SOME EXTENDED SOURCE AND CRACK PROBLEMS IN ELASTODYNAMICS

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CRACK PROBLEMS IN ELASTODYNAMICS

C H A P T E R - III

DIFFRACTION PROBLEMS IN ELASTODYNAMICS

INTRODUCTION

The study of wave and vibration phenomena in elastic solids has a distinguished history of more than hundred years. Some pioneer workers in the field of wave propagation in elastic medium and vibrating bodies are Cauchy, Rayleigh, Love, Poisson, Ostrogadsky, Green, Lame, Stokes, Kelvin.

Seismology has made a tremendous progress during the last three decades, mainly because of the technological developments, which have enabled seismologist to make measurements with far greater precision and sophistication than was previously possible.

Here, some of the major progress in the field of wave propagation are given in chronological order.

1678 : Robert Hooke (England) established the stress-strain relation for elastic bodies.

1821 : Louis Nevier (France) derived the differential equations of the theory of elasticity.

1822 : Cauchy developed most of the aspects of the pure theory of elasticity including the dynamical equations of motion for a solid.

1828 : Simeo-Denis Poisson (France) predicted theoretically the existance of longitudinal and tranverse elastic waves.

1849 : George Gabriel Stokes (England) conceived the first mathematical model of an earthquake source.

1857 : First systematic attempt to apply physical principles to earthquake effects by Robert Mallet (Ireland).

1862 : Clebsch found the general theory for the free vibration of solid bodies using normal modes.

1872 : J. Hopkinson performed the first experiments on plastic waves propagation in wires.

1883 : saint Venant summarized the work on impact of earlier investigators and presented his results on transverse impact.

1883 : Rosi-Forel scale for earthquake effects published.

1885 : C. Somigliana (Italy) produced formal solutions to Navier equations for a wide class of sources and boundary conditions.

1887 : Lord Rayleigh (England) predicted the existance of elastic surface waves.

1899 : C. G. Knott (England) derived the general equations for the reflection and refraction of plane seismic waves at plane boundaries.

1903 : A. E. H. Love (England) developed the fundamental theory of point sources in an infinite elastic space.

1904 : Horance Lamb (England) made the first investigation of pulse propagation in a semi-infinite solid.

1911 : Love developed the theory of waves in a thin layer overlying a solid and showed that such waves accounted for certain anomalies in seismogram records.

1949 : Devies published an extensive theoretical and experimental study on waves in bars.

1959 : Ari Ben-Menahem (Israel) discovered that the energy release in earthquakes takes place through a propagating rupture over the causative fault.

1967 : Global seismicity patterns and earthquake generation linked to plate motions.

During the first two decades of this century the subject was not given so much importance by Mathematicians or Physicists. But later, interest in the study of waves in elastic solids attracted the attention of the researchers because of applications in the field of geophysics and engineering constructions. Since that time in seismology the wave propagation has remained an interesting area because of the need for details information on earthquake phenomena, prospecting techniques and the detection of nuclear explosions. Bullen [1963], Ewing et al [1957], Cagniard [1962], Pilant [1979] and Aki and Richards [1980] have discussed about seismic waves in their books.

During last 30-40 years the development of theory of wave propagation in elasticity has been characterized by a detailed investigation of the classical methods of mathematical analysis and the trends to obtain specific results. The solution of many of the problems in elastodynamics, which are frequently encountered in practice need advance level of mathematical technique, which may roughly be grouped into the following categories:

(a) Theory of analytic function

(b) The Fredholm integral equation

(c) The singular integral equation

(d) Integral transforms and Representations

(e) Dual integral and series equations

(f) Harmonic function. Potential theory

(g) The Drichlet and Neumann problems

(h) Green's functions

(i) The Cauchy problem

(j) Cagniard-deHoop technique

(k) Wiener - Hopf technique

(l) Riemann - Hilbert problem

(m) The method of Matched Asymptotic expansions

(n) Perturbation technique

(o) Variational method, The Ritz method

(p) The method of finite element

(q) The method of boundary element

and others.

While earlier investigation in the theory of elasticity was essentially reduced to the construction of particular solution; the invention of computer technology has led to the development of general and quite universal methods of solving the problems of this theory, namely, the boundary value problems and initial boundary value problems for systems of differential equations having partial derivatives of a definite structure.

Most of the experimental works carried out on the wave propagation are concerned with studying propagation in specimens of comparatively simple geometrical shape. The results of this experiment could be compared directly with exact or approximate theoretical predictions. The agreement, with experimental results and theoretical predictions, inspires confidence in taking up complicated problems and makes possible theoretical predictions and interpretations of observations.

The propagation of waves through homogeneous isotropic elastic materials of unbounded extension is not a subject of very complexity. The waves are either dilatiational or distortional or a combination there of. The picture changes radically as soon as there is a boundary. Interaction of two types of waves occurs, when boundary is present and this interaction presents an inherent difficulty in the solution of elastodynamic problems.

More over the effect of a free surface on the generation and propagation of waves in elastic medium has been the subject of many investigations ever since the discovery of existance of surface waves by LORD RAYLEIGH.

In general, problems which mostly attract the researchers both theoretical and experimental, in relation to the generation and propagation of waves in an elastic medium may be classified as follows;

(i) diffraction of propagating waves through the medium due to any obstacle, cavity or a crack of any shape situated some where in the medium;

(ii) reflection, refraction and diffraction of propagating waves due to mixed boundary conditions;

(iii) wave motion generated due to a punch on some bounded region of the medium;

(iv) radiation of waves i.e. the wave motions generated due to some fixed external disturbance and propagating away from the source of disturbance;

(v) wave motion generated in a medium when a source of disturbance moves along the medium.

Depending on the nature of the source of disturbance, shape of the punch or normal loading on the free surface and the presence of discontinuities in the medium, different complicated problems arise. The solution of these problems need an advance level of sophisticated mathematical techniques some of which have been mentioned earlier.

The dynamic response of an elastic half space due to an external load or punch on the free surface and also the scattering of elastic waves by a finite crack or a strip inside an elastic medium may be investigated by the use of integral transform technique.

The propagation of waves due to the application of loads at the boundary of a semi-infinite medium was first considered by Lamb [1904], who studied the axisymmetric propagation of a pulse created by transient normal point load on the surface of the half-space. By means of Fourier Synthesis of steady state solutions, Lamb showed the predominant character of the Rayleigh wave response. Later, Sauter [1950] derived a closed form solution by means of an integral superposition of plane harmonic waves. Many authors have subsequently viewed and reviewed the problems which deal with the disturbance produced by a point or line source acting on the surface or buried in an elastic half-space by means of Laplace transform. Pekeris [1955] derived the exact expression for the vertical and horizontal components of the displacement on the surface of a uniform elastic half-space due to a point load with step function time variation, situated on the surface and also at a finite depth below the surface. Thiruvenkatachar [1955] derived the exact expression for the Laplace transform of the displacement over a circular region which is more realistic physically. Knopoff and Gilbert [1959] and Lang [1961] derived the wave front approximation by the application of saddle point method to the Laplace transformed solution and limit theorems of Tauberian type. While

Cagniard [1962] developed powerful technique of finding the Laplace inversion for this class of problems. Mitra [1964] investigated this type of problem in detail, verified Pekaris's result and pointed out that Cagniard's method can be applied more widely than either Pekaris's or Chao's method. This type of problem was then investigated by Eason [1964, 1966], Mitra [1964], Chakraborty and De [1971], Gakenheimer [1971], Ghosh [1971] and many others. All these are axisymmetric problems.

Very few wave propagation problems of non-axisymmetric type have been solved. Chao [1960] derived the exact solution for the half-space problem in which the disturbance is due to a tangential surface point load. Pekeris and Longman [1958] investigated the motion of the surface of a uniform elastic half-space produced by the application of torque pulse at a point below the surface. Using a modification of Cagniard's method, Gakenheimer and Miklowitz [1969] analysed transient excitation of the elastic half-space by a point load travelling on the surface. All these non-axisymmetric problems deal with the point load.

For the problems dealing with the ring load we refer Maiti [1978], Ghosh [1980-81] and others. Maiti [1978] treated the problem of asymmetric finite source, examined the effect of a half-space of impulsive shearing traction over a circular portion of the surface. The formal solution is obtained by expressing the displacement components in terms of scalar and vector potentials and using Laplace and double Fourier transforms. The inverse transforms are evaluated by modified Cagniard's techinque which yields the solution within and on the half-space in a closed integral form. Ghosh [1980-81] treated the problem of disturbance in an elastic semi-infinite medium due to the torsonal motion of a circular ring source on the free surface of homogeneous and inhomogeneous medium. Using Laplace transform and the Hankel transform and the Laplace inversion by Cagniard's method the integrals for displacement are evaluated numerically.

On the other hand Pal and Ghosh [1987] considered the elliptic ring load propagating over the free surface of a semi-infinite medium. The expression for displacement at points on the free surface has been derived in integral form by the application of Cagniard-de-Hoop technique for different values of the rate of increase of the major and minor axes of the elliptic ring source The displacement jumps across the different wave fronts have also been derived. A comprehensive survey of the field due to extended source problems has been given by Scott and Miklowitz [1964].

The problems relating to the propagation of elastic waves, due to applied boundary tractions, in semi-infinite media containing internal boundaries are of immense importance in seismology and geophysics rather than of point source problems in homogeneous semi-infinite medium. This type of problem was first considered by Johnson and Parnes [1977]. The problem, they treated, is that of a semi-infinite elastic body containing a rigid lubricted inclusion whose axis is perpendicular to the plane surface subjected to an axisymmetric concentric line load applied dynamically as a step function in time at the plane surface. The dynamic problem was formulated interms of two potential functions which satisfy uncoupled two dimensional wave equations with coupled boundary conditions. Using Laplace transform, the integral solution for the transformed stress and displacement fields throughout the medium are obtained. The behaviour near the wave fronts was analyzed and singularities at the load were determined.

This type of work has been treated by Pal, Ghosh and Chowdhuri [1985]. They solved the problem of SH-type of elastic wave propagating in the semi-infinite medium due to a ring source producing SH-waves in presence of circular cylindrical cavity as well as circular cylindrical inclusion in the semi-infinite medium.

The diffraction of elastic waves by cracks is the most interesting branch of elastodynamics. Normally cracks are present in all structural materials, either as natural defects or as a result of fabrication processes. In many cases, the cracks are sufficiently small so that their presence does not significantly reduce the strength of the material. In other cases, however, the cracks are large enough, or they may become large enough through fatigue, stress corrosion cracking, etc., so that they must be taken into account in determining the strength. The body of knowledge which has been developed for the analysis of stresses in cracked solids is known generally as fracture mechanics. In recent years problems of diffraction of elastic waves by cracks are of considerable importance in view of their application in seismology and geophysics. Indeed in geophysical stratifications, faults occur at the interfaces while in manufactured laminates imperfections occur at the interface of the adjoining layers. This stress singularity near the edge of finite crack at the bimaterial interface is important in view of its practical application. Also the results of research on dynamic crack propagation are relevent in numerous applications. For example, a primary objective in engineering structures is to avoid a running fracture, or at least to arrest a running crack once it is initiated. Indeed there are several kinds of large engineering structures in which rapid crack growth is a definite possibility. In earth science, study of the elastic field near about the propagating fault is also important from the stand point of earthquake engineering.

