CHAPTER 5

SCALAR DRESSING OF BRANS-DICKE BLACK HOLES: A CUE FROM BEKENSTEIN'S NO-HAIR THEOREM

5.1. INTRODUCTION

This chapter continues the discussion of the no-hair theorem. In particular, we wish to investigate the extent to which BD black hole solutions admit dressing by nonminimally coupled scalar fields. The inquiry is prompted by the existence of black hole solutions having conformally coupled scalar hair. Just like this case, the BD solutions too do not satisfy the conditions laid down in the novel no-hair theorem proposed recently by Bekenstein [76]. The theorem implies that asymptotically flat, static and spherically symmetric black hole exteriors with a minimally coupled scalar field having positive definite energy density ρ do not admit nonconstant scalar hairs. However, inapplicability of Bekenstein's theorem does not, by itself, guarantee that all black hole exteriors with $\rho<0$ would be dressed by nonconstant scalar fields replicating the conformal case. This assertion, which we shall later exemplify, provides a justification as to why explicit calculations are necessary to investigate the existence of such dressings in individual cases. For two families of BD solutions, it is found that no pathological conditions occur at the horizon for a certain range of parameters: The scalar field, curvature scalar and tidal force components all remain bounded. Also, the scalar test charge trajectories interacting with the field approach the horizon asymptotically and in that sense are complete. The scalar field itself becomes constant at infinity. All these features seem to suggest that black holes in BD theory with nonconstant scalar field may after all be admissible just as they are in the conformally coupled case.

Before we deal with specific BD solutions, we must pay attention to three prerequisites: (i) Since conformal scalar hair black holes are already known to exist [73], it is necessary to show that our BD solutions are not exactly the same in content as the former ones. This is done in Sec. 5.2.(ii) Plausible justifications in favor of the BD coupling parameter range $\omega \in(-\infty,\infty)$. Recall that negative values of ω are usually regarded as unsupportable either on the ground of experimental evidences to date or on the ground that it corresponds to a negative energy density. We choose to ignore the former ground on the conviction that, since there is no *a priori* theoretical restriction on ω, it would only be legitimate to leave the values open to future experiments. The latter ground does not obviously apply in our case as we are interested only in $\rho<0$ solutions. A brief remark is presented in Sec. 5.3. (iii) We have to provide the example in favor of an assertion made in the last paragraph. This is done in Sec. 5.4 by means of Brans Class IV solutions. After the above three tasks, Brans Class I solutions are taken up in Sec. 5.5 because of their intrinsic interests. Sec. 5.6 discusses the trajectories of interacting scalar test charges and Sec. 5.7 contains a few concluding remarks.

5.2. BRANS-DICKE vis-a-vis BEKENSTEIN SOLUTIONS

As a prerequisite for the ensuing analysis, it is important to establish that the problem at hand is nontrivial, that is, the two types of solutions mentioned above are really distinct or, to be more precise, unconnected. The task is best performed by referring to a chain of available solution generating techniques that rely either on conformal rescaling of the seed metric or on redefinition of the scalar field or on both.

Let us start from the Einstein conformal scalar field matter-free action given by

$$S_C[g,\psi] = \int d^4x(-g)^{1/2}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - \frac{1}{12}R\psi^2\right]. \qquad (5.2.1)$$

Defining a scalar field ϕ as

$$\phi = (2/3)^{1/2} ln \left(\frac{\sqrt{6}+\psi}{\sqrt{6}-\psi}\right) \text{ or } \phi = (2/3)^{1/2} ln \left(\frac{\sqrt{6}-\psi}{\sqrt{6}+\psi}\right) \qquad (5.2.2)$$

according as $\psi^2<6$ or >6, it is possible to rewrite the action (5.2.1) in its minimally coupled form given by

$$S_{MC}[g,\phi] = \int d^4x(-g)^{1/2}\left[R - g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right]. \qquad (5.2.3)$$

The solution for action (5.2.3) has already been given by Buchdahl [75]. Remarkably, this solution can also be generated by the Janis-Robinson-Winicour [96] technique from the vacuum Einstein equations. One may now start from (5.2.3), as does

Bekenstein [73], and generate solutions for (5.2.1). The type A and B solutions in Ref.[73] correspond to the two cases $\psi^2<6$ or >6 respectively.

