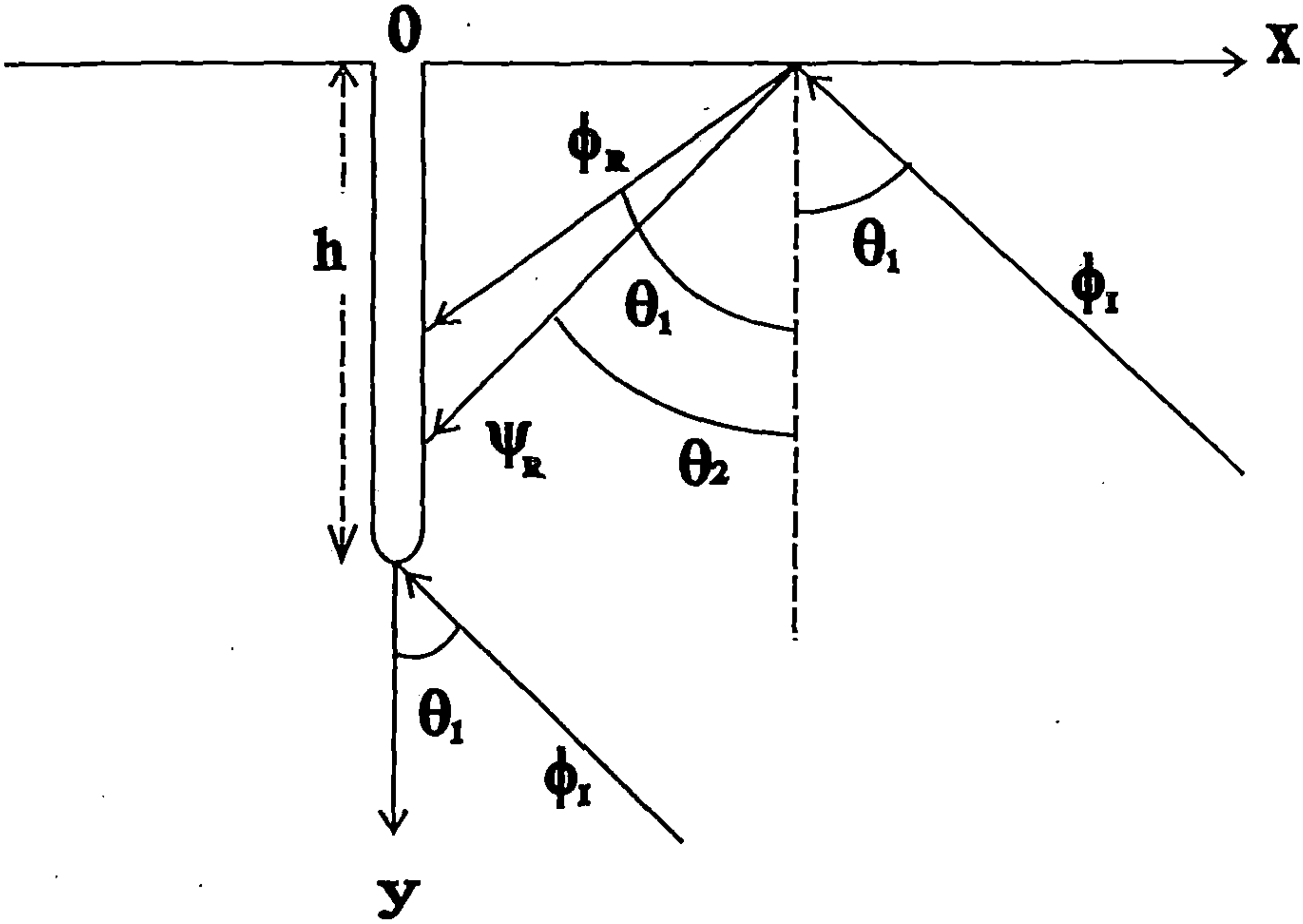

Fig.1 Geometry of the problem.

$$\frac{B_R}{A_0} = \frac{-2\cos 2\theta_2 \sin 2\theta_1}{(\sin 2\theta_2 \sin 2\theta_1 + k^2 \cos^2 2\theta_2)} \tag{2}$$

where k^2 is the ratio of the square of longitudinal and shear wave velocities and is given by

$$k^2 = \frac{c_1^2}{c_2^2} = \frac{k_2^2}{k_1^2}.$$

We shall now determine the scattered field produced by the vertical crack when ϕ_I, ϕ_R and ψ_R are incident on it assuming for the time being that the free surface y=0 is absent.

To this end, firstly we consider the scattered field when the longitudinal wave given by

$$\phi_I = A_0 \exp\left[-ik_1(x\sin\theta_1 + y\cos\theta_1)\right]$$

is incident on a semi-infinite crack in an infinite medium. The crack is on the y-axis extending from y=-∞ to y=h. The diffrection of elastic waves by a semi-infinite crack in an infinite medium has been studied by Maue [1953], De-Hoop [1958] and also by Achenbach [1975] using Wiener-Hopf technique.

We write ϕ_I in the form $\phi_I = \phi_{Ie} + \phi_{Io}$ where

$$\phi_{Ie} = A_0 \exp(-ik_1 y\cos\theta_1)\cos(k_1 x\sin\theta_1)$$

$$\phi_{Io} = -iA_0 \exp(-ik_1 y\cos\theta_1)\sin(k_1 x\sin\theta_1). \tag{3}$$

Clearly ϕ_{Ie} is even in x and ϕ_{Io} is odd in x.

Symmetric problem :

Now consider the interaction of the field given by ϕ_{Ie} with the semi-infinite crack. If $\phi^{(1)}$ and $\psi^{(1)}$ be the displacement potentials of the scattered field due to the incidence of ϕ_{Ie} on the semi-infinite crack, then the displacement components corresponding to the scattered field are

$$u_1^{(e)} = \frac{\partial\phi^{(1)}}{\partial x} + \frac{\partial\psi^{(1)}}{\partial y} \quad \text{and} \quad u_2^{(e)} = \frac{\partial\phi^{(1)}}{\partial y} - \frac{\partial\psi^{(1)}}{\partial x}.$$

From the geometry, it follows that in this symmetric problem $\phi^{(1)}$, $u_2^{(e)}$ and the stress components τ_{11}, τ_{22} are even functions in x while $\psi^{(1)}$, $u_1^{(e)}$ and τ_{12} are odd functions in x. Thus the boundary conditions satisfied by the scattered field due to the incidence of ϕ_{Ie} are

$$u_1^{(e)} = 0; \qquad y>h, \quad x=0 \qquad \text{(i)}$$

$$\tau_{12} = 0; \qquad -\infty<y<\infty, \qquad x=0 \qquad \text{(ii)}$$

$$\tau_{11} = A_0\rho\omega^2\left[1 - \frac{2c_2^2\cos^2\theta_1}{c_1^2}\right]\exp(-ik_1 y\cos\theta_1); \quad y<h, \quad x=0. \qquad \text{(iii)}$$

Introducing Fourier transform defined by

$$\bar{\phi}^{(1)}(x,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi^{(1)}(x,y)e^{i\alpha y}\,dy \qquad (4)$$

we have

$$\bar{\phi}^{(1)} = Ae^{-\gamma_1 x} \text{ and } \bar{\psi}^{(1)} = Be^{-\gamma_2 x} \quad \text{for } x\geq 0 \qquad (5)$$

where $$\gamma_j = \sqrt{\alpha^2 - k_j^2}; \qquad j=1,2. \qquad (6)$$

Branches of γ_j are chosen such that $\mathrm{Re}\gamma_1>0$ and $\mathrm{Re}\gamma_2>0$ for $\mathrm{Im}(-k_1)<\mathrm{Im}(\alpha)<\mathrm{Im}(k_1)$.

