CHAPTER 2

THE VARIATIONAL PRINCIPLE AND THE EQUATIONS OF MOTION

In this chapter, we develop the variational principle, the equations of motion and their solutions. In Sec.2.1, the principle is formulated. Sec.2.2 contains the exact equations of motion in Newtonian form. The crucial role of the Evans-Rosenquist parameter A is elucidated in Sec.2.3. Taking the example of Schwarzschild field of general relavity, we illustrate two methods of solution (energy and force methods) of equations of motion, in Sec.2.4.

2.1. Formulation of the variational principle

We wish to formulate, for the trajectories in general relativity, a variational principle that combines both the principle of Fermat (classical geometric optics) and the principle of Maupertuis (Newtonian mechanics in velocity independent potentials). To do this, it is convenient to cast the spherically symmetric spacetime into isotropic coordinates as

$$ds^2 = \Omega^2(r,t)c_0^2dt^2 - \Phi^{-2}(r,t)|dr|^2, \qquad (2.1.1)$$

$$|dr|^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \qquad (2.1.2)$$

We have the following expressions from the metric eq.(2.1.1):

$$g_{oo} = \Omega^2, \; g_{ij} = -\Phi^{-2}\delta_{ij}, \; g^{ij} = -\Phi^2\delta^{ij}, \; -(-g)^{1/2} = \Omega\Phi^{-3} \qquad (2.1.3)$$

so that we get, from eq.(1.1.22), the dielectric tensors as

$$\mu^{ij} = \varepsilon^{ij} = \Omega^{-1}\Phi^{-1}\delta^{ij}, \; n(r,t) = \Omega^{-1}\Phi^{-1}. \qquad (2.1.4)$$

The expression for the refractive index n(r,t) can also be obtained in an alternative way. The idea is to compute the isotropic cordinate speed of light c(r) at any arbitrary point in the field by putting $ds^2 = 0$. This gives

$$c(r) = |dr/dt| = c_o\Phi(r,t)\Omega(r,t). \qquad (2.1.5)$$

Using $c = c_o/n$, we get the effective index of refraction as $n(r,t) = \Phi^{-1}\Omega^{-1}$ which is precisely the same as eq.(2.1.4). Another way to deduce n is to use the conformal invariance of Maxwell's equations. It can be shown[34,35] that the metric (2.1.1) has the same properties as Gordon's form of optical metric given by

$$ds^2 = [\eta_{\alpha\beta} + (1-n^{-2})\delta^o_\alpha\delta^o_\beta]dx^\alpha dx^\beta$$

from which n can be deduced by comparison.

Light trajectories in the gravitational field can be calculated by using the effective index of refraction, eq.(2.1.4), in any formulation of geometrical optics that happens to be convenient. For example, Wu and Xu have recently shown that the standard differential equation of the ray in classical geometrical optics can be applied to the null geodesic problem[36].

An especially convenient version of geometric optics is the so-called "F = ma" formulation[3] in which the equation governing the optical ray assumes the form of Newton's law of motion (acceleration = - gradient of potential energy):

$$d^2r/dA^2 = \nabla(n^2 c_0^2/2) \tag{2.1.6}$$

r is the position of a pulse moving along the ray. All the usual force and energy methods of elementary mechanics can be brought to bear on geometrical optics.

The effective index of refraction eq.(2.1.4) (for the Schwarzschild metric, for example) can be used in eq.(2.1.6) without modification[37]. In solving problems, one goes into the isotropic coordinates, applies the "F = ma" optics, then transform back to the standard or other coordinates, if desired.

We shall obtain the variational principle by transformation of the geodesic condition for the particle trajectories,

$$\delta \int_{x_1, t_1}^{x_2, t_2} ds = 0 \tag{2.1.7}$$

where δ indicates a variation in the path of integration between two fixed points in spacetime, (x_1, t_1) and (x_2, t_2). If we assume the line element can be written in the form eq.(2.1.1), this becomes

$$\delta \int_{x_1, t_1}^{x_2, t_2} \Omega c_o \, [1 - v^2 n^2 / c_o^2]^{1/2} dt = 0. \tag{2.1.8}$$

This is analogous to Hamilton's principle and the effective Lagrangian is

$$L(x_i, \dot{x}_i) = - c_o^2 \Omega [1 - v^2 n^2 / c_o^2]^{1/2} \tag{2.1.9}$$

where Ω and n are functions of the coordinates alone, where $\dot{x}_i \equiv dx_i/dt$, and where $v^2 = \sum_{i=1}^{3} (dx_i/dt)^2$, if we choose to work in Cartesian coordinates. The expression for the Lagrangian has been multiplied by an extra factor of $-c_o$ for later convenience.

