CHAPTER 6

AN ALTERNATIVE DESCRIPTION OF GRAVITATIONAL REFRACTIVE MEDIUM AND DEDUCTION OF GENERAL RELATIVISTIC EFFECTS: A CRITIQUE

In the foregoing chapters, the gravitational refractive index associated with centrally symmetric field of general relativity played a crucial role in the context of Evans-Rosenquist formulation of the optical-mechanical analogy.[77] We derived the index from the solution of general relativity itself.

There do however exist other ways of deducing a refractive index to be associated with a centrally symmetric gravity field, that have nothing to do with general relativity. One such method is due to Tangherlini[17] who allows an effective mass increase in the corpuscular theory of light and shows that the light particle travels slower rather than faster on entering an optically denser medium in accordance with Huygen's principle. The disagreement between Newtonian corpuscular theory of light and Foucault's experiment is also removed thereby. The relevant Hamiltonian equations then lead to a gravitational refractive index that accounts for half of the general relativistic value for light deflection. The other half is separately calculated as it is due to an altogether new effect, namely, gravitational shift in the photon frequency.

Using the Tangherlini Hamiltonian, Tian and Li[78] obtained an equivalent description of the gravitational refractive medium in terms of expressions giving varying speed and relative mass of a photon on a flat space. They further combined these two quantities to deduce an "apparent" rest mass of photon as well. Using all these in their formalism, Tian and Li obtain the full general relativistic value for light deflection and radar echo delay. The formulation can be regarded as a new addition to the already existing repertoire of alternative deductions of general relativistic effects. Such approaches should however be examined critically as they often turn out to be faulty for one reason or other.[79] By way of doing so, our basic aim in this chapter is to illuminate some of the tricky points. The ensuing analysis is supposed to lead to a better understanding of general relativity *vis-a-vis* an alternative formulation thereto. In this sense, this chapter is somewhat pedagogical in its contents.

The contents are arranged sectionwise as follows. First, in Sec.6.1, we clarify a misleading step in the Hamiltonian procedure of Tian and Li. Then, in Sec.6.2, we examine their equations for light deflection and point out an incompatibility that invalidates their conclusions. A comparison with our approach here provides an opportunity to address an important issue, namely, the meaning of observables in a flat space gravity theory and in general relativity. This is done consecutively in Secs.6.3 and 6.4. Finally, Sec.6.5 concludes the chapter.

6.1. Hamiltonian equations

Tangherlini[17] treats light corpuscles as classical particles with variable mass in a medium having gravitational refractive index n(x,y,z). The Hamiltonian is prescribed as

$$H(x,y,z,p_x,p_y,p_z) = \frac{c_o}{n(x,y,z)} [p_x+p_y+p_z]^{1/2} \qquad (6.1.1)$$

where c_o is the vacuum speed of light; p_x,p_y,p_z are the generalized photon momenta conjugate to x,y,z respectively. Also, as before,

$$c(x,y\ z) = \frac{c_o}{n(x,y,z)} \qquad (6.1.2)$$

represents variable speed of light in a medium. The magnitude of photon momentum is defined as

$$p = \mu(x,y,z)c(x,y,z) \qquad (6.1.3)$$

where $\mu(x,y,z)$ is the variable photon mass. Tangherlini obtains, for n, the following expression for the centrally symmetric field

$$n(r) = \left(1 - \frac{2m}{r}\right)^{-1/2}, \quad m = GMc_0^{-2} \qquad (6.1.4)$$

where G is the Newtonian constant of gravity and M is the gravitating mass.