The boundary conditions (i)-(iii) now become

$$-\gamma_1 A - i\alpha B = \frac{e^{i\alpha h}}{\sqrt{2\pi}}\int_{-\infty}^{h}u_1^{(e)}(0+,y)e^{i\alpha(y-h)}dy = e^{i\alpha h}U_-(\alpha) \qquad (7)$$

$$B(\alpha^2 - k_2^2/2) - i\alpha\gamma_1 A = 0 \qquad (8)$$

and

$$2\rho c_2^2\left[(\alpha_2 - k_2^2/2)A + i\alpha\gamma_2 B\right]$$

$$= \frac{e^{i\alpha h}}{\sqrt{2\pi}}\int_{h}^{\infty}\tau_{11}(0+,y)e^{i\alpha(y-h)}dy + A_0\rho\omega^2\left(1 - \frac{2c_2^2\cos^2\theta_1}{c_1^2}\right)\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h}e^{iy(\alpha-k_1\cos\theta_1)}dy$$

$$= e^{i\alpha h}G_+(\alpha) + \frac{A_0\rho\omega^2}{\sqrt{2\pi}}\left(1 - \frac{2c_2^2\cos^2\theta_1}{c_1^2}\right)\frac{e^{ih(\alpha-k_1\cos\theta_1)}}{i(\alpha-k_1\cos\theta_1)}; \quad \mathrm{Im}(\alpha)<\mathrm{Im}(k_1\cos\theta_1) \qquad (9)$$

where

$$U_-(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h}u_1^{(e)}(0+,y)e^{i\alpha(y-h)}dy$$

and $$G_+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_h^\infty \tau_{11}(0+,y) e^{i\alpha(y-h)} dy. \qquad (10)$$

Eliminating A and B from equation (7), (8) and (9) one obtains the Wiener-Hopf equation

$$CK_2(\alpha)U_-(\alpha) + G_+(\alpha) = \frac{B e^{-i\alpha_0 h}}{(\alpha - \alpha_0)} \qquad (11)$$

where

$$\alpha_0 = k_1 \cos\theta_1; \quad C = \frac{2\rho c_2^2 (k_2^2 - k_1^2)}{k_2^2}; \quad B = -\frac{iA_0 \rho k_1^2 \left[2c_2^2 \cos^2\theta_1 - c_1^2\right]}{\sqrt{2\pi}} \qquad (12)$$

and $$K_j(\alpha) = \frac{2\sqrt{(\alpha^2 - k_j^2)}}{(k_2^2 - k_1^2)} \left[\alpha^2 - \frac{(\alpha^2 - k_2^2/2)^2}{\sqrt{(\alpha^2 - k_1^2)}\sqrt{(\alpha^2 - k_2^2)}}\right] = \sqrt{(\alpha^2 - k_j^2)} R(\alpha), \quad j=1,2. \qquad (13)$$

Following Chang [1971], it can be shown that

$$K_{j\pm}(\alpha) = \sqrt{(\alpha \pm k_j)}\, R_\pm(\alpha) \qquad (14)$$

where

$$R_\pm(\alpha) = \frac{(\alpha \pm k_r)}{(\alpha \pm k_2)} \exp\left[\frac{1}{\pi} \int_{k_1}^{k_2} \tan^{-1}\{f(s)\} \frac{ds}{(s \pm \alpha)}\right] \qquad (15)$$

with

$$f(s) = \frac{(k_2^2/2 - s^2)^2}{s^2 \sqrt{s^2 - k_1^2} \sqrt{k_2^2 - s^2}} \qquad (16)$$

where $K_+(\alpha)$ and $K_-(\alpha)$ are analytic in upper and lower half of the complex α-plane and k_r is the root of Rayleigh wave equation $R(\alpha)=0$.

By the usual Wiener-Hopf argument, equation (11) subsequently yields

$$G_+(\alpha) = \frac{B e^{-i\alpha_0 h} K_{2+}(\alpha)}{(\alpha - \alpha_0)} \left[\frac{1}{K_{2+}(\alpha)} - \frac{1}{K_{2+}(\alpha_0)}\right] \qquad (17)$$

$$U_-(\alpha) = \frac{B e^{-i\alpha_0 h}}{C(\alpha - \alpha_0) K_{2+}(\alpha_0) K_{2-}(\alpha)}; \quad \mathrm{Im}(\alpha) < \mathrm{Im}(\alpha_0). \qquad (18)$$

In order to determine the value of τ_{11} just ahead of crack tip, we need the form of $G_+(\alpha)$ as $\alpha \to \infty$.

Using the fact that $K_{2+}(\alpha) \to \alpha^{1/2}$ as $|\alpha| \to \infty$, we obtain from equation (17)

$$G_+(\alpha) = E\alpha^{-1/2}, \text{ as } |\alpha| \to \infty, \ -\pi/2 < \arg\alpha < 3\pi/2$$

where $E = -\dfrac{Be^{-i\alpha_0 h}}{K_2+(\alpha_0)}$.

Taking Fourier inversion, we obtain

$$\left[\tau_{11}(0+,y)\right]_{y-h+0} = \frac{E}{\sqrt{2\pi}} \int_{-\infty+ic_1}^{\infty+ic_1} \frac{e^{-i\alpha(y-h)}}{\sqrt{\alpha}} d\alpha. \tag{19}$$

Here $\alpha=0$ is the branch point. We draw a cut from $\alpha=0$ along negative imaginary axis. The line of integration is deformed into a loop round the branch cut as shown in Fig.2.

So equation (19) reduces to

$$\left[\tau_{11}(0+,y)\right]_{y-h+0} = \frac{E(1-i)}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\beta(y-h)}}{\sqrt{\beta}} d\beta = \frac{(i-1)Be^{-i\alpha_0 h}}{K_{2+}(\alpha_0)\sqrt{(y-h)}}. \tag{20}$$

Next, in order to determine the scattered field of displacement due to the incidence of ϕ_{Ie} on the semi-infinite crack we consider equation (18) which yields

$$u_1^{(e)}(0+,y) = \frac{Be^{-i\alpha_0 h}}{\sqrt{2\pi}CK_{2+}(\alpha_0)} \int_{-\infty+ic_1}^{\infty+ic_1} \frac{e^{-i\alpha(y-h)}}{(\alpha-\alpha_0)K_{2-}(\alpha)} d\alpha; \quad \text{Im}(\alpha)<\text{Im}(\alpha_0)$$

where the line of integration is in the common region of regularity of $U_-(\alpha)$ and $G_+(\alpha)$. We draw cuts through k_1 and k_2 parallel to the imaginary axis in the upper half of the complex α-plane.

