CHAPTER 4

MODIFIED NO-HAIR CONJECTURE AND THE LIMITING PROCESS

4.1. INTRODUCTION

In theories of gravity coupled to scalar fields, there always arises the question as to whether black hole solutions admit dressing by scalar fields [66-74, 86-88, 92-95]. This question becomes particularly important in the Brans-Dicke (BD) scalar-tensor theory which we are discussing in the thesis. The answer lies in Wheeler's dictum [66] which is supposed to mean that a black hole can be dressed only by fields which are constant everywhere or at most satisfy a Gauss-like law. This is the popular version of what is known as "no-hair" conjecture. An alternative interpretation of such a conjecture is that a stationary black hole with an exterior devoid of matter can be parameterized only by mass, angular momentum and electric charge. However, Bekenstien [73] has shown that scalar charge is also an admissible parameter. In the context of conformal scalar field, it has been shown that the black hole solution is parameterized by electric and scalar charges although the conformal field becomes unbounded at the horizon. This feature was previously regarded as incompatible with black hole interpretation but Zannias and Bekenstein [72,73] have shown that this singularity is harmless from the physical point of view.

Early no-hair theorems excluded scalar [68-70], vector [85] and spinor [8] fields from stationary black holes exterior. But, due to the developments in particle physics, those early theorems have become outdated, and as a consequence, black hole solutions with various "hairs" have been found. Among them are black holes dressed with Yang-Mills, Proca-type Yang-Mills and Skyrme fields in various combinations with Higgs fields [86-88]. Although the perimeter of the early no-hair theorems has been widened to include black holes dressed by different fields, there arises the question as to whether the solutions are unstable. Possibly it is this aspect of instability that distinguishes early no-hair theorem from the modified ones.

In Sec. 4.2 , it is argued, contrary to a certain viewpoint, that Schwarzschild black hole solution follows as a unique limit of the BD Class I solutions, provided the correct iterated limit is taken. Such a uniqueness is essential for the validity of a recent version of the no-hair conjecture. In Sec. 4.3, a non-trivial modification to this version is proposed in order to exclude BD Class IV solutions which appear to represent scalar hair black holes. Sec.4.4 summarizes the contents.

4.2. THE LIMITING PROCESS

We start from the version by Saa [71] which read :

The only asymptotically flat, static and spherically symmetric exterior solution of the system governed by the action

$$S[g,\phi]=\int d^4x\sqrt{-g}\ [f(\phi)\bar{R} - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi], \qquad (4.2.1)$$

where $\bar{R}$ is a scalar curvature formed from $g^{\mu\nu}$; $f(\phi)$, $h(\phi) > 0$ with ϕ finite everywhere, is the Schwarzschild black hole solution. It should be noted that Saa [71] defines a black hole solution as one for which the scalar curvature $\bar{R}$ is finite at the metric singularity. In what follows, we shall retain this definition but, noting that Saa's generation technique remains valid for negative values of ω as well, relax the restrictive condition $h(\phi)>0$ and take $\omega \in(-\infty,+\infty)$. This is the first modification.

The BD field equations follow in the specific case where $h(\phi)=\omega\phi^{-1}$ and $f(\phi)=\phi$. We shall be concerned with the limiting process $|\omega| \to \infty$ that is supposed to provide a passage from BD to Einstein field equations. Another limiting process that will be shown to have a direct relationship with the conjecture is $r\to r_h$, where r is a radial coordinate and r_h is the coordinate horizon radius in a given solution. In the literature, the statements $r=r_h$ and $r\to r_h+$ are often understood to mean the same thing and there arises no essential difficulty as long as only this "single limit" process is considered. However, care must be exercised as soon as "double limit" or "iterated limit" processes come into play. For example, to verify the validity of the no-hair conjecture in the BD theory, it becomes necessary to study the nature of scalar curvature $\overline{R}$ in the limit of the null surface $(r\to r_h)$ and $|\omega| \to \infty$. In this case, it is essential to state exactly what type of limit process one has in mind and why. Unfortunately, in Ref.[71], the exact nature of the limiting process, if any, has been kept obscure although the final conclusions drawn therein are correct.

Incorrect limit processes lead to erroneous conclusions. For example, Matsuda [65] concludes that Schwarzschild exterior solution is not a *unique* limit of the BD Class I solutions of BD theory for ϕ=constant. There is also another solution, having a singular event horizon at the origin, that resembles Schwarzschild solution in the exterior but does not represent a black hole. This result clearly violates the no-hair conjecture formulated above.

In this section, we wish to examine various possible limiting processes and in particular argue that there is no violation of no-hair conjecture as far as BD Class I solutions are concerned, although naked singularities may occur under specific conditions. However, scalar hair black holes seem to occur in the case of Class IV solutions unless the above version of the conjecture is modified still further.