Whithin the framework of a continuum model, such as the homogeneous, isotropic linearly elastic continuum, the classic analytical problem of fracture mechanics consists of the computation of the fields of stress and deformation in the vicinity of the tip of a crack, together with the application of a fracture criterion. In a conventional analysis inertia (or dynamic) effects are neglected and the analytical work is quasi-static in nature.

Because of loading conditions and material characteristics, however, there are many fracture mechanics problems which can not be viewed as being quasi-static and for which the inertia of the material must be taken into account. Also inertia effects become of importance if the propagation of the crack is so fast, as for example in essentially brittle fracture, that rapid motions are generated in the medium. The label "dynamic loading" is attached to the effects of inertia on fracture due to rapidly applied loads.

There are two broad classes of fracture mechanics problems that have to be treated as dynamic problems. These are concerned with

1. cracked bodies subjected to rapidly varying loads,
2. bodies containing rapidly propagating cracks.

In both the cases the crack tip is an environment disturbed by wave motion.

Impact and vibration problems fall into the first class of dynamic problems. In the analysis of such problems it is often found that at inhomogeneities in a body the dynamic stresses are higher than the stresses computed from the corresponding problem of static equilibrium. This effect occurs when a propagating mechanical disturbance strikes a cavity. The dynamic stress "overshoot" is especially pronounced if the cavity contains a sharp edge. For a crack the intensity of the stress field in the vicinity of the crack tip can be significantly affected by dynamic effects.

In view of the dynamic amplification, it is conceivable that there are cases for which fracture at a crack tip does not occur under a gradually applied system of loads, but where a crack does indeed propagate when the same system of loads is rapidly applied, and gives rise to wave which strike the crack tip.

The second class of problems is equally important. Indeed, there are several kinds of large engineering structure in which rapid crack growth is a definite possibility. Examples are gas transmission pipelines, ship hulls, aircraft fuselages and nueclear reactor components. Dynamic effects affect the stress fields near rapidly propagating cracks, and hence the conditions for further unstable crack propagation or for crack arrest. Another area to which the analysis of rapidly propagating cracks is relevant is the study of earthquake mechanisims.

There have been a number of comprehensive review articles in the general area of elastodynamic fracture mechanics. some of them are Achenbach [1972], Freund [1975], Achenbach [1976], Freund [1976] and Kanninen [1978].

At present, dynamic fracture mechanics solutions are largely confined to conditions where Linear Elastic Fracture Mechanics (LEFM) is valid. These are appropriate when the plastic deformation attending the crack tip is small enough to be dominated by the elastic field surrounding it. Problems of crack growth initiation under impact loads and of rapid unstable crack propagation and arrest can be treated with LEFM by using dynamically computed fields of stress and deformation. Engineering structures requring protection against the possibility of large-scale catastropic crack propagation are, however, generally constructed of ductile, tough materials. For the initiation of crack growth, LEFM procedures can give only approximately correct predictions for such materials. The elastic-plastic treatments required to give precise results have not yet been developed in a completely acceptable manner, even under static conditions.

The shapes of the cracks which have been studied uptil now are as follows :

(i) Semi-infinite plane cracks;

(ii) Finite Griffith cracks;

(iii) Penny shaped and annular cracks;

(iv) Non-planar cracks.

A transient problem involving the sudden appearance of a semi-infinite crack in a stretched elastic plate was considered by Maue [1954]. Baker [1962] studied the problem of a semi-infinite crack suddenly appearing and growing at constant velocity in a stretched elastic body. The mixed boundary value problem is solved by transform methods including the Weiner-Hopf and Cagniard techniques. Atkinson and List [1978] considered the steady state semi-infinite crack propagation into media with spatially varying elastic properties. The quasi-static problem of an infinite elastic medium weakened by an infinite number of semi-infinite, rectilinear, parallel and equally spaced cracks which are subjected to identical loads satisfying the conditions of amtiplane state of strain was solved by Matczynski [1973]. Sarkar, Ghosh and Mandal [1991] studied the problem of scattering of horizontally polarized shear wave by a semi-infinite crack running with uniform velocity along the interface of two dissimilar semi-infinite elastic media.

The powerful technique to solve the diffraction problem of semi-infinite crack is the Wiener-Hopf [Noble 1958] technique.

The in-plane problem of finite Griffith crack propagating at a constant velocity under a uniform load was first solved by Yoffe [1951]. Sih [1968] has also provided a Riemann-Hilbert formulation of the same problem where both in-plane extensional and antiplane shear loads were considered.

Other references treating elastodynamic problem involving a single finite Griffith crack are Loeber and Sih [1967]. Ang and Knopoff [1964]. Mal [1970, 1972], Chang [1971], Kassir and Bandyopadhyay [1983], Kassir and Tse [1983]. Loeber and Sih [1967] solved the problem of diffraction of antiplane shear waves by a finite crack by using integral transform method. Kassir and Bondyopadhyay [1983] considered the problem of impact response of a cracked orthotropic medium. Laplace and Fourier transforms were employed to reduce the transient problem to the solution of standard integral equation in the Laplace transform plane and was solved by Laplace inversion technique [Krylov et al, 1957]; Miller and Guy [1966].

The problems of finite Griffith crack lying at the interface of two dissimilar elastic media have been studied by Srivastava, Palaiya and Karaulia [1980], Nishida, Shindo and Atsumi [1984] and Bostrom [1987]. Bostrom [1987] used the method of Krenk and Schmidt [1982] to solve the two-dimensional scalar problem of scattering of elastic waves under antiplane strain from an interface crack between two elastic half-spaces. Sih and Chen [1980] analyzed the dynamic response of a layered composite containing a Griffith crack under normal and shear impact.

The problems of diffraction of elastic waves become more complicated when boundaries are present in the medium. Chen [1978] considered the problem of dynamic response of a central crack in a finite elastic strip. The crack was assumed to appear suddenly when the strip is being stretched at its two ends. The problem was solved by Laplace and Fourier transform technique. Some other references are Srivastava, Gupta and Palaiya [1981], Srivastava, Palaiya and Karaulia [1983], Shindo, Nozaki and Higaki [1986], De and Patra [1990].

High frequency solution of the diffraction of elastic waves by a crack of finite size is interesting in view of the fact that transient solution close to the wave front can be represented by an integral of the high frequency componant of the solution. Green's function method together with a function-theoretic technique based upon an extended Wiener-Hopf argument has been developed by Keogh [1985 a, 1985 b] for solving the problem of high frequency scattering of elastic waves by a Griffith crack situated in an infinite homogeneous elastic medium. Pal and Ghosh [1990] considered the problem of diffraction of normally incident antiplane shear waves by a crack of finite length situated at the interface of two bonded dissimilar elastic half-spaces. The problem is reduced to the solution of a Wiener-Hopf problem. The expressions for the stress intensity factor and the crack opening displacement have been derived for the case of wave-lengths short compared to the length of the crack. Recently Pal and Ghosh [1993] have investigated the high frequency solution of the problem of diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface. Following the method of Chang [1971], the problem has been formulated as an extended Wiener-Hopf equation and the asymptotic solutions for high frequencies or for wave lengths which are short compared to the length of the crack have been derived. Expressions for the dynamic stress intensity factor at the crack tip and the crack opening displacement have been derived.

Vibratory motion of a body on an elastic half-plane was treated by Karasudhi, Keer and Lee [1968]. They considered the vertical, horizontal and rocking vibrations of a body on the surface of an otherwise unloaded half-plane. The problem was formulated so that shearing stress vanishes over the entire surface, and an oscillating displacement is prescribed in the loaded region. The problems were mixed with respect to the prescribed displacement and the remaining stress. Each case led to a mixed boundary value problem represented by dual integral equations which were reduced to a single Fredholm integral equation.

Wickham [1977] studied the problem of the forced two dimensional oscillations of a rigid strip in smooth contact with a semi-infinite elastic solid. He reduced the mixed boundary value problem with the help of Green's function to Fredholm integral equation of the first kind involving displacement boundary conditions. Using Noble's [1962] method, this equation was reduced to Fredholm integral equation of the second kind with a kernel which was small in the low frequency limit. Then applying the method of iteration, a simple explicit long-wave asymptotic formula for the normal stress in terms of the prescribed displacement and dimensionless wave number K was rigorously derived.

Rocking motion of a rigid strip on a semi-infinite elastic medium has been studied by Ghosh and Ghosh [1985] by using a different technique. The forced rocking of the strip about the horizontal axis has been reduced to a solution of a dual integral equation. Following Tranter's [1968] method the dual integral equation was solved for low frequency oscillations by reducing the equation to a system of linear algebraic equations.

Studies of single Griffith crack as well as two parallel and coplanar Griffith cracks have been made by Mal [1970], Jain and Kanwal [1972] and Itou [1978, 1980 a, 1980 b]. The coresponding problems of diffraction by a single and two parallel rigid strips have been solved by Wickham [1977], Jain and Kanwal [1972] and Mandal and Ghosh [1992] respectively. And three dimensional problem of moving crack was considered by Itou [1979]. In most of the cases the problems were solved by integral equation technique.

The problem involving single Griffith crack in orthotropic medium was investigated by Kassir and Bandopadhyay [1983], Shindo et al [1986] and De and Patra [1990]. Sindo et al [1991] have investigated the impact response of symmetric edge cracks in an orthotropic strip. Mandal and Ghosh [1994] considered the problem of interaction of elastic waves with a periodic array of coplanar Griffith cracks in an orthotropic elastic medium.

Recently Mandal, Pal and Ghosh [1996 a] considered the two-dimensional problems of diffraction of elastic waves by four coplanar parallel rigid strips embedded in an infinite orthotropic medium. The five part mixed boundary value problem is reduced to the solution of a set of integral equations. The normal stress under the strips and displacement out side the strips were derived in close analytical form. In another paper, Mandal, Pal and Ghosh [1996 b] considered the vertical vibration of four rigid strips in smooth contact with a semi-infinite elastic medium. The resulting mixed boundary value problem has been reduced to the solution of quadruple integral equations, which have further been reduced to the solution of a integro-differential equations. An iterative solution valid for low frequency has been obtained. From the solution, the stress just below the strips and also the vertical displacement at points outside the trips on the free surface have been found.

In case of low frequency oscillations Noble's [1963] method of solving dual integral equations, Tranter's [1968] technique for solving dual integral equations, Matched Asymptotic Expansion, and variational principle are found to be very useful techniques.

Different techniques have been applied by many authors to tackle these type of crack problems. From these stand point, these problems may be divided into two categories : one for low frequency oscillation of the source or long wave scattering or transmission and the other for high frequency oscillation or short wave scattering or transmission in the medium. The term long and short are used in comparison to the region of the source of distrubance or the size of the crack or strip etc. inside the medium to the wave length of disturbance. The useful techniques for low frequency scattering are due to Noble [1963] and Tranter [1968]. In case of high frequency oscillations Wiener-Hopf [Noble, 1958] technique and Keller's [1958] geometrical theory are found to be most suitable.

Here we briefly discuss some of the useful methods.

GREEN'S FUNCTIONS :

The general theory of linear equations suggests two methods which can be used to solve the equation of the type

$$Lu = f \tag{1}$$

where L is an ordinary linear differential operator, f a known function, and u the unknown function.

