ON THE OPTICAL-MECHANICAL ANALOGY IN GENERAL RELATIVITY AND SOME RELATED STUDIES

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CERTIFICATE

This is to certify that the thesis entitled "ON THE OPTICAL-MECHANICAL ANALOGY IN GENERAL RELATIVITY AND SOME RELATED STUDIES" which is being submitted by Md. Anwarul Islam to the University of North Bengal, Darjeeling, for the degree of Doctor of Philosophy is a record of bonafide research work carried out by him under my supervision. The results embodied in the thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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CHAPTER 1

INTRODUCTION

This introductory chapter contains descriptions of the basic ideas to be employed in the thesis. For convenience, the contents are divided into various sections. In Sec.1.1, we present the basic idea of the optical-mechanical analogy and introduce what we choose to call the Evans-Rosenquist parameter. The intended application of this analogy in the realm of general relativity requires that the gravity field be represented as a refractive optical medium on a flat space. That such a representation is possible is demonstrated in Sec.1.2. The distinctions between gravitational and electromagnetic refractive indices are elucidated in Sec.1.3. Another ingredient in our approach is a suitable variational principle. To that end, a brief description of Fermat's principle and its ramifications in general relativity is provided in Sec.1.4. The final section (Sec.1.5) details the objectives of the present work.

1.1. The optical-mechanical analogy: Evans-Rosenquist parameter A

The historial optical-mechanical analogy[1,2] has recently been cast into a familiar form by Evans and Rosenquist[3-5]. This new formulation, based on Fermat's principle, provides an interesting approach that can be profitably utilized in the solution of many classical problems. The approach works either way: The wellknown ideas and techniques of classical optics can be successfully applied to the problem of classical mechanics and vice versa.

The ray equations from the classical gradient-index optics can be written in the form[1]

$$\nabla n = \frac{d}{ds}\left(n \frac{dr}{ds}\right) \tag{1.1.1}$$

where $r \equiv (x,y,z)$ is the position of a point on the ray, $\nabla \equiv (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$ is the gradient operator, ds is the element of arc length along the ray and $n(r)$ is the index of refraction of the optical medium. This differential equation can be solved either exactly or numerically depending on the form of $n(r)$ and the shape of the ray can be obtained thereby. The variational principle leading to eq.(1.1.1) is that of Fermat. The optical wave disturbance can be regarded as a sum of contributions propagated along all possible null paths from the source to the receiver. Since neighboring paths traverse nearly the same ground in the medium, the magnitudes of the disturbances arriving along each path must be virtually identical. However, since we are concerned only with wave motion in the geometrical limit in which the wavelength is much smaller than any other feature of the problem, even a slight difference in paths means that the phase of the wavelets contributed by neighboring paths will be many cycles different. A bundle of neighboring virtual paths from the source to the receiver will therefore make contributions with equal magnitudes but with a random assortment of phases, and will therefore interfere destructively. The only circumstance in which such a cancellation will not occur is when the phase along the path is stationary. In this case, the phases of the virtual paths are nearly equal and the wavelets interefere constructively.

In a medium with a varying index of refraction n(r), the phase along a path is given by

$$\text{phase} = \frac{\omega}{c_o} \int n(r)\,|dr|. \qquad (1.1.2)$$

where c_o is the speed of light in vacuum. The requirement that the phase be stationary implies a variational principle in the form

$$\delta \int n(r)\,|dr| = 0. \text{ (optics)} \qquad (1.1.3)$$

Newtonian mechanics → ray optics

To give the mechanical equations an optical ray form, we start with the Maupertuis' principle

$$\delta\int mv(r)\left|dr\right| = 0 \quad \text{(mechanics)} \tag{1.1.4}$$

where v(r) is the speed of the particle (= $|v|$) considered to be a function of its position alone. The rules governing the variation of this principle are the same as those of Fermat's. Comparing the two eqs.(1.1.3) and (1.1.4), we can pass from point particle mechanics to geometrical ray optics via the transcription

$$v \rightarrow n. \tag{1.1.5}$$

This transcription applies in a potential that depends on position alone. Hence, from eq.(1.1.1),

$$\nabla v = \frac{d}{ds}\left(v\,\frac{dr}{ds}\right). \tag{1.1.6}$$

This is basically the ray form of Newtonian mechanics. The form can be simplified further by noting that dr/ds is a unit tangent vector to the trajectory, that is, in the direction of v so that

$$\nabla v = \frac{dv}{ds}. \tag{1.1.7}$$

Another particularly interesting derivation can be obtained in terms of the wave number k through the relation $n = c_0 k/\omega$. We get from the eq.(1.1.1)

$$\nabla k = \frac{d}{ds}\left(k \frac{d\mathbf{r}}{ds}\right). \qquad (1.1.8)$$

Assuming that this equation represents the *particle ray* equation in the short wave length limit of quantum mechanics, we can use the de Broglie relation

$$m\mathbf{v} = \hbar \mathbf{k} \Rightarrow mv = \hbar k \qquad (1.1.9)$$

where $|\mathbf{k}| = k$ and obtain, finally, eq.(1.1.7) from eq.(1.1.8). As one can see, de Broglie relations are equivalent to the optical-mechanical analogy $v \equiv k$ upto a proportionality factor $\hbar$, the Planck constant that ushered in a new physics. The basis of the analogy lies in the common wavelike properties of light and material particles (wave-particle duality in quantum mechanics). However, in the present classical treatment of trajectories, it is important to remember that "light is not just like particles"[3].

A couple of remarks about the transcription (1.1.5) is in order. Observe first that the two sides are not dimensionally equivalent. Therefore what is intended here is only the proportionality factor. Second, notice that v is generally a time dependent quantity whereas n is not. But there is no contradiction as v is to be regarded as a symbol for a time independent (only space dependent) potential function. With this understanding, we can define the *refractive index of a particle* *with velocity* v *in free space* as n. But the converse is not true. The reason is that a given medium does not, in general, have the same dispersion as free space. Knowledge of n does not imply the knowledge of $dn/d\omega$, in contradistinction to the case in free space[2].

Ray optics → "F=ma" form of Newtonian mechanics

In order to arrive at the "F=ma" form of optics, it is necessary to identify several mechanical quantities associated with a particle with those associated with a pulse of light. In this identification, the role of time t is played by the Evans-Rosenquist parameter A in the motion of a light pulse. The following table succinctly displays the analogy:

Quantity	Mechanics	Optics
"Position"	$r(t)$	$r(A)$
"Time"	t	A
"Velocity"	$\frac{dr}{dt} \equiv \dot{r}$	$\frac{dr}{dA} \equiv r'$
"Potential energy"	$U(r)$	$-n^2(r)/2$
"Mass"	m	1
"Kinetic energy"	$T = \frac{m}{2}\lvert\dot{r}\rvert^2$	$\frac{1}{2}\lvert r'\rvert^2$
"Total energy"	$\frac{m}{2}\lvert\dot{r}\rvert^2 + U$	$\frac{1}{2}\lvert r'\rvert^2 - \frac{n^2}{2}$
"Equation of motion"	$m\ddot{r} = -\operatorname{grad} U$	$r'' = \operatorname{grad}\left(\frac{n^2}{2}\right)$

Evans[4] interpreted the parameter A, having the dimension of length, as "optical action": Just as a mechanical particle progresses in time t along its trajectory, a light pulse progresses in optical action A along its ray and is related to t by

$$dA = c_o n^{-2} dt. \tag{1.1.10}$$

The identification $U(r) = -n^2/2$ giving the "potential" U bestows some sort of a mechanical character to photon motions; the only restriction being that the motion corresponds to mechanics at "zero total energy". We may look for cases where light and mechanical motions take place under the identical form of "force"/force law so that the same method of solution applies to both cases. An excellent example is probably the Luneberg Lens[6] in optics and the harmonic oscillator in mechanics. In order to apply the above formalism to the motions in general relativity, it is necessary to be assured that a gravity field can really be represented as an optical medium so that we may expect to find a gravitational refractive index n in specific cases.

1.2. Gravity field as an optical medium

The geometric features of general relativity have been investigated to a great depth since its very formulation.[7]

Another aspect of the geometric general relativity is that Einstein's field equations can be reformulated as a "field" on a background Ricci-flat spacetime[8-10]. However, the optical features of general relativity are somewhat less known.

The equivalence of a gravitational field and an optical medium was suggested by Einstein himself and formally developed later, among others, by physicists such as Eddington[11], Plebanski[12], Balazs[13], Skrotskii[14], Winterberg[15] and Bertotti[16]. The idea has been extremely useful in the investigation of electromagnetic phenomena in a general gravity field[17-21]. We here describe the equations briefly.

Let the metric tensor of a given spacetime be $g_{\alpha\beta}$. (Greek indices run from 0 to 3). Source free Maxwell equations are

$$\nabla_\alpha \phi^{\alpha\beta} = 0, \; \nabla_\alpha \overset{*}{\phi}{}^{\alpha\beta} = 0 \tag{1.1.11}$$

where $\phi_{\alpha\beta}$ represent antisymmetric electromagnetic field and ∇_α represent covariant derivative with respect to $g_{\alpha\beta}$. Let u^μ be the 4-velocity of an observer given by $u^\mu = (-g_{oo})^{-1/2}\delta^\mu_o$. Then it is possible to express $\phi_{\alpha\beta}$ as

$$\phi_{\alpha\beta} = \eta_{\alpha\beta\gamma\delta} u^\delta \hat{H}^\gamma + 2u_{[\alpha}\hat{E}_{\beta]}, \tag{1.1.12}$$

$$\hat{H}^{\gamma} = \overset{*}{\phi}^{\gamma\alpha} u_{\alpha}, \quad \hat{E}_{\beta} = \phi_{\alpha\beta} u^{\alpha}, \tag{1.1.13}$$

where $\eta_{\alpha\beta\gamma\delta}$ is the Levi-Civita alternating symbol. We can now rewrite Maxwell equations (1.1.11) as (Latin indices run from 1 to 3):

$$\delta^{ijk} \partial_j \mathbb{H}_k - \partial_o \mathbb{D}^i = 0, \quad \partial_p \mathbb{D}^p = 0 \tag{1.1.14}$$

$$\delta^{ijk} \partial_j \mathbb{E}_k + \partial_o \mathbb{B}^i = 0, \quad \partial_p \mathbb{B}^p = 0 \tag{1.1.15}$$

where the new vector densities $\mathbb{D}^i$ and $\mathbb{B}^i$ induced by $g_{\alpha\beta}$ are

$$\mathbb{H}_k = (-g_{oo})^{1/2} \hat{H}_k \ , \ \mathbb{E}_k = (-g_{oo})^{1/2} \hat{E}_k \tag{1.1.16}$$

$$\mathbb{D}^i = -(-g)^{1/2} \left(\frac{g^{ij}}{g_{oo}} \right) \mathbb{E}_j - \delta^{ijk} \left(\frac{g_{ok}}{g_{oo}} \right) \mathbb{H}_j \tag{1.1.17}$$

$$\mathbb{B}^i = -(-g)^{1/2} \left(\frac{g^{ij}}{g_{oo}} \right) \mathbb{H}_j + \delta^{ijk} \left(\frac{g_{ok}}{g_{oo}} \right) \mathbb{E}_j \tag{1.1.18}$$

and δ^{ijk} are alternating symbols, $-g = \det|g_{\alpha\beta}|$.

Let the equivalent optical medium be defined by the constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ on a flat spacetime $\eta_{\alpha\beta}$. Then the electromagnetic field tensors $F_{\alpha\beta}$ and $Q_{\alpha\beta}$ satisfy[22]

$$Q^{\alpha\beta} = \frac{1}{2}\chi^{\alpha\beta\gamma\delta}F_{\gamma\delta}. \qquad (1.1.19)$$

Finally, the tensor $\chi^{\alpha\beta\gamma\delta}$ can be decomposed into components ε^{ij} and χ^{ij} which are symmetric 3-tensors representing the dielectric and the inverse of magnetic permeability of the equivalent optical medium we have been looking for:

$$\mu^{ij} = \varepsilon^{ij} = -(-g)^{1/2}\left(\frac{g^{ij}}{g_{oo}}\right) \qquad (1.1.20)$$

$$\gamma^{ij} = \bar{\gamma}^{ij} = -\delta^{ijk}\left(\frac{g_{ok}}{g_{oo}}\right) \qquad (1.1.21)$$

The tensors γ^{ij} and $\bar{\gamma}^{ij}$ represent peculiar properties of the medium. It can be verified that a conformal transformation on the metric $g_{\alpha\beta}$ leaves the above two equations invariant.

1.3. Remarks on the equivalent optical medium

From the eqs.(1.1.20) and (1.1.21) it is possible, in specific cases, to deduce a gravitational refractive index n(r,t). It is important that some caution[17] be excercised in interpreting this n(r,t) as the equivalent of that calculated from a *purely* electromagnetic theory on a flat spacetime. The difference becomes clear when one considers the motion of high speed charged particles through the gravitational field. If the gravitational index were indeed entirely equivalent to an electromagnetic index, such particles would emit Čerenkov radiation at the critical velocity c_0/n. The critical energy, calculated from special relativity, is nearly $(rc_0^2/4GM)^{1/2}mc_0^2$, where M is the gravitating mass and m is the rest mass of the particle. This amounts to about 10 GeV for an electron in the Earth's field. Such a low value is quite out of the question because of the energies achieved by high energy electron accelerators and also the energies in cosmic rays. To what extent n(r,t) is or is not equivalent to electromagnetic index can be assessed from Maxwell equations themselves. The homogeneous equations are indifferent to whether one has an ordinary refractive index associated with electric polarization or a modification of the spacetime manifold. The homogeneous case has just been described in Sec.1.2 above. However, for the inhomogeneous equations in the gravitational case, the current source j is a function of $(1-n^2v^2/c_0^2)^{-1/2}$ and this is *not* the case *in the usual equations of special relativistic electromagnetic* theory. Hence, the absence of Čerenkov radiation is indicative of the special and, in some sense, limited role played by the gravitational index n(r,t).

The flat space η_{ij} on which the gravitational optical medium (or, for that matter, "field") is defined is *fictitious*[8-10] in the sense that this space (or the "coordinates" therein) can never be located *observationally* due to the universality of gravitational interactions. There exist no energetic particle neutral to gravity that can trace for us the background flat space.

Finally, notice (from Sec.2.1 below) that the "coordinate" speed of light in the optical medium is c_o/n. Does this mean that the speed of light really slows down for $n > 1$?[23] We have just now pointed out that the flat space "coordinates" are not observable in general relativity. Hence the "coordinate speed" is also not measurable. Actually, the observable quantities, as a rule, have to be independent of coordinate choices. That is, they should be "proper" quantities. The proper velocity of light is $dl/d\tau$ where dl and $d\tau$ are the proper length and time respectively at the site of the observer, and is related by

$$ds^2 = c_o^2 d\tau^2 - dl^2. \tag{1.1.22}$$

Hence, the proper velocity of light passing *through* each observer is just the vacuum speed c_o. On the other hand, if the measurement is made from a different site where the proper time is $d\tau'$, the quantity $dl/d\tau'$ becomes a coordinate dependent quantity as $d\tau/d\tau'$ is so.[24] However, if we are interested only in a *formal* analysis involving coordinate speeds, we can regard the speed of light as slowing down - even vanishing for $n \to \infty$ - in a denser medium. We now have come to a point where, with respect to the speed of *light, another physical distinction must be made between a* gravitational medium and an ordinary *material* medium. Wang and Chen[25] have shown that in a material medium some of the photons are absorbed by the atoms while other noninteracting ones are in a free state. The free photons always have an instanteneous velocity equal to the proper velocity c_o mentioned above. However, the statistical average velocity or the velocity of energy transport is always *less than* c_o. *Such physical effects involving atoms* are unavailable in a gravitational medium.

The two distinctions already pointed out in this section appear to indicate that a gravitational optical medium such as the one contemplated above may just be a mathematical contrivance. Despite this, it is remarkable that, in regard to some events, explored in the sequel, this medium *does* behave as a real physical medium.

1.4. Fermat's principle in general relativity

In Sec.1.1, the eq. (1.1.3) represents Fermat's principle in a flat spacetime. The principle provides, in a concise form, the laws of propagation, reflection and refraction of light. The question is whether it is possible to formulate this principle in the arbitrary spacetimes of general relativity. A survey of relevant literature reveals that, just as in the case of optical-mechanical analogy, the formulation of Fermat's principle in general relativity also has a long history. Early formulations adopted a coordinate system consistent with the Killing vector field admitted by the spacetime in question. The static case was dealt by Weyl[26] and the stationary case by Levi-Civita.[27]

During the last decade, there has been a remarkable revival of interest in Fermat's principle and its reformulation in a general spacetime. This interest has primarily been motivated by the emergence of new ideas/interpretations in Astrophysics. For instance, in the case of gravitational lensing, Blandford and Narayan[28] have utilized Fermat's principle to classify different gravitational lens images. On the theoretical side, Kovner[29] has given a new variant of the principle which extremizes the arrival time at the world line of the receiver of light emitted at a certain point. This formulation is in contrast to the usual one where the end points are two instants of time having an absolute meaning in Newtonian physics. Nityananda and Samuel[30] have established, on rigorous mathematical grounds, the validity of Fermat's principle (and Kovner's formulation thereof) in arbitrary spacetimes of general relativity.

Recently, Bel and Martin[31] have improved upon the versions of Kovner and Perlick[32] and developed Fermat's principle from the causal structure of spacetime. The most recent work, to our knowledge, is that of Nandor and Helliwell[33] concerning the use of this principle in the multiple imaging and in the construction of equivalent lenses for specific gravitational fields.

1.5. Detailed objectives

This thesis extends the application of the Evans-Rosenquist optical-mechanical analogy to the regime of general relativity. To our knowledge, the work being undertaken is the first of its kind. A relevant variational principle is to be developed which requires *that the gravity field be portrayed as an optical medium.* With many metrics of physical interest, it is possible to associate equivalent indices of refraction. We wish to show that, in such spacetimes, the orbit equations are governed by a *unified* variational principle involving the gravitational index of refraction. From this variational principle, we shall derive exact equations of motion of *Newtonian form* which govern both massive and massless particles. Throughout the thesis, we shall be concerned only with the experimentally relevant spherically *symmetric spacetimes where the dielectric tensors satisfy*

$$\mu^{ij} = \varepsilon^{ij} = n(r,t)\delta^{ij} \qquad (1.1.23)$$

$$\gamma^{ij} = \bar{\gamma}^{ij} = 0. \qquad (1.1.24)$$

The form of n(r,t) will be evident in the cases considered. The time dependence will appear only in the cosmological context but its interpretation will be given at the appropriate place.

The approach is intended to cover a reasonably wide class of physical situations. We consider Reissner-Nordström types of spacetimes in general relativity where different choices of parameters correspond to different physical situations. Their effects on observations will be computed and an error in the earlier literature will be corrected. In the regime of cosmology, we shall attempt to describe light motion in the de Sitter and Robertson-Walker universes. It will be shown that these universes correspond to a traditional problem in geometrical optics - the Maxwell "fish-eye lens". Through our optical approach we shall attempt to provide a *unified* interpretation of two redshifts having apparently dissimilar characters: gravitational redshift in general relativity and recessional redshift in cosmology.

As a related study, we shall critically review a recent alternative procedure that equates the *effects of a refractive* medium with those caused by a photon having varying mass and speed on a flat space. By way of doing so, we shall have occasion to clarify the meaning of observables in flat space gravity theory and in general relativity itself.

A chapterwise description of the contents is as follows: In Chapter 2, we shall derive the relevant variational principle and deduce from it the equations of motion of Newtonian form. With these developments in view, we would like to make a brief journey into the past: a little history, in Chapter 3. The application of the equations to the Reissner-Nordström type spacetime will be considered in Chapter 4 while that to cosmology will be taken up in Chapter 5. As a related investigation, we shall critically examine, in Chapter 6, a recent attempt at an alternative deduction of general relativistic effects that uses another form of gravitational index n. There will be an epilogue at the end.

CHAPTER 2

THE VARIATIONAL PRINCIPLE AND THE EQUATIONS OF MOTION

In this chapter, we develop the variational principle, the equations of motion and their solutions. In Sec.2.1, the principle is formulated. Sec.2.2 contains the exact equations of motion in Newtonian form. The crucial role of the Evans-Rosenquist parameter A is elucidated in Sec.2.3. Taking the example of Schwarzschild field of general relavity, we illustrate two methods of solution (energy and force methods) of equations of motion, in Sec.2.4.

2.1. Formulation of the variational principle

We wish to formulate, for the trajectories in general relativity, a variational principle that combines both the principle of Fermat (classical geometric optics) and the principle of Maupertuis (Newtonian mechanics in velocity independent potentials). To do this, it is convenient to cast the spherically symmetric spacetime into isotropic coordinates as

$$ds^2 = \Omega^2(r,t)c_0^2dt^2 - \Phi^{-2}(r,t)|dr|^2, \qquad (2.1.1)$$

$$|dr|^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \qquad (2.1.2)$$

We have the following expressions from the metric eq.(2.1.1):

$$g_{oo} = \Omega^2,\ g_{ij} = -\Phi^{-2}\delta_{ij},\ g^{ij} = -\Phi^2\delta^{ij},\ -(-g)^{1/2} = \Omega\Phi^{-3} \qquad (2.1.3)$$

so that we get, from eq.(1.1.22), the dielectric tensors as

$$\mu^{ij} = \varepsilon^{ij} = \Omega^{-1}\Phi^{-1}\delta^{ij},\ n(r,t) = \Omega^{-1}\Phi^{-1}. \qquad (2.1.4)$$

The expression for the refractive index n(r,t) can also be obtained in an alternative way. The idea is to compute the isotropic cordinate speed of light c(r) at any arbitrary point in the field by putting $ds^2 = 0$. This gives

$$c(r) = |dr/dt| = c_o\Phi(r,t)\Omega(r,t). \qquad (2.1.5)$$

Using $c = c_o/n$, we get the effective index of refraction as $n(r,t) = \Phi^{-1}\Omega^{-1}$ which is precisely the same as eq.(2.1.4). Another way to deduce n is to use the conformal invariance of Maxwell's equations. It can be shown[34,35] that the metric (2.1.1) has the same properties as Gordon's form of optical metric given by

$$ds^2 = [\eta_{\alpha\beta} + (1-n^{-2})\delta^o_\alpha\delta^o_\beta]dx^\alpha dx^\beta$$

from which n can be deduced by comparison.

Light trajectories in the gravitational field can be calculated by using the effective index of refraction, eq.(2.1.4), in any formulation of geometrical optics that happens to be convenient. For example, Wu and Xu have recently shown that the standard differential equation of the ray in classical geometrical optics can be applied to the null geodesic problem[36].

An especially convenient version of geometric optics is the so-called "F = ma" formulation[3] in which the equation governing the optical ray assumes the form of Newton's law of motion (acceleration = - gradient of potential energy):

$$d^2r/dA^2 = \nabla(n^2c_0^2/2) \tag{2.1.6}$$

r is the position of a pulse moving along the ray. All the usual force and energy methods of elementary mechanics can be brought to bear on geometrical optics.

The effective index of refraction eq.(2.1.4) (for the Schwarzschild metric, for example) can be used in eq.(2.1.6) without modification[37]. In solving problems, one goes into the isotropic coordinates, applies the "F = ma" optics, then transform back to the standard or other coordinates, if desired.

We shall obtain the variational principle by transformation of the geodesic condition for the particle trajectories,

$$\delta\int_{x_1,t_1}^{x_2,t_2} ds = 0 \tag{2.1.7}$$

where δ indicates a variation in the path of integration between two fixed points in spacetime, (x_1,t_1) and (x_2,t_2). If we assume the line element can be written in the form eq.(2.1.1), this becomes

$$\delta\int_{x_1,t_1}^{x_2,t_2} \Omega c_o\ [1 - v^2n^2/c_o^2]^{1/2}dt = 0. \tag{2.1.8}$$

This is analogous to Hamilton's principle and the effective Lagrangian is

$$L(x_i,\dot{x}_i) = - c_o^2\Omega[1 - v^2n^2/c_o^2]^{1/2} \tag{2.1.9}$$

where Ω and n are functions of the coordinates alone, where $\dot{x}_i \equiv dx_i/dt$, and where $v^2 = \sum_{i=1}^{3}(dx_i/dt)^2$, if we choose to work in Cartesian coordinates. The expression for the Lagrangian has been multiplied by an extra factor of $-c_o$ for later convenience.

The canonical momenta p_i are

$$p_i \equiv \partial L/\partial \dot{x}_i$$

$$= \Omega n^2 [1 - v^2 n^2 / c_0^2]^{1/2} x_i . \qquad (2.1.10)$$

The Hamiltonian H may be formed in the usual way:

$$H = \sum_{i=1}^{3} p_i \dot{x}_i - L$$

$$= c_0^2 \Omega [1 - v^2 n^2 / c_0^2]^{-1/2} . \qquad (2.1.11)$$

Because $\partial L/\partial t = 0$, H is a constant of the motion. If we express H in terms of the p_i rather than $\dot{x}_i$, we obtain

$$H = c_0^2 [\Omega^2 + p^2 / n^2 c_0^2]^{1/2} \qquad (2.1.12)$$

where $p = |\mathbf{p}|$. From Hamilton's principle,

$$\delta \int_{x_1, t_1}^{x_2, t_2} L dt = o, \qquad (2.1.13)$$

one may derive in the usual way the corresponding action principle (Jacobi's form of Maupertuis' principle)

$$\delta\int_{x_1}^{x_2} \left(\Sigma_{i=1}^{3} p_i \dot{x}_i \right) dt = 0, \qquad (2.1.14)$$

where now the path of integration is varied between two fixed points in space, x_1 and x_2, where the energy must be held constant on the varied paths, but where the times at the end points need not be held fixed. With the canonical momenta from eq.(2.1.10), this becomes

$$\delta\int_{x_1}^{x_2} n^2 v^2 \Omega [\, 1 - v^2 n^2 / c_0^2 \,]^{-1/2} dt = 0. \qquad (2.1.15)$$

We restrict the varied paths to those that satisfy the energy constraint by substituting the constant H for the right side of eq.(2.1.11) where this appears in eq.(2.1.15). Then, putting dt = dl/v, where dl $= |dr| = (\Sigma_{i=1}^{3} dx_i^{\,2})^{1/2}$, we obtain

$$\delta\int_{x_1}^{x_2} n^2 v\, dl = 0. \qquad (2.1.16)$$

This is a variational principle on which an analogy to geometrical optics or to classical mechanics can be constructed. In obtaining eq.(2.1.16) we have preferred, for the sake of directness, clarity and consistency of notation, to begin from the fundamental principle eq.(2.1.7). But eq.(2.1.16) may also be derived from versions of the three-dimensional variational principle for particle orbits in static metrics, for example, the forms first obtained by Weyl[26] and Levi-Civita.[27]

In eq.(2.1.16), $n^2 v$ is to be considered a function of position alone. (This condition is met in the fields of gravitation considered here. Only in cosmological applications, there occurs a cosmic time dependence in the potential. See chapter 5 for discussion). The path of integration is varied between the fixed end points x_1 and x_2, and the value of H is held constant during the variation. Thus, eq.(2.1.16) is of the same form as Fermat's principle, which forms a basis for geometrical optics, and Maupertuis's principle, which forms a basis for classical mechanics (as long as the force can be derived from a velocity-independent potential):

Relativistic gravitational mechanics	Geometrical optics (Fermat)	Classical mechanics (Maupertuis)
$\delta\int n^2 v\,dl = 0$	$\delta\int n\,dl = 0$	$\delta\int v\,dl = 0$

In the context of motion in a gravitational field, both

Fermat's principle and Maupertuis' principle are simply special cases of eq.(2.1.16). For the null geodesics, i.e., for the paths of light, the derivation given above must be slightly modified, to keep each step defined. But the final result is too well known to require detailed discussion here: in static metrics, light obeys Fermat's principle. That is, the path taken by light between two fixed points in space is one for which the coordinate time of travel is stationary. In the language of refractive index, this may be written as $\delta\int n\,dl = 0$. Since, for light, $v = c_0/n$, eq.(2.1.16) does reduce to the appropriate form. To obtain Maupertuis' principle (and hence particle motion in Newtonian gravity), note that in ordinary solar-system dynamics, we may put $n^2 \approx 1$. That is, in the Newtonian limit, n^2 may be treated as constant in the variational calculation and we obtain Maupertuis' principle as the classical limit of eq.(2.1.16).