However, it is also possible to redefine ψ in a different way. Take [62]

$$\varphi = 8\pi - (4\pi/3)\psi^2, \ \omega(\varphi) = (3/2)\frac{\varphi}{8\pi-\varphi} \qquad (5.2.4)$$

then the action (5.2.1) reduces to the BD form given by

$$S_{BD}[g,\varphi] = (1/16\pi)\int d^4x(-g)^{1/2}\left[\varphi R - \varphi^{-1}\omega(\varphi)g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right]. \qquad (5.2.5)$$

Lastly, the above action can be further transformed by the Dicke transformations

$$\bar{g}_{\mu\nu}= (1/16\pi)\varphi g_{\mu\nu}, \ d\bar{\phi} =[\omega(\varphi)+3/2]^{1/2}(d\varphi/\varphi) \qquad (5.2.6)$$

to the one giving $S_{MC}[\bar{g},\bar{\phi}]$. Thus, the entire chain can be diagramatized as follows:

$$S_{MC}[g,\phi] \leftrightarrow S_C[g,\psi] \leftrightarrow S_{BD}[g,\varphi] \leftrightarrow S_{MC}[\bar{g},\bar{\phi}] \leftrightarrow S_C[\bar{g},\bar{\psi}] \leftrightarrow \cdots\cdots$$

(5.2.7)

Notice that, in the route $S_C[g,\psi] \leftrightarrow S_{BD}[g,\varphi]$, the generated coupling parameter ω must necessarily be a function of φ as specified in Eq.(5.2.4). If, on the other hand, ω = constant, the case we are considering, the route is snapped. Therefore, $\tilde{S}_{BD}[\tilde{g},\tilde{\varphi},\omega=\text{const.}] \neq S_{BD}[g,\varphi,\omega=\omega(\varphi)]$. In other words, the corresponding solutions are unconnected.

Note, however, that the demand of total absence of correlation is rather too strong and could actually be relaxed. One could equally well consider the generated BD solutions with $\omega = \omega(\varphi)$ following from action (5.2.5). The reason is that the generated solutions do not necessarily share all the properties of the seed solution. An excellent example is the Buchdahl [75] solution coming from S_{MC} which satisfies the WEC while those coming from S_C or S_{BD} do not. (For a given stress-energy tensor $T_{\mu\nu}$, the WEC states that $T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0$ for arbitrary timelike ξ^{μ}). This apart, there also exist differences in the interpretations given to the scalar fields. In S_{MC} theories, the field is viewed as some kind of matter with WEC satisfying stress-energy, while in the S_C or S_{BD} theories, the fields are regarded as a spin-0 component of gravitational interaction [62]. The gravitational constant is redefined using ψ or φ. The point we wish to make is that the task for searching scalar hair black holes is not a trivial one *even* for $\omega = \omega(\varphi)$. The action (5.2.5) actually constitutes a special case [$\omega(\varphi)$ as specified in Eq.(5.2.4)] of the Bergman-Wagoner-Nordvedt model [58].

5.3. RANGE OF ω

It is generally accepted that the coupling constant ω in the BD theory can take on only positive values, $\omega \in (0,\infty)$ which is an *a posteriori* input coming exclusively from different experiments to date [58]. There is also another ground which owes its origin to the physical requirement of having a positive definite stress-energy tensor that satisfies WEC. Such a $T_{\mu\nu}$ is available only if the scalar field couples minimally to gravity as in S_{MC} and this is the case considered by Bekenstein [76]. But, even then, the transferability from action (5.2.5) to (5.2.3) requires, in virtue of Eq.(5.2.6), that the range be $\omega \in [-3/2,\infty)$ and not $(0,\infty)$ only. The situation is worse for nonminimal coupling such as in S_{BD}. The resultant stress-energy need not even be positive definite. Consequently, there can not be any theoretical restriction on the range $\omega \in (-\infty,\infty)$ in principle, excluding possibly $\omega=0$. One may, in fact, recall that the whole gamut of classical stationary wormhole solutions in the BD theory relies essentially on negative values of ω, which, in turn, entail a negative energy density at least at the throat of the wormhole [23-25,97]. Although classical fields having $\rho<0$ have not yet been discovered experimentally, quantum fields with $\rho<0$ have long been known to exist (Casimir effect).

