ON THE WORMHOLE SOLUTIONS OF THE SCALAR-TENSOR THEORY OF GRAVITATION

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CERTIFICATE

This is to certify that the thesis entitled "ON THE WORMHOLE SOLUTIONS OF THE SCALAR-TENSOR THEORY OF GRAVITATION" which is being submitted by S.M.Khurshed Alam to the University of North Bengal, Darjeeling, for the degree of Doctor of Philosophy is a record of bonafide research work carried out by him under our joint supervision. The results embodied in the thesis have not been submitted to any other University or Institute for the award of any degree or diploma at home or abroad.

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CHAPTER 1

GENERAL INTRODUCTION

The topic of wormhole physics occupies a distinguished place in the frontiers of theoretical physics today. Wormholes are topology changes that connect two distant asymptotically flat regions. Over the last few years, considerable amount of investigation has gone into this area of theoretical physics, following especially the seminal works of Morris, Thorne and Yurtsever [1,2]. There have been efforts that concentrated on the exciting possibility of constructing time machines or of rapid interstellar travel. The standard methods use solutions of Einstein's field equations which are shown to give rise to "exotic" stress-energy distribution at least in some region of spacetime. The exotic materials violate the energy conditions and it may be that fundamental laws of physics forbid the existence of such materials on macroscopic scales. However, with the developments of quantum field theory, the situation has changed. An early example has been provided by Zel'dovich [3] in which a quantum field leads macroscopically to an exotic equation of state. Casimir effect or squeezed vacuum states

[4-6] also lead to a stress-energy tensor that violates energy conditions. Such violations however need not always contradict the positive mass theorem [7]. Quite recently, Vollick [8] has demonstrated a remarkable result that negative energy densities may occur even in the classical regime via interaction of classical fields with gravity. Therefore, all in all, there is good reason to look for wormhole solutions even when the stress-energy is provided by classical fields.

The primary aim of this thesis is to examine the possibility of static, spherically symmetric wormhole solutions in the scalar-tensor theories such as the Brans-Dicke (BD) theory and the string modified four- dimensional gravity. As a related study, we shall also attempt to focus on the "no-hair" theorem attendent upon such theories.

In order to make the contents of the thesis self-contained as far as possible, we shall systematically present relevant basic materials in different sections. In Sec. 1.1, we describe what is known as a Morris-Thorne wormhole. Sec. 1.2 provides a survey of recent works including those that focus on the applications of wormhole physics. In Sec. 1.3, we display energy conditions relevant to wormhole scenarios. This will be

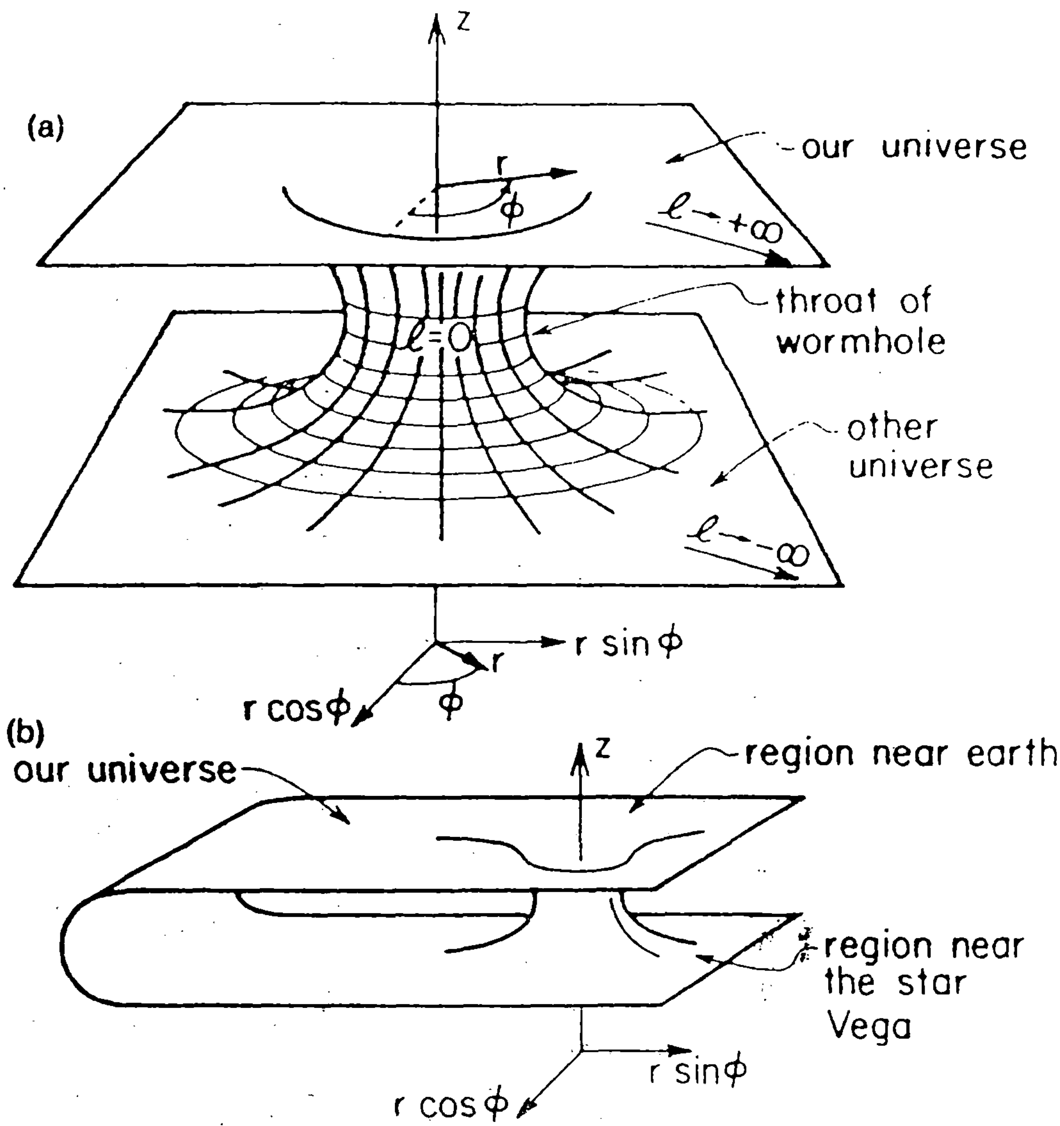

Fig. 1. (a) Embedding diagram for a wormhole that connects two different universes. (b) Embedding diagram for a wormhole that connects two distant regions of our own universe. Each diagram depicts the geometry of an equatorial ($\theta = \pi/2$) slice through space at a specific moment of time ($t =$ const)

followed by a detailed statement in Sec. 1.4 of the objectives pursued in this thesis.

1.1. MORRIS-THORNE TRAVERSABLE WORMHOLE

Wormhole geometries can be embedded into the spacetime that follow as exact solutions of Einstein's equations. The embedding diagram connects two different universes or two distant regions of our own universe (Fig. 1). Only the topologies differ and Einstein field equations do not constrain the topology of a solution. Depending on the physical conditions, it is possible that a traveler may pass from one region to another through what is termed as the "throat" of the wormhole. These are called traversable wormhole. Below we shall quote its essential properties from Ref. [1] :

(i) Basic Wormhole Criteria

(1) The metric should be both spherically symmetric and static (time independent). The requirement is imposed only to simplify the calculations, and one should keep in mind that the wormhole might be unstable to spherical or nonspherical perturbations.

(2) The solution must everywhere obey the Einstein field equations. We assume the correctness of general relativity theory.

(3) To be a wormhole the solution must have a throat that connects two asymptotically flat regions of spacetime.

(4) There should be no horizon, since a horizon, if present, would prevent two-way travel through the wormhole.

(ii) Usability Criteria

(1) The tidal gravitational forces experienced by a traveler must be bearably small.

(2) A traveler must be able to cross through the wormhole in a finite and reasonably small proper time (e.g., less than a year) as measured not only by herself/himself, but also by observers who remained behind or who await her/him outside the wormhole.

(iii) Physical Criteria

The matter and fields that generate the wormhole's spacetime curvature must have a physically reasonable stress-energy tensor. It turns out that the form of the stress-energy tensor is strongly constrained by the preceding six properties. That constrained form in fact violates what we usually mean by "physically reasonable".

(2) The solution should be perturbatively stable (especially as a spaceship passes through). Enforcing this requirement would involve a time-dependent and nonspherical analysis.

(3) It should be possible to assemble the wormhole. For instance, the assembly should require both much less than the mass of the universe and much less than the age of the universe. Although not enough is known to permit a quantitative analysis, present knowledge of quantum gravity suggests that assembly *might* be possible.

(a) Form of the metric

The form of the metric is taken in the Morris-Thorne canonical form given by

$$ds^2=-e^{2\Phi}c^2dt^2+dr^2/(1-b/r)+r^2(d\theta^2+\sin^2\theta\ d\varphi^2). \quad (1.1.1)$$

Here, $\Phi=\Phi(r)$ and $b=b(r)$ are two arbitrary functions of radius only and c is the speed of light in vacuum. The function b(r) determines the spatial shape of the wormhole, so we shall call it the "shape function" and $\Phi(r)$ determines the gravitational redshift, so we shall call it the "redshift function". Notice that the radial coordinate r has special geometric significance: $2\pi r$ is the circumference of a circle centered on the wormhole's throat, and thus r is equal to the embedding-space radial coordinate. As a result, r is nonmonotonic: It decreases from $+\infty$ to a minimum value, b_o, as one moves through the lower universe of Fig. 1 toward the wormhole and into the throat; then it increases from b_o back to $+\infty$ as one moves out of the throat and into the upper universe.

We define an orthonormal static coordinate system for which the basis vectors are

$$e_{\hat{t}} = e^{-\Phi}e_t, \quad e_{\hat{r}} = (1 - b/r)^{1/2}e_r, \quad (1.1.2)$$

$$e_{\hat{\theta}} = r^{-1} e_{\theta}, \qquad e_{\hat{\varphi}} = (r \sin\theta)^{-1} e_{\varphi} \tag{1.1.3}$$

In this basis the metric coefficients take on their standard, special relativity forms,

$$g_{\hat{\alpha}\hat{\beta}} = e_{\hat{\alpha}} \cdot e_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.1.4}$$

The nonzero components of the Einstein tensor in this proper reference frame are

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2}, \tag{1.1.5}$$

$$G_{\hat{r}\hat{r}} = -\frac{b}{r^3} + \frac{2(1 - b/r)\Phi'}{r}, \tag{1.1.6}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = \left(1 - \frac{b}{r}\right)\left(\Phi'' - \frac{b'r - b}{2r(r - b)}\,\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r - b)}\right). \tag{1.1.7}$$

The stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$ must have the same algebraic structure as the $G_{\hat{\mu}\hat{\nu}}$ and we take them as

$$T_{\hat{t}\hat{t}} = \rho(r)c^2, \qquad T_{\hat{r}\hat{r}} = -\tau(r), \qquad \text{and} \qquad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\varphi}\hat{\varphi}} = p(r). \tag{1.1.8}$$

where $\rho(r)$ is the total density of mass-energy that they measure (in units of g/cm^3), $\tau(r)$ is the tension per unit area that they measure in the radial direction (i.e., it is the negative of the radial pressure and has units dyn/cm^2) and $p(r)$ is the pressure (in dyn/cm^2) that they measure in lateral directions (directions orthogonal to radial).

We can rewrite Eqs. (1.1.5)-(1.1.7) in a form

$$\rho = \frac{b'}{8\pi G c^{-2} r^2}, \tag{1.1.9}$$

$$\tau = \frac{b/r - 2(r - b)\Phi'}{8\pi G c^{-4} r^2} \tag{1.1.10}$$

$$p = \frac{r}{2}\left[\left(\rho c^2 - \tau\right)\Phi' - \tau'\right] - \tau. \tag{1.1.11}$$

Clearly, the choices of $b(r)$ and $\Phi(r)$ determine the stress-energy tensor required to construct the Morris-Thorne wormhole.

(b) Embedding

Let us consider the line element for an equatorial slice,

$\theta=\pi/2$, and t=constant from Eq. (1.1.1). It is given by

$$ds^2 = (1 - b/r)^{-1}dr^2 + r^2d\varphi^2. \qquad (1.1.12)$$

We shall remove this slice and embed it in Euclidean space for which the metric has the form

$$ds^2 = dz^2 + dr^2 + r^2d\varphi^2. \qquad (1.1.13)$$

The embedded surface will be axially symmetric described by a single function z=z(r). On that surface the line element will be

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)\right]dr^2 + r^2d\varphi^2. \qquad (1.1.14)$$

Identifying Eq. (1.1.14) with Eq. (1.1.12), we find that the embedded surface z=z(r) satisfies

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-1/2}. \qquad (1.1.15)$$

(c) Wormhole throat

The minimum radius of the wormhole, called its throat, occurs at a radial value $r=r_0$ for which $b(r_0)=r_0$. Also, the proper radial distance is defined by

$$l(r) = \pm \int_{b_0}^{r} \frac{dr}{[1 - b(r)/r]^{1/2}} \qquad (1.1.16)$$

In order that l(r) is well behaved everywhere, we require that

$$(1 - b/r) \geq 0 \text{ throughout spacetime.} \qquad (1.1.17)$$

Far from the throat in both radial directions space must become asymptotically flat; i.e., $dz/dr = \pm(r/b - 1)^{-1/2}$ must approach zero as $l \to \pm \infty$; i.e., $b/r \to 0$, as $l \to \pm \infty$. Note that Eqs. (1.1.15) and (1.1.16) imply that for the embedded wormhole

$$\frac{dz}{dl} = \pm\left(\frac{b}{r}\right)^{1/2} \text{ and } \frac{dr}{dl} = \pm\left(1 - \frac{b}{r}\right)^{1/2}. \qquad (1.1.18)$$

(d) Absence of a horizon and tidal forces

A wormhole will be traversable if it does not possesses any horizon. This implies that $\Phi(r)$ must be everywhere finite. The tidal gravitational forces as measured by a static observer depend on the values of the curvature tensor in the proper reference frame at that point. The equation is given by the well known equation for geodesic deviation. For the metric (1.1.1) the nonvanishing curvature components are given by $R_{\hat{0}\hat{1}\hat{0}\hat{1}}$, $R_{\hat{0}\hat{2}\hat{0}\hat{2}}$, $R_{\hat{0}\hat{3}\hat{0}\hat{3}}$, $R_{\hat{1}\hat{2}\hat{1}\hat{2}}$, $R_{\hat{1}\hat{3}\hat{1}\hat{3}}$ and $R_{\hat{2}\hat{3}\hat{2}\hat{3}}$. In a Lorentz boosted frame (*) the nonvanishing curvature will be

$$R_{0^*1^*0^*1^*}=R_{0101}, \quad R_{0^*k1^*k}=\cosh\alpha\,\sinh\alpha(R_{0k0k}+R_{1k1k}), \tag{1.1.19}$$

$$R_{0^*k0^*k}=R_{0k0k}+\sinh^2\alpha(R_{0k0k}+R_{1k1k}), \tag{1.1.20}$$

$$R_{1^*k1^*k}=R_{1k1k}+\sinh^2\alpha(R_{0k0k}+R_{1k1k}), \tag{1.1.21}$$

$$\cosh\alpha=(1-v^2)^{-1/2}, \quad \sinh\alpha=v(1-v^2)^{-1/2}, \tag{1.1.22}$$

and R_{k1k1} for k=2,3. For a wormhole to be traversable the tidal acceleration must be finite both in the static and Lorentz boosted frame.

(e) Flaring out condition

The requirement that the wormhole be connectible to asymptotically flat spacetime entails at the throat that the embedding surface flare outward. This requires that the embedding function $r(z)$ must satisfy $d^2r/dz^2>0$ at or near the throat, $r=b$. That is

$$\frac{d^2 r}{dz^2} = \frac{b - b' r}{2b^2} >0. \tag{1.1.23}$$

We can define a dimensionless function,

$$\zeta \equiv \frac{\tau - \rho c^2}{|\rho c^2|} = \frac{b/r - b' - 2(r - b)\Phi'}{|b'|}, \tag{1.1.24}$$

and rewrite it in the form

$$\zeta = \frac{2b^2}{r|b'|}\left(\frac{d^2 r}{dz^2}\right) - 2(r - b)\frac{\Phi'}{|b'|}. \tag{1.1.25}$$

The finiteness of b' and the fact that $(r-b)\Phi' \to 0$ at the throat enables us to rewrite the flaring out condition as

$$\zeta_0 = \frac{\tau_0 - \rho_0 c^2}{|\rho_0 c^2|}>0 \tag{1.1.26}$$

at or near the throat, $r=b=b_0$.

Matter field satisfying $\tau>\rho c^2$ is called "exotic". One of the reason is that the signal speed $|\tau/\rho|^{1/2}$ in the exotic material exceeds the vacuum speed of light. An equivalent condition of $\tau > \rho c^2 > 0$ can be obtained by going over to a Lorentz boosted frame. It then follows that it is sufficient to have $\rho c^2 < 0$ for matter to be exotic.

So far we have reproduced the basic outline of the Morris-Thorne wormhole as developed in Ref. [1]. Whenever necessary other aspects will be provided at appropriate places.

1.2. ENERGY CONDITIONS

Energy conditions are supposed to provide constraints on classical or quantum wormholes threaded by classical or quantum fields. There exist several pointwise or global energy conditions [4]. For example, the weak energy condition (WEC) states that $T_{\hat{\mu}\hat{\nu}}\xi^{\mu}\xi^{\nu} \geq 0$ for any timelike vector ξ_{ν}. Written out in full, this implies

$$\rho \geq 0, \quad \rho + \tau \geq 0, \quad \rho + p \geq 0. \tag{1.2.1}$$

For quantum fields, Ford and Roman [9] have proposed, on the basis of certain assumptions, an inequality that constrains the magnitude of negative energy density at the throat of a traversable wormhole. A fundamental assumption for quantum wormholes is that the stress-energy of the spacetime is a renormalized expectation value of the energy momentum operator in some quantum state, say, $|\psi\rangle$. In the literature [10,11], one actually considers field equations of semiclassical gravity in the form $G_{\mu\nu}=8\pi\langle\psi|T_{\mu\nu}|\psi\rangle$. However, some doubts have been raised, notably by Unruh [12], as to whether field equations in this form could be an exact description of gravity [13]. On the other hand, quantized source fields obey well defined uncertainty relations and it is expected that uncertainty in the source would induce uncertainty in the gravidynamic variables and in the light cone structure of spacetime [14,15]. If the source is taken as $\langle T_{\mu\nu}\rangle$, such fluctuations would not occur. Despite these questions, it must be emphasized that field equations in the above form provide a very good approximation in many physical situations, especially in the description of early universe [16].

There also exist classical fields playing the role of "exotic matter" that violates the WEC, at least at the throat of the wormhole. Examples are provided by the stress-energy tensors occurring in theories where the action contains: $R+R^2$ terms [17], an antisymmetric 3-form axion field coupled to scalar fields [18], minimally coupled fields with a self-interacting potential [19]. Other theories include string inspired 4-dimensional gravity coupled nonminimally to a scalar field [20], Zee's induced gravity [21] and the BD scalar-tensor theory [22]. Most of the analyses dealt with dynamic rather than static wormholes. Unfortunately, investigations aimed at finding out static wormhole solutions in the alternative theories of gravity such as BD scalar-tensor theory have been relatively few [23,24]. On the other hand, it is well known that in the limit $|\omega|\to\infty$, BD theory goes over to Einstein's general relativity (GR). Hence, it is desirable that a thorough investigation is carried out with regard to the possible existence of wormhole solutions in the theories of gravity coupled to scalar and vector fields.

It has recently been shown by Vollick [8] that an interaction stress-energy tensor between matter fields does not satisfy WEC although the interacting matter fields individually do. It is this interaction energy that maintains a wormhole with a scalar field. It has also been shown recently by Anchordoqui, Bergliaffa and Torres [25] that the wormhole throat in the BD theory in the presence of matter need not be necessarily threaded with exotic matter.

1.3. APPLICATIONS OF WORMHOLE PHYSICS

There have been attempts to interpret and predict new physical phenomena in terms of wormhole scenarios. Cramer, Forward, Morris, Visser, Benford and Landis [26] attempted to interpret the phenomenon of gravitational lensing in terms of Visser wormholes [4] which are born out of cut-and-paste surgery. Take two copies of Schwarzschild spacetime, and remove from each manifold the four-dimensional regions described by

$$\Omega_{1,2} \equiv (r_{1,2} \leq a \mid a > 2M) \tag{1.3.1}$$

where a is a constant. The resulting manifolds have boundaries given by the timelike hypersurfaces

$$\partial\Omega_{1,2} \equiv (r_{1,2} = a \mid a > 2M). \tag{1.3.2}$$

Now *identify* these two timelike hypersurfaces (i.e., $\partial\Omega_1 \equiv \partial\Omega_2$). The resulting manifold M is geodesically complete and possesses two asymptotically flat regions connected by a traversable Lorentzian wormhole. The throat of the wormhole is at $\partial\Omega$.

Because M is piecewise Schwarzschild, the Einstein tensor is zero everywhere *except* at the throat, where it is formally singular: the Einstein tensor is a Dirac distribution on the manifold. Using the field equations, the surface stress-energy tensor can be calculated in terms of the jump in the second fundamental form across $\partial\Omega$. This prescription is known as the "thin-shell formalism" [4].