Let us consider Eqs. (2.2.1), (2.2.5)-(2.2.11) and (2.3.6) of Chapter 2 and recall that Schwarzschild exterior metric

$$ds^2 = -(1-2M/R)dt^2+(1-2M/R)^{-1}dR^2+R^2d\theta^2+R^2\sin^2\theta d\varphi^2 \qquad (4.2.2)$$

is defined strictly in the range $2M<R<\infty$, and the scalar curvature $\bar{R} \to$ finite value as $R \to 2M$. Therefore, $R=2M$ represents a null regular surface or a non-singular event horizon. In the BD metric (2.2.4) of Chapter 2, let us first choose finite values for ω (or, C)and λ such that $\Omega > 0$ and try to find out the R-coordinate range and the scalar curvature $\bar{R}$. Clearly, we get from equations (2.2.9) and (2.3.6) of Chapter 2, the single limit

$$\lim_{\substack{r \to B \\ \Omega > 0}} R \to 0, \quad \bar{R} \to \infty. \qquad (4.2.3)$$

Therefore, we get $0 < R < \infty$ and that there occurs an irremovable singularity as $R \to 0$. This result is in perfect accordance with the no-hair conjecture as $\phi \neq$ constant and the limiting metric too is not Schwarzschild. The detailed topology of such a point singularity has been studied by Agnese and La Camera [27]. Secondly, let us compute another single limit

$$\lim_{\substack{r > B \\ \Omega \to 0}} \bar{R} \to 0. \qquad (4.2.4)$$

This is an expected result as Class I solutions reduce to the

Schwarzschild exterior metric for which the Ricci tensor $R_{\mu\nu}=0$, and hence $\bar{R}=0$.

Thirdly, let us consider the limit: $r=B$, $\Omega \to 0+$. This is the case considered by Matsuda [65]. In this limit, both the metric component $[1-b(R)/R]^{-1}$ and the scalar curvature $\bar{R}$ are not even *defined*, let alone their existence, as they involve a division exactly by zero. The concept of a limit can not even be applied here. These are evident from Eqs.(2.2.11) and (2.3.6) of Chapter 2 respectively. Hence, the claim that $R=0$ is a singular event horizon is not strictly correct if we accept the definition that a singular surface is one where the scalar curvature diverges. Clearly, the divergence of a limit is distinct from the situation where the limiting process itself is not definable. Thus, using this limiting case, it is not possible to say whether or not $R=0$ constitutes a singular event horizon. The existence of Matsuda's non-black hole solution becomes untenable and our no-hair theorem remains effectively unchallenged.

Saa [71] seems to have calculated the scalar of curvature $\bar{R}$ at the exact equality $r=B$ and $\Omega=0$, and taken at the face value, $\bar{R}$ has an exact form $[0/0]$, which is by itself meaningless.

Clearly, the value has to be understood only in the sense of a limit, if it is defined. But then there arises the question: Should one take a double limit or iterated limits? There is no physical ground to prefer one operation to the other. However, the requirement of a double limit constitutes a much stronger condition than that of an iterated limit. The reason is that the existence of the former does imply the same of the latter, but the converse is not true. If we decide to compute iterated limits only, then the question is which one? Let us write the two iterated limits

$$\lim_{\Omega\to 0}\lim_{r\to B}\bar{R} = \lim_{\Omega\to 0}\infty = ? \tag{4.2.5}$$

$$\lim_{r\to B}\lim_{\Omega\to 0}\bar{R} = \lim_{r\to B} 0 = 0 \tag{4.2.6}$$

The former iterated limit does not exist in the usual sense [89] and therefore we can not say anything about the singularity or otherwise of $\bar{R}$ at $r=B$ ($r\to B$). Saa [71] actually *means* the second iterated limit and thereby arrives at the no-hair formulation. Nonetheless, the reason why we should prefer the limit (4.2.6) is not obvious.

There is also another case: Put $\Omega=0$. Then, $\lambda=C+1$, $C\neq 0$, $\lambda\neq 0$. We then have Schwarzschild exterior metric with $\phi \neq$ constant. Then we have $R\to 4B$ and $\bar{R}\to\infty$ as $r\to B$. This implies that the Schwarzschild sphere itself becomes an irregular null surface. This result is in perfect accordance with the no-hair conjecture. But note that $\Omega=0$ is not strictly a valid equality in the same sense as $\omega=\infty$ is not. In accordance with $\omega\to\infty$, we must take $\Omega \to 0$. Also we should take $r\to B$ in the computation of the scalar curvature $\bar{R}$ and then we have the above iterated limits at our disposal.

Unless we have a definite physical ground to prefer one of the iterated limits, the difference between the two remains an enigma. One plausible but by no means exclusive procedure could be to choose a certain path along which the two limits would be the same.