2.2. Exact equations of motion of Newtonian form

Let the path of the particle be parametrized by a stepping parameter A. That is, at each point on the path, the three space coordinates r (and also the time t) are regarded as functions of A. We defer for the moment choosing A: we shall define A to get the simplest equations of motion. Thus we write eq.(2.1.16) in the form

$$\delta \int_{x_1}^{x_2} n^2 v \left|\frac{dr}{dA}\right| dA = 0, \tag{2.2.1}$$

where $\left|dr/dA\right| = \left[\sum_{i=1}^{3} (dx_i/dA)^2\right]^{1/2}$.

Let r(A) denote the true path. To obtain a varied path, we replace r(A) by r(A) + w(A), where w(A) is an arbitrary, infinitesimal vector function, subject to the condition that w = 0 when A is such that $r = x_1$ or x_2. That is, the variation must vanish at the end points. Now

$$\delta\int (n^2 v)\left|\frac{dr}{dA}\right| dA = \int\left[\delta(n^2 v)\right]\left|\frac{dr}{dA}\right| dA + \int (n^2 v)\delta\left|\frac{dr}{dA}\right| dA + \int (n^2 v)\delta\left|\frac{dr}{dA}\right|\delta dA \tag{2.2.2}$$

Calculating the two variation in the first term on the right-hand side of eq.(2.2.2), we have

$$\delta(n^2 v) = \nabla(n^2 v)w. \tag{2.2.3}$$

In calculating the variation in the second term of eq.(2.2.2) it is important to remember that the change to the varied path will, in general, also produce a change in A. Thus

$$\delta\left|\frac{dr}{dA}\right| = \left|\frac{dr + dw}{dA + \delta dA}\right| - \left|\frac{dr}{dA}\right|$$

$$= \frac{\frac{dr}{dA}\cdot\frac{dw}{dA}}{\left|\frac{dr}{dA}\right|} - \left|\frac{dr}{dA}\right| \frac{\delta dA}{dA} \qquad (2.2.4)$$

to first order in the variation. Substituting eq.(2.2.3) and eq. (2.2.4) into eq.(2.2.2), we find

$$\delta\int n^2 v\left|\frac{dr}{dA}\right| dA = \int \left[\left|\frac{dr}{dA}\right| \nabla(n^2 v)\cdot w + n^2 v\left|\frac{dr}{dA}\right|^{-1}\frac{dr}{dA}\cdot\frac{dw}{dA}\right] dA.$$

Note that the terms involving δdA have cancelled out. This was to be expected, since eq.(2.1.16) shows that the integral does not actually depend upon the range in A or, indeed, on what we select to use as parameter. Integrating the term involving dw/dA by parts, and using the fact that w must vanish at the endpoints, but is otherwise arbitrary, we arrive at the differential equation that must be satisfied by the particle trajectory:

$$\left|\frac{dr}{dA}\right|\nabla(n^2 v) - \frac{d}{dA}\left(n^2 v\left|\frac{dr}{dA}\right|^{-1}\frac{dr}{dA}\right) = 0. \qquad (2.2.5)$$

This differential equation plays the role of an equation of motion. Another way to obtain eq.(2.2.5) is to parametrize the path by one of the Cartesian coordinates (say z), rather than A, since the variation in z must vanish at x_1 and x_2. In this case, one writes

$$\delta\int n^2 v \frac{dl}{dz} dz = 0.$$

One may then simply write down the Euler conditions for the integral to be stationary, and then transform from z to A as independent variable. The result will be the same, that is eq.(2.2.5).

To give the equation of motion the simplest possible form, and to take advantage of the analogy to Newtonian mechanics, let us now define A by

$$\left|\frac{dr}{dA}\right| = n^2 v. \tag{2.2.6}$$

With this definition of A, the equation of motion, eq.(2.2.5), becomes

$$d^2r/dA^2 = \nabla(\tfrac{1}{2}n^4v^2). \tag{2.2.7}$$

Eq.(2.2.7) is the generalization of eq.(2.1.6) that was sought. The left-hand side of eq.(2.2.7) is of the form of an acceleration: it is the second derivative of the position vector with respect to the independent variable. The right-hand side of the equation is of the form of a force: $-\frac{1}{2}n^4v^2$ plays the role of a "potential energy function".[38] The analogue of the velocity is $|dr/dA|$ although it is a dimensionless quantity since A has the dimension of length. Thus the analogue of the kinetic energy is $\frac{1}{2}|dr/dA|^2$. The analogue of the total energy is the sum of the potential and the kinetic. But, by virtue of eq.(2.2.6) these two are guaranteed to sum to zero:

$$\frac{1}{2}|dr/dA|^2 - \frac{1}{2}n^4v^2 = 0. \qquad (2.2.8)$$

Thus the calculation of the paths of light and of massive particles in general relativity reduces to the zero-energy "F = ma" optics of ref.[3]. It is to be noted that the "conservation of energy" condition eq.(2.2.8) amounts to a restatement of the definition of eq.(2.2.6) of A.

The optical-mechanical analogy, embodied in eq.(2.2.7) and eq.(2.2.8), provides an exact treatment in Newtonian form of the motion of massive particles as well as light, in general relativity. The Newtonian form should be thought of as coming from "F = ma" optics (which is exact) and not from Newtonian mechanics (which is, of course, only approximate). Equations (2.2.7) and (2.2.8) allow one to handle the paths of light and of planets as if they existed in a flat three-dimensional space on which is superimposed a refractive medium. Other approaches to this goal are, of course, possible, but the treatment presented here has three advantages: simplicity, complete conformity to the equations of Newtonian mechanics, and a uniform treatment to both light and massive particles. This treatment has a reasonably high degree of generality and is applicable whenever the line element can be written in the form of (2.1.1).

The only difference between the treatment of light and that of particles resides in the choice of v(r), which forms a part of the effective potential energy $-n^4v^2/2$. For light,

$$v = c_0 n^{-1}. \qquad \text{(light)} \qquad (2.2.9)$$

But for massive particles, eq.(2.1.11) gives

$$v = c_0 n^{-1}[1 - c_0^4 \Omega^2/H^2]^{1/2}. \qquad \text{(particles)} \qquad (2.2.10)$$

In eq.(2.2.10), H is a constant parameter determined by the initial conditions, while n and Ω are functions of the spatial coordinates determined by the metric. Because the particle expression for v(r) contains the parameter H, the particle problem has an extra degree of freedom: we may specify the initial speed of the particle. This we can not do for light which is always c_0, the proper velocity through any observer. Thus, in general, more types of orbits exist for massive particles than for light in the same metric.

For a particle in empty space devoid of gravitational influences, $\Omega \approx 1$, $n \approx 1$, and eq.(2.1.11) becomes

$$H \approx c_o^2(1 - v^2/c_o^2)^{-1/2} = c_o^2\gamma. \qquad (2.2.11)$$

In the solar-system dynamics of the Schwarzschild metric, $v/c_o \ll 1$ and (See eqs. (4.2.10) and (4.2.14) below) $\Omega \approx 1 - m/r$ so eq. (2.1.11) becomes

$$H \approx c_o^2 + \frac{1}{2}v^2 - m/r. \qquad (2.2.12)$$

That is, in classical planetary orbits, H is approximately equal to $c_o^2 + E$, the rest-mass energy plus the classical kinetic and potential energy per unit mass.

2.3. On the parameter A in general relativity

In solving problems with eqs.(2.2.7) or (2.2.8), one may use without modification all the familiar methods of Newtonian mechanics. One simply thinks of A as if it were the time. Moreover, rather than beginning from eqs.(2.2.7) and (2.2.8) in every case, one may often simply write down an exact general relativistic formula by analogy to the corresponding classical formula. For the motion of both light and massive particles in spherically symmetric metrics, one may begin from any classical Newtonian formula describing the motion of particles in velocity-independent potentials. The correct general relativistic expressions will be obtained if one makes the following transcriptions in the classical formulas:

$$
\begin{aligned}
t &\to A, \\
U &\to -n^4 v^2/2, \qquad (2.3.1) \\
E &\to 0.
\end{aligned}
$$

The spatial coordinates (x,y,z), or (r,θ,ϕ), etc., transcribe as themselves. Of course, it must be kept in mind that the formulas written down in this way apply only in the isotropic coordinate system. After the equations governing the situation are obtained, or after they are solved, one may transform back to other coordinate systems, such as the standard or harmonic systems, if desired. A second point of vital importance is that the analogue of the classical energy E is always the number zero. Thus, the constant of the motion H plays no role in the analogy. H should rather be regarded as a parameter: the potential energy function depends upon H as well as the coordinates. A third point to be stressed is that the exact general relativistic formulas written down by analogy to the classical formulas will always apply with equal validity to both massive and massless particles.

For both light and particles, the stepping parameter A is defined by eq.(2.2.6). In many calculations, e.g., in finding the shape of an orbit, the stepping parameter is virtually eliminated. Eq.(2.2.6) will suffice for this purpose. In other situations (for example, in a radar echo-delay calculation), it may be necessary to have an explicit connection between A and t. Note that

$$\left|\frac{dr}{dA}\right| = \left|\frac{dr}{dt}\right|\frac{dt}{dA} = v\frac{dt}{dA}.$$

Substituting in eq.(2.2.6) *and restoring* c_0, *we get* (Note that the integrand of eq.(2.1.16) is actually multiplied by c_0^{-2} and the r.h.s. of eq.(2.2.6) is multiplied by c_0^{-1}. Those were suppressed before for neatness.),

$$dA = c_0 dt/n^2. \tag{2.3.2}$$

Thus the stepping parameter in general relativity is the same as that used in "F=ma" formulation of classical geometrical optics. As before, A is called the optical action because dA is proportional to c(r)dl, and thus is analogous to the action v(r)dl of classical mechanics.

The variational principle (2.1.16) has permitted us to extend the analogy to the geodesic problem for both light and particles in isotropic metrics. Our discussion of the optical analogy in general relativity has stressed Newtonian forms. But, corresponding to every formulation of either classical particle mechanics or of classical geometrical optics, there will be an analogous formulation of the geodesic problem in general relativity. Few of these classical models for the reformulation of the geodesic equations of motion lead to any special insight or simplification. The economy of expression and simplicity of the form embodied in eqs.(2.2.7) and (2.2.8) depend, not so much on the formulation of mechanics (or of geometrical optics) that is *chosen as model, as on the use of A rather than t as independent* variable.

The parameter A plays a key role in the optical- mechanical analogy in that it beautifully synthesizes two oppositely directed propositions: One is the Newtonian corpuscular hypothesis according to which light speeds up upon entering a denser medium because of the influence of some attractive force exerted by the medium. This attractive force has actually been experimentally verified long ago by Poynting[39] by an optical-mechanical device. After an extraordinary gap of nearly half a century, one finds more accurate experiments under different conditions by Jones[40], Jones and Richards[41] who verified an *increase* in pressure on a vane immersed in a fluid of index n, in perfect accordance with the Newtonian hypothesis. Even more sophisticated experiments with the aid of a laser beam on a liquid surface have been conducted by

Ashkin and Dziedzic[42] and they also confirmed the earlier results to an even greater accuracy. Moreover, the corpuscular hypothesis explained the laws of refraction of light. The other proposition stems from the Foucault experiment which shows that light propagates more slowly in water than in air.

One way to reconcile this increase in pressure with the diminishing speed of light in a denser medium is to postulate a varying mass of a photon. The consequences of this postulate have been investigated by Tangherlini.[17,43] The other way to reconcile the two is to use the parameter A. Let us consider eqs.(2.2.8) (with c_o restored) and (2.2.9) which give $|dr/dA| = n$. When interpreted *formally* as light progressing in optical action A, we see that the above equation mimics the behavior of a Newtonian corpuscule or a particle to the extent that its "velocity" = n, just like that of an electron in a medium: $v_{electron} \propto n$. In the real experiment with light progressing *in time* t, on the other hand, the equation $|dr/dt| = c_o/n$ corresponds to Foucault's experiment supporting Huyghen's principle. We shall see that the former equation, when A is eliminated, provides the exact orbit equation and the latter equation gives us the exact equation for the radar echo delay in general relativity. That is to say, the roles of the two equations are complementary in that one takes care of the spatial shape of the trajectory while the other keeps track of the real time elapsed during the motion along the trajectory. In another language, we may say that light displays its particle character (the "velocity" being proportional to n) in space whereas it displays wave chracter in time as revealed by its phase velocity c_o/n. The same reasonings apply also for material particles, where, in the expression for $n^2 v$, one has only to use the expression for v given in eq.(2.2.10). For yet another logical alternative resolving the crisis, see the works of Michels, Correl and Patterson.[44]

2.4.(a) Energy methods for the orbits

To illustrate energy methods, we shall calculate the path of a planet in the Schwarzschild field[37]. If we write out the "kinetic energy" in polar coordinates, the equation for "total energy" becomes

$$\text{"total energy"} = \frac{1}{2}\left(\frac{dr}{dA}\right)^2 - \frac{1}{2} n^4 v^2 = 0. \tag{2.4.1}$$

Writing out in full, we have

$$\left(\frac{dr}{dA}\right)^2 + r^2\left(\frac{d\phi}{dA}\right)^2 - n^4 v^2 = 0. \tag{2.4.2}$$

Now, the fact that the "potential energy" is a function of radial coordinate alone leads (just as in classical central-force motion) to a conserved "angular momentum":

$$h = r^2 d\phi/dA = \text{constant}. \tag{2.4.3}$$

(This may be easily seen by writing out the ϕ-component of eq.(2.2.7)). We use eq.(2.4.3) to pass over from A to ϕ as independent variable in eq.(2.4.2), obtaining

$$r^{-4}\left(\frac{d\phi}{dA}\right)^2 + r^{-2} - h^{-2}n^4v^2 = 0. \tag{2.4.4}$$

In the usual way, put

$$u = r^{-1}. \tag{2.4.5}$$

Then $dr/d\phi = - u^{-2}du/d\phi$ and eq.(2.4.4) may be written as

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - h^{-2}n^4v^2 = 0. \tag{2.4.6}$$

Note that eq.(2.4.6) still applies both to planets and to light.

To calculate the path of a planet, we invoke eq.(2.2.10) so that eq.(2.4.6) becomes

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - n^2c_0^2h^{-2}[1 - c_0^4\Omega^2/H^2] = 0. \qquad \text{(planet)} \tag{2.4.7}$$

To calculate the path of light, we use eq.(2.2.9). Thus eq.(2.4.6) becomes

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - n^2 c_o^2 h^{-2} = 0. \qquad \text{(light)} \qquad (2.4.8)$$

The only fact about the metric we have used so far is its spherical symmetry. (That is, $n^4 v^2$ is a function of r alone.) Let us now focus on the Schwarzschild problem: Φ, Ω and n are given by

$$\Phi(u) = \left(1 + \frac{mu}{2}\right)^{-2} \qquad (2.4.9)$$

$$\Omega(u) = \left(1 + \frac{mu}{2}\right)^{-1}\left(1 - \frac{mu}{2}\right) \qquad (2.4.10)$$

$$n(u) = \left(1 + \frac{mu}{2}\right)^{3}\left(1 - \frac{mu}{2}\right)^{-1}. \qquad (2.4.11)$$

To return to the original (non-isotropic) metric, we need to invert the coordinate transformation given by:

$$u = u'/\Phi, \qquad (2.4.12)$$

where $u' = 1/r'$ and r' is the standard radial coordinate. It is not hard to show that

$$du/du' = \Phi^{-1}\Omega^{-1} = n. \qquad (2.4.13)$$

Also, it will eventually be helpful to have explicit forms for Φ, Ω, and n as functions of u′ rather than u. A little algebra gives

$$\Phi = \frac{1}{4}\,[1 + (1 - 2mu')^{1/2}]^2 \tag{2.4.14}$$

$$\Omega = (1 - 2mu')^{1/2} \tag{2.4.15}$$

$$n = 4(1 - 2mu')^{-1/2}[1 + (1 - 2mu')^{1/2}]^{-2}. \tag{2.4.16}$$

(We do not need all of these relations for the present calculation, but it is convenient to group them in one place.) With the use of eq.(2.4.12) and eq.(2.4.13), eq.(2.4.7) becomes

$$\left(\frac{du'}{d\phi}\right)^2 + \Omega^2 u'^2 - c_o^2 h^{-2}[1 - c_o^4\Omega^2/H^2] = 0. \qquad \text{(planet)} \tag{2.4.17}$$

Only now do we need the explicit form of eq.(2.4.15) for Ω. Then, eq.(2.4.17) becomes

$$\left(\frac{du'}{d\phi}\right)^2 - 2mc_o^6h^{-2}H^{-2}u' + u'^2 - 2mu'^3 + c_o^6h^{-2}H^{-2} - c_o^2h^{-2} = 0.$$

Differentiating with respect to ϕ, we get

$$d^2u'/d\phi^2 + u' - 3mu'^2 - mc_o^6h^{-2}H^{-2} = 0. \tag{2.4.18}$$

Now, the constant term in eq.(2.4.18) is already of first order in

m. Thus, by virtue of eq.(2.2.12), we may safely put $H \approx c_o^2$, and we will ignore terms of order $(m/r)^2$ (weak field). The other constant of motion is $h = r^2(d\phi/dA)$. With the use of eq.(2.3.2) this may be written as

$$h = r^2 n^2 d\phi/dt$$
$$= n^2 h_o, \qquad (2.4.19)$$

where h_o is the mechanical angular momentum per unit mass, $h_o = r^2 d\phi/dt$, from classical mechanics. Because the constant term in eq.(2.4.18) is already of first order in m, we may replace n by unity. Thus, eq.(2.4.18) becomes

$$d^2u'/d\phi^2 + u' = mc_o^2/h_o^2 + 3mu'^2 \qquad \text{(planet)} \qquad (2.4.20)$$

This is the usual equation obtained in general relativity for a precessing elliptical orbit. The perihelion advance per revolution of the orbit as calculated in a standard way[45] is $6\pi m^2 c_o^2/h_o^2 = 6\pi G^2 M^2/(h_o c_o)^2$. No approximations have been made in deriving eq.(2.4.20), except, of course, in evaluating the constants of motion H and h.

The usual light-orbit equation for the Schwarzschild metric is obtained in a similar fashion, by using eqs. (2.4.12), (2.4.13) and (2.4.15) in eq.(2.4.8), with the result:

$$d^2u'/d\phi^2 + u' = 3mu'^2. \qquad \text{(light)} \qquad (2.4.21)$$

We could also have written this down immediately after having solved the particle case, simply by noting that for light we let H approach ∞ (see eq.(2.1.11). Thus, eq.(2.4.18) immediately reduces to eq.(2.4.21).

(b) Force methods for the orbits

In force methods, we shall calculate the gravitational deflection of star light, or of an ultra-relativistic ($v \approx c_o$) particle, passing near the Sun. These problems could, of course, be solved by beginning from eqs.(2.4.20) and (2.4.21). But, because we want to illustrate an "F = ma" calculation in general relativity, let us instead begin from eq.(2.2.7), which we write in the following form:

$$dp/dA = \nabla(\tfrac{1}{2} n^4 v^2), \qquad (2.4.22)$$

where the "momentum" p is defined by

$$p \equiv dr/dA.$$

Note that p is not the usual momentum defined in Newtonian mechanics but an optical analogue thereof. As the "potential energy" is a function of r alone, the "force" points in the radial direction, and the x- and y-components of the "equation of motion" are

$$\frac{dp_x}{dA} = \sin\theta \, \frac{d}{dr}\left(\frac{1}{2}n^4v^2\right) \qquad (2.4.23)$$

$$\frac{dp_y}{dA} = \cos\theta \, \frac{d}{dr}\left(\frac{1}{2}n^4v^2\right). \qquad (2.4.24)$$

Because the deflection will be very small, p ($\equiv$ dr/dA) will point almost entirely in the x-direction during the whole course of the motion. That is $|p| \approx p_x$. Thus the general condition expressed by eq.(2.2.6), namely that $|p| = n^2v$, may be written $p_x \approx n^2v$, and the x-component of the motion becomes

$$\frac{d}{dA}(n^2v) = \sin\theta \, \frac{d}{dr}\left(\frac{1}{2}n^4v^2\right). \qquad (2.4.25)$$

We may use this relation to eliminate A from y-component of the "equation of motion". This is just like eliminating t between the x- and y-components of the equation of motion in a Newtonian projectile-motion situation. Dividing eq.(2.4.24) by eq.(2.4.25), we get

$$dp_y = \cot\theta \; d(n^2v),$$

that is,

$$dp_y = \cot\theta \, \frac{d}{dr}(n^2 v)dr. \qquad (2.4.26)$$

Since the deflection is assumed very small,

$$\cot\theta \approx R_o(r^2 - R_o^{\,2})^{-1/2} \qquad (2.4.27)$$

where R_o is the closest radial distance by which the ray passes the gravitating object. With this substitution in eq.(2.4.26), we may calculate the change in the y-"momentum" as r goes from ∞ to R_o and out to ∞ again:

$$\Delta p_y = 2R_o \int_{-\infty}^{R_o} (r^2 - R_o^2)^{-1/2} \, \frac{d}{dr}(n^2 v)dr. \qquad (2.4.28)$$

This expression is valid both for light and for an ultra-relativistic ($v \approx c_o$) particle, since the only approximation we have made so far is that the deflection will be small. To evaluate the integral for light, we note from eq.(2.2.9) that $n^2 v = nc_o$. Furthermore, in the weak field of Sun, eq.(2.4.11) may be expanded with the result $n \approx 1 + 2m/r$. Thus, for light in the weak field of the Sun,

$$\frac{d}{dr}(n^2 v) = \frac{-2mc_o}{r^2}. \qquad \text{(light)} \qquad (2.4.29)$$

Substituting eq.(2.4.29) into eq.(2.4.28) and carrying out the integration, we obtain

$$\Delta p_y = -4mc_o/R_o. \qquad \text{(light)} \qquad (2.4.30)$$

At $r = \infty$, the x-"momentum" is $p_x \approx c_o n\,(r = \infty) = c_o$. Thus, the deflection angle, in radian measure, is

$$\Delta p_y/p_x = -4m/R_o, \qquad \text{(light)} \qquad (2.4.31)$$

the familiar result predicted by Einstein. (Note that as $r \to \infty$, $r \to r'$, and we are working to first order in m/R_o, so in this case there is no need to transform back to the original coordinates.)

To calculate the deflection of an ultra-relativistic particle, note, from eq.(2.2.10), that

$$n^2 v = c_o n[1 - c_o^4 \Omega^2/H^2]^{1/2}, \qquad \text{(particle)} \qquad (2.4.32)$$

Also, by eq.(2.2.11)

$$H \approx c_o^2(1 - v_o^2/c_o^2)^{-1/2} = c_o^2\gamma \qquad (2.4.33)$$

where v_o is the particle's speed when it is (effectively) *infinitely far from the Sun. Substituting* eqs.(2.4.33), (2.4.10) and (2.4.11) into eq.(2.4.32), and expanding for a weak field (to first order in m/r), we obtain

$$n^2 v \approx c_o(1 + 2m/r)[1 - \gamma^{-2}(1 - 2m/r)]^{1/2} \qquad (2.4.34)$$

and thus, again keeping terms only to first order in m,

$$\frac{d}{dr}(n^2 v) = \frac{-mc_o}{r^2}\cdot\frac{c_o}{v_o}(1 + v_o^2/c_o^2) \qquad \text{(particle)} \qquad (2.4.35)$$

which differs from eq.(2.4.29) only in the coefficient multiplying r^{-2}. Integration of *eq.(2.4.28) then gives*

$$\Delta p_y = \frac{-2mc_o}{R_o}\cdot\frac{c_o}{v_o}(1 + v_o^2/c_o^2). \qquad \text{(particle)} \qquad (2.4.36)$$

Now, evaluating p_x at infinity gives $p_x \approx n^2 v \approx v_o$. The angular deflection of the ultra-relativistic particle is therefore[46]

$$\Delta p_y/p_x = -2m/R_o(1 + c_o^2/v_o^2). \qquad \text{(particle)} \qquad (2.4.37)$$

For $v_o < c_o$, the deflection of the particle *is greater* than the deflection of light. But as $v_o \to c_o$, the deflection of the particle becomes equal to that of light. This is indeed an interesting result.

It is sometimes said that Newtonian mechanics implies a deflection of starlight by the Sun, since the deflection of a particle infinitely less massive than the Sun does not depend on mass of incoming particle, and we could think of the photon as a Newtonian particle in the limit of very small mass. We then need only put $v_o = c_o$ and calculate the deflection using Newtonian principles. As is well known, the result is just one half of that given in eq.(2.4.31)[47]. The present calculations show that in general relativity also, the deflections of light and of a material particle coming in at approximately the speed of light are the *same* (both being twice the Newtonian result). The main purpose of these calculations, however, is to show how easily force methods can be applied using eq.(2.2.7), and how easy it is to handle light and particles with exactly the same formalism.

CHAPTER 3

A LITTLE HISTORY

We saw that eq. (2.2.8) and, indeed most of this thesis, is based on the variational principle, eq.(2.1.16), which constitutes the basis of the analogy to geometrical optics and to classical mechanics. To clarify the physical significance of eq. (2.1.16), it will help to place this variational principle in the historical context of the optical-mechanical analogy. This is done in *Sec.3.1. The other section (Sec.3.2) displays the relationship* between the "F=ma" optics and Hamilton's procedure.

3.1. Historical background: a reapprisal of events

The history of the relationship between Fermat's principle and Maupertuis' principle spans the period from the seventeenth to the twentieth century and includes the two-and-a-half century debate over the nature of light and the rise of quantum mechanics. It is a story rich in irony and reversals of fortune. The story begins with Descartes' derivation of the law of refraction from a *particle model for light.*[48] *Descartes likened a particle of* light to a tennis ball that suffers an impulsive blow, in the direction of the normal to the boundary between two media as it passes over that boundary. The component of the velocity parallel to the boundary is therefore the same in both media, while the normal component changes. The constancy of the ratio of the sines of the angles of incidence and refraction immediately follows. Because light bends closer to the normal while passing from air to water, it follows that light must travel more rapidly in the water. Descartes was immediately imbroiled in controversy. For, in about same time that Descartes was applying his mechanical philosophy to light, the constancy of the ratio of the sines was discovered experimentally by Snell. Friends of Snell accused Descartes of plagiarism.

Twenty five years later, Pierre de Fermat succeeded in connecting the law of refraction with an extremum principle. Fermat proposed, in private correspondence, that light travels from a point in one medium to a point in another by the path requiring the least time.[49] From this Fermat was able to obtain the constancy of the ratio of the sines of the angles of incidence and refraction. but if light obeyed Fermat's principle, it must travel more rapidly in the air. Fermat was attacked by the followers of Descartes for advocating a principle which implies that light travels more rapidly in air than in water when the master had taught that just the opposite was true. Worse, Fermat appeared to be reintroducing teleological arguments and final causes into natural philosophy[50]. Fermat replied bitterly that he would leave to Mr. Descartes the glory of having explained the refraction of light and would content himself with offering an abstract mathematical proposition, without asserting that it applied to light.

In 1690, Christiaan Huygens published his *Treatise on Light*, in which he developed a wave theory of light, and applied it to rectilinear propagation, reflection, refraction and double refraction. Huygens also proved that, if light were a wave, it would obey Fermat's principle of least time. Thus, at the close of the seventeenth century, Fermat's principle was attached to the wave theory.[51]

Newton, in his *Opticks* of 1704, advocated a corpuscular theory of light--although his corpuscles had to be endowed with a vibratory character to explain the inflections of light (or, as we would say, diffraction and interference phenomena). The corpuscularity of light was practically the only proposition of physics on which Newtonians and Cartesians could agree. It is not surprising that, under the combined influence of Newton and Descartes, the wave theory practically disappeared. And Fermat's principle of least time fell into disrepute: hardly anyone believed in it. It is not to be found in most of the eighteenth-century textbooks, for example. There were in the eighteenth century always a few advocates of the wave theory, among whom Euler was the most prominent. But the overwhelming majority of physical thinkers believed that light is a stream of particles and that, as both Descartes and Newton had shown, it must therefore travel more quickly in water than in air.

In 1744, Maupertuis announced a minimization principle intended to encompass both mechanics and optics: the principle of least action.[52] Maupertuis called *action* the product of mass and distance. His definition of action was vague and was applied inconsistently, but he succeeded in deducing from his principle the rules governing elastic and inelastic collisions of particles. More to the point for our story, he also applied his principle to light and managed to derive the law of refraction. Maupertuis' principle was consistent with the view of light as a particle and the opinion of Descartes and Newton that the speed of light was greater in denser media. With this fact of nature re-established, Maupertuis exclaimed, "the whole edifice that Fermat had constructed is destroyed: light, when it traverses different media, goes neither by the shortest route, nor by the route of least time .... "[53]. Maupertuis also pointed out that for the cases of reflection and straight-line propagation --cases in which the speed of light does not change--the path of least time is the same as the path of least action, which, according to Maupertuis, explained how Fermat had gone wrong.

