CHAPTER 4

APPLICATION TO REISSNER-NORDSTRÖM TYPE METRICS

In this chapter, we shall illustrate the application of the above approach to the orbit equations in Reissner-Nordström type metrics. In particular, we shall assess the effect of the parameter β [see eq.(4.1.1) below] on three celebrated tests of general relativity. Our calculations supplement those recently performed by Halilsoy[66]. An error in ref.[66] is also corrected. In Sec.4.1, we deduce the equivalent gravitational index by casting the Reissner-Nordström type metric into isotropic form. Sec.4.2 shows how different choices of parameters in the metric give rise to different physical situations. We, then, write out, in Sec.4.3, the general form of the orbit equation. In subsequent sections (Sec.4.4, 4.5 and 4.6) we calculate the general relativistic effects such as the bending of light rays, precession of planetary apsides and the radar echo delay.

4.1. The refractive index

A number of metrics of physical interest assume the following form in standard coordinates (t, r', θ, ϕ):

$$ds^2 = c_o^2\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]dt^2 - \left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2$$

$$- r'^2 d\theta^2 - r'^2 \sin^2\theta d\phi^2, \qquad (4.1.1)$$

where

$$m = MG/c_o^2, \qquad (4.1.2)$$

M is the mass of the central gravitating body, G is the gravitation constant, and β is another parameter. We wish to write the line element in terms of isotropic coordinates (t,r,θ,ϕ). We will indicate briefly how to effect the transformation, using a systematic technique[67]. The idea is to express the spatial part as $-\Phi^{-2}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2]$, where $\Phi(r)$ is yet to be determined. Equating the angular and the radial parts of the two line elements, we have

$$r'^2 = \Phi^{-2} r^2 \qquad (4.1.3)$$

and

$$\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2 = \Phi^{-2} dr^2 . \qquad (4.1.4)$$

If we divide eq.(4.1.4) by eq.(4.1.3) to eliminate Φ, then

integrate and use the condition that at large radial distances r and r' must be asymptotically equal, we obtain

$$2r = (r' - m) + (r^2 - 2mr' + \beta)^{1/2}. \qquad (4.1.5)$$

The inverse transformation is

$$r' = r + m + (m^2 - \beta)/4r. \qquad (4.1.6)$$

Using these transformations, the line element (4.1.1) can be expressed in the form of (2.1.1), with

$$\Omega^2(r) = [1 - (m^2 - \beta)/4r^2]^2[1 + m/r + (m^2 - \beta)/4r^2]^{-2} \qquad (4.1.7)$$

$$\Phi^{-2}(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2. \qquad (4.1.8)$$

The effective refractive index n(r) is

$$n(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2[1 - (m^2 - \beta)/4r^2]^{-1}. \qquad (4.1.9)$$

It is also helpful to have expressions for Φ, Ω and n in terms of the standard radial coordinate r'.

Let $u \equiv 1/r$ and $u' \equiv 1/r'$. Then it is easy to show that

$$\Phi^2(u') = \frac{1}{4}[1 - mu' + (1 - 2mu' + \beta u'^2)^{1/2}]^2 \qquad (4.1.10)$$

$$\Omega^2(u') = 1 - 2mu' + \beta u'^2. \qquad (4.1.11)$$

And of course

$$n(u') = \Phi^{-1}(u')\Omega^{-1}(u'). \qquad (4.1.12)$$

In transforming coordinates, it is often helpful to use

$$du = ndu' \quad \text{or} \quad dr = \Phi\Omega^{-1}dr' \qquad (4.1.13)$$

together with

$$u = \Phi^{-1}u' \quad \text{or} \quad r = \Phi r'. \qquad (4.1.14)$$

The singularities of $n(r)$, or, equivalently, the horizons of the spacetime, occur at $r_s = \frac{m}{2}(1 - \beta/m^2)^{1/2}$, provided that $\beta/m^2 \leq 1$. Therefore, the expression for $n(r)$ is valid in the region $r > r_s$. If $\beta = m^2$, $r_s = 0$; i.e., the event horizon shrinks to zero size. In this case, $n(r) = 1 + m/r$ and is regular everywhere for $r > 0$. If $\beta > m^2$, the function $n(r)$ is not singular anywhere, since r_s becomes imaginary. Let us now examine some special cases of the metric (4.1.1).

