CHAPTER 1

INTRODUCTION

This introductory chapter contains descriptions of the basic ideas to be employed in the thesis. For convenience, the contents are divided into various sections. In Sec.1.1, we present the basic idea of the optical-mechanical analogy and introduce what we choose to call the Evans-Rosenquist parameter. The intended application of this analogy in the realm of general relativity requires that the gravity field be represented as a refractive optical medium on a flat space. That such a representation is possible is demonstrated in Sec.1.2. The distinctions between gravitational and electromagnetic refractive indices are elucidated in Sec.1.3. Another ingredient in our approach is a suitable variational principle. To that end, a brief description of Fermat's principle and its ramifications in general relativity is provided in Sec.1.4. The final section (Sec.1.5) details the objectives of the present work.

1.1. The optical-mechanical analogy: Evans-Rosenquist parameter A

The historial optical-mechanical analogy[1,2] has recently been cast into a familiar form by Evans and Rosenquist[3-5]. This new formulation, based on Fermat's principle, provides an interesting approach that can be profitably utilized in the solution of many classical problems. The approach works either way: The wellknown ideas and techniques of classical optics can be successfully applied to the problem of classical mechanics and vice versa.

The ray equations from the classical gradient-index optics can be written in the form[1]

$$\nabla n = \frac{d}{ds}\left(n\,\frac{dr}{ds}\right) \tag{1.1.1}$$

where $r \equiv (x,y,z)$ is the position of a point on the ray, $\nabla \equiv (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$ is the gradient operator, ds is the element of arc length along the ray and n(r) is the index of refraction of the optical medium. This differential equation can be solved either exactly or numerically depending on the form of n(r) and the shape of the ray can be obtained thereby. The variational principle leading to eq.(1.1.1) is that of Fermat. The optical wave disturbance can be regarded as a sum of contributions propagated along all possible null paths from the source to the receiver. Since neighboring paths traverse nearly the same ground in the medium, the magnitudes of the disturbances arriving along each path must be virtually identical. However, since we are concerned only with wave motion in the geometrical limit in which the wavelength is much smaller than any other feature of the problem, even a slight difference in paths means that the phase of the wavelets contributed by neighboring paths will be many cycles different. A bundle of neighboring virtual paths from the source to the receiver will therefore make contributions with equal magnitudes but with a random assortment of phases, and will therefore interfere destructively. The only circumstance in which such a cancellation will not occur is when the phase along the path is stationary. In this case, the phases of the virtual paths are nearly equal and the wavelets interefere constructively.

In a medium with a varying index of refraction n(r), the phase along a path is given by

$$\text{phase} = \frac{\omega}{c_o}\int n(r)\,|dr|. \tag{1.1.2}$$

where c_o is the speed of light in vacuum. The requirement that the phase be stationary implies a variational principle in the form

$$\delta\int n(r)\,|dr| = 0. \quad \text{(optics)} \tag{1.1.3}$$

Newtonian mechanics → ray optics

To give the mechanical equations an optical ray form, we start with the Maupertuis' principle

$$\delta\int mv(r)\left|dr\right| = 0 \quad \text{(mechanics)} \tag{1.1.4}$$

where v(r) is the speed of the particle (= $|v|$) considered to be a function of its position alone. The rules governing the variation of this principle are the same as those of Fermat's. Comparing the two eqs.(1.1.3) and (1.1.4), we can pass from point particle mechanics to geometrical ray optics via the transcription

$$v \rightarrow n. \tag{1.1.5}$$

This transcription applies in a potential that depends on position alone. Hence, from eq.(1.1.1),

$$\nabla v = \frac{d}{ds}\left(v\ \frac{dr}{ds}\right). \tag{1.1.6}$$

This is basically the ray form of Newtonian mechanics. The form can be simplified further by noting that dr/ds is a unit tangent vector to the trajectory, that is, in the direction of v so that

$$\nabla v = \frac{dv}{ds}. \tag{1.1.7}$$

Another particularly interesting derivation can be obtained in terms of the wave number k through the relation $n = c_o k/\omega$. We get from the eq.(1.1.1)