The canonical momenta p_i are

$$p_i \equiv \partial L/\partial \dot{x}_i$$

$$= \Omega n^2 [1 - v^2 n^2/c_0^2]^{1/2} x_i . \qquad (2.1.10)$$

The Hamiltonian H may be formed in the usual way:

$$H = \sum_{i=1}^{3} p_i \dot{x}_i - L$$

$$= c_0^2 \Omega [1 - v^2 n^2/c_0^2]^{-1/2} . \qquad (2.1.11)$$

Because $\partial L/\partial t = 0$, H is a constant of the motion. If we express H in terms of the p_i rather than $\dot{x}_i$, we obtain

$$H = c_0^2 [\Omega^2 + p^2/n^2 c_0^2]^{1/2} \qquad (2.1.12)$$

where $p = |p|$. From Hamilton's principle,

$$\delta \int_{x_1, t_1}^{x_2, t_2} L dt = o, \qquad (2.1.13)$$

one may derive in the usual way the corresponding action principle (Jacobi's form of Maupertuis' principle)

$$\delta\int_{x_1}^{x_2} \left(\Sigma_{i=1}^{3} p_i \dot{x}_i \right) dt = 0, \qquad (2.1.14)$$

where now the path of integration is varied between two fixed points in space, x_1 and x_2, where the energy must be held constant on the varied paths, but where the times at the end points need not be held fixed. With the canonical momenta from eq.(2.1.10), this becomes

$$\delta\int_{x_1}^{x_2} n^2 v^2 \Omega [\, 1 - v^2 n^2 / c_0^2 \,]^{-1/2} dt = 0. \qquad (2.1.15)$$

We restrict the varied paths to those that satisfy the energy constraint by substituting the constant H for the right side of eq.(2.1.11) where this appears in eq.(2.1.15). Then, putting $dt = dl/v$, where $dl = |dr| = (\Sigma_{i=1}^{3} dx_i^{\,2})^{1/2}$, we obtain

$$\delta\int_{x_1}^{x_2} n^2 v\, dl = 0. \qquad (2.1.16)$$

This is a variational principle on which an analogy to geometrical optics or to classical mechanics can be constructed. In obtaining eq.(2.1.16) we have preferred, for the sake of directness, clarity and consistency of notation, to begin from the fundamental principle eq.(2.1.7). But eq.(2.1.16) may also be derived from versions of the three-dimensional variational principle for particle orbits in static metrics, for example, the forms first obtained by Weyl[26] and Levi-Civita.[27]

In eq.(2.1.16), n^2v is to be considered a function of position alone. (This condition is met in the fields of gravitation considered here. Only in cosmological applications, there occurs a cosmic time dependence in the potential. See chapter 5 for discussion). The path of integration is varied between the fixed end points x_1 and x_2, and the value of H is held constant during the variation. Thus, eq.(2.1.16) is of the same form as Fermat's principle, which forms a basis for geometrical optics, and Maupertuis's principle, which forms a basis for classical mechanics (as long as the force can be derived from a velocity-independent potential):

Relativistic gravitational mechanics	Geometrical optics (Fermat)	Classical mechanics (Maupertuis)
$\delta\int n^2 v\,dl = 0$	$\delta\int n\,dl = 0$	$\delta\int v\,dl = 0$

In the context of motion in a gravitational field, both

Fermat's principle and Maupertuis' principle are simply special cases of eq.(2.1.16). For the null geodesics, i.e., for the paths of light, the derivation given above must be slightly modified, to keep each step defined. But the final result is too well known to require detailed discussion here: in static metrics, light obeys Fermat's principle. That is, the path taken by light between two fixed points in space is one for which the coordinate time of travel is stationary. In the language of refractive index, this may be written as $\delta\int n\,dl = 0$. Since, for light, $v = c_0/n$, eq.(2.1.16) does reduce to the appropriate form. To obtain Maupertuis' principle (and hence particle motion in Newtonian gravity), note that in ordinary solar-system dynamics, we may put $n^2 \approx 1$. That is, in the Newtonian limit, n^2 may be treated as constant in the variational calculation and we obtain Maupertuis' principle as the classical limit of eq.(2.1.16).

