REFERENCES

[1]. As quoted by Evans, the optical-mechanical analogy was formulated by Hamilton (1883) in terms of characteristic functions. See: J. Evans, Am. J. Phys. 61, 347 (1993). In this paper, Evans derives the ray form without the use of characteristic functions.

[2]. Historically, the basic idea of the optical-mechanical analogy was proposed much earlier by none other than Descartes (1637). This information is due to: J. A. Arnaud, Am. J. Phys. 44, 1067 (1976). In this paper are also discussed the limitations and significance of the analogy.

[3]. Essential ingredients of what is referred to as the Evans-Rosenquist approach are contained in: J. Evans and M. Rosenquist, Am. J. Phys. 54, 876 (1986).

[4]. J. Evans, Am. J. Phys., 58, 773 (1990). See especially p.774.

[5]. Newton's laws of motion are obtained directly from Fermat's principle, in: M. Rosenquist and J. Evans, Am. J. Phys., 56, 881 (1988).

[6]. R.P.Luneberg, *Mathematial Theory of Optics*, Brown University mimeographed notes (1944)(University of California, Berkeley, CA, 1964).

[7]. J. Isenberg and M. Jackson, in: *Directions in General Relativity,* vol.1, B. L. Hu and T. A. Jacobson (Eds.) (Cambridge University Press, Cambridge, 1993).

[8]. L. P. Grishchuk, A. N. Petrov and A. D. Popova, Comm. Phys. 94, 379 (1984).

[9]. A. N. Petrov, Class. Quant. Grav., 10, 2663 (1993).

[10]. A. N. Petrov, Int. J. Mod. Phys. D4, 451 (1995).

[11]. Sir A. S. Eddington calculated the bending of light rays round a massive object by assuming that the space around the sun is filled with a medium with refractive index $n(r)= (1- 2m/r)^{-1} \sim 1+ 2m/r$. Incidentally, he was also the first to experimentally observe such a bending in 1919 during the Solar eclipse in Sobral, Brazil. See, A. S. Eddington, *Space, Time and Gravitation* (Cambridge University, Cambridge, 1920), reissued in the Cambridge Science Classical Series, 1987, p.109. However, F.de Felice (ref.19) reports that Einstein himself was the first to suggest the idea of an refractive medium.

[12]. J. Plebanski, Phys. Rev., 118, 1396 (1960).

[13]. N. L. Balazs, Phys. Rev., 110, 236 (1957).

[14]. G. B. Skrotskii, Sov. Phys. Doklady, 2, 226 (1957).

[15]. F. Winterberg, Nuovo Cim., 53B, 264 (1968).

[16]. B. Bertotti, Proc. Roy. Soc. Lond. Ser.A, 269, 195 (1966).

[17]. F. R. Tangherlini, Am. J. Phys., 36, 1001 (1968).

[18]. M. Kline and I. W. Kay, *Electromagnetic Theory and Geometrical Optics*, (Interscience, New York, 1964).

[19]. F. de Felice, Gen. Rel. Grav. 2, 347 (1971).

[20]. C. Cattaneo, Nuovo Cim., 10, 318 (1958).

[21]. L. D. Landau and E. M. Lifschitz, *The Classical Theory of*

Fields, (Pergamon, Oxford, 1975).

[22]. E. J. Post, *Formal Structure of Electromagnetics*, (North Holland Publishing Company, Amsterdam, 1962).

[23]. See, for instance: D. Griffiths and M. B. James, Am. J. Phys., 60, 309 (1992).

[24]. S. K. Bose, *An Introduction To General Relativity*, (Wiley Eastern, New Delhi, 1980).

[25]. W.-C. Wang and C.-M. Chen, Am. J. Phys., 58, 744 (1990).

[26]. H. Weyl, Ann. Phys. (Leipzig), 54, 117 (1917).

[27]. T. Levi-Civita, Nuovo Cim., 16, 105 (1908).

[28]. R. Blandford and R. Narayan, Ap. J., 310, 568 (1986).

[29]. I. Kovner, Ap. J., 351, 114 (1990).

[30]. R. Nityananada and J. Samuel, Phys. Rev., D45, 3862 (1992).

[31]. Ll. Bel and J. Martin, Gen. Rel. Grav., 26, 567 (1994).

[32]. V. Perlick, Class. Quantum Grav., 7, 1319 (1990).

[33]. M. J. Nandor and T. M. Helliwell, Am. J. Phys., 64, 45 (1996).

[34]. J. L. Synge, *The General Theory of Relativity*, (North Holland Publishing Company, Amsterdam, 1960).

[35]. W. Gordon, Ann. Phys. (Leipzig), 77, 421 (1923).

[36]. X. Wu and C. Xu, Comm. Theor. Phys. (Beijing), 9, 119 (1988).

[37]. K. K. Nandi and A. Islam, Am. J. Phys., 63, 251 (1995).

[38]. The classical limit of this expression is $-\frac{1}{2}v^2$, which is equal to U(r) - E. In the classical situation, it might be more

appropriate to refer to $-\frac{1}{2}v^2(r)$ as the "speed function", since it does differ from the potential energy by an additive constant. In classical geometrical optics (Ref.[3]), the "potential energy" is $-\frac{1}{2}n^2c_o^2$ and the additive constant is always zero. In the *geometrical optics of the Schwarzschild field*, by virtue of eq.(2.2.9), the "potential energy" is again $-\frac{1}{2}n^2c_o^2$, with no additive constant. In the case of particle motion in the schwarzschild field, the "potential energy" is given by $-\frac{1}{2}n^4v^2$ -- with no additive constant, but with a constant of integration *(corresponding to the energy) buried in the expression for* $v(r)$. Thus the term "potential energy" for $-\frac{1}{2}n^4v^2$ should be regarded as coming from the optical situations, in which it is perfectly apt. For particle dynamics, whether classical or relativistic, the term is perhaps not a perfect choice.

[39]. J. H. Poynting, *Philos. Mag.*, 9, 393 (1905).

[40]. R. V. Jones, Nature, London, 167, 439 (1951).

[41]. R. V. Jones and J. C. S. Richards, Proc. Roy. Soc. Lond. Ser.A, 221, 480 (1954).

[42]. A. Ashkin and J. M. Dziedzic, Phys. Rev. Lett., 30, 139 (1973).

[43]. F. R. Tangherlini, Nuovo Cim., 4B, 13 (1971).

[44]. W. C. Michels, M. Correl and A. L. Patterson, *Foundations of Physics*, (Englewood Cliffs, New Jersey, 1968).

[45]. See, for instance: J. B. Marion, *Classical Dynamics of Particles and Systems*, (Academic Press, New York, 1970).

[46]. This is in accord with the result obtained by standard methods. See: C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman, San Francisco, 1973).

[47]. To begin the classical calculation, write eq.(2.4.22) as $dp/dt = \mathrm{grad}(\frac{1}{2}v^2)$, where $\frac{1}{2}v^2 = \frac{1}{2}v_o^2 + MG/r$. The angular deflection turns out to be $-(2m/R_o)(c_o^2/v_o^2)$, just half of the general relativistic value.

[48]. René Descartes, *La dioptrique,* published as a supplement (along with the *Geometry* and the *Meteorology*) to *Discours de la méthode* (Leiden, 1637). *Oeuvres de Descartes,* ed. by Charles Adam and Paul Tannery, 2nd ed. (J. Vrin, Paris, 1874-1882); vol. 6, pp.93-105. English translation: *Discourse on Method, Optics, Geometry, and Meteorology,* Paul J. Olscamp, Trans. (Bobbs-Merrill, Indianapolis, 1965). The treatment of refraction is found in the "Second Discourse" of the *Optics.*

[49]. Fermat made his principle of least time known through correspondence, starting around 1662. He wrote two short pieces for this purpose, which were never published in a regular way during his lifetime. They first appeared in print in the published *Correspondence* of Descartes. These two pieces can be found in standard edition of Fermat's works: *Oeuvres de Fermat,* Paul Tannery and Charles Henry, eds. (Gauthier-Villars et Fils, Paris, 1891-1912). See "Analysis ad refractiones," in vol. 1, pp.170-172, with French translation in vol. 3, pp.149-151; and "Synthesis ad refractiones," vol. 1, pp. 173-179, with French translation in

vol. 3, 151-156. For an English translation of the first piece, see William Francis Magie, *A Source Book in physics* (McGraw-Hill, New York, 1935), pp. 278-280.

[50]. On Fermat vs. the Cartesians, see René Dugas, *A History of Mechanics,* J.R. Maddox, trans. (Dover, New York, 1988), pp. 258-259.

[51]. Christiaan Huygens, *Traité de la lumière* (Lieden,1690). Reprinted in in *Oeuvres complète de Christian Huygens* (Martinus Nijhoff, La Haye, 1888-1950), vol. 19, pp. 451-537. English translation: *Treatise on Light... by Christiaan Huygens,* Silvanus P. Thompson, trans. (Dover, New York, 1962; first published in 1912). For Huygens' treatment of Fermat's principle, see *Oeuvres,* vol. 19, pp. 42-45.

[52]. P.L. Moreau de Maupertuis, *Oeuvres,* 4 vols. (Georg Olms, Hildesheim and New York, 1965- 1974). (This is a reprint based on the editions of Lyon, 1768 and Berlin, 1758.) Maupertuis announced his principle of least action in a paper read in 1744 to the Paris Academy of Sciences, and titled, " The Agreement between Different Laws of Nature, Which Had Until Now Appeared Incompatible," (*Oeuvres,* vol. 4, pp. 3-28). This first paper addressed the refraction of light. A second paper, "Search for the laws of motion", read to the Royal Academy of Sciences of Berlin in 1746, treated the collisions of elastic and inelastic bodies by the method of least action (*Oeuvres,* vol. 4, pp. 31-42.) It is one of the many ironies of this story that Maupertuis applied least

action first to light, and only later to mechanics; the principle was applied first to the domain in which it is invalid.

[53]. Fermat, *Oeuvres* (ref. 51), vol. 4, p. 15.

[54]. For an account of Euler's and Lagrange's development of Maupertuis' principle, see Wolfgang Yourgrau and Stanley Mandelstam, *Variational Principles in Dynamics and Quantum Mechanics* (Dover, New York, 1979).

[55]. Pierre-Simon, Marquis de Laplace, *Oeuvres compléte de Laplace,* 13 vols. (Gauthier-Villars, Paris, 1878-1904), Vol. 12, pp. 267-298.

[56]. Hamilton's work is most easily consulted in: *The Mathematical Papers of Sir William Rowan Hamilton,* 3 vols. (Cambridge University Press, Cambridge, 1931-1967). Vol. 1, *Geometrical Optics,* ed. A.W. Conway and J. L. Synge; Vol. 2, *Dynamics,* ed. A.W.Conway and A.J.McConnell; Vol. 3 *Algebra*, ed. H. Halberstam and R. E. Ingram.

[57]. Hamilton, " Theory of System of Rays." *Mathematical papers* (ref. 58), vol. 1, p. 14.

[58]. Hamilton, "Third Supplement to an essay on the theory of Systems of Rays," *Mathematical Papers*, vol. 1, pp. 168-169.

[59]. See, for example: Miles V. Klein, *Optics* (John Wiley & Sons, New York, 1970), p. 28.

[60]. This is clearer if we write V in the form $V = \int(v_x dx + v_y dy + v_z dz)$. Then $\partial V/\partial x = v_x$.

[61]. Reprinted in Hamilton, *Mathematical Papers*, Vol. 1,

pp.311-332. For an excellent summary of Hamilton's further development of his dynamical theory (leading to the canonical equations of motion and the Hamilton-Jacobi equation) see: Thomas L. Hankins, *Sir William Rowan Hamilton* (John Hopkins University Press, Baltimore and London, 1980) pp. 181-198. Hankins also provides a good summary of Hamilton's work in geometrical optics (pp. 61-81).

[62]. Cf. Editor's introduction to Hamilton, *Mathematical Papers* (ref. 56), vol. 1, p.xxi.

[63]. Hankins (ref. 61), p.xx.

[64]. Hankins (ref. 61), p. 87.

[65]. Louis de Broglie, *Recherches sur la théorie des quanta* (Masson, Paris, 1963). (A reprint of the thesis of 1924). Here, we have expressed the de Broglie relation non-relavistically. de Broglie's relation does, of course, hold relativistically: we need only replace mv by the relavistic momentum p.

[66]. M. Halilsoy, Gen. Rel. Grav., 25, 975 (1993).

[67]. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, (McGraw-Hill, New York. 1965).

[68]. B. Bertotti, Phys. Rev., 116, 1331 (1959).

[69]. M. Halilsoy, Gen. Rel. Grav., 25, 275 (1993).

[70]. H. H. Soleng, Gen. Rel. Grav., 26, 149 (1994).

[71]. I. Dymnikova, Gen. Rel. Grav., 24, 235 (1992).

[72]. I. Goldman, J. Pérez-Mercader, F. Cooper and M. M. Nieto, Phys. Lett., B281, 219 (1992).

[73]. R. C. Tolman, *Relativity, Thermodynamics and Cosmology*, (Oxford University Press, Oxford, 1934).

[74]. M. Born and E. Wolf, *Principles of Optics*, (Pergamon Press, Oxford, 1980), 6th ed.

[75]. J. V. Narlikar, Am. J. Phys., 62, 903 (1994).

[76]. L. Kostro, in: *Physical Interpretations of Relativity Theory*, (Proc. British Soc. Philos. Sc. Conf., London, 1990), Ed. M. C. Duffy.

[77]. J. Evans, K. K. Nandi and A. Islam, Gen. Rel. Grav., 28, 413 (1996).

[78]. R. Tian and Z. Li, Am.J.Phys., 58, 890 (1990).

[79]. For a case, see: S. K. Ghosal, K. K. Nandi and T. K. Ghoshal, Ind. J. Pure & Appl. Math., 18, 194 (1987).

[80]. F. R. Tangherlini, Phys. Rev., 12A, 139 (1975).

[81]. E. A. Lord, *Tensors, Relativity and Cosmology*, (Tata-McGraw Hill, New Delhi, 1976).

[82]. T. Ohta and T. Kimura, Nuovo Cim., 105B, 1385 (1990); *ibid*, 106B, 291 (1991).

[83]. K. K. Nandi, Zeits. Naturforsch., 46A, 1026 (1991); *ibid*, 47A, 918 (1992).

[84]. S. Ichinose and Y. Kaminaga, Phys. Rev., D40, 3997 (1989).

this might make a "cleaner" measurement, it is not necessary given the width of the transition, and it might mask the very behavior the students are being asked to observe.