Visser [4] suggested a particular thin-shell wormhole configuration, a flat-space wormhole that is framed by "struts" of an exotic material, a variant of the cosmic string solutions of Einstein's equations [27,28]. To satisfy the Einstein field equations the cosmic string framing Visser wormholes must have a *negative* string tension [29] of -1/4G and therefore a negative mass density. However, for the total mass of the wormhole system, the negative mass density of the struts should be combined with the effective positive mass density of the wormhole's gravitational field. The overall object could, depending on the details of the model, have positive, zero, or negative net external mass. Note that, in hypothesizing the existence of such a wormhole, one has to abandon the averaged null energy condition [1,2]. Indeed, in a generic curved spacetime, it is possible to prove that the averaged null energy condition fails [30]. Therefore, the hypotheses underlying the positive mass theorem no longer apply and there is nothing, in principle, to prevent the occurrence of negative total mass [31].

If a particle with positive electric charge passes through such a wormhole, its lines of force, threading through the wormhole aperture, give the entrance mouth an effective positive charge (flux lines radiating outward) and give the exit mouth an effective negative charge (flux lines converging inward). Similarly, when a massive object passes through the wormhole, the same back-reaction mechanism might cause the entrance mouth to gain mass and the exit mouth to lose mass [32]. Now let us consider a stable Visser wormhole with near-zero mass residing in the mass-energy-rich environment of the early universe. The expected density fluctuations of the early universe suggest that the separated wormhole mouths will reside in regions of differing mass density, leading to a mass flow between the regions they connect. As this mass passes through the wormhole, the entrance mouth will gain mass while the exit mouth will lose mass by the same amount. Soon, if the mass flow continues, the exit wormhole mouth will acquire a net negative mass.

This will lead to a gravitational instability, since the positive mass mouth will attract more mass through its aperture while the negative mass mouth will gravitationally repel nearby mass [33]. Thus the positive-negative imbalance will be fed by gravity and will continue to grow. If this process proceeds without interruption, the exit wormhole mouth might develop a stellar-scale negative mass. Visser wormholes, therefore, provide at least one motivation for seriously considering the possible existence of naturally occurring astronomical objects of sizable negative mass.

Negative mass objects, while repelling all nearby mass (positive or negative), would themselves be attracted [33] by the mass of a nearby galaxy and might form part of a galactic halo. They would have unusual gravitational properties that could produce detectable gravitational lensing effects. These lensing effects, for the same absolute mass, are of the same magnitude as those recently detected for massive cosmic halo objects (MACHO's) [34,35]. Cramer *et al* [26] examined in detail the lensing effects of gravitationally negative anomalous compact halo objects (GNACHO's) and they recommend that MACHO search data be analyzed for GNACHO's.

Another interesting application of wormhole topology lies in the field of cosmology. For example, Hochberg and Kephart [36] attempted to solve cosmological problem in terms of dynamic wormhole "handles" which connect a Friedmann universe onto itself. The present observation of isotropicity and uniformity in temperature of the cosmic microwave background radiation (CMBR) can not be explained by the standard big bang model. The reason is that the CMBR is received from regions which were not in causal contact at the time of last scattering. This is commonly known as the horizon problem and the generally accepted resolution invoke an inflationary period in the early universe. On the other hand, as we saw in Sec. 1.1, wormhole solutions require that the WEC be violated in the vicinity of the wormhole throat. It is also known that WEC violating quantum fields occur in the gravitationally squeezed state [5,6]. Therefore, the best place to explore wormhole physics is in the quantum or semiclassical regime during the planck epoch in the early universe. On these grounds, Hochberg and Kephart [36] suggest another resolution which envisages that the universe was populated with traversable wormholes connecting otherwise causally disconnected regions of the early universe leading to a rapid thermalization which has persisted in the observed CMBR today.

Most recently González-Díaz [37] has proposed the idea of spacetime tunnels which are wormholes and "ringholes" existing naturally in the universe. Two-sphere tunnel topology give rise to wormholes and two-torus topologies give rise to what are called ringholes. Embedding them in the Friedmann universe leads to many possible observable effects including lensing, frequency shift of emitting sources, discontinuous change of background temperature, broadening and intensity enhancement of spectral lines, as well as an unexpected increase of luminosity of any object at the tunnel's throat.

Cavaglià, de Alfaro and de Felice [38] have suggested that gravitational Euclidean wormholes (+,+,+,+) obtained through coupling of gravity to electromagnetism can be interpreted as a tunneling to a hyperbolic baby universe. This Euclidean instanton can also be interpreted as a static wormhole that joins two asymptotically flat spaces of a Reissner-Nordström type solution. Euclidean wormholes in the early universe lead to consequences such as quantum decoherence [39-41] and the vanishing of cosmological constant [42,43].

On the theoretical side some notable works are worth mentioning. Coule and Maeda [18] have discussed wormhole solutions in a theory with an axion field coupled to scalar fields. Earlier works involving the 3-form axion field carried out by Giddings and Strominger [44]. Lee [45] obtained wormhole solutions for a spontaneously broken complex field. Brown, Bergess, Kshirsagar, Whiting and York [46] used an Euclideanised scalar field and Hosoya and Ogura [47] used SU(2) Yang-Mills theory to obtain wormhole solution. Hochberg, Popov and Sushkov [10] obtained Lorentzian wormhole solutions of semiclassical Euclidean field equations coupled to a quantum scalar field. Schein and Aichelburg [11] constructed static axisymmetric traversable wormhole solutions of two charged shells kept in equilibrium by their electromagnetic repulsion. Kar and Sahdev [48] have obtained evolving Lorentzian wormholes for which the time spent over which the WEC satisfied can be made as large as one wishes. Evolving wormholes were also considered by Roman [49], Wang and Letelier [50]. Barceló, Garay, Gonzàlez-Díaz and Marugán [51] have constructed a quantum theory for asymptotically anti-de Sitter wormholes. Dynamic Euclidean wormhole solutions have been obtained as a solution of the Wheeler-DeWitt equation in the context of BD theory have been obtained by Xiao, Carr and Liu [52]. Kar [53] has obtained exact solutions of the string equations of motion in a specific Lorentzian wormhole background. Hochberg and

Visser [54] have investigated into the geometric structure of generic static traversable wormhole throat. They have also derived generalized theorems regarding violations of the energy conditions which apply to a wide-ranging class of wormholes. Balbinot, Barrabès and Fabbri [55] have discussed the traversability and stability of wormholes consisting of closed Friedmann-like regions connected by Reissner-Nordström black holes.

1.4. DETAILED OBJECTIVES

As already mentioned before, our primary concern in this thesis would be to examine the possibility of static, spherically symmetric wormhole solutions in the scalar-tensor theories of gravitation. For the vacuum BD theory, we consider both the original (Jordan) frame and the conformally rescaled (Einstein frame) frame. Our aim is to find out the exact range of the coupling parameter ω for which such wormholes are possible. We shall also investigate the question of traversability. These are accomplished in Chapter 2. A further realm of investigation is provided by the now well known string theory which, in the low-energy limit, provides a four-dimensional theory of gravity characterized by an action containing couplings among gravity, dilaton and Maxwell fields. We shall be particularly interested in the wormhole solutions when the kinetic term in the action has a negative sign. These stringy wormholes are investigated in Chapter 3.

In a theory where gravity couples to scalar fields, a pertinent inquiry is whether the no-hair theorem is satisfied. In the context of BD theory, we shall examine, in Chapter 4, the no-hair theorem and focus on the limiting process involved. Carrying the analysis further, we shall also take a cue from Bekenstein's no-hair theorem and compare the scalar dressing of black holes in the BD and conformally coupled scalar field theory. This is carried out in Chapter 5.

Very recently, Horowitz and Ross [56,57] have proposed the idea of naked black holes (NBH). We shall present, in Chapter 6, two NBH solutions coming from dilaton-Maxwell gravity which satisfy WEC.

For convenience, the chapters are divided into subsections as displayed in the CONTENTS . Each chapter is provided with an introduction and concluded with a summary and/or remarks. All the references are appended in the end. A general summary of the contents is also annexed.

CHAPTER 2

BRANS-DICKE WORMHOLES IN THE JORDAN AND EINSTEIN FRAMES

2.1. INTRODUCTION

The utility of Brans-Dicke (BD) theory in interpreting the solar system experiments is wellknown. The precision of perihelion of a planet sets a lower bound to the coupling parameter $\omega \gtrsim 6$. Recent laser ranging experiments have increased the limit to $\omega > 500$ [58]. The usefullness of BD theory in the interpretation of different astrophysical phenomena beyond the solar system is also wellknown. Recent works indicate that the presence of scalar field ϕ may explain the observed flat rotation curves in the vast domain of dark galactic halos and thereby solve the dark matter problem [59,60]. Harada *et al* [61] have investigated the possibility of scalar gravitational waves in the Oppenheimer-Synder collapse of a dust ball in the scalar-tensor theory.

In this chapter, we intend to examine the possibility of BD wormhole solutions in the Jordan and Einstein frames which are defined as follows [62]: The pair of variables (metric $g_{\mu\nu}$, scalar φ) defined originally in the BD theory constitute what is called Jordan frame. Consider now a conformal rescaling

$$\tilde{g}_{\mu\nu} = f(\varphi) g_{\mu\nu}, \ \phi = g(\varphi) \tag{2.1.1}$$

such that, in the redefined action, ϕ becomes minimally coupled to $\tilde{g}_{\mu\nu}$ for some functions $f(\varphi)$ and $g(\varphi)$. Then the new pair $(\tilde{g}_{\mu\nu}, \phi)$ is said to constitute an Einstein frame. There exist different viewpoints as to the question of which of these two frames are physical, but the arguments of Magnano and Sokołowski [62] seem convincing enough in favor of the physicality of the Einstein frame.

In what follows, we shall be concerned only with static, spherically symmetric solutions of the BD theory. For this purpose, only Class I type of solution is considered; other Classes (II-IV) of solutions can be dealt with in a similar way. Our results are sectionwise stated as follows: In Sec. 2.2, we consider the Jordan frame and derive the general condition for the existence of wormholes. This condition is then used to find wormhole ranges of ω in specific cases. Sec. 2.3 shows that these wormholes are not traversable due to the occurrence of naked singularity. Einstein frame is considered in Sec. 2.4, and it is shown that wormhole solutions do not exist at all in that frame. The last section (Sec. 2.5) summarizes the contents.

2.2. JORDAN FRAME

In order to investigate the possibility of wormholes in the vacuum (matter free) BD theory, it is convenient to cast the spacetime metric in the Morris-Thorne canonical form (1.1.1), repeated here :

$$d\tau^2 = -e^{2\Phi(R)}dt^2 + \left[1 - \frac{b(R)}{R}\right]^{-1}dR^2 + R^2 d\Omega_2^2 \quad (2.2.1)$$

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$$

where $\Phi(R)$ and $b(R)$ are defined to be the redshift and shape functions respectively. These functions are required to satisfy some constraints, enumerated in Sec. 1.1, in order that they represent a wormhole. It is however important to stress that the choice of coordinates (Morris-Thorne) is purely a matter of convenience and not a physical necessity. For instance, one could equally well work directly with isotropic coordinates using the analyses of Visser [4] but the final conclusions would be the same. Nonetheless, it must be understood that a more appropriate procedure should involve coordinate independent proper quantities.

The matter free action in the Jordan variables is (G=c=1):

$$S = \frac{1}{16\pi}\int d^4x \; (-g)^{1/2}\left(\varphi R - \varphi^{-1}\omega(\varphi) g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right) \qquad (2.2.2)$$

The field equations are

$$\Box^2\varphi = 0$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = -\frac{\omega}{\varphi^2}\left[\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{\mu\nu}\varphi_{,\rho}\varphi^{,\rho}\right]$$

$$- \frac{1}{\varphi}\left[\varphi_{;\mu;\nu} - g_{\mu\nu}\Box^2\varphi\right], \qquad (2.2.3)$$

where $\Box^2 \equiv (\varphi^{;\rho})_{;\rho}$ and ω is a dimensionless coupling parameter. The general solution, in isotropic coordinates (r,θ,φ,t), is given by:

$$d\tau^2 = - e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\nu(r)}r^2 d\Omega_2^2. \qquad (2.2.4)$$

Brans class I solutions [63] correspond to the gauge $\beta - \nu = 0$, and are given by

$$e^{\alpha(r)} = e^{\alpha_0}\left[\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right]^{\frac{1}{\lambda}} \tag{2.2.5}$$

$$e^{\beta(r)} = e^{\beta_0}\left[1 + \frac{B}{r}\right]^2 \left[\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right]^{\frac{\lambda-C-1}{\lambda}} \tag{2.2.6}$$

$$\varphi(r) = \varphi_0\left[\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right]^{\frac{C}{\lambda}} \tag{2.2.7}$$

$$\lambda^2 \equiv (C+1)^2 - C(1 - \frac{\omega C}{2}) > 0, \tag{2.2.8}$$

α_0, β_0, B, C, φ_0 are constants. The constants α_0, β_0 are determined by asymptotic flatness condition as $\alpha_0 = \beta_0 = 0$.

Redefining the radial coordinate $r \to R$ in the metric (2.2.4) as

$$R = re^{\beta_0}\left[1 + \frac{B}{r}\right]^2\left[\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right]^{\Omega}, \quad \Omega = 1 - \frac{C+1}{\lambda} \tag{2.2.9}$$

we obtain the following functions for $\Phi(R)$ and b(R):

$$\Phi(R) = \alpha_0 + \frac{1}{\lambda}\left[ln\left\{1 - \frac{B}{r(R)}\right\} - ln\left\{1 + \frac{B}{r(R)}\right\}\right] \qquad (2.2.10)$$

$$b(R) = R\left[1 - \left\{ \frac{\lambda\{r^2(R)+B^2\} - 2r(R)B(C+1)}{\lambda\{r^2(R)-B^2\}} \right\}^2\right]. \qquad (2.2.11)$$

The throat of the wormhole occurs at $R=R_0$ such that $b(R_0)=R_0$. This gives minimum allowed r-coordinate radii $r_0^{\pm}$ as

$$r_0^{\pm} = \alpha^{\pm} B \qquad (2.2.12)$$

$$\alpha^{\pm} = (1 - \Omega) \pm \sqrt{\Omega(\Omega - 2)} \qquad (2.2.13)$$

The values $R_0^{\pm}$ can be obtained from Eq.(2.2.9) using this $r_0^{\pm}$. Noting that $R \to \infty$ as $r \to \infty$, we find that $b(R)/R \to 0$ as $R \to \infty$. Also $b(R)/R \leq 1$ for all $R \geq R_0^{\pm}$. The redshift function $\Phi(R)$ has a singularity at $r = r_s = B$. In order that a wormhole be just geometrically traversable, the minimum allowed values $r_0^{\pm}$ must exceed $r_s = B$. It can be immediately verified from Eq.(2.2.9) that $r_0^{\pm} \geq B \Rightarrow R_0^{\pm} \geq 0$. This is possible only if the range of Ω is chosen either as $-\infty<\Omega\leq 0$ or as $2<\Omega<\infty$. We shall not consider the latter range here.

The energy density of the wormhole material is given by Eq. (1.1.9)

$$\rho(R) = (8\pi R^{-2})(db/dR) \tag{2.2.14}$$

and a straightforward calculation gives

$$db/dR = 4r^2(R)B^2[r^2(R)-B^2]^{-2}\Omega(2-\Omega).$$

$$=4r^2(R)B^2[r^2(R)-B^2]^{-2}[1-(\frac{C+1}{\lambda})^2]. \tag{2.2.15}$$

Therefore, the most general condition for the violation of WEC is that

$$C(\omega)+1>\lambda(\omega), \tag{2.2.16}$$

where the real function $C(\omega)$ is as yet unspecified. As long as the general condition (2.2.16), which ensures $R_o^{\pm} > 0$, is satisfied, it follows that

$$b_o' = \frac{db}{dR}\Big|_{R=R_o^{\pm}} = -1 \tag{2.2.17}$$

so that $\rho_o = \rho|_{R=R_o^{\pm}} < 0$, and a violation of WEC at the throat is achieved thereby. In the limit $r_o^{\pm} \to B+$, or, equivalently, $R_o^{\pm} \to 0+$, one obtains $\rho_o \to -\infty$. This means that there occurs an infinitely large concentration of exotic matter at the throat when its r-radius is in the vicinity of the Schwarzschild radius $r_s = B$. No upper limit to this classical negative energy density is known to us. The general profile for $\rho(R)$ for a given wormhole configuration is that $\rho(R)$ attains its maximum at the throat, and falls off in an inverse square law as one moves away from the throat to the asymptotic region.

The constraint (2.2.16) can be rephrased, using (2.2.8), as

$$C(\omega)\left[1-\frac{\omega C(\omega)}{2}\right] > 0 \tag{2.2.18}$$

and depending on the form of $C(\omega)$, this inequality fixes the range of wormhole values of ω, provided one excludes the forbidden range coming from the requirement that $\lambda^2>0$. A further exclusion of the range $\omega\leq-3/2$ comes from a "physical" requirement that the theory be transferrable to Einstein frame [62]. In the limiting case, $C(\omega)\to0$, $\lambda(\omega)\to1$ as $\omega\to\infty$, one simply recovers Schwarzschild exterior metric in standard coordinates from Eqs.(2.2.10) and (2.2.11), so that $b(R)=2M$ and $b'_0=0$. The inequality (2.2.18) is violated, and there occurs no traversable wormhole, as expected [1].

The analysis of Agnese and La Camera [23] corresponds, as pointed out earlier [24], to the choice

$$C(\omega) = -\frac{1}{\omega+2} \tag{2.2.19}$$

which suggests, via (2.2.18), a wormhole range $\omega<-4/3$. The forbidden range turns out to be $-2<\omega<-3/2$ which is already a part of the unphysical range $\omega\leq-3/2$. Therefore, one is left with a very narrow actual interval for wormhole solutions, viz., $-3/2<\omega<-4/3$. It appears that the authors just missed this interval [64].

We should recall here that Eq.(2.2.19) is derived on the basis of a weak field (post Newtonian) approximation and there is no reason for Eq. (2.2.19) to hold for the stars with a strong field such as neutron stars. In reality, if we assume such a restriction as Eq.(2.2.19) the junction conditions for the metric and the scalar field are not satisfied at the boundary of the stars [65]. Evidently, any form for $C(\omega)$ different from Eq.(2.2.19) would lead to a different wormhole interval for ω. For example, in the context of gravitational collapse in the BD theory, Matsuda [65] chose $C(\omega)\propto -\omega^{-1/2}$. Let us take $C(\omega)=-q\omega^{-1/2}$ and choose $q<0$ such that $C(\omega)>0$. Then the constraint (2.2.18) will be satisfied only if $\omega > 4/q^2$. The exact form of $C(\omega)$ should be known *a priori* from other physical considerations. However, this is just a tentative example and is meant to highlight how crucially the wormhole range for ω depends on the form of $C(\omega)$.

The constraint (2.2.16) is based only on the requirement of geometric traversability i.e., on the requirement that the throat radii be larger than the event horizon radius r=B. Therefore, an immediate inquiry is whether such wormholes are traversable in practice. We discuss this issue in the following section.

2.3. TRAVERSABILITY

In order to get a first hand idea about traversability in the Jordan frame, a convenient procedure is to calculate the scales over which wormhole functions change. Ford and Roman [9] defined the following quantities at the throat $R=R_0$ of a traversable wormhole:

$$\bar{r}_0 = R_0, \; r_1 = \frac{R_0}{|b'_0|}, \; R_2 = \frac{1}{|\Phi'_0|}, \; r_3 = \left|\frac{\Phi'_0}{\Phi''_0}\right|. \quad (2.3.1)$$

These quantities are a measure of coordinate length scales at the throat over which the functions $b(R)$, $\Phi(R)$ and $\Phi'(R)$ change respectively. For the Class I solutions, they become

$$\bar{r}_o = R_o^{\pm}, \quad r_1 = R_o^{\pm}, \quad R_2 = 0, \quad r_3 = 0. \tag{2.3.2}$$

The vanishing of R_2 and r_3 implies that both $\Phi(R)$ and $\Phi'(R)$ exhibit an abrupt jump at the throat. It is therefore expected that the tidal forces at the throat would be large. That this is indeed so can be verified by calculating, for example, the differential of the radial tidal acceleration [1] given in an orthonormal frame $(\hat{e}_t, \hat{e}_R, \hat{e}_\theta, \hat{e}_\varphi)$ by

$$\Delta a^R = - \mathcal{R}_{\hat{R}\hat{t}\hat{R}\hat{t}} \xi^R \tag{2.3.3}$$

where ξ^R is the radial component of the separation vector and

$$|R_{\hat{R}\hat{t}\hat{R}\hat{t}}| = \left| (1-b/R)\left[-\Phi'' + \frac{b'R-b}{2R(R-b)}\Phi' - (\Phi')^2\right]\right| \tag{2.3.4}$$

For the metric given by Eqs.(2.2.10) and (2.2.11), we find

$$|R_{\hat{R}\hat{t}\hat{R}\hat{t}}| = \left|\frac{Br}{\lambda R^2(r^2-B^2)}\left[2(1-b/R)^{1/2}+(1-b/R)^{-1/2}b' + \frac{2\lambda(r^2+B^2)-4Br}{\lambda(r^2-B^2)}\right]\right| \tag{2.3.5}$$

At the throat where $b(R_0^{\pm})=R_0^{\pm}$, we have $|R_{\hat{R}\hat{t}\hat{R}\hat{t}}|\to\infty$, and this implies $\Delta a^R\to\infty$. As we march away from the throat to the asymptotic limit $r\to\infty$, or, $R\to\infty$, we find $|R_{\hat{R}\hat{t}\hat{R}\hat{t}}|\to 0$, as is to be expected.

Such an infinitely large tidal force at the throat is presumably related to the presence of singular null surface or naked singularity in the wormhole spacetime. These wormholes - to use a phrase by Visser [4] - are "badly diseased".

