CHAPTER 5

APPLICATION TO COSMOLOGY

This chapter contains applications of an formalism to the realm of cosmology. We demonstrate in Sec.5.1 and 5.2 that anti-de Sitter and Robertson-Walker universes constitute Maxwell "fish-eye lens." Sec.5.3 deals with light motion in these two universes. The arguments will allow us to see how one should interpret a time dependent gravity index in these contents. The developments of Sec.5.4 indicate how the redshifts arise in the language of a refractive index and how a unified view could be achieved in regard to two apparently dissimilar redshifts.

5.1. The de Sitter Universe and the Maxwell's "fish-eye lens"

The de Sitter line element in standard coordinates is

$$ds^2 = (1 - \Lambda r'^2/3)c_o^2 dt^2 - (1 - \Lambda r'^2/3)^{-1} dr'^2$$

$$- r'^2 d\theta^2 - r'^2 \sin^2\theta d\phi^2, \qquad (5.1.1)$$

where Λ is the cosmological constant, which is proportional to the

space curvature. Λ can be positive or negative, corresponding to a closed or an open (or, anti-) de Sitter universe.[73]

To pass over to isotropic coordinates, we may use the method outlined above, together with the requirement that for small radial distances the new radial variable r should asymptotically approach r'. The result is the well-known transformation

$$r' = r(1 + \Lambda r^2/12)^{-1}. \qquad (5.1.2)$$

Then, in the isotropic coordinates,

$$ds^2 = (1 - \Lambda r^2/12)^2(1 + \Lambda r^2/12)^{-2}c_o^2dt^2$$

$$-(1 + \Lambda r^2/12)^{-2}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2). \qquad (5.1.3)$$

The effective index of refraction is

$$n(r) = (1 - \Lambda r^2/12)^{-1} \qquad (5.1.4)$$

This index of refraction is valid for either positive or negative Λ, with r defined through eq.(5.1.2) let us examine the case $\Lambda < 0$, corresponding to the anti-de Sitter universe. Let us write $\Lambda = -K$, where K is then a positive constant. The effective index of refraction of the anti-de Sitter universe, in the

isotropic coordinates, is then

$$n(r) = (1 + Kr^2/12)^{-1} \tag{5.1.5}$$

This index of refraction is of exactly the same form as the index encountered in a traditional problem of classical geometrical optics — the Maxwell "fish-eye lens". The index of refraction in the Maxwell fish-eye is

$$n_M(r) = n_o(1 + r^2/a^2)^{-1}, \tag{5.1.6}$$

in which a and n_o are constants. Comparing eqs.(5.1.5) and (5.1.6) we note that the open version of the de Sitter universe is a Maxwell "fish-eye lens" with $n_o = 1$ and $a^2 = 12/K$.

5.2. Robertson-Walker Universe

The Robertson-Walker (RW) metric represents the gravitational field in a homogeneous and isotropic universe. In the standard comoving coordinates (t, r', θ, ϕ), the RW line element is given by

$$ds^2 = c_o^2 dt^2 - R^2(t)\left[\frac{dr'^2}{1 - kr'^2} + r'^2 d\theta^2 + r'^2 \sin^2\theta \, d\phi^2\right], \tag{5.2.1}$$

in which R(t) is a dimensionless scale factor and k is a constant

with dimensions of $(\text{length})^{-2}$. We may pass over to isotropic coordinates by the usual method and requiring that for small radial distances, the new radial coordinates should asymptotically be equal to r'. The result is the well-known transformation

$$r' = r(1 + Kr^2/4)^{-1}. \tag{5.2.2}$$

In the isotropic coordinates (t, r, θ, ϕ), the line element is

$$ds^2 = c_0^2 dt^2 - R^2(t)(1 + Kr^2/4)^{-2}|dr|^2. \tag{5.2.3}$$

Defining the refractive index n in the usual way, we obtain

$$n = \frac{R(t)}{1 + Kr^2/4}. \tag{5.2.4}$$

For the case $k > 0$, corresponding to a closed RW universe, and for a fixed cosmological epoch $t = t_0$, this corresponds to the index of refraction (5.1.6). Thus the closed Robertson-Walker universe is a Maxwell fish-eye lens with $n_0 = R(t_0)$ and $a^2 = 4/k$. We shall see below that the correspondence between the Maxwell fish-eye and the Robertson-Walker universe does not actually demand that we restrict the later to a particular moment t_0.