D. Suggested samples

The device was constructed to use amorphous Fe–Ni–P–B alloy samples with an 80% metal to 20% metalloid composition ratio in the form of thin ribbons produced by rapid quenching from the melt. These materials have easily accessible Curie temperatures (ranging from about −50 to 300 °C, depending upon composition). They also have accessible melting points and can be produced in the laboratory or purchased inexpensively from commercial sources. They have the added advantage of being somewhat novel substances.

E. Original usage

This instrument was used originally to make precise measurements of Curie temperatures for the purpose of discerning small changes in Curie temperature induced by particle bombardments or thermal annealing of thin ribbons of amorphous metallic alloys.[10,11] Some had T_c values below room temperature, and some above. The precision obtainable with the instrument (tenths of a Celsius degree) and the reproducibility of the measurements (typically under 1 °C unless heating during measurement provided further annealing of the substance) were crucial for identifying changes of a few Celsius degrees in transition temperature.

III. CONCLUSION

A device has been described that is capable of making precise measurements of the Curie temperatures of ferromagnetic materials, which can be used for demonstration purposes, or as an undergraduate laboratory experiment, such as determining the effect that thermal annealing has upon the Curie point and the ferromagnetic-to-paramagnetic transition width. It requires a vacuum pump, a lock-in amplifier, a small Dewar of liquid nitrogen, a strip-chart recorder, and miscellaneous small items typically found in any undergraduate physics department.

The thermometric technique described herein may also be adapted to measure other transition temperatures, such as the critical temperature for superconducting materials.

[1]Paul G. Hewitt, *Conceptual Physics*, 7th ed. (Harper Collins, New York, 1992), pp. 286 ff.

[2]Paul W. Zitzewitz and Robert F. Neff, *Merrill Physics: Principles and Problems* (Glencoe Division of Macmillan/McGraw-Hill, New York, 1992).

[3]David Halliday, Robert Resnick, and Jearl Walker, *Fundamentals of Physics: Extended, with Modern Physics*, 4th ed. (Wiley, New York, 1993).

[4]Francis W. Sears, Mark W. Zemansky, and Hugh D. Young, *University Physics* (Addison-Wesley, Reading, MA, 1987).

[5]Raymond A. Serway and Jerry Faughn, *College Physics* (Saunders College, Philadelphia, 1992).

[6]H. B. G. Casimir, *Magnetism and Very Low Temperatures* (Dover, New York, 1961), pp. 13–23.

[7]B. R. Sood, R. Kamal, and S. Sikri, "Simple experiment for studying the properties of a ferromagnetic material," Am. J. Phys. 48(6), June, 481–482 (1980).

[8]B. D. Cullity, *Introduction to Magnetic Materials* (Addison-Wesley, Reading, MA, 1972), pp. 67–69.

[9]B. D. Cullity, *Introduction to Magnetic Materials* (Addison-Wesley, Reading, MA, 1972), pp. 341–347.

[10]D. G. Fisher, R. B. Murray, and C. P. Swann, "Changes in Curie temperature due to irradiation and thermal annealing in amorphous Fe20Ni60P14B6," J. Appl. Phys. 58(1), 1 July (1985), pp. 460–465.

[11]T. A. Donnelly, D. G. Fisher, R. B. Murray, and C. P. Swann, "Electron and proton irradiation effects in amorphous Fe–Ni–P–B alloys," J. Appl. Phys. 53(11), November, 7801–7803 (1982).

On the optical–mechanical analogy in general relativity

Kamal K. Nandi and Anwarul Islam[a)]

Department of Mathematics, University of North Bengal, Darjeeling (W.B.)-734 430, India

(Received 31 March 1994; accepted 14 July 1994)

We demonstrate that the Evans–Rosenquist formulation of the optical–mechanical analogy, so successful in the application to classical problems, also describes the motion of massless particles in the Schwarzschild field of general relativity. It is possible to obtain the well-known equations for light orbit and radar echo delay which account for two exclusive tests of Einstein's field equations. Some remarks including suggestions for future work are also added. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

The historical optical–mechanical analogy[1,2] has recently been cast into a familiar form by Evans and Rosenquist.[3–5] This new formulation, based on Fermat's principle, provides an interesting approach that can be profitably utilized in the solution of many classical problems. The approach works either way: The well-known ideas and techniques of classical mechanics can be successfully applied to the problem of classical optics and vice versa. Such success naturally prompts further inquiry as to whether the applicability of the optical–mechanical analogy could be extended even beyond the classical regimes. More specifically, a curious student might ask: Can the Evans–Rosenquist (ER) formulation be used to describe the phenomena of light propagation and massive particle motion in general relativity (GR)? In order to decide this question, let us note that it addresses two distinct areas of investigation: One has to first examine if the

"$F=ma$" optics of ER does at all describe light propagation in the GR scenario. If it does, then the first hurdle is overcome and the question of applying the optical analogy to massive particle motion becomes meaningful. To what extent that analogy would describe planetary motion in GR is a matter of separate investigation.

In this paper, we shall examine only the first part of the question that relates to the motion of massless particles in GR. We choose Schwarzschild field of gravity primarily because of its experimental importance. Besides, on many occasions, it is used as an ideal example for illustrating fundamental notions of GR.

The exterior Schwarzschild metric, which is a solution of Einstein's GR field equations, has been astoundingly successful in describing various gravitational phenomena. Famous experimental tests of GR include the bending of light rays, perihelion advance of a planet, radar echo delay, and gravitational redshift in the environment of the Schwarzschild gravity field generated by a static, spherically symmetric material source like the sun.[6]

In the present approach, the only ingredient of GR to be used is the isotropic form of the metric relevant to the gravity field (here the Schwarzschild field). No sophisticated mathematics involving four vectors, tensors, Christoffel symbols, or geodesic equations will be necessary. In addition to being simpler and hence understandable to a wide range of readers, the ensuing exposition offers a different window through which the GR events can be visualized. From the instructional point of view, it is always useful to look at familiar things from as many different perspectives as possible.

It would be worthwhile to display the relevant formulas of the optical–mechanical analogy that we are going to use in our paper. This is done in Sec. II. Thereafter, as a necessary step, the Schwarzschild gravity field is portrayed as a refractive optical medium in Sec. III. GR equations for the light orbit and the radar echo delay in Schwarzschild gravity are derived in Secs. IV and V, respectively. Finally, some relevant remarks including a few suggestions for future work are made in Sec. VI.

II. OPTICAL–MECHANICAL ANALOGY

We need not go into the detailed formulation of the "$F=ma$" optics of ER. Instead, for our present purpose, we reproduce only a table which is self-explanatory[3]

Quantity	Mechanics	Optics
Position	$\mathbf{r}(t)$	$\mathbf{r}(a)$
"Time"	t	a
"Velocity"	$\frac{d\mathbf{r}}{dt}\equiv\dot{\mathbf{r}}$	$\frac{d\mathbf{r}}{da}\equiv\mathbf{r}'$
"Potential energy"	$U(\mathbf{r})$	$-n^2(\mathbf{r})/2$
"Mass"	m	1
"Kinetic energy"	$T=\frac{m}{2}\lvert\dot{\mathbf{r}}\rvert^2$	$\frac{1}{2}\lvert\mathbf{r}'\rvert^2$
"Total energy"	$\frac{m}{2}\lvert\dot{\mathbf{r}}\rvert^2+U$	$\frac{1}{2}\lvert\mathbf{r}'\rvert^2-\frac{n^2}{2}$
"Equation of motion"	$m\ddot{\mathbf{r}}=-\operatorname{grad} U$	$\mathbf{r}''=\operatorname{grad}\left(\frac{n^2}{2}\right)$

As can be seen, the role of time t is played by the ER stepping parameter a having the dimension of length so that $\mathbf{r}'$ is not a velocity. It is a dimensionless quantity. The transition between t and a can be accomplished by using the relation

$$da=\frac{c_0}{n^2}\,dt, \tag{1}$$

where n denotes the refractive index of the optical medium, c_0 denotes the vacuum speed of light. Evans[4] has shown that the stepping parameter can be physically interpreted as "optical action." Just as a mechanical particle progresses in time along its trajectory, light progresses in optical action along its ray.

The identification

$$U(\mathbf{r})=-\frac{n^2}{2} \tag{2}$$

giving the "potential" U essentially bestows a mechanical character to photon motions; the only restriction being that the motion corresponds to mechanics at "zero total energy." We can also imagine the possibility that, at least in the classical regime, optical and mechanical motions may take place under the same form of "force"/force law. An excellent example is probably the Luneburg Lens in optics and the harmonic oscillator in mechanics.[5] In order to use the ER formulation in the GR regime, it would be necessary to associate a single scalar function, the optical refractive index n, with the gravity field under consideration. The "force" on massless particles moving in that gravity field can then be explicitly obtained[7] via the ER expression $\operatorname{grad}(-n^2/2)$.

From now on, we shall use primes (′) only to designate a radial coordinate (r') and not differentiation with respect to the stepping parameter a. All the differentiations will be displayed in full.

III. GRAVITY FIELD AS A REFRACTIVE MEDIUM

One might wonder what relationship could there possibly be between two entities apparently as diverse as a gravity field and a refractive optical medium? But, indeed, there is one! It was guessed by Einstein and Eddington,[8] formally developed by Plebanski and others[9] and utilized in the investigation of specific problems by many physicists.[9,10] We shall only quote the results in a form that is easily intelligible. Plebanski[9] has shown that, in a curved spacetime with a metric tensor $g_{\alpha\beta}$, Maxwell's electromagnetic equations can be rewritten as if they were valid in a flat space–time in which there is an optical medium with a constitutive equation. More specifically, with regard to light propagation, the gravity field behaves as a refractive optical medium. For example, the gravitational field exterior to a static, spherically symmetric mass M is equivalent to an isotropic, nonhomogeneous optical medium with a refractive index n given by[10]

$$n^2(r)=\left(1+\frac{m}{2r}\right)^6\left(1-\frac{m}{2r}\right)^{-2}, \quad m=GM/c_0^2, \tag{3}$$

where, as usual, G is the gravitational constant, r is the Euclidean radial coordinate.

Advanced students who are likely to have a fair grasp on the two field theories, Einstein's and Maxwell's, along with the details of algebraic manipulations should have no difficulty in pursuing the analysis of Plebanski and de Felice.[9,10] However, there is a simpler alternative derivation of Eq. (3), given below, that demands only the knowledge of the form

the Schwarzschild exterior metric. One may also take Eq. (3) at its face value and proceed by treating this $n(r)$ as just a given choice of the refractive index in the ER formulation. We shall essentially investigate the consequences of such a choice in the succeeding sections.

Consider the exterior Schwarzschild metric in standard coordinates (r',θ,φ,t)

$$ds^2=\left(1-\frac{2m}{r'}\right)c_0^2\,dt^2-\left(1-\frac{2m}{r'}\right)^{-1}dr'^2-r'^2\times(d\theta^2+\sin^2\theta\,d\varphi^2)$$
$$=B(r')c_0^2\,dt^2-A(r')dr'^2-r'^2(d\theta^2+\sin^2\theta\,d\varphi^2), \tag{4}$$

where $r'>2m$. Redefine the radial coordinate as $r'=r\phi^{-1}(r)=r(1+(m/2r))^2$, so that if $r=u^{-1}$ and $r'=u'^{-1}$, then

$$u'=u\phi(u), \tag{5}$$

$$\phi(u)=\left(1+\frac{mu}{2}\right)^{-2}. \tag{6}$$

The metric (4) can then be rewritten in the so-called isotropic coordinates (r,θ,φ,t) as

$$ds^2=\Omega^2(r)c_0^2\,dt^2-\phi^{-2}(r)[dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\varphi^2]=\Omega^2(r)c_0^2\,dt^2-\phi^{-2}(r)|d\mathbf{r}|^2, \tag{7}$$

where $\Omega(r)=(1+(m/2r))^{-1}(1-(m/2r))$.

The above form of the metric has a conformally Euclidean spatial part.[11] Therefore, the isotropic coordinate speed of light $c(r)$ at any arbitrary point in the gravitational field, obtained by putting $ds^2=0$, is

$$c(r)=\left|\frac{d\mathbf{r}}{dt}\right|=c_0\phi(r)\Omega(r). \tag{8}$$

Defining the refractive index as $n(r)=c_0/c(r)$, we have

$$n(r)=\Omega^{-1}\phi^{-1}=\left(1+\frac{m}{2r}\right)^3\left(1-\frac{m}{2r}\right)^{-1}, \tag{9}$$

which is precisely the same as Eq. (3) above. It is easy to see that

$$\phi^2(u')=2^{-4}[1+(1-2mu')^{1/2}]^4, \tag{10}$$

$$\Omega^2(u')=1-2mu', \tag{11}$$

and

$$du'=\phi(u)\Omega(u)du. \tag{12}$$

All the above relationships will be used throughout the paper. It should be mentioned that Sjödin[12] has also derived the expressions for ϕ, Ω and n by a different method that is based on Rindler's operational definitions of length and time in a gravitational field.

The refractive index $n(r)$ above corresponds to a radial "force law" having a magnitude

$$F=\frac{dU}{dr}=\left[2mr^{-2}-\frac{m}{2}r^{-3}\right]\left[1+\frac{m}{2}r^{-1}\right]^5\left[1-\frac{m}{2}r^{-1}\right]^{-3}.$$

IV. LIGHT ORBIT EQUATION

Within the ER framework, the optical analog of the mechanical zero total energy or in short "zero total energy" (see the entry in the table in Sec. II) is given by

$$\frac{1}{2}\left|\frac{d\mathbf{r}}{da}\right|^2-\frac{n^2}{2}=0. \tag{13}$$

We straightaway claim that this very equation is the GR light orbit equation in Schwarzschild gravity, provided n is given by Eq. (9). In order to see that, we proceed as follows. Since there is spherical symmetry in the problem, we can, without loss of generality, fix a plane in which the orbiting has to take place. It is customary[13] to choose $\theta=\pi/2$. Writing out Eq. (13) in plane polar coordinates, we have

$$\left(\frac{dr}{da}\right)^2+r^2\left(\frac{d\varphi}{da}\right)^2-n^2(r)=0. \tag{14}$$

Notice that Eq. (13) is the first integral of the "equation of motion" $\mathbf{r}''=\text{grad}\,(n^2/2)$. Since n does not depend on φ, the φ component of the equation of motion yields a conserved quantity called by ER the "angular momentum" h, given by

$$h=r^2\frac{d\varphi}{da}=\text{constant}. \tag{15}$$

Eliminating the stepping parameter a, and writing $r=u^{-1}$, we have

$$u^2+\left(\frac{du}{d\varphi}\right)^2-n^2h^{-2}=0. \tag{16}$$

Evans and Rosenquist have solved a number of problems in classical optics/mechanics using different choices for $n(\mathbf{r})$. Below we shall discuss, in the context of GR, some aspects of the Eqs. (13)–(16):

(i) As ER have already discussed, the optical quantity h is completely different from the corresponding conserved mechanical quantity $h_0=r^2\,d\varphi/dt$ which is proportional to the areal velocity. In the optical formalism, we have $h=c_0^{-1}n^2r^2\,d\varphi/dt$ so that $h_0\propto n^{-2}$ and $d\varphi/dt\propto n^{-2}r^{-2}$. Hence, with our form for $n(r)$, Eq. (9), neither the areal velocity h_0 nor the angular velocity $d\varphi/dt$ remains constant. Of course, for the same force law, the optical and mechanical zero energy orbits must have the same form, since a or t are ultimately eliminated. We see that the GR light orbit equation has the form of well-known classical optics equation,[14] although this interesting fact is not very widely noticed. The reason for this is that in most text books on GR, the light orbit equation is given in a completely different form.