One method is to find the operator inverse to L, that is, to find an operator L^{-1} such that the product $L^{-1}L$ is the identity operator. We shall find that the inverse of a differential operator is an integral operator. The kernel of that integral operator will be called the Green's function of the differential operator. The techniques which we shall provide for finding the Green's function use a tool which has proved valuable in many branches of applied mathematics, namely, the Dirac δ-function.

Inverse of a differential operator can be obtained, following Friedman [1966], Roach [1982], as follows:

Suppose that ψ and ϕ are testing functions and consider the equation

$$L\psi = \phi \tag{2}$$

Here we assume that the inverse operator L^{-1} is an integral operator with some kernel G(x,t) such that

$$L^{-1}\phi = \int G(x,t)\ \phi(t)\ dt. \tag{3}$$

Now we permit G(x,t) to be symbolic function. Applying the differential operator L to both sides of this equation, we get

$$LL^{-1}\phi = \phi = \int L\ G\ \phi\ dt. \tag{4}$$

This equation will be satisfied if we find g such that

$$LG = \delta(x-t), \tag{5}$$

where the differentiation is to be understood as symbolic differentiation.

To illustrate the method of inverting an operator, we consider the special case when

$$L = \frac{d^2}{dx^2}\ ;$$

then (5) becomes

$$\frac{d^2}{dx^2} G(x,t) = \delta(x-t) \tag{6}$$

This equation can be solved by straightforward integration and using the fact that the δ-function is the derivative of the Heaviside unit function and we get

$$\frac{d}{dx} G(x,t) = H(x-t) + \alpha(t) \tag{7}$$

where $\alpha(t)$ is an arbitrary function.

Integrating again, we get

$$G(x,t) = \int H(x-t)dt + x\alpha(t) + \beta(t)$$

$$= (x-t)H(x-t) + x\alpha(t) + \beta(t), \tag{8}$$

where $\beta(t)$ is another arbitrary function. It can be proved that any symbolic function which is a solution of (6) may be written in the form (8). Note that $G(x,t)$ is a continuous, piecewise, differentiable function, and note also that if $f(x)$ is an integrable function which vanishes outside a finite interval, then it is easy to show that the function

$$u(x) = \int G(x,t)\ f(t)\ dt \tag{9}$$

satisfies the differential equation

$$\frac{d^2u}{dx^2} = f(x) \qquad (10)$$

By the suitable choice of the function $\alpha(t)$ and $\beta(t)$ we can in general find a solution of (10) which satisfies two conditions. Thus, to find a solution of (10) which satisfies the conditions $u(0) = u(1) = 0$, we proceed as follows :

From (9) we have

$$u(x) = \int_{-\infty}^{x} (x-t)\, f(t)\, dt + x \int_{-\infty}^{\infty} \alpha(t)\, f(t)\, dt + \int_{-\infty}^{\infty} \beta(t)\, f(t)\, dt. \qquad (11)$$

Substituting $x = 0$ and $x = 1$ in (11) we get

$$0 = -\int_{-\infty}^{0} t\, f(t)\, dt + 0 + \int_{-\infty}^{\infty} \beta(t)\, f(t)\, dt \qquad (12)$$

$$0 = \int_{-\infty}^{1} (1-t)\, f(t)\, dt + \int_{-\infty}^{\infty} \alpha(t)\, f(t)\, dt + \int_{-\infty}^{\infty} \beta(t)\, f(t)\, dt. \qquad (13)$$

From equation (12) we get

$$\beta(t) = t\, H(-t), \tag{14}$$

and then from (13) we obtain

$$\alpha(t) = -1 + t\, H(t), \qquad -\infty \leq t \leq 1 \tag{15}$$

$$= 0\,, \qquad \text{for all other values of } t.$$

Substituting (14) and (15) in (9) we get

$$u(x) = \int_0^x (x-t)\, f(t)\, dt - x \int_0^1 (1-t)\, f(t)\, dt. \tag{16}$$

So, in this case the kernel (Green's function)

$$G(x,t) = (x-t)\, H(x-t) - x\,(1-t), \qquad 0 \leq x, \quad t \leq 1 \tag{17}$$

also satisfies the boundary conditions

$$G(0,t) = G(1,t) = 0 \tag{18}$$

The Other Method is to find the spectral representation of L by studing the solution of the equation

$$Lu = \lambda u, \tag{19}$$

where λ is an arbitrary constant.

Let L be an ordinary self-adjoint differential operator and suppose that u_1, u_2, ... are its eigenfunctions and λ_1, λ_2, ... the corresponding eigenvalues. We shall also suppose that the eigenfunctions span the domain of the given operator, and that, in consequence, any square integrable function u(x) may be expanded as

$$u(x) = \sum \alpha_k u_k(x), \tag{20}$$

where $\alpha_k = (u_k, u).$ (21)

Now, it follows that

$$Lu(x) = \sum \alpha_k \lambda_k u_k(x) \tag{22}$$

and if f(x) denotes a function which is analytic in a region containing the eigenvalues, we define

$$f(L)u(x) = \sum f(\lambda_k)\alpha_k u_k(x). \tag{23}$$

For the particular case when

$f(t) = (\lambda - t)^{-1}$ we obtain

$$\left(\frac{1}{\lambda - L}\right)u(x) = \sum \frac{\alpha_k u_k(x)}{\lambda - \lambda_k}. \tag{24}$$

The left hand side of (24) can be expressed in terms of the Green's function for the differential operator L-λ. Therefore, we put

$$w(x) = (\lambda - L)^{-1} u(x);$$

and we have $(L - \lambda)w = - u.$

If $G(x,\xi,\lambda)$ is the Green's function for the operator $L-\lambda$, we have

$$w(x) = -\int G(x,\xi,\lambda)\ u(\xi)d\xi\ , \tag{25}$$

and consequently,

$$\left(\frac{1}{\lambda - L}\right)u(x) = -\int G(x,\xi,\lambda)\ u(\xi)d\xi \tag{26}$$

Now, integrating (24) over a large circle of radius R in the complex λ-plane, we get

$$\frac{1}{2\pi i}\int \frac{u(x)}{\lambda - L}\, d\lambda = \sum \frac{1}{2\pi i}\int \frac{\alpha_k u_k(x)}{\lambda - \lambda_k}\, d\lambda\ . \tag{27}$$

Now, as the radius of the circle approaches infinity, the right-hand side of (27) includes more and more residues, and we obtain, bearing in mind that necessarily u is also a function of λ,

$$\underset{R \to \infty}{\mathrm{Lt}}\ \frac{1}{2\pi i}\int \frac{u(x)}{L - \lambda}\, d\lambda = -\sum \alpha_k u_k(x) = -\ u(x)\ . \tag{28}$$

This result, which connects the Green's function with the eigenfunctions, was obtained, by making a great many assumptions, such as that the eigenfunctions were known and that they were complete. In practice, we try to work it backwards. We start with a knowledge of the Green's function $G(x,\xi;\lambda)$ for the operator $L-\lambda$;

then we consider the following integral in the complex λ-plane;

$$\frac{1}{2\pi i}\int \frac{u(x)}{L-\lambda}\, d\lambda = \frac{1}{2\pi i}\int d\lambda \int G(x,\xi;\lambda)u(\xi)d\xi, \qquad (29)$$

and then, by evaluating it in terms of residues, we hope to get (28), that is, an expansion of u(x) in terms of the eigenfunctions of L.

CAGNIARD-DEHOOP TRANSFORMATION :

Following Pilant [1979] Cagniard-deHoop technique can better be explined taking an example. We find a solution of the inhomogeneous scalar wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2\phi}{\partial t^2} = -2\pi\delta(x)\delta(z)\delta(t)$$

$$= -\frac{\delta(r)\delta(t)}{r} \qquad (30)$$

Taking a Laplace transform with respect to time, we get

$$\frac{\partial^2\overline{\phi}}{\partial x^2} + \frac{\partial^2\overline{\phi}}{\partial z^2} - \frac{s^2}{V^2}\overline{\phi} = -2\pi\delta(x)\delta(z), \qquad (31)$$

where $\qquad \overline{\phi} = \int_0^\infty \phi(x,z,t)\, e^{-st}\, dt. \qquad (32)$

In order to simplify what is to come, we shall take a slightly modified Fourier transform with respect to x, i.e.,

$$\bar{\bar{\phi}}\,(q,z,s) = \int_{-\infty}^{\infty} \bar{\phi}(x,z,s)\, e^{-isqx/v}\, dx \ , \tag{33}$$

with the inverse

$$\bar{\phi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\bar{\phi}}(q,z,s)\, e^{isqx/v}\, d(sq/v). \tag{34}$$

This gives

$$-(sq/v)^2\, \bar{\bar{\phi}} + \partial^2 \bar{\bar{\phi}} / \partial z^2 - (s/v)^2\, \bar{\bar{\phi}} = -2\pi\delta(z) \tag{35}$$

Finally, taking a two-sided Laplace transform with respect to z, we have

$$\left\{ p^2 - (s/v)^2\,(q^2 + 1) \right\} \bar{\bar{\bar{\phi}}} = -2\pi, \tag{36}$$

where $\bar{\bar{\bar{\phi}}} = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \bar{\bar{\phi}}(q,z,s)\, e^{-pz}\, dz$

Inverting with respect to p, we have

$$\bar{\bar{\phi}} = (\pi v / s)\, e^{-(s/v)(q^2+1)^{1/2}|z|}\, (q^2+1)^{-1/2} \tag{37}$$

Inverting with respect to q, we obtain

$$\bar{\phi}(x,z,s) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-(s/v)(q^2+1)^{1/2}|z|}\,(q^2+1)^{-1/2}\,e^{isqx/v}\,dq$$

$$= K_0(sr/v) \qquad (38)$$

The expression (38) is just the integral representation of the Macdonald function $K_0(sr/v)$.