The occurrence of singular null surface in the scalar-tensor theories is directly related to the "no-hair" theorem which commonly means that "black holes have no scalar hair" [66]. Early investigations into no-hair theorem in the BD theory are due to Hawking [67], Chase [68], Teitelboim [69] and Bekenstein [70]. Recently, Saa [71] has formulated a new no-hair theorem which basically relies on the assessment of the behavior of scalar curvature R, which, for the metric (2.2.5) and (2.2.6), turns out to be

$$R(r) = \frac{4\omega C^2 B^2 r^4 (r+B)^{2\Omega-6}}{\lambda^2 (r-B)^{2\Omega+2}} . \tag{2.3.6}$$

Then it follows that $R \to \infty$ as $r \to B+$ for $C \neq 0$. In other words, the scalar curvature diverges as $R \to 0+$, implying that this shrunk surface does not represent a black hole for $\varphi \neq$ constant. It is instead a naked singularity [71]. On the other hand, if $C \to 0$ and $\lambda \to 1$, we have a finite value of R as $r \to B$. This means that we have a black hole solution for φ=constant, in total accordance with the no-hair theorem.

Generally speaking, wormhole solutions obtain in the Jordan frame because the sign of the energy density is indefinite in that frame. The sign is positive or negative according as $C(\omega)+1<\lambda$ or $C(\omega)+1>\lambda$. Let us examine the situation in the Einstein frame, defined earlier.

2.4. EINSTEIN FRAME

Under the conformal transformation

$$\tilde{g}_{\mu\nu} = p g_{\mu\nu}, \quad p = \frac{1}{16\pi}\varphi \tag{2.4.1}$$

and a redefinition of the BD scalar

$$d\phi = \left(\frac{\omega + \frac{3}{2}}{\alpha} \right)^{1/2} \frac{d\varphi}{\varphi} \tag{2.4.2}$$

in which we have intentionally introduced an arbitrary parameter α, the action (2.2.2) in the Einstein variables $(\tilde{g}_{\mu\nu},\phi)$ becomes

$$S = \int d^4x(-\tilde{g})^{1/2}\left[\tilde{R} - \alpha\tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right]. \qquad (2.4.3)$$

The field equations are

$$\tilde{R}_{\mu\nu} = \alpha\phi_{,\mu}\phi_{,\nu} \qquad (2.4.4)$$

$$\tilde{\Box}^2\phi = 0. \qquad (2.4.5)$$

The solutions of equations (2.4.4) and (2.4.5) can be obtained, using the transformations (2.4.1) and (2.4.2), as

$$d\tau^2 = -\left(1-\frac{B}{r}\right)^{2\beta}\left(1+\frac{B}{r}\right)^{-2\beta}dt^2 + \left(1-\frac{B}{r}\right)^{2(1-\beta)}\left(1+\frac{B}{r}\right)^{2(1+\beta)}[dr^2+r^2d\Omega_2^2] \qquad (2.4.6)$$

$$\phi = \left[\left(\frac{\omega+\frac{3}{2}}{\alpha}\right)\left(\frac{C^2}{\lambda^2}\right)\right]^{1/2} ln\left[\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right] \qquad (2.4.7)$$

$$\beta = \frac{1}{\lambda}\left(1+\frac{C}{2}\right). \qquad (2.4.8)$$

The expression for λ^2, of course, continues to be the same as

Eq.(2.2.8) and using this, we can rewrite Eq.(2.4.7) as

$$\phi = \left[\frac{2(1-\beta^2)}{\alpha}\right]^{1/2} ln \left[\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right] \qquad (2.4.9)$$

Redefining the radial coordinate $r \div R$ in the metric (2.4.6) as

$$R = re^{\beta_0}\left(1 - \frac{B}{r}\right)^{(1-\beta)}\left(1 + \frac{B}{r}\right)^{(1+\beta)}, \qquad (2.4.10)$$

we obtain the following functions of $\Phi(R)$ and $b(R)$:

$$\Phi(R) = \alpha_0 + \frac{1}{\beta}\left[ln\left\{1 - \frac{B}{r(R)}\right\} - ln\left\{1 + \frac{B}{r(R)}\right\}\right] \qquad (2.4.11)$$

$$b(R) = R\left[1 - \left\{\frac{(r^2 + B^2) - 2Br(R)\beta}{[r^2(R) - B^2]}\right\}\right] \qquad (2.4.12)$$

Casting the metric (2.4.6) into the Morris-Thorne form, we can find the wormhole throat r-radii to be

$$r_0^{\pm} = B\left[\beta \pm (\beta^2-1)^{1/2}\right]. \qquad (2.4.13)$$

For real $r_0^{\pm}$, we must have $\beta^2 \geq 1$. But $\beta^2=1$ corresponds to a non-traversable wormhole since $r_0^{\pm}$ coincides with the singular radius $r_s=B$. From Eq.(2.4.9), it follows that, if $\alpha>0$ and $\beta^2>1$, then no wormhole is possible as ϕ becomes imaginary. This result is quite consistent with the fact that the stress-energy tensor for massless minimally coupled scalar field ϕ, viz.,

$$T_{\mu\nu} = \alpha\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\phi^{,\sigma}\phi_{,\sigma}\right) \qquad (2.4.14)$$

satisfies all energy conditions [4]. The Einstein frame is thus called "physical" for which the restriction $\omega>-3/2$ follows from Eq.(2.4.7).

On the other hand, if we choose $\alpha<0$, which amounts to violating all energy conditions by brute force, one may find wormholes for $\beta^2>1$ in Eq.(2.4.9), or, equivalently, for $\omega<-3/2$.

We wish to point out a few more relevant points:

(i) Just as in the Jordan frame, the "no-hair" theorem holds also in the Einstein frame. This can be seen from the expression for scalar curvature $\tilde{R}$ computed from the metric (2.4.6):

$$\tilde{R} = \frac{8B^2 r^4 (1-\beta^2)}{(r-B)^{2(2-\beta)}(r+B)^{2(2+\beta)}} . \qquad (2.4.15)$$

One can see that $\tilde{R}$ is negative for wormhole solutions. In the Schwarzschild limit $\beta\to 1$, $\tilde{R}$ is finite for $r\to B$, and a black hole solution results, in complete accordance with the no-hair theorem [71]. The divergence of ϕ at r=B has been shown to be physically innocuous [72,73]. Generally, for $\beta\neq 1$, $\tilde{R}\to\infty$ as $r\to B$. This implies that the surface r=B (or, R=0) is not a black hole surface for nonconstant ϕ. This conclusion is in agreement with that reached by Agnese and La Camera [74] in a different way.

(ii) The ADM mass of the configuration is defined by

$$M = \frac{1}{16\pi} \lim_{S\to\infty} \int_S \sum_{i,j=1}^{3} (\partial_j g_{ij} - \partial_i g_{jj}) n^i dS \qquad (2.4.16)$$

where S is a 2-surface enclosing the active region, n^i denotes the unit outward normal. For the metric (2.4.6), we get

$$M = 2B\beta \qquad (2.4.17)$$

and using this value, the metric can be expanded in the weak field as

$$d\tau^2 = -(1+2Mr^{-1}+\cdots)dt^2+(1-2Mr^{-1}+2Mr^{-2}+\cdots)[dr^2+r^2 d\Omega_2^2], \qquad (2.4.18)$$

that is, it predicts exactly the same results for a neutral test particle as does Einstein's general relativity. The factor α does not appear in the metric although it does appear in the scalar field ϕ. Hence, α can not be determined by any metric test of gravity.

(iii) It should be remarked that if we replace B by another integration constant m/2, the solutions (2.4.6) and (2.4.9) become those proposed by Buchdahl [75] long ago. Defining the field strength σ for the scalar field ϕ in analogy with "electrostatic field", one obtains

$$\sigma = -2\delta m, \ \delta = [(1-\beta^2)/2\alpha]^{1/2}. \tag{2.4.19}$$

Then, from Eqs.(2.4.17) and (2.4.19), it follows that the gravitation producing mass M is given by

$$M^2 = m^2 - \frac{1}{2}\alpha\sigma^2 \tag{2.4.20}$$

where m can be regarded as the strength of the source excluding the scalar field. For $\beta \to 0$, we have $M \to 0$. The situation in this case is that, for $\alpha > 0$, we can have both m and σ nonzero but with their effects mutually annulled. In other words, we obtain a configuration which is indifferent to gravitational interaction with distant bodies. The reason is that the stresses of the ϕ field contribute an amount of negative gravitational potential energy (attractive) just sufficient to make the total energy zero [75]. On the other hand, if $\alpha<0$, the ϕ field has a positive gravitational potential energy (repulsive). We can not take $\beta\to0$ owing to Eq.(2.4.20), but it is possible to make $m\to0$ so that $M\to0$. In this case, we have $\sigma=0$. That is, the vanishing of total energy implies vanishing of individual source contributions.

2.5. SUMMARY

The foregoing analysis reveals that spherically symmetric static vacuum BD wormholes exist in the Jordan frame only in a very narrow interval $-3/2<\omega<-4/3$, corresponding to a physical situation where the post-Newtonian approximation is valid. In general, the wormhole range for ω depends entirely on the form of $C(\omega)$ supposed to be dictated by physical conditions. Wormhole solutions do not exist at all in the conformally rescaled (Einstein) frame unless one is willing to violate the energy conditions by choice ($\alpha<0$). However, such a manipulation is not always necessary. For example, there exist theories where one adds to the Einstein frame vacuum action other fields (such as the axion field [18]) or potentials [76] and obtains dynamic wormhole solutions in a natural way.

It is evident that the factor α does not appear in the metric (2.4.6), although it does appear in the expression for the scalar field ϕ. In particular, for local tests of gravity, the predictions are exactly the same as those of Einstein's general relativity where the Robertson parameters take on values $\alpha=\beta=\gamma=1$. In contrast, in the Jordan frame, one has $\alpha=\beta=1$, $\gamma=(\omega+1)/(\omega+2)$. For finite ω, is is evident that the predictions deviate somewhat from the actually observed values.

The ADM mass of the configuration is positive in both the frames. In the Jordan frame, it is $M=\frac{2B}{\lambda}(C+1)$ while in the Einstein frame it is $M=2B/\beta$. It is also shown that gravitationally indifferent real configuration with zero total energy ($M=0$) does or does not exist in the Einstein frame according as $\alpha>0$ or $\alpha<0$.

An interesting feature of BDwormholes is that infinitely large radial tidal accelerations occur at the throat so that these wormholes are not traversable in practice. This feature is reflected in the absence of a black hole surface at r=B, or, in the Morris-Thorne coordinates, at R=0.

We have not addressed the question of stability of BD wormholes in this chapter. With regard to classical perturbations, it should be pointed out that the results of Anchordoqui, Bergliaffa and Torres [25] indicate that addition of extra ordinary matter does not destroy the wormhole. The effect of quantum backreaction of the scalar field on stability will be consider elsewhere.

CHAPTER 3

STRINGY WORMHOLES

3.1. INTRODUCTION

In this chapter, we shall examine the possibility of static, spherically symmetric wormhole solutions in the string modified four-dimensional gravity [77]. We may term these as stringy wormholes. The action corresponds to Einstein-Maxwell gravity coupled to a dilaton field with negative kinetic term. The reason for taking negative kinetic term has been provided by Berkin and Hellings [78], Damour and Taylor [79], Damour and Esposito-Farese [80]. It is believed that in spite of strong resemblance to GR in our solar system, gravity may be radically different in other regimes. The gravitational field in our solar system is weak, severely limiting the parameter space of gravity tested. Theories which differ from GR in strong gravitational fields, but agree with solar system constraints, are needed to further test GR. These theories can satisfy the solar system criteria to arbitrary accuracy, but still be diverse from GR in other limits, for example, in the strong field regime around binary pulsars. Thus, such theories provide an important test of GR in a previously sparsely tested regime.

Berkin and Hellings [78] consider a model with an arbitrary number N of scalar fields and derive the conditions necessary to meet the solar system constraints. Two constraints result from these conditions. First, the scalar field kinetic terms must be non-positive-definite. Second, the scalars must take on the correct asymptotic values far from the solar system. In theories like this, with negative kinetic terms, the cosmological solutions which meet the solar system constraints are unstable as they evolve in time, and thus cannot be maintained. If these solutions can be stabilized by some means such as an explicit potential term or a more complete theory of gravity, then they will be a useful probe of GR. One way to ensure probable stabilization is to choose constants in such a way that the theory agrees with GR and cosmology only in the post-Newtonian (pN) approximation, but not exactly with either.

The action for N fields model which the following

$$S = \int d^4x\sqrt{-g}\left[f(\phi)R - G_{AB}g^{\alpha\beta}\phi_{A,\alpha}\phi_{B,\beta} - 2U(\phi) + 16\pi L_m\right], \quad (3.1.1)$$

where ϕ is an N component scalar field whose individual components will be denoted with capital Roman letters, U is a potential, G_{AB} describes the kinetic coupling of the fields, L_m is the Lagrangian for other matter and summation over A and B is implicit. The scalar field enters the Lagrangian with metric coupling to matter. By an appropriate linear transformation and rescaling of ϕ, G_{AB} may be taken to be diagonal with entries of ± 1 for each scalar field.

In Sec. 3.2, we shall consider the simplest extension of gravity coupled to interacting electromagnetic and massless scalar fields and discuss the charged Neveu-Schwarz solution. Sec. 3.3 considers the recently proposed Turyshev solution. In Sec. 3.4, we shall display, for convenience, the WEC explicitly and deduce a criterion about the nature of wormholes. In Sec. 3.5, Ford-Roman length scales and tidal force components are calculated and Sec. 3.6 provides a summary of the results.

3.2. NEVEU-SCHWARZ SOLUTION

The generic action is suggested by the low-energy limit of the string theory and might be written as

$$S = -\frac{1}{16\pi}\int(-g)^{1/2}\left[R+\alpha_1 g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi+\alpha_2 e^{-2\phi}g^{\mu\nu}g^{\gamma\delta}F_{\mu\gamma}F_{\nu\delta}\right]d^4x, \quad (3.2.1)$$

where α_1 and α_2 are constants. This represents an action where gravity is coupled to the dilaton field ϕ and electromagnetic field $F_{\mu\nu}$ interacting with the dilaton field. The kinetic term is positive or negative according as $\alpha_1 < 0$ or > 0. Let us take $\alpha_1=-2$ and $\alpha_2 = -1$, then the black hole solution of action (3.2.1) in the string frame [81] with Neveu-Schwarz charges associated with internal momentum and string winding number is given by [82,56]

$$ds^2 = -\Delta^{-1}\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \quad (3.2.2)$$

$$\Delta = \left(1 + 2mr^{-1}\sinh^2\gamma_1\right)\left(1 + 2mr^{-1}\sinh^2\gamma_p\right), \quad (3.2.3)$$

$$e^{2\phi}=\Delta^{-1/2}. \quad (3.2.4)$$

The redshift function and the shape function are respectively given by

$$\Phi(r)= (1/2)[\ln \Delta^{-1}+ \ln(1-2m/r)], \quad (3.2.5)$$

$$b(r)=2m. \quad (3.2.6)$$

The throat of the wormhole is given by $r_o = 2m$ and all the wormhole conditions are satisfied except that $\Phi(r)$ has a logarithmic singularity at $r = r_o$. Therefore, the wormhole is not traversable and much of the interest is lost consequently. Nonetheless, it can be verified by straightforward computation that the third of the WEC inequalities is violated for large r where the string coupling e^{ϕ} is also large [82]. This wormhole is attractive since $a^r>0$. We also computed the length scales at or around the throat and found that Φ and Φ' vary rather abruptly there (since both R_2 and r_3 vanish at the throat). Surprisingly enough, all the tidal force components remain finite for an observer travelling through the hole. The fact that Ψ becomes large near the throat implies that signals emanating from there are largely red/blue shifted for an observer outside the throat. This is certainly not a very desirable feature if any wormhole has to be traversable. Similar analysis in the case of the dual solution known as the magnetic black hole reveals that it also does not constitute a traversable wormhole.

3.3. TURYSHEV SOLUTION

This new special solution of dilaton-Maxwell gravity which follows from action (3.2.1) for $\alpha_1=2$ and $\alpha_2=1$, has been proposed by Turyshev [83] and is given by

$$ds^2 = -\left(1-\frac{2\mu}{r}\right)e^{-2\phi(r)}dt^2+\left(1-\frac{2\mu}{r}\right)^{-1}e^{2\phi(r)}dr^2+r^2e^{2\phi(r)}d\Omega^2, \qquad (3.3.1)$$

$$\phi(r) = \frac{Q^2}{2\mu r}, \qquad (3.3.2)$$

$$E(r) = \frac{Q}{r^2}, \qquad (3.3.3)$$

We must point out two important features of this solution. First, the dilaton charge is given by $D = Q^2/2\mu$, where Q is the electric charge and μ is related to physical mass. This is evident from Eq.(3.3.2) above and it shows that D is not an independent parameter. This fact is in perfect accordance with the "no-hair" conjecture [84] which states that a black hole can be completely described by its charge, mass and angular momentum. Second, the appearance of negative kinetic term $-2g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$ is dictated by the requirement of having a generic theory which agrees in the post Newtonian limit with general theory of relativity but differs significantly either in the strong gravity field or in the realm of cosmology. Such possibilities are currently under investigation [77].

Under the radial transformation

$$R = re^{\phi(r)} \qquad (3.3.4)$$

which ensures that $R \to \infty$ as $r \to \infty$, the metric (3.3.1) deduces to Morris-Thorne form for which

$$b(R) = R\left[1 - \left(1 - \frac{2\mu}{r(R)}\right)\left(1 - \frac{Q^2}{2\mu r(R)}\right)^2\right], \qquad (3.3.5)$$

$$\Phi(R) = \frac{1}{2}\ln\left(1 - \frac{2\mu}{r(R)}\right) - \frac{Q^2}{2\mu r(R)}. \qquad (3.3.6)$$

The throat radii $r_o^{\pm}$ are given by

$$r_o^+ = \frac{Q^2}{2\mu}\ , \ r_o^- = 2\mu. \qquad (3.3.7)$$

The R-radii turn out to be

$$R_o^+ = \frac{eQ^2}{2\mu}\ , \ R_o^- = 2\mu e^{Q^2/4\mu^2}. \qquad (3.3.8)$$

For $r=r_o^-$, the wormhole is not traversable as $\Phi(R)$ becomes singular there. However, we can take $r_o^+ > r_o^-$ so that an observer does not have to encounter the horizon at r_o^-. In this case, at the throat radius $R=R_o^+$, $\Phi(R)$ is finite. This is equivalent to the condition that

$$Q^2 > 4\mu^2. \tag{3.3.9}$$

From Eq.(3.3.5), we find that $|b(R)/R| \leq 1$ and $b(R)/R \to 0$ as $R \to \infty$, and the wormhole becomes geometrically traversable. The scalar curvature R is given by

$$R(r) = \frac{Q^4}{2\mu^2 r^4}\left(1 - \frac{2\mu}{r}\right)e^{-\frac{Q^2}{\mu r}}, \tag{3.3.10}$$

which is also finite for $r = r_o^{\pm}$. Let us now examine the WEC. The energy density works out to

$$\rho = \frac{1}{R^2}\left[\frac{8\mu^2}{Q^2} - 1\right]. \tag{3.3.11}$$

We can choose the parameters μ and Q such that

$$\frac{1}{2} < \frac{4\mu^2}{Q^2} < 1, \tag{3.3.12}$$

which is in accordance with Eq. (3.3.9), then $\rho > 0$. Also,

$$\tau = \frac{1}{R^2}\left[\frac{b}{R} - \frac{2}{R}\left(1 - \frac{b}{R}\right)\frac{d\psi}{dR}\right] \qquad (3.3.13)$$

gives

$$\tau_o = \frac{1}{\left(R_o^{\pm}\right)^2} \qquad (3.3.14)$$

since $\left(1 - \frac{b}{R}\right)\frac{d\psi}{dR} \to 0$ as $R \to R_0^{\pm}$ for which $b(R) \to R$. Under the condition (3.3.12), we find that

$$\rho_0 - \tau_0 < 0 \qquad (3.3.15)$$

and the second inequality is violated, although $\rho_0 > 0$, as is to be expected. In the extremal limit $Q^2 = 4\mu^2$, we have $R_0^+ = R_0^-$ and $\Phi(R)$ becomes infinite there, rendering the wormhole non-traversable.

3.4. NATURE OF THE WORMHOLE

For latter computation we explicitly writeout the WEC which given by the three inequalities together:

$$\rho(r) = \frac{b'}{r^2} \geq 0, \qquad (3.4.1)$$

$$\rho(r)-\tau(r) = \frac{b'r-b}{r^3} + \frac{2\Phi'}{r}\left[1 - \frac{b(r)}{r}\right] \geq 0, \qquad (3.4.2)$$

$$\rho(r)+p(r) = \frac{b + b'r}{2r^3} + \left[1 - \frac{b(r)}{r}\right]\left[\Phi'' + (\Phi')^2 + \frac{\Phi'}{r} + \frac{b-b'r}{2r(r-b)}\Phi'\right] \geq 0,$$

$$(3.4.3)$$

where $T_{00} = \rho$, $T_{11} = -\tau$ and $T_{22} = T_{33} = p$. Violation of any one of the inequalities can be interpreted as the violation of WEC.

To obtain a knowledge about other features of a traversable wormhole, a convenient method is to compute the length scales over which the wormhole parameters $\Phi(r)$ and $b(r)$ vary at or near the throat. These scales have been recently proposed by Ford and Roman [9] and has already been shown before.