(ii) Let us obtain that familiar text book form. Using Eqs. (5), (11), and (12) in Eq. (16), we get

$$\phi^{-2}(1-2mu')^{-1}\left(\frac{du'}{d\varphi}\right)^2+u'^2\phi^{-2}-n^2h^{-2}=0. \tag{17}$$

Using Eq. (9) for n, we find

$$u'^2+\left(\frac{du'}{d\varphi}\right)^2-2mu'^3-h^{-2}=0. \tag{18}$$

Differentiating with respect to φ, we get

$$u'+\frac{d^2u'}{d\varphi^2}-3mu'^2=0. \tag{19}$$

This is precisely the light orbit equation in Schwarzschild gravity in standard coordinates. Therefore, one can say that the familiar Eq. (19) represents in disguise just the optical analog of the classical zero total energy mechanics. This conclusion will find a further justification in the developments of Sec. V.

(iii) It would be of interest to see how the "angular momentum" h is related to the conserved GR quantities E and L, associated with the killing fields $\partial/\partial t$ and $\partial/\partial\varphi$, respectively. This would also provide a relationship between the ER stepping parameter a and the geodesic affine parameters used by relativists. Integration of the GR null geodesic equations gives

$$\left|\frac{d\mathbf{r}}{dt}\right|^2=c_0^2\Omega^2\phi^2,\quad \Omega^2\frac{dt}{dp}=E,\quad \phi^{-2}r^2\frac{d\varphi}{dp}=L, \tag{20}$$

where $ds^2=\lambda\ dp^2$, λ is a constant and p is a new geodesic affine parameter such that $ds^2=0\Rightarrow\lambda=0$, $dp^2\neq0$. Eliminating t from Eqs. (20), we get

$$c_0^{-2}E^{-2}\phi^{-4}\left|\frac{d\mathbf{r}}{dp}\right|^2-n^2=0. \tag{21}$$

If we now define

$$dp=c_0^{-1}E^{-1}\phi^{-2}\ da \tag{22}$$

then Eq. (13) follows immediately from Eq. (21). We also find that $h=L/c_0E$. From Eqs. (20) and (22), there also follows the ER relation (1) connecting dt and da. Further, Eq. (22) implies that $ds^2=\lambda c_0^{-2}E^{-2}\phi^{-4}\ da^2$, giving the connection between a and the affine parameter s.

(iv) By integrating Eq. (19), one obtains all allowable light orbits;[15] bound, unbound, or even the so-called *boomerang* orbits.[16] There have been many observations confirming the GR predictions of the bending of light rays just grazing the sun, the amount being $\Delta\varphi\sim4m/R_0$, where R_0 is the solar radius. Interested readers may consult any textbook on GR. We shall only make a relevant remark here. Møller[17] has shown that the bending of light rays is due partly to the geometrical curvature of space and partly to the variation of light speed in a Newtonian potential. In fact, the ratio of the parts is 50:50. The GR null trajectory equations can be integrated, once assuming a Euclidean space with a variable light speed and again a curved space with a constant light speed; both contributing just half the amount of the observed bending. In the present approach, on the other hand, we are describing light motion by means of a scalar function $n(r)$. Thus with regard to the light propagation, it looks as if spatial curvature and Newtonian potential lost their separate identities and merged into an equivalent refractive medium. There is, of course, no point in asking which one has a physical reality and which one has not; all these are mathematical constructs [like the $n(r)$ here] designed only to interpret our physical observations.

(v) Finally, some words of caution. From the similarity of Eqs. (16) and (18), one might be tempted to conclude that in the u' coordinates, n is given by $n(u')=(1+2mh^2u'^3)^{1/2}$, but that would be incorrect. The correct expression for $n(u')$ is obtained by using the fact that $n(u)$ transforms as a scalar. Hence, one obtains[18]

$$n(u')=4(1-2mu')^{-1/2}[1+(1-2mu')^{1/2}]^{-2}. \tag{23}$$

Also, it must be understood that, with this $n(u')$, it is not possible to define an isotropic coordinate velocity of light in the u' coordinates. From a direct calculation with the metric (4), it will turn out that the coordinate velocities of light are different in radial and cross radial directions.

V. RADAR ECHO DELAY

Let us now go beyond the geometrical shape of the light orbit and consider the dynamics along its path. In other words, we shall derive the GR equation of motion involving the time t for the propagation of light rays around a static, spherically symmetric gravitating mass M.

Once again we start from the ER "zero energy" equation $(1/2)|d\mathbf{r}/da|^2-(n^2/2)=0$ and claim that it *is* the GR equation we are looking for provided n is given by Eq. (9). To see this, consider Eqs. (13)–(15) and write

$$\left(\frac{dr}{da}\right)^2+h^2r^{-2}-n^2(r)=0. \tag{24}$$

Utilizing the redefinition $u\rightarrow u'$ and the expression for n from Eq. (9), we have

$$\phi^4u'^{-4}\left(\frac{du'}{da}\right)^2+h^2(1-2mu')u'^2-1=0. \tag{25}$$

At the position $r_0'[=u_0'^{-1}]$ of the closest approach to the gravitating mass M, we have $dr'/da=0\Rightarrow du'/da|_{u_0'}=0$, giving $h^2=(1-2mu_0')^{-1}u_0'^{-2}$. Putting it in Eq. (25), we find

$$\phi^4\Omega^{-2}u'^{-4}\left(\frac{du'}{da}\right)^2+u_0'^{-2}u'^2(1-2mu_0')^{-1}-\Omega^{-2}=0. \tag{26}$$

Noting from the metric (4) that $B(r')=\Omega^2$ and $A(r')=\Omega^{-2}$ and using Eq. (1), we finally have

$$c_0^{-2}A(r')B^{-2}(r')\left(\frac{dr'}{dt}\right)^2+r_0'^2r'^{-2}B^{-1}(r_0')-B^{-1}(r')=0. \tag{27}$$

This is precisely the textbook form of what is known as the radar echo delay equation in Schwarzschild gravity—and, once again, we see that it is still the same ER "zero energy" mechanics, only buried in the (r',t) language. Equation (27) is integrated to obtain the time t required by the light signal (in practice, a radar signal) to travel from one point of space to another. It is also evident that the coordinate speed of light dr'/dt is less than what it would be if the gravitating mass were absent. In other words, light is slowed down and the travel time is longer. All observers will nonetheless measure a photon's speed to be c_0 through their positions. In a round trip journey around the mass M, there would be a net GR delay in the radar echo reception. This GR prediction has been confirmed to a great accuracy by sending radar signals from Earth to Mercury at superior conjunction and back.[19]

VI. SOME REMARKS

The contents of the entire paper vividly demonstrate that light propagation in Schwarzschild gravity is indeed describable by the language of "$F=ma$" optics, a shorthand for the optical–mechanical analogy. It is remarkable that the ER for-

mulation, emerging basically from the investigations of classical problems, works equally well also in the GR regime.

We are concerned with the predictions for light propagation that follow exclusively from Einstein's field equations. On the other hand, the formula for gravitational redshift can be derived from special relativity and the principle of equivalence alone. The field equations or their solutions need not be used. In that sense, the redshift is not a prediction following exclusively from the field equations, albeit the effect *does* follow also from the latter. In the present approach, the usual Principle of Equivalence that equates the effects of gravity and artificial forces, is translated into an equivalence of the effects of gravitation and refractive medium. Using this idea, the gravitational redshift formula can be obtained in its familiar form. We shall present the deductions in a separate paper.

It is evident that the considered approach can deal with a reasonably general class of gravity fields where the metric can be cast into an isotropic form. The latter makes it possible to construct a scalar function n which acts as a representative of the gravity field in the matter of light propagation. However, the whole range of GR solutions corresponding to different field distributions can not be made to correspond to such single scalar functions. At best, a detailed constitutive tensor for the equivalent medium may be developed.[10] In this case, the challenging task would be to generalize the ER equations appropriately. Nonetheless, we can list some immediate future works:

(i) The motion of a light pulse inside and across the body of a spherical star can be tackled quite comfortably. The interior of a spherical star (like the sun) is described by the Schwarzschild interior metric. It corresponds to a refractive medium with index, say, n_1 while the exterior field corresponds to n, Eq. (9). Therefore, the whole problem is reduced to one of light propagation in a composite media with indices n and n_1. At the interface, the matching condition is provided by none other than good old Snell's law. The result will provide a theoretical idea about the total path of light rays between the sun's core and the Earth.

(ii) For the sake of completeness, we would expect the "$F=ma$" optics to describe also the motion of massive particles (planetary motion) in the Schwarzschild gravity field. To that end, efforts are underway to extend the present treatment. The answer to the second part of the question raised in Sec. I would then be available.

Finally, it must be emphasized that there is neither any substitute for nor shortcut to the beauty, generality, and richness of Einstein's general relativity theory. One must eventually grasp all the details of the physics, mathematics, and the philosophy of this magnificent never ending edifice. On the other hand, the language of the ER optical–mechanical analogy has the power to stimulate the interests of a wide cross section of readers who do not have a formal training in the sophistications of GR. Those who have the training may, however, regard the preceding developments as providing yet another avenue to the same experimentally verified tests of light motion in GR. The educational importance of such alternative points of view can not be mistaken.

ACKNOWLEDGMENTS

We are deeply indebted to Professor J. Evans for his criticisms and suggestions that have led to a considerable improvement of the paper. One of us (A.I.) is grateful to the Indian Council for Cultural Relations (I.C.C.R.), Azad Bhawan, Indraprastha, New Delhi for a fellowship under an Exchange Program of the Government of India.

[a]On leave from Department of Mathematics, Tolaram Government College, Narayanganj, Dhaka, Bangladesh.

[1]As quoted by Evans, the optical–mechanical analogy was formulated by Hamilton (1833) in terms of characteristic functions. See, J. Evans, "The ray form of Newton's law of motion," Am. J. Phys. 61, 347–350 (1993). In this paper, Evans derives the ray form without the use of characteristic functions.

[2]Historically, the basic idea of the optical–mechanical analogy was proposed much earlier by none other than Descartes (1637). This information is due to J. A. Arnaud, "Analogy between optical rays and nonrelativistic particle trajectories: A comment," Am. J. Phys. 44, 1067–1069 (1976). In this paper are also discussed the limitations and significance of the analogy.

[3]Essential ingredients of what is referred to as the ER approach are contained in: J. Evans and M. Rosenquist, "'$F=ma$" optics," Am. J. Phys. 54, 876–883 (1986).

[4]J. Evans, "Simple forms for equations of rays in gradient-index lenses," Am. J. Phys. 58, 773–778 (1990), see especially p. 774.

[5]Newton's laws of motion are obtained directly from Fermat's principle, in M. Rosenquist and J. Evans, "The classical limit of quantum mechanics from Fermat's principle and the de Broglie relation," Am. J. Phys. 56, 881–882 (1988).

[6]For a general information: Latest confirmations of GR come from the observations of the Hulse–Taylor binary pulsar PSR1913+16. There is a rate of decrease in the orbital period $[P\sim(3.2\pm0.6)\times10^{-12}\mathrm{ss}^{-1}]$ as well as a precession of the order of 4° per year. Observations are in excellent agreement with the GR predictions. See, J. H. Taylor, L. A. Fowler and P. M. McCulloch, "Measurements of general relativistic effects in the binary pulsar PSR 1913+16," Nature (London) 277, 437–440 (1979).

[7]To repeat, the "force" is not the force in the mechanical sense. Nonetheless, a material particle acted on by a mechanical force of the same form will follow the same null track. See Ref. 16 below for the citation of an interesting example.

[8]Sir A. S. Eddington calculated the bending of light rays round a massive object by assuming that the space around the sun is filled with a medium with a refractive index $n(r)=(1-2m/r)^{-1}\sim1+2m/r$. Incidentally, he was also the first to experimentally observe such a bending in 1919 during the Solar eclipse in Sobral, Brazil. See, A. S. Eddington, *Space, Time and Gravitation* (Cambridge University, Cambridge, 1920), reissued in the Cambridge Science Classics Series, 1987, p. 109. However, F. de Felice (Ref. 10) reports that Einstein himself was the first to suggest the idea of an equivalent refractive medium.

[9]Among a number of works in this direction, we list only a few, just to give an idea: J. Plebanski, "Electromagnetic waves in gravitational fields," Phys. Rev. 118, 1396–1408 (1960). B. Bertotti, "The luminosity of distant galaxies," Proc. R. Soc. 294, 195–207 (1966). F. Winterberg, "Deflection of gravitational waves by stellar scintillation in space," Nuovo Cim. B 53, 264–279 (1968). See also Ref. 10 for a collection of other references.

[10]F. de Felice shows that the equivalence of a gravity field with the refractive medium can be successfully employed as a method of investigation: F. de Felice, "On the gravitational field acting as an optical medium," Gen. Rel. Grav. 2, 347–357 (1971).

[11]Because of the general coordinate covariance of Einstein's GR, its solutions can be freely expressed in any coordinate system we like. Such a freedom, however, does not affect the unique observable predictions of GR. This important point is further illuminated in some recent papers. See, Ya. B. Zel'dovich and L. P. Grishchuk, "The general relativity theory is correct," Sov. Phys. Usp. 31, 666–671 (1988), and references therein. T. Ohta and T. Kimura, "Coordinate independence of physical observables in the classical tests of general relativity," Nuovo Cim. B 106, 291–305 (1991); A. N. Petrov, "New harmonic coordinates for the Schwarzschild geometry and the field approach," Astron. Astrophys. Trans. 1, 195–205 (1992). We would particularly recommend these papers to the advanced graduate students and researchers in GR.