Cagniard-deHoop transformation involves the following change of variable :

$$\cos\theta\,(q^2+1)^{1/2} - iq\sin\theta = \tau = vt/r, \qquad (39)$$

where $r\cos\theta = z$, $r\sin\theta = x$, and τ is the reduced time variable as shown in Fig. 1. Note that r-θ system is not standard cylindrical co-ordinates. The inverse of this transformation is

$$q(\tau) = i\tau\sin\theta + \cos\theta\,(\tau^2-1)^{1/2}; \qquad (40)$$

Therefore

$$\frac{dq}{d\tau} = i\sin\theta + \frac{\tau\cos\theta}{(\tau^2-1)^{1/2}} = \frac{(q^2+1)^{1/2}}{(\tau^2-1)^{1/2}}, \qquad (41)$$

The last expression comes from solving for $(q^2+1)^{1/2}$ from (39) while substituting (40) for q. Taking account of the symmetry

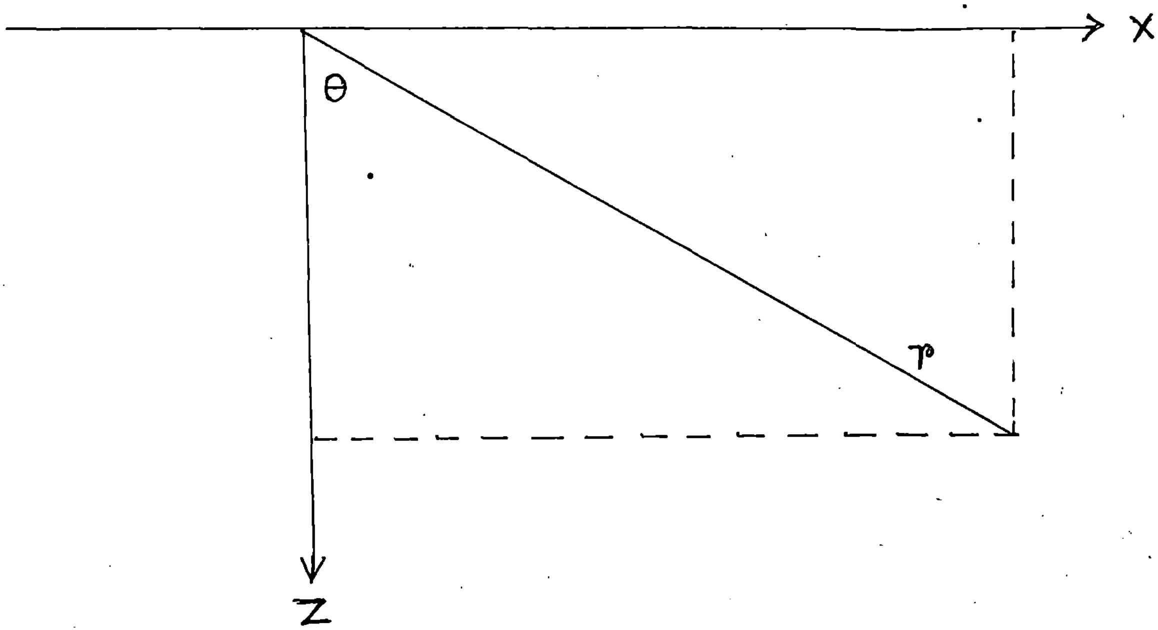

Fig. 1. Two dimensional co-ordinate systems.

of the real and imaginary parts of $\exp(isqx/v)$, we can write (38) as

$$\bar{\phi} = \mathrm{Re}\left[\int_0^{\infty} \frac{e^{-(s/v)(q^2+1)^{1/2}|z| + isqx/v}}{(q^2+1)^{1/2}} dq \right] \qquad (42)$$

we can now write this using (41) in terms of the new variable "τ" and obtain

$$\bar{\phi}(x,z,s) = \mathrm{Re}\left[\int_?^? \frac{e^{-st}}{(q^2+1)^{1/2}} \frac{dq}{dt} \frac{v}{r} dt \right] \qquad (43)$$

$$= \mathrm{Re}\left[\int_?^? \frac{e^{-st}}{(\tau^2-1)^{1/2}} \frac{v}{r} dt \right]$$

Equation (43) can now be recognized as the Laplace transform of the function

$$= \mathrm{Re}\left[\frac{1}{(\tau^2-1)^{1/2}} \frac{v}{r} \right]$$

looked at as a function of the time variable "τ". However, we have to look at a few details before we can say that this identification is valid and place proper limits on the integral. First of all, we want to look at the path q takes as we let the variable τ run from

0 to ∞ . For $\tau = 0$, we have that $q = -i\cos\theta$ where the sign has been choosen in (40) to satisfy (39). The variable q then moves up the imaginary axis to $q = i\sin\theta$, and then branches out into the first quadrant along a hyperbola as defined by (40) and along an asymptote at an angle θ as in Fig. 2(a). Inasmuch as the singularities of (42) are branch points at $q = \pm i$, we see that the original path can be deformed into the dashed line path as in Fig. 2(b). However, on the vertical segment from 0 to $i\sin\theta$ we see that the integrand of (42) has no real part. Consequently the limits on (43) may be written

$$\bar{\phi}(x,z,s) = \mathrm{Re}\left[\int_{r/v}^{\infty} \frac{e^{-st}}{(\tau^2-1)^{1/2}} \frac{v}{r}\, dt \right] \tag{44}$$

By inspection we have

$$\phi = \frac{1}{(t^2 - r^2/v^2)^{1/2}} H (t - r/v), \tag{45}$$

where H is the Heaviside Unit Step Function defined by

$$\begin{aligned} H(x) &= 1, & x > 0 \\ &= 1/2, & x = 0 \\ &= 0, & x < 0 \end{aligned} \tag{46}$$

There is a sharp wavefront associated with the response to a delta-function source, but in two dimensions we also have a tail

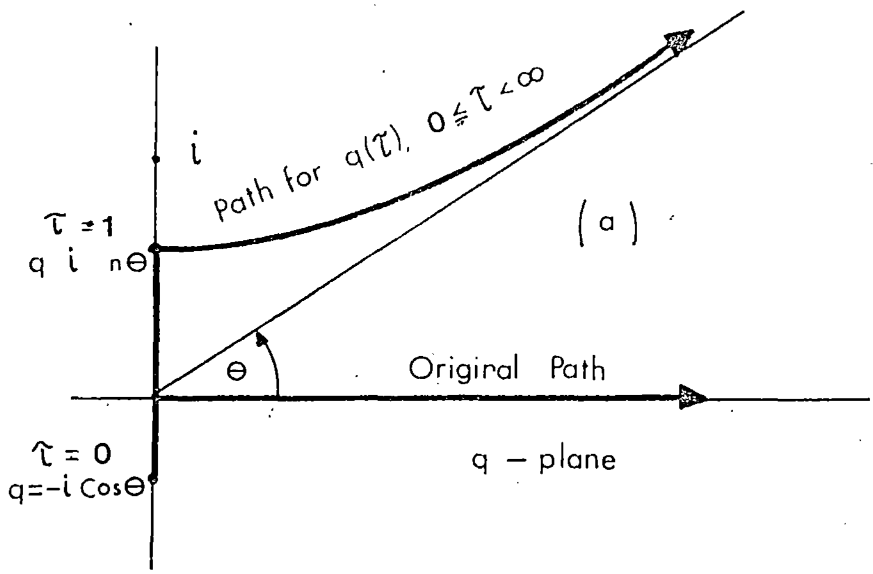

Fig. 2(a). The relationship between the original path of integration in (42) and the path which q takes as τ varies between zero and infinity.

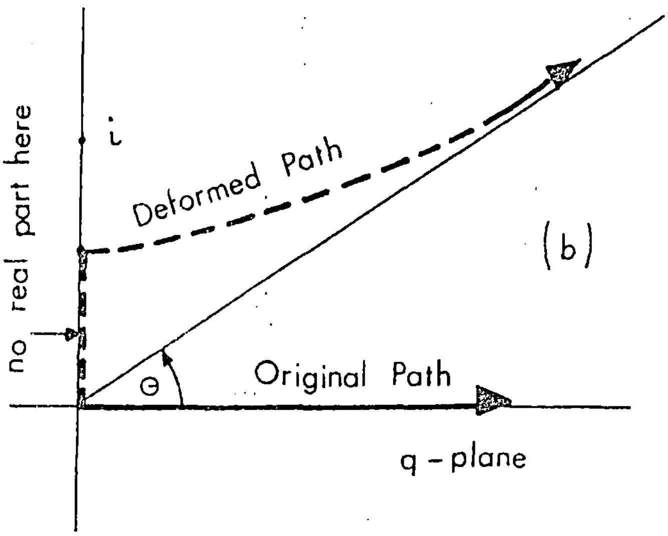

Fig. 2(b) The relationship between the original path and the deformed path (Cagniard Path) in the complex q-plane.

associated with the waveform in contrast to the delta-function which has zero width.

INTEGRAL TRANSFORM TECHNIQUE :

As the equations of motion in the theory of elasticity are partial differential equations which may be discussed with reference either to Helmholtz equation or to Laplace's equation, the method of integral transform is one of the most effective methods for solving such equations as application of this method to such equations results in the lowering of the dimension of an equation by one. There are several forms of integral transform and the choice of an integral transform depends on the structure of the equation and the geometry of the domain.

The integral transform $\bar{f}(\alpha)$ of a function f(x) defined on an interval (a,∞) is an expression of the form

$$\bar{f}(\alpha) = \int_a^\infty f(x)\, K(x,\alpha)\, dx \qquad (47)$$

where a is a real number and α is a complex parameter varying over some region D of the complex plane. $K(x,\alpha)$ is called the kernel of the transformation. The transformation (47) becomes particularly useful if it possesses inverse mapping. In that case one can express f(x) in terms of its integral transform by

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma} \bar{f}(\alpha)\; M(x,\alpha)\; d\alpha \qquad (48)$$

where $M(x,\alpha)$ is a suitable function defined in $a < x < \infty$ and $\alpha \in D$ and is called the kernel of the inverse transform, which is defined for all x in the interval (a,∞). The complex Γ is a suitable path of integration in D. After reducing the governing partial differential equation, the reduced problem can be solved for $\bar{f}(\alpha)$. The solution of the original equation can be expressed in terms of the inverse integral, which may then be evaluated. The inversion from the transformed space to the space of actual variables usually involved very complicated integrations. In many cases even the numerical integration can not be performed successfully because of the highly oscillatory character of the integrands (cf. Eringen and Suhubi [1975], chap.7; Achenbach [1975], chap.7). In particular, mixed boundary value problems like the dynamic response of a punch on an elastic half-space and the problem involving the presence of a crack or a strip inside an elastic medium may be reduced to Fredholm integral equation of first kind or to dual integral equations.

HILBERT TRANSFORM TECHNIQUE :

If $P(y) \in L_2(a,b)$, then the equation

$$\int_a^b \frac{h(x)}{x-y}\,dx = \pi\, P(y), \qquad y \in (a,b) \tag{49}$$

has the solution

$$h(x) = \frac{1}{\pi}\left[\frac{x-a}{b-x}\right]^{1/2} \int_a^b \left[\frac{b-y}{y-a}\right]^{1/2} \frac{P(y)}{x-y}\,dy + \frac{C}{\sqrt{(x-a)(b-x)}} \tag{50}$$

where C is an arbitrary constant, and the first term belongs to the class $L_2(a,b)$.

Using the above theorem, we find that the solution to the integral equation

$$\int_a^b \frac{2xh(x^2)}{x^2-y^2}\,dx = \pi P(y), \qquad y \in (a,b) \tag{51}$$

(provided that P satisfies the conditions of the above theorem) is given by

$$h(x^2) = \frac{1}{\pi}\left[\frac{x^2-a^2}{b^2-x^2}\right]^{1/2} \int_a^b \left[\frac{b^2-y^2}{y^2-a^2}\right]^{1/2} \frac{2yP(y)}{x^2-y^2}\,dy + \frac{C}{\sqrt{(x^2-a^2)(b^2-x^2)}}$$

where C is an arbitrary constant.

THE WIENER-HOPF TECHNIQUE :

Let a function $\phi(z)$ analytic in the interval $y_- < \text{Im } z < y_+$ be defined in the plane of a complex variable z. It is required to express $\phi(z)$ in the form

$$\phi(z) = \phi_+(z)\ \phi_-(z) \qquad (52)$$

where $\phi_+(z)$ and $\phi_-(z)$ are functions analytic in the half-plane Im z $> y_-$ and the half-plane Im z $< y_+$ respectively. The problem is called factorization problem. In a more general case, it is required to define two functions $\phi_+(z)$ and $\phi_-(z)$ which are analytic in the same half-planes respectively and which satisfy the following relation in the interval

$$A(z)\phi_+(z) + B(z)\phi_-(z) + C(z) = 0 \qquad (53)$$

where A(z), B(z) and C(z) are given analytic functions in the interval. It is obvious that if C(z) = 0, we obtain the representation (52) after the corresponding changes in the notation.