Thus, by the middle of the eighteenth century, there were two competing variational principles, least time and least action, each of which could serve as a basis for geometrical optics. One of these was associated with the wave theory and one with the particle theory. The *shape* of a ray in a given situation was predicted to be exactly the same by the two theories. The theories differed only in the progress in time of light along the ray; but, of course, this could not be measured.

The story told so far is the prehistory of the optical-mechanical analogy. A major issue at stake was the nature of light. Proponents of the particle theory were not making an analogy between geometrical optics and mechanics but rather *claiming* that they were one and the same. At the close of the eighteenth century, majority opinion held that one variational principle--that of Maupertuis--covered both. The minority school of wave theorists recognized two principles--those of Fermat and Maupertuis--which applied to different domains of physics.

The principle of least action *as it applies to mechanics* was elaborated and clarified by Euler, and especially by Lagrange. Lagrange formulated the least action principle more precisely, developed a calculus of variations for exploring its consequences and demonstrated that it could serve as a foundation for mechanics, a welcome alternative to the philosophically disturbing *force-based physics of Newton. Among other things, Lagrange showed* that principle of least action and conservation of energy were together equivalent to Newton's law of motion ($F = ma$)[54]. This also served to strip Maupertuis' principle of its metaphysical significance. For Maupertuis, the fact that both light and particles follow the paths of least action had been a sure sign of the wisdom of the Creator. For Lagrange, the principle of least action was just an alternative foundation for mechanics.

The principle of "least action" (really, least time) *as it applies to optics* was explored by Laplace,[55] but most notably by William Rowan Hamilton, who was largely responsible for the creation of the optical-mechanical analogy.[56] Hamilton's early work in mathematical physics was devoted to geometrical optics. When Hamilton began this work, in the 1820's, French physicists *were already abondoning the particle theory of light, as a result* of the success of Fresnel's mathematical wave theory. In England and Ireland (Hamilton's native land), where Newton's influence was stronger, the conservation required another decade or so. Thus, in his first papers on systems of rays (1824-1830), Hamilton always called the integral $\int n dl$ the *action*, in analogy with the action integral $\int v\, dl$ of mechanics. The analogy would be most *straightforward for the particle theory, in which n is directly* proportional to v. However, Hamilton did not necessarily subscribe to the particle theory. In his first paper, he refrained from stating a position on the nature of light and wrote that he used the term *action*, "intending only to express a remarkable analogy, and not assuming any hypothesis about the nature of light".[57] For the integrand of the "action" integral (essentially the index of refraction), Hamilton sometimes used m, sometimes μ, and sometimes v. The use of v for the refractive index might seem to imply acceptance of the particle theory, but this was not the case.

Indeed, by the time (1832) of his "Third Supplement", Hamilton had definitely adopted the wave theory. This is apparent in the softening of his vocabulary; for he now referred to a "law of least action, or of swiftest propagation", i.e., he began to adopt the language of the wave theory and to use it side-by-side with, and as an alternative to, the language of the particle theory.[58] The integrand v he called the *medium-function* and characterized it as a "a molecular velocity or an undulatory slowness". In the "Third Supplement", moreover, Hamilton applied his methods directly to the wave theory of Fresnel. The most spectacular result was Hamilton's theoretical prediction of the phenomenon of conical refraction, which was observed shortly afterward by Lloyd.

The essence of Hamilton's geometrical optics is his *characteristic function*

$$V(a,\ b,\ c,\ x,\ y,\ z) \equiv \int_{a,\ b,\ c}^{x,\ y,z} n\,dl \qquad (3.1.1)$$

which is nothing other than Hamilton's "action" integral or, (as we would say) the optical path length between point (a, b, c) and point (x, y, z) on the same ray. Hamilton showed that V satisfies the partial differential equation

$$\left|\frac{\partial V}{\partial x}\right|^2 + \left|\frac{\partial V}{\partial y}\right|^2 + \left|\frac{\partial V}{\partial z}\right|^2 = n^2(x,\ y,\ z). \qquad (3.1.2)$$

If this differential equation can be solved, knowledge of V constitutes a complete knowledge of the optical system. In particular, the direction of the ray at point can be found by differentiation of V. Hamilton's characteristic function V is today usually called the *eikonal*, a term introduced by H. Bruns in 1895. eq.(3.1.2) is called the *eikonal equation*, and is one of the fundamental equations of modern geometrical optics.[59] The surfaces V = constant are surfaces of equal travel time--the wavefronts.

In the same way, a central feature of Hamilton's mechanics is the characteristic function

$$V(a, b, c, x, y, z) \equiv \int_{a,\, b,\, c}^{x,\, y,\, z} v\, dl, \tag{3.1.3}$$

which is the action for a particle moving between points (a, b, c) and (x, y, z). Hamilton showed that, for a unit mass, V satisfies the partial differential equation

$$\left|\frac{\partial V}{\partial x}\right|^2 + \left|\frac{\partial V}{\partial y}\right|^2 + \left|\frac{\partial V}{\partial z}\right|^2 = v^2(x, y, z) = 2(E - U) \tag{3.1.4}$$

where U is a potential energy function and E is the energy, which is constant on a given trajectory. If this differential equation can be solved, the characteristic function V constitutes a complete description of the motion of the particle. For example, the velocity components of the particle at (x, y, z) can be found by differentiation.[60] Thus, in Hamilton's formulation of mechanics, it is the characteristic function that embodies the optical-mechanical analogy. In mechanics, the surfaces V = constant are surfaces of constant action, analogous to the surfaces of constant phase or travel time in optics. The particle trajectories correspond to the optical rays. Hamilton's paper on the optical-mechanical analogy, "On a General Method of Expressing the Paths of Light, and of Planets, by the Coefficients of a Characteristic Function", was published, obscurely, in the *Dublin University Review* for October, 1833.[61]

Hamilton has struggled long and hard with geometrical optics. The development of general theory of rays occupied him from 1824 to 1832. By contrast the works on mechanics came quickly and easily in the years 1833-34, for it was largely but an extension of the optical theory. It is one of the many ironies of this story that there exists today a Hamiltonian system of dynamics because Hamilton happened to develop his geometrical optics at a time when the debate over particle and wave theories of light had not yet been resolved.[62]

Hamilton's papers on dynamics won him lasting recognition. His methods in dynamics was further developed by Jacobi and remained a part of the main stream of nineteenth-century theoretical physics. In contrast, Hamilton's optical papers were read and understood by very few and soon dropped almost completely out of sight. There were several reasons for this. Hamilton's Irish national sentiment had led him to publish his optical papers in an obscure and little-circulated journal, the *Transactions* of the Royal Irish Academy, even though his uncle had warned him that this would be little better than committing them to a tomb.[63] (The papers on mechanics appeared in the *Philosophical Transactions* of the Royal Society of London). Perhaps an even greater obstacle was Hamilton's difficult style. The papers are very densely written. Moreover, Hamilton strove constantly for higher abstraction and greater generality, and rarely concerned himself with practical applications. He created a general system of geometrical optics, but failed to teach his prospective readers how to use it.

It is a further irony of this story that Bruns, the modern rediscoverer of Hamilton's method in optics, had seen Hamilton's optical papers. Bruns essentially worked backward from Hamilton's theory of characteristic function in mechanics to reach an optical analogy--the reverse of the route that Hamilton had traveled some seventy years before.[64]

By the 1830's, the wave theory had triumphed. Fermat was vindicated at last, and one could confidently express Fermat's principle in the form

$$\delta\int dl/v = 0. \qquad \text{(optics)} \qquad (3.1.5)$$

Now there were two well-established variational principles, eq.(3.1.5) for light and Maupertuis' principle for particles:

$$\delta\int v\,dl = 0. \qquad \text{(mechanics)} \qquad (3.1.6)$$

The fact that they were so similar in form was only a curiosity, and Hamilton's optical-mechanical analogy was regarded as an elaborate and sterile oddity, until the development of quantum theory in the first few decades of our own century.

The action integral of eq.(3.1.6) played a major role in the old quantum theory of Bohr and Sommerfeld. But it was Louis de Broglie who first brought the principles of Fermat and Maupertuis together and showed that they were one. de Broglie argued that atoms displayed effects involving integer numbers (as in the Balmer formula for the frequencies of light emitted by the hydrogen atom). Practically the only place in physics where such numbers turn up is in wave phenomena. But the orbiting electrons are also particles. Thus, de Broglie began by trying to make the electron in the atom simultaneously obey both the principle of Fermat and the principle of Maupertuis. The result of this attempt was the de Broglie relation

$$mv = \hbar k \tag{3.1.7}$$

where k is the wave number of the electron wave, $\hbar$ is Planck's constant divided by 2π, m is the mass of the electron (now considered as a particle), and v is the particle's speed (which de Broglie was also able to identify with the group velocity of the waves).[65] Working backwards from the de Broglie relation, it is easy to see that two variational principles (as applied to electrons) are really one and the same. Let us express Fermat's principle in terms of the wave number k (rather than n or the phase velocity v).

$$\delta\int k\,dl = 0. \qquad \text{(optics)} \qquad (3.1.8)$$

Eq. (3.1.8) expresses the condition that, in the geometrical optics limit, small variations in the path do not affect the optical path length. This is why the ray is where it is: only along the actual ray can neighboring bundles of virtual paths interfere constructively. Eq. (3.1.8) must apply, in the geometrical optics limit, to all wave disturbances that obey the superposition principle. To apply eq.(3.1.8), which is very general, to the particular case of matter waves, we use the de Broglie relation, eq. (3.1.7). Eq. (3.1.8) immediately becomes Maupertuis' principle (eq.(3.1.6)). Thus, classical mechanics can be understood as the geometrical-optics limit of wave-mechanics: the particle trajectories are the rays of the matter waves. Maupertuis' principle therefore amounts to a special case of Fermat's principle--the special case involving the matter waves of quantum mechanics. In the context of general relativity, this result may assume deep significance.(See the epilogue at the end).

3.2. On the relation between "F=ma" optics and Hamilton's formulation of the optical-mechanical analogy

We can, if we wish, formulate the problem of the motion of photons and massive particles in the Schwarzschild field in the same way as Hamilton formulated the motion of classical particles and the paths of light in refractive media. In analogy to eq.(3.1.1) or (3.1.3), we define a characteristic function

$$V(y,\ x) \equiv \int_y^x n^2 v\,dl \qquad (3.2.1)$$

where y and x are two points in space. V will then satisfy the partial differential equation

$$\sum_i \left(\frac{\partial V}{\partial x}\right)_i = n^4 v^2. \qquad (3.2.2)$$

If eq. (3.2.2) can be solved, V will contain all the information we require about the trajectories of photons and particles in the Schwarzschild field. This would lead us into a realm of high abstraction, with no increase in calculating power over eq.(2.2.7).

But it is worth a moment's reflection to see that eq.(3.2.2) is actually equivalent to eq. (2.2.7). We may write

$$dl = \sum_i \cos\theta_i dx_i \qquad (3.2.3)$$

where the $\cos\theta_i$ are the direction cosines of dl, i.e., the cosines of the angles dl makes with the Cartesian axes. If we now parameterize the path by a stepping parameter A (yet to be defined), then

$$\cos\theta_i = (dx_i/dA)/(dl/dA). \tag{3.2.4}$$

Then V can be written as

$$V = \sum_i \int n^2 v\left(\frac{dl}{dA}\right)^{-1} \frac{dx_i}{dA} dx_i \tag{3.2.5}$$

from which it follows that

$$\frac{\partial V}{\partial x_i} = n^2 v\left(\frac{dl}{dA}\right)^{-1} \frac{dx_i}{dA}. \tag{3.2.6}$$

Substituting eq.(3.2.5) into eq. (3.2.1), then differentiating both sides of eq.(3.2.1) with respect to A gives

$$\sum_i \frac{d}{dA}\left[n^2 v\left(\frac{dl}{dA}\right)^{-1} \frac{dx_i}{dA}\right]^2 = \sum_i \frac{\partial}{\partial x_i}(n^4 v^2)\ \frac{dx_i}{dA}. \tag{3.2.7}$$

The left-hand side of eq. (3.2.7) cries out for us to define A by

$$\frac{dl}{dA} = n^2 v \tag{3.2.8}$$

just as in eq.(2.2.6). For then eq.(3.2.7) becomes

$$2\sum_i \frac{d^2 x_i}{dA^2}\frac{dx_i}{dA} = \sum_i \frac{\partial}{\partial x_i}(n^4 v^2)\frac{dx_i}{dA}. \tag{3.2.9}$$

The direction of the "velocity" dx/dA at a given point can be chosen at will, in the form of initial conditions. Thus, for eq. (3.2.9) to hold in all circumstances, the coefficients of the "velocity" components dx_i/dA must be equal term-by-term:

$$d^2x_i/dA^2 = \partial(1/2\ n^4v^2)/\partial x_i$$

or, in the vector form, eq.(2.2.7) precisely.

Thus, eq. (2.1.6) (and its generalization, eq.(2.2.7)) can be regarded as equivalent to Hamilton's formulation of the optical-mechanical analogy: we need only make the right choice for the stepping parameter. Again, it is to be noted both eq. (3.2.2) and eq. (2.2.7), written in the generalized forms required for motion in the Schwarzschild metric, include classical mechanics (for which $n \to 1$ and $A \to t$) and classical geometrical optics (for which $v \to c_o/n$) as special cases.

Since $dx_i/dA = \partial V/\partial x_i$, the generalized version of Hamilton's differential equation (eq.(3.2.2) becomes, for us, a first integral of the motion (eq.(2.2.8), equivalent to "conservation of energy". Hamilton, of course, would have regarded $dx_i/dA = \partial V/\partial x_i$ as the prescription for extracting answers, in the form of velocity components (dx_i/dA) by differentiation of the solution V, to the differential equation!

CHAPTER 4

APPLICATION TO REISSNER-NORDSTRÖM TYPE METRICS

In this chapter, we shall illustrate the application of the above approach to the orbit equations in Reissner-Nordström type metrics. In particular, we shall assess the effect of the parameter β [see eq.(4.1.1) below] on three celebrated tests of general relativity. Our calculations supplement those recently performed by Halilsoy[66]. An error in ref.[66] is also corrected. In Sec.4.1, we deduce the equivalent gravitational index by casting the Reissner-Nordström type metric into isotropic form. Sec.4.2 shows how different choices of parameters in the metric give rise to different physical situations. We, then, write out, in Sec.4.3, the general form of the orbit equation. In subsequent sections (Sec.4.4, 4.5 and 4.6) we calculate the general relativistic effects such as the bending of light rays, precession of planetary apsides and the radar echo delay.

4.1. The refractive index

A number of metrics of physical interest assume the following form in standard coordinates (t, r', θ, ϕ):

$$ds^2 = c_o^2\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]dt^2 - \left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1}dr'^2$$

$$- r'^2 d\theta^2 - r'^2 \sin^2\theta d\phi^2, \qquad (4.1.1)$$

where

$$m = MG/c_o^2, \qquad (4.1.2)$$

M is the mass of the central gravitating body, G is the gravitation constant, and β is another parameter. We wish to write the line element in terms of isotropic coordinates (t, r, θ, ϕ). We will indicate briefly how to effect the transformation, using a systematic technique[67]. The idea is to express the spatial part as $-\Phi^{-2}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2]$, where $\Phi(r)$ is yet to be determined. Equating the angular and the radial parts of the two line elements, we have

$$r'^2 = \Phi^{-2} r^2 \qquad (4.1.3)$$

and

$$\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2 = \Phi^{-2} dr^2. \qquad (4.1.4)$$

If we divide eq.(4.1.4) by eq.(4.1.3) to eliminate Φ, then integrate and use the condition that at large radial distances r and r′ must be asymptotically equal, we obtain

$$2r = (r' - m) + (r'^2 - 2mr' + \beta)^{1/2}. \qquad (4.1.5)$$

The inverse transformation is

$$r' = r + m + (m^2 - \beta)/4r. \qquad (4.1.6)$$

Using these transformations, the line element (4.1.1) can be expressed in the form of (2.1.1), with

$$\Omega^2(r) = [1 - (m^2 - \beta)/4r^2]^2[1 + m/r + (m^2 - \beta)/4r^2]^{-2} \qquad (4.1.7)$$

$$\Phi^{-2}(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2. \qquad (4.1.8)$$

The effective refractive index n(r) is

$$n(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2[1 - (m^2 - \beta)/4r^2]^{-1}. \qquad (4.1.9)$$

It is also helpful to have expressions for Φ, Ω and n in terms of the standard radial coordinate r′.

Let $u \equiv 1/r$ and $u' \equiv 1/r'$. Then it is easy to show that

$$\Phi^2(u') = \frac{1}{4}[1 - mu' + (1 - 2mu' + \beta u'^2)^{1/2}]^2 \qquad (4.1.10)$$

$$\Omega^2(u') = 1 - 2mu' + \beta u'^2. \qquad (4.1.11)$$

And of course

$$n(u') = \Phi^{-1}(u')\Omega^{-1}(u'). \qquad (4.1.12)$$

In transforming coordinates, it is often helpful to use

$$du = ndu' \qquad \text{or} \qquad dr = \Phi\Omega^{-1}dr' \qquad (4.1.13)$$

together with

$$u = \Phi^{-1}u' \qquad \text{or} \qquad r = \Phi r'. \qquad (4.1.14)$$

The singularities of $n(r)$, or, equivalently, the horizons of the spacetime, occur at $r_s = \frac{m}{2}(1 - \beta/m^2)^{1/2}$, provided that $\beta/m^2 \leq 1$. Therefore, the expression for $n(r)$ is valid in the region $r > r_s$. If $\beta = m^2$, $r_s = 0$; i.e., the event horizon shrinks to zero size. In this case, $n(r) = 1 + m/r$ and is regular everywhere for $r > 0$. If $\beta > m^2$, the function $n(r)$ is not singular anywhere, since r_s becomes imaginary. Let us now examine some special cases of the metric (4.1.1).

4.2. Some special cases

Schwarzschild Exterior Metric

The Schwarzschild exterior metric applies to the spacetime around an electrically neutral, static, spherical mass M. In this case, eqs.(4.1.1) through (4.1.14) apply with

$$\beta = 0. \tag{4.2.1}$$

Reissner-Nordström Metric

The gravitational field due to an electrically charged, static spherical mass M is given by the Reissner-Nordström solution of Einstein's field equations. In this case, eqs.(4.1.1)-(4.1.14) apply with

$$\beta = GQ^2/c_0^4, \tag{4.2.2}$$

where Q is the charge on the central body.

Bertotti-Robinson Metric

This metric describes a universe filled with electromagnetic radiation of uniform density and uniformly random direction[68]. In this case, eqs.(4.1.1)-(4.1.14) apply with

$$\beta = m^2, \tag{4.2.3}$$

where m is now a nonphysical effective point mass. The BR solution may also be obtained as a special case of the metric obtained recently by Halilsoy.

Halilsoy Metric

The Halilsoy metric describes spacetime around a static, uncharged, spherically symmetric mass M which is embedded in an externally created electromagnetic field.[66,69] Once again, eqs.(4.1.1)-(4.1.14) apply with

$$\beta = q^2 m^2 \tag{4.2.4}$$

where $0 \leq q \leq 1$, and where q represents the measure of the external electromagnetic field.

Soleng Metric

The Soleng metric represents the gravitational field due to a central mass M surrounded by a field having a traceless energy-momentum tensor $T^{\mu}{}_{\nu}$ = f(r) diag[1, 1, -1, -1]. Recently, such a $T^{\mu}{}_{\nu}$ has been interpreted as the energy-momentum tensor associated with an anisotropic vacuum.[70-72] Here eqs.(4.1.1)-(4.1.14) apply with

$$\beta = 6\delta m^2, \tag{4.2.5}$$

where δ is the Soleng parameter, which determines the effective energy density of the anisotropic vacuum.

4.3. Central force motion

In the cases under consideration, n, v, Ω, and Φ are functions of the radial coordinate alone. The orbit (whether of light or of a massive particle) lies in a plane containing the force center and there is a constant of the motion analogous to the angular momentum. Let ϕ be measured in the plane of the motion $\theta = \pi/2$. Then, from eq.(2.2.7)

$$r^2 d\phi/dA \equiv h = \text{constant}. \qquad (4.3.1)$$

Note that h is related to the classical-mechanical angular momentum per unit mass $h_0 (\equiv r^2 d\phi/dt)$ by

$$h = n^2 h_0. \qquad (4.3.2)$$

Now we may easily obtain general-relativistic analogues of the standard formulas of classical central-force motion. In eq.(2.2.8) which is the analogue of the classical conservation of energy condition, we may write out $|dr/dA|^2$ in plane-polar coordinates, then eliminate A by means of eq.(4.3.1). The orbit shape $\phi(r)$ is thereby reduced to an integration:

$$\phi = h \int^{r} r^{-2}[n^4 v^2 - h^2/r^2]^{-1/2} dr. \qquad (4.3.3)$$

The classical limit of eq.(4.3.3) is the familiar equation

$$\phi = h_o \int^{r} r^{-2}[2(E - U) - h_o^2/r^2]^{-1/2} dr.$$

Note that we could have immediately written down eq.(4.3.3), which is an exact general-relativistic expression, on the model of the classical expression, simply by using the transcriptions (2.3.1) together with $h_o \to h$ (which follows from $t \to A$). Moreover, eq.(4.3.3) applies both to light and to massive particles. To apply eq.(4.3.3) to either massless or massive particles, we need only insert the appropriate specific form (2.2.9) or (2.2.10) for v(r).

Another form of the orbit equation is frequently useful. Let u = *1/r*. *Then, in analogy to the classical formula*

$$\frac{d^2u}{d\phi^2} + u = -h_o^{-2} \frac{dU}{du},$$

we must have in general relativity

$$\frac{d^2u}{d\phi^2} + u = -h^{-2} \frac{d}{du}(n^4 v^2/2), \qquad (4.3.4)$$

which, again, applies to both particles and photons. Figures 1-14 display the form of the potential $n^4v^2/2$ for the Reissner-Nordström case for different values of m and β. We have written down eq.(4.3.4) simply by analogy to classical mechanics. But it may also be obtained by beginning with the radial component of eq.(2.2.1) and eliminating A.

A third useful form of the orbit equation is

$$h^2[(du/d\phi)^2 + u^2] - n^4v^2 = 0. \tag{4.3.5}$$

This equation is now in a familiar form. Specific choices of n and v *will lead to corresponding solutions by standard methods.*

4.4. Bending of light rays

We may begin from eq.(4.3.5). Inserting eq.(2.2.10) for v(r) in the second term, we obtain

$$\left[\frac{du}{d\phi}\right]^2 + u^2 - (c_o/h)^2 n^2 \left[1 - c_o^4 H^{-2} \Omega^2\right]. \tag{4.4.1}$$

This differential equation is exact, but it may not appear very familiar. We may transform back to the original (standard) coordinates by using eqs.(4.1.13) and (4.1.14) in the first term of eq.(4.4.1), with the result.

$$\left[\frac{du}{d\phi}\right]^2 + u'^2\Omega^2 - (c_o/h)^2[1 - c_o^4H^{-2}\Omega^2] = 0. \qquad (4.4.2)$$

Substituting (4.1.11) for $\Omega^2(u')$, then differentiating with respect to ϕ, we obtain

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_o^6}{h^2H^2} = -\frac{\beta c_o^6}{h^2H^2}u' + 3mu'^2 - 2\beta u'^3. \qquad (4.4.3)$$

The equation for the shape of a light ray results from letting $H \longrightarrow \infty$ in eq.(4.4.3)

$$\frac{d^2u'}{d\phi^2} + u' = 3mu'^2 - 2\beta u'^3. \qquad (4.4.4)$$

This equation may be solved by the usual perturbative method. If the right side of eq.(4.4.4) is temporarily put equal zero, we obtain the straight-line solution

$$u' = \frac{\sin\phi}{R},$$

where R is the distance of closest approach to the origin. Substituting the zeroth-order solution $\sin\phi/R$ for u' on the right side of eq.(4.4.4) and solving the resulting differential equation for $u'(\phi)$, we obtain the solution of first order in m and β:

$$u' = \frac{\sin\phi}{R} + \frac{3m}{2R^2}\left[1 + \frac{1}{3}\cos2\phi\right] + \frac{3\beta}{4R^3}\phi\cos\phi - \frac{\beta}{16R^3}\sin3\phi. \quad (4.4.5)$$

(This differs sightly from Halilsoy's solution, which is missing the last term.) As $r' \to \infty$, $u' \to 0$, and $\phi \to \phi_\infty$, which may be assumed small. Thus eq.(4.4.5) reduces to

$$0 = \frac{\phi_\infty}{R} + \frac{2m}{R^2} + \frac{9\beta\phi_\infty}{16R^3}.$$

The total deflection is $\Delta\phi_\infty = 2|\phi_\infty|$ or

$$\Delta\phi_\infty \approx \frac{4m}{R}\left(1 - \frac{9\beta}{16R^2}\right). \quad (4.4.6)$$

The coefficient of β/R^2 differs from the factor $\frac{3}{4}$ obtained by Halilsoy, the difference being the contribution of the last term in eq.(4.4.5).

For some of the metrics under consideration [see eqs.(4.2.4) and (4.2.5)], β can be of order m^2. Thus, expressions for light orbit eq.(4.4.5) and for the bending eq.(4.4.6) should be carried to higher order in m/R to provide a fair assessment of the importance of the contributions due to β. This may be done by iteration. That is, we substitute eq.(4.4.5) for u' on the right-hand side of eq.(4.4.4) and proceed as before. The result is that the following terms should be added to right-hand side of eq.(4.4.4):

$$-\frac{15m^2}{4R^3}\phi\cos3\phi - \frac{3m^2}{16R^3}\sin3\phi. \qquad (4.4.7)$$

The expression (4.4.6) for the deflection of the light ray becomes

$$\Delta\phi_\infty \approx \frac{4m}{R}\left[1 - \frac{9\beta}{16R^2} + \frac{69m^2}{16R^2}\right]. \qquad (4.4.8)$$

4.5. Precession of planetary apsides

For a planet, we return to eq.(4.4.3). This equation is exact and may be handled as it stands. However, since we will treat some of the terms on the right side of the equation as perturbations, no precision will be lost by replacing the constants of the motion h and H by their classical limits. For a planet moving at non-relativistic speed, we may, by eq.(2.2.11), put $H^2 \approx c_0^4$. Also, at sufficiently large r (i.e., at the site of a planetary orbit), we may put $h \approx h_0$, the classical angular momentum per unit mass. Thus we have

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_0^2}{h_0^2} = -\frac{\beta c_0^2}{h_0^2}u' + 3mu'^2 - 2\beta u'^3. \qquad (4.5.1)$$

This differential equation is not quite the same as recently obtained through other means by Halilsoy.[66] In particular, Halilsoy's equation is missing the term $-\beta(c_o/h_o)^2u'$ [ref.[66], eq.(25)].

If we temporarily put the terms in u'^2 and u'^3 equal to zero, we obtain a differential equation that may solved exactly:

$$\frac{d^2u'}{d\phi^2} + s^2u' = \frac{1}{\alpha_o}, \tag{4.5.2}$$

where

$$s^2 = 1 + \beta c_o^2/h_o^2, \tag{4.5.3}$$

and

$$\alpha_o = h_o^2/mc_o^2. \tag{4.5.4}$$

The solution is the precessing ellipse

$$u' = \alpha^{-1}(1 + e\cos s\phi) \tag{4.5.5}$$

where the eccentricity e is arbitrary and where the semi-latus rectum α is

$$\alpha = \alpha_o\left[1 + \frac{\beta c_o^2}{h_o^2}\right]. \tag{4.5.6}$$

The precession of the apsides, per revolution of the planet on the orbit, due to the term in $\beta u'$, is then

$$\Delta_1 = \frac{-\pi\beta c_o^2}{h_o^2} = \frac{-\pi m}{\alpha_o}\frac{\beta}{m^2} . \tag{4.5.7}$$

The terms in u'^2 and u'^3 may be treated as perturbations. Thus, one inserts eq.(4.5.5) in these two terms on the right side of eq.(4.5.1) and solves the resulting equation. The term in u'^2, acting alone, produces the usual precession associated with Schwarzschild problem:

$$\Delta_2 = \frac{6\pi m^2}{h_o^2}c_o^2 = \frac{6\pi m}{\alpha_o} \tag{4.5.8}$$

As shown by Halilsoy,[66] the term in u'^3, acting alone, produces the precession

$$\Delta_3 = -\frac{6\pi\beta m^2}{h_o^4}c_o^4 = -6\pi\left[\frac{m}{\alpha_o}\right]^2\frac{\beta}{m^2} \tag{4.5.9}$$

However, this term is smaller than (4.5.7) by a factor of m/α_o and is thus entirely negligible. To lowest order in m/α_o. then, the total procession Δ of the apsides per revolution is just $\Delta_1 + \Delta_2$:

$$\Delta = \frac{6\pi m}{\alpha_o}\left(1 - \frac{\beta}{6m^2}\right). \tag{4.5.10}$$

4.6. Radar echo delay

We now consider the propagation in time of light in the Reissner-Nordström type metric. Again, let the motion take place in the $\theta = \pi/2$ plane. Writing out the conservation of energy equation (2.2.8) in plane polar coordinates and making use of eqs.(2.2.9) and (4.3.1), we have

$$\left(\frac{dr}{dA}\right)^2 + \frac{h^2}{r^2} - n^2 c_o^2 = 0 \qquad (4.6.1)$$

Let us now evaluate the constant of the motion, h. Let r_o denote the distance of closest approach of the ray to the center of the gravitating body. When $r = r_o$, we have $dr/dA = 0$. Thus eq.(4.6.1) gives

$$h = r_o n(r_o) c_o, \qquad (4.6.2)$$

which is analogous to the classical-mechanical expression $r_o v(r_o)$.