$$\nabla k = \frac{d}{ds}\left(k \frac{dr}{ds}\right). \qquad (1.1.8)$$

Assuming that this equation represents the *particle ray* equation in the short wave length limit of quantum mechanics, we can use the de Broglie relation

$$mv = \hbar k \Rightarrow mv = \hbar k \qquad (1.1.9)$$

where $|k| = k$ and obtain, finally, eq.(1.1.7) from eq.(1.1.8). As one can see, de Broglie relations are equivalent to the optical-mechanical analogy $v \equiv k$ upto a proportionality factor $\hbar$, the Planck constant that ushered in a new physics. The basis of the analogy lies in the common wavelike properties of light and material particles (wave-particle duality in quantum mechanics). However, in the present classical treatment of trajectories, it is important to remember that "light is not just like particles"[3].

A couple of remarks about the transcription (1.1.5) is in order. Observe first that the two sides are not dimensionally equivalent. Therefore what is intended here is only the proportionality factor. Second, notice that v is generally a time dependent quantity whereas n is not. But there is no contradiction as v is to be regarded as a symbol for a time independent (only space dependent) potential function. With this understanding, we can define the *refractive index of a particle* *with velocity* v *in free space* as n. But the converse is not true. The reason is that a given medium does not, in general, have the same dispersion as free space. Knowledge of n does not imply the knowledge of $dn/d\omega$, in contradistinction to the case in free space[2].

Ray optics → "F=ma" form of Newtonian mechanics

In order to arrive at the "F=ma" form of optics, it is necessary to identify several mechanical quantities associated with a particle with those associated with a pulse of light. In this identification, the role of time t is played by the Evans-Rosenquist parameter A in the motion of a light pulse. The following table succinctly displays the analogy:

Quantity	Mechanics	Optics
"Position"	$r(t)$	$r(A)$
"Time"	t	A
"Velocity"	$\frac{dr}{dt} \equiv \dot{r}$	$\frac{dr}{dA} \equiv r'$
"Potential energy"	$U(r)$	$-n^2(r)/2$
"Mass"	m	1
"Kinetic energy"	$T = \frac{m}{2}\lvert\dot{r}\rvert^2$	$\frac{1}{2}\lvert r'\rvert^2$
"Total energy"	$\frac{m}{2}\lvert\dot{r}\rvert^2 + U$	$\frac{1}{2}\lvert r'\rvert^2 - \frac{n^2}{2}$
"Equation of motion"	$m\ddot{r} = -\operatorname{grad} U$	$r'' = \operatorname{grad}\left(\frac{n^2}{2}\right)$

Evans[4] interpreted the parameter A, having the dimension of length, as "optical action": Just as a mechanical particle progresses in time t along its trajectory, a light pulse progresses in optical action A along its ray and is related to t by

$$dA = c_o n^{-2} dt. \qquad (1.1.10)$$

The identification $U(r) = -n^2/2$ giving the "potential" U bestows some sort of a mechanical character to photon motions; the only restriction being that the motion corresponds to mechanics at "zero total energy". We may look for cases where light and mechanical motions take place under the identical form of "force"/force law so that the same method of solution applies to both cases. An excellent example is probably the Luneberg Lens[6] in optics and the harmonic oscillator in mechanics. In order to apply the above formalism to the motions in general relativity, it is necessary to be assured that a gravity field can really be represented as an optical medium so that we may expect to find a gravitational refractive index n in specific cases.

1.2. Gravity field as an optical medium

The geometric features of general relativity have been investigated to a great depth since its very formulation.[7]

Another aspect of the geometric general relativity is that Einstein's field equations can be reformulated as a "field" on a background Ricci-flat spacetime[8-10]. However, the optical features of general relativity are somewhat less known.

The equivalence of a gravitational field and an optical medium was suggested by Einstein himself and formally developed later, among others, by physicists such as Eddington[11], Plebanski[12], Balazs[13], Skrotskii[14], Winterberg[15] and Bertotti[16]. The idea has been extremely useful in the investigation of electromagnetic phenomena in a general gravity field[17-21]. We here describe the equations briefly.