4.2. Some special cases

Schwarzschild Exterior Metric

The Schwarzschild exterior metric applies to the spacetime around an electrically neutral, static, spherical mass M. In this case, eqs.(4.1.1) through (4.1.14) apply with

$$\beta = 0. \tag{4.2.1}$$

Reissner-Nordström Metric

The gravitational field due to an electrically charged, static spherical mass M is given by the Reissner-Nordström solution of Einstein's field equations. In this case, eqs.(4.1.1)-(4.1.14) apply with

$$\beta = GQ^2/c_o^4, \tag{4.2.2}$$

where Q is the charge on the central body.

Bertotti-Robinson Metric

This metric describes a universe filled with electromagnetic radiation of uniform density and uniformly random direction[68]. In this case, eqs.(4.1.1)-(4.1.14) apply with

$$\beta = m^2, \tag{4.2.3}$$

where m is now a nonphysical effective point mass. The BR solution may also be obtained as a special case of the metric obtained recently by Halilsoy.

Halilsoy Metric

The Halilsoy metric describes spacetime around a static, uncharged, spherically symmetric mass M which is embedded in an externally created electromagnetic field.[66,69] Once again, eqs.(4.1.1)-(4.1.14) apply with

$$\beta = q^2 m^2 \tag{4.2.4}$$

where $0 \leq q \leq 1$, and where q represents the measure of the external electromagnetic field.

Soleng Metric

The Soleng metric represents the gravitational field due to a central mass M surrounded by a field having a traceless energy-momentum tensor $T^{\mu}{}_{\nu} = f(r)\ \mathrm{diag}[1,\ 1,\ -1,\ -1]$. Recently, such a $T^{\mu}{}_{\nu}$ has been interpreted as the energy-momentum tensor associated with an anisotropic vacuum.[70-72] Here eqs.(4.1.1)-(4.1.14) apply with

$$\beta = 6\delta m^2, \tag{4.2.5}$$

where δ is the Soleng parameter, which determines the effective energy density of the anisotropic vacuum.

4.3. Central force motion

In the cases under consideration, n, v, Ω, and Φ are functions of the radial coordinate alone. The orbit (whether of light or of a massive particle) lies in a plane containing the force center and there is a constant of the motion analogous to the angular momentum. Let ϕ be measured in the plane of the motion $\theta = \pi/2$. Then, from eq.(2.2.7)

$$r^2 d\phi/dA \equiv h = \text{constant}. \tag{4.3.1}$$

Note that h is related to the classical-mechanical angular momentum per unit mass $h_o (\equiv r^2 d\phi/dt)$ by

$$h = n^2 h_o . \tag{4.3.2}$$

Now we may easily obtain general-relativistic analogues of the standard formulas of classical central-force motion. In eq.(2.2.8) which is the analogue of the classical conservation of energy condition, we may write out $|dr/dA|^2$ in plane-polar coordinates, then eliminate A by means of eq.(4.3.1). The orbit shape $\phi(r)$ is thereby reduced to an integration:

$$\phi = h \int^{r} r^{-2}[n^4 v^2 - h^2/r^2]^{-1/2} dr. \qquad (4.3.3)$$

The classical limit of eq.(4.3.3) is the familiar equation

$$\phi = h_o \int^{r} r^{-2}[2(E - U) - h_o^2/r^2]^{-1/2} dr.$$

Note that we could have immediately written down eq.(4.3.3), which is an exact general-relativistic expression, on the model of the classical expression, simply by using the transcriptions (2.3.1) together with $h_o \to h$ (which follows from $t \to A$). Moreover, eq.(4.3.3) applies both to light and to massive particles. To apply eq.(4.3.3) to either massless or massive particles, we need only insert the appropriate specific form (2:2.9) or (2.2.10) for v(r).