In the context of no-hair theorem, Bekenstein [76] has also pointed out that the theorem fails whenever the energy density is not positive definite ($\rho<0$) or WEC is violated. This failure provides a rationale for the conformal scalar hair black holes to exist [76]. Several examples of weak energy violating black holes are now available [82]. Most well known are the Neveu-Schwarz charged black hole and its dual in the string modified gravity. Our interest here is in BD theory with ω = constant and in what follows, we take $\omega \in (-\infty,\infty)$ to be in conformity with $\rho<0$.

5.4. AN EXAMPLE: BRANS CLASS IV SOLUTIONS

We wish to exemplify here the statement that the inapplicability of Bekenstein's theorem does not necessarily guarantee that the negation of its conclusion would be true. In other words, we have to show that $\rho<0$ black hole exteriors need not always be covered by non- constant scalar fields. The best example is provided by what is known as Brans Class IV solutions given by Eqs. (4.3.1)-(4.3.4).

There is a singularity in the metric at r = 0. In the static proper orthonormal frame defined by $e_{\hat{0}}=e^{-\alpha}e_t$, $e_{\hat{1}}=e^{-\beta}e_r$, $e_{\hat{2}}=r^{-1}e^{-\beta}e_\theta$, $e_{\hat{3}}=(r\sin\theta)^{-1}e^{-\beta}e_\varphi$, $\eta_{\hat{\alpha}\hat{\beta}}=e_{\hat{\alpha}}.e_{\hat{\beta}}=[-1,1,1,1]$, $e_t=\partial/\partial t$, $e_r=\partial/\partial r$, $e_\theta=\partial/\partial\theta$, $e_\varphi=\partial/\partial\varphi$, the tidal force components are given by

$$\frac{\partial^2\zeta^{\hat{\alpha}}}{\partial s^2} = R^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}\hat{\delta}}\, u^{\hat{\beta}}\zeta^{\hat{\gamma}}u^{\hat{\delta}} \tag{5.4.1}$$

where $u^{\hat{\alpha}}$ are the 4-velocities and $\zeta^{\hat{\alpha}}$ are the components of separation vector between two neighboring geodesics. For the solutions (4.3.1) - (4.3.4) of Chapter 4, the nonvanishing Riemann curvature components turn out to be

$$\underset{\sim}{R}_{1010} = \frac{1}{Br^3 e^{2(C+1)/Br}} \left[\frac{C + 2}{Br} - 2 \right], \qquad (5.4.2)$$

$$\underset{\sim}{R}_{2020} = \underset{\sim}{R}_{3030} = \frac{1}{Br^3 e^{2(C+1)/Br}} \left[1 - \frac{C + 1}{Br} \right], \qquad (5.4.3)$$

$$\underset{\sim}{R}_{2121} = \underset{\sim}{R}_{3131} = - \frac{C + 1}{Br^3 e^{2(C+1)/Br}}, \qquad (5.4.4)$$

$$\underset{\sim}{R}_{3232} = \frac{C + 1}{Br^3 e^{2(C+1)/Br}} \left[2 - \frac{C + 1}{Br} \right]. \qquad (5.4.5)$$

All of these tend to zero as $r \to 0$ or ∞, for $C+1>0$. For a diagonal stress-energy tensor $T_{\mu\nu} \equiv [\rho, \tau, p, p]$, we get the energy density $\rho(r)$ and the curvature scalar R as

$$\rho(r) = - \frac{(C + 1)^2}{B^2 r^4} e^{- 2(C+1)/Br}, \qquad (5.4.6)$$

$$\bar{R} = \frac{\omega C^2}{B^2 r^4} e^{- 2(C+1)/Br}. \qquad (5.4.7)$$

Clearly, $\rho<0$ and Bekenstein's theorem becomes inapplicable. One might easily verify that all curvature components including both $\rho(r)$ and $\bar{R}$ tend to zero as $r \to 0$. Therefore. the degenerate surface $r = 0$ is nonsingular and thus truly represent a black hole horizon. If $C\neq0$, then we do have black holes with nonconstant scalar field $\tilde{\varphi}$. But if C=0, which comes from the Einstein limit $\omega \to -\infty$ in Eq.(4.3.4) of Chapter 4, then ρ and $\bar{R}$ both continue to remain finite, though negative, as $\omega C^2 = -2$ while $\tilde{\varphi} = \tilde{\varphi}_0$ = constant. This completes what we set out to show.