2.2. Exact equations of motion of Newtonian form

Let the path of the particle be parametrized by a stepping parameter A. That is, at each point on the path, the three space coordinates r (and also the time t) are regarded as functions of A. We defer for the moment choosing A: we shall define A to get the simplest equations of motion. Thus we write eq.(2.1.16) in the form

$$\delta \int_{x_1}^{x_2} n^2 v \left|\frac{dr}{dA}\right| dA = 0, \qquad (2.2.1)$$

where $\left|dr/dA\right| = \left[\sum_{i=1}^{3} (dx_i/dA)^2 \right]^{1/2}$.

Let r(A) denote the true path. To obtain a varied path, we replace r(A) by r(A) + w(A), where w(A) is an arbitrary, infinitesimal vector function, subject to the condition that w = 0 when A is such that $r = x_1$ or x_2. That is, the variation must vanish at the end points. Now

$$\delta\int (n^2 v)\left|\frac{dr}{dA}\right| dA = \int \left[\delta(n^2 v)\right] \left|\frac{dr}{dA}\right| dA + \int (n^2 v)\delta\left|\frac{dr}{dA}\right| dA + \int (n^2 v)\delta\left|\frac{dr}{dA}\right| \delta dA \qquad (2.2.2)$$

Calculating the two variation in the first term on the right-hand side of eq.(2.2.2), we have

$$\delta(n^2 v) = \nabla(n^2 v)w. \qquad (2.2.3)$$

In calculating the variation in the second term of eq.(2.2.2) it is important to remember that the change to the varied path will, in general, also produce a change in A. Thus

$$\delta\left|\frac{dr}{dA}\right| = \left|\frac{dr + dw}{dA + \delta dA}\right| - \left|\frac{dr}{dA}\right|$$

$$= \frac{\frac{dr}{dA}\cdot\frac{dw}{dA}}{\left|\frac{dr}{dA}\right|} - \left|\frac{dr}{dA}\right| \frac{\delta dA}{dA} \qquad (2.2.4)$$

to first order in the variation. Substituting eq.(2.2.3) and eq. (2.2.4) into eq.(2.2.2), we find

$$\delta\int n^2 v\left|\frac{dr}{dA}\right| dA = \int \left[\left|\frac{dr}{dA}\right| \nabla(n^2 v)\cdot w + n^2 v\left|\frac{dr}{dA}\right|^{-1}\frac{dr}{dA}\cdot\frac{dw}{dA}\right] dA.$$

Note that the terms involving δdA have cancelled out. This was to be expected, since eq.(2.1.16) shows that the integral does not actually depend upon the range in A or, indeed, on what we select to use as parameter. Integrating the term involving dw/dA by parts, and using the fact that w must vanish at the endpoints, but is otherwise arbitrary, we arrive at the differential equation that must be satisfied by the particle trajectory:

$$\left|\frac{dr}{dA}\right|\nabla(n^2 v) - \frac{d}{dA}\left(n^2 v\left|\frac{dr}{dA}\right|^{-1}\frac{dr}{dA}\right) = 0. \qquad (2.2.5)$$

This differential equation plays the role of an equation of motion. Another way to obtain eq.(2.2.5) is to parametrize the path by one of the Cartesian coordinates (say z), rather than A, since the variation in z must vanish at x_1 and x_2. In this case, one writes

$$\delta\int n^2 v\frac{dl}{dz}dz = 0.$$

One may then simply write down the Euler conditions for the integral to be stationary, and then transform from z to A as independent variable. The result will be the same, that is eq.(2.2.5).