We may now transform from r back to r′ using eqs.(4.1.13) and (4.1.14). Also, because we are interested in the propagation of light in time, we use eq.(1.1.10) to pass over from A to t as independent variable. Thus, with substitution and transformation of eq.(4.6.2), eq.(4.6.1) becomes

$$\left(\frac{dr'}{dt}\right)^2 = \Omega^4(r')\ c_o^2\ \left[1 - \frac{r_o'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_o')}\right]. \tag{4.6.3}$$

The time of travel from r_o' to r' is then

$$\Delta t = c_o^{-1}\int_{r_o'}^{r'} \Omega^{-2}(r')\left[1 - \frac{r_o'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_o')}\right]^{-1/2} dr'$$

$$= c_o^{-1}\int_{r_o'}^{r'} I(r')dr'\ . \tag{4.6.4}$$

Now,

$$I = \Omega^{-2}\left(1 - \frac{r_o'^2}{r'^2}\right)^{-1/2}\left[1 + \frac{[1 - \Omega^2(r')/\Omega^2(r_o')]}{(r'^2/r_o'^2 - 1)}\right] \tag{4.6.5}$$

Using eq.(4.1.11) to write out $\Omega(r')$ and $\Omega(r_o')$, then expanding to first order in m and β, we obtain

$$I = \left(1 - \frac{r_o'^2}{r'^2}\right)^{-1/2}\left[1 + \frac{2m}{r'} + \frac{mr_o'}{r'(r' + r'_o)} - \frac{3\beta}{2r'^2}\right] \tag{4.6.6}$$

The total time travel $\Delta t(r_o', r')$ from r_o' to r' is obtained by substituting eq.(4.6.6) into eq.(4.6.4) and integrating:

$$\Delta t(r_o', r') \approx c_o^{-1}(r'^2 - r_o'^2)^{1/2}$$

$$+\frac{2m}{c_o}\ln\left[\frac{r'}{r'_o}+\frac{(r'^2-r_o'^2)^{1/2}}{r'_o}\right]$$

$$+\frac{m}{c_o}\left[\frac{r'-r'_o}{r'+r'_o}\right]^{1/2}+\frac{3\beta}{2r'_o c_o}\sin^{-1}(r'_o/r')-\frac{3\pi\beta}{4r'_o c_o}. \qquad (4.6.7)$$

The first term on the right side of eq.(4.6.7) is the transit time of light in Euclidean space. The delay $\Delta T(r'_o,\ r')$ due to general-relativistic effects is the sum of the remaining terms.

As an example, let us estimate the radar echo delay for a signal from the Earth at radius $r'_\oplus$ to an inferior planet at radius r'_p when that planet is near superior conjunction with the Sun. Let the distance of closest approach to the center of the Sun be r'_o. If we suppose that the signal passes very near the Sun, so that r'_o is much smaller than either $r'_\oplus$ or r'_p, then

$$\Delta T(r'_o, r'_\oplus) \approx \frac{2m}{c_o}\ln\left[\frac{2r'_\oplus}{r'_o}\right]+\frac{m}{c_o}\left[1-\frac{r'_o}{r'_\oplus}\right]-\frac{3\pi\beta}{4c_o r'_o}\left[1-\frac{2r'_o}{\pi r'_\oplus}\right]. \qquad (4.6.8)$$

and the total delay in the signal for the round trip is

$$2\left[\Delta T(r'_o, r'_\oplus)+\Delta T(r'_o, r'_p)\right]. \qquad (4.6.9)$$

The effect of β on the delay is quite evident from the above.

CHAPTER 5

APPLICATION TO COSMOLOGY

This chapter contains applications of an formalism to the realm of cosmology. We demonstrate in Sec.5.1 and 5.2 that anti-de Sitter and Robertson-Walker universes constitute Maxwell "fish-eye lens." Sec.5.3 deals with light motion in these two universes. The arguments will allow us to see how one should interpret a time dependent gravity index in these contents. The developments of Sec.5.4 indicate how the redshifts arise in the language of a refractive index and how a unified view could be achieved in regard to two apparently dissimilar redshifts.

5.1. The de Sitter Universe and the Maxwell's "fish-eye lens"

The de Sitter line element in standard coordinates is

$$ds^2 = (1 - \Lambda r'^2/3)c_o^2dt^2 - (1 - \Lambda r'^2/3)^{-1}dr'^2$$

$$- r'^2d\theta^2 - r'^2\sin^2\theta d\phi^2, \qquad (5.1.1)$$

where Λ is the cosmological constant, which is proportional to the space curvature. Λ can be positive or negative, corresponding to a closed or an open (or, anti-) de Sitter universe.[73]

To pass over to isotropic coordinates, we may use the method outlined above, together with the requirement that for small radial distances the new radial variable r should asymptotically approach r'. The result is the well-known transformation

$$r' = r(1 + \Lambda r^2/12)^{-1}. \tag{5.1.2}$$

Then, in the isotropic coordinates,

$$ds^2 = (1 - \Lambda r^2/12)^2(1 + \Lambda r^2/12)^{-2}c_o^2dt^2$$

$$-(1 + \Lambda r^2/12)^{-2}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2). \tag{5.1.3}$$

The *effective index of refraction is*

$$n(r) = (1 - \Lambda r^2/12)^{-1} \tag{5.1.4}$$

This index of refraction is valid for either positive or negative Λ, with r defined through eq.(5.1.2) let us examine the case $\Lambda < 0$, corresponding to the anti-de Sitter universe. Let us write $\Lambda = -K$, where K is then a positive constant. The effective index of refraction of the anti-de Sitter universe, in the isotropic coordinates, is then

$$n(r) = (1 + Kr^2/12)^{-1} \tag{5.1.5}$$

This index of refraction is of exactly the same form as the index encountered in a traditional problem of classical geometrical optics — the Maxwell "fish-eye lens". The index of refraction in the Maxwell fish-eye is

$$n_M(r) = n_o(1 + r^2/a^2)^{-1}, \tag{5.1.6}$$

in which a and n_o are constants. Comparing eqs.(5.1.5) and (5.1.6) we note that the open version of the de Sitter universe is a Maxwell "fish-eye lens" with $n_o = 1$ and $a^2 = 12/K$.

5.2. Robertson-Walker Universe

The Robertson-Walker (RW) metric represents the gravitational field in a homogeneous and isotropic universe. In the standard comoving coordinates (t, r', θ, ϕ), the RW line element is given by

$$ds^2 = c_o^2 dt^2 - R^2(t)\left[\frac{dr'^2}{1 - kr'^2} + r'^2 d\theta^2 + r'^2 \sin^2\theta \, d\phi^2\right], \tag{5.2.1}$$

in which R(t) is a dimensionless scale factor and k is a constant with dimensions of $(\text{length})^{-2}$. We may pass over to isotropic coordinates by the usual method and requiring that for small radial distances, the new radial coordinates should asymptotically be equal to r'. The result is the well-known transformation

$$r' = r(1 + Kr^2/4)^{-1}. \tag{5.2.2}$$

In the isotropic coordinates (t, r, θ, ϕ), the line element is

$$ds^2 = c_0^2 dt^2 - R^2(t)(1 + Kr^2/4)^{-2}|dr|^2. \tag{5.2.3}$$

Defining the refractive index n in the usual way, we obtain

$$n = \frac{R(t)}{1 + Kr^2/4} . \tag{5.2.4}$$

For the case $k > 0$, corresponding to a closed RW universe, and for a fixed cosmological epoch $t = t_0$, this corresponds to the index of refraction (5.1.6). Thus the closed Robertson-Walker universe is a Maxwell fish-eye lens with $n_0 = R(t_0)$ and $a^2 = 4/k$. We shall see below that the correspondence between the Maxwell fish-eye and the Robertson-Walker universe does not actually demand that we restrict the later to a particular moment t_0.

5.3. Light ray in the de Sitter and Robertson-Walker Universes

As noted above, the open de Sitter universe is equivalent to a traditional problem in classical geometrical optics--Maxwell's fish-eye lens. It follows (1) that the anti-de Sitter universe constitutes an absolute, optical instrument and (2) that, in the system of isotropic coordinates, the rays are eccentric circles.

Beginning from the orbit equation (4.3.3) and the index of refraction (5.1.5) and integrating, we obtain the polar equation for the light ray in the anti-de Sitter universe:

$$\sin(\phi - \alpha) = \frac{h(Kr^2 - 12)}{r(144c_o^2 - 48h^2K)^{1/2}}, \qquad (5.3.1)$$

where α is a constant of integration. In effecting this calculation, we can follow step-for-step the calculation of ray shapes in the classical Maxwell fish-eye.[74]

Since $(Kr^2 - 12)/r\sin(\phi - \alpha)$ = constant, we can write the equation for a family of light rays passing through a fixed point $P_o(r_o, \phi_o)$ as (See fig.1)

$$\frac{Kr^2 - 12}{r\sin(\phi - \alpha)} = \frac{Kr_o^2 - 12}{r_o\sin(\phi_o - \alpha)} \qquad (5.3.2)$$

for any value of α, this equation is satisfied at point $P_1 = (r_1,$

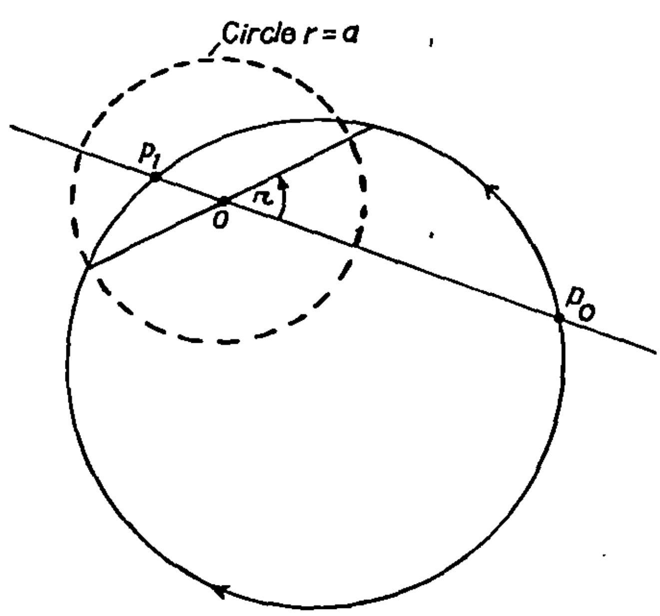

Fig. 1 Rays in Maxwell's "Fish-eye"

ϕ_1), where $r_1 = 12/Kr_0$ and $\phi_1 = \phi_0 + \pi$. This shows that all the rays from an arbitrary point P_0 meet at a point P_1 on the line joining P_0 to the origin O such that $OP_0 . OP_1 = 12/K$. Hence the imaging in Maxwell's fish-eye lens is an inversion. From any point P_0 in the three-dimensional space an infinity of rays originate which are then focused at an image point P_1. The images are therefore *sharp* (stigmatic). (In most real optical instruments, of the infinity of points passing through an object point, *only a finite number pass* through the image point, the other rays only passing near the image point. Such images are not sharp ones.) Now, an instrument which sharply focuses an image of a three-dimensional region of space is called an absolute optical instrument. Thus, the anti- de Sitter universe constitutes an absolute optical instrument. All *the theorems* pertaining to absolute *optical instruments* apply. For example, the optical length of a line segment in the image must be equal to the optical length of the corresponding line segment in the object[74].

Moreover, by analogy to the Maxwell fish-eye, a ray in the de *Sitter universe is a circle in the isotropic coordinate system.* This goes exactly as in the classical geometrical optics treatment of the Maxwell fish-eye. On the left-hand side of eq.(5.3.1) write $\sin(\phi - \alpha) = \sin\phi\cos\alpha - \cos\phi\sin\alpha$, then put $x = r\cos\phi$ and $y = r\sin\phi$. Then it may be written in the form

$$(x + b\ \sin\alpha)^2 + (y - b\ \cos\alpha)^2 = 12/K + b^2 \qquad (5.3.3)$$

where $b = (hK)^{-1}(36c_0^2 - 12h^2K)^{1/2}$. Thus each ray is a circle. Note that in eq.(5.3.1) if we put $K = 0$, we obtain a straight-line ray, as we would expect:

$$r\sin(\phi - \alpha) = \text{constant}.$$

These results may be extended with little modification to the closed Robertson-Walker universe. The formalism developed in this paper was designed for static metrics, so it may seem at first sight that we can not deal with the Robertson-Walker case. However, in the case of null geodesic, a time-dependent conformal factor in the metric need not affect the basic procedure or the most important conclusions. (The same can not be said of the particle trajectories.) Let us see how this works out in the language of refractive indices.

As noted above, the closed Robertson-Walker universe yields a factorable index of refraction:

$$n = n_s(r)n_t(t),$$

in which n_s is a function of the spatial coordinates alone and n_t

is a function of the time alone. For the RW metric in isotropic *comoving coordinates*

$$n_s = (1 + kr^2/4)^{-1} \tag{5.3.5}$$

$$n_t = R(t). \tag{5.3.6}$$

The bending of light rays depends only on the spatial gradient of n. The fact that n varies everywhere in space with the same multiplicative function of time R(t) does not affect the shape of a light ray. One way to see this is to consider Snell's law. If *the index of refraction factors into the form* (5.3.4), then whenever Snell's law is applied at an interface between regions of different n, the common factor $n_t(t)$ will cancel from the two sides of equation. More formally, we can define a new time coordinate τ by $dt = Rd\tau$. Then the line element (5.2.3) becomes

$$ds^2 = c_0^2R^2d\tau^2 - R^2(1 + kr^2/4)^{-2}|dr|^2.$$

and the effective index of refraction becomes simply

$$n = n_s = (1 + kr^2/4)^{-1} \tag{5.3.7}$$

The scale factor $R(\tau)$ or $R(t)$ thus can have no effect on the shape of a ray in the isotropic, comoving coordinates. R only influences the progress in time of light along the ray.

As far as ray shapes are concerned, then, the RW universe is *entirely* analogous to the Maxwell *fish-eye lens*. It *follows* (i) that the closed Robertson-Walker universe also constitutes an absolute optical instrument and (ii) that, in the system of isotropic, comoving coordinates, the rays are eccentric circles.

5.4. Redshifts

The gravitational and cosmological redshifts are not dynamical effects; i.e., we do not need to solve an equation of motion in order to calculate them. However, it may be of some interest to see how the redshifts arise in the language of an effective index of refraction.

In ordinary optics, if a light wave travels from a region of high index of refraction n_2 to a region of low index of refraction n_1, the wavelength λ increases because the leading part of the wave enters n_1 first and speeds up while the trailing part is still in n_2. Thus the wave begins to stretch out. λ and c change but the frequency ν does not. This holds even if the index varies continuously with the spatial coordinates and even if the ray crosses obliquely through the contours of constant n.

Thus, in general,

$$\lambda(r_1)n(r_1) = \lambda(r_2)n(r_2). \tag{5.4.1}$$

Now consider a situation in which n does not depend on the spatial coordinates, but does vary with time. An example can be imagined: let the air slowly be pumped from a chamber. Then, as long as the wave does not leave the chamber, n is everywhere the same, but it is a decreasing function of time. The wavelength will not change, since the leading edge of the wave never encounters a new value of n before the trailing edge does. Thus, in this case c and ν change, but λ does not. We have

$$\lambda(t_1) = \lambda(t_2) \tag{5.4.2}$$

Suppose now that the index of refraction can be written as a product of two functions--a function n_s of the spatial coordinates alone and a function n_t of the time alone, as in eq.(5.3.4). For the reasons just mentioned, n_t does not affect the wavelength and we have

$$\lambda(r_1)n_s(r_1) = \lambda(r_2)n_s(r_2). \tag{5.4.3}$$

We wish to apply these rules of ordinary optics to the propagation of light in general relativity. Our effective index of refraction eq.(2.1.5) is based upon the isotropic coordinate speed of light. Thus the quantity analogous to the wavelength of classical optics is the *coordinate distance* between successive crests of the wave. Coordinate distances are not, of course, directly measurable in general relativity. The physically measurable metric length is obtained from the coordinate length by means of the metric (2.1.1).

Gravitational Redshift

Let $|\Delta r_1|$ be the coordinate distance between successive crests of a light wave located at r_1. Similarly, let $|\Delta r_2|$ be the coordinate distance between successive crests at a different point r_2 located on the same ray. Then, in analogy to the condition (5.4.1) from ordinary optics, we must have

$$|\Delta r_1| n(r_1) = |\Delta r_2| n(r_2). \tag{5.4.5}$$

The metric length λ of the wave at point r_1 is obtained by applying the metric (2.1.1) to the coordinate length $|\Delta r_1|$ of the wave: $\lambda(r_1) = \Phi^{-1}(r_1)|\Delta r_1|$. A similar expression holds for $\lambda(r_2)$. Thus we have

$$\Phi(r_1)\lambda(r_1)n(r_1) = \Phi(r_2)\lambda(r_2)n(r_2), \tag{5.4.6}$$

or, using eq.(2.1.5),

$$\lambda(r_1)\Omega^{-1}(r_1) = \lambda(r_2)\Omega^{-1}(r_2) \qquad (5.4.7)$$

the usual gravitational redshift relation.

As an example, let us take the case of the Schwarzschild metric. Let a source of light be located at r_1 and an observer at r_1, sufficiently far from the central gravitating body so that we may put $\Omega(r_2) \approx 1$. Then, with the use of eqs.(4.1.11) and (4.2.1), (5.4.5) gives

$$z \equiv (\lambda_{observed} - \lambda_{emitted})/\lambda_{emitted}$$

$$= (1 - 2m/r')^{-1/2} - 1. \qquad (5.4.8)$$

This results may, of course, be derived by many other methods.

Cosmological Redshift

In the expanding universe of the Robertson-Walker metric, we have an index of refraction (5.2.4) that factors like (5.3.4), with n_s and n_t given by eqs.(5.3.5) and (5.3.6). Let a light wave of coordinate length $|\Delta r_1|$ be emitted at (t_1, r_1) and received at (t_2, r_2). By analogy to eq.(5.4.3), the coordinate length $|\Delta r_2|$ of the received wave is determined by

$$|\Delta r_1| n_s(r_1) = |\Delta r_2| n_s(r_2). \tag{5.4.9}$$

The metric length λ of the wave is obtained by applying the metric (5.2.3) to the coordinate length $|\Delta r|$. Thus eq.(5.4.9) becomes

$$\frac{(1 + kr_1^2/4)}{R(t_1)} \lambda(t_1, r_1) n_s(r_1) = \frac{(1 + kr_2^2/4)}{R(t_2)} \lambda(t_2, r_2) n_s(r_2),$$

or, with use of eq.(5.3.5), we get

$$\lambda_1/R(t_1) = \lambda_2/R(t_2) \tag{5.4.10}$$

which is precisely the usual cosmological relation.

While many writers have stressed the fundamentally different natures of the gravitational and cosmological redshifts, others have argued that it is possible to treat them with a single unified approach[75]. In the effective optical-medium formulation persued here, it is interesting to note that both spectral shifts depend on a single optical principle eq.(5.4.3). The following remarks are in order.

In the context of redshifts, we recall that we mean redshift in frequency. We do not specify whether it is the frequency of an elementary wave or an elementary particle or a material clock. Kostro[76] has raised a very interesting question with respect to the definition of a clock. He argues that the total energy of a moving particle increases according to the equation

$$E = E_0(1 - \beta^2)^{-1/2}, \quad \beta = v/c_0 \qquad (5.4.11)$$

de Broglie has written this as $h\nu = h\nu_0(1 - \beta^2)^{-1/2}$ and concluded that with the *elementary particle* must be associated a *wave* for which the frequency transforms like this. On the other hand, Einstein considered a photon clock in which a photon moves between two mirrors and has shown that $\nu = \nu_0(1 - \beta^2)^{1/2}$. So, we see that for Einstein photon clock, the frequeny slows down and for de Broglie's standing wave clock, the frequency increases as $v \to c_0$. Are we dealing with two kinds of time? Are we to exclude de Broglie's clocks as time measuring devices? Kostro[76] argues that the ideal relativistic clock is a massless clock having greatest possible frequency. However, note that the opposite behavior of clocks is reminiscnt of the conflict between Newton's corpuscular hypothesis and Foucault's experiment, addressed in Sec.2.3.

CHAPTER 6

AN ALTERNATIVE DESCRIPTION OF GRAVITATIONAL REFRACTIVE MEDIUM AND DEDUCTION OF GENERAL RELATIVISTIC EFFECTS: A CRITIQUE

In the *foregoing chapters*, the *gravitational refractive index* associated with centrally symmetric field of general relativity played a crucial role in the context of Evans–Rosenquist formulation of the optical–mechanical analogy.[77] We derived the index from the solution of general relativity itself.

There do however exist other ways of deducing a refractive index to be associated with a centrally symmetric gravity field, *that have nothing to do with general relativity. One such method* is due to Tangherlini[17] who allows an effective mass increase in the corpuscular theory of light and shows that the light particle travels slower rather than faster on entering an optically denser medium in accordance with Huygen's principle. The disagreement between Newtonian corpuscular theory of light and Foucault's experiment is also removed thereby. The relevant Hamiltonian equations then lead to a gravitational refractive index that accounts *for half of the general relativistic value for* light deflection. The other half is separately calculated as it is due to an altogether new effect, namely, gravitational shift in the photon frequency.

Using the Tangherlini Hamiltonian, Tian and Li[78] obtained an equivalent description of the gravitational refractive medium in terms of expressions giving varying speed and relative mass of a photon on a flat space. They further combined these two quantities to deduce an "apparent" rest mass of photon as well. Using all these in their formalism, Tian and Li obtain the full general relativistic value for light deflection and radar echo delay. The formulation can be regarded as a new addition to the already existing repertoire of alternative deductions of general relativistic effects. Such approaches should however be examined critically as they often turn out to be faulty for one reason or other.[79] By way of doing so, our basic aim in this chapter is to illuminate some of the tricky points. The ensuing analysis is supposed to lead to a better understanding of general relativity *vis-a-vis* an alternative formulation thereto. In this sense, this chapter is somewhat pedagogical in its contents.

The contents are arranged sectionwise as follows. First, in Sec.6.1, we clarify a misleading step in the Hamiltonian procedure of Tian and Li. Then, in Sec.6.2, we examine their equations for light deflection and point out an incompatibility that invalidates their conclusions. A comparison with our approach here provides an opportunity to address an important issue, namely, the meaning of observables in a flat space gravity theory and in general relativity. This is done consecutively in Secs.6.3 and 6.4. Finally, Sec.6.5 concludes the chapter.

6.1. Hamiltonian equations

Tangherlini[17] treats light corpuscles as classical particles with variable mass in a medium having gravitational refractive index n(x,y,z). The Hamiltonian is prescribed as

$$H(x,y,z,p_x,p_y,p_z) = \frac{c_o}{n(x,y,z)}\,[p_x+p_y+p_z]^{1/2} \qquad (6.1.1)$$

where c_o is the vacuum speed of light; p_x,p_y,p_z are the generalized photon momenta conjugate to x,y,z respectively. Also, as before,

$$c(x,y\ z) = \frac{c_o}{n(x,y,z)} \qquad (6.1.2)$$

represents variable speed of light in a medium. The magnitude of photon momentum is defined as

$$p = \mu(x,y,z)c(x,y,z) \qquad (6.1.3)$$

where $\mu(x,y,z)$ is the variable photon mass. Tangherlini obtains, for n, the following expression for the centrally symmetric field

$$n(r) = \left(1 - \frac{2m}{r}\right)^{-1/2}, \quad m = GMc_0^{-2} \qquad (6.1.4)$$

where G is the Newtonian constant of gravity and M is the gravitating mass.

To describe the effects of gravity, Tian and Li[78] emphasize more on the variation of photon's speed than on the gravitational refractive index. The same expressions for H, p and c as those of Tangherlini are used and the calculation is carried out in polar coordinates (r,ϕ).

$$p^2 = p_r^2 + p_\phi^2 = (\mu\dot{r})^2 + (\mu r\dot{\phi})^2 \qquad (6.1.5)$$

where p_r , p_ϕ represent radial and cross-radial components of ordinary momentum of a photon and, of course, $\mu = \mu(r)$. The Hamiltonian

$$H = c(r)[p_r^2 + p_\phi^2]^{1/2} = cp \qquad (6.1.6)$$

yields, according to Tian and Li, the equation

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -p\frac{dc}{dr} \qquad (6.1.7)$$

With $\dot{p}_r = -\frac{\mu m c_o^2}{r^2}$ and $p = \mu c$, TL obtained by integrating eq.(6.1.7) from $r = \infty$ to $r = r$ the following equation

$$c(r) = c_o\left(1 - \frac{2m}{r}\right)^{1/2}. \tag{6.1.8}$$

Using eq.(6.1.2) we see that eq.(6.1.4) indeed follows from eq.(6.1.8). Note that the Hamiltonian ϕ-equation has not been explicitly used by Tian and Li so far. On the other hand, in order to check the validity of their basic equation, eq.(6.1.7), it is necessary to express the Hamiltonian properly. The specific form of the Hamiltonian H (= cp) in polar coordinates depends on whether p is expressed in non-Euclidean or isotropic coordinates. For a photon traveling with isotropic speed c(r) [see eq.(6.2.2) below] on a flat space, the Tangherlini Hamiltonian[80] is given by

$$H = c(r)[P_r^2 + r^{-2}P_\phi^2]^{1/2} \tag{6.1.9}$$

which is, of course, numerically the same as eq.(6.1.6) above but the generalized momenta are now given by

$$P_r = \mu\dot{r} \tag{6.1.10}$$

$$P_\phi = \mu r^2 \dot{\phi}. \tag{6.1.11}$$

The canonical equations of motion are

$$\dot{P}_r = -\frac{\partial H}{\partial r}, \quad \dot{P}_\phi = \frac{\partial H}{\partial \phi} = 0. \tag{6.1.12}$$

The last equation implies

$$P_\phi = \mu r^2 \dot{\phi} = \text{constant}. \tag{6.1.13}$$

We can verify from the correct Hamiltonian eq.(6.1.9) that eq.(6.1.7), in general, does not hold! It holds only in the special case when $P_\phi = 0$. There is a tricky point here by which one is most likely to be misled into believing that eq.(6.1.8) for c(r) holds only in the case of radial photon motion and not for motion in all directions. Thereafter, one is confused by the expression for c(r) [Tian-Li eq.(6.1.14)] which states that the velocity of light is c(r) in *all* spatial directions! Then, one should remember that the Newtonian inverse square law operates not only for radial motion but also for motion in other directions. Hence, if the calculation is limited only to radial motion, as purportedly done by Tian and Li, the conclusions are likely to lack general validity. We must clarify these points immediately.

Consider arbitrary photon motion such that $P_\phi \neq 0$. Then, from eq.(6.1.9) and first of eq.(6.1.12) we get

$$\mu \ddot{r} = - \left[\mu c \frac{\partial c}{\partial r} + c \frac{\partial p}{\partial r} \right]. \tag{6.1.14}$$

Since in the Hamiltonian H, we treat r, ϕ, P_r, P_ϕ as independent variables, it can be verified from eq.(6.1.9) that

$$\frac{\partial p}{\partial r} = - \mu r \dot{\phi}^2 / c \tag{6.1.15}$$

so that from eq.(6.1.14) we get

$$\mu(\ddot{r} - r\dot{\phi}^2) = - \mu c \frac{dc}{dr} . \tag{6.1.16}$$

It is remarkable that $\mu(r)$ cancels out and the remaining l.h.s. is just the Newtonian expression for the radial acceleration which, by Tian-Li assumption, is

$$\ddot{r} - r\dot{\phi}^2 = - m c_0^2 r^{-2}. \tag{6.1.17}$$

Multiplying both sides of eq.(6.1.17) by $\mu(r)$ and then equating its r.h.s. with that of eq.(6.1.16), eq.(6.1.8) for $c(r)$ follows under usual conditions. It is now clear that the expression for $c(r)$, eq.(6.1.8), holds not only for $P_\phi = 0$, as Tian-Li calculations would have us believe, but also for $P_\phi \neq 0$. The confusion in the TL calculation results from not recognizing that, in polar coordinates, the independent variables in H are *not* p_r, p_ϕ but P_r, P_ϕ although, incidentally, $p_r \equiv P_r$. Inspite of the fact that eq.(6.1.8) has a general validity, as demonstrated just now, there are other difficulties that make the conclusions of Tian and Li unacceptable. In the context of the problem of light deflection, we shall demonstrate this.