Let us assume that the function $\phi(z)$ which is to be factorised does not have any zeros in the interval $y_- < \text{Im } z < y_+$ and tends to infinity as $x \longrightarrow \infty$. In this case, neither of the functions $\phi_+(z)$ and $\phi_-(z)$ will have any zero, and we can take the logarithm of both sides of the relation (52)

$$\log \phi(z) = \log \phi_+(z) + \log \phi_-(z) \tag{54}$$

The function $F(z) = \log \phi(z)$ satisfies the condition

$$\left| F(x+iy) \right| < C \left|x\right|^{-P}, \quad (P > 0 \quad \text{for} \quad x \longrightarrow \infty) \tag{55}$$

and hence the relation (54) can always be solved with the help of the transformation

$$F(z) = F_+(z) + F_-(z) \tag{56}$$

Finally, we get

$$\phi(z) = e^{F_+(z)} \cdot e^{F_+(z)} = \phi_+(z)\phi_-(z) \tag{57}$$

If the function $\phi(z)$ has zeros in the intervals we must consider a new function

$$\phi_1(z) = \frac{(z^2+b^2)^{N/2} \quad \phi(z)}{\prod_{i=1}^{N_1} (z - z_i)^{\alpha_i}} \tag{58}$$

where z_i and α_i are the zeros, their multiplicity in the interval $N_1 \leq N$, where N is the total number of zeros, $b > (y_+, y_-)$. The factor in the numerator of (58) ensures that the properties of auxiliary functions are conserved at infinity.

Let us now consider the relation (53) and carry out its factorisation into L_+ and $1/L_-$ for the same interval of the ratio A/B. The relation (53) can be represented in the form

$$L_+(z)\phi_+(z) + L_-(z)\phi_-(z) + L_-(z)C(z)/B(z) = 0 \qquad (59)$$

The expression $L_-(z)C(z)/B(z)$ can be represented in the following form in accordance with (56)

$$E_+(z) + E_-(z)$$

where $\phi_+(z)$ and $\phi_-(z)$ are functions analytic in the half-plane $y > y_-$ and the half-plane $y < y_+$ respectively. Taking this into account, we get

$$L_+(z)\phi_+(z) + E_+(z) = - L_-(z)\phi_-(z) - E_-(z) \qquad (60)$$

It follows from the generalized Liouville's theorem that the left as well as right hand side of (60) represents the same polynomial $P_n(z)$ of nth degree.

Wiener-Hopf technique and different other techniques for solving partial differential equation arising in Solid Mechanics have been elaborately discussed by Duffy [1994] in his book.

The thesis presented here consists of some boundary value problems in elastodynamics involving wave propagation due to some finite source or cracks. The work has been presented in three chapters. The first chapter deals with problems on moving source on the free surface.

The problems on scattering of waves by moving interface crack have been presented in the second chapter.

The third chapter deals with the diffraction problems in elastic medium.

The summary of the thesis is presented here chapter wise.

The first problem of chapter-1 has been formulated as follows:

We have considered the problem of the SH-type of elastic wave propagation in the semi-infinite medium due to a ring source producing SH-waves in the presence of a circular cylindrical cavity and the problem of SH-wave propagation in the presence of rigid circular cylindrical inclusion in the semi-infinite medium due a ring source.

An integral representation of the Dirac delta function required for solving the above axisymmetric boundary value problem has been derived first.

In the second problem of chapter-1, an elliptic ring load emanating from the origin of co-ordinates at $t = 0$ is assumed to expand on the free-surface of an elastic half-space. The displacement at points on the free-surface has been derived in integral form by Cagniard-De Hoop technique. Displacement jumps across different wave fronts have also been derived.

In chapter-2, the problem of diffraction of normally incident antiplane shear wave by a crack of finite length situated at the interface of two bonded dissimilar elastic half-spaces has been considered in the first problem. The problem is reduced to the solution of a Wiener-Hopf equation. The expressions for the stress intensity factor (SIF) and the crack opening displacement have been derived for the case of wave length short compared to the length of the crack. The numerical results for two different pairs of samples have been presented graphically.

In the second problem of this chapter, the diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface is studied. In order to obtain a high frequency solution, the problem is formulated as an extended Wiener-Hopf problem. The expressions for the dynamic stress intansity factor at the crack tip and the crack opening displacement are derived for the case of wave lengths which are short compared to the length of the crack. The dynamic stress intensity factor for high frequencies is illustrated graphically for two pairs of different types of material for different crack velocities and angles of incidence.

In chapter-3, first paper deals with the problem of two dimansional oscillations of four rigid strips, situated on a homogeneous isotropic semi-infinite elasitic solid and forced by a specified normal component of the displacement. The mixed boundary value problem of determining the unknown stress distribution just below the strips and vertical displacement outside the strips has been converted to the determination of the solution of quadruple integral equations by the use of Fourier transform. An iterative solution of these integral equations valid for low frequency has been found by the application of the finite Hilbert transform. The normal stress just below the strips and the vertical displacement away from the strips have been obtained. Finally graphs are presented which illustrate the salient features of the displacement and stress intensity factors at the edges of the strips.

The last problem of this chapter deals with the elastodynamic response of four coplanar rigid strips embedded in an infinite orthotropic medium due to elastic waves incident normally on the strips. The resulting mixed boundary value problem has been solved by Integral Equation method. The normal stress and the vertical displacement have been derived in closed analytic form. Numerical values of stress intensity factors at the edges of the strips and the vertical displacement at points in the plane of the strips for several orthotropic materials have been calculated and plotted graphically.

With this much of introduction, we now present the thesis chapterwise. References given in the thesis do not include all the previous workers in this line. But attempt has been made to include most of them.

CHAPTER - I

RING SOURCE PROBLEMS

SPECTRAL REPRESENTATION OF A CERTAIN CLASS OF SELF-ADJOINT DIFFERENTIAL OPERATORS AND ITS APPLICATION TO AXISYMMETRIC BOUNDARY VALUE PROBLEMS IN ELASTODYNAMICS

1. INTRODUCTION

In this work an inategral representation of the Dirac delta function required for solving the axisymmetric boundary value problem has been derived first. This representation is particularly suitable for problems where mixed boundary conditions are encountered. Following Friedmann [1966], by contour integration of a suitable Green's function, integral representation of $\delta(R - R_0)$ $(R, R_0 > 1)$ has been derived. This representation has been used to solve a particular type of axisymmetric problem in elastodynamics.

The problem treated is that of a semi-infinite elastic body containing a circular cylindrical cavity, whose axis is perpendicular to the plane surface. The semi-infinite medium is subjected to an axisymmetric concentric torque applied dynamically as a step function in time at the plane surface.

At first Lamb [1904] investigated the classical normal loading problem of an elastic half-space. Similar type of problem was

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investigated by Eason [1964], Mitra [1964], Chakraborty and De [1971] and many others. They are all point source problems in a homogeneous semi-infinite medium.

The propagation of elastic waves, due to applied boundary tractions, in semi-infinite media containing internal boundaries has as yet not been studied to any large extent.

An earlier and comprehensive survey of the field is given by Scott and Miklowitz [1964]. Recently this type of work has been done by Johnson and Parnes [1977].

We have solved the problem of the SH-type of elastic wave propagation in the semi-infinite medium due to a ring source producing SH-waves in the presence of a circular cylindrical cavity (case 1). The problem of SH-wave propagation in the presence of rigid circular cylindrical inclusion in the semi-infinite medium due to the ring source has also been treated in the case 2.

2. INTEGRAL REPRESENTATION OF A DIRAC DELTA FUNCTION

Consider the operator L with λ as a complex parameter, where

$$L \equiv \frac{d}{dr}\left[r\,\frac{d}{dr} \right] + \lambda r - \frac{1}{r} \tag{1}$$

whose domain, D, is the set of all twice-differentiable functions $u(r)$, $a < r < \infty$ such that

(i) $\quad r\dfrac{du}{dr} - u = 0 \qquad \text{at } r = a > 0$

(ii) the behaviour of u as $r \longrightarrow \infty$ is that of an outgoing wave.

The solutions of $LG_1 = 0$ which satisfy (i) are

$$G_1 = A_1\left[J_1\left(\sqrt{\lambda}\,r\right)Y_2\left(\sqrt{\lambda}\,a\right) - Y_1\left(\sqrt{\lambda}\,r\right)J_2\left(\sqrt{\lambda}\,a\right)\right], \quad a<r<r_0, \qquad (2)$$

Where A_1 is an arbitrary constant and J_n and Y_n are the Bessel functions of the first and second kind, respectively.

Again the function G_2 which will satisfy $LG_2 = 0$ and the condition (ii) can be written as

$$G_2 = A_2 H_1^{(1)}\left(\sqrt{\lambda}\,r\right) \qquad (a < r_0 < r < \infty), \qquad (3)$$

where A_2 is an arbitrary constant and $H_n^{(1)}$ is the Hankel function of the first kind of order n.

From Eqs. (2) and (3) the Green's function G satisfying the equation $LG = -\delta(r - r_0)$ and the conditions (i) and (ii) mentioned above is given by (e.f. Friedmann [1966])

$$G(r,r_0;\lambda) =$$

$$= \frac{\pi H_1^{(1)}(\sqrt{\lambda} r_0)}{2H_2^{(1)}(\sqrt{\lambda} a)} \left[J_1(\sqrt{\lambda} r)Y_2(\sqrt{\lambda} a) - Y_1(\sqrt{\lambda} r)J_2(\sqrt{\lambda} a) \right] H(r_0-r) -$$

$$- \frac{\pi H_1^{(1)}(\sqrt{\lambda} r)}{2H_2^{(1)}(\sqrt{\lambda} a)} \left[J_1(\sqrt{\lambda} r_0)Y_2(\sqrt{\lambda} a) - Y_1(\sqrt{\lambda} r_0)J_2(\sqrt{\lambda} a) \right] H(r-r_0),$$

$$0 < \arg\lambda < 2\pi . \qquad (4)$$

Now consider

$$\frac{1}{2\pi i} \oint G(r,r_0;\lambda) r d\lambda, \qquad (5)$$

where the contour of integration in the λ-plane is shown in Fig. 1. Since G has a branch point at $\lambda = 0$, we introduce a branch cut in the complex λ-plane along the positive real axis and then take the contour as a large circle of radius R_1^2, having the centre at $\lambda = 0$, not crossing the branch cut. In terms of Hankel functions Eq. (4) can be written as

$$\frac{\pi}{4i}\left[H_1^{(1)}(\sqrt{\lambda} r_0)H_1^{(1)}(\sqrt{\lambda} r) \frac{H_2^{(2)}(\sqrt{\lambda} a)}{H_2^{(1)}(\sqrt{\lambda} a)} - H_1^{(1)}(\sqrt{\lambda} r_0)H_1^{(2)}(\sqrt{\lambda} r)\right] H(r_0-r) +$$

$$+ \frac{\pi}{4i}\left[H_1^{(1)}(\sqrt{\lambda} r_0)H_1^{(1)}(\sqrt{\lambda} r) \frac{H_2^{(2)}(\sqrt{\lambda} a)}{H_2^{(1)}(\sqrt{\lambda} a)} - H_1^{(1)}(\sqrt{\lambda} r)H_1^{(2)}(\sqrt{\lambda} r_0)\right] H(r-r_0).$$