To give the equation of motion the simplest possible form, and to take advantage of the analogy to Newtonian mechanics, let us now define A by

$$\left|\frac{dr}{dA}\right| = n^2 v. \tag{2.2.6}$$

With this definition of A, the equation of motion, eq.(2.2.5), becomes

$$d^2r/dA^2 = \nabla(\frac{1}{2}n^4 v^2). \tag{2.2.7}$$

Eq.(2.2.7) is the generalization of eq.(2.1.6) that was sought. The left-hand side of eq.(2.2.7) is of the form of an acceleration: it is the second derivative of the position vector with respect to the independent variable. The right-hand side of the equation is of the form of a force: $-\frac{1}{2}n^4 v^2$ plays the role of a "potential energy function".[38] The analogue of the velocity is $|dr/dA|$ although it is a dimensionless quantity since A has the dimension of length. Thus the analogue of the kinetic energy is $\frac{1}{2}|dr/dA|^2$. The analogue of the total energy is the sum of the potential and the kinetic. But, by virtue of eq.(2.2.6) these two are guaranteed to sum to zero:

$$\frac{1}{2}|dr/dA|^2 - \frac{1}{2}n^4 v^2 = 0. \tag{2.2.8}$$

Thus the calculation of the paths of light and of massive particles in general relativity reduces to the zero-energy "F = ma" optics of ref.[3]. It is to be noted that the "conservation of energy" condition eq.(2.2.8) amounts to a restatement of the definition of eq.(2.2.6) of A.

The optical-mechanical analogy, embodied in eq.(2.2.7) and eq.(2.2.8), provides an exact treatment in Newtonian form of the motion of massive particles as well as light, in general relativity. The Newtonian form should be thought of as coming from "F = ma" optics (which is exact) and not from Newtonian mechanics (which is, of course, only approximate). Equations (2.2.7) and (2.2.8) allow one to handle the paths of light and of planets as if they existed in a flat three-dimensional space on which is superimposed a refractive medium. Other approaches to this goal are, of course, possible, but the treatment presented here has three advantages: simplicity, complete conformity to the equations of Newtonian mechanics, and a uniform treatment to both light and massive particles. This treatment has a reasonably high degree of generality and is applicable whenever the line element can be written in the form of (2.1.1).

The only difference between the treatment of light and that of particles resides in the choice of v(r), which forms a part of the effective potential energy $-n^4v^2/2$. For light,

$$v = c_o n^{-1}. \qquad \text{(light)} \qquad (2.2.9)$$

But for massive particles, eq.(2.1.11) gives

$$v = c_o n^{-1}[1 - c_o^4\Omega^2/H^2]^{1/2}. \qquad \text{(particles)} \qquad (2.2.10)$$

In eq.(2.2.10), H is a constant parameter determined by the initial conditions, while n and Ω are functions of the spatial coordinates determined by the metric. Because the particle expression for v(r) contains the parameter H, the particle problem has an extra degree of freedom: we may specify the initial speed of the particle. This we can not do for light which is always c_o, the proper velocity through any observer. Thus, in general, more types of orbits exist for massive particles than for light in the same metric.

For a particle in empty space devoid of gravitational influences, $\Omega \approx 1$, $n \approx 1$, and eq.(2.1.11) becomes

$$H \approx c_o^2(1 - v^2/c_o^2)^{-1/2} = c_o^2\gamma. \qquad (2.2.11)$$

In the solar-system dynamics of the Schwarzschild metric, $v/c_o \ll 1$ and (See eqs. (4.2.10) and (4.2.14) below) $\Omega \approx 1 - m/r$ so eq. (2.1.11) becomes

$$H \approx c_o^2 + \frac{1}{2}v^2 - m/r. \qquad (2.2.12)$$

That is, in classical planetary orbits, H is approximately equal to $c_o^2 + E$, the rest-mass energy plus the classical kinetic and potential energy per unit mass.

2.3. On the parameter A in general relativity

In solving problems with eqs.(2.2.7) or (2.2.8), one may use without modification all the familiar methods of Newtonian mechanics. One simply thinks of A as if it were the time. Moreover, rather than beginning from eqs.(2.2.7) and (2.2.8) in every case, one may often simply write down an exact general relativistic formula by analogy to the corresponding classical formula. For the motion of both light and massive particles in spherically symmetric metrics, one may begin from any classical Newtonian formula describing the motion of particles in velocity-independent potentials. The correct general relativistic expressions will be obtained if one makes the following transcriptions in the classical formulas:

$$t \to A,$$

$$U \to -n^4 v^2/2, \qquad (2.3.1)$$

$$E \to 0.$$

The spatial coordinates (x,y,z), or (r,θ,ϕ), etc., transcribe as themselves. Of course, it must be kept in mind that the formulas written down in this way apply only in the isotropic coordinate system. After the equations governing the situation are obtained, or after they are solved, one may transform back to other coordinate systems, such as the standard or harmonic systems, if desired. A second point of vital importance is that the analogue of the classical energy E is always the number zero. Thus, the constant of the motion H plays no role in the analogy. H should rather be regarded as a parameter: the potential energy function depends upon H as well as the coordinates. A third point to be stressed is that the exact general relativistic formulas written down by analogy to the classical formulas will always apply with equal validity to both massive and massless particles.