Taking a semi circular contour with loops round the cuts in the upper half plane as shown in Fig.3, we get (for y<h)

$$u_1^{(e)}(0+,y) = \frac{2\pi i Be^{-i\alpha_0 y}}{\sqrt{2\pi}CK_2(\alpha_0)} + \frac{2\pi i Be^{-i\alpha_0 h}\sqrt{(k_r-k_2)}e^{-ik_r(y-h)}}{\sqrt{2\pi}C(k_r-\alpha_0)K_{2+}(\alpha_0)S(k_r)} + \text{contribution from the branch line integrals} \tag{21}$$

where

$$S(k_r) = \exp\left[\frac{1}{\pi}\int_{k_1}^{k_2} \tan^{-1}\{f(s)\}\frac{ds}{(s-k_r)}\right]. \tag{22}$$

Now consider the contribution from the branch line integral along the loop L_{k_1} round the branch point $\alpha=k_1$. This is equal to

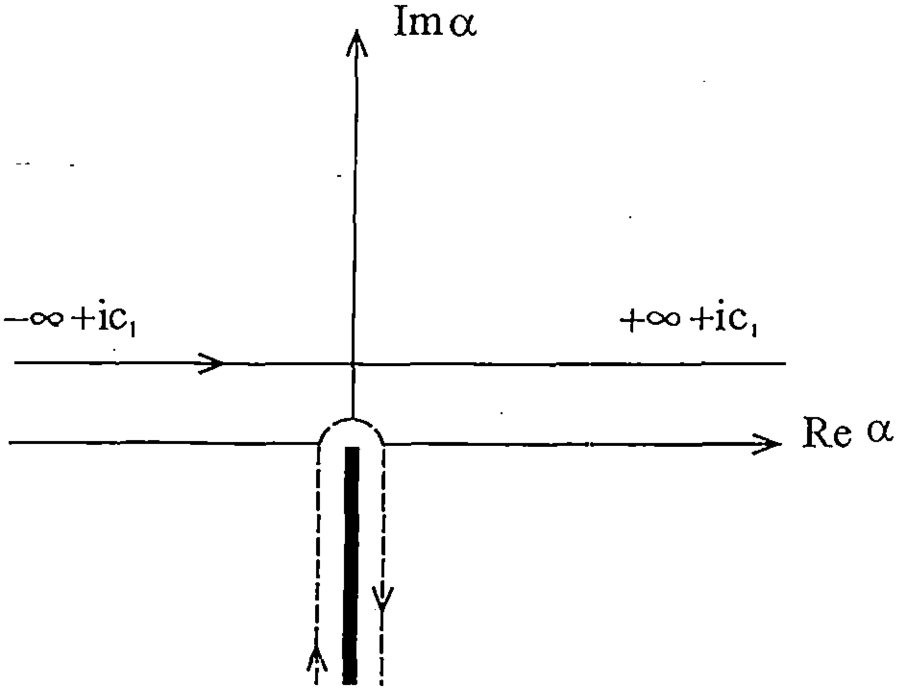

Fig.2 Path of integration.

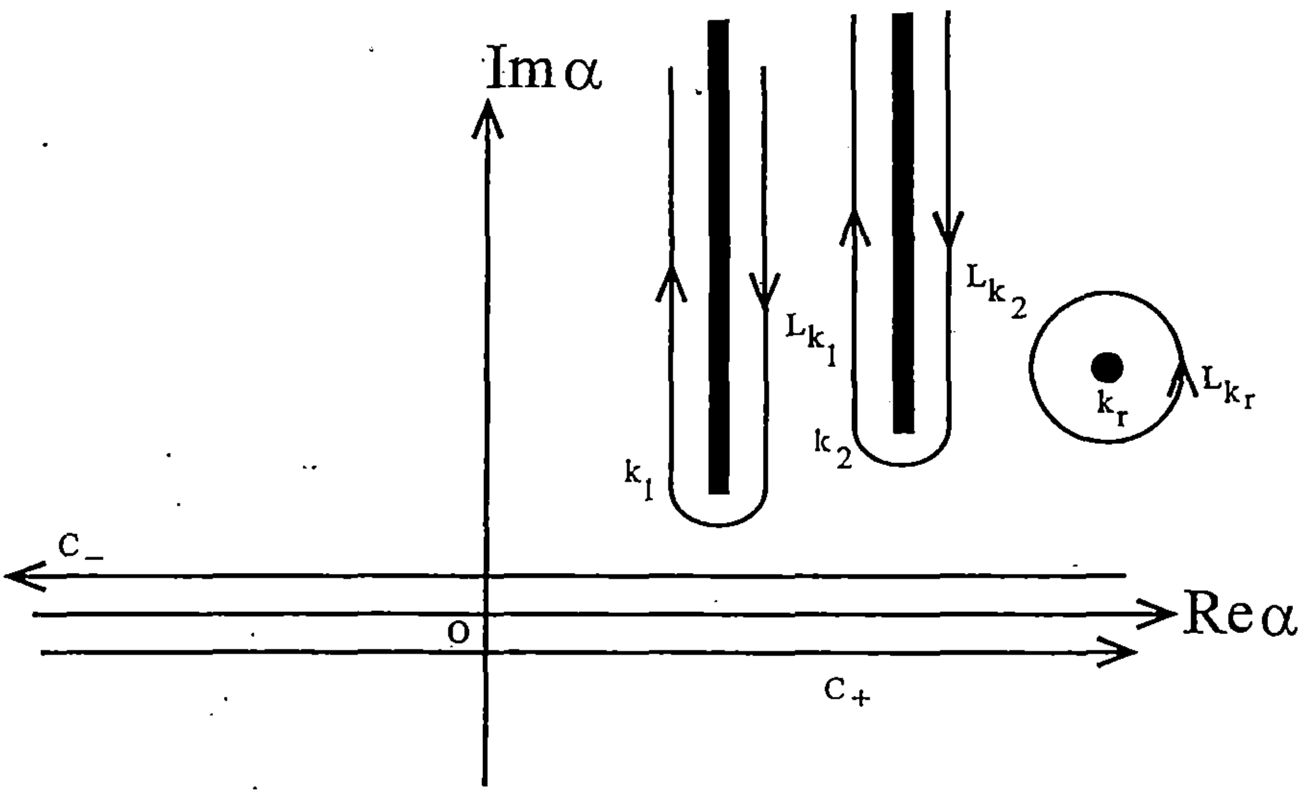

Fig.3 Region of regularity in the transformed plane.

$$\frac{Be^{-i\alpha_0 h}}{\sqrt{2\pi}CK_{2+}(\alpha_0)}\int_{L_{k_1}}\frac{e^{-i\alpha(y-h)}}{(\alpha-\alpha_0)\sqrt{(\alpha-k_2)}R_-(\alpha)}d\alpha$$

$$= \frac{Bie^{-i\alpha_0 h - ik_1(y-h)}}{\sqrt{2\pi}CK_{2+}(\alpha_0)}\int_0^\infty\frac{e^{-k_1(h-y)u}}{\sqrt{(k_2-k_1-ik_1u)}\left[u-i(1-\alpha_0/k_1)\right]R_-^{(1)}(k_1+ik_1u)}\times$$

$$\times\left[1-\frac{R_-^{(1)}(k_1+ik_1u)}{R_-^{(2)}(k_1+ik_1u)}\right]du \qquad (y<h) \tag{24}$$

where $R_-^{(1)}(k_1+ik_1u)$ is the value of $R_-^{(1)}$ on the right hand side of the cut whereas $R_-^{(2)}(k_1+ik_1u)$ is its value on the left hand side.