[12]T. Sjödin, in *Physical Interpretations of Relativity Theory*, edited by M. C. Duffy (British Soc. Philos. Sc., London, 1990), pp. 515–521. Sjödin also obtains the light orbit equation starting from Rindler's prescription. See,

W. Rindler, *Essential Relativity* (Springer, New York, 1977), especially Secs. 8.3 and 8.4.

[13]The reason for fixing the $\theta=\pi/2$ plane is that $\partial/\partial\varphi$ is a Killing field on the entire manifold while $\partial/\partial\theta$ is not.

[14]Equation (16) is the classical optics equation. See, A. Maréchal, *Optique Géometriqué Générale*, edited by S. Flügge, Handbuch der Physik, Vol. 34 (Springer, Berlin, 1956), p. 44. M. Born and E. Wolf, *Principles of Optics*, 2nd ed. (Pergamon, New York, 1964), pp. 121–124.

[15]A number of light orbits have been plotted in S. Chandrasekhar, *The Mathematical Theory Of Black Holes* (Oxford University, Oxford, 1983). See also the classic treatise L. D. Landau and E. M. Lifshitz, *The Classical Theory Of Fields* (Pergamon, New York, 1975), pp. 313–316.

[16]W. M. Stuckey, "The Schwarzschild black hole as a gravitational mirror," Am. J. Phys. 61, 448–456 (1993). The possibility of a boomerang shaped light orbit is interesting in its own right. We can say that the path of a real massive boomerang around a central mass is also obtained by the same force law in mechanics. The total energy, however, need not be zero.

[17]The detailed calculations appear in C. Møller, *The Theory of Relativity*, 2nd ed. (Oxford University, Oxford, 1972), pp. 498–501. The variable speed of light is given by $c^*=c_0(1+2\chi/c_0^2)^{1/2}$, where $\chi=GM/r'$. Purely spatial curvature is represented by the metric $ds^2=c_0^2\,dt^2-(1-2m/r')^{-1}\,dr'^2-r'^2(d\theta^2+\sin^2\theta\,d\varphi^2)$. Møller expresses the null trajectory in terms of c^* and the metric tensor $(g_{\alpha\beta})$ to demonstrate his result.

[18]Notice that at the Schwarzschild black hole radius $r'=2m$, $n(u')=\infty$. Also from Eq. (9), at $r=m/2$, $n(u)=\infty\Rightarrow c(r)=0$. This result reflects the fact that, to a distant observer, even light comes to a standstill! To a local observer, however, light always travels at a speed c_0.

[19]I. I. Shapiro *et al.*, "Fourth test of general relativity: New radar result," Phys. Rev. Lett. 26, 1132–1135 (1971). The GR predicted value of time delay for the Earth–Mercury system is ~240 μs while the observed value is (245±12)μs. A remarkable agreement indeed.

Potentials and bound states

Walter F. Buell[a)]
Department of Physics, The University of Texas at Austin, Austin, Texas 78712-1081

B. A. Shadwick
Department of Physics and Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712-1081

(Received 23 May 1994; accepted 18 August 1994)

We discuss several quantum mechanical potential problems, focusing on those which highlight commonly held misconceptions about the existence of bound states. We present a proof, based on the variational principle, that certain one dimensional potentials always support at least one bound state, regardless of the potential's strength. We examine arguments concerning the existence of bound states based on the uncertainty principle and demonstrate, by explicit calculations, that such arguments must be viewed with skepticism. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

One of the first types of problems encountered by students beginning a study of quantum mechanics is that of finding the eigenstates of a potential. Such problems form the basis of understanding for a great many physical systems, and so are important not just as pedagogical exercises, but also as real world models in solid state, nuclear, atomic and molecular physics. In addition, simple one and two dimensional potentials form the basis of our understanding of low dimensional structures such as quantum well devices.[1]

There is a substantial folklore concerning these simple potential problems. In surveying a variety of standard introductory (or even advanced) quantum mechanics texts, one finds various fragments of this folklore but rarely are they presented in a comprehensive fashion which would allow the reader to apply them to more general problems or, for that matter, to understand their physical and mathematical origin. This situation is made worse by the fact that some of these so-called standard results are wrong. Our purpose here is to present an organized view of a selection of this folklore, expunging the erroneous results along the way.

Before proceeding, the reader is asked to apply his or her knowledge of this folklore to the following questions: How would you modify the statement, "Every potential has at least one bound state," in order to make it true? (Not comprehensive, only correct.) How would you prove it? Can the condition for the existence of at least one bound state in a spherical step well be related to the Heisenberg uncertainty relation? If so, does such a relation also apply to the one dimensional step well? How does $\Delta x\Delta p$ behave as a potential well is made deeper and additional eigenstates appear?

To focus the discussion of the issues raised above, we consider the spherically symmetric step well

$$V(r)=\begin{cases}-V_0 & r\leq a;\\ 0 & r>a,\end{cases} \tag{1}$$

which supports a bound state only for

$$V_0>\frac{\hbar^2\pi^2}{8ma^2}. \tag{2}$$

This is a generic feature of three dimensional central potentials. Numerous authors[2,3] have attempted to give a physical explanation of this by means of the uncertainty principle. The essence of the argument is as follows: Assuming $\Delta x\sim a$, from the uncertainty relation one obtains

$$\Delta p\sim\frac{\hbar}{2a}. \tag{3}$$

For the states under consideration it is reasonable to assume

General Relativity and Gravitation, Vol. 28, No. 4, 1996

The Optical-Mechanical Analogy in General Relativity: Exact Newtonian Forms for the Equations of Motion of Particles and Photons

James Evans,[1] Kamal K. Nandi[2] and Anwarul Islam[3]

Received March 15, 1995. Rev. version October 2, 1995

In many metrics of physical interest, the gravitational field can be represented as an optical medium with an effective index of refraction. We show that, in such a metric, the orbits of both massive and massless particles are governed by a variational principle which involves the index of refraction and which assumes the form of Fermat's principle or of Maupertuis's principle. From this variational principle we derive exact equations of motion of Newtonian form which govern both massless and massive particles. These equations of motion are applied to some problems of physical interest.

1. INTRODUCTION

The representation of the gravitational field as an optical medium is an old idea, which was exploited by Eddington (Ref. 1, p.109) and which has been developed in more detail by others [2,3]. In many metrics of physical interest, one may find a coordinate transformation that renders the space part of the line element isotropic. If, in addition, the metric has no explicit

[1] Department of Physics, University of Puget Sound, Tacoma, Washington 98416, USA. E-mail: JCEvans@UPS.edu

[2] Department of Mathematics, University of North Bengal, Darjeeling (W.B.) 734 430, India

[3] Department of Mathematics, Government Tolaram College, Narayanganj, Dhaka, Bangladesh

0001-7701/96/0400-0413$09.50/0

time dependence, the line element may be written in the form

$$\begin{aligned} ds^2 &= \Omega^2(\mathbf{r})c_0^2\,dt^2 - \Phi^{-2}(\mathbf{r})\,[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2] \\ &= \Omega^2(\mathbf{r})c_0^2\,dt^2 - \Phi^{-2}(\mathbf{r})\,|d\mathbf{r}|^2, \end{aligned} \tag{1}$$

where Ω and Φ are functions of the so-called isotropic coordinates r, θ, and ϕ (which are related to the standard coordinates by a transformation) and c_0 is the vacuum speed of light. Boldface $\mathbf{r}$ is an abbreviation for (r, θ, ϕ). The isotropic coordinate speed of light $c(\mathbf{r})$ at any point in the field may be obtained by putting $ds = 0$:

$$c(\mathbf{r}) = |d\mathbf{r}/dt| = c_0\Phi(\mathbf{r})\Omega(\mathbf{r}). \tag{2}$$

Thus the effective index of refraction is

$$n = \Phi^{-1}(\mathbf{r})\Omega^{-1}(\mathbf{r}). \tag{3}$$

Light trajectories in the gravitational field can be calculated by using the effective index of refraction (3) in any formulation of geometrical optics that happens to be convenient. For example, Wu and Xu have recently shown that the standard differential equation of the ray in classical geometrical optics can be applied to the null geodesic problem [4].

An especially convenient version of geometrical optics is the so-called "$F = ma$" formulation [5,6] in which the equation governing the optical ray assumes the form of Newton's law of motion (acceleration = − gradient of potential energy):

$$d^2\mathbf{r}/dA^2 = \nabla(n^2c_0^2/2). \tag{4}$$

$\mathbf{r}$ is the position of a light pulse moving along the ray. The independent variable (analogous to the time) is the stepping parameter, or optical action A. The effective potential energy function is $-n^2c_0^2/2$. All the usual force and energy methods of elementary mechanics can be brought to bear on geometrical optics.

The effective index of refraction (3) (for the Schwarzschild metric, for example) can be used in (4) without modification [7]. In solving problems, one goes into the isotropic coordinates, applies the $F = ma$ optics, then transforms back to the standard coordinates, if desired. The goal of the present paper is to extend these methods to the motion of massive particles.

Fermat's principle has been the subject of renewed interest in general relativity [8–11]. The present paper differs from other recent work in this area in that (i) it focuses on the analogy between the principles of Maupertuis and Fermat in the context of general relativity and (ii) it leads to

a remarkable simplification of the equations of motion for both particles and photons.

In Section 2, we derive from the geodesic condition a variational principle which takes on the form of Fermat's principle or Maupertuis' principle. The variational principle, which governs the trajectories of both massive and massless particles, implies that, in isotropic metrics, the particle equations of motion can be cast into the form of Newtonian mechanics or of classical geometrical optics. In Section 3, we derive equations of motion from the variational principle. These equations of motion, which represent a *generalization of* (4), *are exact, apply equally to massive and massless* particles, but are nevertheless of Newtonian form. In Section 4, we present effective indices of refraction for a number of metrics of physical and cosmological significance. In Section 5, we illustrate the use of the new equations of motion in some concrete applications: we demonstrate an analogy between two cosmological models and the Maxwell fish-eye lens, we extend some recent calculations involving tests of general relativity in Reissner–Nordström-type metrics, and we present novel derivations of the gravitational and cosmological redshifts.

2. TRANSFORMATION OF THE GEODESIC CONDITION

Our goal is to apply the classical optical-mechanical analogy to particle orbits in general relativity. In order to set up the analogy, it will be convenient to begin from a variational principle for the trajectories that can be considered analogous to the principle of Fermat (classical geometrical optics) and the principle of Maupertuis (Newtonian mechanics in velocity-independent potentials).

We shall obtain the variational principle by transformation of the geodesic condition for the particle trajectories,

$$\delta \int_{\mathbf{x}_1, t_1}^{\mathbf{x}_2, t_2} ds = 0, \tag{5}$$

where δ indicates a variation in the path of integration between two fixed points in spacetime, $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_2)$. If we assume the line element can be written in the form (1) this becomes

$$\delta \int_{\mathbf{x}_1, t_1}^{\mathbf{x}_2, t_2} \Omega c_0 [1 - v^2 n^2 / c_0^2]^{1/2} \, dt = 0 \,. \tag{6}$$

This is analogous to Hamilton's principle and the effective Lagrangian is

$$L(x_i, \dot{x}_i) = -c_0^2 \Omega [1 - v^2 n^2 / c_0^2]^{1/2}, \tag{7}$$

where Ω and n are functions of the coordinates alone, where $\dot{x}_i \equiv dx_i/dt$, and where $v^2 = \sum_{i=1}^{3} (dx_i/dt)^2$, if we choose to work in Cartesian coordinates. The expression for the Lagrangian has been multiplied by an extra factor of $-c_0$ for later convenience. (Note: We will always write c_0 explicitly. This paper is concerned with an analogy linking geodesic motion, classical geometrical optics, and classical Newtonian mechanics. c_0 is not usually suppressed in the latter two fields. Thus the clarity of the analogy is enhanced by retaining classical units of measure.)

The canonical momenta p_i are

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{x}_i} \\ &= \Omega n^2 [1 - v^2 n^2 / c_0^2]^{-1/2} \, \dot{x}_i \, . \end{aligned} \tag{8}$$

The Hamiltonian H may be formed in the usual way,

$$\begin{aligned} H &= \sum_{i=1}^{3} p_i \dot{x}_i - L \\ &= c_0^2 \Omega [1 - v^2 n^2 / c_0^2]^{-1/2}. \end{aligned} \tag{9}$$

Because $\partial L / \partial t = 0$, H is a constant of the motion. If we express H in terms of the p_i rather than the x_i we obtain

$$H = c_0^2 [\Omega^2 + p^2 / n^2 c_0^2]^{1/2}, \tag{10}$$

where $p = |\vec{p}|$. From Hamilton's principle,

$$\delta \int_{\mathbf{x}_1, t_1}^{\mathbf{x}_2, t_2} L \, dt = 0, \tag{11}$$

one may derive in the usual way the corresponding action principle (Jacobi's form of Maupertuis' principle) (Ref. 12, p.125-8,132-4),

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} \left(\sum_{i=1}^{3} p_i \dot{x}_i \right) \, dt = 0, \tag{12}$$

where now the path of integration is varied between two fixed points in space, $\mathbf{x}_1$ and $\mathbf{x}_2$, where the energy must be held constant on the varied paths, but where the times at the end points need not be held fixed. With the canonical momenta from (8), this becomes

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v^2 \Omega [1 - v^2 n^2 / c_0^2]^{-1/2} \, dt = 0 . \tag{13}$$

We restrict the varied paths to those that satisfy the energy constraint by substituting the constant H for the right side of (9) where this appears in (13). Then, putting $dt = dl/v$, where $dl = |d\mathbf{r}| = (\sum_{i=1}^{3} dx_i^2)^{1/2}$ we obtain

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v \, dl = 0 . \tag{14}$$

This is a variational principle on which an analogy to geometrical optics or to classical mechanics can be constructed. In obtaining (14) we have preferred, for the sake of directness, clarity and and consistency of notation, to begin from the fundamental principle (5). But (14) may also be derived from other versions of the three-dimensional variational principle for particle orbits in static metrics, for example, the forms first obtained by Weyl [13] and Levi-Civita [14].