Let us consider the four-velocity of a static observer in the Morris-Thorne spacetime. This is given by

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \left[u^{t},0,0,0\right] = \left[e^{-\Phi(r)},0,0.0\right]. \qquad (3.4.4)$$

The observer's four-acceleration is

$$a^{\mu} = \frac{Du^{\mu}}{d\tau} = u^{\mu}{}_{;\nu}u^{\nu} = \left(u^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\beta\nu}u^{\beta}\right)u^{\nu}. \qquad (3.4.5)$$

For the metric (1.1.1) of Chapter 1, we have

$$a^t = 0 \tag{3.4.6}$$

$$a^r = \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 = \Phi'(1 - b/r) \tag{3.4.7}$$

where $\Phi' = d\Phi/dr$. From the geodesic equation, a radially moving test particle which starts from rest initially has the equation of motion

$$\frac{d^2r}{d\tau^2} = - \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 = - a^r. \tag{3.4.8}$$

Hence a^r is the radial component of proper acceleration that an observer must maintain in order to remain at rest at constant r,θ,φ. Note that from Eq.(3.4.7), a static observer at the throat of any wormhole is a geodesic observer. For $\Phi'(r) \neq 0$ wormholes, static observers are not geodesic (except at the throat), whereas for $\Phi'(r)=0$ wormholes they are. A wormhole is "attractive" if $a^r>0$ (observers must maintain an outward-directed radial acceleration to keep from being pulled into the wormhole) and "repulsive" if $a^r<0$ (observers must maintain an inward-directed radial acceleration to avoid being pushed away from the wormhole). From Eq. (3.4.7), this distinction depends on the sign of Φ'. For $a^{r}=o$, the wormhole is neither attractive nor repulsive.

3.5. LENGTH SCALES AND TIDAL FORCES

Although the geometrical conditions for wormholes are satisfied by Turyshev solution, traversability from the dynamical point of view could still be a problem. In order to examine this aspect, let us compute the Ford-Roman length scales defined earlier. In the present case, these are given by

$$\bar{r}_0 = R\left[1 - \left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{Q^2}{2\mu r}\right)^2\right], \qquad (3.5.1)$$

$$r_1 = \left|\frac{R\left[1 - \left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{Q^2}{2\mu r}\right)^2\right]}{\frac{Q^2}{r^2} + \frac{Q^2}{2\mu r^2}\left(\frac{1}{r} - \frac{1}{2\mu}\right)}\right|, \qquad (3.5.2)$$

$$R_2 = \left|\frac{2\mu r R\left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{Q^2}{2\mu r}\right)}{2\mu^2 + \left(1 - \frac{2\mu}{r}\right)Q^2}\right|, \qquad (3.5.3)$$

$$r_3 = \left|\frac{R\left\{2\mu^2 + \left(1 - \frac{2\mu}{r}\right)Q^2\right\}}{2\mu Y}\right|. \qquad (3.5.4)$$

where

$$
Y= \frac{2\mu}{\left(1- \frac{Q^2}{2\mu r}\right)} + \frac{2\mu^2}{r\left(1- \frac{2\mu}{r}\right)\left(1- \frac{Q^2}{2\mu r}\right)} + \frac{Q^2\left(1 - \frac{2\mu}{r}\right)}{\mu\left(1 - \frac{Q^2}{2\mu r}\right)} + \frac{Q^4}{4\mu r^2} + \frac{Q^6\left(1 - \frac{2\mu}{r}\right)}{8\mu^9 r^2}. \tag{3.5.5}
$$

At the throat $r=r_0^+$, both R_2 and r_3 vanish while $\bar{r}_0$ and r_1 remain finite. This implies that the redshift function $\Phi(R)$ varies abruptly at the throat and hence it is likely that the wormhole is not traversable dynamically. To be more concrete, suppose that the velocity of an observer O, that starts from one mouth and goes to the other via the wormhole throat, is $v(r)$. Let $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ be the curvature components in a static orthonormal frame $(e_{\hat{t}}, e_{\hat{R}}, e_{\hat{\theta}}, e_{\hat{\varphi}})$. The nonvanishing curvature components are

$$
\begin{aligned}
|R_{\hat{R}\hat{t}\hat{R}\hat{t}}| = \Bigg| \frac{1}{R^2}\Bigg[& \frac{2\mu}{r} + \frac{2\mu^2}{r^2\left(1- \frac{2\mu}{r}\right)} + \frac{Q^2\left(1- \frac{2\mu}{r}\right)}{\mu r} \\
& + \frac{Q^4}{4\mu r\left(1- \frac{Q^2}{2\mu r}\right)}\left\{\frac{1}{r^2} + \frac{Q^2\left(1- \frac{2\mu}{r}\right)}{2\mu^2 r^2}\right\} + \frac{\left(\frac{3Q^2}{r^2} - \frac{Q^2}{\mu r} - \frac{2\mu}{r}\right)\left\{2\mu^2 + \left(1- \frac{2\mu}{r}\right)Q^2\right\}}{4\mu r\left(1 - \frac{2\mu}{r}\right)\left(1- \frac{Q^2}{2\mu r}\right)} \\
& - \frac{\left\{2\mu^2 + \left(1 - \frac{2\mu}{r}\right)Q^2\right\}^2}{4\mu^2 r^2\left(1 - \frac{2\mu}{r}\right)}\Bigg]\Bigg|
\end{aligned} \tag{3.5.6}
$$

$$\left|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}\right| = \left|R_{\hat{\varphi}\hat{t}\hat{\varphi}\hat{t}}\right| = \left|\frac{1}{R^2}\left[\left\{\frac{\mu}{r} + \left(1 - \frac{2\mu}{r}\right)\frac{Q^2}{2\mu r}\right\}\left(1 - \frac{Q^2}{2\mu r}\right)\right]\right|, \qquad (3.5.7)$$

$$\left|R_{\hat{\theta}\hat{R}\hat{\theta}\hat{R}}\right| = \left|R_{\hat{\varphi}\hat{R}\hat{\varphi}\hat{R}}\right| = \left|\frac{1}{R^2}\left[\frac{3Q^2}{2r^2} - \frac{Q^2}{2\mu r} - \frac{\mu}{r}\right]\right|, \qquad (3.5.8)$$

$$\left|R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}}\right| = \left|\frac{1}{R^2}\left[1 - \left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{Q^2}{2\mu r}\right)^2\right]\right|. \qquad (3.5.9)$$

The curvature components in the observer's orthonormal frame ($e_{\hat{0}^*}$, $e_{\hat{1}^*}$, $e_{\hat{2}^*}$, $e_{\hat{3}^*}$) are obtained by the Lorentz boost [1]:

$$e_{\hat{0}^*} = \gamma e_{\hat{t}} \pm \gamma v e_{\hat{R}} \ , \ e_{\hat{1}^*} = \mp\gamma e_{\hat{R}} + \gamma v e_{\hat{t}} \ , \ e_{\hat{2}^*} = e_{\hat{\theta}} \ , \ e_{\hat{3}^*} = e_{\hat{\varphi}} \qquad (3.5.10)$$

where $\gamma = (1-v^2)^{-1/2}$. The only nonvanishing tidal acceleration components for Turyshev wormhole are given by

$$\Delta a^{\hat{j}^*} = - R_{\hat{j}^*\hat{0}^*\hat{j}^*\hat{0}^*}\,\xi^{\hat{j}^*} \qquad (3.5.11)$$

where j = 1,2,3 and $\xi^{\hat{j}^*}$ are the components of the separation vector between two parts of the traveller's body. The curvature components are explicitly given by

$$\left|R_{\hat{1}^*\hat{0}^*\hat{1}^*\hat{0}^*}\right| = \left|R_{\hat{R}\hat{t}\hat{R}\hat{t}}\right|, \tag{3.5.12}$$

$$\left|R_{\hat{2}^*\hat{0}^*\hat{2}^*\hat{0}^*}\right| = \left|R_{\hat{3}^*\hat{0}^*\hat{3}^*\hat{0}^*}\right| = \left|\frac{\gamma^2}{R^2}\left[v^2\left(\frac{3Q^2}{2r^2} - \frac{Q^2}{2\mu r} - \frac{\mu}{r}\right)\right.\right.$$

$$\left.\left. + \left\{\frac{\mu}{r} + \left(1 - \frac{2\mu}{r}\right)\frac{Q^2}{2\mu r}\right\}\left(1 - \frac{Q^2}{2\mu r}\right)\right]\right|. \tag{3.5.13}$$

In the limit $R \to R_0^+$, it follows that $\Delta a^{\hat{1}^*}$ is infinite for any v, while $\Delta a^{\hat{2}^*}$ and $\Delta a^{\hat{3}^*}$ remain finite. This implies that, as the traveler approaches the throat, the radial tidal force increases indefinitely while the cross radial forces remain finite. Clearly, the traveler is torn into pieces at or near the throat even for velocity $v \ll 1$. This concludes what is wished to demonstrate, namely, that the wormhole is traversable only in principle but not in practice.

3.6. SUMMARY

The possibility of practically traversable static wormhole solutions in the string-inspired gravity appears to be rather slim. In the Neveu-Schwarz zero-density wormhole, there occurs a singularity in $\Phi(r)$ at $r_0=2m$, although, surprisingly, the tidal forces remain finite at the throat. In order to avoid the singularity in $\Phi(r)$, one may contemplate a "proximal" Neveu-Schwarz solution obtained by the formal replacement $(1-\frac{2m}{r}) \rightarrow (1 - \frac{2m}{r} + \frac{\varepsilon}{r^2})$ only in $\Phi(r)$. The tidal forces would continue to remain finite thereby offering the possibility of safe travel across.

As to the Turyshev solution (obtained by specifically choosing the value of a constant), it is unstable due to the presence of a negative kinetic term in the action. However, this solution does offer a nonsingular $\Phi(r)$. But the problem comes from the fact that the tidal forces are indefinitely large at or near the throat. As a consequence, no observer can cross the wormhole throat.

It is interesting to see from the above that the Neveu-Schwarz and Turyshev solutions offer exactly opposite features with regard to their interpretations in terms of wormhole geometries.

CHAPTER 4

MODIFIED NO-HAIR CONJECTURE AND THE LIMITING PROCESS

4.1. INTRODUCTION

In theories of gravity coupled to scalar fields, there always arises the question as to whether black hole solutions admit dressing by scalar fields [66-74, 86-88, 92-95]. This question becomes particularly important in the Brans-Dicke (BD) scalar-tensor theory which we are discussing in the thesis. The answer lies in Wheeler's dictum [66] which is supposed to mean that a black hole can be dressed only by fields which are constant everywhere or at most satisfy a Gauss-like law. This is the popular version of what is known as "no-hair" conjecture. An alternative interpretation of such a conjecture is that a stationary black hole with an exterior devoid of matter can be parameterized only by mass, angular momentum and electric charge. However, Bekenstien [73] has shown that scalar charge is also an admissible parameter. In the context of conformal scalar field, it has been shown that the black hole solution is parameterized by electric and scalar charges although the conformal field becomes unbounded at the horizon. This feature was previously regarded as incompatible with black hole interpretation but Zannias and Bekenstein [72,73] have shown that this singularity is harmless from the physical point of view.

Early no-hair theorems excluded scalar [68-70], vector [85] and spinor [8] fields from stationary black holes exterior. But, due to the developments in particle physics, those early theorems have become outdated, and as a consequence, black hole solutions with various "hairs" have been found. Among them are black holes dressed with Yang-Mills, Proca-type Yang-Mills and Skyrme fields in various combinations with Higgs fields [86-88]. Although the perimeter of the early no-hair theorems has been widened to include black holes dressed by different fields, there arises the question as to whether the solutions are unstable. Possibly it is this aspect of instability that distinguishes early no-hair theorem from the modified ones.

In Sec. 4.2 , it is argued, contrary to a certain viewpoint, that Schwarzschild black hole solution follows as a unique limit of the BD Class I solutions, provided the correct iterated limit is taken. Such a uniqueness is essential for the validity of a recent version of the no-hair conjecture. In Sec. 4.3, a non-trivial modification to this version is proposed in order to exclude BD Class IV solutions which appear to represent scalar hair black holes. Sec.4.4 summarizes the contents.

4.2. THE LIMITING PROCESS

We start from the version by Saa [71] which read :

The only asymptotically flat, static and spherically symmetric exterior solution of the system governed by the action

$$S[g,\phi]=\int d^4x\sqrt{-g}\ [f(\phi)\overline{R} - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi], \qquad (4.2.1)$$

where $\overline{R}$ is a scalar curvature formed from $g^{\mu\nu}$; $f(\phi)$, $h(\phi) > 0$ with ϕ finite everywhere, is the Schwarzschild black hole solution. It should be noted that Saa [71] defines a black hole solution as one for which the scalar curvature $\overline{R}$ is finite at the metric singularity. In what follows, we shall retain this definition but, noting that Saa's generation technique remains valid for negative values of ω as well, relax the restrictive condition $h(\phi)>0$ and take $\omega \in(-\infty,+\infty)$. This is the first modification.

The BD field equations follow in the specific case where $h(\phi)=\omega\phi^{-1}$ and $f(\phi)=\phi$. We shall be concerned with the limiting process $|\omega| \to \infty$ that is supposed to provide a passage from BD to Einstein field equations. Another limiting process that will be shown to have a direct relationship with the conjecture is $r \to r_h$, where r is a radial coordinate and r_h is the coordinate horizon radius in a given solution. In the literature, the statements $r=r_h$ and $r \to r_h+$ are often understood to mean the same thing and there arises no essential difficulty as long as only this "single limit" process is considered. However, care must be exercised as soon as "double limit" or "iterated limit" processes come into play. For example, to verify the validity of the no-hair conjecture in the BD theory, it becomes necessary to study the nature of scalar curvature $\bar{R}$ in the limit of the null surface $(r \to r_h)$ and $|\omega| \to \infty$. In this case, it is essential to state exactly what type of limit process one has in mind and why. Unfortunately, in Ref.[71], the exact nature of the limiting process, if any, has been kept obscure although the final conclusions drawn therein are correct.

Incorrect limit processes lead to erroneous conclusions. For example, Matsuda [65] concludes that Schwarzschild exterior solution is not a *unique* limit of the BD Class I solutions of BD theory for ϕ=constant. There is also another solution, having a singular event horizon at the origin, that resembles Schwarzschild solution in the exterior but does not represent a black hole. This result clearly violates the no-hair conjecture formulated above.

In this section, we wish to examine various possible limiting processes and in particular argue that there is no violation of no-hair conjecture as far as BD Class I solutions are concerned, although naked singularities may occur under specific conditions. However, scalar hair black holes seem to occur in the case of Class IV solutions unless the above version of the conjecture is modified still further.

Let us consider Eqs. (2.2.1), (2.2.5)-(2.2.11) and (2.3.6) of Chapter 2 and recall that Schwarzschild exterior metric

$$ds^2 = -(1-2M/R)dt^2+(1-2M/R)^{-1}dR^2+R^2d\theta^2+R^2\sin^2\theta d\varphi^2 \qquad (4.2.2)$$

is defined strictly in the range $2M<R<\infty$, and the scalar curvature $\bar{R} \to$ finite value as $R \to 2M$. Therefore, $R=2M$ represents a null regular surface or a non-singular event horizon. In the BD metric (2.2.4) of Chapter 2, let us first choose finite values for ω (or, C)and λ such that $\Omega > 0$ and try to find out the R-coordinate range and the scalar curvature $\bar{R}$. Clearly, we get from equations (2.2.9) and (2.3.6) of Chapter 2, the single limit

$$\lim_{\substack{r \to B \\ \Omega > 0}} R \to 0, \quad \bar{R} \to \infty. \tag{4.2.3}$$

Therefore, we get $0<R<\infty$ and that there occurs an irremovable singularity as $R \to 0$. This result is in perfect accordance with the no-hair conjecture as $\phi \neq$ constant and the limiting metric too is not Schwarzschild. The detailed topology of such a point singularity has been studied by Agnese and La Camera [27]. Secondly, let us compute another single limit

$$\lim_{\substack{r > B \\ \Omega \to 0}} \bar{R} \to 0. \tag{4.2.4}$$

This is an expected result as Class I solutions reduce to the

Schwarzschild exterior metric for which the Ricci tensor $R_{\mu\nu}=0$, and hence $\bar{R}=0$.

Thirdly, let us consider the limit: $r=B$, $\Omega\to0+$. This is the case considered by Matsuda [65]. In this limit, both the metric component $[1-b(R)/R]^{-1}$ and the scalar curvature $\bar{R}$ are not even *defined*, let alone their existence, as they involve a division exactly by zero. The concept of a limit can not even be applied here. These are evident from Eqs.(2.2.11) and (2.3.6) of Chapter 2 respectively. Hence, the claim that $R=0$ is a singular event horizon is not strictly correct if we accept the definition that a singular surface is one where the scalar curvature diverges. Clearly, the divergence of a limit is distinct from the situation where the limiting process itself is not definable. Thus, using this limiting case, it is not possible to say whether or not $R=0$ constitutes a singular event horizon. The existence of Matsuda's non-black hole solution becomes untenable and our no-hair theorem remains effectively unchallenged.

Saa [71] seems to have calculated the scalar of curvature $\bar{R}$ at the exact equality $r=B$ and $\Omega=0$, and taken at the face value, $\bar{R}$ has an exact form [0/0], which is by itself meaningless.

Clearly, the value has to be understood only in the sense of a limit, if it is defined. But then there arises the question: Should one take a double limit or iterated limits? There is no physical ground to prefer one operation to the other. However, the requirement of a double limit constitutes a much stronger condition than that of an iterated limit. The reason is that the existence of the former does imply the same of the latter, but the converse is not true. If we decide to compute iterated limits only, then the question is which one? Let us write the two iterated limits

$$\lim_{\Omega\to 0}\lim_{r\to B}\bar{R} = \lim_{\Omega\to 0}\infty = ? \tag{4.2.5}$$

$$\lim_{r\to B}\lim_{\Omega\to 0}\bar{R} = \lim_{r\to B} 0 = 0 \tag{4.2.6}$$

The former iterated limit does not exist in the usual sense [89] and therefore we can not say anything about the singularity or otherwise of $\bar{R}$ at $r=B$ $(r\to B)$. Saa [71] actually *means* the second iterated limit and thereby arrives at the no-hair formulation. Nonetheless, the reason why we should prefer the limit (4.2.6) is not obvious.

There is also another case: Put $\Omega=0$. Then, $\lambda=C+1$, $C\neq 0$, $\lambda\neq 0$. We then have Schwarzschild exterior metric with $\phi \neq$ constant. Then we have $R\to 4B$ and $\bar{R}\to\infty$ as $r\to B$. This implies that the Schwarzschild sphere itself becomes an irregular null surface. This result is in perfect accordance with the no-hair conjecture. But note that $\Omega=0$ is not strictly a valid equality in the same sense as $\omega=\infty$ is not. In accordance with $\omega\to\infty$, we must take $\Omega \to 0$. Also we should take $r\to B$ in the computation of the scalar curvature $\bar{R}$ and then we have the above iterated limits at our disposal.

Unless we have a definite physical ground to prefer one of the iterated limits, the difference between the two remains an enigma. One plausible but by no means exclusive procedure could be to choose a certain path along which the two limits would be the same.

4.3. CLASS IV SOLUTIONS

These solutions are given by

$$\alpha(r) = \alpha_0 - \frac{1}{Br} \qquad (4.3.1)$$

$$\beta(r) = \beta_0 + \frac{(C+1)}{Br} \tag{4.3.2}$$

$$\phi = \phi_0 \, e^{-\frac{C}{Br}} \tag{4.3.3}$$

$$C = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2} \tag{4.3.4}$$

Usual asymptotic flatness and weak field conditions fix α_0, β_0 and B as

$$\alpha_0 = \beta_0 = 0, \; B = \frac{1}{M} > 0 \tag{4.3.5}$$

There is a singularity in the metric at r=0. The solutions (4.3.1)-(4.3.4) represent asymptotically flat, static and spherically symmetric solution of a system governed by the action (4.2.1). Also, $C \to 0$ as $\omega \to -\infty$. The scalar curvature turns out to be

$$\bar{R} = -2(1+C+C^2)(B^2 r^4)^{-1} e^{-2(C+1)/(Br)} \tag{4.3.6}$$

In the limit

$$\lim_{r\to 0} \bar{R} \to 0. \tag{4.3.7}$$

This implies that the degenerate surface r=o is nonsingular, no matter whether $C \to 0$ or not. Therefore, Class IV solutions may represent $\phi \neq$ constant black holes violating the modified no-hair conjecture. In the limit $C \to 0$, the metric does not become exactly Schwarzschild, although it is approximately so. It is given by

$$ds^2 = - e^{-2M/r} dt^2 + e^{2M/r}\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\varphi^2 \right]. \qquad (4.3.8)$$

By comparing this metric with the Robertson expansion for the static, spherically symmetric problem [90]:

$$ds^2 = -\left(1 - 2\alpha \frac{M}{r} + 2\beta \frac{M^2}{r^2} + \ldots\right) dt^2 +$$

$$\left(1 + 2\gamma \frac{M}{r} + \ldots\right)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\varphi^2\right] \qquad (4.3.9)$$

we find that the parameters have the values

$$\alpha = \beta = \gamma = 1. \qquad (4.3.10)$$

Hence, the metric (4.3.8) is indistinguishable from the Schwarzschild metric in the sense of many experimental predictions. In this connexion, it is tempting to point out that the metric (4.3.8) is exactly of the same form as Rosen's metric in the bimetric theory of gravity [91]. Whether it is just a remarkable coincidence or there is a deeper connexion is a matter of further investigation.