To describe the effects of gravity, Tian and Li[78] emphasize more on the variation of photon's speed than on the gravitational refractive index. The same expressions for H, p and c as those of Tangherlini are used and the calculation is carried out in polar coordinates (r,ϕ).

$$p^2 = p_r^2 + p_\phi^2 = (\mu\dot{r})^2 + (\mu r\dot{\phi})^2 \qquad (6.1.5)$$

where p_r , p_ϕ represent radial and cross-radial components of ordinary momentum of a photon and, of course, $\mu = \mu(r)$. The Hamiltonian

$$H = c(r)[p_r^2 + p_\phi^2]^{1/2} = cp \qquad (6.1.6)$$

yields, according to Tian and Li, the equation

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -p\frac{dc}{dr} \qquad (6.1.7)$$

With $\dot{p}_r = -\frac{\mu m c_o^2}{r^2}$ and $p = \mu c$, TL obtained by integrating eq.(6.1.7) from $r = \infty$ to $r = r$ the following equation

$$c(r) = c_o\left(1 - \frac{2m}{r}\right)^{1/2}. \qquad (6.1.8)$$

Using eq.(6.1.2) we see that eq.(6.1.4) indeed follows from eq.(6.1.8). Note that the Hamiltonian ϕ–equation has not been explicitly used by Tian and Li so far. On the other hand, in order to check the validity of their basic equation, eq.(6.1.7), it is necessary to express the Hamiltonian properly. The specific form of the Hamiltonian H (= cp) in polar coordinates depends on whether p is expressed in non-Euclidean or isotropic coordinates. For a photon traveling with isotropic speed c(r) [see eq.(6.2.2) below] on a flat space, the Tangherlini Hamiltonian[80] is given by

$$H = c(r)[P_r^2 + r^{-2}P_\phi^2]^{1/2} \qquad (6.1.9)$$

which is, of course, numerically the same as eq.(6.1.6) above but the generalized momenta are now given by

$$P_r = \mu\dot{r} \qquad (6.1.10)$$

$$P_\phi = \mu r^2 \dot{\phi}. \tag{6.1.11}$$

The canonical equations of motion are

$$\dot{P}_r = -\frac{\partial H}{\partial r}, \quad \dot{P}_\phi = \frac{\partial H}{\partial \phi} = 0. \tag{6.1.12}$$

The last equation implies

$$P_\phi = \mu r^2 \dot{\phi} = \text{constant}. \tag{6.1.13}$$

We can verify from the correct Hamiltonian eq.(6.1.9) that eq.(6.1.7), in general, does not hold! It holds only in the special case when $P_\phi = 0$. There is a tricky point here by which one is most likely to be misled into believing that eq.(6.1.8) for c(r) holds only in the case of radial photon motion and not for motion in all directions. Thereafter, one is confused by the expression for c(r) [Tian-Li eq.(6.1.14)] which states that the velocity of light is c(r) in *all* spatial directions! Then, one should remember that the Newtonian inverse square law operates not only for radial motion but also for motion in other directions. Hence, if the calculation is limited only to radial motion, as purportedly done by Tian and Li, the conclusions are likely to lack general validity. *We must clarify these points immediately.*

Consider arbitrary photon motion such that $P_\phi \neq 0$. Then, from eq.(6.1.9) and first of eq.(6.1.12) we get

$$\mu\dot{r}^{\cdot} = - \left[\mu c\frac{\partial c}{\partial r} + c\frac{\partial p}{\partial r} \right]. \tag{6.1.14}$$

Since in the Hamiltonian H, we treat r, ϕ, P_r, P_ϕ as independent variables, it can be verified from eq.(6.1.9) that

$$\frac{\partial p}{\partial r} = - \mu r\dot{\phi}^2/c \tag{6.1.15}$$

so that from eq.(6.1.14) we get

$$\mu(\dot{r}^{\cdot} - r\dot{\phi}^2) = - \mu c\frac{dc}{dr} . \tag{6.1.16}$$

It is remarkable that $\mu(r)$ cancels out and the remaining l.h.s. is just the Newtonian expression for the radial acceleration which, by Tian-Li assumption, is