The reason for such a behavior of $\tilde{\varphi}$ can be understood from the BD field equations themselves that follow from action (5.2.5). Take any component of field equations, say (00), given by

$$2\beta'' + 4r^{-1}\beta' + (\beta')^2 = (\tilde{\varphi})^{-1}[\tilde{\varphi}'\alpha' - (2\tilde{\varphi})^{-1}\omega(\tilde{\varphi}')^2], \qquad (5.4.8)$$

where (′) denotes derivatives w.r.t. r. Note that the r.h.s. never vanishes in the Einstein limit as $\omega C^2 \neq 0$. That means one does not recover the vacuum Einstein equations. There is a way to avoid this situation in the case of Brans Class I solutions (see Sec.5.5) as there are three parameters ω, λ and C instead of just two (ω and C), as in the present case.

5.5. BRANS CLASS I SOLUTIONS

Note that actually four classes of static spherically symmetric solutions from action (5.2.5) with ω = constant are known. Of these, Class II solutions have no coordinate singularity and Class III solutions are not asymptotically flat. Hence, we disregard them in the present analysis. Class IV solutions have already been dealt with. We are left with Class I solutions which are given by Eqs. (2.2.5)-(2.2.8).

In the same orthonormal frame as defined in Sec. 5.4, we can work out the nonvanishing Riemann curvature components which turn out to be

$$R_{\hat{1}\hat{0}\hat{1}\hat{0}} = \frac{4Br^3[Br(C + 2) - \lambda(r^2 + B^2)]}{\lambda^2(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}, \tag{5.5.1}$$

$$R_{\hat{2}\hat{0}\hat{2}\hat{0}} = R_{\hat{3}\hat{0}\hat{3}\hat{0}} = \frac{2Br^3[\lambda(r^2 + B^2) - 2Br(C + 1)]}{\lambda^2(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}, \tag{5.5.2}$$

$$R_{\hat{2}\hat{1}\hat{2}\hat{1}} = R_{\hat{3}\hat{1}\hat{3}\hat{1}} = \frac{2Br^3[2\lambda Br - (r^2 + B^2)(C + 1)]}{\lambda(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}, \tag{5.5.3}$$

$$R_{\hat{3}\hat{2}\hat{3}\hat{2}} =- \frac{4Br^3[Br(C + 1)^2 - \lambda(r^2 + B^2)(C + 1) + \lambda^2 Br]}{\lambda^2(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}. \tag{5.5.4}$$

For a diagonal stress energy tensor $T_{\mu\nu} \equiv [\rho,\tau,p,p]$, we get, after straightforward calculations, the energy density as

$$\rho(r) = \frac{4B^2 r^4}{(r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}\left[1-\frac{C+1}{\lambda}\right] \qquad (5,5.5)$$

and the curvature scalar as

$$\bar{R} = \frac{4\omega C^2 B^2 r^4}{\lambda^2 (r + B)^{4[1+(C+1)/2\lambda]}(r - B)^{4[1-(C+1)/2\lambda]}}\,. \qquad (5.5.6)$$

Using the above expressions, let us find out the conditions for which the surface $r = B$ is nonsingular. To this end, note that the parameters ω, λ and C are connected by a single expression (2.2.8) of Chapter 2. Therefore, to arrive at a specific solution, we must specify any two of the parameters. For example, in the case C=0 and $\lambda = 1$, which ensures $\omega C^2=0$, both $\rho(r)$ and $\bar{R}$ are finite (zero!) at the horizon $r = B$ and therefore the black hole solution turns out to be the one with a constant $\tilde{\varphi}$ or without hair. In fact, one just lands up with the trivial Schwarzschild exterior metric of vacuum Einstein equations.