For both light and particles, the stepping parameter A is defined by eq.(2.2.6). In many calculations, e.g., in finding the shape of an orbit, the stepping parameter is virtually eliminated. Eq.(2.2.6) will suffice for this purpose. In other situations (for example, in a radar echo-delay calculation), it may be necessary to have an explicit connection between A and t. Note that

$$\left|\frac{dr}{dA}\right| = \left|\frac{dr}{dt}\right|\frac{dt}{dA} = v\frac{dt}{dA}.$$

Substituting in eq.(2.2.6) *and restoring* c_o, *we get* (Note that the integrand of eq.(2.1.16) is actually multiplied by c_o^{-2} and the r.h.s. of eq.(2.2.6) is multiplied by c_o^{-1}. Those were suppressed before for neatness.),

$$dA = c_o dt/n^2. \qquad (2.3.2)$$

Thus the stepping parameter in general relativity is the same as that used in "F=ma" formulation of classical geometrical optics. As before, A is called the optical action because dA is proportional to c(r)dl, and thus is analogous to the action v(r)dl of classical mechanics.

The variational principle (2.1.16) has permitted us to extend the analogy to the geodesic problem for both light and particles in isotropic metrics. Our discussion of the optical analogy in general relativity has stressed Newtonian forms. But, corresponding to every formulation of either classical particle mechanics or of classical geometrical optics, there will be an analogous formulation of the geodesic problem in general relativity. Few of these classical models for the reformulation of the geodesic equations of motion lead to any special insight or simplification. The economy of expression and simplicity of the form embodied in eqs.(2.2.7) and (2.2.8) depend, not so much on the formulation of mechanics (or of geometrical optics) that is *chosen as model, as on the use of A rather than t as independent* variable.

The parameter A plays a key role in the optical- mechanical analogy in that it beautifully synthesizes two oppositely directed propositions: One is the Newtonian corpuscular hypothesis according to which light speeds up upon entering a denser medium because of the influence of some attractive force exerted by the medium. This attractive force has actually been experimentally verified long ago by Poynting[39] by an optical-mechanical device. After an extraordinary gap of nearly half a century, one finds more accurate experiments under different conditions by Jones[40], Jones and Richards[41] who verified an *increase* in pressure on a vane immersed in a fluid of index n, in perfect accordance with the Newtonian hypothesis. Even more sophisticated experiments with the aid of a laser beam on a liquid surface have been conducted by

Ashkin and Dziedzic[42] and they also confirmed the earlier results to an even greater accuracy. Moreover, the corpuscular hypothesis explained the laws of refraction of light. The other proposition stems from the Foucault experiment which shows that light propagates more slowly in water than in air.

One way to reconcile this increase in pressure with the diminishing speed of light in a denser medium is to postulate a varying mass of a photon. The consequences of this postulate have been investigated by Tangherlini.[17,43] The other way to reconcile the two is to use the parameter A. Let us consider eqs.(2.2.8) (with c_o restored) and (2.2.9) which give $|dr/dA| = n$. When interpreted *formally* as light progressing in optical action A, we see that the above equation mimics the behavior of a Newtonian corpuscule or a particle to the extent that its "velocity" = n, just like that of an electron in a medium: $v_{electron} \propto n$. In the real experiment with light progressing *in time* t, on the other hand, the equation $|dr/dt| = c_o/n$ corresponds to Foucault's experiment supporting Huyghen's principle. We shall see that the former equation, when A is eliminated, provides the exact orbit equation and the latter equation gives us the exact equation for the radar echo delay in general relativity. That is to say, the roles of the two equations are complementary in that one takes care of the spatial shape of the trajectory while the other keeps track of the real time elapsed during the motion along the trajectory. In another language, we may say that light displays its particle character (the "velocity" being proportional to n) in space whereas it displays wave chracter in time as revealed by its phase velocity c_o/n. The same reasonings apply also for material particles, where, in the expression for $n^2 v$, one has only to use the expression for v given in eq.(2.2.10). For yet another logical alternative resolving the crisis, see the works of Michels, Correl and Patterson.[44]