$$\dot{r}^{\cdot} - r\dot{\phi}^2 = - mc_0^2 r^{-2}. \tag{6.1.17}$$

Multiplying both sides of eq.(6.1.17) by $\mu(r)$ and then equating its r.h.s. with that of eq.(6.1.16), eq.(6.1.8) for $c(r)$ follows under usual conditions. It is now clear that the expression for $c(r)$, eq.(6.1.8), holds not only for $P_\phi = 0$, as Tian-Li calculations would have us believe, but also for $P_\phi \neq 0$. The confusion in the TL calculation results from not recognizing that, in polar coordinates, the independent variables in H are *not* p_r, p_ϕ but P_r, P_ϕ although, incidentally, $p_r \equiv P_r$. Inspite of the fact that eq.(6.1.8) has a general validity, as demonstrated just now, there are other difficulties that make the conclusions of Tian and Li unacceptable. In the context of the problem of light deflection, we shall demonstrate this.

6.2. Deflection of light

The problem of light deflection around a spherically symmetric massive gravitating body has been tackled also by Tangherlini through the formalism of classical optics. With the expression n(r), eq.(6.1.4), he employs Fermat's principle, to repeat:

$$\delta \int \frac{n dl}{c_o} = 0 \qquad (6.2.1)$$

where dl is the Euclidean element of distance, to show that the resulting deflection Δ_N is just half of the full GR value $\Delta(\sim 4m/R)$ in which R is the distance of closest approach. The remaining half, one may call it Einstein contribution Δ_E, is separately accounted for by considering the gravitational frequency shift of a photon.

On the other hand, Tian and Li choose to combine the frequency shift and the variable speed of a photon in such a way that it leads to a specific expression for the variation of photon mass. Using this form suitably, they obtain full general relativistic value Δ. Let us closely examine the procedure below.

In the process of obtaining the exact path equation for light rays, Tian and Li use the equations denoting the isotropic velocity of light [their eq.(6.1.14)] in flat space:

$$c^2(r) = \dot{r}^2 + r^2\dot{\phi}^2 \qquad (6.2.2)$$

and the constancy of angular momentum, eq.(6.1.13) [Tian-Li eq.(6.1.15)],

$$\mu(r)r^2\dot{\phi} = A. \qquad (6.2.3)$$

But we must also remember eqs.(6.1.16) and (6.1.17) of the preceding section. It is a logical requirement that all the four equations be compatible. To verify this, it is sufficient to check if c(r) is a solution of eq.(6.1.16). From eq. (6.2.2), with u =1/r, we get

$$c^2(r) = \dot{r}^2 + r^2\dot{\phi}^2 = A^2\mu^{-2}\left[u^2 + \left(\frac{du}{d\phi} \right)^2 \right] \qquad (6.2.4)$$

and computing l.h.s. of eq. (6.1.16), we find

$$-\frac{1}{2}\frac{dc^2}{dr} = \ddot{r} - r\dot{\phi}^2$$

$$= -A^2u^2\mu^{-2}\Big[\,u + \frac{d^2u}{d\phi^2} + \mu\Big(\frac{du}{d\phi}\Big)^2\,\frac{d}{du}\big(\mu^{-1}\big)\Big]. \qquad (6.2.5)$$

Putting the expression for c^2 from eq.((6.2.4) into the l.h.s of eq.(6.2.5), we do not obtain its r.h.s.! Instead we find

$$-\frac{1}{2}\frac{dc^2}{dr} = A^2u^2\mu^{-2}\Big[u + \frac{d^2u}{d\phi^2} + \frac{\mu^2}{2}\Big[\,u^2 + \Big(\frac{du}{d\phi}\Big)^2\Big]\frac{d}{du}\big(\mu^{-2}\big)\Big]. \quad (6.2.6)$$

This implies that $c^2(r)$ is *not* a solution of eq.(6.2.5) and we conclude that the eqs.(6.1.16), (6.2.2) and (6.2.3) are not compatible. This inconsistency renders the claims of Tian and *Li* unacceptable.