(6)

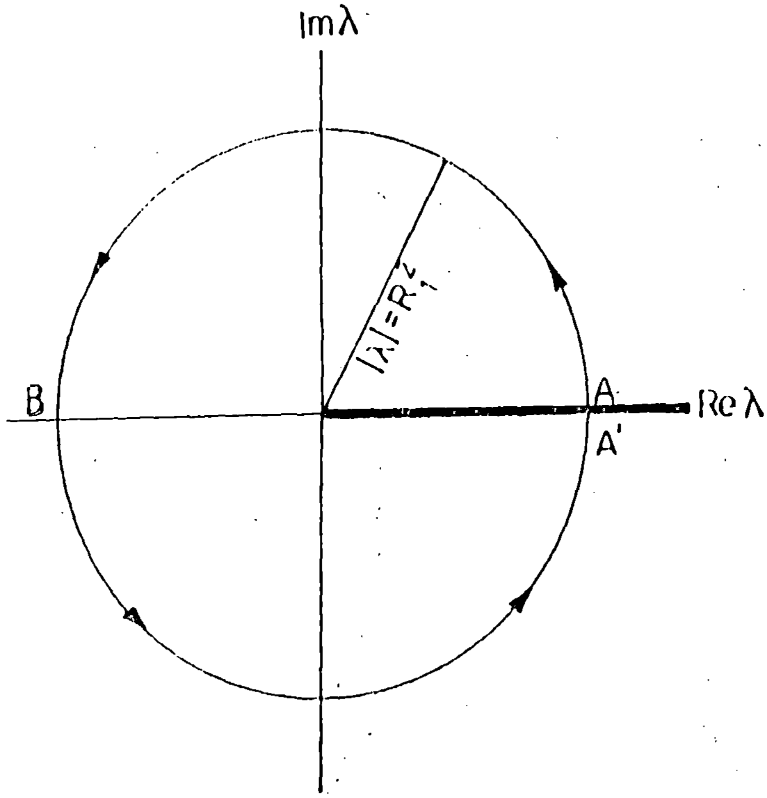

FIG. 1. Circular contour of integration ABA' in the λ-plane.

For large $|z|$, the asymptotic behaviour of $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ are (Lebedev [1965])

$$H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\ \exp\left[\ i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right],$$
$$H_n^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\ \exp\left[\ -i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]. \tag{7}$$

Thus, for large values of $|\lambda|$, from the relations (7) we obtain

$$H_1^{(1)}(\sqrt{\lambda} r_0)H_1^{(1)}(\sqrt{\lambda} r)\frac{H_2^{(2)}(\sqrt{\lambda} a)}{H_2^{(1)}(\sqrt{\lambda} a)} \sim \frac{2}{\pi\sqrt{\lambda r r_0}}\ \exp\left[i\sqrt{\lambda}(r + r_0 - 2a) + i\pi\right],$$
$$H_1^{(1)}(\sqrt{\lambda} r_0)H_1^{(2)}(\sqrt{\lambda} r) \sim \frac{2}{\pi\sqrt{\lambda r r_0}}\ \exp\left[\ i\sqrt{\lambda}\ (r_0 - r)\right], \tag{8}$$
$$H_1^{(1)}(\sqrt{\lambda} r)H_1^{(2)}(\sqrt{\lambda} r_0) \sim \frac{2}{\pi\sqrt{\lambda r r_0}}\ \exp\left[\ i\sqrt{\lambda}\ (r - r_0)\right].$$

If we put $\lambda = k^2$, then the circle in the λ-plane becomes a semi-circular arc C of radius R_1 in the upper half of the k-plane (shown in Fig.2.) Consequently, for large values of R_1 the integral

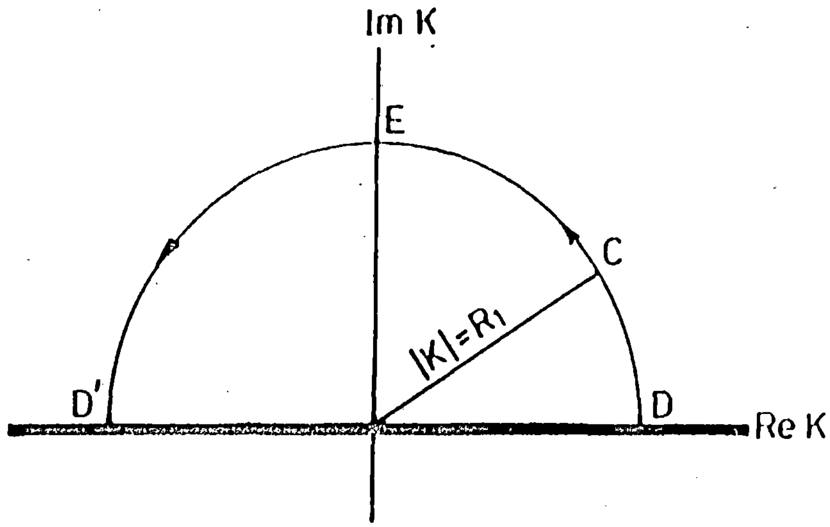

Fig. 2. DED′ - the semi-circular path of integration C in the K-plane.

(5) can be written as

$$\frac{1}{2\pi}\sqrt{\frac{r}{r_o}}\int_C\Big[\exp\{ik(r_o - r)\}H(r_o - r) + \exp\{ik(r-r_o)\}H(r-r_o)\Big]dk -$$

$$-\frac{1}{2\pi}\int_C\sqrt{\frac{r}{r_o}}\ \exp\{ik(r + r_o - 2a)\}dk$$

$$= -\frac{1}{2\pi}\sqrt{\frac{r}{r_o}}\int_{-R_1}^{R_1}\exp(ik|r-r_o|)dk +$$

$$+\frac{1}{2\pi}\sqrt{\frac{r}{r_o}}\int_{-R_1}^{R_1}\exp\{ik(r + r_o - 2a)\}dk$$

$$= -\frac{1}{\pi}\sqrt{\frac{r}{r_o}}\ \frac{\sin R_1(r-r_o)}{r - r_o} + \frac{1}{\pi}\sqrt{\frac{r}{r_o}}\ \frac{\sin R_1(r + r_o - 2a)}{r + r_o - 2a}\ . \quad (9)$$

Our object is to show that the integral (5) represents $-\delta(r-r_o)$ when $R_1 \to \infty$. To justify the statement, consider a testing function $\phi(r)$, in D which is continuous, has a continuous derivative of order two and vanishes outside a finite interval. Then, from the relations (5) and (9)

$$\lim_{R_1 \to \infty} \int_a^\infty \phi(r) \frac{1}{2\pi i} \oint G(r, r_0; \lambda) r \, d\lambda \, dr$$

$$= - \lim_{R_1 \to \infty} \frac{1}{\pi} \int_a^\infty \phi(r) \sqrt{\frac{r}{r_0}} \frac{\sin R_1 (r - r_0) dr}{(r - r_0)} +$$

$$+ \lim_{R_1 \to \infty} \frac{1}{\pi} \int_a^\infty \phi(r) \sqrt{\frac{r}{r_0}} \frac{\sin R_1 (r + r_0 - 2a) dr}{(r + r_0 - 2a)}$$

$$= - \phi(r_0),$$

where we have used the result of Dirichlet integral and Riemann-Lebesgue Lemma (Whittaker and Watson [1963]).

Therefore

$$\lim_{R_1 \to \infty} \frac{1}{2\pi i} \oint G(r, r_0; \lambda) r \, d\lambda = - \delta(r - r_0).$$

To obtain an alternative integral representation, which will be useful for our subsequent application in physical problems, we consider the contour Γ (Fig.3) consisting of the real axis from $k = \rho$ to $k = R_1$, where $0 < \rho < R_1$; a semi-circle C of radius R_1 above the real axis; the real axis again from $- R_1$ to $- \rho$; and finally a semi-circle γ of radius ρ above the real axis with the centre at the origin. We take ρ small and R_1 large.

The integrand $2G(r, r_0, k^2)\, kr$ has no singularity inside the contour Γ , and so the value of the integral

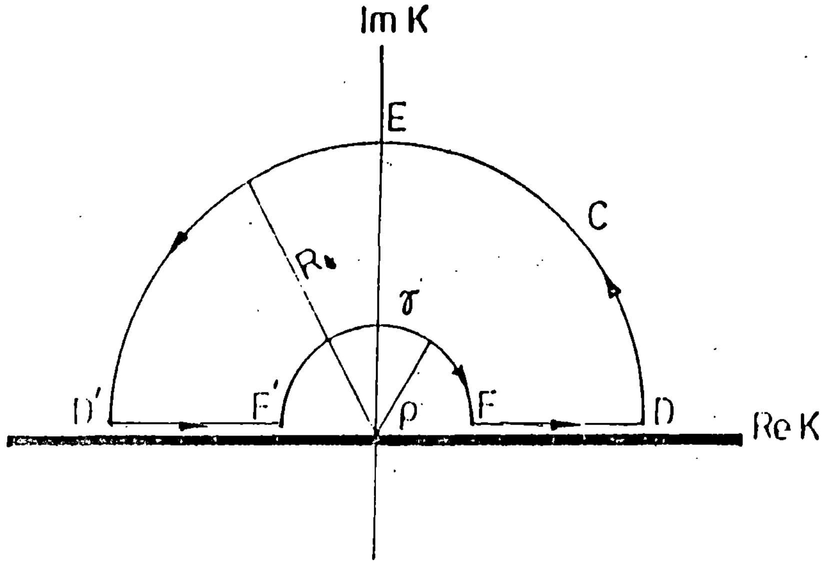

Fig. 3. FDED' F' F- thr path of integration Γ in the K-plane.

$$\frac{1}{2\pi i}\int_{\Gamma} G(r,r_0;k^2)2krdk = 0,$$

i.e.
$$\frac{1}{2\pi i}\int_{C} G(r,r_0;k^2)2krdk = -\frac{1}{2\pi i}\int_{\rho}^{R_1} G(r,r_0;u^2)2urdu +$$

$$+\frac{1}{2\pi i}\int_{\rho}^{R_1} G(r,r_0;e^{2\pi i}u^2)2rudu -$$

$$-\frac{1}{2\pi}\int_{0}^{\pi} G(r,r_0;\rho^2 e^{2i\theta})2r\rho^2 e^{2i\theta}d\theta. \qquad (10)$$

The behaviour of $Y_n(z)$ for small values of $|z|$ is described by the formula (Lebedev [1965])

$$Y_n(z) \sim -\frac{2^n\Gamma(n)}{\pi z^n}$$

and $J_n(z)$ is bounded for small values of $|z|$ when n is a positive integer. Using these results we conclude

$$|\ G(r,r_0;\rho^2 e^{2i\theta})\rho\ |$$

is bounded for small values of ρ. Hence

$$\lim_{\rho\to 0}\frac{1}{\pi}\int_{0}^{\pi} G(r,r_0;\rho^2 e^{2i\theta})e^{2i\theta}\rho^2 rd\theta = 0.$$