Let us return to the no-hair conjecture. From the calculations above including the limit (4.3.7), modified conjecture is violated, whereas in the case of Class I solutions, it is not. In the latter case, recall that it is necessary to fix *two* conditions independently in order to go to the Schwarzschild metric: $C=C(\omega)$, $\lambda \to 1$. In the present case, on the other hand, there is no λ but just two parameters ω and C connected by Eq.(4.3.4). From this equation, it already follows that $C \to 0$ as $\omega \to -\infty$. That is ϕ=constant as $|\omega| \to \infty$ and the metric does not contain any unfixed parameter whatsoever. At first sight, it may appear that no other condition is necessary for the passage to the Einstein limit $G_{\mu\nu}=0$. But that is not so! In order that the r.h.s. of the matter-free BD Eqs. (2.2.3) vanish identically, we must impose, in addition to $|\omega| \to \infty$, an extra condition that $\omega C^2 \to 0$, without which the passage from BD to Einstein equations can not be ensured. But this condition is *not* satisfied in the case of Class IV solutions as it can be verified that $G_{0}^{0} \neq 0$. This feature shows itself up in the form of scalar hair ($C \neq 0$) black hole solutions. Therefore, we must add a second modification to the version proposed by Saa [71] in order to exclude Class IV types of solutions. The modified version of the conjecture should now read:

The only asymptotically flat, static and spherically symmetric exterior solution of the system governed by the action

$$S[g,\phi] = \int d^4x\sqrt{-g}\ [f(\phi)R - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi], \qquad (4.3.11)$$

$f(\phi)>0$ with ϕ finite everywhere, *satisfying vacuum Einstein equations*, is the Schwarzschild solution.

4.4. CONCLUDING REMARKS

The computation of the limiting processes has a direct relevance to the no-hair conjecture. It is necessary to be specific about the type of limit process considered. Converting the iterated limit to a single limit process, one might land up with a conclusion that seemingly violates the conjecture. We are aware that some recent investigations imply that the conjecture *is* anyway violated under different conditions [86-88]. However, those cases do not correspond to the non-minimally coupled scalar field considered here.

We saw that Class IV solutions may represent black holes dressed by $\phi\neq$constant scalar field. The cause is analyzed and a modification to the version is proposed. The modification is by no means trivial, as the counter example in the form of Class IV solutions testify.

CHAPTER 5

SCALAR DRESSING OF BRANS-DICKE BLACK HOLES: A CUE FROM BEKENSTEIN'S NO-HAIR THEOREM

5.1. INTRODUCTION

This chapter continues the discussion of the no-hair theorem. In particular, we wish to investigate the extent to which BD black hole solutions admit dressing by nonminimally coupled scalar fields. The inquiry is prompted by the existence of black hole solutions having conformally coupled scalar hair. Just like this case, the BD solutions too do not satisfy the conditions laid down in the novel no-hair theorem proposed recently by Bekenstein [76]. The theorem implies that asymptotically flat, static and spherically symmetric black hole exteriors with a minimally coupled scalar field having positive definite energy density ρ do not admit nonconstant scalar hairs. However, inapplicability of Bekenstein's theorem does not, by itself, guarantee that all black hole exteriors with $\rho<0$ would be dressed by nonconstant scalar fields replicating the conformal case. This assertion, which we shall later exemplify, provides a justification as to why explicit calculations are necessary to investigate the existence of such dressings in individual cases. For two families of BD solutions, it is found that no pathological conditions occur at the horizon for a certain range of parameters: The scalar field, curvature scalar and tidal force components all remain bounded. Also, the scalar test charge trajectories interacting with the field approach the horizon asymptotically and in that sense are complete. The scalar field itself becomes constant at infinity. All these features seem to suggest that black holes in BD theory with nonconstant scalar field may after all be admissible just as they are in the conformally coupled case.

Before we deal with specific BD solutions, we must pay attention to three prerequisites: (i) Since conformal scalar hair black holes are already known to exist [73], it is necessary to show that our BD solutions are not exactly the same in content as the former ones. This is done in Sec. 5.2.(ii) Plausible justifications in favor of the BD coupling parameter range $\omega \in(-\infty,\infty)$. Recall that negative values of ω are usually regarded as unsupportable either on the ground of experimental evidences to date or on the ground that it corresponds to a negative energy density. We choose to ignore the former ground on the conviction that, since there is no *a priori* theoretical restriction on ω, it would only be legitimate to leave the values open to future experiments. The latter ground does not obviously apply in our case as we are interested only in $\rho<0$ solutions. A brief remark is presented in Sec. 5.3. (iii) We have to provide the example in favor of an assertion made in the last paragraph. This is done in Sec. 5.4 by means of Brans Class IV solutions. After the above three tasks, Brans Class I solutions are taken up in Sec. 5.5 because of their intrinsic interests. Sec. 5.6 discusses the trajectories of interacting scalar test charges and Sec. 5.7 contains a few concluding remarks.

5.2. BRANS-DICKE vis-a-vis BEKENSTEIN SOLUTIONS

As a prerequisite for the ensuing analysis, it is important to establish that the problem at hand is nontrivial, that is, the two types of solutions mentioned above are really distinct or, to be more precise, unconnected. The task is best performed by referring to a chain of available solution generating techniques that rely either on conformal rescaling of the seed metric or on redefinition of the scalar field or on both.

Let us start from the Einstein conformal scalar field matter-free action given by

$$S_C[g,\psi] = \int d^4x(-g)^{1/2}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - \frac{1}{12}R\psi^2\right]. \qquad (5.2.1)$$

Defining a scalar field ϕ as

$$\phi = (2/3)^{1/2} ln \left(\frac{\sqrt{6}+\psi}{\sqrt{6}-\psi}\right) \text{ or } \phi = (2/3)^{1/2} ln \left(\frac{\sqrt{6}-\psi}{\sqrt{6}+\psi}\right) \qquad (5.2.2)$$

according as $\psi^2<6$ or >6, it is possible to rewrite the action (5.2.1) in its minimally coupled form given by

$$S_{MC}[g,\phi] = \int d^4x(-g)^{1/2}\left[R - g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right]. \qquad (5.2.3)$$

The solution for action (5.2.3) has already been given by Buchdahl [75]. Remarkably, this solution can also be generated by the Janis-Robinson-Winicour [96] technique from the vacuum Einstein equations. One may now start from (5.2.3), as does

Bekenstein [73], and generate solutions for (5.2.1). The type A and B solutions in Ref.[73] correspond to the two cases $\psi^2<6$ or >6 respectively.

However, it is also possible to redefine ψ in a different way. Take [62]

$$\varphi = 8\pi - (4\pi/3)\psi^2, \quad \omega(\varphi) = (3/2)\frac{\varphi}{8\pi-\varphi} \tag{5.2.4}$$

then the action (5.2.1) reduces to the BD form given by

$$S_{BD}[g,\varphi] = (1/16\pi)\int d^4x(-g)^{1/2}\left[\varphi R - \varphi^{-1}\omega(\varphi)g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right]. \tag{5.2.5}$$

Lastly, the above action can be further transformed by the Dicke transformations

$$\bar{g}_{\mu\nu} = (1/16\pi)\varphi g_{\mu\nu}, \quad d\bar{\phi} = [\omega(\varphi)+3/2]^{1/2}(d\varphi/\varphi) \tag{5.2.6}$$

to the one giving $S_{MC}[\bar{g},\bar{\phi}]$. Thus, the entire chain can be diagramatized as follows:

$$S_{MC}[g,\phi] \leftrightarrow S_C[g,\psi] \leftrightarrow S_{BD}[g,\varphi] \leftrightarrow S_{MC}[\bar{g},\bar{\phi}] \leftrightarrow S_C[\bar{g},\bar{\psi}] \leftrightarrow \cdots\cdots$$

(5.2.7)

Notice that, in the route $S_C[g,\psi] \leftrightarrow S_{BD}[g,\varphi]$, the generated coupling parameter ω must necessarily be a function of φ as specified in Eq.(5.2.4). If, on the other hand, ω = constant, the case we are considering, the route is snapped. Therefore, $\tilde{S}_{BD}[g,\tilde{\varphi},\omega=\text{const.}] \neq S_{BD}[g,\varphi,\omega=\omega(\varphi)]$. In other words, the corresponding solutions are unconnected.

Note, however, that the demand of total absence of correlation is rather too strong and could actually be relaxed. One could equally well consider the generated BD solutions with $\omega = \omega(\varphi)$ following from action (5.2.5). The reason is that the generated solutions do not necessarily share all the properties of the seed solution. An excellent example is the Buchdahl [75] solution coming from S_{MC} which satisfies the WEC while those coming from S_C or S_{BD} do not. (For a given stress-energy tensor $T_{\mu\nu}$, the WEC states that $T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0$ for arbitrary timelike ξ^{μ}). This apart, there also exist differences in the interpretations given to the scalar fields. In S_{MC} theories, the field is viewed as some kind of matter with WEC satisfying stress-energy, while in the S_C or S_{BD} theories, the fields are regarded as a spin-0 component of gravitational interaction [62]. The gravitational constant is redefined using ψ or φ. The point we wish to make is that the task for searching scalar hair black holes is not a trivial one *even* for $\omega = \omega(\varphi)$. The action (5.2.5) actually constitutes a special case [$\omega(\varphi)$ as specified in Eq.(5.2.4)] of the Bergman-Wagoner-Nordvedt model [58].

5.3. RANGE OF ω

It is generally accepted that the coupling constant ω in the BD theory can take on only positive values, $\omega \in (0,\infty)$ which is an *a posteriori* input coming exclusively from different experiments to date [58]. There is also another ground which owes its origin to the physical requirement of having a positive definite stress-energy tensor that satisfies WEC. Such a $T_{\mu\nu}$ is available only if the scalar field couples minimally to gravity as in S_{MC} and this is the case considered by Bekenstein [76]. But, even then, the transferability from action (5.2.5) to (5.2.3) requires, in virtue of Eq.(5.2.6), that the range be $\omega \in [-3/2,\infty)$ and not $(0,\infty)$ only. The situation is worse for nonminimal coupling such as in S_{BD}. The resultant stress-energy need not even be positive definite. Consequently, there can not be any theoretical restriction on the range $\omega \in (-\infty,\infty)$ in principle, excluding possibly $\omega=0$. One may, in fact, recall that the whole gamut of classical stationary wormhole solutions in the BD theory relies essentially on negative values of ω, which, in turn, entail a negative energy density at least at the throat of the wormhole [23-25,97]. Although classical fields having $\rho<0$ have not yet been discovered experimentally, quantum fields with $\rho<0$ have long been known to exist (Casimir effect).

In the context of no-hair theorem, Bekenstein [76] has also pointed out that the theorem fails whenever the energy density is not positive definite ($\rho<0$) or WEC is violated. This failure provides a rationale for the conformal scalar hair black holes to exist [76]. Several examples of weak energy violating black holes are now available [82]. Most well known are the Neveu-Schwarz charged black hole and its dual in the string modified gravity. Our interest here is in BD theory with ω = constant and in what follows, we take $\omega \in (-\infty,\infty)$ to be in conformity with $\rho<0$.

5.4. AN EXAMPLE: BRANS CLASS IV SOLUTIONS

We wish to exemplify here the statement that the inapplicability of Bekenstein's theorem does not necessarily guarantee that the negation of its conclusion would be true. In other words, we have to show that $\rho<0$ black hole exteriors need not always be covered by non- constant scalar fields. The best example is provided by what is known as Brans Class IV solutions given by Eqs. (4.3.1)-(4.3.4).

There is a singularity in the metric at r = 0. In the static proper orthonormal frame defined by $e_{\hat{0}}=e^{-\alpha}e_t$, $e_{\hat{1}}=e^{-\beta}e_r$, $e_{\hat{2}}=r^{-1}e^{-\beta}e_\theta$, $e_{\hat{3}}=(r\sin\theta)^{-1}e^{-\beta}e_\varphi$, $\eta_{\hat{\alpha}\hat{\beta}}=e_{\hat{\alpha}}.e_{\hat{\beta}}=[-1,1,1,1]$, $e_t=\partial/\partial t$, $e_r=\partial/\partial r$, $e_\theta=\partial/\partial\theta$, $e_\varphi=\partial/\partial\varphi$, the tidal force components are given by

$$\frac{\partial^2\zeta^{\hat{\alpha}}}{\partial s^2} = R^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}\hat{\delta}}\, u^{\hat{\beta}}\zeta^{\hat{\gamma}}u^{\hat{\delta}} \tag{5.4.1}$$

where $u^{\hat{\alpha}}$ are the 4-velocities and $\zeta^{\hat{\alpha}}$ are the components of separation vector between two neighboring geodesics. For the solutions (4.3.1) - (4.3.4) of Chapter 4, the nonvanishing Riemann curvature components turn out to be

$$\tilde{R}_{1010} = \frac{1}{Br^3 e^{2(C+1)/Br}}\left[\frac{C + 2}{Br} - 2\right], \qquad (5.4.2)$$

$$\tilde{R}_{2020} = \tilde{R}_{3030} = \frac{1}{Br^3 e^{2(C+1)/Br}}\left[1 - \frac{C + 1}{Br}\right], \qquad (5.4.3)$$

$$\tilde{R}_{2121} = \tilde{R}_{3131} = - \frac{C + 1}{Br^3 e^{2(C+1)/Br}}, \qquad (5.4.4)$$

$$\tilde{R}_{3232} = \frac{C + 1}{Br^3 e^{2(C+1)/Br}}\left[2 - \frac{C + 1}{Br}\right]. \qquad (5.4.5)$$

All of these tend to zero as $r \to 0$ or ∞, for $C+1>0$. For a diagonal stress-energy tensor $T_{\mu\nu} \equiv [\rho,\tau,p,p]$, we get the energy density $\rho(r)$ and the curvature scalar R as

$$\rho(r) = - \frac{(C + 1)^2}{B^2 r^4} e^{- 2(C+1)/Br}, \qquad (5.4.6)$$

$$\bar{R} = \frac{\omega C^2}{B^2 r^4} e^{- 2(C+1)/Br}. \qquad (5.4.7)$$

Clearly, $\rho<0$ and Bekenstein's theorem becomes inapplicable. One might easily verify that all curvature components including both $\rho(r)$ and $\bar{R}$ tend to zero as $r \to 0$. Therefore. the degenerate surface $r = 0$ is nonsingular and thus truly represent a black hole horizon. If $C\neq 0$, then we do have black holes with nonconstant scalar field $\tilde{\varphi}$. But if C=0, which comes from the Einstein limit $\omega \to -\infty$ in Eq.(4.3.4) of Chapter 4, then ρ and $\bar{R}$ both continue to remain finite, though negative, as $\omega C^2 = -2$ while $\tilde{\varphi} = \tilde{\varphi}_0$ = constant. This completes what we set out to show.

The reason for such a behavior of $\tilde{\varphi}$ can be understood from the BD field equations themselves that follow from action (5.2.5). Take any component of field equations, say (00), given by

$$2\beta'' + 4r^{-1}\beta' + (\beta')^2 = (\tilde{\varphi})^{-1}[\tilde{\varphi}'\alpha' - (2\tilde{\varphi})^{-1}\omega(\tilde{\varphi}')^2], \qquad (5.4.8)$$

where ($'$) denotes derivatives w.r.t. r. Note that the r.h.s. never vanishes in the Einstein limit as $\omega C^2 \neq 0$. That means one does not recover the vacuum Einstein equations. There is a way to avoid this situation in the case of Brans Class I solutions (see Sec.5.5) as there are three parameters ω, λ and C instead of just two (ω and C), as in the present case.

5.5. BRANS CLASS I SOLUTIONS

Note that actually four classes of static spherically symmetric solutions from action (5.2.5) with ω = constant are known. Of these, Class II solutions have no coordinate singularity and Class III solutions are not asymptotically flat. Hence, we disregard them in the present analysis. Class IV solutions have already been dealt with. We are left with Class I solutions which are given by Eqs. (2.2.5)-(2.2.8).

In the same orthonormal frame as defined in Sec. 5.4, we can work out the nonvanishing Riemann curvature components which turn out to be

$$R_{\hat{1}\hat{0}\hat{1}\hat{0}} = \frac{4Br^3[Br(C + 2) - \lambda(r^2 + B^2)]}{\lambda^2(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}, \qquad (5.5.1)$$

$$R_{\hat{2}\hat{0}\hat{2}\hat{0}} = R_{\hat{3}\hat{0}\hat{3}\hat{0}} = \frac{2Br^3[\lambda(r^2 + B^2) - 2Br(C + 1)]}{\lambda^2(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}, \qquad (5.5.2)$$

$$R_{\hat{2}\hat{1}\hat{2}\hat{1}} = R_{\hat{3}\hat{1}\hat{3}\hat{1}} = \frac{2Br^3[2\lambda Br - (r^2 + B^2)(C + 1)]}{\lambda(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}, \qquad (5.5.3)$$

$$R_{\hat{3}\hat{2}\hat{3}\hat{2}} = -\frac{4Br^3[Br(C + 1)^2 - \lambda(r^2 + B^2)(C + 1) + \lambda^2 Br]}{\lambda^2(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}. \qquad (5.5.4)$$

For a diagonal stress energy tensor $T_{\mu\nu} \equiv [\rho,\tau,p,p]$, we get, after straightforward calculations, the energy density as

$$\rho(r)= \frac{4B^2 r^4}{(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}\left[1-\frac{C+1}{\lambda}\right] \qquad (5,5.5)$$

and the curvature scalar as

$$\bar{R} = \frac{4\omega C^2 B^2 r^4}{\lambda^2 (r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}. \qquad (5.5.6)$$

Using the above expressions, let us find out the conditions for which the surface r = B is nonsingular. To this end, note that the parameters ω, λ and C are connected by a single expression (2.2.8) of Chapter 2. Therefore, to arrive at a specific solution, we must specify any two of the parameters. For example, in the case C=0 and λ = 1, which ensures $\omega C^2=0$, both $\rho(r)$ and $\bar{R}$ are finite (zero!) at the horizon r = B and therefore the black hole solution turns out to be the one with a constant $\tilde{\varphi}$ or without hair. In fact, one just lands up with the trivial Schwarzschild exterior metric of vacuum Einstein equations.

There is also another possibility. Consider the case

$$\frac{C + 1}{\lambda} = 2. \qquad (5.5.7)$$

Then R and $\rho(r)$ become finite at the horizon r=B. Also, the curvature components $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ are finite implying that the tidal forces do not tear apart an extended test particle at the horizon. Nonetheless, choices of C and λ satisfying Eq.(5.5.7) lead to negative values for ω via Eq.(2.2.8) and hence, the scalar curvature $\bar{R}$ and the energy density ρ also become negative. Due to this, once again, Bekenstein's theorem does not apply here. As a consequence, we have the following results: One is that weak energy condition violating ($\rho<0$) black holes exist just as they do in string modified gravity theories [82], and the other is that, since $C\neq0 \Rightarrow \tilde{\varphi} \neq$ constant, scalar hairs also exist for such black holes. What happens for $C=0$ and $\tilde{\varphi}$ = constant? From Eq. (5.5.7), we have $\lambda=1/2$ and rewriting Eq.(2.2.8), we get

$$\omega C^2 = 2[\lambda^2 - (1+C+C^2)]. \tag{5.5.8}$$

This equation makes sense only in the Einstein limit $\omega \to -\infty$, in which case we have $\omega C^2 = -3/2$. Thus, we land up with exactly the same situation as that occurred in Sec.5.4, that is, a $\rho<0$, $\tilde{\varphi}$ = constant black hole.

Admittedly, for $C \neq 0$ and $R<0$, the spacetime has a topology that differs from that of an ordinary black hole. But this feature is common to practically all WEC violating black holes including the conformal Bekenstein one. We have to analyze test particle motions to see if there arises any practical problem due to such a geometry. This is done in Sec. 5.6.

Two questions appear pertinent. Are the finiteness of curvature components preserved under radial Lorentz boost? By working out the nonvanishing components in the orthonormal frame propagated parallely along a radial geodesic, it can be shown that, under Eq.(5.5.7), some of the components for Class I solutions diverge at the horizon. Consider a typical component in the boosted (*) frame

$$R_{\hat{0}^*\hat{k}^*\hat{0}^*\hat{k}^*} = R_{\hat{0}\hat{k}\hat{0}\hat{k}} + v^2(1-v^2)^{-1}(R_{\hat{0}\hat{k}\hat{0}\hat{k}} + R_{\hat{1}\hat{k}\hat{1}\hat{k}}) \qquad (5.5.9)$$

where k=2,3 and v is the instantaneous velocity expressed in terms of a constant of motion E as $v^2=1-E^{-2}e^{2\alpha}$ [see Eq.(5.6.4) below]. At the horizon: $e^{2\alpha}\to 0$ and hence $v\to 1$, but for Class I solutions, $(R_{\hat{0}\hat{k}\hat{0}\hat{k}}+R_{\hat{1}\hat{k}\hat{1}\hat{k}})$ is finite due to Eq.(5.5.7) so that $R_{\hat{0}^*\hat{k}^*\hat{0}^*\hat{k}^*}$ diverges due to the Lorentz factor. Incidentally, this particular feature of Class I solutions is shared also by Bekenstein's conformal scalar hair solution [73], as we have similarly verified. In contrast, for Class IV solutions, all the transformed components are finite at the horizon as $(R_{\hat{0}\hat{k}\hat{0}\hat{k}}+R_{\hat{1}\hat{k}\hat{1}\hat{k}})$ tends to zero there. The conclusion is that the finiteness of curvature is not Lorentz invariant for Class I solutions while for Class IV solutions, it is. This is just an interesting observation. In order to find if a solution represents a black hole, it is enough to show that the horizon is nonsingular, that is, the tidal forces are finite there.

The second question relates to test particle motion which we consider in the next section. Does an interacting test charge approach the horizon asymptotically? The answer is in the affirmative.