Let the metric tensor of a given spacetime be $g_{\alpha\beta}$. (Greek indices run from 0 to 3). Source free Maxwell equations are

$$\nabla_{\alpha}\phi^{\alpha\beta} = 0, \ \nabla_{\alpha}\overset{*}{\phi}{}^{\alpha\beta} = 0 \tag{1.1.11}$$

where $\phi_{\alpha\beta}$ represent antisymmetric electromagnetic field and ∇_{α} represent covariant derivative with respect to $g_{\alpha\beta}$. Let u^{μ} be the 4-velocity of an observer given by $u^{\mu} = (-g_{oo})^{-1/2}\delta^{\mu}_{o}$. Then it is possible to express $\phi_{\alpha\beta}$ as

$$\phi_{\alpha\beta} = \eta_{\alpha\beta\gamma\delta}u^{\delta}\hat{H}^{\gamma} + 2u_{[\alpha}\hat{E}_{\beta]}, \tag{1.1.12}$$

$$\hat{H}^{\gamma} = \overset{*}{\phi}{}^{\gamma\alpha}u_{\alpha}, \quad \hat{E}_{\beta} = \phi_{\alpha\beta}u^{\alpha}, \tag{1.1.13}$$

where $\eta_{\alpha\beta\gamma\delta}$ is the Levi-Civita alternating symbol. We can now rewrite Maxwell equations (1.1.11) as (Latin indices run from 1 to 3):

$$\delta^{ijk}\partial_j\mathbb{H}_k - \partial_o\mathbb{D}^i = 0, \quad \partial_p\mathbb{D}^p = 0 \tag{1.1.14}$$

$$\delta^{ijk}\partial_j\mathbb{E}_k + \partial_o\mathbb{B}^i = 0, \quad \partial_p\mathbb{B}^p = 0 \tag{1.1.15}$$

where the new vector densities $\mathbb{D}^i$ and $\mathbb{B}^i$ induced by $g_{\alpha\beta}$ are

$$\mathbb{H}_k = (-g_{oo})^{1/2}\hat{H}_k \ , \ \mathbb{E}_k = (-g_{oo})^{1/2}\hat{E}_k \tag{1.1.16}$$

$$\mathbb{D}^i = -(-g)^{1/2}\left(\frac{g^{ij}}{g_{oo}}\right)\mathbb{E}_j - \delta^{ijk}\left(\frac{g_{ok}}{g_{oo}}\right)\mathbb{H}_j \tag{1.1.17}$$

$$\mathbb{B}^i = -(-g)^{1/2}\left(\frac{g^{ij}}{g_{oo}}\right)\mathbb{H}_j + \delta^{ijk}\left(\frac{g_{ok}}{g_{oo}}\right)\mathbb{E}_j \tag{1.1.18}$$

and δ^{ijk} are alternating symbols, $-g = \det|g_{\alpha\beta}|$.

Let the equivalent optical medium be defined by the constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ on a flat spacetime $\eta_{\alpha\beta}$. Then the electromagnetic field tensors $F_{\alpha\beta}$ and $Q_{\alpha\beta}$ satisfy[22]

$$Q^{\alpha\beta} = \frac{1}{2}\chi^{\alpha\beta\gamma\delta}F_{\gamma\delta}. \tag{1.1.19}$$

Finally, the tensor $\chi^{\alpha\beta\gamma\delta}$ can be decomposed into components ε^{ij} and χ^{ij} which are symmetric 3-tensors representing the dielectric and the inverse of magnetic permeability of the equivalent optical medium we have been looking for:

$$\mu^{ij} = \varepsilon^{ij} = -(-g)^{1/2}\left(\frac{g^{ij}}{g_{oo}}\right) \tag{1.1.20}$$

$$\gamma^{ij} = \bar{\gamma}^{ij} = -\,\delta^{ijk}\left(\frac{g_{ok}}{g_{oo}}\right) \tag{1.1.21}$$

The tensors γ^{ij} and $\bar{\gamma}^{ij}$ represent peculiar properties of the medium. It can be verified that a conformal transformation on the metric $g_{\alpha\beta}$ leaves the above two equations invariant.