In (14), $n^2 v$ is to be considered a function of position alone. The path of integration is varied between the fixed end points $\mathbf{x}_1$ and $\mathbf{x}_2$, and the value of H is held constant during the variation. Thus, (14) is of the same form as Fermat's principle, which forms a basis for classical geometrical optics, and Maupertuis' principle, which forms a basis for classical mechanics (as long as the force can be derived from a velocity-independent potential):

Relativistic gravitational mechanics	Geometrical optics (Fermat)	Classical mechanics (Maupertuis)
$\delta \int n^2 v \, dl = 0$	$\delta \int n \, dl = 0$	$\delta \int v \, dl = 0 .$

In the context of motion in a static gravitational field, both Fermat's principle and Maupertuis' principle are simply special cases of (14). For the

null geodesics, i.e., for the paths of light, the derivation given above must be slightly modified, to keep each step well defined. But the final result is too well known to require detailed discussion here: in static metrics, light obeys Fermat's principle. That is, the path taken by light between two fixed points in space is one for which the coordinate time of travel is stationary (Ref. 15, Ref. 16, p.99-100). In the language of a refractive index, this may be written $\delta \int n\, dl = 0$. Since, for light, $v = c_0/n$, (14) does reduce to the appropriate form. To obtain Maupertuis' principle (and hence Newtonian gravitational motion), note that in ordinary solar-system dynamics, we may put $n^2 \approx 1$. That is, in the Newtonian limit, n^2 may be treated as constant in the variational calculation and we obtain Maupertuis' principle as the classical limit of (14).

3. EXACT EQUATIONS OF MOTION OF NEWTONIAN FORM

Let the path of the particle be parametrized by a stepping parameter A. That is, at each point on the path, the three space coordinates $\mathbf{r}$ (and also the time t) are regarded as functions of A. We defer for the moment choosing A: we shall define A to get the simplest equations of motion. Thus we write (14) in the form

$$\delta \int_{\mathbf{x}_1}^{\mathbf{x}_2} n^2 v \left| \frac{d\mathbf{r}}{dA} \right| dA = 0,$$

where $|d\mathbf{r}/dA| = [\sum_{i=1}^{3} (dx_i/dA)^2]^{1/2}$.

Let $\mathbf{r}(A)$ denote the true path. To obtain a varied path, we replace $\mathbf{r}(A)$ by $\mathbf{r}(A) + \mathbf{w}(A)$, where $\mathbf{w}(A)$ is an arbitrary, infinitesimal vector function, subject to the constraint that $\mathbf{w} = 0$ when A is such that $\mathbf{r} = \mathbf{x}_1$ or $\mathbf{x}_2$. That is, the variation must vanish at the end points. Now

$$\delta \int n^2 v \left| \frac{d\mathbf{r}}{dA} \right| dA = \int [\delta(n^2 v)] \left| \frac{d\mathbf{r}}{dA} \right| dA + \int (n^2 v) \left(\delta \left| \frac{d\mathbf{r}}{dA} \right| \right) dA + \int n^2 v \left| \frac{d\mathbf{r}}{dA} \right| \delta dA. \tag{16}$$

Calculating the two variation in the first term on the right-hand side of (16), we have

$$\delta(n^2 v) = \nabla(n^2 v) \cdot \mathbf{w}. \tag{17}$$

In calculating the variation in the second term of (16) it is important to remember that the change to the varied path will, in general, also produce

a change in A. Thus

$$\delta\left|\frac{d\mathbf{r}}{dA}\right| = \left|\frac{d\mathbf{r}+d\mathbf{w}}{dA+\delta dA}\right| - \left|\frac{d\mathbf{r}}{dA}\right|$$
$$= \frac{d\mathbf{r}}{dA}\cdot\frac{d\mathbf{w}}{dA}\left|\frac{d\mathbf{r}}{dA}\right|^{-1} - \left|\frac{d\mathbf{r}}{dA}\right|\frac{\delta dA}{dA}, \tag{18}$$

to first order in the variation. Substituting (17) and (18) into (16), we find

$$\delta\int n^2 v\left|\frac{d\mathbf{r}}{dA}\right| dA = \int\left[\left|\frac{d\mathbf{r}}{dA}\right|\nabla(n^2v)\cdot\mathbf{w} + n^2 v\left|\frac{d\mathbf{r}}{dA}\right|^{-1}\frac{d\mathbf{r}}{dA}\cdot\frac{d\mathbf{w}}{dA}\right] dA.$$

Note that the terms involving δdA have cancelled. This was to be expected, since (14) shows that the integral does not actually depend upon the range in A or, indeed, on what we select to use as parameter. Integrating the term involving $d\mathbf{w}/dA$ by parts, and using the fact that $\mathbf{w}$ must vanish at the endpoints, but is otherwise arbitrary, we arrive at the differential equation that must be satisfied by the particle trajectory:

$$\left|\frac{d\mathbf{r}}{dA}\right|\nabla(n^2v) - \frac{d}{dA}\left(n^2 v\left|\frac{d\mathbf{r}}{dA}\right|^{-1}\frac{d\mathbf{r}}{dA}\right) = 0. \tag{19}$$

This differential equation plays the role of an equation of motion. Another way to obtain (19) is to parametrize the path by one of the Cartesian coordinates (say z), rather than A, since the variation in z must vanish at $\mathbf{x}_1$ and $\mathbf{x}_2$. In this case, one writes

$$\delta\int n^2 v\frac{dl}{dz}\,dz = 0.$$

One may then simply write down the Euler conditions for the integral to be stationary, and then transform from z to A as independent variable. The result will be the same, that is (19).

To give the equation of motion the simplest possible form, and to take advantage of the analogy to Newtonian mechanics, let us now define A by

$$\left|\frac{d\mathbf{r}}{dA}\right| \equiv n^2 v. \tag{20}$$

With this definition of A, the equation of motion (19) becomes

$$\frac{d^2\mathbf{r}}{dA^2} = \nabla\left(\frac{1}{2}n^4v^2\right). \tag{21}$$

Equation (21) is the generalization of (4) that was sought. The left-hand side of (21) is of the form of an acceleration: it is the second derivative of the position vector with respect to the independent variable. The right-hand side of the equation is of the form of a force: $-\frac{1}{2}n^4v^2$ plays the role of a "potential energy function." The analogue of the velocity is $d\mathbf{r}/dA$. Thus the analogue of the kinetic energy is $|d\mathbf{r}/dA|^2$. The analogue of the total energy is the sum of the potential and the kinetic. But, by virtue of eq. (20), these two are guaranteed to sum to zero:

$$\tfrac{1}{2}|d\mathbf{r}/dA|^2 - \tfrac{1}{2}n^4v^2 = 0\,. \tag{22}$$

Thus the calculation of the paths of light and of massive particles in general relativity reduces to the zero-energy $F = ma$ optics of [5]. It is to be noted that the "conservation of energy" condition (22) amounts to a restatement of the definition (20) of A.

The optical-mechanical analogy, embodied in (21) and (22), provides an exact treatment in Newtonian form of the motion of massive particles, as well as light, in general relativity. The Newtonian form should be thought of as coming from $F = ma$ optics (which is exact) and not from Newtonian mechanics (which is, of course, only approximate). Equations (21) and (22) allow one to handle the paths of light and of the planets as if they existed in a flat three-dimensional space. Other approaches to this goal are, of course, possible [17], but the treatment presented here has three advantages: simplicity, complete conformity to the equations of *Newtonian mechanics, and a uniform treatment of both light and massive* particles. This treatment has a reasonably high degree of generality and is applicable whenever the line element can be written in the form (1).

In solving problems with (21) or (22), one may use without modification all the familiar methods of Newtonian mechanics. One simply thinks of A as if it were the time. Moreover, rather than beginning from (21) or (22) in every case, one may often simply write down an exact general-relativistic formula by analogy to the corresponding classical formula. For the motion of both light and massive particles in static metrics, one may begin from any classical Newtonian formula describing the motion of particles in static, velocity-independent potentials. The correct general-relativistic expressions will be obtained if one makes the following transcriptions in the classical formulas:

$$t \to A, \qquad U \to -n^4v^2/2, \qquad E \to 0\,. \tag{23}$$

The spatial coordinates (x, y, z), or (r, θ, ϕ), etc., transcribe as themselves. Of course, it must be kept in mind that the formulas written down in

this way apply in the isotropic coordinate system. After the equations governing the situation are obtained, or after they are solved, one may transform back to the original metric, if desired. A second point of vital importance is that the analogue of the classical energy E is always the number zero. Thus, the constant of the motion H plays no role in the analogy. H should rather be regarded as a parameter: the potential energy function depends upon H as well as the coordinates. A third point to be stressed is that the exact general-relativistic formulas written down by analogy to the classical formulas will always apply with equal validity to both massive and massless particles.

For both light and particles, the stepping parameter A is defined by (20). In many calculations, e.g., in finding the shape of an orbit, the stepping parameter is ultimately eliminated. Equation (20) will suffice for this purpose. In other situations (for example, in a radar-echo delay calculation), it may be necessary to have an explicit connection between A and t. Note that

$$\left|\frac{d\mathbf{r}}{dA}\right| = \left|\frac{d\mathbf{r}}{dt}\right|\frac{dt}{dA} = v\,\frac{dt}{dA}\,.$$

Substituting in (20) gives

$$dA = dt/n^2. \tag{24}$$

Thus the stepping parameter is the same as that used in the $F = ma$ formulation of geometrical optics. A is called the optical action because dA is proportional to $c(\mathbf{r})\,dl$, and thus is analogous to the action $v(\mathbf{r})\,dl$ of classical mechanics.

The only difference between the treatment of light and that of particles resides in the choice of $v(\mathbf{r})$, which forms a part of the effective potential energy $-n^4v^2/2$. For light,

$$v = c_0 n^{-1} \qquad \text{(light)}, \tag{25}$$

But for massive particles, (9) gives

$$v = c_0 n^{-1}[1 - c_0^4\Omega^2/H^2]^{1/2} \qquad \text{(particles)}. \tag{26}$$

In (26), H is a constant parameter determined by the initial conditions, while n and Ω are functions of the spatial coordinates determined by the metric. Because the particle expression for $v(\mathbf{r})$ contains the parameter H, the particle problem has an extra degree of freedom: we may specify the initial speed of the particle. Thus, in general, more types of orbits exist for massive particles than for light in the same metric.

For a particle in empty space devoid of gravitational influences, $\Omega \approx 1$, $n \approx 1$, and (9) becomes

$$H \approx c_0^2(1 - v^2/c_0^2)^{-1/2} = c_0^2\gamma. \tag{27}$$

In the solar-system dynamics of the Schwarzschild metric, $v/c_0 \ll 1$ and [see (39) and (43)] $\Omega \approx 1 - m/r$ so (9) becomes

$$H \approx c_0^2 + \tfrac{1}{2}v^2 - m/r. \tag{28}$$

That is, in classical planetary orbits, H is approximately equal to $c_0^2 + E$, the rest-mass energy plus the classical kinetic and potential energy per unit mass.

The classical optical-mechanical analogy is based upon the similarity of form of the principles of Fermat and Maupertuis. Usually this analogy is expressed in terms of nonlinear partial differential equations. In the form of the analogy originally due to William Rowan Hamilton, the eikonal equation of geometrical optics corresponds to the (time-independent) Hamilton–Jacobi equation of particle mechanics. But in fact the analogy is far more general. Corresponding to every possible formulation of classical mechanics, there is an analogous formulation of geometrical optics (and vice versa). Thus, as in (4), we can cast geometrical optics into the form of Newton's law of motion.

The variational principle (14) has permitted us to extend the analogy to the geodesic problem for both light and particles in isotropic metrics. Our discussion of the optical-mechanical analogy in general relativity has stressed Newtonian forms. But, corresponding to every formulation of *either* classical particle mechanics *or* of classical geometrical optics, there will be an analogous formulation of the geodesic problem in general relativity. Few of these classical models for the reformulation of the geodesic equations of motion lead to any special insight or simplification. The economy of expression and simplicity of form embodied in (21) and (22) depend, not so much on the formulation of mechanics (or of geometrical optics) that is chosen as model, as on the use of A rather than t as independent variable. The remainder of the paper is devoted to the development of calculation techniques based on the Newtonian formulation embodied in (21) and (22).

4. INDICES OF REFRACTION FOR SOME IMPORTANT METRICS

4.1. Metrics of the Reissner–Nordström (RN) type

A number of line elements of physical interest assume the following form in standard coordinates (t, r', θ, ϕ):

$$ds^2 = c_0^2\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]dt^2 - \left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2\theta \phi^2, \tag{29}$$

where

$$m = MG/c_0^2, \tag{30}$$

M is the mass of the central gravitating body, G is the gravitation constant, and β is another parameter. We wish to write the line element in terms of isotropic coordinates (t, r, θ, ϕ). We will indicate briefly how to effect the transformation, using a systematic technique (Ref. 18, p.174-177). The idea is to express the spatial part as $-\Phi^{-2}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2]$, where $\Phi(r)$ is yet to be determined. Equating the angular and the radial parts of the two line elements, we have

$$r'^2 = \Phi^{-2} r^2 \tag{31}$$

and

$$\left[1 - \frac{2m}{r'} + \frac{\beta}{r'^2}\right]^{-1} dr'^2 = \Phi^{-2} dr^2. \tag{32}$$

If we divide (32) by (31) to eliminate Φ, then integrate and use the condition that at large radial distances r and r' must be asymptotically equal, we obtain

$$2r = (r' - m) + (r'^2 - 2mr' + \beta)^{1/2}. \tag{33}$$

The inverse transformation is

$$r' = r + m + (m^2 - \beta)/4r. \tag{34}$$

Using these transformations, the line element (29) can be expressed in the form of (1), with

$$\Omega^2(r) = [1 - (m^2 - \beta)/4r^2]^2 [1 + m/r + (m^2 - \beta)/4r^2]^{-2} \tag{35}$$

$$\Phi^{-2}(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2. \tag{36}$$

The effective refractive index $n(r)$ is

$$n(r) = [1 + m/r + (m^2 - \beta)/4r^2]^2[1 - (m^2 - \beta)/4r^2]^{-1}. \tag{37}$$

It is also helpful to have expressions for Φ, Ω and n in terms of the standard radial coordinate r'. Let $u \equiv 1/r$ and $u' \equiv 1/r'$. Then it is easy to show that

$$\Phi^2(u') = \tfrac{1}{4}[1 - mu' + (1 - 2mu' + \beta u'^2)^{1/2}]^2 \tag{38}$$

$$\Omega^2(u') = 1 - 2mu' + \beta u'^2. \tag{39}$$

And of course

$$n(u') = \Phi^{-1}(u')\Omega^{-1}(u'). \tag{40}$$

In transforming coordinates, it is often helpful to use

$$du = n\,du' \qquad \text{or} \qquad dr = \Phi\Omega^{-1}dr', \tag{41}$$

together with

$$u = \Phi^{-1}u' \qquad \text{or} \qquad r = \Phi r'. \tag{42}$$

The singularities of $n(r)$, or, equivalently, the horizons of the spacetime, occur at $r_s = (m/2)(1 - \beta/m^2)^{1/2}$, provided that $\beta/m^2 \leq 1$. Therefore, the expression for $n(r)$ is valid in the region $r > r_s$. If $\beta = m^2$, $r_s = 0$; i.e., the event horizon shrinks to zero size. In this case, $n(r) = 1 + m/r$ and is regular everywhere for $r > 0$. If $\beta > m^2$, the function $n(r)$ is not singular anywhere, since r_s becomes imaginary. Let us now examine some special cases of the metric (29).