Letting $\rho \to 0$ and $R_1 \to \infty$ in (10), we get

$$\delta(r - r_0) = - \lim_{R_1 \to \infty} \frac{1}{2\pi i} \int_c G(r, r_0; k^2) 2krdk$$

$$= \frac{1}{2\pi i} \int_0^{\infty} \left[G(r, r_0; k^2) - G(r, r_0; k^2 e^{2i\pi}) \right] 2krdk. \quad (11)$$

From Eq. (4)

$$G(r, r_0; k^2) - G(r, r_0; k^2 e^{2i\pi}) =$$

$$= - \frac{\pi}{2} \left[\frac{J_1(kr_0) + iY_1(kr_0)}{J_2(ka) + iY_2(ka)} - \frac{J_1(kr_0) - iY_1(kr_0)}{J_2(ka) - iY_2(ka)} \right] \times$$

$$\times \left[J_1(kr) Y_2(ka) - Y_1(kr) J_2(ka) \right] H(r_0 - r) -$$

$$- \frac{\pi}{2} \left[\frac{J_1(kr) + iY_1(kr)}{J_2(ka) + iY_2(ka)} - \frac{J_1(kr) - iY_1(kr)}{J_2(ka) - iY_2(ka)} \right] \times$$

$$\times \left[J_1(kr_0) Y_2(ka) - Y_1(kr_0) J_2(ka) \right] H(r - r_0)$$

$$= i\pi \frac{\left[J_1(kr) Y_2(ka) - Y_1(kr) J_2(ka) \right] \left[J_1(kr_0) Y_2(ka) - Y_1(kr_0) J_2(ka) \right]}{J_2^2(ka) + Y_2^2(ka)}$$

Substituting this expression in Eq. (11), we get

$$\delta(r - r_0) =$$

$$= \int_0^\infty \frac{\left[J_1(kr_0)Y_2(ka) - Y_1(kr_0)J_2(ka)\right]\left[J_1(kr)Y_2(ka) - Y_1(kr)J_2(ka)\right]}{J_2^2(ka) + Y_2^2(ka)} rkdk \tag{12}$$

Substituting $r/a = R$, $r_0/a = R_0$ and $ka = \gamma$, Eq.(12) can be written as

$$\delta(R - R_0) =$$

$$= \int_0^\infty \frac{\left[J_1(\gamma R_0)Y_2(\gamma) - Y_1(\gamma R_0)J_2(\gamma)\right]\left[J_1(\gamma R)Y_2(\gamma) - Y_1(\gamma R)J_2(\gamma)\right]}{J_2^2(\gamma) + Y_2^2(\gamma)} R\gamma d\gamma \tag{13}$$

Since $\delta(R - R_0)$ is symmetric with respect to R and R_0, then, on the right hand side of Eq. (13), R and R_0 can be interchanged. So we write

$$\delta(R - R_0) =$$

$$= R_0 \int_0^\infty \frac{\gamma\left[J_1(\gamma R_0)Y_2(\gamma) - Y_1(\gamma R_0)J_2(\gamma)\right]\left[J_1(\gamma R)Y_2(\gamma) - Y_1(\gamma R)J_2(\gamma)\right]}{J_2^2(\gamma) + Y_2^2(\gamma)} d\gamma . \tag{14}$$

3. FORMULATION AND GENERAL SOLUTION (CASE - 1)

Case 1. We shall now use the integral representation of the delta function given by Eq. (13) to derive the time dependent response of an isotropic linearly elastic half-space containing a cylindrical cavity of radius a due to a ring source. The axis of the cylider considered as the z-axis, which is perpendicular to the plane surface, is directed downwards (Fig.4). A torque is applied on the free surface of the half-space over the rim of a concentric circle of radius $r = r_0$ ($r_0 > a$) for $t \geq 0$. Therefore on the cavity surface $r = a$

$$\tau_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) = 0 \tag{15}$$

and on the plane surface $z = 0$

$$\tau_{\theta z} = \mu \frac{\partial u_\theta}{\partial z} = \delta(r - r_0)H(t) \qquad (a<r<\infty,\ r_0>a), \tag{16}$$

where μ is Lame's constant, δ is the Dirac delta function and H is the unit step function.

Now the only non-zero equation of motion is

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = \frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial t^2}, \tag{17}$$

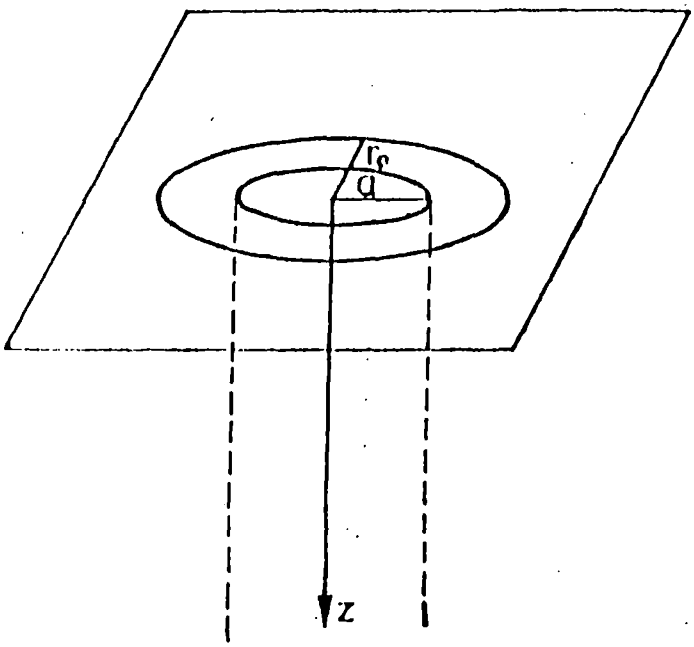

Fig. 4. Geometry of the problem.

where $\beta = \sqrt{\mu/\rho}$ is the shear wave velocity.

Changing the independent variables (r,z,t) to the no-dimensional variables (R,Z,τ) defined by

$$R = \frac{r}{a}\ ,\ Z = \frac{z}{a}\ ,\ \tau = \frac{\beta t}{a}\ ,\ \ R_o = \frac{r_o}{a} \tag{18}$$

the above equation reduces to

$$\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R}\frac{\partial u_\theta}{\partial R} + \frac{\partial^2 u_\theta}{\partial Z^2} - \frac{u_\theta}{R^2} = \frac{\partial^2 u_\theta}{\partial \tau^2} \tag{19}$$

and boundary conditions become

$$\tau_{r\theta} = \frac{\mu}{a}\left(\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right) = 0 \quad \text{on } R = 1 \tag{20}$$

and

$$\tau_{\theta z} = \frac{\mu}{a}\frac{\partial u_\theta}{\partial z} = \frac{1}{a}\,\delta(R - R_o)H(t) \quad \text{on} \quad Z = 0. \tag{21}$$

Now, taking the Laplace transform with respect to nondimensional time (τ) and assuming the homogeneous initial conditions

$$u_\theta(R,Z,0) = \frac{\partial u_\theta(R,Z,0)}{\partial t} = 0 \quad \text{at} \quad t = 0$$

Eq. (19) takes the form

$$\frac{\partial^2 \tilde{u}_\Theta}{\partial R^2} + \frac{1}{R}\frac{\partial \tilde{u}_\Theta}{\partial R} + \frac{\partial^2 \tilde{u}_\Theta}{\partial Z^2} - \frac{\tilde{u}_\Theta}{R^2} = s^2 \tilde{u}_\Theta , \tag{22}$$

where $$\tilde{u}_\Theta = \int_0^\infty u_\Theta e^{-s\tau} d\tau . \tag{23}$$

Take solution of Eq. (22) in the form

$$\tilde{u}_\Theta(R,Z,s) = \int_0^\infty \left[A_1(\gamma)J_1(\gamma R)+B_1(\gamma)Y_1(\gamma R) \right] e^{-\sqrt{s^2+\gamma^2}\, Z} d\gamma , \tag{24}$$

where γ is real, J_1 and Y_1 are Bessel functions of the first and second kind respectively.

Using the boundary condition (20), we obtain

$$B_1(\gamma) = - A_1(\gamma) \frac{J_2(\gamma)}{Y_2(\gamma)} . \tag{25}$$

Substituting the value of $B_1(\gamma)$ in Eq. (24), we have

$$\tilde{u}_\Theta(R,Z,s) = \int_0^\infty A(\gamma)\left[J_1(\gamma R)Y_2(\gamma)-J_2(\gamma)Y_1(\gamma R)\right]e^{-\sqrt{s^2+\gamma^2}\, Z} d\gamma , \tag{26}$$

where $$A(\gamma) = \frac{A_1(\gamma)}{Y_2(\gamma)} . \tag{27}$$

Therefore the transformed stress component reduces to

$$\tilde{\tau}_{\theta Z} = \frac{\mu}{a} \int_0^\infty A(\gamma)\ (\gamma^2+s^2)^{1/2}\ C_2(\gamma R)\ e^{-\sqrt{\gamma^2+s^2}\ Z}\ d\gamma, \qquad (28)$$

where $\quad C_2(\gamma R) = J_2(\gamma)\ Y_1(\gamma R) - Y_2(\gamma)\ J_1(\gamma R). \qquad (29)$

Now, using the representation (29), Eq. (14) becomes

$$\delta(R-R_0) = R_0 \int_0^\infty \frac{\gamma C_2(\gamma R)\ C_2(\gamma R_0)}{J_2^2(\gamma) + Y_2^2(\gamma)}\ d\gamma. \qquad (30)$$

Using Eqs. (21), (28) and (30), the value of $A(\gamma)$ is obtained as

$$A(\gamma) = \frac{R_0}{\mu s} \frac{\gamma C_2(\gamma R_0)}{(s^2+\gamma^2)^{1/2}\ \{J_2^2(\gamma) + Y_2^2(\gamma)\}}\ . \qquad (31)$$

Therefore $\tilde{u}_\theta$ becomes

$$\tilde{u}_\theta(R,Z,s) = -\frac{R_0}{\mu s} \int_0^\infty \frac{\gamma C_2(\gamma R)\ C_2(\gamma R_0)}{(\gamma^2+s^2)^{1/2}\ \{J_2^2(\gamma) + Y_2^2(\gamma)\}}\ e^{-\sqrt{\gamma^2+s^2}\ Z} d\gamma. \qquad (32)$$

On the plane boundary Z = 0

$$\tilde{u}_\theta(R,0,s) = -\frac{R_0}{\mu s} \int_0^\infty \frac{\gamma C_2(\gamma R)\ C_2(\gamma R_0)}{(\gamma^2+s^2)^{1/2}\ \{J_2^2(\gamma) + Y_2^2(\gamma)\}}\ d\gamma. \qquad (33)$$

Now, introducing the change of the variable $\gamma = s\zeta$ into the above expression (33), we obtain

$$\tilde{u}_\theta(R,0,s) = -\frac{R_o}{\mu}\int_0^\infty \frac{\zeta C_2(s\zeta R)\, C_2(s\zeta R_o)}{(\zeta^2+1)^{1/2}\,\{J_2^2(s\zeta) + Y_2^2(s\zeta)\}}\, d\zeta . \tag{34}$$