2.4.(a) Energy methods for the orbits

To illustrate energy methods, we shall calculate the path of a planet in the Schwarzschild field[37]. If we write out the "kinetic energy" in polar coordinates, the equation for "total energy" becomes

$$\text{"total energy"} = \frac{1}{2}\left(\frac{dr}{dA}\right)^2 - \frac{1}{2} n^4 v^2 = 0. \tag{2.4.1}$$

Writing out in full, we have

$$\left(\frac{dr}{dA}\right)^2 + r^2\left(\frac{d\phi}{dA}\right)^2 - n^4 v^2 = 0. \tag{2.4.2}$$

Now, the fact that the "potential energy" is a function of radial coordinate alone leads (just as in classical central-force motion) to a conserved "angular momentum":

$$h = r^2 d\phi/dA = \text{constant}. \tag{2.4.3}$$

(This may be easily seen by writing out the ϕ-component of eq.(2.2.7)). We use eq.(2.4.3) to pass over from A to ϕ as independent variable in eq.(2.4.2), obtaining

$$r^{-4}\left(\frac{d\phi}{dA}\right)^2 + r^{-2} - h^{-2}n^4v^2 = 0. \tag{2.4.4}$$

In the usual way, put

$$u = r^{-1}. \tag{2.4.5}$$

Then $dr/d\phi = - u^{-2}du/d\phi$ and eq.(2.4.4) may be written as

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - h^{-2}n^4v^2 = 0. \tag{2.4.6}$$

Note that eq.(2.4.6) still applies both to planets and to light.

To calculate the path of a planet, we invoke eq.(2.2.10) so that eq.(2.4.6) becomes

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - n^2c_0^2h^{-2}[1 - c_0^4\Omega^2/H^2] = 0. \qquad \text{(planet)} \tag{2.4.7}$$

To calculate the path of light, we use eq.(2.2.9). Thus eq.(2.4.6) becomes

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - n^2 c_o^2 h^{-2} = 0. \qquad \text{(light)} \qquad (2.4.8)$$

The only fact about the metric we have used so far is its spherical symmetry. (That is, $n^4 v^2$ is a function of r alone.) Let us now focus on the Schwarzschild problem: Φ, Ω and n are given by

$$\Phi(u) = \left(1 + \frac{mu}{2}\right)^{-2} \qquad (2.4.9)$$

$$\Omega(u) = \left(1 + \frac{mu}{2}\right)^{-1}\left(1 - \frac{mu}{2}\right) \qquad (2.4.10)$$

$$n(u) = \left(1 + \frac{mu}{2}\right)^{3}\left(1 - \frac{mu}{2}\right)^{-1}. \qquad (2.4.11)$$

To return to the original (non-isotropic) metric, we need to invert the coordinate transformation given by:

$$u = u'/\Phi, \qquad (2.4.12)$$

where $u' = 1/r'$ and r' is the standard radial coordinate. It is not hard to show that

$$du/du' = \Phi^{-1}\Omega^{-1} = n. \qquad (2.4.13)$$

Also, it will eventually be helpful to have explicit forms for Φ, Ω, and n as functions of u′ rather than u. A little algebra gives

$$\Phi = \frac{1}{4}\,[1 + (1 - 2mu')^{1/2}]^2 \qquad (2.4.14)$$

$$\Omega = (1 - 2mu')^{1/2} \qquad (2.4.15)$$

$$n = 4(1 - 2mu')^{-1/2}[1 + (1 - 2mu')^{1/2}]^{-2}. \qquad (2.4.16)$$

(We do not need all of these relations for the present calculation, but it is convenient to group them in one place.) With the use of eq.(2.4.12) and eq.(2.4.13), eq.(2.4.7) becomes

$$\left(\frac{du'}{d\phi}\right)^2 + \Omega^2 u'^2 - c_o^2 h^{-2}[1 - c_o^4\Omega^2/H^2] = 0. \qquad \text{(planet)} \qquad (2.4.17)$$

Only now do we need the explicit form of eq.(2.4.15) for Ω. Then, eq.(2.4.17) becomes

$$\left(\frac{du'}{d\phi}\right)^2 - 2mc_o^6 h^{-2}H^{-2}u' + u'^2 - 2mu'^3 + c_o^6 h^{-2}H^{-2} - c_o^2 h^{-2} = 0.$$