5.3. Light ray in the de Sitter and Robertson-Walker Universes

As noted above, the open de Sitter universe is equivalent to a traditional problem in classical geometrical optics--Maxwell's fish-eye lens. It follows (1) that the anti-de Sitter universe constitutes an absolute, optical instrument and (2) that, in the system of isotropic coordinates, the rays are eccentric circles.

Beginning from the orbit equation (4.3.3) and the index of refraction (5.1.5) and integrating, we obtain the polar equation for the light ray in the anti-de Sitter universe:

$$\sin(\phi - \alpha) = \frac{h(Kr^2 - 12)}{r(144c_0^2 - 48h^2K)^{1/2}}, \qquad (5.3.1)$$

where α is a constant of integration. In effecting this calculation, we can follow step-for-step the calculation of ray shapes in the classical Maxwell fish-eye.[74]

Since $(Kr^2 - 12)/r\sin(\phi - \alpha)$ = constant, we can write the equation for a family of light rays passing through a fixed point $P_0(r_0, \phi_0)$ as (See fig.1)

$$\frac{Kr^2 - 12}{r\sin(\phi - \alpha)} = \frac{Kr_0^2 - 12}{r_0\sin(\phi_0 - \alpha)} \qquad (5.3.2)$$

for any value of α, this equation is satisfied at point $P_1 = (r_1,$

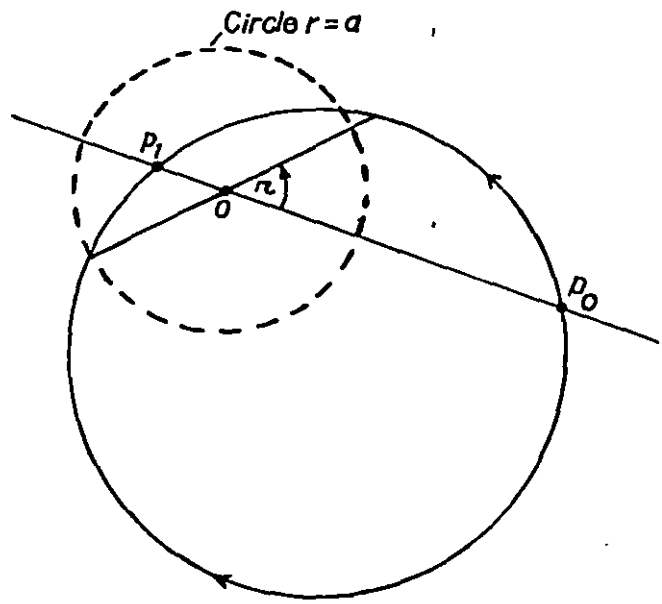

Fig. 1 Rays in Maxwell's "Fish-eye"

ϕ_1), where $r_1 = 12/Kr_0$ and $\phi_1 = \phi_0 + \pi$. This shows that all the rays *from an arbitrary point* P_0 *meet at a point* P_1 on the line joining P_0 to the origin O such that $OP_0 . OP_1 = 12/K$. Hence the imaging in Maxwell's fish-eye lens is an inversion. From any point P_0 in the three-dimensional space an infinity of rays originate which are then focused at an image point P_1. The images are therefore *sharp* (stigmatic). (In most real optical instruments, of the infinity of points passing through an object point, *only a finite number pass* through the image point, the other rays only passing near the image point. Such images are not sharp ones.) Now, an instrument which sharply focuses an image of a three-dimensional region of space is called an absolute optical instrument. Thus, the anti- de Sitter universe constitutes an absolute optical instrument. All *the theorems pertaining to absolute optical instruments apply. For* example, the optical length of a line segment in the image must be equal to the optical length of the corresponding line segment in the object[74].