There is also another possibility. Consider the case

$$\frac{C + 1}{\lambda} = 2. \qquad (5.5.7)$$

Then R and $\rho(r)$ become finite at the horizon r=B. Also, the curvature components $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ are finite implying that the tidal forces do not tear apart an extended test particle at the horizon. Nonetheless, choices of C and λ satisfying Eq.(5.5.7) lead to negative values for ω via Eq.(2.2.8) and hence, the scalar curvature $\bar{R}$ and the energy density ρ also become negative. Due to this, once again, Bekenstein's theorem does not apply here. As a consequence, we have the following results: One is that weak energy condition violating ($\rho<0$) black holes exist just as they do in string modified gravity theories [82], and the other is that, since $C\neq0 \Rightarrow \tilde{\varphi} \neq$ constant, scalar hairs also exist for such black holes. What happens for C=0 and $\tilde{\varphi}$ = constant? From Eq. (5.5.7), we have $\lambda=1/2$ and rewriting Eq.(2.2.8), we get

$$\omega C^2 = 2[\lambda^2 - (1+C+C^2)]. \tag{5.5.8}$$

This equation makes sense only in the Einstein limit $\omega \to -\infty$, in which case we have $\omega C^2 = -3/2$. Thus, we land up with exactly the same situation as that occurred in Sec.5.4, that is, a $\rho<0$, $\tilde{\varphi}$ = constant black hole.

Admittedly, for $C \neq 0$ and R<0, the spacetime has a topology that differs from that of an ordinary black hole. But this feature is common to practically all WEC violating black holes including the conformal Bekenstein one. We have to analyze test particle motions to see if there arises any practical problem due to such a geometry. This is done in Sec. 5.6.

Two questions appear pertinent. Are the finiteness of curvature components preserved under radial Lorentz boost? By working out the nonvanishing components in the orthonormal frame propagated parallely along a radial geodesic, it can be shown that, under Eq.(5.5.7), some of the components for Class I solutions diverge at the horizon. Consider a typical component in the boosted (*) frame

$$R_{\hat{0}^*\hat{k}^*\hat{0}^*\hat{k}^*} = R_{\hat{0}\hat{k}\hat{0}\hat{k}} + v^2(1-v^2)^{-1}(R_{\hat{0}\hat{k}\hat{0}\hat{k}} + R_{\hat{1}\hat{k}\hat{1}\hat{k}}) \qquad (5.5.9)$$

where k=2,3 and v is the instantaneous velocity expressed in terms of a constant of motion E as $v^2=1-E^{-2}e^{2\alpha}$ [see Eq.(5.6.4) below]. At the horizon: $e^{2\alpha}\to 0$ and hence $v\to 1$, but for Class I solutions, $(R_{\hat{0}\hat{k}\hat{0}\hat{k}}+R_{\hat{1}\hat{k}\hat{1}\hat{k}})$ is finite due to Eq.(5.5.7) so that $R_{\hat{0}^*\hat{k}^*\hat{0}^*\hat{k}^*}$ diverges due to the Lorentz factor. Incidentally, this particular feature of Class I solutions is shared also by Bekenstein's conformal scalar hair solution [73], as we have similarly verified. In contrast, for Class IV solutions, all the transformed components are finite at the horizon as $(R_{\hat{0}\hat{k}\hat{0}\hat{k}}+R_{\hat{1}\hat{k}\hat{1}\hat{k}})$ tends to zero there. The conclusion is that the finiteness of curvature is not Lorentz invariant for Class I solutions while for Class IV solutions, it is. This is just an interesting observation. In order to find if a solution represents a black hole, it is enough to show that the horizon is nonsingular, that is, the tidal forces are finite there.

The second question relates to test particle motion which we consider in the next section. Does an interacting test charge approach the horizon asymptotically? The answer is in the affirmative.

5.6. TEST SCALAR CHARGE TRAJECTORY

Consider a test charge of rest mass m interacting with the BD scalar field $\tilde{\varphi}$ with a coupling strength f so that the action is given by (We shall follow the developments in Ref.[73]):

$$S = -\int (m+f\tilde{\varphi})\left(-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} \right)^{1/2} d\lambda . \qquad (5.6.1)$$

Since the action is invariant under a change of parameter λ, we define another and call it the proper time τ which is related to λ as

$$-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \left(\frac{d\tau}{d\lambda}\right)^2 = m^{-2}(m + f\tilde{\varphi})^2. \tag{5.6.2}$$