Schwarzschild exterior metric

The Schwarzschild exterior metric applies to the spacetime around an electrically neutral, static, spherical mass M. In this case, (29)–(42) apply (Ref. 19, p.840, Ref. 20, p.515-521) with

$$\beta = 0. \tag{43}$$

Reissner–Nordström (RN) *metric*

The gravitational field due to an electrically charged, static spherical mass M is given by the RN solution of Einstein's field equations. In this case, (29)–(42) apply with

$$\beta = GQ^2/c_0^4, \tag{44}$$

where Q is the charge on the central body.

Bertotti–Robinson (BR) *metric*

The BR metric describes a universe filled with electromagnetic radiation of uniform density and uniformly random direction [21]. In this case, (29)–(42) apply with

$$\beta = m^2 \tag{45}$$

where m is now a nonphysical effective point mass. The BR solution may also be obtained as a special case of the metric obtained recently by Halilsoy.

Halilsoy metric

The Halilsoy metric describes spacetime around a static, uncharged, spherically symmetric mass M which is embedded in an externally created electromagnetic field [22,23]. Equations (29)–(42) apply with

$$\beta = q^2 m^2 \tag{46}$$

where $0 \leq q \leq 1$, and where q represents the measure of the external electromagnetic field.

Soleng metric

The Soleng metric represents the gravitational field due to a central mass M surrounded by a field having a traceless energy-momentum tensor $T^{\mu}{}_{\nu} = f(r)\,\mathrm{diag}\,[1,1,-1,-1]$. Recently, such a $T^{\mu}{}_{\nu}$ has been interpreted as the energy-momentum tensor associated with an anisotropic vacuum [24–26]. Here (29)–(42) apply with

$$\beta = 6\,\delta\, m^2 \tag{47}$$

where δ is the Soleng parameter, which determines the effective energy density of the anisotropic vacuum.

4.2. de Sitter universe and the Maxwell fish-eye

The de Sitter line element in standard coordinates is

$$ds^2 = (1-\Lambda r'^2/3)c_0^2\,dt^2 - (1-\Lambda r'^2/3)^{-1}dr'^2 - r'^2 d\theta^2 - r'^2 \sin^2\theta\, d\phi^2, \tag{48}$$

where Λ is the cosmological constant, which is proportional to the space curvature. Λ can be positive or negative, corresponding to a closed or an open de Sitter universe (Ref. 27, p.346-349).

To pass over to isotropic coordinates, we may use the method outlined above, together with the requirement that for small radial distances the

new radial variable r should asympotically approach r'. The result is the well-known transformation

$$r' = r(1 + \Lambda r^2/12)^{-1}. \tag{49}$$

Then, in the isotropic coordinates,

$$\begin{aligned} ds^2 = {} & (1 - \Lambda r^2/12)^2(1 + \Lambda r^2/12)^{-2}C_0^2 dt^2 \\ & - (1 + \Lambda r^2/12)^{-2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2). \end{aligned} \tag{50}$$

The effective index of refraction is

$$n(r) = (1 - \Lambda r^2/12)^{-1}. \tag{51}$$

This index of refraction is valid for either positive or negative Λ, with r defined through (49). Let us examine the case $\Lambda < 0$, corresponding to the open de Sitter universe. Let us write $\Lambda = -K$, where K is then a positive constant. The effective index of refraction of the open de Sitter universe, in the isotropic coordinates, is then

$$n(r) = (1 + Kr^2/12)^{-1}. \tag{52}$$

This index of refraction is of exactly the same form as the index encountered in a traditional problem of classical geometrical optics — the Maxwell fish-eye lens. The index of refraction in the Maxwell fish-eye is

$$n_{\mathrm{M}}(r) = n_0(1 + r^2/a^2)^{-1}. \tag{53}$$

in which a and n_0 are constants. Comparing (52) and (53), we note that the open version of the de Sitter universe is a Maxwell fish-eye lens with $n_0 = 1$ and $a^2 = 12/K$.

4.3. Robertson–Walker universe

The Robertson–Walker (RW) metric represents the gravitational field in a homogeneous and isotropic universe. In the standard comoving coordinates (t, r', θ, ϕ), the RW line element is given by

$$ds^2 = c_0^2\, dt^2 - R^2(t)\left[\frac{dr'^2}{1 - kr'^2} + r'^2 d\theta^2 + r'^2 \sin^2\theta\, d\phi^2\right], \tag{54}$$

in which $R(t)$ is a dimensionless scale factor and k is a constant with dimensions of (length)$^{-2}$. We may pass over to isotropic coordinates by the usual method and requiring that for small radial distances the new

radial coordinate r should asymptotically be equal to r'. The result is the well-known transformation

$$r' = r(1 + kr^2/4)^{-1}. \tag{55}$$

In the isotropic coordinates (t, r, θ, ϕ), the line element is

$$ds^2 = c_0^2\, dt^2 - R^2(t)(1 + kr^2/4)^{-2}|d\mathbf{r}|^2. \tag{56}$$

Defining the refractive index n in the usual way, we obtain

$$n = \frac{R(t)}{1 + kr^2/4}. \tag{57}$$

For the case $k > 0$, corresponding to a closed RW universe, and for a fixed cosmological epoch $t = t_0$, this corresponds to the index of refraction (53). Thus the closed Robertson–Walker universe is a Maxwell fish-eye lens with $n_0 = R(t_0)$ and $a^2 = 4/k$. We shall see below that the correspondence between the Maxwell fish-eye and the Robertson–Walker universe does not actually demand that we restrict the latter to a particular moment t_0.

5. SOME APPLICATIONS

5.1. Central-force motion

Many metrics of interest — including all those discussed in Section 4 — are spherically symmetric. In such a case, n, v, Ω and Φ are functions of the radial coordinate alone. The orbit (whether of light or of a massive particle) lies in a plane containing the force center and there is a constant of the motion analogous to the angular momentum. Let ϕ be measured in the plane of the motion. Then, from (21),

$$r^2 d\phi/dA \equiv h = \text{constant}. \tag{58}$$

Note that h is related to the classical-mechanical angular momentum per unit mass h_0 $(\equiv r^2 d\phi/dt)$ by

$$h = n^2 h_0\,. \tag{59}$$

Now we may easily obtain general-relativistic analogues of the standard formulas of classical central-force motion. In (22), which is the analogue of the classical conservation of energy condition, we may write out

$|dr/dA|^2$ in plane-polar coordinates, then eliminate A by means of (58). The orbit shape $\phi(r)$ is thereby reduced to an integration:

$$\phi = h \int^r r^{-2}[n^4 v^2 - h^2/r^2]^{-1/2}\, dr. \tag{60}$$

The classical limit of (60) is the familiar equation

$$\phi = h_0 \int^r r^{-2}[2(E-U) - h_0^2/r^2]^{-1/2}\, dr.$$

Note that we could have immediately written down (60), which is an exact general-relativistic expression, on the model of the classical expression, simply by using the transcriptions (23), together with $h_0 \to h$ (which follows from $t \to A$). Moreover, (60) applies both to light and to massive particles. To apply (60) to either massless or massive particles, we need only insert the appropriate specific form (25) or (26) for $v(r)$.

Another form of the orbit equation is frequently useful. Let $u = 1/r$. Then, in analogy to the classical formula

$$\frac{d^2u}{d\phi^2} + u = -h_0^{-2}\frac{dU}{du},$$

we must have in general relativity

$$\frac{d^2u}{d\phi^2} + u = h^{-2}\frac{d}{du}(n^4 v^2/2), \tag{61}$$

which, again, applies to both particles and photons. We have written down (61) simply by analogy to classical mechanics. But it may also be obtained by beginning with the radial component of (21) and eliminating A.

A third useful form of the orbit equation is

$$h^2[(du/d\phi)^2 + u^2] - n^4 v^2 = 0. \tag{62}$$

5.2. Light rays in the de Sitter and Robertson–Walker universes

As noted above, the open de Sitter universe is equivalent to a traditional problem in classical geometrical optics — Maxwell's fish-eye lens. It follows (i) that the open de Sitter universe constitutes an absolute optical instrument and (ii) that, in the system of isotropic coordinates, the rays are eccentric circles.

Beginning from the orbit equation (60) and the index of refraction (52) and integrating, we obtain the polar equation for the light ray in the open de Sitter universe:

$$\sin(\phi - \alpha) = \frac{h(Kr^2 - 12)}{r(144c_0^2 - 48h^2K)^{1/2}}, \tag{63}$$

where α is a constant of integration. In effecting this calculation, we can follow step-for-step the calculation of ray shapes in the classical Maxwell fish-eye (Ref. 28, p.147-149). Since $(Kr^2 - 12)/r\sin(\phi - \alpha)$ =constant, we can write the equation for a family of light rays passing through a fixed point $P_0(r_0, \phi_0)$ as

$$\frac{Kr^2 - 12}{r\sin(\phi - \alpha)} = \frac{Kr_0^2 - 12}{r_0\sin(\phi_0 - \alpha)}. \tag{64}$$

For any value of α, this equation is satisfied at point $P_1 = (r_1, \phi_1)$ where $r_1 = 12/Kr_0$ and $\phi_1 = \phi_0 + \pi$. This shows that all the rays from an arbitrary point P_0 meet at a point P_1 on the line joining P_0 to the origin O such that $OP_0 \cdot OP_1 = 12/K$. Hence the imaging in Maxwell's fish-eye lens is an inversion. From any point P_0 in the three-dimensional space an infinity of rays originate which are then focused at an image point P_1. The images are therefore *sharp* (stigmatic). (In most real optical instruments, of the infinity of points passing thorugh an object point, only a finite number pass through the image point, the other rays only passing near the image point. Such images are not sharp ones.) Now, an instrument which sharply focuses an image of a three-dimensional region of space is called an absolute optical instrument. Thus, the open de Sitter universe constitutes an absolute optical instrument. All the theorems pertaining to absolute optical instruments apply. For example, the optical length of a line segment in the image must be equal to the optical length of the corresponding line segment in the object (Ref. 28, p.143-147).

Moreover, by analogy to the Maxwell fish-eye, a ray in the de Sitter universe is a circle in the isotropic coordinate system. This goes exactly as in the classical geometrical optics treatment of the Maxwell fish-eye. On the left-hand side of (63), write $\sin(\phi - \alpha) = \sin\phi\cos\alpha - \cos\phi\sin\alpha$, then put $x = r\cos\phi$ and $y = r\sin\phi$. Then (63) may be written in the form

$$(x + b\sin\alpha)^2 + (y - b\cos\alpha)^2 = 12/K + b^2 \tag{65}$$

where $b = (hK)^{-1}(36c_0^2 - 12h^2K)^{1/2}$. Thus each ray is a circle. Note that in (63) if we put $K = 0$, we obtain a straight-line ray, as we would expect:

$$r\sin(\phi - \alpha) = \text{constant}.$$

These results may be extended with a little modification to the closed Robertson–Walker universe. The formalism developed in this paper was designed for static metrics, so it may seem at first sight that we cannot deal with the RW case. However, in the case of the null geodesic, a time-dependent conformal factor in the metric need not affect the basic procedure or the most important conclusions. (The same cannot be said of the particle trajectories.) Let us see how this works out in the language of refractive indices.

As noted above, the closed Robertson–Walker universe yields a factorable index of refraction,

$$\underset{\sim}{n} = n_s(\mathbf{r})n_t(t), \tag{66}$$

in which n_s is a function of the spatial coordinates alone and n_t is a function of the time alone. For the RW metric in isotropic comoving coordinates

$$n_s = (1 + kr^2/4)^{-1} \tag{67}$$

$$n_t = R(t). \tag{68}$$

The bending of light rays depends only on the spatial gradient of n. The fact that n varies everywhere in space with the same multiplicative function of time $R(t)$ does not affect the shape of a light ray. One way to see this is to consider Snell's law. If the index of refraction factors into the form (66), then whenever Snell's law is applied at an interface between regions of different n, the common factor $n_t(t)$ will cancel from the two sides of the equation. More formally, we can define a new time coordinate τ by $dt = Rd\tau$. Then the line element (56) becomes

$$ds^2 = c_0^2 R^2\, d\tau^2 - R^2(1 + kr^2/4)^{-2}|d\mathbf{r}|^2.$$

and the effective index of refraction becomes simply

$$n = n_s = (1 + kr^2/4)^{-1} \tag{69}$$

The scale factor $R(\tau)$ or $R(t)$ thus can have no effect on the shape of a ray in the isotropic, comoving coordinates. R only influences the progress in time of light along the ray.

As far as ray shapes are concerned, then, the RW universe is entirely analogous to the Maxwell fish-eye lens. It follows (i) that the closed Robertson–Walker universe also constitutes an absolute optical instrument and (ii) that, in the system of isotropic, comoving coordinates, the rays are eccentric circles.

5.3. Light and particle motion in RN-type metrics

In this section, we shall illustrate the use of the Newtonian forms of the orbit equations in some applications to RN-type metrics. In particular, we calculate the effect of the parameter β (29) on three tests of general relativity. Our calculations will supplement and extend those of Halilsoy [23].