Another form of the orbit equation is frequently useful. Let u = 1/r. Then, in analogy to the classical formula

$$\frac{d^2u}{d\phi^2} + u = -h_o^{-2} \frac{dU}{du},$$

we must have in general relativity

$$\frac{d^2u}{d\phi^2} + u = -h^{-2} \frac{d}{du}(n^4 v^2/2), \qquad (4.3.4)$$

which, again, applies to both particles and photons. Figures 1-14 display the form of the potential $n^4v^2/2$ for the Reissner-Nordström case for different values of m and β. We have written down eq.(4.3.4) simply by analogy to classical mechanics. But it may also be obtained by beginning with the radial component of eq.(2.2.1) and eliminating A.

A third useful form of the orbit equation is

$$h^2[(du/d\phi)^2 + u^2] - n^4v^2 = 0. \qquad (4.3.5)$$

This equation is now in a familiar form. Specific choices of n and v will lead to corresponding solutions by standard methods.

4.4. Bending of light rays

We may begin from eq.(4.3.5). Inserting eq.(2.2.10) for v(r) in the second term, we obtain

$$\left[\frac{du}{d\phi}\right]^2 + u^2 - (c_o/h)^2 n^2 \left[1 - c_o^4 H^{-2} \Omega^2\right]. \qquad (4.4.1)$$

This differential equation is exact, but it may not appear very familiar. We may transform back to the original (standard) coordinates by using eqs.(4.1.13) and (4.1.14) in the first term

of eq.(4.4.1), with the result

$$\left[\frac{du}{d\phi}\right]^2 + u'^2\Omega^2 - (c_o/h)^2[1 - c_o^4H^{-2}\Omega^2] = 0. \qquad (4.4.2)$$

Substituting (4.1.11) for $\Omega^2(u')$, then differentiating with respect to ϕ, we obtain

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_o^6}{h^2H^2} = -\frac{\beta c_o^6}{h^2H^2}u' + 3mu'^2 - 2\beta u'^3. \qquad (4.4.3)$$

The equation for the shape of a light ray results from letting $H \longrightarrow \infty$ in eq.(4.4.3)

$$\frac{d^2u'}{d\phi^2} + u' = 3mu'^2 - 2\beta u'^3. \qquad (4.4.4)$$

This equation may be solved by the usual perturbative method. If the right side of eq.(4.4.4) is temporarily put equal zero, we obtain the straight-line solution

$$u' = \frac{\sin\phi}{R},$$

where R is the distance of closest approach to the origin. Substituting the zeroth-order solution $\sin\phi/R$ for u' on the right side of eq.(4.4.4) and solving the resulting differential equation for $u'(\phi)$, we obtain the solution of first order in m and β:

$$u' = \frac{\sin\phi}{R} + \frac{3m}{2R^2}\left[1 + \frac{1}{3}\cos 2\phi\right] + \frac{3\beta}{4R^3}\phi\cos\phi - \frac{\beta}{16R^3}\sin 3\phi. \quad (4.4.5)$$

(This differs sightly from Halilsoy's solution, which is missing the last term.) As $r' \to \infty$, $u' \to 0$, and $\phi \to \phi_\infty$, which may be assumed small. Thus eq.(4.4.5) reduces to

$$0 = \frac{\phi_\infty}{R} + \frac{2m}{R^2} + \frac{9\beta\phi_\infty}{16R^3}.$$

The total deflection is $\Delta\phi_\infty = 2|\phi_\infty|$ or

$$\Delta\phi_\infty \approx \frac{4m}{R}\left(1 - \frac{9\beta}{16R^2}\right). \quad (4.4.6)$$

The coefficient of β/R^2 differs from the factor $\frac{3}{4}$ obtained by Halilsoy, the difference being the contribution of the last term in eq.(4.4.5).