If (h-y) > 0, then for large values of k_1(h-y), the main contribution to the integral (24) will be from the value of the integrand near u=0. So, for large values of k_1(h-y), the integral (24) can approximately be written as

$$\frac{Bie^{-i\alpha_0 h - ik_1(y-h)}}{\sqrt{2\pi}\sqrt{(k_2-k_1)}CK_{2+}(\alpha_0)}\int_0^\infty\frac{e^{-k_1(h-y)u}}{\left[u-i(1-\alpha_0/k_1)\right]\underset{u\to 0}{\mathrm{Lt}}\ R_-^{(1)}(k_1+ik_1u)}\times$$

$$\times\underset{u\to 0}{\mathrm{Lt}}\left[1-\frac{R_-^{(1)}(k_1+ik_1u)}{R_-^{(2)}(k_1+ik_1u)}\right]du.$$

Now

$$R_-(k_1+ik_1u) = \frac{(k_r-k_1-ik_1u)}{(k_2-k_1-ik_1u)}\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\tan^{-1}\{f(s)\}\frac{ds}{(s-k_1-ik_1u)}\right].$$

Integrating by parts and taking limit, we obtain

$$\underset{u\to 0}{\mathrm{Lt}}\ R_-(k_1+ik_1u) = \frac{(k_r-k_1)e^{i\pi/4}}{C_1\sqrt{k_1(k_2-k_1)u}} \tag{25}$$

where

$$\frac{1}{C_j} = \exp\left[-\frac{1}{\pi}\int_{k_1}^{k_2}\log(s-k_j)\frac{d}{ds}\tan^{-1}\{f(s)\}\,ds\right]; \qquad j=1,2. \tag{26}$$

Next consider the evaluation of

$$\underset{u\to 0}{\mathrm{Lt}}\ \frac{R_-^{(1)}(k_1+ik_1u)}{R_-^{(2)}(k_1+ik_1u)} = \underset{u\to 0}{\mathrm{Lt}}\ \frac{R_-^{(1)}(t_2)}{R_-^{(2)}(t_1)} \qquad \text{(Say)}.$$

We know

$$R_-(\alpha) = \frac{(k_r-\alpha)}{(k_2-\alpha)}\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\tan^{-1}\{f(s)\}\frac{ds}{(s-\alpha)}\right]$$

where α is either t_1 or t_2 .

Following Chang [1971], it can be shown that $R_-(\alpha)$ can also be written as

$$R_-(\alpha) = \frac{(k_r-\alpha)}{(k_2-\alpha)}\exp\left[\frac{1}{2\pi i}\int_{L'_{k_1}+L'_{k_2}}\log\left\{1-\frac{(z^2-k_2^2/2)^2}{z^2\sqrt{z^2-k_1^2}\sqrt{z^2-k_2^2}}\right\}\frac{ds}{(s-\alpha)}\right]$$

where the paths L'_{k_1} and L'_{k_2} are shown in Fig.4.

The point t_1 and t_2 being just on opposite sides of the cut through k_1, the path L'_{k_1} is deformed so as to enclose t_2 and t_1 as shown in Fig.4.

$$\underset{u\to 0}{\mathrm{Lt}}\ \frac{R_-^{(1)}(t_2)}{R_-^{(2)}(t_1)} = \exp\left[\log\left\{1-\frac{(t_2^2-k_2^2/2)^2}{t_2^2\sqrt{t_2^2-k_1^2}\sqrt{t_2^2-k_2^2}}\right\}-\log\left\{1-\frac{(t_1^2-k_2^2/2)^2}{t_1^2\sqrt{t_1^2-k_1^2}\sqrt{t_1^2-k_2^2}}\right\}\right]$$

$$= \frac{1-\sqrt{\dfrac{1}{u}}\,\dfrac{(k_1^2-k_2^2/2)^2}{(1+i)k_1^3\sqrt{k_1^2-k_2^2}}}{1+\sqrt{\dfrac{1}{u}}\,\dfrac{(k_1^2-k_2^2/2)^2}{(1+i)k_1^3\sqrt{k_1^2-k_2^2}}}$$

$$= -1+\frac{2\sqrt{u}(1-i)k_1^3\sqrt{k_2^2-k_1^2}}{(k_1^2-k_2^2/2)^2} - \dots\ \dots \tag{27}$$

Using the results (25) and (27), the integral (24) reduces to the form

$$\frac{2BC_1e^{\pi i/4}\sqrt{k_1}\,e^{-i\alpha_0h}e^{-ik_1(y-h)}}{\sqrt{2\pi}C(k_r-k_1)k_{2+}(\alpha_0)}\int_0^\infty\frac{e^{-k_1(h-y)u}\sqrt{u}}{\{u-i(1-\alpha_0/k_1)\}}du$$

$$= \frac{2BC_1e^{\pi i/4}\sqrt{k_1}\,e^{-i\alpha_0h}e^{-ik_1(y-h)}}{\sqrt{2\pi}C(k_r-k_1)k_{2+}(\alpha_0)\sqrt{k_1(h-y)}}W_0\{-ik_1(h-y)(1-\cos\theta_1)\} \tag{28}$$