We may begin from (62). Inserting (26) for $v(r)$ in the second term, we obtain

$$\left[\frac{du}{d\phi}\right]^2 + u^2 - (c_0/h)^2 n^2 [1 - c_0^4 H^{-2}\Omega^2] = 0\,. \tag{70}$$

This differential equation is exact, but it may not appear very familiar. We may transform back to the original (nonisotropic) coordinates by using (41) and (42) in the first two terms of (70), with the result

$$\left[\frac{du'}{d\phi}\right]^2 + u'^2\Omega^2 - (c_0/h)^2 [1 - c_0^4 H^{-2}\Omega^2] = 0\,. \tag{71}$$

Substituting (39) for $\Omega^2(u')$, then differentiating with respect to ϕ, we obtain

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_0^6}{h^2H^2} = -\frac{\beta c_0^6}{h^2H^2}\, u' + 3mu'^2 - 2\beta u'^3. \tag{72}$$

Bending of light rays

The equation for the shape of a light ray results from letting $H \to \infty$ in (72):

$$\frac{d^2u'}{d\phi^2} + u' = 3mu'^2 - 2\beta u'^3. \tag{73}$$

This equation may be solved by the usual perturbative method. If the right side of (73) is temporarily put equal to zero, we obtain the straight-line solution

$$u' = \frac{\sin\phi}{R},$$

where R is the distance of closest approach to the origin. Substituting the zeroth-order solution $\sin\phi/R$ for u' on the right side of (73) and solving the resulting differential equation for $u'(\phi)$, we obtain the solution of first order in m and β:

$$u' = \frac{\sin\phi}{R} + \frac{3m}{2R^2}\left[1 + \frac{1}{3}\cos 2\phi\right] + \frac{3\beta}{4R^3}\,\phi\cos\phi - \frac{\beta}{16R^3}\sin 3\phi. \tag{74}$$

(This differs slightly from Halilsoy's solution, which is missing the last term.) As $r' \to \infty$, $u' \to 0$, and $\phi \to \phi_\infty$, which may be assumed small. Thus (74) reduces to

$$0 \doteq \frac{\phi_\infty}{R} + \frac{2m}{R^2} + \frac{9\beta\phi_\infty}{16R^3}\,.$$

The total deflection is $\Delta\phi_\infty = 2|\phi_\infty|$ or

$$\Delta\phi_\infty \approx \frac{4m}{R}\left(1 - \frac{9\beta}{16R^2}\right). \tag{75}$$

The coefficient of β/R^2 differs from the $\frac{3}{4}$ obtained by Halilsoy, the difference being the contribution of the last term in (74).

For some of the metrics under consideration [see (46) and (47)], β can be of order m^2. Thus, the expressions for the light orbit (74) and for the bending (75) should be carried to higher order in m/R to provide a fair assessment of the importance of the contributions due to β. This may be done by iteration. That is, we substitute (74) for u' on the right-hand side of (73) and proceed as before. The result is that the following terms should be added to the right-hand side of (74)

$$\frac{-15m^2}{4R^3}\,\phi\cos\phi - \frac{3m^2}{16R^3}\sin 3\phi. \tag{76}$$

The expression (75) for the deflection of the light ray becomes

$$\Delta\phi_\infty \approx \frac{4m}{R}\left(1 - \frac{9\beta}{16R^2} + \frac{69m^2}{16R^2}\right). \tag{77}$$

Precession of planetary apsides

For a planet, we return to (72). This equation is exact and may be handled as it stands. However, since we will treat some of the terms on the right side of the equation as perturbations, no precision will be lost by replacing the constants of the motion h and H by their classical limits. For a planet moving at non-relativistic speed, we may, by (27), put $H^2 \approx c_0^4$. Also, at sufficiently large r (i.e., at the radius of a planetary orbit), we may put $h \approx h_0$ the classical angular momentum per unit mass. Thus we have

$$\frac{d^2u'}{d\phi^2} + u' - \frac{mc_0^2}{h_0^2} = -\frac{\beta c_0^2}{h_0^2}\,u' + 3mu'^2 - 2\beta u'^3. \tag{78}$$

This differential equation is not quite the same as that recently obtained through other means by Halilsoy. In particular, Halilsoy's equation is missing the term $-\beta(c_0/h_0)^2u'$ [Ref. 23, eq. (5)].

If we temporarily put the terms in u'^2 and u'^3 equal to zero, we obtain a differential equation that may be solved exactly:

$$\frac{d^2u'}{d\phi^2} + s^2u' = \frac{1}{\alpha_0}, \tag{79}$$

where

$$s^2 = 1 + \beta c_0^2/h_0^2, \tag{80}$$

and

$$\alpha_0 = h_0^2/mc_0^2. \tag{81}$$

The solution is the precessing ellipse

$$u' = \alpha^{-1}(1 + e\cos s\phi), \tag{82}$$

where the eccentricity e is arbitrary and where the semi-latus rectum α is

$$\alpha = \alpha_0\left[1 + \frac{\beta c_0^2}{h_0^2}\right]. \tag{83}$$

The precession of the apsides, per revolution of the planet on the orbit, due to the term in $\beta u'$, is then

$$\triangle_1 = \frac{-\pi\beta c_0^2}{h_0^2} = \frac{-\pi m}{\alpha_0}\frac{\beta}{m^2}. \tag{84}$$

The terms in u'^2 and u'^3 may be treated as perturbations. Thus, one inserts (82) in these two terms on the right side of (78) and solves the resulting equation. The term in u'^2, acting alone, produces the usual precession associated with the Schwarzschild problem:

$$\triangle_2 = \frac{6\pi m^2}{h_0^2}c_0^2 = \frac{6\pi m}{\alpha_0}. \tag{85}$$

As shown by Halilsoy, the term in u'^3, acting alone, produces the precession

$$\triangle_3 = -\frac{6\pi\beta m^2}{h_0^4}c_0^4 = -6\pi\left[\frac{m}{\alpha_0}\right]^2\frac{\beta}{m^2}. \tag{86}$$

However, this term is smaller than (84) by a factor of m/α_0 and is therefore negligible. To lowest order in m/α_0, then, the total precession Δ of the apsides per revolution is just $\Delta_1 + \Delta_2$:

$$\Delta = \frac{6\pi m}{\alpha_0}\left(1 - \frac{\beta}{6m^2}\right). \tag{87}$$

Radar echo delay

We consider the propagation in time of light in an RN-type metric. Again, let the motion take place in the $\theta = \pi/2$ plane. Writing out the conservation of energy equation (22) in plane polar coordinates and making use of (25) and (58), we have

$$\left(\frac{dr}{dA}\right)^2 + \frac{h^2}{r^2} - n^2 c_0^2 = 0\,. \tag{88}$$

Let us now evaluate the constant of the motion, h. Let r_0 denote the distance of closest approach of the ray to the center of the gravitating body. When $r = r_0$, we have $dr/dA = 0$. Thus (88) gives

$$h = r_0 n(r_0) c_0\,, \tag{89}$$

which is analogous to the classical-mechanical expression $r_0 v(r_0)$.

We may now transform from r back to r' using (41) and (42). Also, because we are interested in the propagation of light in time, we use (24) to pass over from A to t as independent variable. Thus, with substitution and transformation of (89), (88) becomes

$$\left(\frac{dr'}{dt}\right)^2 = \Omega^4(r') c_0^2 \left[1 - \frac{r_0'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_0')}\right]. \tag{90}$$

The time of travel from r_0 to r' is then

$$\begin{aligned}\Delta t &= c_0^{-1} \int_{r_0'}^{r'} \Omega^{-2}(r') \left[1 - \frac{r_0'^2}{r'^2}\frac{\Omega^2(r')}{\Omega^2(r_0')}\right]^{-1/2} dr' \\ &\equiv c_0^{-1} \int_{r_0'}^{r'} I(r')\, dr'. \end{aligned} \tag{91}$$

Now,

$$I = \Omega^{-2}\left(1 - \frac{r_0'}{r'^2}\right)^{-1/2}\left[1 + \frac{[1 - \Omega^2(r')/\Omega^2(r_0')]}{(r'^2/r_0'^2 - 1)}\right]^{-1/2}. \tag{92}$$

Using (39) to write out $\Omega(r')$ and $\Omega(r_0')$, then expanding to first order in m and first order in β, we obtain

$$I = \left(1 - \frac{r_0'^2}{r'^2}\right)^{-1/2}\left[1 + \frac{2m}{r'} + \frac{mr_0'}{r'(r' + r_0')} - \frac{3\beta}{2r'^2}\right]. \tag{93}$$

The total time of travel $\Delta t(r_0', r')$ from r_0' to r' is obtained by substituting (93) into (91) and integrating:

$$\begin{aligned}\Delta t(r_0', r') \approx\ & c_0^{-1}(r'^2 - r_0'^2)^{1/2} \\ & + \frac{2m}{c_0}\ln\left[\frac{r'}{r_0'} + \frac{(r'^2 - r_0'^2)^{1/2}}{r_0'}\right] \\ & + \frac{m}{c_0}\left[\frac{r' - r_0'}{r' + r_0'}\right]^{1/2} + \frac{3\beta}{2r_0'c_0}\sin^{-1}\left(\frac{r_0'}{r'}\right) - \frac{3\pi\beta}{4r_0'c_0}.\end{aligned} \tag{94}$$

The first term on the right side of (94) is the transit time of light in Euclidean space. The delay $\Delta T(r_0, r')$ due to general-relativisitic effects is the sum of the remaining terms.

As an example, let us estimate the radar echo delay for a signal sent from the Earth at radius r_e' to an inferior planet at radius r_p' when that planet is near superior conjunction with the Sun. Let the distance of closest approach to the center of the Sun be r_0'. If we suppose that the signal passes very near the Sun, so that r_0' is much smaller than either r_e' or r_p', then

$$\Delta T(r_0', r_e') \approx \frac{2m}{c_0}\ln\left[\frac{2r_e'}{r_0'}\right] + \frac{m}{c_0}\left[1 - \frac{r_0'}{r_e'}\right] - \frac{3\pi\beta}{4c_0r_0'}\left[1 - \frac{2r_0'}{\pi r_e'}\right], \tag{95}$$

and the total delay in the signal for the round trip is

$$2[\Delta T(r_0', r_e') + \Delta T(r_0', r_p')]. \tag{96}$$

5.4. Redshifts

The gravitational and cosmological redshifts are not dynamical effects; i.e., we do not need to solve an equation of motion in order to calculate them. However, it may be of some interest to see how the redshifts arise in the language of an effective index of refraction.

In ordinary optics, if a light wave travels from a region of high index of refraction n_2 to a region of low index of refraction n_1, the wavelength λ increases because the leading part of the wave enters n_1 first and speeds up while the trailing part is still in n_2. Thus the wave begins to stretch out. λ and c change but the frequency ν does not. This holds even if the *index varies continuously with the spatial coordinates and even if the ray* crosses obliquely through the contours of constant n. Thus, in general,

$$\lambda(\mathbf{r}_1)n(\mathbf{r}_1) = \lambda(\mathbf{r}_2)n(\mathbf{r}_2). \tag{97}$$

Now consider a situation in which n does not depend on the spatial coordinates, but does vary with time. An example can be imagined: let the air slowly be pumped from a chamber. Then, as long as the wave does not leave the chamber, n is everywhere the same, but is a decreasing function of time. The wavelength will not change, since the leading edge of the wave never encounters a new value of n before the trailing edge does. Thus, in this case ν and c change but λ does not. We have

$$\lambda(t_1) = \lambda(t_2). \tag{98}$$

Suppose now that the index of refraction can be written as a product of two functions — a function n_s of the spatial coordinates alone and a function n_t of the time alone, as in (66). For the reasons just mentioned, n_t *does not affect the wavelength and we have*

$$\lambda(\mathbf{r}_1)n_s(\mathbf{r}_1) = \lambda(\mathbf{r}_2)n_s(\mathbf{r}_2). \tag{99}$$

We wish to apply these rules of ordinary optics to the propagation *of light in general relativity. Our effective index of refraction* (3) *is based* upon the isotropic coordinate speed of light. Thus the quantity analogous to the wavelength of classical optics is the coordinate distance between successive crests of the wave. Coordinate distances are not, of course, directly measurable in general relativity. The physically measurable metric length is obtained from the coordinate length by means of the metric (1).

Gravitational redshift

Let $|\Delta\mathbf{r}_1|$ be the coordinate distance between successive crests of a light wave located at $\mathbf{r}_1$. Similarly, let $|\Delta\mathbf{r}_2|$ be the coordinate distance between successive crests at a different point $\mathbf{r}_2$ located on the same ray. Then, *in analogy to the condition (97) from ordinary optics, we must have*

$$|\Delta\mathbf{r}_1|n(\mathbf{r}_1) = |\Delta\mathbf{r}_2|n(\mathbf{r}_2). \tag{100}$$

The metric length λ of the wave at point $\mathbf{r}_1$ is obtained by applying the metric (1) to the coordinate length $|\Delta\mathbf{r}_1|$ of the wave:

$$\lambda(\mathbf{r}_1) = \Phi^{-1}(\mathbf{r}_1)|\Delta\mathbf{r}_1|.$$

A similar expression holds for $\lambda(\mathbf{r}_2)$. Thus we have

$$\Phi(\mathbf{r}_1)\lambda(\mathbf{r}_1)n(\mathbf{r}_1) = \Phi(\mathbf{r}_2)\lambda(\mathbf{r}_2)n(\mathbf{r}_2), \tag{101}$$

or, using (3),

$$\lambda(\mathbf{r}_1)\Omega^{-1}(\mathbf{r}_1) = \lambda(\mathbf{r}_2)\Omega^{-1}(\mathbf{r}_2), \tag{102}$$

the usual gravitational redshift relation.

As an example, let us take the case of the Schwarzschild metric. Let a source of light be located at $\mathbf{r}_1$ and an observer at $\mathbf{r}_2$, sufficiently far from the central gravitating body so that we may put $\Omega(\mathbf{r}_2) \approx 1$. Then, with the use of (39) and (43), (102) gives

$$\begin{aligned} z &\equiv (\lambda_{observed} - \lambda_{emitted})/\lambda_{emitted} \\ &= (1 - 2m/r')^{-1/2} - 1. \end{aligned} \tag{103}$$

This result may, of course, be derived by many other methods.

Cosmological redshift

In the expanding universe of the Robertson–Walker metric, we have an index of refraction (57) that factors like (66), with n_s and n_t given by (67) and (68). Let a light wave of coordinate length $|\Delta\mathbf{r}_1|$ be emitted at $(t_1, \mathbf{r}_1)$ and received at $(t_2, \mathbf{r}_2)$. By analogy to (99), the coordinate length $|\Delta\mathbf{r}_1|$ of the received wave is determined by

$$|\Delta\mathbf{r}_1|n_s(\mathbf{r}_1) = |\Delta\mathbf{r}_2|n_s(\mathbf{r}_2). \tag{104}$$

The metric length λ of the wave is obtained by applying the metric (56) to the coordinate length $|\Delta\mathbf{r}|$. Thus (104) becomes

$$\frac{(1 + kr_1^2/4)}{R(t_1)}\lambda(t_1, \mathbf{r}_1)n_s(\mathbf{r}_1) = \frac{(1 + kr_2^2/4)}{R(t_2)}\lambda(t_2, \mathbf{r}_2)n_s(\mathbf{r}_2),$$

or, with use of (67),

$$\frac{\lambda_1}{R(t_1)} = \frac{\lambda_2}{R(t_2)}, \tag{105}$$

the usual cosmological redshift relation.