Moreover, by analogy to the Maxwell fish-eye, a ray in the de *Sitter universe is a circle in the isotropic coordinate system.* This goes exactly as in the classical geometrical optics treatment of the Maxwell fish-eye. On the left-hand side of eq.(5.3.1) write $\sin(\phi - \alpha) = \sin\phi\cos\alpha - \cos\phi\sin\alpha$, then put $x = r\cos\phi$ and $y = r\sin\phi$. Then it may be written in the form

$$(x + b\sin\alpha)^2 + (y - b\cos\alpha)^2 = 12/K + b^2 \qquad (5.3.3)$$

where $b = (hK)^{-1}(36c_0^2 - 12h^2K)^{1/2}$. Thus each ray is a circle. Note that in eq.(5.3.1) if we put K = 0, we obtain a straight-line ray, as we would expect:

$$r\sin(\phi - \alpha) = \text{constant}.$$

These results may be extended with little modification to the closed Robertson-Walker universe. The formalism developed in this paper was designed for static metrics, so it may seem at first sight that we can not deal with the Robertson-Walker case. However, in the case of null geodesic, a time-dependent conformal factor in the metric need not affect the basic procedure or the most important conclusions. (The same can not be said of the particle trajectories.) Let us see how this works out in the language of refractive indices.

As noted above, the closed Robertson-Walker universe yields a factorable index of refraction:

$$n = n_s(r)n_t(t),$$

in which n_s is a function of the spatial coordinates alone and n_t

is a function of the time alone. For the RW metric in isotropic comoving coordinates

$$n_s = (1 + kr^2/4)^{-1} \tag{5.3.5}$$

$$n_t = R(t). \tag{5.3.6}$$

The bending of light rays depends only on the spatial gradient of n. The fact that n varies everywhere in space with the same multiplicative function of time R(t) does not affect the shape of a light ray. One way to see this is to consider Snell's law. If the index of refraction factors into the form (5.3.4), then whenever Snell's law is applied at an interface between regions of different n, the common factor $n_t(t)$ will cancel from the two sides of equation. More formally, we can define a new time coordinate τ by $dt = Rd\tau$. Then the line element (5.2.3) becomes

$$ds^2 = c_o^2 R^2 d\tau^2 - R^2(1 + kr^2/4)^{-2} |dr|^2.$$

and the effective index of refraction becomes simply

$$n = n_s = (1 + kr^2/4)^{-1} \tag{5.3.7}$$

The scale factor $R(\tau)$ or R(t) thus can have no effect on the shape of a ray in the isotropic, comoving coordinates. R only influences

the progress in time of light along the ray.

As far as ray shapes are concerned, then, the RW universe is entirely analogous to the Maxwell fish-eye lens. It follows (i) that the closed Robertson-Walker universe also constitutes an absolute optical instrument and (ii) that, in the system of isotropic, comoving coordinates, the rays are eccentric circles.

5.4. Redshifts

The gravitational and cosmological redshifts are not dynamical effects; i.e., we do not need to solve an equation of motion in order to calculate them. However, it may be of some interest to see how the redshifts arise in the language of an effective index of refraction.

In ordinary optics, if a light wave travels from a region of high index of refraction n_2 to a region of low index of refraction n_1, the wavelength λ increases because the leading part of the wave enters n_1 first and speeds up while the trailing part is still in n_2. Thus the wave begins to stretch out. λ and c change but the frequency ν does not. This holds even if the index varies continuously with the spatial coordinates and even if the ray crosses obliquely through the contours of constant n.

Thus, in general,

$$\lambda(r_1)n(r_1) = \lambda(r_2)n(r_2). \tag{5.4.1}$$

Now consider a situation in which n does not depend on the spatial coordinates, but does vary with time. An example can be imagined: let the air slowly be pumped from a chamber. Then, as long as the wave does not leave the chamber, n is everywhere the same, but it *is a decreasing function of time. The wavelength will not change,* since the leading edge of the wave never encounters a new value of n before the trailing edge does. Thus, in this case c and ν change, but λ does not. We have

$$\lambda(t_1) = \lambda(t_2) \tag{5.4.2}$$

Suppose now that the index of refraction can be written as a product of two functions--a function n_s of the spatial coordinates alone and a function n_t of the time alone, as in eq.(5.3.4). For the reasons just mentioned, n_t does not affect the wavelength and we have

$$\lambda(r_1)n_s(r_1) = \lambda(r_2)n_s(r_2). \tag{5.4.3}$$

We wish to apply these rules of ordinary optics to the propagation of light in general relativity. Our effective index of refraction eq.(2.1.5) is based upon the isotropic coordinate speed

of light. Thus the quantity analogous to the wavelength of classical optics is the *coordinate distance* between successive crests of the wave. Coordinate distances are not, of course, directly measurable in general relativity. The physically measurable metric length is obtained from the coordinate length by means of the metric (2.1.1).