6.2. Deflection of light

The problem of light deflection around a spherically symmetric massive gravitating body has been tackled also by Tangherlini through the formalism of classical optics. With the expression n(r), eq.(6.1.4), he employs Fermat's principle, to repeat:

$$\delta \int \frac{n dl}{c_o} = 0 \tag{6.2.1}$$

where dl is the Euclidean element of distance, to show that the resulting deflection Δ_N is just half of the full GR value $\Delta(\sim 4m/R)$ in which R is the distance of closest approach. The remaining half, one may call it Einstein contribution Δ_E, is separately accounted for by considering the gravitational frequency shift of a photon.

On the other hand, Tian and Li choose to combine the frequency shift and the variable speed of a photon in such a way that it leads to a specific expression for the variation of photon mass. Using this form suitably, they obtain full general relativistic value Δ. Let us closely examine the procedure below.

In the process of obtaining the exact path equation for light rays, Tian and Li use the equations denoting the isotropic velocity of light [their eq.(6.1.14)] in flat space:

$$c^2(r) = \dot{r}^2 + r^2\dot{\phi}^2 \tag{6.2.2}$$

and the constancy of angular momentum, eq.(6.1.13) [Tian-Li eq. (6.1.15)],

$$\mu(r)r^2\dot{\phi} = A. \tag{6.2.3}$$

But we must also remember eqs.(6.1.16) and (6.1.17) of the preceding section. It is a logical requirement that all the four equations be compatible. To verify this, it is sufficient to check if c(r) is a solution of eq.(6.1.16). From eq. (6.2.2), with $u = 1/r$, we get

$$c^2(r) = \dot{r}^2 + r^2\dot{\phi}^2 = A^2\mu^{-2}\left[u^2 + \left(\frac{du}{d\phi} \right)^2 \right] \tag{6.2.4}$$

and computing l.h.s. of eq. (6.1.16), we find

$$-\frac{1}{2}\frac{dc^2}{dr} = \ddot{r} - r\dot{\phi}^2$$

$$= - A^2u^2\mu^{-2}\Big[\, u + \frac{d^2u}{d\phi^2} + \mu\Big(\frac{du}{d\phi}\Big)^2\, \frac{d}{du}\big(\mu^{-1}\big)\Big]. \qquad (6.2.5)$$

Putting the expression for c^2 from eq.((6.2.4) into the l.h.s of eq.(6.2.5), we do not obtain its r.h.s.! Instead we find

$$-\frac{1}{2}\frac{dc^2}{dr} = A^2u^2\mu^{-2}\Big[u + \frac{d^2u}{d\phi^2} + \frac{\mu^2}{2}\Big[\, u^2 + \Big(\frac{du}{d\phi}\Big)^2\Big]\frac{d}{du}\big(\mu^{-2}\big)\Big]. \quad (6.2.6)$$

This implies that $c^2(r)$ is *not* a solution of eq.(6.2.5) and we conclude that the eqs.(6.1.16), (6.2.2) and (6.2.3) are not compatible. This inconsistency renders the claims of Tian and Li unacceptable.

Nonetheless, the approach by Tian and Li offers an opportunity to address an important question, namely, the meaning of observables in a gravity field. For a comparison, notice that, in our approach, we did not depart from the solutions of general relativity - only a suitable reformulation of the orbit equations was involved. Therefore, essentially, the same operational meanings of length and time as in general relativity applied there. On the other hand, as evidenced from eqs.(6.1.17) and

(6.2.2), Tian and Li adhere to the background flat space so that the *corresponding operational definitions of length and time apply* there. This feature distinguishes the Tian-Li approach from that of general relativity. However, the notion of length and time in flat space gravity often lead to erroneous conclusions. In order to illustrate the issue from a familiar vantage point, we pick up the wellknown Kepler problem which is interesting in its own right.

6.3. The Kepler problem

Consider the usual Kepler problem of a massive test particle (say, a planet) moving around a spherical gravitating mass M *(say, the Sun) under Newtonian inverse square law.* Let T,V,E and h_o be the kinetic, potential, constant total energies and angular momentum per unit mass of the test particle respectively. Then T + V = E implies

$$\frac{1}{2}\,[\dot{r}^2 + r^2\dot{\phi}^2] - mc_o^2 r^{-1} = E \qquad (6.3.1)$$

and the central nature of force implies a constancy of angular momentum such that

$$r^2\dot{\phi} = h_o. \qquad (6.3.2)$$

We can rewrite eq.(6.3.1) as

$$h_o^2\left[u^2 + \left(\frac{du}{d\phi}\right)^2\right] - 2muc_o^2 = E_o = \text{const.}, \qquad (6.3.3)$$

where $E_o = 2E$ is a new constant. As is customary, by differentiating the above equation we get the differential equation of the Keplerian ellipse in a familiar form.

Let us now redefine the radial variable $u \rightarrow u'$ via

$$u' = u\Phi(u), \quad u' = 1/r' \qquad (6.3.4)$$

$$\Phi(u) = (1 + \frac{mu}{2})^{-2}. \qquad (6.3.5)$$

After some algebra, we get

$$du' = \Omega(u)\Phi(u)du \qquad (6.3.6)$$

$$\Omega(u) = \left(1 + \frac{mu}{2}\right)^{-1}\left(1 - \frac{mu}{2}\right) \qquad (6.3.7)$$

$$\Omega(u') = (1 - 2mu')^{1/2} \qquad (6.3.8)$$

$$\Phi(u') = \frac{1}{4}\,[1 + (1 - 2mu')^{1/2}]^2. \qquad (6.3.9)$$

In fact, these are the same expressions as used in Sec.2.4 above but we group them here for an easy view. The following can also be directly verified:

$$2mu = 2mu' + 2m^2u'^2 + 5m^3u'^3 + \ldots = 2mu' + 0(m^2u'^2). \quad (6.3.10)$$

This implies that, to first order, $r \approx r'$. Also,

$$\Phi^2(u')\Omega^2(u') = 1 - 4mu' + 0(m^2u'^2). \quad (6.3.11)$$

We shall now express eq.(6.3.3) in terms of the new variable u'. To do this, multiply both sides of eq.(6.3.3) by $\Phi^2\Omega^2$, which yields, through the use of eqs.(6.3.4) —(6.3.11), the following equation:

$$h_o^2\ [\Omega^2u'^2 + (\frac{du'}{d\phi})^2] = c_o^2\left[E_oc_o^{-2} + 2mu' + 0(m^2u'^2)\right]\left[1-4mu' + 0(m^2u'^2)\right]$$

$$(6.3.12)$$

Simplifying further, using eq.(6.3.8), we get

$$h_o^2[u'^2 + (\frac{du'}{d\phi})^2 - 2mu'^3] = c_o^2[E_oc_o^{-2} + 2mu' - 4mu'E_oc_o^{-2} + 0(m^2u'^2)].$$

$$(6.3.13)$$

Let us now estimate the relative strength of the terms appearing inside the bracket on the r.h.s. of eq.(6.3.13). Note that, even at the site of Mercury, the planet nearest to the Sun, we have $mu \approx mu' \approx 2.5\times10^{-8}$. Further ,for a bound orbit, we know that $E < 0$. Using from Newtonian mechanics the approximate result for a bound system, namely, $v^2 \sim mc_0^2/r$ where $v\ [\equiv (\dot{r}^2 + r^2\dot{\phi}^2)^{1/2}]$, we find that the absolute magnitude of the first constant term is given by $|E_0c_0^{-2}| \approx mu \approx mu' \approx 2.5\times10^{-8}$. Therefore the second term is of comparable order of magnitude to the first constant term while the third and $m^2u'^2$ terms are 10^{-8} times smaller. The next higher order term is 10^{-16} times smaller and so on. In view of this practical numerology, it looks quite reasonable to disregard terms that are 10^{-8} times smaller or less than the first term. We are then left with following differential path equation:

$$h_0^2[u'^2+(\frac{du'}{d\phi})^2 - 2mu'^3] = E_0 + 2mu'c_0^2. \qquad (6.3.14)$$

Differentiating with respect to ϕ, we get

$$u' + \frac{d^2u'}{d\phi^2} = mc_0^2h_0^{-2} + 3mu'^2 \qquad (6.3.15)$$

and Lo - we have the equation for a processing ellipse of Einstein's general relativity giving the perihelic shift of Mercury, the famous value 43"/century! We began with a closed Kepler ellipse and have now ended up with an open orbit! Moreover, the light ray equation follows if the customary limit $h_0 \to \infty$ is taken. So, let us pause a little and look back. The neglect of smaller terms seems perfectly justified, as we have just seen. The radial coordinate transformation $u \to u'$ is quite harmless since a rearithmetization of space is just an abstract mathematical operation and physical conclusions should not depend on it. Should we then accept the above procedure as an alternative way of deducing the famous general relativistic predictions?

One may guess that the answer will be in the negative but the reason is not obvious. Some plausible arguments arise. One may straightaway point out that, after all, it is the same old classical inverse square force law on a flat space and a mere spatial coordinate transformation has no physics in it. Therefore there is no reason to expect any deviation from the usual Kepler elliptic orbit. This is a perfectly valid physical argument. But, mathematically, *eq.(6.3.13) does* ***not*** *yield any Keplerian ellipse* $u' = a + b \cos \phi$, where a and b are constants, as a solution. Also, there is no reason whatsoever to return to the (u,ϕ) coordinate system as all systems are *equally* preferable. The (fictitious) nature of the above ambiguity will be understood only after a discussion, presented at the end, of the meaning of observables in a given gravity theory.

However, the question above relates not to the exact eq.(6.3.13) but to the status of the procedure leading eq.(6.3.15). In that procedure, one should not underestimate the innocent looking approximations. The resulting equation, eq.(6.3.15), when reverted to the (u,ϕ) language, does no longer correspond to the inverse square force law we started with. This means that the operation of approximation amounts to introducing *extra* forces that modify the original force law and consequently, the particle orbit. Thus, the Keplerian elliptic orbit is modified into a precessing ellipse. Recall that there is nothing wrong with the operation of approximation itself. In similar circumstances, *such approximations are essentially employed also in* general relativity or in other approaches (like that of Tian and Li) whenever nonlinear orbit equations are solved. In general relativity, the exact geodesic equations do relate to a given spacetime metric (analogy: given inverse square law here) but the approximate equations do not. These latter equations may, at best, relate to a modified metric (analogy: modified force law here). These analogies seem to suggest that the physics and mathematics of the procedure may be justified. Even then the procedure can not be treated as a genuine alternative for the single reason that it is based on a coordinate transformation eq.(6.3.4) which is more an *accidental* choice than a result of some physical postulate!

6.4. Observables in a gravity field

In order to motivate the readers, let us start with a simpler example. Consider the light path equation [Tian- Li eq.(6.1.17)]:

$$u + \frac{d^2u}{d\phi^2} = 2mR^{-2} \tag{6.4.1}$$

and its solution

$$u = \frac{1}{d}\left[1 + \varepsilon \cos(\phi - \phi') \right] \tag{6.4.2}$$

where $d = R^2/2m$, $\varepsilon = bd$; b, ϕ' are arbitrary constants of integration and R is the perihelion distance. Change the coordinate $u \to y$ such that

$$y = u - \frac{1}{d} . \tag{6.4.3}$$

Then eq.(6.4.1) reduces to

$$y + \frac{d^2 y}{d\phi^2} = 0 \tag{6.4.4}$$

which is the differential equation of a straight line having a solution

$$y = h \cos (\phi - \phi'') \tag{6.4.5}$$

where h, ϕ'' are arbitrary constants. Again, a rearithmetization eq.(6.4.3) is of no physical consequence and both the systems (u,ϕ) and (y,ϕ) are equally preferable. Yet, the predictions look ambiguous: An elliptical path in the (u,ϕ) space implies that there *is* a deflection of the light ray while a straight path in the (y,ϕ) space implies that there is none. Which one of the predictions is to be really observed? Such (apparent) ambiguities do arise also in general relativity. Depending on the choice of coordinate systems, general relativity equations may lead to totally opposite predictions.[81]

To answer the question posed above, again one must understand what is meant by "observables" in a gravity field. The most reasonable definition of observables is that they should be scalars, independent of the choice of coordinate systems.[82] Such a definition is indeed very general and applies to any theory. In keeping with this idea, the observable predictions in general relativity are required to be expressed in terms of "proper" quantities defined in a coordinate independent manner. Let us illustrate how this should be achieved in the wellknown Schwarzschild problem.

The proper length in the "standard" coordinates (r,θ,ϕ) is defined by

$$dl^2 = (1 - \frac{2m}{r})^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \qquad (6.4.6)$$

The surface area of a sphere is, of course, $4\pi r^2$ which endows usual meanings to θ and ϕ coordinates. In principle, the exact procedure is to express the general relativistic orbit equation in terms of proper quantities, say, in the (l,ϕ) language and then solve it. But, unfortunately, dl, in general, is not integrable. Therefore, the first step is to write the geodesic equation in (u,ϕ) coordinates and then reexpress it in the (l,ϕ) language using the expression for proper radial length derived from the metric eq.(6.4.6) as

$$l = \int^{r} (1 - \frac{2m}{r})^{-1/2} dr$$

$$= \left[r(1 - \frac{2m}{r})^{1/2} + 2m \, ln \, ([r - 2m]^{1/2} + r^{1/2}) \right]^{r}. \quad (6.4.7)$$

As one can see, the choice of a different radial coordinate $r' : r \to r'$ amounts only to a substitution of the running variable under an integral sign, an operation that does not alter the numerical value of l. However, there is the great problem that eq.(6.4.7) can not be inverted, that is, an exact expression of the form $r = f(l)$ is not available in general. Fortunately, in the Solar system scenario, it is possible to take $u \approx l^{-1}$ without committing much error. Putting it in the (u,ϕ) geodesic equation, we achieve our goal, that is, a coordinate independent form. That is also the reason why in text books geodesic equations in the Solar system are analyzed manifestly not in terms of l but only in terms of coordinate r since $l \approx r$. But, in the field of strong gravity, say, in the vicinity of a black hole, it is essential to take the (u,ϕ) orbit equation together with eq.(6.4.7) *per se* and solve the problem numerically to obtain orbits in the (l,ϕ) space.

In the Tian-Li approach, the spatial metric is flat, given by

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (6.4.8)$$

and accordingly, the proper radial length is

$$l = \int^r dr. \qquad (6.4.9)$$

There is some advantage in this approach. The problem of inversion just does not arise as we have simply $r = l$. Putting $u = l^{-1}$ in eq.(6.4.1), we get a coordinate independent form of equation. The prediction is also unique now: there *will* occur a deflection of light ray at the perihelion by an amount $4\mu/L$, where $L = \int^R dr$, which is the same as in general relativity.

From this it follows that if a flat space gravity theory is free of any internal inconsistency (Unfortunately, this is not the case with the approach of Tian and Li, as demonstrated in Sec.6.1.), there does not seem to be anything to distinguish it from general relativity as far as *first* order predictions for light motion are concerned. For a detailed investigation into this topic from the point of view of a minimally relativistic gravity, see ref.[83]. However, we must be aware of a basic difficulty, already noted in Sec.1.3: The universality of gravitational interaction prevents an observational location of the background flat space. Consequently, an *exact* identification $l = r$, as in eq.(6.4.9), at *all* points in the field is not achievable. Only in the weak field region can one reasonably take $l \approx r$. Therefore, any new approach, besides being mathematically consistent, must possess definitions of observable physical length and time. In this connection, Ichinose and Kaminaga[84] point out a more fundamental "inevitable ambiguity" - not necessarily a defect - of general relativity that owes its origin to the geometric formulation. For instance, in order to obtain a perturbative formula for the radar echo travel time, which is not what is defined in the asymptotic region, one is forced to compare points in two *different* geometries, curved and flat - which is a *geometrically nonsense procedure. We do not wish to go into* details here as it will take us out of the main scope of the present work, but interested readers may have a look at ref.[84].

Let us return to the Kepler problem. The differential equation of the classical orbit is

$$u + \frac{d^2u}{d\phi^2} = mc_o^2 h_o^{-2}. \qquad (6.4.10)$$

The proper radial length, in this case, too, is given by eq.(6.4.9). Replacing u by l^{-1}, we obtain a coordinate independent form of eq.(6.4.10) which is just that of an ellipse in the (l,ϕ) space. Observe that, a transition to u′ via eq.(6.3.4) introduces a corresponding change in the metric eq.(6.4.8) and thereby results a different expression, f(r′), on the r.h.s. of eq.(6.4.9). Consequently, the functional form for $r' = f'(l)$ will also be different. Using this in the exact eq.(6.3.13), we are able to retrieve the same equation (6.4.10) but in the coordinate independent (l,ϕ) language. Hence, we have the same old Kepler orbit, whatever be the coordinate choice - and the ambiguity is resolved.

6.5. Concluding remarks

In the approach of Tian and Li, the effects of gravity (or equivalently, refractive medium) is incorporated into the notion of variable photon "mass" and speed, $\mu(r)$ and $c(r)$ respectively, on a flat space. The (weak) principle of equivalence is also accommodated through their assumption, eq.(6.1.17). It seems that there are sufficiently plausible ingredients in the Tian-Li approach and a consistent procedure, supplemented with definitions of observable length and time, could possibly lead to interesting alternative deductions of effects - at least on light motion in a gravity field.

After correcting a misleading step, it is shown that there is an inconsistency inherent in the approach that invalidates their conclusions. A further analysis with regard to the meaning of observables in general relativity and in flat space gravity clarifies the question of coordinate independent predictions. We saw that "coordinate" descriptions may vary - even lead to opposite predictions - but "proper" descriptions do not. The theoretical scheme of achieving the latter description is also indicated. It is perhaps pointless to ask whether the notion of c(r), $\mu(r)$ or a refractive medium on a flat space is more/less real than the notion of general relativistic spacetime curvature. These are just different artifacts to describe actual observations - light propagation in the present case - the basic condition being that the description must be free of internal inconsistencies.

EPILOGUE

The approach developed in the thesis is mathematically simpler and physically intuitive as it employs the familiar language of optics/mechanics. Indeed, eq.(2.2.7) - with the Evans-Rosenquist parameter A defined by eq.(1.1.10) - covers a lot of physics: It applies to geometrical optics of isotropic inhomogeneous media, to classical mechanics in velocity-independent potentials (for which we put $n = 1$ and $A \rightarrow t$), as well as to the orbits of light and particles in metrics representing a fairly wide class of physical situations. The expression of particle mechanics and geometrical optics in the same mathematical language is an aspect of the optical-mechanical analogy. There are several different formulations of this analogy, but eq.(2.1.6) (with its generalization, eq.(2.2.8)) is the simplest.

The Newtonian forms, eqs.(2.2.7) and (2.2.8), for the geodesic equations of motion offer some practical advantages for calculation. In particular, they facilitate the writing down of exact general relativistic expressions simply by analogy to classical formulas. Thus, they constitute one more tool for the relativist's tool kit. But the most interesting consequence of *extending the optical-mechanical analogy to general relativity is* that one simple equation of motion, eq.(2.2.7), now summarizes three fields of study: classical geometrical optics, classical particle mechanics and geodesic motion of both massive and massless particles in general relativity. Of course, our treatment is restricted to isotropic fields and media. Nevertheless, this *unified* approach, based on the use of optical action, possesses considerable flexibility and scope. A single variational principle (2.1.16) governs all three domains.

When the equations governing separate domains of physics can be cast into similar forms, two possibilities arise. The similarity might be due to mere contrivance on the part of the physicist. In this case, the mathematical analogy can be exploited to transfer methods and results from the more to the less familiar field, but *the underlying physics of the two fields could be quite different.* The second possibility is that the mathematical analogy results from a deeper connection between the two apparently disparate fields. In the case of the optical-mechanical analogy, this is certainly the situation.

The variational principle (2.1.16) puts a new twist on a conclusion mentioned before, in Sec.3.1: Maupertuis' principle amounts to a special case of Fermat's principle - the special case involving the matter waves of quantum mechanics. In the case of the propagation of light and the motion of particles in a gravitational field, both Fermat's principle and Maupertuis' principle may be regarded as special, or limiting cases of eq.(2.1.16), which, in turn, follows from the geodesic property of the trajectories. There is *nothing* of waves involved in eq.(2.1.16), which follows entirely from considerations of the behavior of particles in general relativity. How can it be the case that (1) Maupertuis' principle is a special case of Fermat's in the context of wave mechanics and that (2) both Maupertuis' principle and Fermat's principle are special cases of a variational principle deduced from general relativity?

The optical-mechanical analogy, as developed by Hamilton, can now be seen as providing a crucial hint about the wave-like nature of material particles, a hint that lay unnoticed until de Bròglie took it up a century later. A great, and so far unrealized, goal of late-twentieth-century physics is the unification of gravity with other forces of nature. The program of unification has been profoundly successful, beginning from the nineteenth century unification of optics, radiant heat, electricity and magnetism. But one could still maintain the possibility that gravity has nothing to do with other forces. The program of unification with it might represent only a unjustifiable act of faith by physicists in the unity of nature. The fact that Maupertuis' principle can simultaneously be the geometrical-optics limit of wave mechanics and a special case of the geodesic equation in general relativity seems to argue strongly in favor of an underlying unity of quantum mechanics and gravitation theory.

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D. Suggested samples

The device was constructed to use amorphous Fe–Ni–P–B alloy samples with an 80% metal to 20% metalloid composition ratio in the form of thin ribbons produced by rapid quenching from the melt. These materials have easily accessible Curie temperatures (ranging from about −50 to 300 °C, depending upon composition). They also have accessible melting points and can be produced in the laboratory or purchased inexpensively from commercial sources. They have the added advantage of being somewhat novel substances.

E. Original usage

This instrument was used originally to make precise measurements of Curie temperatures for the purpose of discerning small changes in Curie temperature induced by particle bombardments or thermal annealing of thin ribbons of amorphous metallic alloys.[10,11] Some had T_c values below room temperature, and some above. The precision obtainable with the instrument (tenths of a Celsius degree) and the reproducibility of the measurements (typically under 1 °C unless heating during measurement provided further annealing of the substance) were crucial for identifying changes of a few Celsius degrees in transition temperature.

III. CONCLUSION

A device has been described that is capable of making precise measurements of the Curie temperatures of ferromagnetic materials, which can be used for demonstration purposes, or as an undergraduate laboratory experiment, such as determining the effect that thermal annealing has upon the Curie point and the ferromagnetic-to-paramagnetic transition width. It requires a vacuum pump, a lock-in amplifier, a small Dewar of liquid nitrogen, a strip-chart recorder, and miscellaneous small items typically found in any undergraduate physics department.

The thermometric technique described herein may also be adapted to measure other transition temperatures, such as the critical temperature for superconducting materials.

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On the optical–mechanical analogy in general relativity

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We demonstrate that the Evans–Rosenquist formulation of the optical–mechanical analogy, so successful in the application to classical problems, also describes the motion of massless particles in the Schwarzschild field of general relativity. It is possible to obtain the well-known equations for light orbit and radar echo delay which account for two exclusive tests of Einstein's field equations. Some remarks including suggestions for future work are also added. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

The historical optical–mechanical analogy[1,2] has recently been cast into a familiar form by Evans and Rosenquist.[3-5] This new formulation, based on Fermat's principle, provides an interesting approach that can be profitably utilized in the solution of many classical problems. The approach works either way: The well-known ideas and techniques of classical mechanics can be successfully applied to the problem of classical optics and vice versa. Such success naturally prompts further inquiry as to whether the applicability of the optical–mechanical analogy could be extended even beyond the classical regimes. More specifically, a curious student might ask: Can the Evans–Rosenquist (ER) formulation be used to describe the phenomena of light propagation and massive particle motion in general relativity (GR)? In order to decide this question, let us note that it addresses two distinct areas of investigation: One has to first examine if the

"$F=ma$" optics of ER does at all describe light propagation in the GR scenario. If it does, then the first hurdle is overcome and the question of applying the optical analogy to massive particle motion becomes meaningful. To what extent that analogy would describe planetary motion in GR is a matter of separate investigation.

In this paper, we shall examine only the first part of the question that relates to the motion of massless particles in GR. We choose Schwarzschild field of gravity primarily because of its experimental importance. Besides, on many occasions, it is used as an ideal example for illustrating fundamental notions of GR.

The exterior Schwarzschild metric, which is a solution of Einstein's GR field equations, has been astoundingly successful in describing various gravitational phenomena. Famous experimental tests of GR include the bending of light rays, perihelion advance of a planet, radar echo delay, and gravitational redshift in the environment of the Schwarzschild gravity field generated by a static, spherically symmetric material source like the sun.[6]

In the present approach, the only ingredient of GR to be used is the isotropic form of the metric relevant to the gravity field (here the Schwarzschild field). No sophisticated mathematics involving four vectors, tensors, Christoffel symbols, or geodesic equations will be necessary. In addition to being simpler and hence understandable to a wide range of readers, the ensuing exposition offers a different window through which the GR events can be visualized. From the instructional point of view, it is always useful to look at familiar things from as many different perspectives as possible.

It would be worthwhile to display the relevant formulas of the optical–mechanical analogy that we are going to use in our paper. This is done in Sec. II. Thereafter, as a necessary step, the Schwarzschild gravity field is portrayed as a refractive optical medium in Sec. III. GR equations for the light orbit and the radar echo delay in Schwarzschild gravity are derived in Secs. IV and V, respectively. Finally, some relevant remarks including a few suggestions for future work are made in Sec. VI.

II. OPTICAL–MECHANICAL ANALOGY

We need not go into the detailed formulation of the "$F=ma$" optics of ER. Instead, for our present purpose, we reproduce only a table which is self-explanatory[3]

Quantity	Mechanics	Optics
Position	$\mathbf{r}(t)$	$\mathbf{r}(a)$
"Time"	t	a
"Velocity"	$\frac{d\mathbf{r}}{dt}\equiv\dot{\mathbf{r}}$	$\frac{d\mathbf{r}}{da}\equiv\mathbf{r}'$
"Potential energy"	$U(\mathbf{r})$	$-n^2(\mathbf{r})/2$
"Mass"	m	1
"Kinetic energy"	$T=\frac{m}{2}\lvert\dot{\mathbf{r}}\rvert^2$	$\frac{1}{2}\lvert\mathbf{r}'\rvert^2$
"Total energy"	$\frac{m}{2}\lvert\dot{\mathbf{r}}\rvert^2+U$	$\frac{1}{2}\lvert\mathbf{r}'\rvert^2-\frac{n^2}{2}$
"Equation of motion"	$m\ddot{\mathbf{r}}=-\operatorname{grad}U$	$\mathbf{r}''=\operatorname{grad}\left(\frac{n^2}{2}\right)$

As can be seen, the role of time t is played by the ER stepping parameter a having the dimension of length so that $\mathbf{r}'$ is not a velocity. It is a dimensionless quantity. The transition between t and a can be accomplished by using the relation

$$da=\frac{c_0}{n^2}\,dt, \tag{1}$$

where n denotes the refractive index of the optical medium, c_0 denotes the vacuum speed of light. Evans[4] has shown that the stepping parameter can be physically interpreted as "optical action." Just as a mechanical particle progresses in time along its trajectory, light progresses in optical action along its ray.

The identification

$$U(\mathbf{r})=-\frac{n^2}{2} \tag{2}$$

giving the "potential" U essentially bestows a mechanical character to photon motions; the only restriction being that the motion corresponds to mechanics at "zero total energy." We can also imagine the possibility that, at least in the classical regime, optical and mechanical motions may take place under the same form of "force"/force law. An excellent example is probably the Luneburg Lens in optics and the harmonic oscillator in mechanics.[3] In order to use the ER formulation in the GR regime, it would be necessary to associate a single scalar function, the optical refractive index n, with the gravity field under consideration. The "force" on massless particles moving in that gravity field can then be explicitly obtained[7] via the ER expression $\operatorname{grad}(-n^2/2)$.