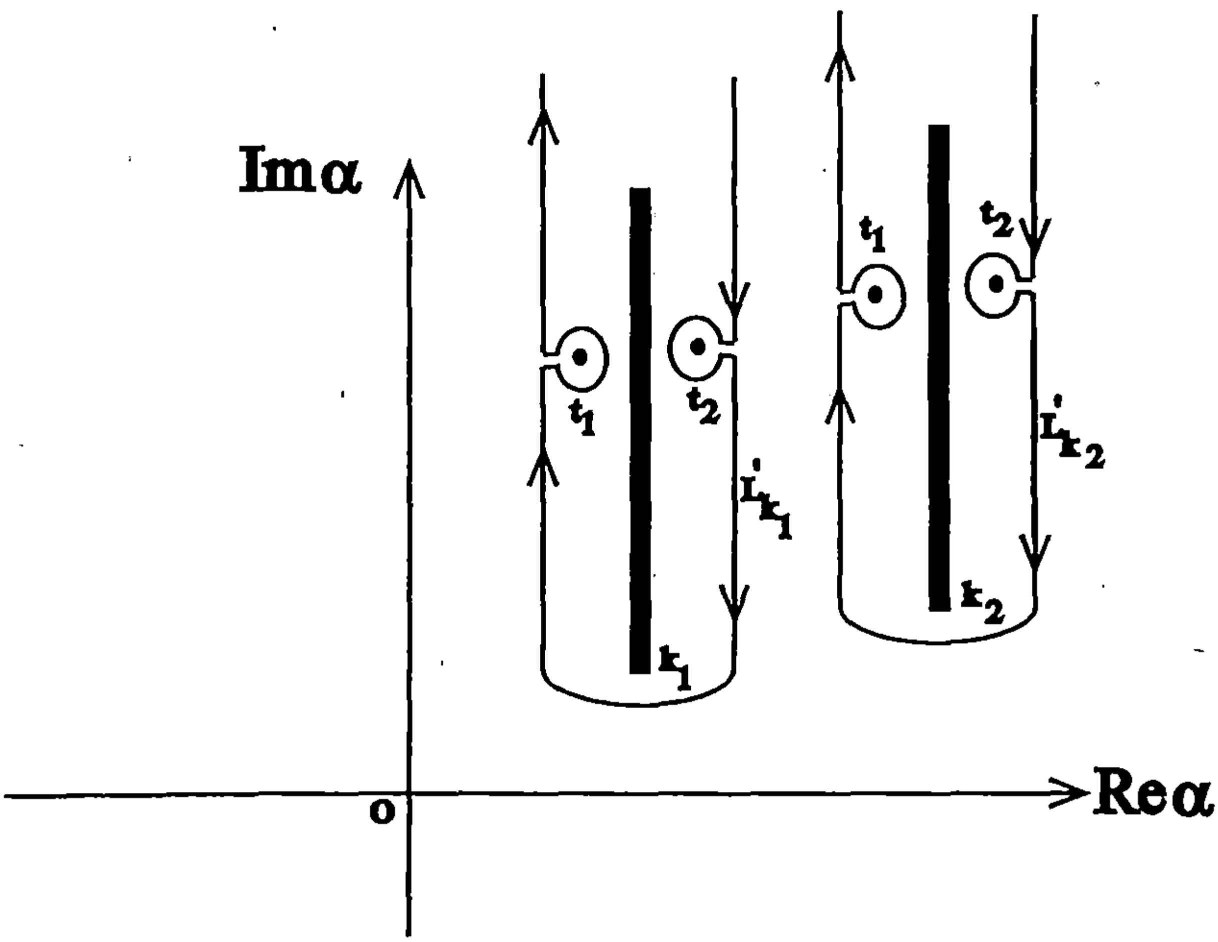

Fig.4 Deformation of the contour for the calculation of $R_-^{(1)}(t_2)$ and $R_-^{(2)}(t_1)$.

where

$$W_0(-iz) = \int_0^\infty \frac{e^{-u}\sqrt{u}}{u-iz}du = \sqrt{\pi} + 2\sqrt{\pi}ie^{-iz}\sqrt{z}\{F(\sqrt{z})\}$$

where F(y) is the Fresnel integral given by

$$F(y) = \int_y^\infty e^{iu^2}du \qquad (\text{c.f. Noble [1958]}).$$

Using the asymptotic value of

$$W_0(-iz) = \frac{\sqrt{\pi}}{-2iz} \quad \text{as} \quad |z|\to\infty$$

expression (28) becomes

$$\frac{BC_1 i e^{\pi i/4}\sqrt{k_1}e^{-i\alpha_0 h}e^{-ik_1(y-h)}}{\sqrt{2}C(k_r-k_1)k_{2+}'(\alpha_0)\{k_1(h-y)\}^{3/2}(1-\cos\theta_1)} \qquad (29)$$

provided $k_1(h-y)$ is large and $\cos\theta_1$ is neither equal to 1 nor nearly equal to 1.

Expression (29) is the diffracted P-wave originating from the edge of the crack due to the incidence of ϕ_{Ie}.

Next consider the contribution to $u_1^{(e)}(0+,y)$, (y<h) from the branch line integral along the loop L_{k_2} round k_2. This is equal to

$$\frac{Be^{-i\alpha_0 h}}{\sqrt{2\pi}CK_{2+}(\alpha_0)}\int_{L_{k_2}}\frac{e^{-i\alpha(y-h)}}{(\alpha-\alpha_0)\sqrt{\alpha-k_2}R_-(\alpha)}d\alpha \qquad (30)$$

Using similar procedure as used while evaluating (23), it can be shown that for large value of $k_1(h-y)$, the expression (30) becomes

$$\frac{4iBC_2e^{i\pi/4}(k_2-k_1)\sqrt{k_1(k_2+k_1)}e^{-i\alpha_0 h}e^{-ik_2(y-h)}}{C(k_r-k_2)K_{2+}(\alpha_0)k_2^{3/2}\{k_1(h-y)\}^{3/2}(k_2/k_1-\cos\theta_1)}. \qquad (31)$$

Therefore in the symmetric problem involving ϕ_{Ie}, the x-component of displacement $u_1^{(e)}(0+,y)$, (y<h) due to scattering by semi-infinite crack is given by

$$u_1^{(e)}(0+,y) = \frac{2\pi i B e^{-i\alpha_0 y}}{\sqrt{2\pi} C K_2(\alpha_0)} + \frac{2\pi i B\sqrt{k_r - k_2}\, e^{-i\alpha_0 h} e^{-ik_r(y-h)}}{\sqrt{2\pi} C K_{2+}(\alpha_0)(k_r - \alpha_0)\, S(k_r)} +$$

$$+ \frac{i B C_1 e^{\pi i/4}\sqrt{k_1}\, e^{-i\alpha_0 h} e^{-ik_1(y-h)}}{\sqrt{2} C(k_r - k_1) K_{2+}(\alpha_0)\, \{k_1(h-y)\}^{3/2}(1-\cos\theta_1)} +$$

$$+ \frac{4 i B C_2 e^{i\pi/4}(k_2 - k_1)\sqrt{k_1(k_2+k_1)}\, e^{-i\alpha_0 h} e^{-ik_2(y-h)}}{C(k_r - k_2) K_{2+}(\alpha_0)\, k_2^{3/2}\{k_1(h-y)\}^{3/2}(k_2/k_1 - \cos\theta_1)}. \tag{32}$$

The first term of $u_1^{(e)}(0+,y)$ in (32) arises due to the reflection of ϕ_{Ie} on the surface of the crack. The second term gives the Rayleigh wave and the last two terms are difffracted body waves due to the diffraction of ϕ_{Ie} from the edge of the crack. While evaluating body wave contribution, it has been assumed that $k_1(h-y)$ is large and that $(1 - \cos\theta_1)$ is different from zero. For $\cos\theta_1=1$, the third term of (32) giving diffracted P-wave is $O\{[k_1(h-y)]^{1/2}\}$ for large values of $[k_1(h-y)]$.

The diffracted body waves near the edge of the crack are of the same order as the Rayleigh wave terms since their joint contribution must vanish at the edge. However, at high frequencies the order of these body wave terms change to $O\{[k_1(h-y)]^{-3/2}\}$ within the distance of a few wave lengths from the edge. Therefore at distances away from the crack, the displacement on the crack surface can be approximated very well by Rayleigh wave contribution in (32) (c. f. Achenbach et al [1982])

$$u_1^{(e)}(0+,y) \approx \frac{2\pi i B e^{-i\alpha_0 y}}{\sqrt{2\pi} C K_2(\alpha_0)} + 2\pi i D_1 M_2 e^{-ik_r(y-h)} \tag{33}$$

where

$$D_1 = \frac{B e^{-i\alpha_0 h}}{\sqrt{2\pi}\, C K_{2+}(\alpha_0)(k_r - \alpha_0)} \quad \text{and} \quad M_j = \frac{(k_r - k_2)}{\sqrt{(k_r - k_j)}\, S(k_r)}; \quad j=1,2. \tag{34}$$

The first term of (33) is due to the geometrical reflection of ϕ_{Ie} from the surface of the crack.