Gravitational Redshift

Let $|\Delta r_1|$ be the coordinate distance between successive crests of a light wave located at r_1. Similarly, let $|\Delta r_2|$ be the coordinate distance between successive crests at a different point r_2 located on the same ray. Then, in analogy to the condition (5.4.1) from ordinary optics, we must have

$$|\Delta r_1| n(r_1) = |\Delta r_2| n(r_2). \tag{5.4.5}$$

The metric length λ of the wave at point r_1 is obtained by applying the metric (2.1.1) to the coordinate length $|\Delta r_1|$ of the wave: $\lambda(r_1) = \Phi^{-1}(r_1)|\Delta r_1|$. A similar expression holds for $\lambda(r_2)$. Thus we have

$$\Phi(r_1)\lambda(r_1)n(r_1) = \Phi(r_2)\lambda(r_2)n(r_2), \tag{5.4.6}$$

or, using eq.(2.1.5),

$$\lambda(r_1)\Omega^{-1}(r_1) = \lambda(r_2)\Omega^{-1}(r_2) \qquad (5.4.7)$$

the usual gravitational redshift relation.

As an example, let us take the case of the Schwarzschild metric. Let a source of light be located at r_1 and an observer at r_1, sufficiently far from the central gravitating body so that we may put $\Omega(r_2) \approx 1$. Then, with the use of eqs.(4.1.11) and (4.2.1), (5.4.5) gives

$$z \equiv (\lambda_{observed} - \lambda_{emitted})/\lambda_{emitted}$$

$$= (1 - 2m/r')^{-1/2} - 1. \qquad (5.4.8)$$

This results may, of course, be derived by many other methods.

Cosmological Redshift

In the expanding universe of the Robertson-Walker metric, we have an index of refraction (5.2.4) that factors like (5.3.4), with n_s and n_t given by eqs.(5.3.5) and (5.3.6). Let a light wave of coordinate length $|\Delta r_1|$ be emitted at (t_1, r_1) and received at (t_2, r_2). By analogy to eq.(5.4.3), the coordinate length $|\Delta r_2|$ of the received wave is determined by

$$|\Delta r_1| n_s(r_1) = |\Delta r_2| n_s(r_2). \tag{5.4.9}$$

The metric length λ of the wave is obtained by applying the metric (5.2.3) to the coordinate length $|\Delta r|$. Thus eq.(5.4.9) becomes

$$\frac{(1 + kr_1^2/4)}{R(t_1)} \lambda(t_1, r_1) n_s(r_1) = \frac{(1 + kr_2^2/4)}{R(t_2)} \lambda(t_2, r_2) n_s(r_2),$$

or, with use of eq.(5.3.5), we get

$$\lambda_1/R(t_1) = \lambda_2/R(t_2) \tag{5.4.10}$$

which is precisely the usual cosmological relation.

While many writers have stressed the fundamentally different natures of the gravitational and cosmological redshifts, others have argued that it is possible to treat them with a single unified approach[75]. In the effective optical-medium formulation persued here, it is interesting to note that both spectral shifts depend on a single optical principle eq.(5.4.3). The following remarks are in order.

In the context of redshifts, we recall that we mean redshift in frequency. We do not specify whether it is the frequency of an

elementary wave or an elementary particle or a material clock. Kostro[76] has raised a very interesting question with respect to the definition of a clock. He argues that the total energy of a moving particle increases according to the equation

$$E = E_0(1 - \beta^2)^{-1/2}, \quad \beta = v/c_0 \qquad (5.4.11)$$

de Broglie has written this as $h\nu = h\nu_0(1 - \beta^2)^{-1/2}$ and concluded that with the elementary particle must be associated a wave for which the frequency transforms like this. On the other hand, Einstein considered a photon clock in which a photon moves between two mirrors and has shown that $\nu = \nu_0(1 - \beta^2)^{1/2}$. So, we see that for Einstein photon clock, the frequeny slows down and for de Broglie's standing wave clock, the frequency increases as $v \to c_0$. Are we dealing with two kinds of time? Are we to exclude de Broglie's clocks as time measuring devices? Kostro[76] argues that the ideal relativistic clock is a massless clock having greatest possible frequency. However, note that the opposite behavior of clocks is reminiscnt of the conflict between Newton's corpuscular hypothesis and Foucault's experiment, addressed in Sec.2.3.