Differentiating with respect to ϕ, we get

$$d^2u'/d\phi^2 + u' - 3mu'^2 - mc_o^6 h^{-2}H^{-2} = 0. \qquad (2.4.18)$$

Now, the constant term in eq.(2.4.18) is already of first order in

m. Thus, by virtue of eq.(2.2.12), we may safely put $H \approx c_o^2$, and we will ignore terms of order $(m/r)^2$ (weak field). The other constant of motion is $h = r^2(d\phi/dA)$. With the use of eq.(2.3.2) this may be written as

$$h = r^2 n^2 d\phi/dt$$
$$= n^2 h_o, \qquad (2.4.19)$$

where h_o is the mechanical angular momentum per unit mass, $h_o = r^2 d\phi/dt$, from classical mechanics. Because the constant term in eq.(2.4.18) is already of first order in m, we may replace n by unity. Thus, eq.(2.4.18) becomes

$$d^2u'/d\phi^2 + u' = mc_o^2/h_o^2 + 3mu'^2 \qquad \text{(planet)} \qquad (2.4.20)$$

This is the usual equation obtained in general relativity for a precessing elliptical orbit. The perihelion advance per revolution of the orbit as calculated in a standard way[45] is $6\pi m^2 c_o^2/h_o^2 = 6\pi G^2 M^2/(h_o c_o)^2$. No approximations have been made in deriving eq.(2.4.20), except, of course, in evaluating the constants of motion H and h.

The usual light-orbit equation for the Schwarzschild metric is obtained in a similar fashion, by using eqs. (2.4.12), (2.4.13) and (2.4.15) in eq.(2.4.8), with the result:

$$d^2u'/d\phi^2 + u' = 3mu'^2. \qquad \text{(light)} \qquad (2.4.21)$$

We could also have written this down immediately after having solved the particle case, simply by noting that for light we let H approach ∞ (see eq.(2.1.11). Thus, eq.(2.4.18) immediately reduces to eq.(2.4.21).

(b) Force methods for the orbits

In force methods, we shall calculate the gravitational deflection of star light, or of an ultra-relativistic ($v \approx c_o$) particle, passing near the Sun. These problems could, of course, be solved by beginning from eqs.(2.4.20) and (2.4.21). But, because we want to illustrate an "F = ma" calculation in general relativity, let us instead begin from eq.(2.2.7), which we write in the following form:

$$dp/dA = \nabla(\tfrac{1}{2}\, n^4 v^2), \qquad (2.4.22)$$

where the "momentum" p is defined by

$$p \equiv dr/dA.$$

Note that p is not the usual momentum defined in Newtonian mechanics but an optical analogue thereof. As the "potential energy" is a function of r alone, the "force" points in the radial direction, and the x- and y-components of the "equation of motion" are

$$\frac{dp_x}{dA} = \sin\theta \, \frac{d}{dr}\left(\frac{1}{2}n^4 v^2\right) \qquad (2.4.23)$$

$$\frac{dp_y}{dA} = \cos\theta \, \frac{d}{dr}\left(\frac{1}{2}n^4 v^2\right). \qquad (2.4.24)$$

Because the deflection will be very small, p ($\equiv$ dr/dA) will point almost entirely in the x-direction during the whole course of the motion. That is $|p| \approx p_x$. Thus the general condition expressed by eq.(2.2.6), namely that $|p| = n^2 v$, may be written $p_x \approx n^2 v$, and the x-component of the motion becomes

$$\frac{d}{dA}(n^2 v) = \sin\theta \, \frac{d}{dr}\left(\frac{1}{2}n^4 v^2\right). \qquad (2.4.25)$$

We may use this relation to eliminate A from y-component of the "equation of motion". This is just like eliminating t between the x- and y-components of the equation of motion in a Newtonian projectile-motion situation. Dividing eq.(2.4.24) by eq.(2.4.25), we get

$$dp_y = \cot\theta \, d(n^2 v),$$

that is,

$$dp_y = \cot\theta \, \frac{d}{dr}(n^2 v)dr. \qquad (2.4.26)$$

Since the deflection is assumed very small,

$$\cot\theta \approx R_o(r^2 - R_o^{\,2})^{-1/2} \qquad (2.4.27)$$

where R_o is the closest radial distance by which the ray passes the gravitating object. With this substitution in eq.(2.4.26), we may calculate the change in the y-"momentum" as r goes from ∞ to R_o and out to ∞ again:

$$\Delta p_y = 2R_o \int_{-\infty}^{R_o} (r^2 - R_o^2)^{-1/2} \frac{d}{dr}(n^2 v)dr. \qquad (2.4.28)$$