The geometrically reflected rays from the surface of the crack after striking the free surface y=0 generate reflected rays again. These reflected rays do not reach the surface of the crack and therefore do not make any new contribution to $u_1^{(e)}(0+,y)$. The second term of (33) is due to diffracted Rayleigh wave arising from the edge of the crack occuring due to the incidence of ϕ_{Ie}. The corresponding displacement component in the y-direction which is even in x can easily be determined and is found to be equal to

$$\frac{2\pi i D_1 M_2 (2k_r^2 - k_2^2) e^{-ik_r(y-h)}}{2ik_r\sqrt{(k_r^2 - k_1^2)}}. \qquad (35)$$

The Rayleigh wave with displacement components given by (35) and the second term of (33) when incident on the free surface y=0 gives rise to reflected Rayleigh wave and body waves.

The displacement on the surface of the crack arising from the reflected body waves can again be neglected as in the case of direct incidence. The reflection co-efficient of the Rayleigh waves when Rayleigh wave is incident on a wedge has been determined by Hudson and Knopoff [1964] and by Mal and Knopoff [1965] theoretically and also recently by Li, Achenbach et al [1992]. We denote the complex reflection co-efficient for a 90^0 corner by $A_r^{2\pi i\delta}$ where A_r=0.32, δ=0.106 for Poisson ratio equal to 1/4 (c. f. Li, Achenbach et al [1992]). Therefore the reflected Rayleigh wave components on the surface of the crack corresponding to the incident Rayleigh wave given by (35) and second term of (33) are

$$u_{1R}^{(e)}(0+,y) = 2\pi i D_1 M_2 A_r e^{2\pi i\delta} e^{ik_r(y+h)}$$

and $$u_{2R}^{(e)}(0+,y) = \frac{2\pi i D_1 M_2 A_r e^{2\pi i\delta} (2k_r^2 - k_2^2) e^{ik_r(y+h)}}{2ik_r\sqrt{(k_r^2 - k_1^2)}}. \qquad (36)$$

These reflected Rayleigh waves when incident on the crack tip will again generate diffracted Rayleigh waves which can be determined by the usual Wiener-Hopf technique. While carrying out the Wiener-Hopf procedure, it should be remembered that Rayleigh waves from the corner given by (36) which are incident on the crack tip are such that $u_{1R}^{(e)}$ is odd in x whereas $u_{2R}^{(e)}$ is even in x.

Clearly the shear stress $\tau_{12}^{(R)}$ of the first reflected Rayleigh wave given by (36) is odd in x and the stress $\tau_{11}^{(R)}(0+,y)$ for that Rayleigh wave is even in x. Since the total x-component of displacement for y>h, x=0 is zero, the scattered field due to incidence of the Rayleigh wave given by equation (36) on the crack must satisfy the boundary conditions

$$u_{1RD}(0+,y) = -2\pi i D_1 M_2 A_r e^{2\pi i\delta} e^{ik_r(y+h)}, \qquad y>h, \quad x=0 \qquad \text{(iv)}$$

$$\tau_{12}^{(1)}(0+,y) = 0; \qquad -\infty<y<\infty, \qquad x=0 \qquad \text{(v)}$$

$$\tau_{11}^{(1)}(0+,y) = 0; \qquad y<h, \qquad x=0. \qquad \text{(vi)}$$

where $\tau_{12}^{(1)}(0+,y)$ and $\tau_{11}^{(1)}(0+,y)$ are stresses of the scattered field due to the incidence of the Rayleigh wave given by (36) on the crack tip.

The boundary conditions yield the Wiener-Hopf equation

$$G_{1+}(\alpha) = -CK_2(\alpha)\left[U_{1-}(\alpha) + \frac{\sqrt{2\pi}\,D_1 M_2 A_r e^{2\pi i\delta} e^{2ik_r h}}{(\alpha+k_r)}\right] \qquad (37)$$

where

$$G_{1+}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_h^{\infty} \tau_{11}^{(1)}(0+,y)\, e^{i\alpha(y-h)}\,dy$$

$$U_{1-}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h} U_{1RD}(0+,y)\, e^{i\alpha(y-h)}\,dy. \qquad (38)$$

By the usual Wiener-Hopf argument, equation (37) subsequently yields

$$\frac{G_{1+}(\alpha)}{K_{2+}(\alpha)} = -\frac{C\sqrt{2\pi}D_1 M_2\,|A_r|\,e^{2\pi i\delta}e^{2ik_r h}}{(\alpha+k_r)}K_{2-}(-k_r); \qquad \mathrm{Im}(\alpha)>-\mathrm{Im}(k_r) \qquad (39)$$

$$U_{1-}(\alpha) = -\frac{\sqrt{2\pi}D_1 M_2\,|A_r|\,e^{2\pi i\delta}e^{2ik_r h}}{(\alpha+k_r)K_{2-}(\alpha)}\left[K_{2-}(\alpha) - K_{2-}(-k_r)\right]. \qquad (40)$$

In order to determine the value of $\tau_{11}^{(1)}(0+,y)$ just ahead of the crack tip, we need the form

$G_{1+}(\alpha)$ as $\alpha \to \infty$. Using the fact $K_{2+}(\alpha) \to \alpha^{1/2}$ as $|\alpha| \to \infty$, we obtain from equation (39)

$$G_{1+}(\alpha) = -\frac{C\sqrt{2\pi}D_1 M_2 |A_r| e^{2\pi i\delta} e^{2ik_r h}}{\sqrt{\alpha}} K_{2-}(-k_r); \quad \text{as } |\alpha| \to \infty.$$

Fourier inversion gives

$$\tau_{11}^{(1)}(0+,y) = -\frac{B e^{-i\alpha_0 h} M_2 |A_r| e^{2\pi i\delta} e^{2ik_r h} K_{2-}(-k_r)(1-i)}{(k_r - \alpha_0) K_{2+}(\alpha_0)\sqrt{(y-h)}} \tag{41}$$

just ahead of the crack tip.

Assuming the Rayleigh waves given by equation (36) to be incident on the semi-infinite crack extending from y= -∞ to y=h on the y-axis in an infinite medium, the x-component of the displacement on the cracked surface due to the diffracted Rayleigh wave from the crack tip is obtained by taking the Fourier inversion of equation (40) and considering only the contribution to the integral from the Rayleigh pole $\alpha = k_r$ in the upper half of the complex α-plane for y<h, the Rayleigh wave part of the diffracted wave is given by

$$u_{1RD}(0+,y) = 2\pi i |A_r| e^{2\pi i\delta} D_1 M_2 \left[\frac{M_2 K_{2-}(-k_r) e^{2ik_r h}}{2k_r}\right] e^{-ik_r(y-h)}.$$

This waves on reflection from the corner (x=0, y=0) again gives rise to Rayleigh waves with displacement on the crack surface

$$u_{1RDR}(0+,y) = 2\pi i |A_r|^2 e^{4\pi i\delta} D_1 M_2 \left[\frac{M_2 K_{2-}(-k_r) e^{2ik_r h}}{2k_r}\right] e^{ik_r(y+h)} \tag{42}$$

and the corresponding stress $\tau_{11}^{(11)}$ due to the incidence of the Rayleigh wave given by equation (42) just ahead of the crack tip is

$$\tau_{11}^{(11)}(0+,y) = -\frac{B e^{-i\alpha_0 h} M_2 |A_r| e^{2\pi i\delta} e^{2ik_r h} K_{2-}(-k_r)}{(k_r - \alpha_0) K_{2+}(\alpha_0)} \left[\frac{|A_r| e^{2\pi i\delta} e^{2ik_r h} M_2 K_{2-}(-k_r)}{2k_r}\right] \frac{(1-i)}{\sqrt{(y-h)}}. \tag{43}$$