5.6. TEST SCALAR CHARGE TRAJECTORY

Consider a test charge of rest mass m interacting with the BD scalar field $\tilde{\varphi}$ with a coupling strength f so that the action is given by (We shall follow the developments in Ref.[73]):

$$S = -\int (m+f\tilde{\varphi})\left(-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\right)^{1/2} d\lambda . \qquad (5.6.1)$$

Since the action is invariant under a change of parameter λ, we define another and call it the proper time τ which is related to λ as

$$-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \left(\frac{d\tau}{d\lambda}\right)^2 = m^{-2}(m + f\tilde{\varphi})^2. \qquad (5.6.2)$$

We shall use Synge's procedure [98] that starts with the Lagrangian, for Class I solutions,

$$L = - e^{2\alpha(r)}\dot{t}^2 + e^{2\beta(r)}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2] \qquad (5.6.3)$$

where $\dot{x}^{\alpha} = dx^{\alpha}/d\lambda$. Then the Euler-Lagrange t-equation gives

$$e^{2\alpha(r)}\frac{dt}{d\lambda} = E = \text{constant}. \qquad (5.6.4)$$

We can also find the θ and φ equations in the same way. The first one shows that the motion is confined to an equatorial plane ($\theta = \pi/2$) and the second one gives us another constant of motion which is identified as the angular momentum h. We shall consider for simplicity only radial motions for which h = 0. For the r-equation, we use Eq.(5.6.2) and write

$$\frac{dr}{d\lambda} = \pm\, e^{-(\alpha+\beta)}\left[E^2 - e^{2\alpha}\left(1 + \frac{f\tilde{\varphi}}{m}\right)^2\right]^{1/2} . \qquad (5.6.5)$$

For the condition (2.2.8), motion is possible only if

$$E \geq \left[\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right]^{\frac{1}{\lambda}} + m^{-1} f\tilde{\varphi}_0 \left[\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right]^2 . \qquad (5.6.6)$$

and that $e^{(\alpha+\beta)} \to \infty$ as $r \to B$ so long as $\lambda > 1$. This gives $C > 1$, in consequence of Eq.(2.2.8), which in turn implies that the BD scalar field $\tilde{\varphi}$ does not diverge at the horizon $r = B$. It is also clear from Eq.(5.6.5) that the test charge overcomes any potential barrier on the way and approaches the horizon asymptotically or $dr/d\lambda \to 0$ as $r \to B$. The proper time recorded by a clock situated away from the horizon can be calculated by using equations (5.6.2) and (5.6.5) and is given by

$$\tau = \pm\int_r^B e^{(\alpha+\beta)}\left(1 + \frac{f\tilde{\varphi}}{m}\right)\left[E^2 - e^{2\alpha}\left(1 + \frac{f\tilde{\varphi}}{m}\right)^2\right]^{-1/2} dr$$

$$\simeq \pm\int_r^B e^{(\alpha+\beta)} dr + \cdots\cdots . \qquad (5.6.7)$$

As the horizon is approached, this term becomes divergent. Hence, the radial trajectories extend to infinite proper time. In other words, they are complete.

It is remarkable that we have here exactly the same picture of radial motion as in the conformal case despite the fact that, as argued in Sec.5.2, the two types of solutions are not correlated. Class IV solutions can also be dealt with similarly, under the condition (C+1) >0.

5.7. CONCLUDING REMARKS

Let us sum up. The foregoing analysis was an effort to find a parallel between the conformal scalar field ψ and BD scalar field $\tilde{\varphi}$. The cue has been provided by Bekenstein's no-hair theorem, actually its inapplicability, to both the cases under WEC violating conditions. It was argued how exactly the solutions in the two types of theories are different or unconnected. Nevertheless, on comparison, one finds similar behavior in that the tidal accelerations are finite at the horizon and that the test particle travels all the way asymptotically to the horizon. The dissimilarity is manifested only in the behavior of the scalar fields. The ψ field is divergent while the $\tilde{\varphi}$ field is convergent on the horizon. The crucial condition was provided by Eq.(2.2.8) for Class I solutions. The range that $\lambda > 1 \Rightarrow C > 1$ came actually from the demand of having the test charge approach the horizon asymptotically. For Class IV solutions, the range C > -1 came from the requirement of having a finite curvature at r = 0. All these demonstrate that, for the specified ranges of parameters, a nonconstant BD scalar hair black hole interpretation may be accorded to the considered solutions just as it is done in the conformal scalar case. We have also completed the relevant calculations in the string modified gravity and the result, to be reported elsewhere, is that the Neveu-Schwarz and its dual solutions also represent black holes with scalar hair.

All the above results not only elucidate the close connection that exists between the no-hair theorem and the behavior of related stress-energy but also buttresses Bekenstein's theorem in the sense that none of the solutions provides a counter example in the form of $\rho>0$, $\tilde{\varphi} \neq$ constant black hole.

CHAPTER 6

ON NAKED BLACK HOLES

6.1. INTRODUCTION

In the context of scalar-tensor theory of gravity, a new effect has recently been discovered by Horowitz and Ross [56,57]. This discovery exposes yet another secret of GR. They assumed that there exist what may be called naked black hole (NBH) solutions in a wide variety of theories involving scalar fields including the low-energy limit of the string theory. These are black holes for which the area of the event horizon is large and all curvature invariants are small near the horizon. Nevertheless, any object which falls in experiences enormous tidal forces outside the horizon. The tidal forces are given by the Riemann tensor in a frame associated with ingoing geodesics. Thus, for these black holes, the geodesic components are much larger than the invariants constructed from them.

The examples considered by Horowitz and Ross [56,57] are charged black holes either at or near extremality. These black holes are called NBH as they violate the spirit of cosmic censorship in that large curvatures are visible outside of the event horizon. The analysis by Horowitz and Ross [56,57] also reveal that the size of the components of the Riemann tensor in the geodesic frame is determined by proper time remaining along the geodesic before the singularity is reached. Also, the large contributions to the Riemann tensor come from the Ricci tensor and hence from the stress-energy of the matter field. In the case of a spherically symmetric field, only the dilaton field in dilaton gravity can have the nearly null stress-energy needed to produce the type of curvature required. It is therefore expected that NBH solutions occur only in theories with scalar fields. The occurrence of large tidal forces outside the horizon may have considerable implications, both for cosmic censorship hypothesis and black hole information puzzle.

The examples of NBH considered in Ref. [56] include Neveu-Schwarz charged black hole solution of the string modified four-dimensional gravity. The dilatonic and $U(1)^2$ black holes follow as a special case. In arriving at the general conditions for NBH, Horowitz and Ross had to invoke the condition of positivity of energy: $G_{oo} = \rho \geq 0$. On this point, a remark is in order. Recently, Kar [82] has shown that both Neveu-Schwarz and its dual solution known as the magnetic black hole violate the WEC given by $T_{\mu\nu}u^{\mu}u^{\nu} \geq 0$ for any timelike vector u^{μ}. For a diagonal $T_{\mu\nu} \equiv [\rho(r), \tau(r), p(r), p(r)]$, the WEC reduces to the inequalities [4] $\rho \geq 0$, $\rho+\tau \geq 0$, $\rho+p \geq 0$. Violation of any of the inequalities must be regarded as a violation of WEC. Therefore, in a static frame, it may so happen that $\rho \geq 0$ but either $\rho+\tau<0$ or $\rho+p<0$. It is then possible to go over to a Lorentz boosted frame in which the transformed energy density $\rho'<0$. Thus the positivity of energy density is no longer a Lorentz covariant statement if WEC is violated. According to Morris and Thorne [1], it is perhaps only a small step to arrive at the conclusion that the static observer also sees a negative energy density. This situation warrants that some caution be excercised in using the condition of positivity of energy in the search for NBH solutions under WEC violating circumstances. The above ambiguity immediately provokes a natural inquiry: Does there exist NBH solutions for which WEC is already satisfied? The aim of this chapter is to address this question and the answer appears to be in the affirmative which obviously places the idea of NBH on firmer grounds.

In Sec. 6.2, we shall rewrite the conditions for NBH in a slightly more convenient way that allows us to discern relevant factors. In Sec. 6.3, we shall consider the Chan-Mann-Horne solution of dilaton-Maxwell gravity in the Einstein frame and show that it does represent NBH. The same solution in the string frame also represents NBH and is dealt with in Sec. 6.4. Finally, some concluding remarks are appended in Sec. 5.5.

6.2. NBH CONDITIONS

Horowitz and Ross consider the generic metric in d dimensions which also include black p-branes ($p=d-n-3$):

$$ds^2 = -\frac{F(r)}{G(r)}\,dt^2 + \frac{dr^2}{F(r)} + R^2(r)\,d\Omega_{n+1} + H^2(r)\,dy^i dy_i \tag{6.2.1}$$

where $i=n+3,\cdots,d-1$. We ignore p-branes in this chapter and take $d=4$, $n=1$.

As mentioned before, the basic idea is to obtain conditions which ensure that the curvature components in a static frame are vanishingly small at or near the horizon and that these are large in a freely falling frame near the horizon. The nonvanishing curvature components in a static orthonormal basis are R_{0101}, R_{0202}, R_{0303}, R_{1212}, R_{1313} and R_{2323}. It has been shown in Ref. [56] that the difference between the Lorentz boosted frame and the static frame is the same for all curvature components and is proportional to $(R_{0k0k}+R_{1k1k})$, where k=2,3. Hence, it is enough to consider the increment in just one of the curvature components, say, in R_{0202}. In the boosted (^) frame, it becomes,

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}} = -\frac{1}{R}\left[R''(E^2G-F) + \frac{R'}{2}(E^2G'-F')\right] \qquad (6.2.2)$$

where $E^2=(F/G)(1-v^2)^{-1}$ and primes on the right denote derivatives with respect to r. It has been stated in Ref. [56] that the terms proportional to E^2 correspond to the enhancement of the curvature in the geodesic frame over the static frame. This statement may be slightly rephrased in order to find out exactly which piece we need to have enlarged. Note that the conserved energy E^2 can be decomposed as

$$E^2 = (F/G) + v^2(1-v^2)^{-1}(F/G) = E_s^2 + E_{ex}^2. \qquad (6.2.3)$$

The first term represents the value of E^2 in the static frame (E_s^2) and the second term E_{ex}^2 represents the enhancement in E_s^2 due to geodesic motion. Incorporating this, we can also decompose $R_{\hat{0}\hat{2}\hat{0}\hat{2}}$ as follows:

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}} = -\frac{1}{R}\left[\frac{R}{2}\left(E_s^2 G' - F'\right)\right] - \frac{1}{R}\left[R''G + \frac{R'G'}{2}\right]E_{ex}^2$$

$$= R^{(s)}_{0202} + R^{(ex)}_{0202}. \qquad (6.2.4)$$

It is easy to verify that the term $|R^{(s)}_{0202}|$ actually represents the curvature component in the static frame, viz., $R^{(s)}_{0202} = R_{0202}$. Thus, only the term $|R^{(ex)}_{0202}|$ represents overall enhancement in curvature in the Lorentz boosted frame over the static frame. This overall enhancement consists of the product of two factors. The first, which comes from the geodesic motion itself, is in the form of E_{ex}^2. The second is the coefficient of E_{ex}^2 and it is precisely *this* part that needs to be enlarged in order to make $|R^{(ex)}_{0202}|$ as large as we please. Thus, one has the NBH condition (in Planck units)

$$\left|\frac{1}{R}\left(R''G + \frac{R'G'}{2}\right)\right| > 1 \tag{6.2.5}$$

outside the horizon. The other NBH condition can not come from the imposition of positive energy density criterion as we want to investigate solutions for which the positivity requirement is already Lorentz covariantly satisfied, whatever be the values of the free parameters. It has to come directly from the definition of NBH itself: The free parameters in the solution be so choosen as to make the static curvature components small at the horizon. We must emphasize that this rephrasing does not essentially alter the original analysis of Horowitz and Ross but helps us to clearly identify the factors contributing to the enhancement in curvature.

In fact, the expression on the left side of the inequality (6.2.5) is exactly the one that occurs also in the coefficient of E^2 in Eq.(6.2.2), but the difference now is that, in Eq.(6.2.4), E^2_{ex} is *not* a conserved quantity. The enhanced tidal force in the proper orthonormal frame can be made large simply by making E^2_{ex} large. Therefore, in order to make the analysis meaningful, we must restrict the class of geodesics, as has actually been done in Ref.[56], to the one for which the conserved energy E^2 is of order unity. In that case, the maximum values of E_s^2 and E_{ex}^2 are also unity. This may be achieved by recalling that $v^2=1-E^{-2}(F/G)$ and considering geodesics that start with small velocity ($v\cong 0$) at some point ($r=r_o$), where (F/G) is of order unity. As the object approaches the horizon, $(F/G)\to 0$ so that $v\to 1$, as expected. With this understanding, let us now investigate a couple of specific examples.

6.3. CHAN-MANN-HORNE (CMH) SOLUTION IN THE EINSTEIN FRAME

This electric solution [99] represents asymptotically non-flat black hole solutions in the dilaton-Maxwell gravity. We shall consider the solution in the Einstein as well as in the string frames. The two frames are related to each other by $g^E_{\mu\nu}=g^S_{\mu\nu}e^{-2\phi}$ where ϕ is the massless dilaton field. The string frame is defined as the one in which the variables $(g^S_{\mu\nu},\phi)$ appear in the action given originally by Brans and Dicke in their scalar-tensor theory [63].

In the Einstein frame, the metric is

$$ds^2 = -\frac{F(r)}{G(r)}\, dt^2 + \frac{dr^2}{F(r)} + R^2(r)d\Omega^2 \qquad (6.3.1)$$

$$F = \left(\frac{1}{4} - \frac{\gamma^2 M}{r^2}\right), \quad G = \frac{\gamma^4}{4r^2}, \quad R = r \qquad (6.3.2)$$

while the dilaton and Maxwell fields are

$$\phi(r) = (-1/2)\ ln\ (2Q^2) + ln\ r \qquad (6.3.3)$$

$$F_{01} = \frac{Qe^{2\phi}}{r\gamma^2}. \qquad (6.3.4)$$

The dilaton rolls from $-\infty$ to $+\infty$ as r changes its value from 0 to ∞. The horizon radius occurs at $r_h^2 = 4\gamma^2 M$ where $(F/G)=0$, and it is easy to verify that all the WEC inequalities are satisfied for $r \geq r_h$. We are considering geodesics that start at $r = r_o = (\gamma^4 + 4\gamma^2 M)^{1/2}$ where $(F/G)=1$.

The non-vanishing components of the static curvature are

$$R_{0202} = R_{0303} = \frac{1}{4r^2} \qquad (6.3.5)$$

$$R_{1212} = R_{1313} = -\frac{\gamma^2 M}{r^4} \qquad (6.3.6)$$

$$R_{2323} = \frac{3}{4r^2} + \frac{\gamma^2 M}{r^4} . \qquad (6.3.7)$$

At the horizon, all curvature components are of the order of r_h^{-2}. Hence, the horizon area will be large and the static curvature will be small if $r_h^2 >> 1/2$.

It may be seen from Eqs.(6.3.5)-(6.3.7) that the Chan-Mann-Horne solution has a remarkable feature: $R_{0k0k} + R_{1k1k} = 0$ at the horizon and non-zero outside. This implies that the magnitudes of tidal forces are more outside the horizon than what they are at the horizon. This is already an indication that the solution may serve as a natural candidate for NBH.

The enhancement condition, inequality (6.2.5), gives, near the horizon,

$$\left|\frac{G'}{2r}\right| > 1 \rightarrow r_h^2 < \frac{\gamma^2}{2} . \qquad (6.3.8)$$

Thus, finally, the NBH conditions together yield $(1/2) << r_h^2 < \frac{\gamma^2}{2}$. This range automatically implies that $\gamma >> 1$ indicating a large value of r_h or a large static black hole for a given M.

6.4. CMH SOLUTION IN THE STRING FRAME

By performing conformal rescaling via the dilaton field ϕ, the CMH solution in the string frame can be written as [82]

$$ds^2=-(r^2/\gamma^4)(1-Ar^{-1})dt^2+(1-Ar^{-1})^{-1}dr^2+r^2d\Omega^2, \tag{6.4.1}$$

where $A=2\sqrt{2}\gamma^2M/Q$. The string coupling e^{ϕ} changes from 0 to ∞ as r goes from 0 to ∞. The functions F,G and R are given by

$$F=1-Ar^{-1},\ G=\gamma^4r^{-2},\ R=r. \tag{6.4.2}$$

The horizon occurs at $r_h = A$ and it has been shown in Ref.[82] that all the WEC inequalities are satisfied for $r\geq r_h$. The geodesics start from $r = r_o= (1/2)[A\pm(A^2+4\gamma^4)^{1/2}]$ and the non-vanishing static curvature components turn out to be

$$R_{0101}= -\frac{A}{2r^3} \tag{6.4.3}$$

$$R_{0202}= R_{0303}= \frac{1}{r^2} - \frac{A}{2r^3} \tag{6.4.4}$$

$$R_{1212}=R_{1313}= -\frac{A}{2r^3} \tag{6.4.5}$$

$$R_{2323} = \frac{A}{r^2} . \tag{6.4.6}$$

These components are of the order of r_h^{-2} near the horizon, just as in the previous case and they will be small if $r_h^2 >> 1$. To compute the enhancement in the boosted frame, notice once again that there follows the same remarkable result, viz., $R_{0k0k} + R_{1k1k} = 0$ at $r = r_h$, where $k = 2,3$. The inequality (6.2.5) gives $r_h^2 < \gamma^2$. So, finally, one lands up with the range $1 << r_h^2 < \gamma^2$. What happens to the horizon area $4\pi A^2$? It is certainly enlarged provided that the large value of γ^2 is choosen in such a manner that its product with (M/Q) remains large. For an exclusive γ^2 enlargement effect on A, the most reasonable choice of (M/Q) would be of the order of unity which corresponds to a near extremal black hole. The aspect of near extremality is in perfect accordance with the considerations of Ref.[56].

6.5. CONCLUDING REMARKS

We have shown that at least two WEC satisfying NBH solutions exist in the dilaton-Maxwell gravity. The first is the CMH electric solution and the second is the one obtained by conformal rescaling via the massless dilaton field ϕ. Clearly, the NBH character is not destroyed by such rescalings. Both the solutions exhibit a remarkable feature that $R_{0k0k}+R_{1k1k}=0$ at the horizon and non-zero outside. The analysis reveals that, for an infalling object, the tidal forces are vanishingly small at the horizon while these can be enlarged outside the horizon by choosing the parameter γ such that the source is a large static black hole. The fact that we need no longer bother about WEC violating "exotic" materials, lends a more realistic possibility to the idea of NBH.

As a further study, it would be interesting to examine if NBH solutions occur also in Brans-Dicke theory. Generally, the energy momentum tensor in this theory has an indefinite sign, leading to the possibility of WEC violation. However, it is always possible to go over to its conformally rescaled form where the WEC is not violated [62,97]. Work is underway.

SUMMARY

In the introductory chapter, we have attempted to provide basic materials in order to make the thesis self-contained as far as possible. The focus has entirely been on classical wormholes. Since each chapter ends with a summary of its own, we shall only report here the salient features of our results.

For the first time, we have found the correct range of ω in the Jordan frame viz., $-3/2<\omega<-4/3$ for static, spherically symmetric wormholes in the BD theory. This range is consistent with the pN value of $C(\omega)$ and allows one to go over to the conformally rescaled Einstein frame. In this frame, wormhole solutions are not permitted at all since all the energy conditions are satisfied. Magnano and Sokołowski [62] have argued that only the Einstein frame variables are physical due to the fact the solutions are stable. Our results indicate that everything is not lost in the Jordan frame : There *does* exist a narrow interval of ω for which stable wormhole solutions are available also in the Jordan frame.

Recent investigations by Vollick [8] and Anchordoqui *et al* [25] do buttress our conclusion above. Ref. [8] shows that interaction between gravity and matter fields gives rise to negative energies which maintain a classical wormhole. This is certainly the case in the Jordan frame where $\rho<0$. Ref. [25] shows that the presence of extra matter fields do not destabilize a static wormhole. This result is quite consistent with the fact that the range $-3/2<\omega<-4/3$ leaves the option to go over to a stable Einstein frame version open. Ref. [25] also contains a number of differing viewpoints about the physicality of Jordan or Einstein frame variables. At least in the context of wormhole physics, we now see that there is a common ground of agreement symbolized in the range of ω derived above.

An exceedingly important branch of physics is the string theory, which, in the low-energy limit gives rise to a scalar-tensor theory of 4-dimensional gravity with additional electromagnetic fields. We have studied charged Neveu-Schwarz and Turyshev solutions and shown that they represent wormholes with exactly opposite features. This result gives us an idea as to how the wormhole solutions in string theory would behave. In two consecutive Chapters (4,5), we have studied the "no-hair" theorems which find relevance in any type of scalar-tensor theory. First, we have focused on a problem relating to the limit processes involved and the need to remain aware of it in order to avoid meaningless conclusions. Secondly, we have proposed a modified version of Saa's "no-hair" conjecture which we believe to be more exact.

We next go over to a re-examination of the status of no-hair theorem in the case of BD solutions. The cue has been provided by Bekenstein's novel no-hair theorem stated in Chapter 5. Its inapplicability in the conformal scalar field can allow such hairs to exist around black holes. However, conformal solutions are distinct from BD solutions with ω=constant. Nonetheless, these two types of solutions show remarkable similarity in that both violate Bekenstein's theorem and consequently, both allow scalar hairs to exist provided the right range of values for the parameters are chosen. This result is interesting in its own right and it poses a more basic question : For $(C+1)/\lambda = 2$, can we regard the BD Class I solution as representing a WEC violating black hole? If we insist that the finiteness of curvature has to be Lorentz invariant, then the answer is in the negative. If we relax Lorentz invariance, then the answer is positive.