Nonetheless, the approach by Tian and Li offers an opportunity to address an important question, namely, the meaning of observables in a gravity field. For a comparison, notice that, in our approach, we did not depart from the solutions of general relativity - only a suitable reformulation of the orbit equations was involved. Therefore, essentially, the same operational meanings of length and time as in general relativity applied there. On the other hand, as evidenced from eqs.(6.1.17) and

(6.2.2), Tian and Li adhere to the background flat space so that the *corresponding operational definitions of length and time apply* there. This feature distinguishes the Tian-Li approach from that of general relativity. However, the notion of length and time in flat space gravity often lead to erroneous conclusions. In order to illustrate the issue from a familiar vantage point, we pick up the wellknown Kepler problem which is interesting in its own right.

6.3. The Kepler problem

Consider the usual Kepler problem of a massive test particle (say, a planet) moving around a spherical gravitating mass M *(say, the Sun) under Newtonian inverse square law.* Let T,V,E and h_0 be the kinetic, potential, constant total energies and angular momentum per unit mass of the test particle respectively. Then T + V = E implies

$$\frac{1}{2}\,[\dot{r}^2 + r^2\dot{\phi}^2] - mc_0^2 r^{-1} = E \qquad (6.3.1)$$

and the central nature of force implies a constancy of angular momentum such that

$$r^2\dot{\phi} = h_0. \qquad (6.3.2)$$

We can rewrite eq.(6.3.1) as

$$h_o^2\left[\, u^2 + \left(\frac{du}{d\phi}\right)^2\right] - 2muc_o^2 = E_o = \text{const.}, \qquad (6.3.3)$$

where $E_o = 2E$ is a new constant. As is customary, by differentiating the above equation we get the differential equation of the Keplerian ellipse in a familiar form.

Let us now redefine the radial variable $u \to u'$ via

$$u' = u\Phi(u), \; u' = 1/r' \qquad (6.3.4)$$

$$\Phi(u) = (1 + \frac{mu}{2})^{-2}. \qquad (6.3.5)$$

After some algebra, we get

$$du' = \Omega(u)\Phi(u)du \qquad (6.3.6)$$

$$\Omega(u) = \left(1 + \frac{mu}{2}\right)^{-1}\left(1 - \frac{mu}{2}\right) \qquad (6.3.7)$$

$$\Omega(u') = (1 - 2mu')^{1/2} \qquad (6.3.8)$$

$$\Phi(u') = \frac{1}{4}\,[1 + (1 - 2mu')^{1/2}]^2. \qquad (6.3.9)$$

In fact, these are the same expressions as used in Sec.2.4 above but we group them here for an easy view. The following can also be directly verified:

$$2mu = 2mu' + 2m^2u'^2 + 5m^3u'^3 + \ldots = 2mu' + 0(m^2u'^2). \quad (6.3.10)$$

This implies that, to first order, $r \approx r'$. Also,

$$\Phi^2(u')\Omega^2(u') = 1 - 4mu' + 0(m^2u'^2). \quad (6.3.11)$$

We shall now express eq.(6.3.3) in terms of the new variable u'. To do this, multiply both sides of eq.(6.3.3) by $\Phi^2\Omega^2$, which yields, through the use of eqs.(6.3.4) —(6.3.11), the following equation:

$$h_o^2 \, [\Omega^2u'^2 + (\frac{du'}{d\phi})^2] = c_o^2\left[E_o c_o^{-2} + 2mu' + 0(m^2u'^2)\right]\left[1-4mu' + 0(m^2u'^2)\right]$$

$$(6.3.12)$$

Simplifying further, using eq.(6.3.8), we get

$$h_o^2[u'^2 + (\frac{du'}{d\phi})^2 - 2mu'^3] = c_o^2[E_o c_o^{-2} + 2mu' - 4mu' E_o c_o^{-2} + 0(m^2u'^2)].$$