In presence of the free surface at y=0 Rayleigh wave which originates at the crack tip and moves along the surface of the crack is reflected back to the crack tip which again gives rise to the Rayleigh wave that is subsequently reflected from the free surface. This process continues again and again. Considering the contribution to τ_{11} at the crack tip due to the incidence of all Rayleigh waves which are reflected from the free surface and summing up and adding with (20), the total stress just ahead of the crack tip due to the incidence of ϕ_{Ie} is obtained as

$$\left[\tau_{11}(0+,y)\right]_{y\to h+0} = -\frac{B e^{-i\alpha_0 h}}{K_{2+}(\alpha_0)}\left[1 + \frac{N_2}{(k_r-\alpha_0)}\right]\frac{(1-i)}{\sqrt{(y-h)}} \tag{44}$$

where

$$N_j = \frac{|A_r| e^{2\pi i\delta} M_j e^{2ik_r h} K_{j-}(-k_r)}{\left[1 - (-1)^j \dfrac{|A_r| e^{2\pi i\delta} M_j e^{2ik_r h} K_{j-}(-k_r)}{2k_r}\right]}. \tag{45}$$

Next suppose that $\phi_{Io} = -iA_0\exp(-ik_1 y\cos\theta_1)\sin(k_1 x\sin\theta_1)$ is incident on the semi-infinite crack given by $-\infty<y<h$, $x=0$ in an infinite medium.

Antisymmetric problem :

From geometry, it follows that in the antisymmetric problem involving ϕ_{Io}, the scattered field given by $\phi^{(2)}$, $u_2^{(2)}$, $\tau_{11}^{(2)}$, $\tau_{22}^{(2)}$ will be odd functions of x whereas $\psi^{(2)}$, $u_1^{(2)}$, $\tau_{12}^{(2)}$ Will be even functions of x.

So the conditions to be satisfied are

$$u_2^{(2)} = 0; \quad y>h, \quad x=0 \tag{vii}$$

$$\tau_{11}^{(2)} = 0; \quad -\infty<y<\infty, \quad x=0 \tag{viii}$$

$$\tau_{12}^{(2)} = \rho c_2^2 k_1^2 A_0 \sin 2\theta_1 e^{-i\alpha_0 y}; \quad y<h, \quad x=0. \tag{ix}$$

Using these conditions we obtain the Wiener-Hopf equation

$$H_+(\alpha) + C K_1(\alpha) V_-(\alpha) = \frac{P e^{-i\alpha_0 h}}{(\alpha-\alpha_0)} \tag{46}$$

where

$$P = \frac{iA_0 \mu k_1^2 \sin 2\theta_1}{\sqrt{2\pi}} \tag{47}$$

$$H_+(\alpha) = \frac{1}{\sqrt{2\pi}}\int_h^{\infty} \tau_{12}^{(2)}(0+,y)\, e^{i\alpha(y-h)}\, dy$$

and

$$V_-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h} u_2^{(2)}(0+,y)\, e^{i\alpha(y-h)}\, dy. \tag{48}$$

Using the usual Wiener-Hopf argument, equation (46) subsequently yields

$$\frac{H_+(\alpha)}{K_{1+}(\alpha)} = \frac{Pe^{-i\alpha_0 h}}{(\alpha-\alpha_0)}\left[\frac{1}{K_{1+}(\alpha)} - \frac{1}{K_{1+}(\alpha_0)}\right] \tag{49}$$

and

$$V_-(\alpha) = \frac{Pe^{-i\alpha_0 h}}{CK_{1+}(\alpha_0)K_{1-}(\alpha)(\alpha-\alpha_0)}. \tag{50}$$

From (49) we can derive the shear stress just ahead of the crack tip due to the incidence of ϕ_{Io} as

$$\tau_{12}^{(2)}(0+,y) = -\frac{Pe^{-i\alpha_0 h}}{K_{1+}(\alpha_0)}\frac{(1-i)}{\sqrt{(y-h)}} \qquad |\alpha| \to \infty. \tag{51}$$

Taking Fourier inversion of (50) we obtain (neglecting the diffracted body waves)

$$u_2^{(2)}(0+,y) = \frac{2\pi i Pe^{-i\alpha_0 y}}{\sqrt{2\pi}CK_1(\alpha_0)} + 2\pi i D_2 M_1 e^{-ik_r(y-h)} \tag{52}$$

where

$$D_2 = \frac{Pe^{-i\alpha_0 h}}{\sqrt{2\pi}CK_{1+}(\alpha_0)(k_r-\alpha_0)}. \tag{53}$$

The first term of (52) is due to the geometrical reflection of ϕ_{Io} from the surface of the crack and the second term is due to diffracted Rayleigh wave arising from the crack tip. Taking account of the shear stress contribution due to successive reflection and diffraction of the Rayleigh wave given by the second term of (52) from the corner of the wedge and the edge of the crack respectively and summing up, we obtain finally the shear stress just ahead of the crack tip due to the incidence of ϕ_{Io} as

$$\left[\tau_{12}(\phi_{Io})\right]_{y-h+0} = -\frac{Pe^{-i\alpha_0 h}}{K_{1+}(\alpha_0)}\left[1 - \frac{N_1}{(k_r-\alpha_0)}\right]\frac{(1-i)}{\sqrt{(y-h)}}. \tag{54}$$

Following the same procedure, the stresses just ahead of the crack tip due to the incidence of $\phi_R = \phi_{Re} + \phi_{Ro}$ are given by

$$\left[\tau_{11}(\phi_{Re})\right]_{y\to h+0} = -\frac{B_1 e^{i\alpha_0 h}}{K_{2+}(-\alpha_0)}\left[1 + \frac{N_2}{(k_r+\alpha_0)}\right]\frac{(1-i)}{\sqrt{(y-h)}} \tag{55}$$

where

$$B_1 = -\frac{iA_R\mu k_1^2(2\cos^2\theta_1 - c_1^2/c_2^2)}{\sqrt{2\pi}} \tag{56}$$

and

$$\left[\tau_{12}(\phi_{Ro})\right]_{y\to h+0} = -\frac{P_1 e^{i\alpha_0 h}}{K_{1+}(-\alpha_0)}\left[1 - \frac{N_1}{(k_r+\alpha_0)}\right]\frac{(1-i)}{\sqrt{(y-h)}} \tag{57}$$

where

$$P_1 = -\frac{iA_R\mu k_1^2\sin 2\theta_1}{\sqrt{2\pi}}. \tag{58}$$

Similarly the stresses just ahead of the crack tip due to the incidence of $\psi_R=\psi_{Ro}+\psi_{Re}$ are given by

$$\left[\tau_{11}(\psi_{Ro})\right]_{y\to h+0} = -\frac{B_2 e^{i\beta_0 h}}{K_{2+}(-\beta_0)}\left[1 + \frac{N_2}{(k_r+\beta_0)}\right]\frac{(1-i)}{\sqrt{(y-h)}} \tag{59}$$