For some of the metrics under consideration [see eqs.(4.2.4) and (4.2.5)], β can be of order m^2. Thus, expressions for light orbit eq.(4.4.5) and for the bending eq.(4.4.6) should be carried to higher order in m/R to provide a fair assessment of the importance of the contributions due to β. This may be done by iteration. That is, we substitute eq.(4.4.5) for u' on the right-hand side of eq.(4.4.4) and proceed as before. The result is

that the following terms should be added to right-hand side of eq.(4.4.4):

$$\frac{-15m^2}{4R^3}\phi\cos 3\phi - \frac{3m^2}{16R^3}\sin 3\phi. \qquad (4.4.7)$$

The expression (4.4.6) for the deflection of the light ray becomes

$$\Delta\phi_\infty \approx \frac{4m}{R}\left[1 - \frac{9\beta}{16R^2} + \frac{69m^2}{16R^2}\right]. \qquad (4.4.8)$$

4.5. Precession of planetary apsides

For a planet, we return to eq.(4.4.3). This equation is exact and may be handled as it stands. However, since we will treat some of the terms on the right side of the equation as perturbations, no precision will be lost by replacing the constants of the motion h and H by their classical limits. For a planet moving at non-relativistic speed, we may, by eq.(2.2.11), put $H^2 \approx c_0^4$. Also, at sufficiently large r (i.e., at the site of a planetary orbit), we may put $h \approx h_0$, the classical angular momentum per unit mass. Thus we have

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_0^2}{h_0^2} = -\frac{\beta c_0^2}{h_0^2}u' + 3mu'^2 - 2\beta u'^3. \qquad (4.5.1)$$

This differential equation is not quite the same as recently

obtained through other means by Halilsoy.[66] In particular, Halilsoy's equation is missing the term $-\beta(c_o/h_o)^2u'$ [ref.[66], eq.(25)].

If we temporarily put the terms in u'^2 and u'^3 equal to zero, we obtain a differential equation that may solved exactly:

$$\frac{d^2u'}{d\phi^2} + s^2u' = \frac{1}{\alpha_o}, \tag{4.5.2}$$

where

$$s^2 = 1 + \beta c_o^2/h_o^2, \tag{4.5.3}$$

and

$$\alpha_o = h_o^2/mc_o^2. \tag{4.5.4}$$

The solution is the precessing ellipse

$$u' = \alpha^{-1}(1 + e\cos s\phi) \tag{4.5.5}$$

where the eccentricity e is arbitrary and where the semi-latus rectum α is

$$\alpha = \alpha_o\left[1 + \frac{\beta c_o^2}{h_o^2}\right]. \tag{4.5.6}$$

The precession of the apsides, per revolution of the planet on the orbit, due to the term in $\beta u'$, is then

$$\Delta_1 = \frac{-\pi\beta c_o^2}{h_o^2} = \frac{-\pi m}{\alpha_o}\frac{\beta}{m^2} . \qquad (4.5.7)$$

The terms in u'^2 and u'^3 may be treated as perturbations. Thus, one inserts eq.(4.5.5) in these two terms on the right side of eq.(4.5.1) and solves the resulting equation. The term in u'^2, acting alone, produces the usual precession associated with Schwarzschild problem:

$$\Delta_2 = \frac{6\pi m^2}{h_o^2}c_o^2 = \frac{6\pi m}{\alpha_o} \qquad (4.5.8)$$

As shown by Halilsoy,[66] the term in u'^3, acting alone, produces the precession

$$\Delta_3 = -\frac{6\pi\beta m^2}{h_o^4}c_o^4 = -6\pi\left[\frac{m}{\alpha_o}\right]^2\frac{\beta}{m^2} \qquad (4.5.9)$$

However, this term is smaller than (4.5.7) by a factor of m/α_o and is thus entirely negligible. To lowest order in m/α_o. then, the total procession Δ of the apsides per revolution is just $\Delta_1 + \Delta_2$:

$$\Delta = \frac{6\pi m}{\alpha_o}\left(1 - \frac{\beta}{6m^2}\right). \qquad (4.5.10)$$

4.6. Radar echo delay

We now consider the propagation in time of light in the Reissner-Nordström type metric. Again, let the motion take place in the $\theta = \pi/2$ plane. Writing out the conservation of energy equation (2.2.8) in plane polar coordinates and making use of eqs.(2.2.9) and (4.3.1), we have

$$\left(\frac{dr}{dA}\right)^2 + \frac{h^2}{r^2} - n^2 c_o^2 = 0 \qquad (4.6.1)$$

Let us now evaluate the constant of the motion, h. Let r_o denote the distance of closest approach of the ray to the center of the gravitating body. When $r = r_o$, we have $dr/dA = 0$. Thus eq.(4.6.1) gives

$$h = r_o n(r_o) c_o, \qquad (4.6.2)$$

which is analogous to the classical-mechanical expression $r_o v(r_o)$.