$$(6.3.13)$$

Let us now estimate the relative strength of the terms appearing inside the bracket on the r.h.s. of eq.(6.3.13). Note that, even at the site of Mercury, the planet nearest to the Sun, we have $mu \approx mu' \approx 2.5\times10^{-8}$. Further ,for a bound orbit, we know that $E < 0$. Using from Newtonian mechanics the approximate result for a bound system, namely, $v^2 \sim mc_o^2/r$ where $v\ [\equiv (\dot{r}^2 + r^2\dot{\phi}^2)^{1/2}]$, we find that the absolute magnitude of the first constant term is given by $|E_o c_o^{-2}| \approx mu \approx mu' \approx 2.5\times10^{-8}$. Therefore the second term is of comparable order of magnitude to the first constant term while the third and $m^2u'^2$ terms are 10^{-8} times smaller. The next higher order term is 10^{-16} times smaller and so on. In view of this practical numerology, it looks quite reasonable to disregard terms that are 10^{-8} times smaller or less than the first term. We are then left with following differential path equation:

$$h_o^2[u'^2+(\frac{du'}{d\phi})^2 - 2mu'^3] = E_o + 2mu' c_o^2. \qquad (6.3.14)$$

Differentiating with respect to ϕ, we get

$$u' + \frac{d^2u'}{d\phi^2} = mc_o^2h_o^{-2} + 3mu'^2 \qquad (6.3.15)$$

and Lo - we have the equation for a processing ellipse of Einstein's general relativity giving the perihelic shift of Mercury, the famous value 43"/century! We began with a closed Kepler ellipse and have now ended up with an open orbit! Moreover, the light ray equation follows if the customary limit $h_o \to \infty$ is taken. So, let us pause a little and look back. The neglect of smaller terms seems perfectly justified, as we have just seen. The radial coordinate transformation $u \to u'$ is quite harmless since a rearithmetization of space is just an abstract mathematical operation and physical conclusions should not depend on it. Should we then accept the above procedure as an alternative way of deducing the famous general relativistic predictions?

One may guess that the answer will be in the negative but the reason is not obvious. Some plausible arguments arise. One may straightaway point out that, after all, it is the same old classical inverse square force law on a flat space and a mere spatial coordinate transformation has no physics in it. Therefore there is no reason to expect any deviation from the usual Kepler elliptic orbit. This is a perfectly valid physical argument. But, mathematically, *eq.(6.3.13) does not yield any Keplerian ellipse* $u' = a + b \cos \phi$, where a and b are constants, as a solution. Also, there is no reason whatsoever to return to the (u,ϕ) coordinate system as all systems are *equally* preferable. The (fictitious) nature of the above ambiguity will be understood only after a discussion, presented at the end, of the meaning of observables in a given gravity theory.

However, the question above relates not to the exact eq.(6.3.13) but to the status of the procedure leading eq.(6.3.15). In that procedure, one should not underestimate the innocent looking approximations. The resulting equation, eq.(6.3.15), when reverted to the (u,ϕ) language, does no longer correspond to the inverse square force law we started with. This means that the operation of approximation amounts to introducing *extra* forces that modify the original force law and consequently, the particle orbit. Thus, the Keplerian elliptic orbit is modified into a precessing ellipse. Recall that there is nothing wrong with the operation of approximation itself. In similar circumstances, *such approximations are essentially employed also in* general relativity or in other approaches (like that of Tian and Li) whenever nonlinear orbit equations are solved. In general relativity, the exact geodesic equations do relate to a given spacetime metric (analogy: given inverse square law here) but the approximate equations do not. These latter equations may, at best, relate to a modified metric (analogy: modified force law here). These analogies seem to suggest that the physics and mathematics of the procedure may be justified. Even then the procedure can not be treated as a genuine alternative for the single reason that it is based on a coordinate transformation eq.(6.3.4) which is more an *accidental* choice than a result of some physical postulate!