where

$$B_2 = -\frac{iB_R\mu k_1^2(c_1^2/c_2^2)\sin 2\theta_2}{\sqrt{2\pi}}; \qquad \beta_0 = k_2\cos\theta_2 \tag{60}$$

and

$$\left[\tau_{12}(\psi_{Re})\right]_{y\to h+0} = -\frac{P_2 e^{i\beta_0 h}}{K_{1+}(-\beta_0)}\left[1 - \frac{N_1}{(k_r+\beta_0)}\right]\frac{(1-i)}{\sqrt{(y-h)}} \tag{61}$$

where

$$P_2 = \frac{iB_R\mu k_1^2(c_1^2/c_2^2)\cos 2\theta_2}{\sqrt{2\pi}}. \tag{62}$$

Therefore considering the contribution from ϕ_I, ϕ_R and ψ_R together, the resultant stress components just ahead of the crack tip are given by

$$\left[\tau_{11}(0+,y)\right]_{y\to h+0} = -\left[\frac{Be^{-i\alpha_0 h}}{K_{2+}(\alpha_0)}\left\{1 + \frac{N_2}{(k_r-\alpha_0)}\right\} + \right.$$

$$\left. + \frac{B_1 e^{i\alpha_0 h}}{K_{2+}(-\alpha_0)}\left\{1 + \frac{N_2}{(k_r+\alpha_0)}\right\} + \frac{B_2 e^{i\beta_0 h}}{K_{2+}(-\beta_0)}\left\{1 + \frac{N_2}{(k_r+\beta_0)}\right\}\right]\frac{(1-i)}{\sqrt{(y-h)}} \tag{63}$$

and

$$\left[\tau_{12}(0+,y)\right]_{y\to h+0} = -\left[\frac{Pe^{-i\alpha_0 h}}{K_{1+}(\alpha_0)}\left\{1 - \frac{N_1}{(k_r-\alpha_0)}\right\} + \right.$$

$$\left. + \frac{P_1 e^{i\alpha_0 h}}{K_{1+}(-\alpha_0)}\left\{1 - \frac{N_1}{(k_r+\alpha_0)}\right\} + \frac{P_2 e^{i\beta_0 h}}{K_{1+}(-\beta_0)}\left\{1 - \frac{N_1}{(k_r+\beta_0)}\right\}\right]\frac{(1-i)}{\sqrt{(y-h)}}. \qquad (64)$$

3. STRESS INTENSITY FACTOR

The singular parts of the stress components τ_{11} and τ_{12} just ahead of the crack tip may be expressed in the form

$$\tau_{11} = \frac{K_I}{\sqrt{y/h-1}} \quad \text{and} \quad \tau_{12} = \frac{K_{II}}{\sqrt{y/h-1}}$$

and the corresponding stress intensity factors at the crack tip are defined by

$$S_1 = \left|\frac{\sqrt{2\pi}\,K_I}{(1-i)A_0\mu k_1^2}\right| \qquad (65) \qquad \text{and} \qquad S_2 = \left|\frac{\sqrt{2\pi}\,K_{II}}{(1-i)A_0\mu k_1^2}\right| \qquad (66)$$

4. NUMERICAL RESULTS AND DISCUSSION

The absolute values of the complex dynamic stress intensity factors S_1 and S_2 as defined by equations (65) and (66) have been plotted against the dimensionless wave number k_1h for different values of the angle of incidence $\theta_1=30^0$, 40^0, and 50^0. Numerical results have been computed for Poisson ratio $\gamma=1/4$ so that $\frac{c_1}{c_2}=\sqrt{3}$ and $\frac{c_1}{c_R}=\frac{\sqrt{3}}{0.9194}$. In the figure, values of k_1h have been taken to vary from $k_1h=1$ to 30.

Both stress intensity factors show a tendency to decrease, though oscillatory for increasing k_1h. The general oscillatory feature for the curves in Fig.5 and Fig.6 is due to the effect of interaction between the incident plane waves and Rayleigh waves which originate at the crack tip and are reflected back and forth between the crack tip and the corner of the free surface. In both the figures Fig.5 and Fig.6, S_1 and S_2 show distinct peaks, which indicate constructive interference phenomena.

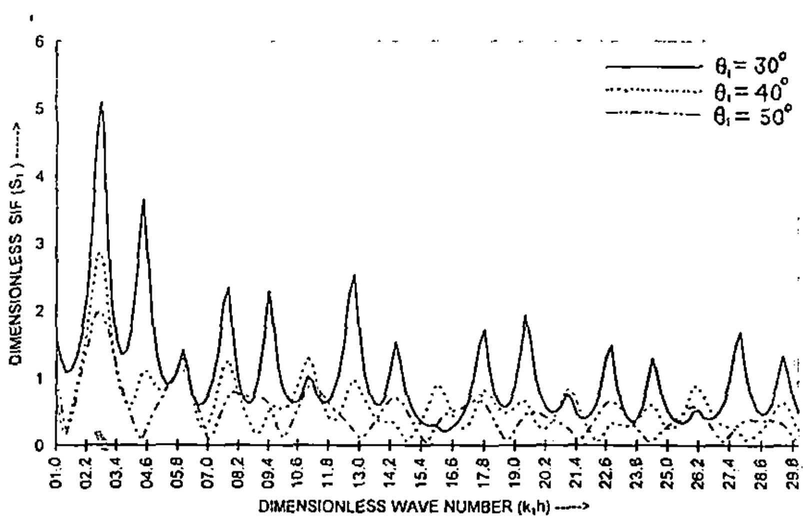

Fig.5 Stress intensity factor S_1 versus dimensionless wave number k_1h for $\theta_1=30^0$, 40^0, and 50^0.

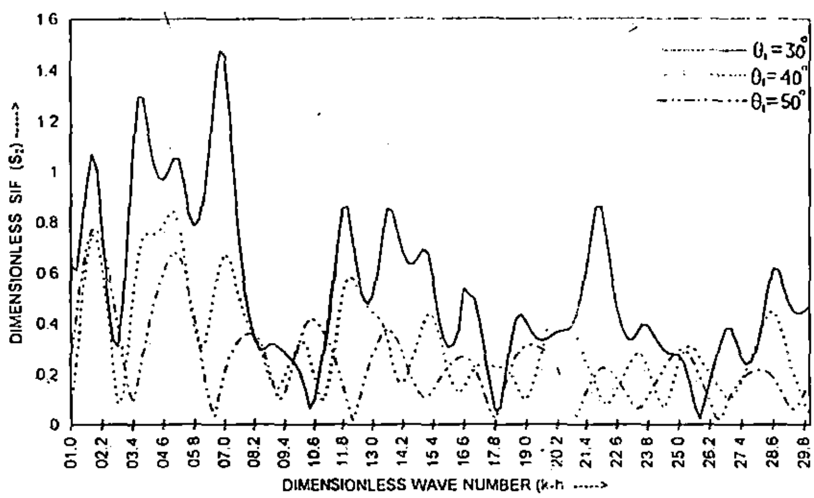

Fig.6 Stress intensity factor S_2 versus dimensionless wave number k_1h for $\theta_1=30^0$, 40^0, and 50^0.