From now on, we shall use primes (′) only to designate a radial coordinate (r') and not differentiation with respect to the stepping parameter a. All the differentiations will be displayed in full.

III. GRAVITY FIELD AS A REFRACTIVE MEDIUM

One might wonder what relationship could there possibly be between two entities apparently as diverse as a gravity field and a refractive optical medium? But, indeed, there is one! It was guessed by Einstein and Eddington,[8] formally developed by Plěbanski and others[9] and utilized in the investigation of specific problems by many physicists.[9,10] We shall only quote the results in a form that is easily intelligible. Plebanski[9] has shown that, in a curved spacetime with a metric tensor $g_{\alpha\beta}$, Maxwell's electromagnetic equations can be rewritten as if they were valid in a flat space–time in which there is an optical medium with a constitutive equation. More specifically, with regard to light propagation, the gravity field behaves as a refractive optical medium. For example, the gravitational field exterior to a static, spherically symmetric mass M is equivalent to an isotropic, nonhomogeneous optical medium with a refractive index n given by[10]

$$n^2(r)=\left(1+\frac{m}{2r}\right)^6\left(1-\frac{m}{2r}\right)^{-2},\quad m=GM/c_0^2, \tag{3}$$

where, as usual, G is the gravitational constant, r is the Euclidean radial coordinate.

Advanced students who are likely to have a fair grasp on the two field theories, Einstein's and Maxwell's, along with the details of algebraic manipulations should have no difficulty in pursuing the analysis of Plebanski and de Felice.[9,10] However, there is a simpler alternative derivation of Eq. (3), given below, that demands only the knowledge of the form of the Schwarzschild exterior metric. One may also take Eq. (3) at its face value and proceed by treating this $n(r)$ as just a given choice of the refractive index in the ER formulation. We shall essentially investigate the consequences of such a choice in the succeeding sections.

Consider the exterior Schwarzschild metric in standard coordinates (r',θ,φ,t)

$$ds^2=\left(1-\frac{2m}{r'}\right)c_0^2\,dt^2-\left(1-\frac{2m}{r'}\right)^{-1}dr'^2-r'^2\times(d\theta^2+\sin^2\theta\,d\varphi^2)$$
$$=B(r')c_0^2\,dt^2-A(r')dr'^2-r'^2(d\theta^2+\sin^2\theta\,d\varphi^2), \tag{4}$$

where $r'>2m$. Redefine the radial coordinate as $r'=r\phi^{-1}(r)=r(1+(m/2r))^2$, so that if $r=u^{-1}$ and $r'=u'^{-1}$, then

$$u'=u\phi(u), \tag{5}$$

$$\phi(u)=\left(1+\frac{mu}{2}\right)^{-2}. \tag{6}$$

The metric (4) can then be rewritten in the so-called isotropic coordinates (r,θ,φ,t) as

$$ds^2=\Omega^2(r)c_0^2\,dt^2-\phi^{-2}(r)[dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\varphi^2]=\Omega^2(r)c_0^2\,dt^2-\phi^{-2}(r)|d\mathbf{r}|^2, \tag{7}$$

where $\Omega(r)=(1+(m/2r))^{-1}(1-(m/2r))$.

The above form of the metric has a conformally Euclidean spatial part.[11] Therefore, the isotropic coordinate speed of light $c(r)$ at any arbitrary point in the gravitational field, obtained by putting $ds^2=0$, is

$$c(r)=\left|\frac{d\mathbf{r}}{dt}\right|=c_0\phi(r)\Omega(r). \tag{8}$$

Defining the refractive index as $n(r)=c_0/c(r)$, we have

$$n(r)=\Omega^{-1}\phi^{-1}=\left(1+\frac{m}{2r}\right)^3\left(1-\frac{m}{2r}\right)^{-1}, \tag{9}$$

which is precisely the same as Eq. (3) above. It is easy to see that

$$\phi^2(u')=2^{-4}[1+(1-2mu')^{1/2}]^4, \tag{10}$$

$$\Omega^2(u')=1-2mu', \tag{11}$$

and

$$du'=\phi(u)\Omega(u)du. \tag{12}$$

All the above relationships will be used throughout the paper. It should be mentioned that Sjödin[12] has also derived the expressions for ϕ, Ω and n by a different method that is based on Rindler's operational definitions of length and time in a gravitational field.

The refractive index $n(r)$ above corresponds to a radial "force law" having a magnitude

$$F=\frac{dU}{dr}=\left[2mr^{-2}-\frac{m}{2}r^{-3}\right]\left[1+\frac{m}{2}r^{-1}\right]^5\left[1-\frac{m}{2}r^{-1}\right]^{-3}.$$

IV. LIGHT ORBIT EQUATION

Within the ER framework, the optical analog of the mechanical zero total energy or in short "zero total energy" (see the entry in the table in Sec. II) is given by

$$\frac{1}{2}\left|\frac{d\mathbf{r}}{da}\right|^2-\frac{n^2}{2}=0. \tag{13}$$

We straightaway claim that this very equation is the GR light orbit equation in Schwarzschild gravity, provided n is given by Eq. (9). In order to see that, we proceed as follows. Since there is spherical symmetry in the problem, we can, without loss of generality, fix a plane in which the orbiting has to take place. It is customary[13] to choose $\theta=\pi/2$. Writing out Eq. (13) in plane polar coordinates, we have

$$\left(\frac{dr}{da}\right)^2+r^2\left(\frac{d\varphi}{da}\right)^2-n^2(r)=0. \tag{14}$$

Notice that Eq. (13) is the first integral of the "equation of motion" $\mathbf{r}''=\text{grad}\,(n^2/2)$. Since n does not depend on φ, the φ component of the equation of motion yields a conserved quantity called by ER the "angular momentum" h, given by

$$h=r^2\frac{d\varphi}{da}=\text{constant}. \tag{15}$$

Eliminating the stepping parameter a, and writing $r=u^{-1}$, we have

$$u^2+\left(\frac{du}{d\varphi}\right)^2-n^2h^{-2}=0. \tag{16}$$

Evans and Rosenquist have solved a number of problems in classical optics/mechanics using different choices for $n(\mathbf{r})$. Below we shall discuss, in the context of GR, some aspects of the Eqs. (13)–(16):

(i) As ER have already discussed, the optical quantity h is completely different from the corresponding conserved mechanical quantity $h_0=r^2\,d\varphi/dt$ which is proportional to the areal velocity. In the optical formalism, we have $h=c_0^{-1}n^2r^2\,d\varphi/dt$ so that $h_0\propto n^{-2}$ and $d\varphi/dt\propto n^{-2}r^{-2}$. Hence, with our form for $n(r)$, Eq. (9), neither the areal velocity h_0 nor the angular velocity $d\varphi/dt$ remains constant. Of course, for the same force law, the optical and mechanical zero energy orbits must have the same form, since a or t are ultimately eliminated. We see that the GR light orbit equation has the form of well-known classical optics equation,[14] although this interesting fact is not very widely noticed. The reason for this is that in most text books on GR, the light orbit equation is given in a completely different form.

(ii) Let us obtain that familiar text book form. Using Eqs. (5), (11), and (12) in Eq. (16), we get

$$\phi^{-2}(1-2mu')^{-1}\left(\frac{du'}{d\varphi}\right)^2+u'^2\phi^{-2}-n^2h^{-2}=0. \tag{17}$$

Using Eq. (9) for n, we find

$$u'^2+\left(\frac{du'}{d\varphi}\right)^2-2mu'^3-h^{-2}=0. \tag{18}$$

Differentiating with respect to φ, we get

$$u'+\frac{d^2u'}{d\varphi^2}-3mu'^2=0. \quad (19)$$

This is precisely the light orbit equation in Schwarzschild gravity in standard coordinates. Therefore, one can say that the familiar Eq. (19) represents in disguise just the optical analog of the classical zero total energy mechanics. This conclusion will find a further justification in the developments of Sec. V.

(iii) It would be of interest to see how the "angular momentum" h is related to the conserved GR quantities E and L, associated with the killing fields $\partial/\partial t$ and $\partial/\partial\varphi$, respectively. This would also provide a relationship between the ER stepping parameter a and the geodesic affine parameters used by relativists. Integration of the GR null geodesic equations gives

$$\left|\frac{dr}{dt}\right|^2=c_0^2\Omega^2\phi^2, \quad \Omega^2\frac{dt}{dp}=E, \quad \phi^{-2}r^2\frac{d\varphi}{dp}=L, \quad (20)$$

where $ds^2=\lambda\ dp^2$, λ is a constant and p is a new geodesic affine parameter such that $ds^2=0\Rightarrow\lambda=0$, $dp^2\neq 0$. Eliminating t from Eqs. (20), we get

$$c_0^{-2}E^{-2}\phi^{-4}\left|\frac{dr}{dp}\right|^2-n^2=0. \quad (21)$$

If we now define

$$dp=c_0^{-1}E^{-1}\phi^{-2}\ da \quad (22)$$

then Eq. (13) follows immediately from Eq. (21). We also find that $h=L/c_0E$. From Eqs. (20) and (22), there also follows the ER relation (1) connecting dt and da. Further, Eq. (22) implies that $ds^2=\lambda c_0^{-2}E^{-2}\phi^{-4}\ da^2$, giving the connection between a and the affine parameter s.

(iv) By integrating Eq. (19), one obtains all allowable light orbits;[15] bound, unbound, or even the so-called *boomerang* orbits.[16] There have been many observations confirming the GR predictions of the bending of light rays just grazing the sun, the amount being $\Delta\varphi\sim 4m/R_0$, where R_0 is the solar radius. Interested readers may consult any textbook on GR. We shall only make a relevant remark here. Møller[17] has shown that the bending of light rays is due partly to the geometrical curvature of space and partly to the variation of light speed in a Newtonian potential. In fact, the ratio of the parts is 50:50. The GR null trajectory equations can be integrated, once assuming a Euclidean space with a variable light speed and again a curved space with a constant light speed; both contributing just half the amount of the observed bending. In the present approach, on the other hand, we are describing light motion by means of a scalar function $n(r)$. Thus with regard to the light propagation, it looks as if spatial curvature and Newtonian potential lost their separate identities and merged into an equivalent refractive medium. There is, of course, no point in asking which one has a physical reality and which one has not; all these are mathematical constructs [like the $n(r)$ here] designed only to interpret our physical observations.

(v) Finally, some words of caution. From the similarity of Eqs. (16) and (18), one might be tempted to conclude that in the u' coordinates, n is given by $n(u')=(1+2mh^2u'^3)^{1/2}$, but that would be incorrect. The correct expression for $n(u')$ is obtained by using the fact that $n(u)$ transforms as a scalar. Hence, one obtains[18]

$$n(u')=4(1-2mu')^{-1/2}[1+(1-2mu')^{1/2}]^{-2}. \quad (23)$$

Also, it must be understood that, with this $n(u')$, it is not possible to define an isotropic coordinate velocity of light in the u' coordinates. From a direct calculation with the metric (4), it will turn out that the coordinate velocities of light are different in radial and cross radial directions.

V. RADAR ECHO DELAY

Let us now go beyond the geometrical shape of the light orbit and consider the dynamics along its path. In other words, we shall derive the GR equation of motion involving the time t for the propagation of light rays around a static, spherically symmetric gravitating mass M.

Once again we start from the ER "zero energy" equation $(1/2)|dr/da|^2-(n^2/2)=0$ and claim that it *is* the GR equation we are looking for provided n is given by Eq. (9). To see this, consider Eqs. (13)–(15) and write

$$\left(\frac{dr}{da}\right)^2+h^2r^{-2}-n^2(r)=0. \quad (24)$$

Utilizing the redefinition $u\rightarrow u'$ and the expression for n from Eq. (9), we have

$$\phi^4u'^{-4}\left(\frac{du'}{da}\right)^2+h^2(1-2mu')u'^2-1=0. \quad (25)$$

At the position $r_0'[=u_0'^{-1}]$ of the closest approach to the gravitating mass M, we have $dr'/da=0\Rightarrow du'/da]_{u_0'}=0$, giving $h^2=(1-2mu_0')^{-1}u_0'^{-2}$. Putting it in Eq. (25), we find

$$\phi^4\Omega^{-2}u'^{-4}\left(\frac{du'}{da}\right)^2+u_0'^{-2}u'^2(1-2mu_0')^{-1}-\Omega^{-2}=0. \quad (26)$$

Noting from the metric (4) that $B(r')=\Omega^2$ and $A(r')=\Omega^{-2}$ and using Eq. (1), we finally have

$$c_0^{-2}A(r')B^{-2}(r')\left(\frac{dr'}{dt}\right)^2+r_0'^2r'^{-2}B^{-1}(r_0')-B^{-1}(r')=0. \quad (27)$$

This is precisely the textbook form of what is known as the radar echo delay equation in Schwarzschild gravity—and, once again, we see that it is still the same ER "zero energy" mechanics, only buried in the (r',t) language. Equation (27) is integrated to obtain the time t required by the light signal (in practice, a radar signal) to travel from one point of space to another. It is also evident that the coordinate speed of light dr'/dt is less than what it would be if the gravitating mass were absent. In other words, light is slowed down and the travel time is longer. All observers will nonetheless measure a photon's speed to be c_0 through their positions. In a round trip journey around the mass M, there would be a net GR delay in the radar echo reception. This GR prediction has been confirmed to a great accuracy by sending radar signals from Earth to Mercury at superior conjunction and back.[19]

VI. SOME REMARKS

The contents of the entire paper vividly demonstrate that light propagation in Schwarzschild gravity is indeed describable by the language of "$F=ma$" optics, a shorthand for the optical–mechanical analogy. It is remarkable that the ER formulation, emerging basically from the investigations of classical problems, works equally well also in the GR regime.

We are concerned with the predictions for light propagation that follow exclusively from Einstein's field equations. On the other hand, the formula for gravitational redshift can be derived from special relativity and the principle of equivalence alone. The field equations or their solutions need not be used. In that sense, the redshift is not a prediction following exclusively from the field equations, albeit the effect *does* follow also from the latter. In the present approach, the usual Principle of Equivalence that equates the effects of gravity and artificial forces, is translated into an equivalence of the effects of gravitation and refractive medium. Using this idea, the gravitational redshift formula can be obtained in its familiar form. We shall present the deductions in a separate paper.

It is evident that the considered approach can deal with a reasonably general class of gravity fields where the metric can be cast into an isotropic form. The latter makes it possible to construct a scalar function n which acts as a representative of the gravity field in the matter of light propagation. However, the whole range of GR solutions corresponding to different field distributions can not be made to correspond to such single scalar functions. At best, a detailed constitutive tensor for the equivalent medium may be developed.[10] In this case, the challenging task would be to generalize the ER equations appropriately. Nonetheless, we can list some immediate future works:

(i) The motion of a light pulse inside and across the body of a spherical star can be tackled quite comfortably. The interior of a spherical star (like the sun) is described by the Schwarzschild interior metric. It corresponds to a refractive medium with index, say, n_1 while the exterior field corresponds to n, Eq. (9). Therefore, the whole problem is reduced to one of light propagation in a composite media with indices n and n_1. At the interface, the matching condition is provided by none other than good old Snell's law. The result will provide a theoretical idea about the total path of light rays between the sun's core and the Earth.

(ii) For the sake of completeness, we would expect the "$F=ma$" optics to describe also the motion of massive particles (planetary motion) in the Schwarzschild gravity field. To that end, efforts are underway to extend the present treatment. The answer to the second part of the question raised in Sec. I would then be available.

Finally, it must be emphasized that there is neither any substitute for nor shortcut to the beauty, generality, and richness of Einstein's general relativity theory. One must eventually grasp all the details of the physics, mathematics, and the philosophy of this magnificent never ending edifice. On the other hand, the language of the ER optical–mechanical analogy has the power to stimulate the interests of a wide cross section of readers who do not have a formal training in the sophistications of GR. Those who have the training may, however, regard the preceding developments as providing yet another avenue to the same experimentally verified tests of light motion in GR. The educational importance of such alternative points of view can not be mistaken.

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[1]As quoted by Evans, the optical–mechanical analogy was formulated by Hamilton (1833) in terms of characteristic functions. See, J. Evans, "The ray form of Newton's law of motion," Am. J. Phys. 61, 347–350 (1993). In this paper, Evans derives the ray form without the use of characteristic functions.

[2]Historically, the basic idea of the optical–mechanical analogy was proposed much earlier by none other than Descartes (1637). This information is due to J. A. Arnaud, "Analogy between optical rays and nonrelativistic particle trajectories: A comment," Am. J. Phys. 44, 1067–1069 (1976). In this paper are also discussed the limitations and significance of the analogy.

[3]Essential ingredients of what is referred to as the ER approach are contained in: J. Evans and M. Rosenquist, ""$F=ma$" optics," Am. J. Phys. 54, 876–883 (1986).

[4]J. Evans, "Simple forms for equations of rays in gradient-index lenses," Am. J. Phys. 58, 773–778 (1990), see especially p. 774.

[5]Newton's laws of motion are obtained directly from Fermat's principle, in M. Rosenquist and J. Evans, "The classical limit of quantum mechanics from Fermat's principle and the de Broglie relation," Am. J. Phys. 56, 881–882 (1988).

[6]For a general information: Latest confirmations of GR come from the observations of the Hulse–Taylor binary pulsar PSR1913+16. There is a rate of decrease in the orbital period $[P\sim(3.2\pm0.6)\times10^{-12}\mathrm{ss}^{-1}]$ as well as a precession of the order of 4° per year. Observations are in excellent agreement with the GR predictions. See, J. H. Taylor, L. A. Fowler and P. M. McCulloch, "Measurements of general relativistic effects in the binary pulsar PSR 1913+16," Nature (London) 277, 437–440 (1979).

[7]To repeat, the "force" is not the force in the mechanical sense. Nonetheless, a material particle acted on by a mechanical force of the same form will follow the same null track. See Ref. 16 below for the citation of an interesting example.

[8]Sir A. S. Eddington calculated the bending of light rays round a massive object by assuming that the space around the sun is filled with a medium with a refractive index $n(r)=(1-2m/r)^{-1}\sim1+2m/r$. Incidentally, he was also the first to experimentally observe such a bending in 1919 during the Solar eclipse in Sobral, Brazil. See, A. S. Eddington, *Space, Time and Gravitation* (Cambridge University, Cambridge, 1920), reissued in the Cambridge Science Classics Series, 1987, p. 109. However, F. de Felice (Ref. 10) reports that Einstein himself was the first to suggest the idea of an equivalent refractive medium.

[9]Among a number of works in this direction, we list only a few, just to give an idea: J. Plebanski, "Electromagnetic waves in gravitational fields," Phys. Rev. 118, 1396–1408 (1960). B. Bertotti, "The luminosity of distant galaxies," Proc. R. Soc. 294, 195–207 (1966). F. Winterberg, "Deflection of gravitational waves by stellar scintillation in space," Nuovo Cim. B 53, 264–279 (1968). See also Ref. 10 for a collection of other references.

[10]F. de Felice shows that the equivalence of a gravity field with the refractive medium can be successfully employed as a method of investigation: F. de Felice, "On the gravitational field acting as an optical medium," Gen. Rel. Grav. 2, 347–357 (1971).

[11]Because of the general coordinate covariance of Einstein's GR, its solutions can be freely expressed in any coordinate system we like. Such a freedom, however, does not affect the unique observable predictions of GR. This important point is further illuminated in some recent papers. See, Ya. B. Zel'dovich and L. P. Grishchuk, "The general relativity theory is correct," Sov. Phys. Usp. 31, 666–671 (1988), and references therein. T. Ohta and T. Kimura, "Coordinate independence of physical observables in the classical tests of general relativity," Nuovo Cim. B 106, 291–305 (1991); A. N. Petrov, "New harmonic coordinates for the Schwarzschild geometry and the field approach," Astron. Astrophys. Trans. 1, 195–205 (1992). We would particularly recommend these papers to the advanced graduate students and researchers in GR.

[12]T. Sjödin, in *Physical Interpretations of Relativity Theory*, edited by M. C. Duffy (British Soc. Philos. Sc., London, 1990), pp. 515–521. Sjödin also obtains the light orbit equation starting from Rindler's prescription. See,

W. Rindler, *Essential Relativity* (Springer, New York, 1977), especially Secs. 8.3 and 8.4.

[13]The reason for fixing the $\theta=\pi/2$ plane is that $\partial/\partial\varphi$ is a Killing field on the entire manifold while $\partial/\partial\theta$ is not.

[14]Equation (16) is the classical optics equation. See, A. Maréchal, *Optique Géometriqué Générale*, edited by S. Flügge, Handbuch der Physik, Vol. 34 (Springer, Berlin, 1956), p. 44. M. Born and E. Wolf, *Principles of Optics*, 2nd ed. (Pergamon, New York, 1964), pp. 121–124.

[15]A number of light orbits have been plotted in S. Chandrasekhar, *The Mathematical Theory Of Black Holes* (Oxford University, Oxford, 1983). See also the classic treatise L. D. Landau and E. M. Lifshitz, *The Classical Theory Of Fields* (Pergamon, New York, 1975), pp. 313–316.

[16]W. M. Stuckey, "The Schwarzschild black hole as a gravitational mirror," Am. J. Phys. 61, 448–456 (1993). The possibility of a boomerang shaped light orbit is interesting in its own right. We can say that the path of a real massive boomerang around a central mass is also obtained by the same force law in mechanics. The total energy, however, need not be zero.

[17]The detailed calculations appear in C. Møller, *The Theory of Relativity*, 2nd ed. (Oxford University, Oxford, 1972), pp. 498–501. The variable speed of light is given by $c^{*}=c_0(1+2\chi/c_0^2)^{1/2}$, where $\chi=-GM/r'$. Purely spatial curvature is represented by the metric $ds^2=c_0^2\,dt^2-(1-2m/r')^{-1}\,dr'^2-r'^2(d\theta^2+\sin^2\theta\,d\varphi^2)$. Møller expresses the null trajectory in terms of c^{*} and the metric tensor $(g_{\alpha\beta})$ to demonstrate his result.

[18]Notice that at the Schwarzschild black hole radius $r'=2m$, $n(u')=\infty$. Also from Eq. (9), at $r=m/2$, $n(u)=\infty\Rightarrow c(r)=0$. This result reflects the fact that, to a distant observer, even light comes to a standstill! To a local observer, however, light always travels at a speed c_0.

[19]I. I. Shapiro *et al.*, "Fourth test of general relativity: New radar result," Phys. Rev. Lett. 26, 1132–1135 (1971). The GR predicted value of time delay for the Earth–Mercury system is ~240 μs while the observed value is $(245\pm12)\mu$s. A remarkable agreement indeed.

Potentials and bound states

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We discuss several quantum mechanical potential problems, focusing on those which highlight commonly held misconceptions about the existence of bound states. We present a proof, based on the variational principle, that certain one dimensional potentials always support at least one bound state, regardless of the potential's strength. We examine arguments concerning the existence of bound states based on the uncertainty principle and demonstrate, by explicit calculations, that such arguments must be viewed with skepticism.

I. INTRODUCTION

One of the first types of problems encountered by students beginning a study of quantum mechanics is that of finding the eigenstates of a potential. Such problems form the basis of understanding for a great many physical systems, and so are important not just as pedagogical exercises, but also as real world models in solid state, nuclear, atomic and molecular physics. In addition, simple one and two dimensional potentials form the basis of our understanding of low dimensional structures such as quantum well devices.[1]

There is a substantial folklore concerning these simple potential problems. In surveying a variety of standard introductory (or even advanced) quantum mechanics texts, one finds various fragments of this folklore but rarely are they presented in a comprehensive fashion which would allow the reader to apply them to more general problems or, for that matter, to understand their physical and mathematical origin. This situation is made worse by the fact that some of these so-called standard results are wrong. Our purpose here is to present an organized view of a selection of this folklore, expunging the erroneous results along the way.

Before proceeding, the reader is asked to apply his or her knowledge of this folklore to the following questions: How would you modify the statement, "Every potential has at least one bound state," in order to make it true? (Not comprehensive, only correct.) How would you prove it? Can the condition for the existence of at least one bound state in a spherical step well be related to the Heisenberg uncertainty relation? If so, does such a relation also apply to the one dimensional step well? How does $\Delta x\Delta p$ behave as a potential well is made deeper and additional eigenstates appear?

To focus the discussion of the issues raised above, we consider the spherically symmetric step well

$$V(r)=\begin{cases}-V_0 & r\leqslant a;\\ 0 & r>a,\end{cases}\tag{1}$$

which supports a bound state only for

$$V_0>\frac{\hbar^2\pi^2}{8ma^2}.\tag{2}$$

This is a generic feature of three dimensional central potentials. Numerous authors[2,3] have attempted to give a physical explanation of this by means of the uncertainty principle. The essence of the argument is as follows: Assuming $\Delta x\sim a$, from the uncertainty relation one obtains

$$\Delta p\sim\frac{\hbar}{2a}.\tag{3}$$

For the states under consideration it is reasonable to assume

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The Optical-Mechanical Analogy in General Relativity: Exact Newtonian Forms for the Equations of Motion of Particles and Photons

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In many metrics of physical interest, the gravitational field can be represented as an optical medium with an effective index of refraction. We show that, in such a metric, the orbits of both massive and massless particles are governed by a variational principle which involves the index of refraction and which assumes the form of Fermat's principle or of Maupertuis's principle. From this variational principle we derive exact equations of motion of Newtonian form which govern both massless and massive particles. These equations of motion are applied to some problems of physical interest.

1. INTRODUCTION

The representation of the gravitational field as an optical medium is an old idea, which was exploited by Eddington (Ref. 1, p.109) and which has been developed in more detail by others [2,3]. In many metrics of physical interest, one may find a coordinate transformation that renders the space part of the line element isotropic. If, in addition, the metric has no explicit

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time dependence, the line element may be written in the form

$$\begin{aligned} ds^2 &= \Omega^2(\mathbf{r})c_0^2\,dt^2 - \Phi^{-2}(\mathbf{r})\,[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2] \\ &= \Omega^2(\mathbf{r})c_0^2\,dt^2 - \Phi^{-2}(\mathbf{r})\,|d\mathbf{r}|^2, \end{aligned} \tag{1}$$

where Ω and Φ are functions of the so-called isotropic coordinates r, θ, and ϕ (which are related to the standard coordinates by a transformation) and c_0 is the vacuum speed of light. Boldface $\mathbf{r}$ is an abbreviation for (r, θ, ϕ). The isotropic coordinate speed of light $c(\mathbf{r})$ at any point in the field may be obtained by putting $ds = 0$:

$$c(\mathbf{r}) = |d\mathbf{r}/dt| = c_0\Phi(\mathbf{r})\Omega(\mathbf{r}). \tag{2}$$

Thus the effective index of refraction is

$$n = \Phi^{-1}(\mathbf{r})\Omega^{-1}(\mathbf{r}). \tag{3}$$

Light trajectories in the gravitational field can be calculated by using the effective index of refraction (3) in any formulation of geometrical optics that happens to be convenient. For example, Wu and Xu have recently shown that the standard differential equation of the ray in classical geometrical optics can be applied to the null geodesic problem [4].

An especially convenient version of geometrical optics is the so-called "$F = ma$" formulation [5,6] in which the equation governing the optical ray assumes the form of Newton's law of motion (acceleration = − gradient of potential energy):

$$d^2\mathbf{r}/dA^2 = \nabla(n^2c_0^2/2). \tag{4}$$

$\mathbf{r}$ is the position of a light pulse moving along the ray. The independent variable (analogous to the time) is the stepping parameter, or optical action A. The effective potential energy function is $-n^2c_0^2/2$. All the usual force and energy methods of elementary mechanics can be brought to bear on geometrical optics.

The effective index of refraction (3) (for the Schwarzschild metric, for example) can be used in (4) without modification [7]. In solving problems, one goes into the isotropic coordinates, applies the $F = ma$ optics, then transforms back to the standard coordinates, if desired. The goal of the present paper is to extend these methods to the motion of massive particles.

Fermat's principle has been the subject of renewed interest in general relativity [8–11]. The present paper differs from other recent work in this area in that (i) it focuses on the analogy between the principles of Maupertuis and Fermat in the context of general relativity and (ii) it leads to a remarkable simplification of the equations of motion for both particles and photons.

In Section 2, we derive from the geodesic condition a variational principle which takes on the form of Fermat's principle or Maupertuis' principle. The variational principle, which governs the trajectories of both massive and massless particles, implies that, in isotropic metrics, the particle equations of motion can be cast into the form of Newtonian mechanics or of classical geometrical optics. In Section 3, we derive equations of motion from the variational principle. These equations of motion, which represent a *generalization of (4), are exact, apply equally to massive and massless* particles, but are nevertheless of Newtonian form. In Section 4, we present effective indices of refraction for a number of metrics of physical and cosmological significance. In Section 5, we illustrate the use of the new equations of motion in some concrete applications: we demonstrate an analogy between two cosmological models and the Maxwell fish-eye lens, we extend some recent calculations involving tests of general relativity in Reissner–Nordström-type metrics, and we present novel derivations of the gravitational and cosmological redshifts.