6.4. Observables in a gravity field

In order to motivate the readers, let us start with a simpler example. Consider the light path equation [Tian- Li eq.(6.1.17)]:

$$u + \frac{d^2 u}{d\phi^2} = 2mR^{-2} \qquad (6.4.1)$$

and its solution

$$u = \frac{1}{d}\left[1 + \varepsilon \cos(\phi - \phi') \right] \tag{6.4.2}$$

where $d = R^2/2m$, $\varepsilon = bd$; b, ϕ' are arbitrary constants of integration and R is the perihelion distance. Change the coordinate $u \to y$ such that

$$y = u - \frac{1}{d} . \tag{6.4.3}$$

Then eq.(6.4.1) reduces to

$$y + \frac{d^2 y}{d\phi^2} = 0 \tag{6.4.4}$$

which is the differential equation of a straight line having a solution

$$y = h \cos (\phi - \phi'') \tag{6.4.5}$$

where h, ϕ'' are arbitrary constants. Again, a rearithmetization eq.(6.4.3) is of no physical consequence and both the systems (u,ϕ) and (y,ϕ) are equally preferable. Yet, the predictions look ambiguous: An elliptical path in the (u,ϕ) space implies that there *is* a deflection of the light ray while a straight path in the (y,ϕ) space implies that there is none. Which one of the predictions is to be really observed? Such (apparent) ambiguities do arise also in general relativity. Depending on the choice of coordinate systems, general relativity equations may lead to totally opposite predictions.[81]

To answer the question posed above, again one must understand what is meant by "observables" in a gravity field. The most reasonable definition of observables is that they should be scalars, independent of the choice of coordinate systems.[82] Such a definition is indeed very general and applies to any theory. In keeping with this idea, the observable predictions in general relativity are required to be expressed in terms of "proper" quantities defined in a coordinate independent manner. Let us illustrate how this should be achieved in the wellknown Schwarzschild problem.

The proper length in the "standard" coordinates (r,θ,ϕ) is defined by

$$dl^2 = (1 - \frac{2m}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \qquad (6.4.6)$$

The surface area of a sphere is, of course, $4\pi r^2$ which endows usual meanings to θ and ϕ coordinates. In principle, the exact procedure is to express the general relativistic orbit equation in terms of proper quantities, say, in the (l,ϕ) language and then solve it. But, unfortunately, dl, in general, is not integrable. Therefore, the first step is to write the geodesic equation in (u,ϕ) coordinates and then reexpress it in the (l,ϕ) language using the expression for proper radial length derived from the metric eq.(6.4.6) as

$$l = \int^{r}(1 - \frac{2m}{r})^{-1/2}dr$$

$$= \Big[r(1 - \frac{2m}{r})^{1/2} + 2m\, ln\, ([r - 2m]^{1/2} + r^{1/2})\Big]^{r}. \quad (6.4.7)$$

As one can see, the choice of a different radial coordinate $r' : r \to r'$ amounts only to a substitution of the running variable under an integral sign, an operation that does not alter the numerical value of l. However, there is the great problem that eq.(6.4.7) can not be inverted, that is, an exact expression of the form $r = f(l)$ is not available in general. Fortunately, in the Solar system scenario, it is possible to take $u \approx l^{-1}$ without committing much error. Putting it in the (u,ϕ) geodesic equation, we achieve our goal, that is, a coordinate independent form. That is also the reason why in text books geodesic equations in the Solar system are analyzed manifestly not in terms of l but only in terms of coordinate r since $l \approx r$. But, in the field of strong gravity, say, in the vicinity of a black hole, it is essential to take the (u,ϕ) orbit equation together with eq.(6.4.7) *per se* and solve the problem numerically to obtain orbits in the (l,ϕ) space.