The final Chapter (Chapter 6) deals with a very recent proposal of naked black holes (NBH) by Horowitz and Ross

[56,57]. We have presented two examples from string-inspired scalar-tensor theory in the form of Chan-Mann-Horne solutions. The interesting point is that this solution satisfies WEC. Its importance lies in the fact that in the string modified gravity, appropriate singularity theorems for black holes are not yet available. One may adopt a conservative stance and adhere to the old singularity theorems and attempt to find out as many WEC satisfying NBH as possible.

The contents of this thesis reveal that scalar-tensor theories of gravity have more secrets in their fold than have so far been discovered. These theories are not only useful for the interpretation of many physical phenomena but also extremely interesting from the theoretical point of view.

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Brans-Dicke wormholes in the Jordan and Einstein frames

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We examine the possibility of static wormhole solutions in the vacuum Brans-Dicke theory both in the original (Jordan) frame and in the conformally rescaled (Einstein) frame. It turns out that, in the former frame, wormholes exist only in a very narrow interval of the coupling parameter, viz., $-3/2<\omega<-4/3$. It is shown that these wormholes are not traversable in practice. In the latter frame, wormhole solutions do not exist at all unless energy conditions are violated by hand.
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I. INTRODUCTION

Over the last few years, considerable interest has grown in the field of wormhole physics, following especially the seminal works of Morris, Thorne, and Yurtsever [1,2]. Wormholes are topology changes that connect two asymptotically flat regions. Potential applications of wormhole physics range from the interpretation of gravitational lensing effects to the resolution of several outstanding problems in cosmology [3–5].

In the context of traversable wormholes, a crucial issue is the constraint upon the violation of energy conditions by the stress tensor of quantum or classical fields. There exist several pointwise and average energy conditions [6]. Specifically, for quantum fields, Ford and Roman [7] have proposed, on the basis of certain assumptions, an inequality that constrains the magnitude of the negative energy density at the throat of a traversable wormhole. A fundamental assumption for quantum wormholes is that the stress energy of the spacetime is a renormalized expectation value of the energy-momentum operator in some quantum state, say, $|\psi\rangle$. In the literature [8], one actually considers field equations of semiclassical gravity in the form $G_{\mu\nu}=8\pi\langle\psi|T_{\mu\nu}|\psi\rangle$. However, some doubts have been raised, notably by Unruh [9], as to whether field equations in this form could be an exact description of gravity [10]. On the other hand, quantized source fields obey well-defined uncertainty relations and it is expected that uncertainty in the source would induce uncertainty in the gravidynamic variables and in the light cone structure of spacetime [11,12]. If the source is taken as $\langle T_{\mu\nu}\rangle$, such fluctuations would not occur. Despite these questions, it must be emphasized that field equations in the above form provide a very good approximation in many physical situations, especially in the description of the early universe [13].

There also exist classical fields playing the role of "exotic matter" that violates the weak energy condition (WEC), at least at the throat of the wormhole. Examples are provided by the stress-energy tensors occurring in theories where the action contains $R+R^2$ terms [14], an antisymmetric 3-form axion field coupled to scalar fields [15], and minimally coupled fields with a self-interacting potential [16]. Other theories include string-inspired four-dimensional gravity coupled nonminimally to a scalar field [17], Zee's induced gravity [18], and the Brans-Dicke scalar-tensor theory [19]. Most of the works concentrate on dynamic wormholes, while work on static wormholes is relatively scarce. In particular, in the Brans-Dicke theory, a search for static wormholes has been initiated only recently [20,21], followed by Anchordoqui, Bergliaffa, and Torres [22]. Considering the importance of Brans-Dicke theory in the interpretation of various physical phenomena [23–25] and owing to the fact that, in the limit $\omega\to\infty$, one recovers general relativity, it is only desirable that a thorough study of classical wormhole solutions be undertaken in this theory.

In this paper, we intend to examine wormhole solutions in the Jordan and Einstein frames which are defined as follows [26]: The pair of variables (metric $g_{\mu\nu}$, scalar φ) defined originally in the Brans-Dicke theory constitute what is called a Jordan frame. Consider now a conformal rescaling

$$\tilde{g}_{\mu\nu}=f(\varphi)g_{\mu\nu}, \quad \phi=g(\varphi), \tag{1}$$

such that, in the redefined action, ϕ becomes minimally coupled to $\tilde{g}_{\mu\nu}$ for some functions $f(\varphi)$ and $g(\varphi)$. Then the new pair $(\tilde{g}_{\mu\nu},\phi)$ is said to constitute an Einstein frame. There exist different viewpoints as to the question of which of these two frames is physical, but the arguments of Magnano and Sokołowski [26] seem convincing enough in favor of the physicality of the Einstein frame.

In what follows, we shall be concerned only with static spherically symmetric solutions of the Brans-Dicke theory. For this purpose, only a class I type of solution is considered; other classes (II–IV) of solutions can be dealt with in a similar way. Our results are stated as follows. In Sec. II, we consider the Jordan frame and derive the general condition for the existence of wormholes. This condition is then used to find wormhole ranges of ω in specific cases. Section III shows that these wormholes are not traversable due to the occurrence of a naked singularity. The Einstein frame is considered in Sec. IV, and it is shown that wormhole solutions do not exist at all in that frame. The last section, Sec. V, is a summary.

II. JORDAN FRAME

In order to investigate the possibility of wormholes in the vacuum (matter-free) Brans-Dicke theory, it is convenient to cast the spacetime metric in the Morris-Thorne canonical form

$$d\tau^2=-e^{2\Phi(R)}dt^2+\left[1-\frac{b(R)}{R}\right]^{-1}dR^2+R^2d\Omega_2^2,$$

$$d\Omega_2^2=d\theta^2+\sin^2\theta d\varphi^2, \tag{2}$$

where $\Phi(R)$ and $b(R)$ are redshift and shape functions, respectively. These functions are required to satisfy some constraints, enumerated in [1], in order that they represent a wormhole. It is, however, important to stress that the choice of coordinates (Morris-Thorne) is purely a matter of convenience and not a physical necessity. For instance, one could equally well work directly with isotropic coordinates using the analyses of Visser [6], but the final conclusions would be the same. Nonetheless, it must be understood that a more appropriate procedure should involve coordinate-independent proper quantities.

The matter-free action in the Jordan variables is ($G=c=1$)

$$S=\frac{1}{16\pi}\int d^4x(-g)^{1/2}[\varphi\mathbf{R}-\varphi^{-1}\omega(\varphi)g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}]. \tag{3}$$

The field equations are

$$\Box^2\varphi=0,$$

$$\mathbf{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\mathbf{R}=-\frac{\omega}{\varphi^2}\left[\varphi_{,\mu}\varphi_{,\nu}-\frac{1}{2}g_{\mu\nu}\varphi_{,\rho}\varphi^{,\rho}\right]$$
$$-\frac{1}{\varphi}[\varphi_{;\mu;\nu}-g_{\mu\nu}\Box^2\varphi], \tag{4}$$

where $\Box^2\equiv(\varphi^{;\rho})_{;\rho}$ and ω is a dimensionless coupling parameter. The general solution, in isotropic coordinates (r,θ,φ,t), is given by

$$d\tau^2=-e^{2\alpha(r)}dt^2+e^{2\beta(r)}dr^2+e^{2\nu(r)}r^2d\Omega_2^2. \tag{5}$$

Brans class I solutions [27] correspond to the gauge $\beta-\nu=0$ and are given by

$$e^{\alpha(r)}=e^{\alpha_0}\left[\frac{1-B/r}{1+B/r}\right]^{1/\lambda}, \tag{6}$$

$$e^{\beta(r)}=e^{\beta_0}\left[1+\frac{B}{r}\right]^2\left[\frac{1-B/r}{1+B/r}\right]^{(\lambda-C-1)/\lambda}, \tag{7}$$

$$\varphi(r)=\varphi_0\left[\frac{1-B/r}{1+B/r}\right]^{C/\lambda}, \tag{8}$$

$$\lambda^2\equiv(C+1)^2-C\left(1-\frac{\omega C}{2}\right)>0, \tag{9}$$

where α_0, β_0, B, C, and φ_0 are constants. The constants α_0 and β_0 are determined by asymptotic flatness condition as $\alpha_0=\beta_0=0$.

Redefining the radial coordinate $r\rightarrow R$ in the metric (5) as

$$R=re^{\beta_0}\left[1+\frac{B}{r}\right]^2\left[\frac{1-B/r}{1+B/r}\right]^{\Omega},\quad \Omega=1-\frac{C+1}{\lambda}, \tag{10}$$

we obtain the following functions for $\Phi(R)$ and $b(R)$:

$$\Phi(R)=\alpha_0+\frac{1}{\lambda}\left[\ln\left\{1-\frac{B}{r(R)}\right\}-\ln\left\{1+\frac{B}{r(R)}\right\}\right], \tag{11}$$

$$b(R)=R\left[1-\left\{\frac{\lambda\{r^2(R)+B^2\}-2r(R)B(C+1)}{\lambda\{r^2(R)-B^2\}}\right\}^2\right]. \tag{12}$$

The throat of the wormhole occurs at $R=R_0$ such that $b(R_0)=R_0$. This gives minimum allowed r-coordinate radii $r_0^{\pm}$ as

$$r_0^{\pm}=\alpha^{\pm}B, \tag{13}$$

$$\alpha^{\pm}=(1-\Omega)\pm\sqrt{\Omega(\Omega-2)}. \tag{14}$$

The values $R_0^{\pm}$ can be obtained from Eq. (10) using this $r_0^{\pm}$. Noting that $R\rightarrow\infty$ as $r\rightarrow\infty$, we find that $b(R)/R\rightarrow 0$ as $R\rightarrow\infty$. Also, $b(R)/R\leqslant 1$ for all $R\geqslant R_0^{\pm}$. The redshift function $\Phi(R)$ has a singularity at $r=r_S=B$. In order that a wormhole be just geometrically traversable, the minimum allowed values $r_0^{\pm}$ must exceed $r_S=B$. It can be immediately verified from Eq. (10) that $r_0^{\pm}\geqslant B\Rightarrow R_0^{\pm}\geqslant 0$. This is possible only if the range of Ω is chosen either as $-\infty<\Omega\leqslant 0$ or as $2<\Omega<\infty$. We shall not consider the latter range here.

The energy density of the wormhole material is given by [1]

$$\rho(R)=(8\pi R^{-2})(db/dR), \tag{15}$$

and a straightforward calculation gives

$$db/dR=4r^2(R)B^2[r^2(R)-B^2]^{-2}\Omega(2-\Omega)$$
$$=4r^2(R)B^2[r^2(R)-B^2]^{-2}\left[1-\left(\frac{C+1}{\lambda}\right)^2\right]. \tag{16}$$

Therefore, the most general condition for the violation of the WEC is that

$$C(\omega)+1>\lambda(\omega), \tag{17}$$

where the real function $C(\omega)$ is as yet unspecified. As long as the general condition (17), which ensures $R_0^{\pm}>0$, is satisfied, it follows that

$$b_0'=\left.\frac{db}{dR}\right|_{R=R_0^{\pm}}=-1. \tag{18}$$

so that $\rho_0=\rho|_{R=R_0^\pm}<0$, and a violation of the WEC at the throat is achieved thereby. In the limit $r_0^\mp\to B+$, or, equivalently, $R_0^\mp\to 0+$, one obtains $\rho_0\to-\infty$. This means that there occurs an infinitely large concentration of exotic matter at the throat when its r radius is in the vicinity of the Schwarzschild radius $r_s=B$. No upper limit to this classical negative energy density is known to us. The general profile for $\rho(R)$ for a given wormhole configuration is that $\rho(R)$ attains its maximum at the throat and falls off in an inverse square law as one moves away from the throat to the asymptotic region.

The constraint (17) can be rephrased, using Eq. (9), as

$$C(\omega)\left[1-\frac{\omega C(\omega)}{2}\right]>0, \tag{19}$$

and depending on the form of $C(\omega)$, this inequality fixes the range of wormhole values of ω, provided one excludes the forbidden range coming from the requirement that $\lambda^2>0$. A further exclusion of the range $\omega\leqslant-3/2$ comes from a "physical" requirement that the theory be transferrable to Einstein frame [26]. In the limiting case, $C(\omega)\to 0$, $\lambda(\omega)\to 1$ as $\omega\to\infty$, one simply recovers the Schwarzschild exterior metric in standard coordinates from Eqs. (11) and (12), so that $b(R)=2M$ and $b_0'=0$. The inequality (19) is violated, and there occurs no traversable wormhole, as is well known [1].

The analysis of Agnese and La Camera [20] corresponds, as pointed out earlier [21], to the choice

$$C(\omega)=-\frac{1}{\omega+2}, \tag{20}$$

which suggests, via Eq. (19), a wormhole range $\omega<-4/3$. The forbidden range turns out to be $-2<\omega<-3/2$, which is already a part of the unphysical range $\omega\leqslant-3/2$. Therefore, one is left with a very narrow actual interval for wormhole solutions, viz., $-3/2<\omega<-4/3$. It appears that the authors just missed this interval.

We should recall here that Eq. (20) is derived on the basis of a weak field (post Newtonian) approximation and there is no reason for Eq. (20) to hold for stars with a strong field such as neutron stars. In reality, if we assume such a restriction as Eq. (20), the junction conditions for the metric and scalar field are not satisfied at the boundary of the stars [28]. Evidently, any form for $C(\omega)$ different from Eq. (20) would lead to a different wormhole interval for ω. For example, in the context of gravitational collapse in the Brans-Dicke theory, Matsuda [28] chose $C(\omega)\propto-\omega^{-1/2}$. Let us take $C(\omega)=-q\omega^{-1/2}$ and choose $q<0$ such that $C(\omega)>0$. Then the constraint (19) will be satisfied only if $\omega>4/q^2$. The exact form of $C(\omega)$ should be known *a priori* from other physical considerations. However, this is just a tentative example and is meant to highlight how crucially the wormhole range for ω depends on the form of $C(\omega)$.

The constraint (17) is based only on the requirement of geometric traversability, i.e., on the requirement that the throat radii be larger than the event horizon radius $r=B$. Therefore, an immediate inquiry is whether such wormholes are traversable in practice. We discuss this issue in the following section.

III. TRAVERSABILITY

In order to get a firsthand idea about traversability in the Jordan frame, a convenient procedure is to calculate the scales over which wormhole functions change. Ford and Raman [7] defined the following quantities at the throat $R=R_0$ of a traversable wormhole:

$$\bar{r}_0=R_0,\quad r_1=\frac{R_0}{|b_0'|},\quad R_2=\frac{1}{|\Phi_0'|},\quad r_3=\left|\frac{\Phi_0'}{\Phi_0''}\right|. \tag{21}$$

These quantities are a measure of coordinate length scales at the throat over which the functions $b(R)$, $\Phi(R)$, and $\Phi'(R)$ change, respectively. For the class I solutions, they become

$$\bar{r}_0=R_0^\mp,\quad r_1=R_0^\mp,\quad R_2=0,\quad r_3=0. \tag{22}$$

The vanishing of R_2 and r_3 implies that both $\Phi(R)$ and $\Phi'(R)$ exhibit an abrupt jump at the throat. It is therefore expected that the tidal forces at the throat would be large. That this is indeed so can be verified by calculating, for example, the differential of the radial tidal acceleration [1] given in an orthonormal frame $(\hat{e}_t,\hat{e}_R,\hat{e}_\theta,\hat{e}_\varphi)$ by

$$\Delta a^R=-\mathbf{R}_{\hat{R}\hat{t}\hat{R}\hat{t}}\xi^R, \tag{23}$$

where ξ^R is the radial component of the separation vector and

$$|\mathbf{R}_{\hat{R}\hat{t}\hat{R}\hat{t}}|=\left|(1-b/R)\left[-\Phi''+\frac{b'R-b}{2R(R-b)}\Phi'-(\Phi')^2\right]\right|. \tag{24}$$

For the metric given by Eqs. (11) and (12), we find

$$|\mathbf{R}_{\hat{R}\hat{t}\hat{R}\hat{t}}|=\left|\frac{Br}{\lambda R^2(r^2-B^2)}\left[2(1-b/R)^{1/2}+(1-b/R)^{-1/2}b'\right.\right.$$
$$\left.\left.+\frac{2\lambda(r^2+B^2)-4Br}{\lambda(r^2-B^2)}\right]\right|. \tag{25}$$

At the throat where $b(R_0^\mp)=R_0^\mp$, we have $|\mathbf{R}_{\hat{R}\hat{t}\hat{R}\hat{t}}|\to\infty$, and this implies $\Delta a^R\to\infty$. As we march away from the throat to the asymptotic limit $r\to\infty$ or, $R\to\infty$, we find $|\mathbf{R}_{\hat{R}\hat{t}\hat{R}\hat{t}}|\to 0$, as is to be expected.

Such an infinitely large tidal force at the throat is presumably related to the presence of singular null surface or naked singularity in the wormhole spacetime. These wormholes, to use a phrase by Visser [6], are "badly diseased."

The occurrence of singular null surface in the scalar-tensor theories is directly related to the "no-hair theorem," which commonly means that "black holes have no scalar hair" [29]. Early investigations into the no-hair theorem in the Brans-Dicke theory are due to Hawking [30], Chase [31], Teitelboim [32], and Bekenstein [33]. Recently, Saa [34] has formulated a new no-hair theorem which basically relies on the assessment of the behavior of scalar curvature $\mathbf{R}$, which, for the metric (6) and (7), turns out to be

$$\mathbf{R}(r)=\frac{4\omega C^2B^2r^4(r+B)^{2\Omega-6}}{\lambda^2(r-B)^{2\Omega+2}}. \tag{26}$$

Then it follows that $\mathbf{R}\to\infty$ as $r\to B+$ for $C\neq 0$. In other words, the scalar curvature diverges as $R\to 0+$, implying that this shrunk surface does not represent a black hole for $\varphi\neq$ const. It is instead a naked singularity [34]. On the other hand, if $C\to 0$ and $\lambda\to 1$, we have a finite value of $\mathbf{R}$ as $r\to B$. This means that we have a black hole solution for φ = const, in total accordance with the no-hair theorem.

Generally speaking, wormhole solutions obtain in the Jordan frame because the sign of the energy density is indefinite in that frame. The sign is positive or negative according as $C(\omega)+1<\lambda$ or $C(\omega)+1>\lambda$. Let us examine the situation in the Einstein frame, defined earlier.

IV. EINSTEIN FRAME

Under the conformal transformation

$$\tilde{g}_{\mu\nu}=pg_{\mu\nu},\quad p=\frac{1}{16\pi}\varphi, \tag{27}$$

and a redefinition of the Brans-Dicke scalar

$$d\phi=\left(\frac{\omega+\frac{3}{2}}{\alpha}\right)^{1/2}\frac{d\varphi}{\varphi}, \tag{28}$$

in which we have intentionally introduced an arbitrary parameter α, the action (3) in the Einstein variables $(\tilde{g}_{\mu\nu},\phi)$ becomes

$$S=\int d^4x(-\tilde{g})^{1/2}[\tilde{\mathbf{R}}-\alpha\tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu}]. \tag{29}$$

The field equations are

$$\tilde{\mathbf{R}}_{\mu\nu}=\alpha\phi_{,\mu}\phi_{,\nu}, \tag{30}$$

$$\Box^2\phi=0. \tag{31}$$

The solutions of Eqs. (30) and (31) can be obtained, using the transformations (27) and (28), as

$$d\tau^2=-\left(1+\frac{B}{r}\right)^{2\beta}\left(1-\frac{B}{r}\right)^{-2\beta}dt^2+\left(1-\frac{B}{r}\right)^{2(1-\beta)}\times\left(1+\frac{B}{r}\right)^{2(1+\beta)}[dr^2+r^2d\Omega_2^2], \tag{32}$$

$$\phi=\left[\left(\frac{\omega+\frac{3}{2}}{\alpha}\right)\left(\frac{C^2}{\lambda^2}\right)\right]^{1/2}\ln\left[\frac{1-B/r}{1+B/r}\right], \tag{33}$$

$$\beta=\frac{1}{\lambda}\left(1+\frac{C}{2}\right). \tag{34}$$

The expression for λ^2, of course, continues to be the same as Eq. (9), and using this, we can rewrite Eq. (33) as

$$\phi=\left[\frac{2(1-\beta^2)}{\alpha}\right]^{1/2}\ln\left[\frac{1-B/r}{1+B/r}\right]. \tag{35}$$

Casting the metric (32) into the Morris-Thorne form, we can find the wormhole throat r radii to be

$$r_0^{\pm}=B[\beta\pm(\beta^2-1)^{1/2}]. \tag{36}$$

For real $r_0^{\pm}$, we must have $\beta^2\geq 1$. But $\beta^2=1$ corresponds to a nontraversable wormhole since $r_0^{\pm}$ coincides with the singular radius $r_S=B$. From Eq. (35), it follows that, if $\alpha>0$ and $\beta^2>1$, then no wormhole is possible as ϕ becomes imaginary. This result is quite consistent with the fact that the stress-energy tensor for massless minimally coupled scalar field ϕ: viz.,

$$T_{\mu\nu}=\alpha(\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}\tilde{g}_{\mu\nu}\phi^{,\sigma}\phi_{,\sigma}) \tag{37}$$

satisfies all energy conditions [6]. The Einstein frame is thus called "physical" for which the restriction $\omega>-3/2$ follows from Eq. (33).