4.3. CLASS IV SOLUTIONS

These solutions are given by

$$\alpha(r) = \alpha_0 - \frac{1}{Br} \tag{4.3.1}$$

$$\beta(r) = \beta_0 + \frac{(C + 1)}{Br} \tag{4.3.2}$$

$$\phi = \phi_0 \, e^{-\frac{C}{Br}} \tag{4.3.3}$$

$$C = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2} \tag{4.3.4}$$

Usual asymptotic flatness and weak field conditions fix α_0, β_0 and B as

$$\alpha_0 = \beta_0 = 0, \; B = \frac{1}{M} > 0 \tag{4.3.5}$$

There is a singularity in the metric at r=0. The solutions (4.3.1)-(4.3.4) represent asymptotically flat, static and spherically symmetric solution of a system governed by the action (4.2.1). Also, $C \to 0$ as $\omega \to -\infty$. The scalar curvature turns out to be

$$\overline{R} = -2(1+C+C^2)(B^2 r^4)^{-1} e^{-2(C+1)/(Br)} \tag{4.3.6}$$

In the limit

$$\lim_{r \to 0} \overline{R} \to 0. \tag{4.3.7}$$

This implies that the degenerate surface r=o is nonsingular, no matter whether $C \to 0$ or not. Therefore, Class IV solutions may represent $\phi \neq$ constant black holes violating the modified no-hair conjecture. In the limit $C \to 0$, the metric does not become exactly Schwarzschild, although it is approximately so. It is given by

$$ds^2 = -e^{-2M/r} dt^2 + e^{2M/r}\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right]. \qquad (4.3.8)$$

By comparing this metric with the Robertson expansion for the static, spherically symmetric problem [90]:

$$ds^2 = -\left(1 - 2\alpha \frac{M}{r} + 2\beta \frac{M^2}{r^2} + \dots\right) dt^2 +$$

$$\left(1 + 2\gamma \frac{M}{r} + \dots\right)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right] \qquad (4.3.9)$$

we find that the parameters have the values

$$\alpha = \beta = \gamma = 1. \qquad (4.3.10)$$

Hence, the metric (4.3.8) is indistinguishable from the Schwarzschild metric in the sense of many experimental predictions. In this connexion, it is tempting to point out that the metric (4.3.8) is exactly of the same form as Rosen's metric in the bimetric theory of gravity [91]. Whether it is just a remarkable coincidence or there is a deeper connexion is a matter of further investigation.

Let us return to the no-hair conjecture. From the calculations above including the limit (4.3.7), modified conjecture is violated, whereas in the case of Class I solutions, it is not. In the latter case, recall that it is necessary to fix *two* conditions independently in order to go to the Schwarzschild metric: $C=C(\omega)$, $\lambda \to 1$. In the present case, on the other hand, there is no λ but just two parameters ω and C connected by Eq.(4.3.4). From this equation, it already follows that $C \to 0$ as $\omega \to -\infty$. That is ϕ=constant as $|\omega| \to \infty$ and the metric does not contain any unfixed parameter whatsoever. At first sight, it may appear that no other condition is necessary for the passage to the Einstein limit $G_{\mu\nu}=0$. But that is not so! In order that the r.h.s. of the matter-free BD Eqs. (2.2.3) vanish identically, we must impose, in addition to $|\omega| \to \infty$, an extra condition that $\omega C^2 \to 0$, without which the passage from BD to Einstein equations can not be ensured. But this condition is *not* satisfied in the case of Class IV solutions as it can be verified that $G^{0}_{0} \neq 0$. This feature shows itself up in the form of scalar hair ($C \neq 0$) black hole solutions. Therefore, we must add a second modification to the version proposed by Saa [71] in order to exclude Class IV types of solutions. The modified version of the conjecture should now read:

The only asymptotically flat, static and spherically symmetric exterior solution of the system governed by the action

$$S[g,\phi] = \int d^4x\sqrt{-g}\ [f(\phi)R - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi], \tag{4.3.11}$$

$f(\phi)>0$ with ϕ finite everywhere, *satisfying vacuum Einstein equations*, is the Schwarzschild solution.

4.4. CONCLUDING REMARKS

The computation of the limiting processes has a direct relevance to the no-hair conjecture. It is necessary to be specific about the type of limit process considered. Converting the iterated limit to a single limit process, one might land up with a conclusion that seemingly violates the conjecture. We are aware that some recent investigations imply that the conjecture *is* anyway violated under different conditions [86-88]. However, those cases do not correspond to the non-minimally coupled scalar field considered here.

We saw that Class IV solutions may represent black holes dressed by $\phi \neq$ constant scalar field. The cause is analyzed and a modification to the version is proposed. The modification is by no means trivial, as the counter example in the form of Class IV solutions testify.