We shall use Synge's procedure [98] that starts with the Lagrangian, for Class I solutions,

$$L = - e^{2\alpha(r)}\dot{t}^2 + e^{2\beta(r)}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2] \tag{5.6.3}$$

where $\dot{x}^{\alpha} = dx^{\alpha}/d\lambda$. Then the Euler-Lagrange t-equation gives

$$e^{2\alpha(r)}\frac{dt}{d\lambda} = E = \text{constant}. \tag{5.6.4}$$

We can also find the θ and φ equations in the same way. The first one shows that the motion is confined to an equatorial plane ($\theta = \pi/2$) and the second one gives us another constant of motion which is identified as the angular momentum h. We shall consider for simplicity only radial motions for which h = 0. For the r-equation, we use Eq.(5.6.2) and write

$$\frac{dr}{d\lambda} = \pm\ e^{-(\alpha+\beta)}\left[E^2 - e^{2\alpha}\left(1 + \frac{f\tilde{\varphi}}{m}\right)^2\right]^{1/2} . \qquad (5.6.5)$$

For the condition (2.2.8), motion is possible only if

$$E \geq \left[\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right]^{\frac{1}{\lambda}} + m^{-1} f\tilde{\varphi}_0 \left[\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right]^2 . \qquad (5.6.6)$$

and that $e^{(\alpha+\beta)} \to \infty$ as $r \to B$ so long as $\lambda > 1$. This gives $C > 1$, in consequence of Eq.(2.2.8), which in turn implies that the BD scalar field $\tilde{\varphi}$ does not diverge at the horizon $r = B$. It is also clear from Eq.(5.6.5) that the test charge overcomes any potential barrier on the way and approaches the horizon asymptotically or $dr/d\lambda \to 0$ as $r \to B$. The proper time recorded by a clock situated away from the horizon can be calculated by using equations (5.6.2) and (5.6.5) and is given by

$$\tau = \pm\int_r^B e^{(\alpha+\beta)}\left(1 + \frac{f\tilde{\varphi}}{m}\right)\left[E^2 - e^{2\alpha}\left(1 + \frac{f\tilde{\varphi}}{m}\right)^2\right]^{-1/2} dr$$

$$\simeq \pm\int_r^B e^{(\alpha+\beta)} dr + \cdots\cdots\cdots . \qquad (5.6.7)$$

As the horizon is approached, this term becomes divergent. Hence, the radial trajectories extend to infinite proper time. In other words, they are complete.

It is remarkable that we have here exactly the same picture of radial motion as in the conformal case despite the fact that, as argued in Sec.5.2, the two types of solutions are not correlated. Class IV solutions can also be dealt with similarly, under the condition (C+1) >0.

5.7. CONCLUDING REMARKS

Let us sum up. The foregoing analysis was an effort to find a parallel between the conformal scalar field ψ and BD scalar field $\tilde{\varphi}$. The cue has been provided by Bekenstein's no-hair theorem, actually its inapplicability, to both the cases under WEC violating conditions. It was argued how exactly the solutions in the two types of theories are different or unconnected. Nevertheless, on comparison, one finds similar behavior in that the tidal accelerations are finite at the horizon and that the test particle travels all the way asymptotically to the horizon. The dissimilarity is manifested only in the behavior of the scalar fields. The ψ field is divergent while the $\tilde{\varphi}$ field is convergent on the horizon. The crucial condition was provided by Eq.(2.2.8) for Class I solutions. The range that $\lambda > 1 \Rightarrow C > 1$ came actually from the demand of having the test charge approach the horizon asymptotically. For Class IV solutions, the range $C > -1$ came from the requirement of having a finite curvature at $r = 0$. All these demonstrate that, for the specified ranges of parameters, a nonconstant BD scalar hair black hole interpretation may be accorded to the considered solutions just as it is done in the conformal scalar case. We have also completed the relevant calculations in the string modified gravity and the result, to be reported elsewhere, is that the Neveu-Schwarz and its dual solutions also represent black holes with scalar hair.

All the above results not only elucidate the close connection that exists between the no-hair theorem and the behavior of related stress-energy but also buttresses Bekenstein's theorem in the sense that none of the solutions provides a counter example in the form of $\rho>0$, $\tilde{\varphi} \neq$ constant black hole.