We may now transform from r back to r′ using eqs.(4.1.13) and (4.1.14). Also, because we are interested in the propagation of light in time, we use eq.(1.1.10) to pass over from A to t as independent variable. Thus, with substitution and transformation of eq.(4.6.2), eq.(4.6.1) becomes

$$\left(\frac{dr'}{dt}\right)^2 = \Omega^4(r')\ c_o^2 \left[1 - \frac{r_o'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_o')}\right]. \tag{4.6.3}$$

The time of travel from r_o' to r' is then

$$\Delta t = c_o^{-1}\int_{r_o'}^{r'} \Omega^{-2}(r')\left[1 - \frac{r_o'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_o')}\right]^{-1/2} dr'$$

$$= c_o^{-1}\int_{r_o'}^{r'} I(r')dr'. \tag{4.6.4}$$

Now,

$$I = \Omega^{-2}\left(1 - \frac{r_o'^2}{r'^2}\right)^{-1/2}\left[1 + \frac{[1 - \Omega^2(r')/\Omega^2(r_o')]}{(r'^2/r_o'^2 - 1)}\right] \tag{4.6.5}$$

Using eq.(4.1.11) to write out $\Omega(r')$ and $\Omega(r_o')$, then expanding to first order in m and β, we obtain

$$I = \left(1 - \frac{r_o'^2}{r'^2}\right)^{-1/2}\left[1 + \frac{2m}{r'} + \frac{mr_o'}{r'(r' + r'_o)} - \frac{3\beta}{2r'^2}\right] \tag{4.6.6}$$

The total time travel $\Delta t(r_o', r')$ from r_o' to r' is obtained by substituting eq.(4.6.6) into eq.(4.6.4) and integrating:

$$\Delta t(r_o', r') \approx c_o^{-1}(r'^2 - r_o'^2)^{1/2}$$

$$+ \frac{2m}{c_o} \ln\left[\frac{r'}{r'_o} + \frac{(r'^2 - r'^2_o)^{1/2}}{r'_o}\right]$$

$$+ \frac{m}{c_o}\left[\frac{r' - r'_o}{r' + r'_o}\right]^{1/2} + \frac{3\beta}{2r'_o c_o}\sin^{-1}(r'_o/r') - \frac{3\pi\beta}{4r'_o c_o}. \qquad (4.6.7)$$

The first term on the right side of eq.(4.6.7) is the transit time of light in Euclidean space. The delay $\Delta T(r'_o, r')$ due to general-relativistic effects is the sum of the remaining terms.

As an example, let us estimate the radar echo delay for a signal from the Earth at radius r'_e to an inferior planet at radius r'_p when that planet is near superior conjunction with the Sun. Let the distance of closest approach to the center of the Sun be r'_o. If we suppose that the signal passes very near the Sun, so that r'_o is much smaller than either r'_e or r'_p, then

$$\Delta T(r'_o, r'_e) \approx \frac{2m}{c_o} \ln\left[\frac{2r'_e}{r'_o}\right] + \frac{m}{c_o}\left[1 - \frac{r'_o}{r'_e}\right] - \frac{3\pi\beta}{4c_o r'_o}\left[1 - \frac{2r'_o}{\pi r'_e}\right]. \qquad (4.6.8)$$

and the total delay in the signal for the round trip is

$$2\left[\Delta T(r'_o, r'_e) + \Delta T(r'_o, r'_p)\right]. \qquad (4.6.9)$$

The effect of β on the delay is quite evident from the above.