1.3. Remarks on the equivalent optical medium

From the eqs.(1.1.20) and (1.1.21) it is possible, in specific cases, to deduce a gravitational refractive index n(r,t). It is important that some caution[17] be excercised in interpreting this n(r,t) as the equivalent of that calculated from a *purely* electromagnetic theory on a flat spacetime. The difference becomes clear when one considers the motion of high speed charged particles through the gravitational field. If the gravitational index were indeed entirely equivalent to an electromagnetic index, such particles would emit Čerenkov radiation at the critical velocity c_o/n. The critical energy, calculated from special relativity, is nearly $(rc_o^2/4GM)^{1/2}mc_o^2$, where M is the gravitating mass and m is the rest mass of the particle. This amounts to about 10 GeV for an electron in the Earth's field. Such a low value is quite out of the question because of the energies achieved by high energy electron accelerators and also the energies in cosmic rays. To what extent n(r,t) is or is not equivalent to electromagnetic index can be assessed from Maxwell equations themselves. The homogeneous equations are indifferent to whether one has an ordinary refractive index associated with electric polarization or a modification of the spacetime manifold. The homogeneous case has just been described in Sec.1.2 above. However, for the inhomogeneous equations in the gravitational case, the current source j is a function of $(1-n^2v^2/c_o^2)^{-1/2}$ and this is *not* the case *in the usual equations of special relativistic electromagnetic* theory. Hence, the absence of Čerenkov radiation is indicative of the special and, in some sense, limited role played by the gravitational index n(r,t).

The flat space η_{ij} on which the gravitational optical medium (or, for that matter, "field") is defined is *fictitious*[8-10] in the sense that this space (or the "coordinates" therein) can never be located *observationally* due to the universality of gravitational interactions. There exist no energetic particle neutral to gravity that can trace for us the background flat space.

Finally, notice (from Sec.2.1 below) that the "coordinate" speed of light in the optical medium is c_o/n. Does this mean that the speed of light really slows down for $n > 1$?[23] We have just now pointed out that the flat space "coordinates" are not observable in general relativity. Hence the "coordinate speed" is also not measurable. Actually, the observable quantities, as a rule, have to be independent of coordinate choices. That is, they should be "proper" quantities. The proper velocity of light is $dl/d\tau$ where dl and $d\tau$ are the proper length and time respectively at the site of the observer, and is related by

$$ds^2 = c_o^2 d\tau^2 - dl^2. \qquad (1.1.22)$$

Hence, the proper velocity of light passing *through* each observer is just the vacuum speed c_o. On the other hand, if the measurement is made from a different site where the proper time is $d\tau'$, the quantity $dl/d\tau'$ becomes a coordinate dependent quantity as $d\tau/d\tau'$ is so.[24] However, if we are interested only in a *formal* analysis involving coordinate speeds, we can regard the speed of light as slowing down - even vanishing for $n \to \infty$ - in a denser medium. We now have come to a point where, with respect to the speed of *light, another physical distinction must be made between a* gravitational medium and an ordinary *material* medium. Wang and Chen[25] have shown that in a material medium some of the photons are absorbed by the atoms while other noninteracting ones are in a free state. The free photons always have an instanteneous velocity equal to the proper velocity c_o mentioned above. However, the statistical average velocity or the velocity of energy transport is always *less than* c_o. *Such physical effects involving atoms* are unavailable in a gravitational medium.