In the Tian-Li approach, the spatial metric is flat, given by

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (6.4.8)$$

and accordingly, the proper radial length is

$$l = \int^r dr. \qquad (6.4.9)$$

There is some advantage in this approach. The problem of inversion just does not arise as we have simply $r = l$. Putting $u = l^{-1}$ in eq.(6.4.1), we get a coordinate independent form of equation. The prediction is also unique now: there *will* occur a deflection of light ray at the perihelion by an amount $4\mu/L$, where $L = \int^R dr$, which is the same as in general relativity.

From this it follows that if a flat space gravity theory is free of any internal inconsistency (Unfortunately, this is not the case with the approach of Tian and Li, as demonstrated in Sec.6.1.), there does not seem to be anything to distinguish it from general relativity as far as *first* order predictions for light motion are concerned. For a detailed investigation into this topic from the point of view of a minimally relativistic gravity, see ref.[83]. However, we must be aware of a basic difficulty, already noted in Sec.1.3: The universality of gravitational interaction prevents an observational location of the background flat space. Consequently, an *exact* identification $l = r$, as in eq.(6.4.9), at *all* points in the field is not achievable. Only in the weak field region can one reasonably take $l \approx r$. Therefore, any new approach, besides being mathematically consistent, must possess definitions of observable physical length and time. In this connection, Ichinose and Kaminaga[84] point out a more fundamental "inevitable ambiguity" - not necessarily a defect - of general relativity that owes its origin to the geometric formulation. For instance, in order to obtain a perturbative formula for the radar echo travel time, which is not what is defined in the asymptotic region, one is forced to compare points in two *different* geometries, curved and flat - which is a *geometrically nonsense procedure. We do not wish to go into* details here as it will take us out of the main scope of the present work, but interested readers may have a look at ref.[84].

Let us return to the Kepler problem. The differential equation of the classical orbit is

$$u + \frac{d^2u}{d\phi^2} = mc_o^2 h_o^{-2}. \tag{6.4.10}$$

The proper radial length, in this case, too, is given by eq.(6.4.9). Replacing u by l^{-1}, we obtain a coordinate independent form of eq.(6.4.10) which is just that of an ellipse in the (l,ϕ) space. Observe that, a transition to u′ via eq.(6.3.4) introduces a corresponding change in the metric eq.(6.4.8) and thereby results a different expression, f(r′), on the r.h.s. of eq.(6.4.9). Consequently, the functional form for $r' = f'(l)$ will also be different. Using this in the exact eq.(6.3.13), we are able to retrieve the same equation (6.4.10) but in the coordinate independent (l,ϕ) language. Hence, we have the same old Kepler orbit, whatever be the coordinate choice - and the ambiguity is resolved.

6.5. Concluding remarks

In the approach of Tian and Li, the effects of gravity (or equivalently, refractive medium) is incorporated into the notion of variable photon "mass" and speed, $\mu(r)$ and $c(r)$ respectively, on a flat space. The (weak) principle of equivalence is also accommodated through their assumption, eq.(6.1.17). It seems that there are sufficiently plausible ingredients in the Tian-Li approach and a consistent procedure, supplemented with definitions of observable length and time, could possibly lead to interesting alternative deductions of effects - at least on light motion in a gravity field.

After correcting a misleading step, it is shown that there is an inconsistency inherent in the approach that invalidates their conclusions. A further analysis with regard to the meaning of observables in general relativity and in flat space gravity clarifies the question of coordinate independent predictions. We saw that "coordinate" descriptions may vary - even lead to opposite predictions - but "proper" descriptions do not. The theoretical scheme of achieving the latter description is also indicated. It is perhaps pointless to ask whether the notion of $c(r)$, $\mu(r)$ or a refractive medium on a flat space is more/less real than the notion of general relativistic spacetime curvature. These are just different artifacts to describe actual observations - light propagation in the present case - the basic condition being that the description must be free of internal inconsistencies.