While many writers have stressed the fundamentally different natures of the gravitational and cosmological redshifts, others have argued that it is possible to treat them with a single unified approach [29]. In the effective optical-medium formulation pursued here, it is interesting to note that both spectral shifts depend on a single optical principle (99).

6. CONCLUSION

The Newtonian forms (21) and (22) for the geodesic equations of motion offer some practical advantages for calculation. In particular, they facilitate the writing down of exact general relativistic expressions simply by analogy to classical formulas. Thus, they constitute one more tool for the relativist's tool kit. But the most interesting consequence of extending the optical-mechanical analogy to general relativity is that one simple equation of motion (21) now summarizes three fields of study: classical geometrical optics, classical particle mechanics, and geodesic motion of both light and particles in general relativity. Of course, our treatment is restricted to isotropic fields and media. Nevertheless, this unified approach, based on the use of the optical action, possesses considerable flexibility and scope. A single variational principle (14) governs all three domains.

ACKNOWLEDGEMENTS

The authors are grateful for the suggestions contributed by two anonymous referees. One of us (JE) would like to thank Martin Jackson and Matt Moelter for useful discussions and Adam Gromko for help in checking one of the calculations.

REFERENCES

1. Eddington, A.S. (1920). *Space, Time and Gravitation* (Cambridge Univ. Press, Cambridge).
2. Plebański, J. (1960). *Phys. Rev.* **118**, 1396.
3. de Felice, F. (1971). *Gen. Rel. Grav.* **2**, 347.
4. Wu X. and Xu C. (1988). *Commun. Theor. Phys. (Beijing)* **9**, 119.
5. Evans, J., and Rosenquist, M. (1986). *Amer. J. Phys.* **54**, 876.
6. Evans, J. (1990). *Amer. J. Phys.* **58**, 773.
7. Nandi, K. and Islam, A. (1995). *Amer. J. Phys.* **63**, 251.
8. Kovner, I. (1990). *Astrophys. J.* **351**, 114.
9. Perlick, V. (1990). *Class. Quant. Grav.* **7**, 1319.
10. Perlick, V. (1990). *Class. Quant. Grav.* **7**, 1849.
11. Bel, Ll., and Martin, J. (1994). *Gen. Rel. Grav.* **26**, 567.
12. Lanczos, C. (1986). *The Variational Principles of Mechanics* (4th ed., Dover, New York).
13. Weyl, H. (1917). *Ann. Phys. (Leipzig)* **54**, 117.
14. Levi-Civita, T. (1917). *Atti della Reale Accademia Dei Lincei, Ser. 5, Rendiconti, Class. di sc. fisiche, mat. e naturali* **26**, 458.
15. Levi-Civita, T. (1918). *Nuovo Cimento* **16**, 105.
16. Straumann, N. (1984). *General Relativity and Relativistic Astrophysics* (Springer-Verlag, Berlin).
17. Atkinson, R. d'E. (1963). *Proc. Roy. Soc. Lond.* **272**, 6 .

18. Adler, R., Bazin, M., and Schiffer, M. (1965). *Introduction to General Relativity* (McGraw-Hill, New York).
19. Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation* (W. H. Freeman, San Francisco).
20. Sjodin, T. (1990). In *Physical Interpretations of Relativity Theory*, M. C. Duffy, ed. (British Soc. Phil. Sc., London).
21. Bertotti, B. (1959). *Phys. Rev.* **116**, 1331.
22. Halilsoy, M. (1993). *Gen. Rel. Grav.* **25**, 275.
23. Halilsoy, M. (1993). *Gen. Rel. Grav.* **25**, 975.
24. Soleng, H. H. (1994). *Gen. Rel. Grav.* **26**, 149.
25. Dymnikova, I. (1992). *Gen. Rel. Grav.* **24**, 235.
26. Goldman, I., Pérez-Mercader, J., Cooper, F., and Nieto, M. M. (1992). *Phys. Lett.* *B*281, 219.
27. Tolman, R. C. (1934). *Relativity, Thermodynamics and Cosmology* (Oxford University Press, Oxford).
28. Born, M., and Wolf, E. (1980). *Principles of Optics* (6th ed., Pergamon Press, Oxford).
29. Narlikar, J. V. (1994). *Amer. J. Phys.* **62**, 903.

NOTE

Indian J. Phys. **68B** (6), 539 – 543 (1994)

I J P B
— an international journal

On the spherically symmetric static solutions of Brans-Dicke field equations in vacuum

Kamal K Nandi and Anwarul Islam*
Department of Mathematics, University of North Bengal, Darjeeling-734 430, West Bengal, India

Received 16 May 1994, accepted 26 September 1994

Abstract : The analytic solutions, obtained by Riazi and Askari, of the approximate and exact vacuum Brans-Dicke equations for the spherically symmetric static case are shown to correspond to unphysical negative values of the coupling parameter w. The present investigation is meant to highlight the pitfalls that one might encounter in the physical interpretation of such solutions.

Keywords : Vacuum Brans-Dicke equations, spherically symmetric static case, analytic solutions

PACS No. : 04.20.Cv

The idea of utilizing the Brans-Dicke (BD) theory in the interpretation of various astrophysical phenomena is quite attractive. With this intention, Riazi and Askari (RA) [1] have recently obtained analytic solutions (eqs. (13), (20) and (22) of [1]) of the approximate BD equations in the spherically symmetric case. We might call these solutions the 'approximate' RA solutions. These solutions have been utilized to interpret a very important astrophysical phenomenon, namely, the observed flat rotation curves in the vast domain of dark galactic haloes. According to the authors, the interpretation requires a rather 'unnatural' large positive value of w ($\sim 10^{12}$). In our opinion, such a requirement by itself does not constitute any inadequacy of the approximate RA solutions as the large mass of astronomical probes indicates only an ascending order of positive values for w ($\geq 6, 29, 140, 500, \ldots$). However, it turns out that the RA solutions correspond only to negative values of the BD parameter w. Consequently, some care should be exercised in the application of the RA solutions to those problems of physical interest that require a positive w.

*On leave from : Department of Mathematics, Tolaram Government College, Narayanganj, Dhaka, Bangladesh

The BD field equations are :

$$\Box^2 \phi = \frac{8\pi}{3+2w} T^{\mu}_{M\mu}, \tag{1}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{\phi} T_{M\mu\nu} - \frac{w}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\, \phi_{;\rho}\, \phi^{\rho}_{;}\right)$$

$$- \frac{1}{\phi}\left(\phi_{;\mu;\nu} - g_{\mu\nu} \Box^2 \phi\right), \tag{2}$$

where $\Box^2 \equiv (\phi_{;}{}^{\rho})_{;\rho}$ and $T_{M\mu\nu}$ is the matter energy momentum tensor excluding the ϕ-field, w is a dimensionless coupling constant.

Riazi and Askari [1] show that, in the asymptotic region, their approximate solutions, $B(r)$ and $A(r)$, behave as follows (speed of light $c = 1$) :

$$B(r) \to 1 - \frac{(c_2 - 2) r_0}{r}, \tag{3}$$

$$A(r) \to \left(1 - \frac{c_2 r_0}{r}\right)^{-1} = 1 + \frac{c_2 r_0}{r} + 0(r^{-2}), \tag{4}$$

where $d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Further, from the above expression for $B(r)$, they identify the total mass M of the configuration (inclusive of the contribution from the ϕ-field) with $\frac{(c_2 - 2)r_0}{2G}$, where r_0 is a constant c_2 is a dimensionless constant of integration. On account of the positivity of energy, we find that two cases are possible : (i) $r_0 > 0$ and $c_2 > 2$, (ii) $r_0 < 0$ and $c_2 < 2$. In their paper, RA consider only case (i) as is reflected in their requirement of fine tuning $c_2 = 2^+$. We shall first investigate case (i).

Consider the Robertson expansion in standard coordinates [2].

$$d\tau^2 = \left(1 - 2\alpha GMr^{-1} + 2(\beta - \alpha\gamma)\, G^2 M^2 r^{-2} + \ldots\right) dt^2$$

$$- \left(1 + 2\gamma GMr^{-1} + \ldots\right) dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5}$$

This represents spherically symmetric static solutions of BD equations in vacuum [3] provided $\alpha = \beta = 1$, $\gamma = \frac{w+1}{w+2}$, together with

$$\phi = \phi_0\left(1 + (w+2)^{-1}\, GMr^{-1} + \ldots\right). \tag{6}$$

In the asymptotic region (*i.e.*, to the first order in r^{-1}), the expressions (3) and (4) must be compatible with the corresponding parts from the expression (5). This requirement should be regarded as a boundary condition to be satisfied by all spherically symmetric solutions of the

BD equations. The physical viability of different solutions can also be judged by the same token. For the approximate RA solutions, we have

$$(c_2-2)r_0=2GM, \tag{7}$$

$$c_2 r_0 = 2GM\left(\frac{w+1}{w+2}\right). \tag{8}$$

Eliminating $2GMr_0^{-1}$ from the above, we get unphysical negative values for w:

$$w = -\left(\frac{c_2+2}{2}\right) < 0 \qquad \text{if } c_2>2, \tag{9}$$

which is what we set out to show.

One might suspect that the appearance of unphysical negative values of w is due somehow to the approximate nature of the RA solutions that we have considered. It will soon turn out that this is not so; the negativity of w persists even if the exact BD equations are considered. The spherical solutions of the exact BD equations have been obtained (RA) through a conformal reparametrization procedure [4–6]. We might call these solutions the 'exact' RA solutions and in standard coordinates, they are given by [4]

$$\phi(r) = \phi_0 + \frac{Q_s}{r} + \left(1+\frac{Q_s}{2M}\right)\frac{GMQ_s}{r^2} + \left(\frac{(w-6)Q_s^2}{6M^2} + \frac{11Q_s}{3M} - \frac{8}{3}\right)\frac{G^2M^2Q_s}{2r^3} + \ldots,$$

$$B(r) = 1 - \frac{2GM}{r} + \frac{2G^2MQ_s}{r^2} + \left(\frac{(w-16)Q_s}{6M} + \frac{5}{3}\right)\frac{G^3M^2Q_s}{r^3} + \ldots,$$

$$A(r) = 1 - \frac{2G(Q_s-M)}{r} + \left(\frac{(8-w)Q_s^2}{2M^2} + \frac{9Q_s}{M} + 4\right)\frac{G^2M^2}{r^2} + \ldots, \tag{10}$$

where $$2M = Q_s(1-\delta) = \frac{r_0}{G}(\delta-1), \tag{11}$$

Q_s and δ are two integration constants. These are related to r_0, c_2, ϕ_0 and c_1 by the relations: $Q_s=-c_1=-r_0/G=-r_0\phi_0$ and $\delta=c_2-1$ so that

$$2M = c_1(c_2-2). \tag{12}$$

From this, we see that $c_2>2$ translates to $\delta>1$ and that eq. (12) is nothing but eq. (7) redefined. Also the so called fine tuning condition is now restated as $\delta = 1^+$. After making this change, the asymptotic expressions for $B(r)$, $A(r)$ and also $\phi(r)$ have to be compared with those from eqs. (5) and (6). In the first order in r^{-1}, these give the equations

$$B(r) \sim 1 - \frac{Q_s(1-\delta)G}{r} = 1 - \frac{2MG}{r}, \tag{13}$$

$$A(r) \sim 1 - \frac{Q_s(1+\delta)G}{r} = 1 + \left(\frac{w+1}{w+2}\right)\frac{2GM}{r}, \tag{14}$$

$$\phi(r) \sim \phi_0 + \frac{Q_s}{r} = \phi_0\left[1 + \frac{GM}{(w+2)r}\right]. \tag{15}$$

From eqs. (13)–(15) that now also include the expression for $\phi(r)$, it follows that

$$w = -\left(\frac{\delta+3}{2}\right) < 0 \qquad \text{since } \delta > 1. \tag{16}$$

Clearly, this result is again the same as eq. (9) above with $c_2 = \delta + 1$. The exact solutions (10) also permit us to go beyond the first order approximation. For instance, consider the second order term in $B(r)$ from eq. (5) and equate it with that from eq. (10) :

$$2(\beta - \alpha\gamma)\frac{G^2M^2}{r^2} = \frac{2G^2MQ_s}{r^2},$$

with $Q_s = \dfrac{2M}{1-\delta}$. With the specified values of α, β, γ, one immediately finds that eq. (16), or, by another symbol, eq. (9), continues to hold good. We should also recall that the second order term in $B(r)$ plays a crucial role in the Solar system tests of gravity. The most significant one is the wellknown test for the precession of planetary orbits. On the other hand, in the standard BD theory, $w \geq 6$, if all classical tests of General Relativity are to be reasonably accounted for. Thus, eqs. (16) or (9) prevent RA solutions to become applicable in the Solar system scenario, although they may be applicable in other physical situations.

Let us now examine case (ii) : $r_0 < 0$ and $c_2 - 2 < 0$. In this case, it is possible to choose a range of values for c_2 such that a positive w is obtainable from eq. (9). For example, $-\infty < c_2 < -2$ guarantees a positive w. However, we still have to leave out the range $-2 < c_2 < 2$ as this leads to a negative w. Anyhow, everything looks nearly fine as far as w is concerned but the problem crops up elsewhere. The rotational velocity in the range of flat rotation curves becomes imaginary! It is given by $v_{\text{rot}}^2 \equiv (c_2 - 2)/(2w)^{1/2}$. Thus, once again, *we run into a physically meaningless conclusion but of a kind different from that* corresponding to case (i).

Riazi and Askari [1] themselves caution against yet another limitation : the vacuum condition may not be tenable in view of the possible presence of luminous matter in the galactic haloes. We agree with their opinion. Probably an interior BD solution ($T_{\mu\nu} \neq 0$) would be closer to the physical situation.

Acknowledgment

One of the authors (AI) would like to thank the Indian Council for Cultural Relations, Azad Bhawan, New Delhi, for a fellowship under an Exchange Programme of the Government of India.

References

[1] N Riazi and H R Askari *Mon. Not. Roy. Astron. Soc.* **261** 229 (1993)

[2] S Weinberg *Gravitation and Cosmology* (New York : John Wiley) p 246 (1972)

[3] S Weinberg *Gravitation and Cosmology* (New York : John Wiley) p 183 (1972)

[4] H R Askari and N Riazi *Exact Spherical Solutions of Brans-Dicke Theory in Standard Coordinates*, Publication No. 40 of the Biruni Observatory, Shiraz University, Iran (Private Circulation) (1993)

[5] R H Dicke *Phys. Rev.* **125** 2163 (1962)

[6] T Singh and L N Rai *Gen. Rel. Grav.* **15** 875 (1983)