The two distinctions already pointed out in this section appear to indicate that a gravitational optical medium such as the one contemplated above may just be a mathematical contrivance. Despite this, it is remarkable that, in regard to some events, explored in the *sequel, this medium does* behave *as a real physical medium.*

1.4. Fermat's principle in general relativity

In Sec.1.1, the eq. (1.1.3) represents Fermat's principle in a flat spacetime. The principle provides, in a concise form, the laws of propagation, reflection and refraction of light. The question is whether it is possible to formulate this principle in the arbitrary spacetimes of general relativity. A survey of relevant literature reveals that, just as in the case of optical-mechanical analogy, the formulation of Fermat's principle in general relativity also has a long history. Early formulations adopted a coordinate system consistent with the Killing vector field admitted by the spacetime in question. The static case was dealt by Weyl[26] and the stationary case by Levi-Civita.[27]

During the last decade, there has been a remarkable revival of interest in Fermat's principle and its reformulation in a general spacetime. This interest has primarily been motivated by the emergence of new ideas/interpretations in Astrophysics. For instance, in the case of gravitational lensing, Blandford and Narayan[28] have utilized Fermat's principle to classify different gravitational lens images. On the theoretical side, Kovner[29] has given a new variant of the principle which extremizes the arrival time at the world line of the receiver of light emitted at a certain point. This formulation is in contrast to the usual one where the end points are two instants of time having an absolute meaning in Newtonian physics. Nityananda and Samuel[30] have established, on rigorous mathematical grounds, the validity of Fermat's principle (and Kovner's formulation thereof) in arbitrary spacetimes of general relativity.

Recently, Bel and Martin[31] have improved upon the versions of Kovner and Perlick[32] and developed Fermat's principle from the causal structure of spacetime. The most recent work, to our knowledge, is that of Nandor and Helliwell[33] concerning the use of this principle in the multiple imaging and in the construction of equivalent lenses for specific gravitational fields.

1.5. Detailed objectives

This thesis extends the application of the Evans-Rosenquist optical-mechanical analogy to the regime of general relativity. To our knowledge, the work being undertaken is the first of its kind. A relevant variational principle is to be developed which requires *that the gravity field be portrayed as an optical medium. With* many metrics of physical interest, it is possible to associate equivalent indices of refraction. We wish to show that, in such spacetimes, the orbit equations are governed by a *unified* variational principle involving the gravitational index of refraction. From this variational principle, we shall derive exact equations of motion of *Newtonian form* which govern both massive and massless particles. Throughout the thesis, we shall be concerned only with the experimentally relevant spherically *symmetric spacetimes where the dielectric tensors satisfy*

$$\mu^{ij} = \varepsilon^{ij} = n(r,t)\delta^{ij} \qquad (1.1.23)$$

$$\gamma^{ij} = \bar{\gamma}^{ij} = 0. \qquad (1.1.24)$$

The form of n(r,t) will be evident in the cases considered. The time dependence will appear only in the cosmological context but its interpretation will be given at the appropriate place.

The approach is intended to cover a reasonably wide class of physical situations. We consider Reissner-Nordström types of spacetimes in general relativity where different choices of parameters correspond to different physical situations. Their effects on observations will be computed and an error in the earlier literature will be corrected. In the regime of cosmology, we shall attempt to describe light motion in the de Sitter and Robertson-Walker universes. It will be shown that these universes correspond to a traditional problem in geometrical optics - the Maxwell "fish-eye lens". Through our optical approach we shall attempt to provide a *unified* interpretation of two redshifts having apparently dissimilar characters: gravitational redshift in general relativity and recessional redshift in cosmology.

As a related study, we shall critically review a recent alternative procedure that equates the *effects of a refractive* medium with those caused by a photon having varying mass and speed on a flat space. By way of doing so, we shall have occasion to clarify the meaning of observables in flat space gravity theory and in general relativity itself.

A chapterwise description of the contents is as follows: In Chapter 2, we shall derive the relevant variational principle and deduce from it the equations of motion of Newtonian form. With these developments in view, we would like to make a brief journey into the past: a little history, in Chapter 3. The application of the equations to the Reissner-Nordström type spacetime will be considered in Chapter 4 while that to cosmology will be taken up in Chapter 5. As a related investigation, we shall critically examine, in Chapter 6, a recent attempt at an alternative deduction of general relativistic effects that uses another form of gravitational index n. There will be an epilogue at the end.