2. TRANSFORMATION OF THE GEODESIC CONDITION

Our goal is to apply the classical optical-mechanical analogy to particle orbits in general relativity. In order to set up the analogy, it will be convenient to begin from a variational principle for the trajectories that can be considered analogous to the principle of Fermat (classical geometrical optics) and the principle of Maupertuis (Newtonian mechanics in velocity-independent potentials).

We shall obtain the variational principle by transformation of the geodesic condition for the particle trajectories,

$$\delta \int_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} ds = 0, \tag{5}$$

where δ indicates a variation in the path of integration between two fixed points in spacetime, $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_2)$. If we assume the line element can be written in the form (1) this becomes

$$\delta \int_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} \Omega c_0 [1 - v^2 n^2 / c_0^2]^{1/2} \, dt = 0 . \tag{6}$$

This is analogous to Hamilton's principle and the effective Lagrangian is

$$L(x_i, \dot{x}_i) = -c_0^2 \Omega [1 - v^2 n^2 / c_0^2]^{1/2}, \tag{7}$$

where Ω and n are functions of the coordinates alone, where $\dot{x}_i \equiv dx_i/dt$, and where $v^2 = \sum_{i=1}^{3} (dx_i/dt)^2$, if we choose to work in Cartesian coordinates. The expression for the Lagrangian has been multiplied by an extra factor of $-c_0$ for later convenience. (Note: We will always write c_0 explicitly. This paper is concerned with an analogy linking geodesic motion, classical geometrical optics, and classical Newtonian mechanics. c_0 is not usually suppressed in the latter two fields. Thus the clarity of the analogy is enhanced by retaining classical units of measure.)

The canonical momenta p_i are

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{x}_i} \\ &= \Omega n^2 [1 - v^2 n^2 / c_0^2]^{-1/2} \, \dot{x}_i \, . \end{aligned} \tag{8}$$

The Hamiltonian H may be formed in the usual way,

$$\begin{aligned} H &= \sum_{i=1}^{3} p_i \dot{x}_i - L \\ &= c_0^2 \Omega [1 - v^2 n^2 / c_0^2]^{-1/2}. \end{aligned} \tag{9}$$

Because $\partial L / \partial t = 0$, H is a constant of the motion. If we express H in terms of the p_i rather than the x_i we obtain

$$H = c_0^2 [\Omega^2 + p^2 / n^2 c_0^2]^{1/2}, \tag{10}$$

where $p = |\vec{p}|$. From Hamilton's principle,

$$\delta \int_{\mathbf{x}_1, t_1}^{\mathbf{x}_2, t_2} L \, dt = 0, \tag{11}$$

one may derive in the usual way the corresponding action principle (Jacobi's form of Maupertuis' principle) (Ref. 12, p.125-8,132-4),

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} \left(\sum_{i=1}^{3} p_i \dot{x}_i \right) \, dt = 0, \tag{12}$$

where now the path of integration is varied between two fixed points in space, $\mathbf{x}_1$ and $\mathbf{x}_2$, where the energy must be held constant on the varied paths, but where the times at the end points need not be held fixed. With the canonical momenta from (8), this becomes

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v^2 \Omega [1 - v^2 n^2 / c_0^2]^{-1/2} \, dt = 0 . \tag{13}$$

We restrict the varied paths to those that satisfy the energy constraint by substituting the constant H for the right side of (9) where this appears in (13). Then, putting $dt = dl/v$, where $dl = |d\mathbf{r}| = (\sum_{i=1}^{3} dx_i^2)^{1/2}$ we obtain

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v \, dl = 0 . \tag{14}$$

This is a variational principle on which an analogy to geometrical optics or to classical mechanics can be constructed. In obtaining (14) we have preferred, for the sake of directness, clarity and and consistency of notation, to begin from the fundamental principle (5). But (14) may also be derived from other versions of the three-dimensional variational principle for particle orbits in static metrics, for example, the forms first obtained by Weyl [13] and Levi-Civita [14].

In (14), $n^2 v$ is to be considered a function of position alone. The path of integration is varied between the fixed end points $\mathbf{x}_1$ and $\mathbf{x}_2$, and the value of H is held constant during the variation. Thus, (14) is of the same form as Fermat's principle, which forms a basis for classical geometrical optics, and Maupertuis' principle, which forms a basis for classical mechanics (as long as the force can be derived from a velocity-independent potential):

Relativistic gravitational mechanics	Geometrical optics (Fermat)	Classical mechanics (Maupertuis)
$\delta \int n^2 v \, dl = 0$	$\delta \int n \, dl = 0$	$\delta \int v \, dl = 0 .$

In the context of motion in a static gravitational field, both Fermat's principle and Maupertuis' principle are simply special cases of (14). For the null geodesics, i.e., for the paths of light, the derivation given above must be slightly modified, to keep each step well defined. But the final result is too well known to require detailed discussion here: in static metrics, light obeys Fermat's principle. That is, the path taken by light between two fixed points in space is one for which the coordinate time of travel is stationary (Ref. 15, Ref. 16, p.99-100). In the language of a refractive index, this may be written $\delta\int n\,dl = 0$. Since, for light, $v = c_0/n$, (14) does reduce to the appropriate form. To obtain Maupertuis' principle (and hence Newtonian gravitational motion), note that in ordinary solar-system dynamics, we may put $n^2 \approx 1$. That is, in the Newtonian limit, n^2 may be treated as constant in the variational calculation and we obtain Maupertuis' principle as the classical limit of (14).

3. EXACT EQUATIONS OF MOTION OF NEWTONIAN FORM

Let the path of the particle be parametrized by a stepping parameter A. That is, at each point on the path, the three space coordinates $\mathbf{r}$ (and also the time t) are regarded as functions of A. We defer for the moment choosing A: we shall define A to get the simplest equations of motion. Thus we write (14) in the form

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v \left| \frac{d\mathbf{r}}{dA} \right| dA = 0,$$

where $|d\mathbf{r}/dA| = [\sum_{i=1}^{3}(dx_i/dA)^2]^{1/2}$.

Let $\mathbf{r}(A)$ denote the true path. To obtain a varied path, we replace $\mathbf{r}(A)$ by $\mathbf{r}(A) + \mathbf{w}(A)$, where $\mathbf{w}(A)$ is an arbitrary, infinitesimal vector function, subject to the constraint that $\mathbf{w} = 0$ when A is such that $\mathbf{r} = \mathbf{x}_1$ or $\mathbf{x}_2$. That is, the variation must vanish at the end points. Now

$$\delta \int n^2 v \left| \frac{d\mathbf{r}}{dA} \right| dA = \int [\delta(n^2 v)] \left| \frac{d\mathbf{r}}{dA} \right| dA + \int (n^2 v) \left(\delta \left| \frac{d\mathbf{r}}{dA} \right| \right) dA + \int n^2 v \left| \frac{d\mathbf{r}}{dA} \right| \delta dA. \tag{16}$$

Calculating the two variation in the first term on the right-hand side of (16), we have

$$\delta(n^2 v) = \nabla(n^2 v) \cdot \mathbf{w}. \tag{17}$$

In calculating the variation in the second term of (16) it is important to remember that the change to the varied path will, in general, also produce a change in A. Thus

$$\delta\left|\frac{d\mathbf{r}}{dA}\right| = \left|\frac{d\mathbf{r}+d\mathbf{w}}{dA+\delta dA}\right| - \left|\frac{d\mathbf{r}}{dA}\right|$$
$$= \frac{d\mathbf{r}}{dA}\cdot\frac{d\mathbf{w}}{dA}\left|\frac{d\mathbf{r}}{dA}\right|^{-1} - \left|\frac{d\mathbf{r}}{dA}\right|\frac{\delta dA}{dA}, \tag{18}$$

to first order in the variation. Substituting (17) and (18) into (16), we find

$$\delta\int n^2 v\left|\frac{d\mathbf{r}}{dA}\right| dA = \int\left[\left|\frac{d\mathbf{r}}{dA}\right|\nabla(n^2v)\cdot\mathbf{w} + n^2v\left|\frac{d\mathbf{r}}{dA}\right|^{-1}\frac{d\mathbf{r}}{dA}\cdot\frac{d\mathbf{w}}{dA}\right]dA.$$

Note that the terms involving δdA have cancelled. This was to be expected, since (14) shows that the integral does not actually depend upon the range in A or, indeed, on what we select to use as parameter. Integrating the term involving $d\mathbf{w}/dA$ by parts, and using the fact that $\mathbf{w}$ must vanish at the endpoints, but is otherwise arbitrary, we arrive at the differential equation that must be satisfied by the particle trajectory:

$$\left|\frac{d\mathbf{r}}{dA}\right|\nabla(n^2v) - \frac{d}{dA}\left(n^2v\left|\frac{d\mathbf{r}}{dA}\right|^{-1}\frac{d\mathbf{r}}{dA}\right) = 0. \tag{19}$$

This differential equation plays the role of an equation of motion. Another way to obtain (19) is to parametrize the path by one of the Cartesian coordinates (say z), rather than A, since the variation in z must vanish at $\mathbf{x}_1$ and $\mathbf{x}_2$. In this case, one writes

$$\delta\int n^2v\frac{dl}{dz}\,dz = 0.$$

One may then simply write down the Euler conditions for the integral to be stationary, and then transform from z to A as independent variable. The result will be the same, that is (19).

To give the equation of motion the simplest possible form, and to take advantage of the analogy to Newtonian mechanics, let us now define A by

$$\left|\frac{d\mathbf{r}}{dA}\right| \equiv n^2v. \tag{20}$$

With this definition of A, the equation of motion (19) becomes

$$\frac{d^2\mathbf{r}}{dA^2} = \nabla\left(\frac{1}{2}n^4v^2\right). \tag{21}$$

Equation (21) is the generalization of (4) that was sought. The left-hand side of (21) is of the form of an acceleration: it is the second derivative of the position vector with respect to the independent variable. The right-hand side of the equation is of the form of a force: $-\frac{1}{2}n^4v^2$ plays the role of a "potential energy function." The analogue of the velocity is $d\mathbf{r}/dA$. Thus the analogue of the kinetic energy is $|d\mathbf{r}/dA|^2$. The analogue of the total energy is the sum of the potential and the kinetic. But, by virtue of eq. (20), these two are guaranteed to sum to zero:

$$\tfrac{1}{2}|d\mathbf{r}/dA|^2 - \tfrac{1}{2}n^4v^2 = 0\,. \tag{22}$$

Thus the calculation of the paths of light and of massive particles in general relativity reduces to the zero-energy $F = ma$ optics of [5]. It is to be noted that the "conservation of energy" condition (22) amounts to a restatement of the definition (20) of A.

The optical-mechanical analogy, embodied in (21) and (22), provides an exact treatment in Newtonian form of the motion of massive particles, as well as light, in general relativity. The Newtonian form should be thought of as coming from $F = ma$ optics (which is exact) and not from Newtonian mechanics (which is, of course, only approximate). Equations (21) and (22) allow one to handle the paths of light and of the planets as if they existed in a flat three-dimensional space. Other approaches to this goal are, of course, possible [17], but the treatment presented here has three advantages: simplicity, complete conformity to the equations of *Newtonian mechanics, and a uniform treatment of both light and massive* particles. This treatment has a reasonably high degree of generality and is applicable whenever the line element can be written in the form (1).

In solving problems with (21) or (22), one may use without modification all the familiar methods of Newtonian mechanics. One simply thinks of A as if it were the time. Moreover, rather than beginning from (21) or (22) in every case, one may often simply write down an exact general-relativistic formula by analogy to the corresponding classical formula. For the motion of both light and massive particles in static metrics, one may begin from any classical Newtonian formula describing the motion of particles in static, velocity-independent potentials. The correct general-relativistic expressions will be obtained if one makes the following transcriptions in the classical formulas:

$$t \to A, \qquad U \to -n^4v^2/2, \qquad E \to 0\,. \tag{23}$$

The spatial coordinates (x, y, z), or (r, θ, ϕ), etc., transcribe as themselves. Of course, it must be kept in mind that the formulas written down in this way apply in the isotropic coordinate system. After the equations governing the situation are obtained, or after they are solved, one may transform back to the original metric, if desired. A second point of vital importance is that the analogue of the classical energy E is always the number zero. Thus, the constant of the motion H plays no role in the analogy. H should rather be regarded as a parameter: the potential energy function depends upon H as well as the coordinates. A third point to be stressed is that the exact general-relativistic formulas written down by analogy to the classical formulas will always apply with equal validity to both massive and massless particles.

For both light and particles, the stepping parameter A is defined by (20). In many calculations, e.g., in finding the shape of an orbit, the stepping parameter is ultimately eliminated. Equation (20) will suffice for this purpose. In other situations (for example, in a radar-echo delay calculation), it may be necessary to have an explicit connection between A and t. Note that

$$\left|\frac{d\mathbf{r}}{dA}\right| = \left|\frac{d\mathbf{r}}{dt}\right|\frac{dt}{dA} = v\,\frac{dt}{dA}.$$

Substituting in (20) gives

$$dA = dt/n^2. \tag{24}$$

Thus the stepping parameter is the same as that used in the $F = ma$ formulation of geometrical optics. A is called the optical action because dA is proportional to $c(\mathbf{r})\,dl$, and thus is analogous to the action $v(\mathbf{r})\,dl$ of classical mechanics.

The only difference between the treatment of light and that of particles resides in the choice of $v(\mathbf{r})$, which forms a part of the effective potential energy $-n^4v^2/2$. For light,

$$v = c_0 n^{-1} \qquad \text{(light)}, \tag{25}$$

But for massive particles, (9) gives

$$v = c_0 n^{-1}[1 - c_0^4\Omega^2/H^2]^{1/2} \qquad \text{(particles)}. \tag{26}$$

In (26), H is a constant parameter determined by the initial conditions, while n and Ω are functions of the spatial coordinates determined by the metric. Because the particle expression for $v(\mathbf{r})$ contains the parameter H, the particle problem has an extra degree of freedom: we may specify the initial speed of the particle. Thus, in general, more types of orbits exist for massive particles than for light in the same metric.

For a particle in empty space devoid of gravitational influences, $\Omega \approx 1$, $n \approx 1$, and (9) becomes

$$H \approx c_0^2(1 - v^2/c_0^2)^{-1/2} = c_0^2\gamma. \tag{27}$$

In the solar-system dynamics of the Schwarzschild metric, $v/c_0 \ll 1$ and [see (39) and (43)] $\Omega \approx 1 - m/r$ so (9) becomes

$$H \approx c_0^2 + \tfrac{1}{2}v^2 - m/r. \tag{28}$$

That is, in classical planetary orbits, H is approximately equal to $c_0^2 + E$, the rest-mass energy plus the classical kinetic and potential energy per unit mass.

The classical optical-mechanical analogy is based upon the similarity of form of the principles of Fermat and Maupertuis. Usually this analogy is expressed in terms of nonlinear partial differential equations. In the form of the analogy originally due to William Rowan Hamilton, the eikonal equation of geometrical optics corresponds to the (time-independent) Hamilton–Jacobi equation of particle mechanics. But in fact the analogy is far more general. Corresponding to every possible formulation of classical mechanics, there is an analogous formulation of geometrical optics (and vice versa). Thus, as in (4), we can cast geometrical optics into the form of Newton's law of motion.

The variational principle (14) has permitted us to extend the analogy to the geodesic problem for both light and particles in isotropic metrics. Our discussion of the optical-mechanical analogy in general relativity has stressed Newtonian forms. But, corresponding to every formulation of *either* classical particle mechanics *or* of classical geometrical optics, there will be an analogous formulation of the geodesic problem in general relativity. Few of these classical models for the reformulation of the geodesic equations of motion lead to any special insight or simplification. The economy of expression and simplicity of form embodied in (21) and (22) depend, not so much on the formulation of mechanics (or of geometrical optics) that is chosen as model, as on the use of A rather than t as independent variable. The remainder of the paper is devoted to the development of calculation techniques based on the Newtonian formulation embodied in (21) and (22).

4. INDICES OF REFRACTION FOR SOME IMPORTANT METRICS

4.1. Metrics of the Reissner–Nordström (RN) type

A number of line elements of physical interest assume the following form in standard coordinates (t, r', θ, ϕ):

$$ds^2 = c_0^2\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]dt^2 - \left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2\theta\phi^2, \qquad (29)$$

where

$$m = MG/c_0^2, \qquad (30)$$

M is the mass of the central gravitating body, G is the gravitation constant, and β is another parameter. We wish to write the line element in terms of isotropic coordinates (t, r, θ, ϕ). We will indicate briefly how to effect the transformation, using a systematic technique (Ref. 18, p.174-177). The idea is to express the spatial part as $-\Phi^{-2}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2]$, where $\Phi(r)$ is yet to be determined. Equating the angular and the radial parts of the two line elements, we have

$$r'^2 = \Phi^{-2}r^2 \qquad (31)$$

and

$$\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2 = \Phi^{-2}dr^2. \qquad (32)$$

If we divide (32) by (31) to eliminate Φ, then integrate and use the condition that at large radial distances r and r' must be asymptotically equal, we obtain

$$2r = (r' - m) + (r'^2 - 2mr' + \beta)^{1/2}. \qquad (33)$$

The inverse transformation is

$$r' = r + m + (m^2 - \beta)/4r. \qquad (34)$$

Using these transformations, the line element (29) can be expressed in the form of (1), with

$$\Omega^2(r) = [1 - (m^2 - \beta)/4r^2]^2[1 + m/r + (m^2 - \beta)/4r^2]^{-2} \qquad (35)$$

$$\Phi^{-2}(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2. \qquad (36)$$

The effective refractive index $n(r)$ is

$$n(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2[1 - (m^2 - \beta)/4r^2]^{-1}. \tag{37}$$

It is also helpful to have expressions for Φ, Ω and n in terms of the standard radial coordinate r'. Let $u \equiv 1/r$ and $u' \equiv 1/r'$. Then it is easy to show that

$$\Phi^2(u') = \tfrac{1}{4}[1 - mu' + (1 - 2mu' + \beta u'^2)^{1/2}]^2 \tag{38}$$

$$\Omega^2(u') = 1 - 2mu' + \beta u'^2. \tag{39}$$

And of course

$$n(u') = \Phi^{-1}(u')\Omega^{-1}(u'). \tag{40}$$

In transforming coordinates, it is often helpful to use

$$du = n\,du' \qquad \text{or} \qquad dr = \Phi\Omega^{-1}dr', \tag{41}$$

together with

$$u = \Phi^{-1}u' \qquad \text{or} \qquad r = \Phi r'. \tag{42}$$

The singularities of $n(r)$, or, equivalently, the horizons of the spacetime, occur at $r_s = (m/2)(1 - \beta/m^2)^{1/2}$, provided that $\beta/m^2 \leq 1$. Therefore, the expression for $n(r)$ is valid in the region $r > r_s$. If $\beta = m^2$, $r_s = 0$; i.e., the event horizon shrinks to zero size. In this case, $n(r) = 1 + m/r$ and is regular everywhere for $r > 0$. If $\beta > m^2$, the function $n(r)$ is not singular anywhere, since r_s becomes imaginary. Let us now examine some special cases of the metric (29).

Schwarzschild exterior metric

The Schwarzschild exterior metric applies to the spacetime around an electrically neutral, static, spherical mass M. In this case, (29)–(42) apply (Ref. 19, p.840, Ref. 20, p.515-521) with

$$\beta = 0. \tag{43}$$

Reissner–Nordström (RN) *metric*

The gravitational field due to an electrically charged, static spherical mass M is given by the RN solution of Einstein's field equations. In this case, (29)–(42) apply with

$$\beta = GQ^2/c_0^4, \tag{44}$$

where Q is the charge on the central body.

Bertotti–Robinson (BR) *metric*

The BR metric describes a universe filled with electromagnetic radiation of uniform density and uniformly random direction [21]. In this case, (29)–(42) apply with

$$\beta = m^2 \tag{45}$$

where m is now a nonphysical effective point mass. The BR solution may also be obtained as a special case of the metric obtained recently by Halilsoy.

Halilsoy metric

The Halilsoy metric describes spacetime around a static, uncharged, spherically symmetric mass M which is embedded in an externally created electromagnetic field [22,23]. Equations (29)–(42) apply with

$$\beta = q^2 m^2 \tag{46}$$

where $0 \leq q \leq 1$, and where q represents the measure of the external electromagnetic field.

Soleng metric

The Soleng metric represents the gravitational field due to a central mass M surrounded by a field having a traceless energy-momentum tensor $T^{\mu}{}_{\nu} = f(r)\,\mathrm{diag}\,[1,1,-1,-1]$. Recently, such a $T^{\mu}{}_{\nu}$ has been interpreted as the energy-momentum tensor associated with an anisotropic vacuum [24–26]. Here (29)–(42) apply with

$$\beta = 6\,\delta\, m^2 \tag{47}$$

where δ is the Soleng parameter, which determines the effective energy density of the anisotropic vacuum.

4.2. de Sitter universe and the Maxwell fish-eye

The de Sitter line element in standard coordinates is

$$ds^2 = (1-\Lambda r'^2/3)c_0^2\,dt^2 - (1-\Lambda r'^2/3)^{-1}dr'^2 - r'^2 d\theta^2 - r'^2\sin^2\theta\, d\phi^2, \tag{48}$$

where Λ is the cosmological constant, which is proportional to the space curvature. Λ can be positive or negative, corresponding to a closed or an open de Sitter universe (Ref. 27, p.346-349).

To pass over to isotropic coordinates, we may use the method outlined above, together with the requirement that for small radial distances the new radial variable r should asympotically approach r'. The result is the well-known transformation

$$r' = r(1 + \Lambda r^2/12)^{-1}. \tag{49}$$

Then, in the isotropic coordinates,

$$\begin{aligned} ds^2 &= (1 - \Lambda r^2/12)^2(1 + \Lambda r^2/12)^{-2}C_0^2 dt^2 \\ &\quad - (1 + \Lambda r^2/12)^{-2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2). \end{aligned} \tag{50}$$

The effective index of refraction is

$$n(r) = (1 - \Lambda r^2/12)^{-1}. \tag{51}$$

This index of refraction is valid for either positive or negative Λ, with r defined through (49). Let us examine the case $\Lambda < 0$, corresponding to the open de Sitter universe. Let us write $\Lambda = -K$, where K is then a positive constant. The effective index of refraction of the open de Sitter universe, in the isotropic coordinates, is then

$$n(r) = (1 + Kr^2/12)^{-1}. \tag{52}$$

This index of refraction is of exactly the same form as the index encountered in a traditional problem of classical geometrical optics — the Maxwell fish-eye lens. The index of refraction in the Maxwell fish-eye is

$$n_M(r) = n_0(1 + r^2/a^2)^{-1}. \tag{53}$$

in which a and n_0 are constants. Comparing (52) and (53), we note that the open version of the de Sitter universe is a Maxwell fish-eye lens with $n_0 = 1$ and $a^2 = 12/K$.

4.3. Robertson–Walker universe

The Robertson–Walker (RW) metric represents the gravitational field in a homogeneous and isotropic universe. In the standard comoving coordinates (t, r', θ, ϕ), the RW line element is given by

$$ds^2 = c_0^2\, dt^2 - R^2(t)\left[\frac{dr'^2}{1 - kr'^2} + r'^2 d\theta^2 + r'^2 \sin^2\theta\, d\phi^2\right], \tag{54}$$

in which $R(t)$ is a dimensionless scale factor and k is a constant with dimensions of $(\text{length})^{-2}$. We may pass over to isotropic coordinates by the usual method and requiring that for small radial distances the new radial coordinate r should asymptotically be equal to r'. The result is the well-known transformation

$$r' = r(1 + kr^2/4)^{-1}. \tag{55}$$

In the isotropic coordinates (t, r, θ, ϕ), the line element is

$$ds^2 = c_0^2\,dt^2 - R^2(t)(1 + kr^2/4)^{-2}|d\mathbf{r}|^2. \tag{56}$$

Defining the refractive index n in the usual way, we obtain

$$n = \frac{R(t)}{1 + kr^2/4}. \tag{57}$$

For the case $k > 0$, corresponding to a closed RW universe, and for a fixed cosmological epoch $t = t_0$, this corresponds to the index of refraction (53). Thus the closed Robertson–Walker universe is a Maxwell fish-eye lens with $n_0 = R(t_0)$ and $a^2 = 4/k$. We shall see below that the correspondence between the Maxwell fish-eye and the Robertson–Walker universe does not actually demand that we restrict the latter to a particular moment t_0.

5. SOME APPLICATIONS

5.1. Central-force motion

Many metrics of interest — including all those discussed in Section 4 — are spherically symmetric. In such a case, n, v, Ω and Φ are functions of the radial coordinate alone. The orbit (whether of light or of a massive particle) lies in a plane containing the force center and there is a constant of the motion analogous to the angular momentum. Let ϕ be measured in the plane of the motion. Then, from (21),

$$r^2 d\phi/dA \equiv h = \text{constant}. \tag{58}$$

Note that h is related to the classical-mechanical angular momentum per unit mass h_0 ($\equiv r^2 d\phi/dt$) by

$$h = n^2 h_0. \tag{59}$$

Now we may easily obtain general-relativistic analogues of the standard formulas of classical central-force motion. In (22), which is the analogue of the classical conservation of energy condition, we may write out

$|dr/dA|^2$ in plane-polar coordinates, then eliminate A by means of (58). The orbit shane $\phi(r)$ is thereby reduced to an integration:

$$\phi = h \int^r r^{-2}[n^4v^2 - h^2/r^2]^{-1/2}\, dr. \tag{60}$$

The classical limit of (60) is the familiar equation

$$\phi = h_0 \int^r r^{-2}[2(E-U) - h_0^2/r^2]^{-1/2}\, dr.$$

Note that we could have immediately written down (60), which is an exact general-relativistic expression, on the model of the classical expression, simply by using the transcriptions (23), together with $h_0 \to h$ (which follows from $t \to A$). Moreover, (60) applies both to light and to massive particles. To apply (60) to either massless or massive particles, we need only insert the appropriate specific form (25) or (26) for $v(r)$.

Another form of the orbit equation is frequently useful. Let $u = 1/r$. Then, in analogy to the classical formula

$$\frac{d^2u}{d\phi^2} + u = -h_0^{-2}\frac{dU}{du},$$

we must have in general relativity

$$\frac{d^2u}{d\phi^2} + u = h^{-2}\frac{d}{du}(n^4v^2/2), \tag{61}$$

which, again, applies to both particles and photons. We have written down (61) simply by analogy to classical mechanics. But it may also be obtained by beginning with the radial component of (21) and eliminating A.

A third useful form of the orbit equation is

$$h^2[(du/d\phi)^2 + u^2] - n^4v^2 = 0. \tag{62}$$

5.2. Light rays in the de Sitter and Robertson–Walker universes

As noted above, the open de Sitter universe is equivalent to a traditional problem in classical geometrical optics — Maxwell's fish-eye lens. It follows (i) that the open de Sitter universe constitutes an absolute optical instrument and (ii) that, in the system of isotropic coordinates, the rays are eccentric circles.

Beginning from the orbit equation (60) and the index of refraction (52) and integrating, we obtain the polar equation for the light ray in the open de Sitter universe:

$$\sin(\phi - \alpha) = \frac{h(Kr^2 - 12)}{r(144c_0^2 - 48h^2K)^{1/2}}, \tag{63}$$

where α is a constant of integration. In effecting this calculation, we can follow step-for-step the calculation of ray shapes in the classical Maxwell fish-eye (Ref. 28, p.147-149). Since $(Kr^2 - 12)/r\sin(\phi - \alpha)$ =constant, we can write the equation for a family of light rays passing through a fixed point $P_0(r_0, \phi_0)$ as

$$\frac{Kr^2 - 12}{r\sin(\phi - \alpha)} = \frac{Kr_0^2 - 12}{r_0 \sin(\phi_0 - \alpha)}. \tag{64}$$

For any value of α, this equation is satisfied at point $P_1 = (r_1, \phi_1)$ where $r_1 = 12/Kr_0$ and $\phi_1 = \phi_0 + \pi$. This shows that all the rays from an arbitrary point P_0 meet at a point P_1 on the line joining P_0 to the origin O *such that* $OP_0 \cdot OP_1 = 12/K$. *Hence the imaging in Maxwell's fish-eye* lens is an inversion. From any point P_0 in the three-dimensional space an infinity of rays originate which are then focused at an image point P_1. The images are therefore *sharp* (stigmatic). (In most real optical instruments, of the infinity of points passing thorugh an object point, only a finite number pass through the image point, the other rays only passing near the image point. Such images are not sharp ones.) Now, an instrument *which sharply focuses an image of a three-dimensional region of space is* called an absolute optical instrument. Thus, the open de Sitter universe constitutes an absolute optical instrument. All the theorems pertaining to absolute optical instruments apply. For example, the optical length of a line segment in the image must be equal to the optical length of the corresponding line segment in the object (Ref. 28, p.143-147).