This expression is valid both for light and for an ultra-relativistic ($v \approx c_o$) particle, since the only approximation we have made so far is that the deflection will be small. To evaluate the integral for light, we note from eq.(2.2.9) that $n^2 v = nc_o$. Furthermore, in the weak field of Sun, eq.(2.4.11) may be expanded with the result $n \approx 1 + 2m/r$. Thus, for light in the weak field of the Sun,

$$\frac{d}{dr}(n^2 v) = \frac{-2mc_o}{r^2}. \qquad \text{(light)} \qquad (2.4.29)$$

Substituting eq.(2.4.29) into eq.(2.4.28) and carrying out the integration, we obtain

$$\Delta p_y = -4mc_o/R_o. \qquad \text{(light)} \qquad (2.4.30)$$

At $r = \infty$, the x-"momentum" is $p_x \approx c_o n\ (r = \infty) = c_o$. Thus, the deflection angle, in radian measure, is

$$\Delta p_y/p_x = -4m/R_o, \qquad \text{(light)} \qquad (2.4.31)$$

the familiar result predicted by Einstein. (Note that as $r \to \infty$, $r \to r'$, and we are working to first order in m/R_o, so in this case there is no need to transform back to the original coordinates.)

To calculate the deflection of an ultra-relativistic particle, note, from eq.(2.2.10), that

$$n^2 v = c_o n[1 - c_o^4 \Omega^2/H^2]^{1/2}, \qquad \text{(particle)} \qquad (2.4.32)$$

Also, by eq.(2.2.11)

$$H \approx c_o^2 (1 - v_o^2/c_o^2)^{-1/2} = c_o^2 \gamma \qquad (2.4.33)$$

where v_o is the particle's speed when it is (effectively) *infinitely far from the Sun. Substituting* eqs.(2.4.33), (2.4.10) and (2.4.11) into eq.(2.4.32), and expanding for a weak field (to first order in m/r), we obtain

$$n^2 v \approx c_o(1 + 2m/r)[1 - \gamma^{-2}(1 - 2m/r)]^{1/2} \qquad (2.4.34)$$

and thus, again keeping terms only to first order in m,

$$\frac{d}{dr}(n^2 v) = \frac{-mc_o}{r^2}\cdot\frac{c_o}{v_o}(1 + v_o^2/c_o^2) \qquad \text{(particle)} \qquad (2.4.35)$$

which differs from eq.(2.4.29) only in the coefficient multiplying r^{-2}. Integration of *eq.(2.4.28) then gives*

$$\Delta p_y = \frac{-2mc_o}{R_o}\cdot\frac{c_o}{v_o}(1 + v_o^2/c_o^2). \qquad \text{(particle)} \qquad (2.4.36)$$

Now, evaluating p_x at infinity gives $p_x \approx n^2 v \approx v_o$. The angular deflection of the ultra-relativistic particle is therefore[46]

$$\Delta p_y/p_x = -2m/R_o(1 + c_o^2/v_o^2). \qquad \text{(particle)} \qquad (2.4.37)$$

For $v_o < c_o$, the deflection of the particle *is greater* than the deflection of light. But as $v_o \rightarrow c_o$, the deflection of the particle becomes equal to that of light. This is indeed an interesting result.

It is sometimes said that Newtonian mechanics implies a deflection of starlight by the Sun, since the deflection of a particle infinitely less massive than the Sun does not depend on mass of incoming particle, and we could think of the photon as a Newtonian particle in the limit of very small mass. We then need only put $v_0 = c_0$ and calculate the deflection using Newtonian principles. As is well known, the result is just one half of that given in eq.(2.4.31)[47]. The present calculations show that in general relativity also, the deflections of light and of a material particle coming in at approximately the speed of light are the *same* (both being twice the Newtonian result). The main purpose of these calculations, however, is to show how easily force methods can be applied using eq.(2.2.7), and how easy it is to handle light and particles with exactly the same formalism.