On the other hand, if we choose $\alpha<0$, which amounts to violating all energy conditions by brute force, one may find wormholes for $\beta^2>1$ in Eq. (35) or, equivalently, for $\omega<-3/2$.

We wish to point out a few more relevant points.

(i) Just as in the Jordan frame, the "no-hair theorem" holds also in the Einstein frame. This can be seen from the expression for scalar curvature $\tilde{\mathbf{R}}$ computed from the metric (32):

$$\tilde{\mathbf{R}}=\frac{8B^2r^4(1-\beta^2)}{(r-B)^{2(2-\beta)}(r+B)^{2(2+\beta)}}. \tag{38}$$

One can see that $\tilde{\mathbf{R}}$ is negative for wormhole solutions. In the Schwarzschild limit $\beta\to 1$, $\tilde{\mathbf{R}}$ is finite for $r\to B$, and a black hole solution results, in complete accordance with the no-hair theorem [34]. The divergence of ϕ at $r=B$ has been shown to be physically innocuous [35,36]. Generally, for $\beta\neq 1$, $\tilde{\mathbf{R}}\to\infty$ as $r\to B$. This implies that the surface $r=B$ (or, $R=0$) is not a black hole surface for nonconstant ϕ. This conclusion is in agreement with that reached by Agnese and La Camera [37] in a different way.

(ii) The Arnowitt-Deser-Misner (ADM) mass of the configuration is defined by

$$M=\frac{1}{16\pi}\lim_{S\to\infty}\int_S\sum_{i,j=1}^{3}(\partial_j g_{ij}-\partial_i g_{jj})n^i dS, \tag{39}$$

where S is a 2-surface enclosing the active region and n^i denotes the unit outward normal. For the metric (32), we get

$$M=2B\beta, \tag{40}$$

and using this value, the metric can be expanded in the weak field as

$$d\tau^2=-(1+2Mr^{-1}+\cdots)dt^2+(1-2Mr^{-1}+2Mr^{-2}+\cdots)[dr^2+r^2d\Omega_2^2]; \tag{41}$$

that is, it predicts exactly the same results for a neutral test particle as does Einstein's general relativity. The factor α does not appear in the metric, although it does appear in the scalar field ϕ. Hence, α cannot be determined by any metric test of gravity.

(iii) It should be remarked that if we replace B by another integration constant $m/2$, the solutions (32) and (35) become those proposed by Buchdahl [38] long ago. Defining the field strength σ for the scalar field ϕ in analogy with an "electrostatic field," one obtains

$$\sigma=-2\,\delta m, \quad \delta=[(1-\beta^2)/2\alpha]^{1/2}. \tag{42}$$

Then, from Eqs. (40) and (42), it follows that the gravitation producing mass M is given by

$$M^2=m^2-\tfrac{1}{2}\alpha\sigma^2, \tag{43}$$

where m can be regarded as the strength of the source excluding the scalar field. For $\beta\to 0$, we have $M\to 0$. The situation in this case is that, for $\alpha>0$, we can have both m and σ nonzero, but with their effects mutually anulled. In other words, we obtain a configuration which is indifferent to a gravitational interaction with distant bodies. The reason is that the stresses of the ϕ field contribute an amount of negative gravitational potential energy (attractive) just sufficient to make the total energy zero [38]. On the other hand, if $\alpha<0$, the ϕ field has a positive gravitational potential energy (repulsive). We cannot take $\beta\to 0$ owing to Eq. (42), but it is possible to make $m\to 0$ so that $M\to 0$. In this case, we have $\sigma=0$. That is, the vanishing of total energy implies a vanishing of individual source contributions.

V. SUMMARY

The foregoing analysis reveals that spherically symmetric static vacuum Brans-Dicke wormholes exist in the Jordan frame only in a very narrow interval $-3/2<\omega<-4/3$, corresponding to a physical situation where the post-Newtonian approximation is valid. In general, the wormhole range for ω depends entirely on the form of $C(\omega)$ supposed to be dictated by physical conditions. Wormhole solutions do not exist at all in the conformally rescaled (Einstein) frame unless one is willing to violate the energy conditions by choice ($\alpha<0$). However, such a manipulation is not always necessary. For example, there exist theories where one adds to the Einstein frame vacuum action other fields (such as the axion field [15]) or potentials [39] and obtains dynamic wormhole solutions in a natural way.

It is evident that the factor α does not appear in the metric (32), although it does appear in the expression for the scalar field ϕ. In particular, for local tests of gravity, the predictions are exactly the same as those of Einstein's general relativity where the Robertson parameters take on values $\alpha=\beta=\gamma=1$. In contrast, in the Jordan frame, one has $\alpha=\beta=1$, $\gamma=(\omega+1)/(\omega+2)$. For finite ω, it is evident that the predictions deviate somewhat from the actually observed values.

The Arnowitt-Deser-Misner (ADM) mass of the configuration is positive in both the frames. In the Jordan frame, it is $M=(2B/\lambda)(C+1)$, while in the Einstein frame it is $M=2B\beta$. It is also shown that a gravitationally indifferent real configuration with zero total energy ($M=0$) does or does not exist in the Einstein frame according as $\alpha>0$ or $\alpha<0$.

An interesting feature of Brans-Dicke wormholes is that infinitely large radial tidal accelerations occur at the throat so that these wormholes are not traversable in practice. This feature is reflected in the absence of a black hole surface at $r=B$ or, in the Morris-Thorne coordinates, at $R=0$.

We have not addressed the question of stability of Brans-Dicke wormholes in this paper. With regard to classical perturbations, it should be pointed out that the results of Anchordoqui, Bergliaffa, and Torres [22] indicate that addition of extra ordinary matter does not destroy the wormhole. The effect of the quantum back reaction of the scalar field on stability will be considered elsewhere.

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Modified No-hair Conjecture and the Limiting Process

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It is argued that the Schwarzschild black hole solution follows as a unique limit of the Brans–Dicke Class I solutions, provided the correct iterated limit is taken. Such a uniqueness is essential for the validity of a recent version of the no-hair conjecture. A non-trivial modification to this version is proposed in order to exclude Brans–Dicke Class IV solutions
(a) → which appear to ~~violate the conjecture.~~

KEY WORDS : Brans–Dicke theory ; its limit in General Relativity

1. INTRODUCTION

The problem of the existence or absence of a scalar hair in a black hole solution has received considerable attention since the no-hair conjecture was
[1–13] / first proposed by Ruffini and Wheeler [1,1–13]. A popular version of the conjecture is that "black holes have no hair," which is supposed to mean that a black hole can be dressed only by fields which can be described by a Gauss-like law. Recent investigations include the conformally coupled scalar field [11,12], or multicomponent scalar hair with non-quadratic Lagrangian [14]. We reproduce the latest version of the no-hair conjecture propsed by Saa [15] in the context of the non-minimally coupled scalar field ϕ, as follows.

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The only asymptotically flat, static and spherically symmetric exterior solution of the system governed by the action

$$S[g,\phi] = \int d^4x \sqrt{-g}\,[f(\phi)\bar{R} - h(\phi)g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi], \tag{1}$$

where $\bar{R}$ is a scalar curvature formed from $g^{\mu\nu}$, and $f(\phi), h(\phi) > 0$ with ϕ finite everywhere, is the Schwarzschild black hole solution. It should be noted that Saa [15] defines a black hole solution as one for which the scalar curvature $\bar{R}$ is finite at the metric singularity. (b)

The Brans–Dicke (BD) field equations follow in the specific case where $h(\phi) = \omega\phi^{-1}$ and $f(\phi) = \phi$. We shall be concerned with the limiting process $|\omega| \to \infty$ that is supposed to provide a passage from BD to Einstein's field equations. Another limiting process that will be shown to have a direct relationship with the conjecture is $r \to r_{\text{h}}$, where r is a radial coordinate and r_{h} is the coordinate horizon radius in a given solution. In the literature, the statements $r = r_{\text{h}}$ and $r \to r_{\text{h}}^+$ are often understood to mean the same and there arises no essential difficulty as long as only this "single limit" process is considered. However, care must be exercised as soon as "double limit" or "iterated limit" processes come into play. For example, to verify the validity of the no-hair conjecture in the BD theory, it becomes necessary to study the nature of the scalar curvature $\bar{R}$ in the limit of the null surface $(r \to r_{\text{h}})$ and $|\omega| \to \infty$. In this case, it is essential to state exactly what type of limit process one has in mind and why. In [15], the exact nature of the limiting process remains obscure although the final conclusions drawn therein are correct.

Incorrect limit processes lead to erroneous conclusions. For example, Matsuda [16] concluded that the Schwarzschild exterior solution is not a *unique* limit of the Brans–Dicke Class I solutions of BD theory for $\phi =$ constant. There is also another solution, having a singular event horizon at the origin, that resembles the Schwarzschild solution in the exterior but does not represent a black hole. This result clearly violates the no-hair conjecture formulated above.

In this paper, we wish to examine various possible limiting processes and in particular argue that there is no violation of the no-hair conjecture as far as Brans–Dicke Class I solutions are concerned, although naked singularities may occur under specific conditions. However, ~~a violation in the case of~~ Class IV solutions seems ~~to occur~~ unless the above version of the conjecture is appropriately modified. (c)

In Section 2 we display the Class I solution to be discussed. Section 3 contains a discussion of the limiting processes involved and Section 4 shows

a) -> an apparent ~~violation~~ of the no-hair conjecture and, accordingly, a relevant ~~modification~~ is proposed. Section 5 concludes the paper.

2. BRANS–DICKE CLASS I SOLUTIONS

The Euler–Lagrange equations following from the action (1) are

$$\Box\,\phi = \frac{8\pi}{3+2\omega}\,T^{\mu}_{\mathrm{M}\mu}\,, \tag{2}$$

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R \\ &= -\frac{8\pi}{\phi}\,T_{\mathrm{M}\mu\nu} - \frac{\omega}{\phi^2}\left[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\,g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}\right] - \frac{1}{\phi}\,[\phi_{;\mu;\nu} - g_{\mu\nu}\Box\,\phi], \end{aligned} \tag{3}$$

where $\Box \equiv (\phi^{;\rho})_{;\rho}$ and $T_{\mathrm{M}\mu\nu}$ is the matter energy-momentum tensor excluding the ϕ field, ω is a dimensionless coupling parameter. The metric is given by ($G = c = 1$)

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2], \tag{4}$$

where

$$e^{\alpha(r)} = e^{\alpha_0}\left[\frac{1-(B/r)}{1+(B/r)}\right]^{1/\lambda}, \tag{5}$$

$$e^{\beta(r)} = e^{\beta_0}\left(1+\frac{B}{r}\right)^2\left[\frac{1-(B/r)}{1+(B/r)}\right]^{(\lambda-C-1)/\lambda}, \tag{6}$$

$$\phi(r) = \phi_0\left[\frac{1-(B/r)}{1+(B/r)}\right]^{C/\lambda}, \tag{7}$$

$$\lambda \equiv (C+1)^2 - C\left(1 - \frac{\omega C}{2}\right) > 0, \tag{8}$$

α_0, β_0, B, C and ϕ_0 being constants. The constants α_0 and β_0 are determined by the asymptotic flatness condition as $\alpha_0 = \beta_0 = 0$.

Note the three limits are required in order to go to the Schwarzschild exterior metric: $\omega \to \infty$, $C \to 0$ and $\lambda \to 1$. Accordingly, $C = C(\omega)$ should be chosen such that as $\omega \to \infty$, $C \to 0$. Then just two independent choices would suffice for the passage viz., $C = C(\omega)$ and $\lambda \to 1$. Under the radial coordinate transformation $r \to R$ defined by

$$R = re^{\beta_0}\left(1+\frac{B}{r}\right)^2\left[\frac{1-(B/r)}{1+(B/r)}\right]^{\Omega}, \qquad \Omega = 1 - \frac{C+1}{\lambda}, \tag{9}$$

the BD metric goes over to [17]

$$ds^2 = -e^{2\Phi(R)}dt^2 + [1 - b(R)/R]^{-1}dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta\, d\varphi^2, \quad (10)$$

where

$$\Phi(R) = \alpha_0 + \lambda^{-1}\left[\ln\left(1 - \frac{B}{r(R)}\right) - \ln\left(1 + \frac{B}{r(R)}\right)\right], \quad (11)$$

$$b(R) = R\left[1 - \left(\frac{\lambda[r^2(R) + B^2] - 2r(R)B(C+1)}{\lambda[r^2(R) - B^2]}\right)^2\right]. \quad (12)$$

The scalar curvature following from the metric (4) is

$$\bar{R} = \frac{4\omega C^2 B^2 r^4 (r+B)^{2\Omega - 6}}{\lambda^2 (r - B)^{2\Omega + 2}}. \quad (13)$$

Let us now examine various limits.

3. THE LIMITS

Let us recall that Schwarzschild exterior metric

$$ds^2 = -(1 - 2M/R)dt^2 + (1 - 2M/R)^{-1}dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta\, d\varphi^2 \quad (14)$$

is defined strictly in the range $2M < R < \infty$, and the scalar curvature $\bar{R} \to$ finite value as $R \to 2M$. Therefore, $R = 2M$ represents a null regular surface or a non-singular event horizon. In the BD metric (4), let us first choose finite values for ω (or C) and λ such that $\Omega > 0$ and try to find the R-coordinate range and the scalar curvature $\bar{R}$. Clearly, we get from eqs. (9) and (13), the single limit

$$\lim_{\substack{r \to B \\ \Omega > 0}} R \to 0 \qquad \bar{R} \to \infty. \quad (15)$$

Therefore, we get $0 < R < \infty$ and that there occurs an irremovable singularity as $R \to 0$. This result is in perfect accordance with the no-hair conjecture as $\phi \neq$ constant and the limiting metric is also not Schwarzschild. The detailed topology of such a point singularity has been studied by Agnese and La Camera [13]. Secondly, let us compute another single limit

$$\lim_{\substack{r > B \\ \Omega \to 0}} \bar{R} \to 0. \quad (16)$$

This is an expected result as Class I solutions reduce to the Schwarzschild exterior metric for which the Ricci tensor $R_{\mu\nu} = 0$, and hence $\bar{R} = 0$.

Thirdly, let us consider the limit $r = B$, $\Omega \to 0_+$. This is the case considered by Matsuda [16]. In this limit, both the metric component $[1 - b(R)/R]^{-1}$ and the scalar curvature $\bar{R}$ are not even *defined*, ~~let alone their existence~~ demonstrated, as they involve a division exactly by zero. The concept of a limit cannot even be applied here. This is evident from eqs. (12) and (13) respectively. Hence, the claim that $R = 0$ is a singular event horizon is not strictly correct if we accept the definition that a singular surface is one where the scalar curvature diverges. Clearly, the divergence of a limit is distinct from the situation where the limiting process itself is not definable. Thus, using this limiting case, it is not possible to say whether or not $R = 0$ constitutes a singular event horizon. The existence of Matsuda's non-black hole solution becomes untenable and our no-hair theorem remains effectively unchallenged.

Saa [15] seems to have calculated the scalar of curvature $\bar{R}$ at the exact equality $r = B$ and $\Omega = 0$ and, taken at face value, $\bar{R}$ has an exact form [0/0], which is by itself meaningless. Clearly, the value has to be understood only in the sense of a limit, if it is defined. But then there arises the question: Should one take a double limit or iterated limits? There is no physical ground to prefer one operation to the other. However, the requirement of a double limit constitutes a much stronger condition than that of an iterated limit. The reason is that the existence of the former does imply the existence of the latter, but the converse is not true. If we decide to compute iterated limits only, then the question is, which one. Let us write the two iterated limits,

$$\lim_{\Omega\to 0} \lim_{r\to B} \bar{R} = \lim_{\Omega\to 0} \infty = ? \tag{17}$$

$$\lim_{r\to B} \lim_{\Omega\to 0} \bar{R} = \lim_{r\to B} 0 = 0\,. \tag{18}$$

The former iterated limit does not exist in the usual sense[5] and therefore we cannot say anything about the singularity of $\bar{R}$ at $r = B$ $(r \to B)$. Saa [15] actually calculated the second iterated limit and thereby arrives at the no-hair formulation. Nonetheless, the reason why we should prefer the limit (18) is not obvious.

There is also another case: Put $\Omega = 0$. Then $\lambda = C + 1$, $C \neq 0$, $\lambda \neq 0$. We then have the Schwarzschild exterior metric with $\phi \neq$ constant.

[5] In the usual sense, the limit is not defined. However, in the context of topological investigations, sometimes the limit is axiomatically taken to be $|infty$. Such considerations are irrelevant in the present case.

Then we have $R \to 4B$ and $\bar{R} \to \infty$ as $r \to B$. This implies that the Schwarzschild sphere itself becomes an irregular null surface. This result is in perfect accordance with the no-hair conjecture. But note that $\Omega = 0$ is not strictly a valid equality in the same sense as $\omega = \infty$ is not. In accordance with $\omega \to \infty$, we must take $\Omega \to 0$. Also we should take $r \to B$ in the computation of the scalar curvature $\bar{R}$ and then we have the above iterated limits at our displosal.

Unless we have definite physical grounds to prefer one of the iterated limits, the difference between the two remains an enigma. One plausible but by no means exclusive procedure could be to choose a certain path along which the two limits would be the same.

4. CLASS IV SOLUTIONS

These solutions are given by

$$\alpha(r) = \alpha_0 - \frac{1}{Br}, \tag{19}$$

$$\beta(r) = \beta_0 + \frac{(C+1)}{Br}, \tag{20}$$

$$\phi = \phi_0 e^{-(C/Br)}, \tag{21}$$

$$C = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2}. \tag{22}$$

The usual asymptotic flatness and weak field conditions fix α_0, β_0 and B as

$$\alpha_0 = \beta_0 = 0, \qquad B = \frac{1}{M} > 0. \tag{23}$$

There is a singularity in the metric at $r = 0$. The metric (19)–(22) represents asymptotically flat, static, spherically symmetric solutions of a system governed by the action (1). Also, $C \to 0$ as $\omega \to -\infty$. The scalar curvature turns out to be

$$\bar{R} = -2(1 + C + C^2)(B^2 r^4)^{-1}. \tag{24}$$

In the same iterated limit as that considered by Saa [15], we get

$$\lim_{r\to 0} \lim_{C\to 0} \bar{R} \to \infty. \tag{25}$$

This implies that there is an irremovable singularity at $r = 0$ for $\phi =$ constant. In the limit $C \to 0$, the metric does not become exactly Schwarzschild, although it is approximately so. It is given by

$$ds^2 = -e^{-2M/r} dt^2 = e^{2M/r}[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2]. \tag{26}$$

By comparing this metric with the Robertson expansion for the static spherically symmetric problem [18],

$$ds^2 = -\left(1 - 2\alpha\frac{M}{r} + 2\beta\frac{M^2}{r^2} + \ldots\right)dt^2 + \left(1 + 2\gamma\frac{M}{r} + \ldots\right)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2], \tag{27}$$

we find that the parameters have the values

$$\alpha = \beta = \gamma = 1. \tag{28}$$

Hence, the metric (26) is indistinguishable from the Schwarzschild metric in the sense of many experimental predictions. In this connexion it is tempting to point out that the metric (26) is of exactly the same form as Rosen's metric in the bimetric theory of gravity [19]. Whether it is just a remarkable coincidence or there is a deeper connexion is a matter for further investigation.

Let us return to the no-hair conjecture. From the calculations above including the limit (25), it appears that the ~~conjecture as stated by Saa [15]~~ is violated, whereas in the case of Class I solutions, it is not. In the latter case, recall that it is necessary to fix *two* conditions independently in order to go to the Schwarzschild metric: $C = C(\omega)$, $\lambda \to 1$. In the present case, on the other hand, there is no λ but just two parameters ω and C connected by eq. (22). From this equation, it already follows that $C \to 0$ as $\omega \to -\infty$. That is, $\phi =$ constant as $|\omega| \to \infty$ and the metric does not contain any unfixed parameter whatsoever. At first sight, it may appear that no other condition is necessary for the passage to the Einstein limit $G_{\mu\nu} = 0$. But that is not so! In order that the r.h.s. of the matter-free BD equation (3) vanish identically, we must impose, in addition to $|\omega| \to \infty$, an extra condition that $\omega C^2 \to 0$, without which the passage from BD to Einstein equations cannot be ensured. But this condition is *not* satisfied in the case of Class IV solutions, as it can be verified that $G^0_0 \neq 0$. This feature shows up in the form of ~~an apparent violation of~~ the conjecture. ←(5)
Therefore, we must add ~~another restriction~~ to the version proposed by Saa [15] in order to exclude Class IV types of solution. The modified version of the conjecture should now read as follows. ←(6)

The only asymptotically flat, static and spherically symmetric exterior solution of the system governed by the action

$$S[g,\phi] = \int d^4x\, \sqrt{-g}\, [f(\phi)R - h(\phi)g^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi], \tag{29}$$

$f(\phi) > 0$, ~~$h(\phi) > 0$~~ with ϕ finite everywhere, *satisfying Einstein's vacuum equations*, is the Schwarzschild solution.

5. CONCLUSIONS

The computation of the limiting processes has a direct relelvance to the no-hair conjecture. It is necessary to be specific about the type of limit process considered. Converting the iterated limit to a single limit process, one might get a conclusion that seemingly violates the conjecture. We are aware that some recent investigations imply that the conjecture *is* violated ~~anywhere~~ under different conditions [20–22]. However, those cases do not correcpond to the non-minimally coupled scalar field considered here.

We saw that Class IV solutions ~~satisfy the conditions laid down in a recent version of the~~ conjecture and ~~yet there is~~ a violation of the conjecture. The cause is analyzed and a modification to the version is proposed. The modification is by no means trivial, as the ~~counterexample in the form of~~ Class IV solutions shows.

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