Moreover, by analogy to the Maxwell fish-eye, a ray in the de Sitter universe is a circle in the isotropic coordinate system. This goes exactly as in the classical geometrical optics treatment of the Maxwell fish-eye. On the left-hand side of (63), write $\sin(\phi - \alpha) = \sin\phi\cos\alpha - \cos\phi\sin\alpha$, then put $x = r\cos\phi$ and $y = r\sin\phi$. Then (63) may be written in the form

$$(x + b\sin\alpha)^2 + (y - b\cos\alpha)^2 = 12/K + b^2 \tag{65}$$

where $b = (hK)^{-1}(36c_0^2 - 12h^2K)^{1/2}$. Thus each ray is a circle. Note that in (63) if we put $K = 0$, we obtain a straight-line ray, as we would expect:

$$r\sin(\phi - \alpha) = \text{constant}.$$

These results may be extended with a little modification to the closed Robertson–Walker universe. The formalism developed in this paper was designed for static metrics, so it may seem at first sight that we cannot deal with the RW case. However, in the case of the null geodesic, a time-dependent conformal factor in the metric need not affect the basic procedure or the most important conclusions. (The same cannot be said of the particle trajectories.) Let us see how this works out in the language of refractive indices.

As noted above, the closed Robertson–Walker universe yields a factorable index of refraction,

$$n = n_s(\mathbf{r})n_t(t), \tag{66}$$

in which n_s is a function of the spatial coordinates alone and n_t is a function of the time alone. For the RW metric in isotropic comoving coordinates

$$n_s = (1 + kr^2/4)^{-1} \tag{67}$$

$$n_t = R(t). \tag{68}$$

The bending of light rays depends only on the spatial gradient of n. The fact that n varies everywhere in space with the same multiplicative function of time $R(t)$ does not affect the shape of a light ray. One way to see this is to consider Snell's law. If the index of refraction factors into the form (66), then whenever Snell's law is applied at an interface between regions of different n, the common factor $n_t(t)$ will cancel from the two sides of the equation. More formally, we can define a new time coordinate τ by $dt = R d\tau$. Then the line element (56) becomes

$$ds^2 = c_0^2 R^2\, d\tau^2 - R^2(1 + kr^2/4)^{-2}|d\mathbf{r}|^2.$$

and the effective index of refraction becomes simply

$$n = n_s = (1 + kr^2/4)^{-1} \tag{69}$$

The scale factor $R(\tau)$ or $R(t)$ thus can have no effect on the shape of a ray in the isotropic, comoving coordinates. R only influences the progress in time of light along the ray.

As far as ray shapes are concerned, then, the RW universe is entirely analogous to the Maxwell fish-eye lens. It follows (i) that the closed *Robertson–Walker universe also constitutes an absolute optical instrument* and (ii) that, in the system of isotropic, comoving coordinates, the rays are eccentric circles.

5.3. Light and particle motion in RN-type metrics

In this section, we shall illustrate the use of the Newtonian forms of the orbit equations in some applications to RN-type metrics. In particular, we calculate the effect of the parameter β (29) on three tests of general relativity. Our calculations will supplement and extend those of Halilsoy [23].

We may begin from (62). Inserting (26) for $v(r)$ in the second term, we obtain

$$\left[\frac{du}{d\phi}\right]^2 + u^2 - (c_0/h)^2 n^2 [1 - c_0^4 H^{-2} \Omega^2] = 0\,. \tag{70}$$

This differential equation is exact, but it may not appear very familiar. We may transform back to the original (nonisotropic) coordinates by using (41) and (42) in the first two terms of (70), with the result

$$\left[\frac{du'}{d\phi}\right]^2 + u'^2 \Omega^2 - (c_0/h)^2 [1 - c_0^4 H^{-2} \Omega^2] = 0\,. \tag{71}$$

Substituting (39) for $\Omega^2(u')$, then differentiating with respect to ϕ, we obtain

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_0^6}{h^2H^2} = -\frac{\beta c_0^6}{h^2H^2}\, u' + 3mu'^2 - 2\beta u'^3. \tag{72}$$

Bending of light rays

The equation for the shape of a light ray results from letting $H \to \infty$ in (72):

$$\frac{d^2u'}{d\phi^2} + u' = 3mu'^2 - 2\beta u'^3. \tag{73}$$

This equation may be solved by the usual perturbative method. If the right side of (73) is temporarily put equal to zero, we obtain the straight-line solution

$$u' = \frac{\sin\phi}{R},$$

where R is the distance of closest approach to the origin. Substituting the zeroth-order solution $\sin\phi/R$ for u' on the right side of (73) and solving the resulting differential equation for $u'(\phi)$, we obtain the solution of first order in m and β:

$$u' = \frac{\sin\phi}{R} + \frac{3m}{2R^2}\left[1 + \frac{1}{3}\cos 2\phi\right] + \frac{3\beta}{4R^3}\,\phi\cos\phi - \frac{\beta}{16R^3}\sin 3\phi. \tag{74}$$

(This differs slightly from Halilsoy's solution, which is missing the last term.) As $r' \to \infty$, $u' \to 0$, and $\phi \to \phi_\infty$, which may be assumed small. Thus (74) reduces to

$$0 \doteq \frac{\phi_\infty}{R} + \frac{2m}{R^2} + \frac{9\beta\phi_\infty}{16R^3}.$$

The total deflection is $\Delta\phi_\infty = 2|\phi_\infty|$ or

$$\Delta\phi_\infty \approx \frac{4m}{R}\left(1 - \frac{9\beta}{16R^2}\right). \tag{75}$$

The coefficient of β/R^2 differs from the $\frac{3}{4}$ obtained by Halilsoy, the difference being the contribution of the last term in (74).

For some of the metrics under consideration [see (46) and (47)], β can be of order m^2. Thus, the expressions for the light orbit (74) and for the bending (75) should be carried to higher order in m/R to provide a fair assessment of the importance of the contributions due to β. This may be done by iteration. That is, we substitute (74) for u' on the right-hand side of (73) and proceed as before. The result is that the following terms should be added to the right-hand side of (74)

$$\frac{-15m^2}{4R^3}\phi\cos\phi - \frac{3m^2}{16R^3}\sin 3\phi. \tag{76}$$

The expression (75) for the deflection of the light ray becomes

$$\Delta\phi_\infty \approx \frac{4m}{R}\left(1 - \frac{9\beta}{16R^2} + \frac{69m^2}{16R^2}\right). \tag{77}$$

Precession of planetary apsides

For a planet, we return to (72). This equation is exact and may be handled as it stands. However, since we will treat some of the terms on the right side of the equation as perturbations, no precision will be lost by replacing the constants of the motion h and H by their classical limits. For a planet moving at non-relativistic speed, we may, by (27), put $H^2 \approx c_0^4$. Also, at sufficiently large r (i.e., at the radius of a planetary orbit), we may put $h \approx h_0$ the classical angular momentum per unit mass. Thus we have

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_0^2}{h_0^2} = -\frac{\beta c_0^2}{h_0^2}u' + 3mu'^2 - 2\beta u'^3. \tag{78}$$

This differential equation is not quite the same as that recently obtained through other means by Halilsoy. In particular, Halilsoy's equation is missing the term $-\beta(c_0/h_0)^2u'$ [Ref. 23, eq. (5)].

If we temporarily put the terms in u'^2 and u'^3 equal to zero, we obtain a differential equation that may be solved exactly:

$$\frac{d^2u'}{d\phi^2} + s^2u' = \frac{1}{\alpha_0}, \tag{79}$$

where

$$s^2 = 1 + \beta c_0^2/h_0^2, \tag{80}$$

and

$$\alpha_0 = h_0^2/mc_0^2. \tag{81}$$

The solution is the precessing ellipse

$$u' = \alpha^{-1}(1 + e\cos s\phi), \tag{82}$$

where the eccentricity e is arbitrary and where the semi-latus rectum α is

$$\alpha = \alpha_0\left[1 + \frac{\beta c_0^2}{h_0^2}\right]. \tag{83}$$

The precession of the apsides, per revolution of the planet on the orbit, due to the term in $\beta u'$, is then

$$\Delta_1 = \frac{-\pi\beta c_0^2}{h_0^2} = \frac{-\pi m}{\alpha_0}\frac{\beta}{m^2}. \tag{84}$$

The terms in u'^2 and u'^3 may be treated as perturbations. Thus, one inserts (82) in these two terms on the right side of (78) and solves the resulting equation. The term in u'^2, acting alone, produces the usual precession associated with the Schwarzschild problem:

$$\Delta_2 = \frac{6\pi m^2}{h_0^2}c_0^2 = \frac{6\pi m}{\alpha_0}. \tag{85}$$

As shown by Halilsoy, the term in u'^3, acting alone, produces the precession

$$\Delta_3 = -\frac{6\pi\beta m^2}{h_0^4}c_0^4 = -6\pi\left[\frac{m}{\alpha_0}\right]^2\frac{\beta}{m^2}. \tag{86}$$

However, this term is smaller than (84) by a factor of m/α_0 and is therefore negligible. To lowest order in m/α_0, then, the total precession Δ of the apsides per revolution is just $\Delta_1 + \Delta_2$:

$$\Delta = \frac{6\pi m}{\alpha_0}\left(1 - \frac{\beta}{6m^2}\right). \tag{87}$$

Radar echo delay

We consider the propagation in time of light in an RN-type metric. Again, let the motion take place in the $\theta = \pi/2$ plane. Writing out the conservation of energy equation (22) in plane polar coordinates and making use of (25) and (58), we have

$$\left(\frac{dr}{dA}\right)^2 + \frac{h^2}{r^2} - n^2 c_0^2 = 0\,. \tag{88}$$

Let us now evaluate the constant of the motion, h. Let r_0 denote the distance of closest approach of the ray to the center of the gravitating body. When $r = r_0$, we have $dr/dA = 0$. Thus (88) gives

$$h = r_0 n(r_0) c_0\,, \tag{89}$$

which is analogous to the classical-mechanical expression $r_0 v(r_0)$.

We may now transform from r back to r' using (41) and (42). Also, because we are interested in the propagation of light in time, we use (24) to pass over from A to t as independent variable. Thus, with substitution and transformation of (89), (88) becomes

$$\left(\frac{dr'}{dt}\right)^2 = \Omega^4(r')c_0^2\left[1 - \frac{r_0'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_0')}\right]. \tag{90}$$

The time of travel from r_0 to r' is then

$$\begin{aligned}\Delta t &= c_0^{-1}\int_{r_0'}^{r'} \Omega^{-2}(r')\left[1 - \frac{r_0'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_0')}\right]^{-1/2} dr' \\ &\equiv c_0^{-1}\int_{r_0'}^{r'} I(r')\,dr'.\end{aligned} \tag{91}$$

Now,

$$I = \Omega^{-2}\left(1 - \frac{r_0'}{r'^2}\right)^{-1/2}\left[1 + \frac{[1 - \Omega^2(r')/\Omega^2(r_0')]}{(r'^2/r_0'^2 - 1)}\right]^{-1/2}. \tag{92}$$

Using (39) to write out $\Omega(r')$ and $\Omega(r_0')$, then expanding to first order in m and first order in β, we obtain

$$I = \left(1 - \frac{r_0'^2}{r'^2}\right)^{-1/2}\left[1 + \frac{2m}{r'} + \frac{mr_0'}{r'(r'+r_0')} - \frac{3\beta}{2r'^2}\right]. \tag{93}$$

The total time of travel $\Delta t(r_0', r')$ from r_0' to r' is obtained by substituting (93) into (91) and integrating:

$$\begin{aligned}\Delta t(r_0', r') \approx{} & c_0^{-1}(r'^2 - r_0'^2)^{1/2} \\ & + \frac{2m}{c_0}\ln\left[\frac{r'}{r_0'} + \frac{(r'^2 - r_0'^2)^{1/2}}{r_0'}\right] \\ & + \frac{m}{c_0}\left[\frac{r' - r_0'}{r' + r_0'}\right]^{1/2} + \frac{3\beta}{2r_0'c_0}\sin^{-1}\left(\frac{r_0'}{r'}\right) - \frac{3\pi\beta}{4r_0'c_0}.\end{aligned} \tag{94}$$

The first term on the right side of (94) is the transit time of light in Euclidean space. The delay $\Delta T(r_0, r')$ due to general-relativisitic effects is the sum of the remaining terms.

As an example, let us estimate the radar echo delay for a signal sent from the Earth at radius r_e' to an inferior planet at radius r_p' when that planet is near superior conjunction with the Sun. Let the distance of closest approach to the center of the Sun be r_0'. If we suppose that the signal passes very near the Sun, so that r_0' is much smaller than either r_e' or r_p', then

$$\Delta T(r_0', r_e') \approx \frac{2m}{c_0}\ln\left[\frac{2r_e'}{r_0'}\right] + \frac{m}{c_0}\left[1 - \frac{r_0'}{r_e'}\right] - \frac{3\pi\beta}{4c_0r_0'}\left[1 - \frac{2r_0'}{\pi r_e'}\right], \tag{95}$$

and the total delay in the signal for the round trip is

$$2[\Delta T(r_0', r_e') + \Delta T(r_0', r_p')]. \tag{96}$$

5.4. Redshifts

The gravitational and cosmological redshifts are not dynamical effects; i.e., we do not need to solve an equation of motion in order to calculate them. However, it may be of some interest to see how the redshifts arise in the language of an effective index of refraction.

In ordinary optics, if a light wave travels from a region of high index of refraction n_2 to a region of low index of refraction n_1, the wavelength λ increases because the leading part of the wave enters n_1 first and speeds up while the trailing part is still in n_2. Thus the wave begins to stretch out. λ and c change but the frequency ν does not. This holds even if the index varies continuously with the spatial coordinates and even if the ray crosses obliquely through the contours of constant n. Thus, in general,

$$\lambda(\mathbf{r}_1)n(\mathbf{r}_1) = \lambda(\mathbf{r}_2)n(\mathbf{r}_2). \tag{97}$$

Now consider a situation in which n does not depend on the spatial coordinates, but does vary with time. An example can be imagined: let the air slowly be pumped from a chamber. Then, as long as the wave does not leave the chamber, n is everywhere the same, but is a decreasing function of time. The wavelength will not change, since the leading edge of the wave never encounters a new value of n before the trailing edge does. Thus, in this case ν and c change but λ does not. We have

$$\lambda(t_1) = \lambda(t_2). \tag{98}$$

Suppose now that the index of refraction can be written as a product of two functions — a function n_s of the spatial coordinates alone and a function n_t of the time alone, as in (66). For the reasons just mentioned, n_t does not affect the wavelength and we have

$$\lambda(\mathbf{r}_1)n_s(\mathbf{r}_1) = \lambda(\mathbf{r}_2)n_s(\mathbf{r}_2). \tag{99}$$

We wish to apply these rules of ordinary optics to the propagation of light in general relativity. Our effective index of refraction (3) is based upon the isotropic coordinate speed of light. Thus the quantity analogous to the wavelength of classical optics is the coordinate distance between successive crests of the wave. Coordinate distances are not, of course, directly measurable in general relativity. The physically measurable metric length is obtained from the coordinate length by means of the metric (1).

Gravitational redshift

Let $|\Delta\mathbf{r}_1|$ be the coordinate distance between successive crests of a light wave located at $\mathbf{r}_1$. Similarly, let $|\Delta\mathbf{r}_2|$ be the coordinate distance between successive crests at a different point $\mathbf{r}_2$ located on the same ray. Then, in analogy to the condition (97) from ordinary optics, we must have

$$|\Delta\mathbf{r}_1|n(\mathbf{r}_1) = |\Delta\mathbf{r}_2|n(\mathbf{r}_2). \tag{100}$$

The metric length λ of the wave at point $\mathbf{r}_1$ is obtained by applying the metric (1) to the coordinate length $|\Delta\mathbf{r}_1|$ of the wave:

$$\lambda(\mathbf{r}_1) = \Phi^{-1}(\mathbf{r}_1)|\Delta\mathbf{r}_1|.$$

A similar expression holds for $\lambda(\mathbf{r}_2)$. Thus we have

$$\Phi(\mathbf{r}_1)\lambda(\mathbf{r}_1)n(\mathbf{r}_1) = \Phi(\mathbf{r}_2)\lambda(\mathbf{r}_2)n(\mathbf{r}_2), \tag{101}$$

or, using (3),

$$\lambda(\mathbf{r}_1)\Omega^{-1}(\mathbf{r}_1) = \lambda(\mathbf{r}_2)\Omega^{-1}(\mathbf{r}_2), \tag{102}$$

the usual gravitational redshift relation.

As an example, let us take the case of the Schwarzschild metric. Let a source of light be located at $\mathbf{r}_1$ and an observer at $\mathbf{r}_2$, sufficiently far from the central gravitating body so that we may put $\Omega(\mathbf{r}_2) \approx 1$. Then, with the use of (39) and (43), (102) gives

$$\begin{aligned} z &\equiv (\lambda_{observed} - \lambda_{emitted})/\lambda_{emitted} \\ &= (1 - 2m/r')^{-1/2} - 1. \end{aligned} \tag{103}$$

This result may, of course, be derived by many other methods.

Cosmological redshift

In the expanding universe of the Robertson–Walker metric, we have an index of refraction (57) that factors like (66), with n_s and n_t given by (67) and (68). Let a light wave of coordinate length $|\Delta\mathbf{r}_1|$ be emitted at $(t_1, \mathbf{r}_1)$ and received at $(t_2, \mathbf{r}_2)$. By analogy to (99), the coordinate length $|\Delta\mathbf{r}_1|$ of the received wave is determined by

$$|\Delta\mathbf{r}_1|n_s(\mathbf{r}_1) = |\Delta\mathbf{r}_2|n_s(\mathbf{r}_2). \tag{104}$$

The metric length λ of the wave is obtained by applying the metric (56) to the coordinate length $|\Delta\mathbf{r}|$. Thus (104) becomes

$$\frac{(1 + kr_1^2/4)}{R(t_1)}\lambda(t_1, \mathbf{r}_1)n_s(\mathbf{r}_1) = \frac{(1 + kr_2^2/4)}{R(t_2)}\lambda(t_2, \mathbf{r}_2)n_s(\mathbf{r}_2),$$

or, with use of (67),

$$\frac{\lambda_1}{R(t_1)} = \frac{\lambda_2}{R(t_2)}, \tag{105}$$

the usual cosmological redshift relation.

While many writers have stressed the fundamentally different natures of the gravitational and cosmological redshifts, others have argued that it is possible to treat them with a single unified approach [29]. In the effective optical-medium formulation pursued here, it is interesting to note that both spectral shifts depend on a single optical principle (99).

6. CONCLUSION

The Newtonian forms (21) and (22) for the geodesic equations of motion offer some practical advantages for calculation. In particular, they facilitate the writing down of exact general relativistic expressions simply by analogy to classical formulas. Thus, they constitute one more tool for the relativist's tool kit. But the most interesting consequence of extending the optical-mechanical analogy to general relativity is that one simple equation of motion (21) now summarizes three fields of study: classical geometrical optics, classical particle mechanics, and geodesic motion of both light and particles in general relativity. Of course, our treatment is restricted to isotropic fields and media. Nevertheless, this unified approach, based on the use of the optical action, possesses considerable flexibility and scope. A single variational principle (14) governs all three domains.

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On the spherically symmetric static solutions of Brans-Dicke field equations in vacuum

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Abstract : The analytic solutions, obtained by Riazi and Askari, of the approximate and exact vacuum Brans-Dicke equations for the spherically symmetric static case are shown to correspond to unphysical negative values of the coupling parameter w. The present investigation is meant to highlight the pitfalls that one might encounter in the physical interpretation of such solutions.

Keywords : Vacuum Brans-Dicke equations, spherically symmetric static case, analytic solutions

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The idea of utilizing the Brans-Dicke (BD) theory in the interpretation of various astrophysical phenomena is quite attractive. With this intention, Riazi and Askari (RA) [1] have recently obtained analytic solutions (eqs. (13), (20) and (22) of [1]) of the approximate BD equations in the spherically symmetric case. We might call these solutions the 'approximate' RA solutions. These solutions have been utilized to interpret a very important astrophysical phenomenon, namely, the observed flat rotation curves in the vast domain of dark galactic haloes. According to the authors, the interpretation requires a rather 'unnatural' large positive value of w ($\sim 10^{12}$). In our opinion, such a requirement by itself does not constitute any inadequacy of the approximate RA solutions as the large mass of astronomical probes indicates only an ascending order of positive values for w (≥ 6, 29, 140, 500, ...). However, it turns out that the RA solutions correspond only to negative values of the BD parameter w. Consequently, some care should be exercised in the application of the RA solutions to those problems of physical interest that require a positive w.

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The BD field equations are :

$$\Box^2 \phi = \frac{8\pi}{3+2w} T^{\mu}_{M\mu}, \tag{1}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{\phi} T_{M\mu\nu} - \frac{w}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\phi_{;\rho}\phi^{\rho}_{;}\right)$$
$$- \frac{1}{\phi}\left(\phi_{;\mu;\nu} - g_{\mu\nu}\Box^2\phi\right), \tag{2}$$

where $\Box^2 \equiv (\phi;^{\rho})_{;\rho}$ and $T_{M\mu\nu}$ is the matter energy momentum tensor excluding the ϕ-field, w is a dimensionless coupling constant.

Riazi and Askari [1] show that, in the asymptotic region, their approximate solutions, $B(r)$ and $A(r)$, behave as follows (speed of light $c = 1$) :

$$B(r) \to 1 - \frac{(c_2-2)r_0}{r}, \tag{3}$$

$$A(r) \to \left(1 - \frac{c_2 r_0}{r}\right)^{-1} = 1 + \frac{c_2 r_0}{r} + 0(r^{-2}), \tag{4}$$

where $d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Further, from the above expression for $B(r)$, they identify the total mass M of the configuration (inclusive of the contribution from the ϕ-field) with $\frac{(c_2-2)r_0}{2G}$, where r_0 is a constant c_2 is a dimensionless constant of integration. On account of the positivity of energy, we find that two cases are possible : (i) $r_0 > 0$ and $c_2 > 2$, (ii) $r_0 < 0$ and $c_2 < 2$. In their paper, RA consider only case (i) as is reflected in their requirement of fine tuning $c_2 = 2^+$. We shall first investigate case (i).

Consider the Robertson expansion in standard coordinates [2].

$$d\tau^2 = \left(1 - 2\alpha GMr^{-1} + 2(\beta-\alpha\gamma)G^2M^2r^{-2} + \ldots\right)dt^2$$
$$-\left(1 + 2\gamma GMr^{-1} + \ldots\right)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5}$$

This represents spherically symmetric static solutions of BD equations in vacuum [3] provided $\alpha = \beta = 1$, $\gamma = \frac{w+1}{w+2}$, together with

$$\phi = \phi_0\left(1 + (w+2)^{-1} GMr^{-1} + \ldots\right). \tag{6}$$

In the asymptotic region (*i.e.*, to the first order in r^{-1}), the expressions (3) and (4) must be compatible with the corresponding parts from the expression (5). This requirement should be regarded as a boundary condition to be satisfied by all spherically symmetric solutions of the

BD equations. The physical viability of different solutions can also be judged by the same token. For the approximate RA solutions, we have

$$(c_2-2)r_0=2GM, \tag{7}$$

$$c_2 r_0 = 2GM\left(\frac{w+1}{w+2}\right). \tag{8}$$

Eliminating $2GMr_0^{-1}$ from the above, we get unphysical negative values for w :

$$w = -\left(\frac{c_2+2}{2}\right) < 0 \qquad \text{if } c_2>2, \tag{9}$$

which is what we set out to show.

One might suspect that the appearance of unphysical negative values of w is due somehow to the approximate nature of the RA solutions that we have considered. It will soon turn out that this is not so; the negativity of w persists even if the exact BD equations are considered. The spherical solutions of the exact BD equations have been obtained (RA) through a conformal reparametrization procedure [4–6]. We might call these solutions the 'exact' RA solutions and in standard coordinates, they are given by [4]

$$\phi(r) = \phi_0 + \frac{Q_s}{r} + \left(1+\frac{Q_s}{2M}\right)\frac{GMQ_s}{r^2} + \left(\frac{(w-6)Q_s^2}{6M^2} + \frac{11Q_s}{3M} - \frac{8}{3}\right)\frac{G^2M^2Q_s}{2r^3} + \ldots,$$

$$B(r) = 1 - \frac{2GM}{r} + \frac{2G^2MQ_s}{r^2} + \left(\frac{(w-16)Q_s}{6M} + \frac{5}{3}\right)\frac{G^3M^2Q_s}{r^3} + \ldots.,$$

$$A(r) = 1 - \frac{2G(Q_s-M)}{r} + \left(\frac{(8-w)Q_s^2}{2M^2} + \frac{9Q_s}{M} + 4\right)\frac{G^2M^2}{r^2} + \ldots., \tag{10}$$

where $$2M = Q_s(1-\delta) = \frac{r_0}{G}(\delta-1), \tag{11}$$

Q_s and δ are two integration constants. These are related to r_0, c_2, ϕ_0 and c_1 by the relations : $Q_s=-c_1=-r_0/G=-r_0\phi_0$ and $\delta=c_2-1$ so that

$$2M = c_1(c_2-2). \tag{12}$$

From this, we see that $c_2>2$ translates to $\delta>1$ and that eq. (12) is nothing but eq. (7) redefined. Also the so called fine tuning condition is now restated as $\delta=1^+$. After making this change, the asymptotic expressions for $B(r)$, $A(r)$ and also $\phi(r)$ have to be compared with those from eqs. (5) and (6). In the first order in r^{-1}, these give the equations

$$B(r) \sim 1 - \frac{Q_s(1-\delta)G}{r} = 1 - \frac{2MG}{r}, \tag{13}$$

$$A(r) \sim 1 - \frac{Q_s(1+\delta)G}{r} = 1 + \left(\frac{w+1}{w+2}\right)\frac{2GM}{r}, \tag{14}$$

$$\phi(r) \sim \phi_0 + \frac{Q_s}{r} = \phi_0\left[1 + \frac{GM}{(w+2)r}\right]. \tag{15}$$

From eqs. (13)–(15) that now also include the expression for $\phi(r)$, it follows that

$$w = -\left(\frac{\delta+3}{2}\right) < 0 \qquad \text{since } \delta > 1. \tag{16}$$

Clearly, this result is again the same as eq. (9) above with $c_2 = \delta + 1$. The exact solutions (10) also permit us to go beyond the first order approximation. For instance, consider the second order term in $B(r)$ from eq. (5) and equate it with that from eq. (10) :

$$2(\beta - \alpha\gamma)\frac{G^2M^2}{r^2} = \frac{2G^2MQ_s}{r^2},$$

with $Q_s = \frac{2M}{1-\delta}$. With the specified values of α, β, γ, one immediately finds that eq. (16), or, by another symbol, eq. (9), continues to hold good. We should also recall that the second order term in $B(r)$ plays a crucial role in the Solar system tests of gravity. The most significant one is the wellknown test for the precession of planetary orbits. On the other hand, in the standard BD theory, $w \geq 6$, if all classical tests of General Relativity are to be reasonably accounted for. Thus, eqs. (16) or (9) prevent RA solutions to become applicable in the Solar system scenario, although they may be applicable in other physical situations.

Let us now examine case (ii) : $r_0 < 0$ and $c_2 - 2 < 0$. In this case, it is possible to choose a range of values for c_2 such that a positive w is obtainable from eq. (9). For example, $-\infty < c_2 < -2$ guarantees a positive w. However, we still have to leave out the range $-2 < c_2 < 2$ as this leads to a negative w. Anyhow, everything looks nearly fine as far as w is concerned but the problem crops up elsewhere. The rotational velocity in the range of flat rotation curves becomes imaginary! It is given by $v^2_{\text{rot}} \equiv (c_2 - 2)/(2w)^{1/2}$. Thus, once again, *we run into a physically meaningless conclusion but of a kind different from that* corresponding to case (i).

Riazi and Askari [1] themselves caution against yet another limitation : the vacuum condition may not be tenable in view of the possible presence of luminous matter in the galactic haloes. We agree with their opinion. Probably an interior BD solution ($T_{\mu\nu} \neq 0$) would be closer to the physical situation.

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