Next, using

$$J_n(s\zeta R) = \frac{H_n^{(1)}(s\zeta R) + H_n^{(2)}(s\zeta R)}{2} \tag{35}$$

and

$$Y_n(s\zeta R) = \frac{H_n^{(1)}(s\zeta R) - H_n^{(2)}(s\zeta R)}{2i}, \tag{35'}$$

we obtain

$$C_2(s\zeta R) = J_2(s\zeta)Y_1(s\zeta R) - Y_2(s\zeta)J_1(s\zeta R)$$

$$= \frac{1}{2i}\left[H_1^{(1)}(s\zeta R)\, H_2^{(2)}(s\zeta) - H_1^{(2)}(s\zeta R)\, H_2^{(1)}(s\zeta)\right] \tag{36}$$

and

$$C_2(s\zeta R_o) = \frac{1}{2i}\left[H_1^{(1)}(s\zeta R_o)\, H_2^{(2)}(s\zeta) - H_1^{(2)}(s\zeta R_o)\, H_2^{(1)}(s\zeta)\right]. \tag{36'}$$

Also

$$J_2^2(s\zeta) + Y_2^2(s\zeta) = H_2^{(1)}(s\zeta)\, H_2^{(2)}(s\zeta). \tag{36''}$$

Therefore, Eq.(34) becomes

$$\tilde{u}_\theta(R,0,s) = -\frac{R_o}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}}\, F(R,R_o,s\zeta)\, d\zeta , \tag{37}$$

where

$$F(R,R_0,s\zeta) = F_1(R,R_0,s\zeta) + F_2(R,R_0,s\zeta)$$

$$= F_1(R_0,R,s\zeta) + F_2(R_0,R.s\zeta)$$

$$= F(R_0,R,s\zeta) \qquad (38)$$

and

$$F_1(\alpha,\beta,s\zeta) = H_1^{(2)}(s\zeta\beta)\left\{H_1^{(1)}(s\zeta\alpha) - H_1^{(2)}(s\zeta\alpha)\,\frac{H_2^{(1)}(s\zeta)}{H_2^{(2)}(s\zeta)}\right\}, \qquad (38')$$

$$F_2(\alpha,\beta,s\zeta) = H_1^{(1)}(s\zeta\beta)\left\{H_1^{(2)}(s\zeta\alpha) - H_1^{(1)}(s\zeta\alpha)\,\frac{H_2^{(2)}(s\zeta)}{H_2^{(1)}(s\zeta)}\right\}. \qquad (38'')$$

Using the asymptotic values of the Hankel functions for a large argument, it can be shown that

$$\frac{\zeta\ F_1(R,R_0,s\zeta)}{\sqrt{(\zeta^2+1)}} \longrightarrow \frac{2}{\pi s\zeta\ \sqrt{RR_0}}\left[e^{-is\zeta(R_0-R)} + e^{-is\zeta(R+R_0-2)}\right] \qquad (39)$$

as $|s\zeta| \longrightarrow \infty$, showing that $\dfrac{\zeta\ F_1(R,R_0,s\zeta)}{\sqrt{(\zeta^2+1)}}$ vanishes over a large circular arc in the forth quadrant of the complex ζ-plane for $R < R_0$.

Also

$$\frac{\zeta\ F_2(R,R_0,s\zeta)}{\sqrt{(\zeta^2+1)}} \longrightarrow \frac{2}{\pi s\zeta\ \sqrt{RR_0}}\left[e^{is\zeta(R_0-R)} + e^{is\zeta(R+R_0-2)} \right] \tag{39'}$$

showing that $\frac{\zeta\ F_2(R,R_0,s\zeta)}{\sqrt{(\zeta^2+1)}}$ vanishes over a large circular arc in the first quadrant of the complex ζ-plane for $R<R_0$. Therefore, for $R > R_0$,

$$\frac{\zeta\ F_2(R_0,R,s\zeta)}{\sqrt{(\zeta^2+1)}} \quad \text{and} \quad \frac{\zeta\ F_1(R_0,R,s\zeta)}{\sqrt{(\zeta^2+1)}}$$

vanish over large circular arcs in the first and fourth quadrants, respectively, of the complex ζ-plane.

Denoting the responses for field points inside ($R < R_0$) and outside ($R > R_0$) the source by the subscripts I and O respectively, we have for points inside the source ($R < R_0$)

$$\tilde{u}_{\theta I}(R,0,s) = -\frac{R_0}{4\mu}\int_0^{\infty} \frac{\zeta}{\sqrt{(\zeta^2+1)}}\left[F_2(R,R_0,s\zeta) + F_1(R,R_0,s\zeta)\right] d\zeta \tag{40}$$

and for points outside the source ($R > R_0$)

$$\tilde{u}_{\theta O}(R,0,s) = -\frac{R_0}{4\mu}\int_0^{\infty} \frac{\zeta}{\sqrt{(\zeta^2+1)}}\left[F_2(R_0,R,s\zeta) + F_1(R_0,R,s\zeta)\right] d\zeta . \tag{40'}$$

In order to evaluate

$$-\frac{R_0}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_2(R,R_0,s\zeta)\, d\zeta, \qquad (41)$$

which is the first part of $\tilde{u}_{\theta I}(R,0,s)$ we note first that the integrand has branch points at $\zeta = \pm i$ and also has a branch point at the origin of coordinates due to the presence of Hankel functions in the integrand. The integrand has also poles which correspond to the zeros of $H_2^{(1)}(s\zeta)$. From Eq. (32) we note that in order that $\tilde{u}_\theta(R,Z,s)$ may be finite for large positive values of Z, $(\zeta^2+1)^{1/2}$ should have a positive real part on the path of integration. Accordingly, we draw cuts parallel to the real axis from +i to $-\infty+i$ and from -i to $\infty-i$ to satisfy our requirement. A cut along the negative real axis from the origin is also drawn to make Hankel functions single valued

$$-\frac{R_0}{4\mu}\frac{\zeta}{\sqrt{(\zeta^2+1)}} F_2(R,R_0,s\zeta)$$

is now integrated along the quadrant of a large circle lying in the first quadrant of the complex ζ-plane as shown in Fig. 5a. Since poles of the integrand are out side the path of integration, the integral (41) becomes

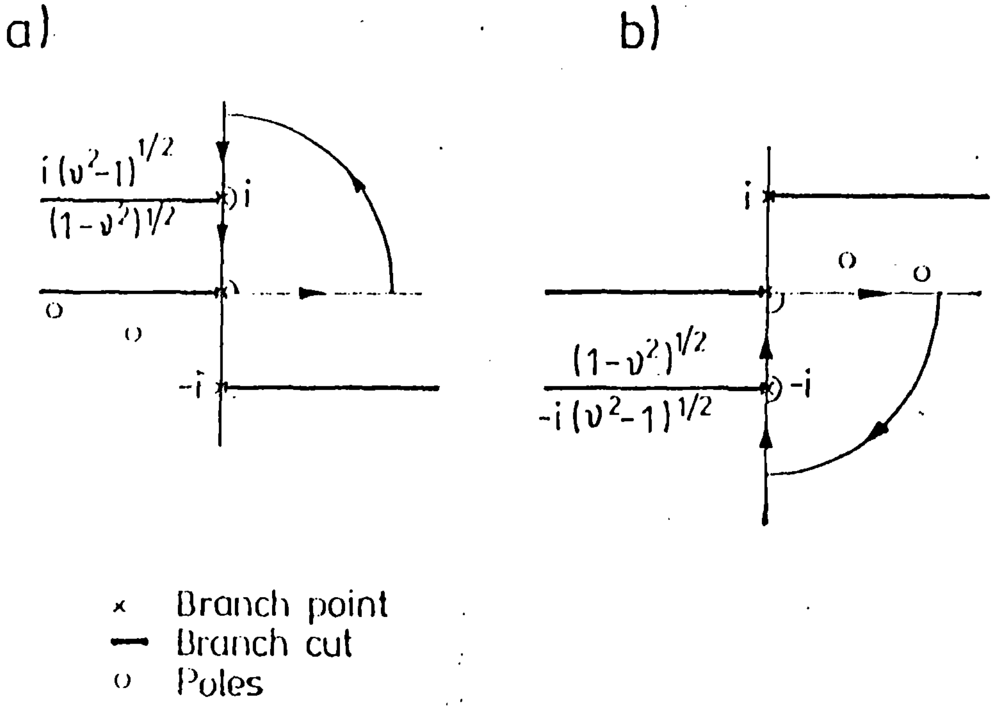

Fig. 5. Integration paths in the complex ζ-plane.

$$\frac{R_0}{4\mu}\left[\int_0^1 \frac{v}{\sqrt{(1-v^2)}} F_2(R,R_0,isv)dv + \right.$$

$$\left. + \int_1^{\infty} \frac{v}{i\sqrt{(v^2-1)}} F_2(R,R_0,isv)dv \right] . \qquad (42)$$

Using the relations

$$H_1^{(1)}(iv) = -\frac{2}{\pi} K_1(v),$$

$$H_1^{(2)}(iv) = \frac{2}{\pi} K_1(v) + 2iI_1(v),$$

$$H_2^{(1)}(iv) = \frac{2i}{\pi} K_2(v), \qquad (43)$$

$$H_2^{(2)}(iv) = -2I_2(v) - \frac{2i}{\pi} K_2(v),$$

we have

$$F_2(R,R_0,isv) = -\frac{4i}{\pi} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}. \qquad (44)$$

Therfore, the expression (42) becomes

$$-\frac{iR_0}{\mu\pi}\int_0^1 \frac{v}{\sqrt{(1-v^2)}} K_1(svR_0)\left\{ I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv \; -$$

$$-\frac{R_0}{\mu\pi}\int_1^{\infty} \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{ I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv. \qquad (45)$$

The second part of $\tilde{u}_{\theta I}(R,0,s)$ is equal to

$$-\frac{R_0}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_1(R,R_0,s\zeta)d\zeta \qquad (46)$$

we draw cuts from +i to ∞+i and from -i to -∞-i as shown in Fig. (5b). A cut from the origin along the negative real axis is also drawn to make Hankel functions single valued.

Taking a quadrant of a large circular contour in the fourth quadrant (Fig. (5b)) and noting that the poles of $F_1(R,R_0,s\zeta)$ lie outside the contour, the integral (46) takes the form

$$\frac{R_0}{4\mu}\left[\int_0^1 \frac{v}{\sqrt{(1-v^2)}} F_1(R,R_0,-isv)dv - \right.$$

$$\left. - \int_1^\infty \frac{v}{i\sqrt{(v^2-1)}} F_1(R,R_0,-isv)dv \right]. \qquad (47)$$

Using the relations

$$H_1^{(1)}(-iv) = \frac{2}{\pi} K_1(v) - 2iI_1(v),$$

$$H_1^{(2)}(-iv) = -\frac{2}{\pi} K_1(v),$$

$$H_2^{(1)}(-iv) = -\ 2I_2(v) + \frac{2i}{\pi} K_2(v),$$

$$H_2^{(2)}(-iv) = -\frac{2i}{\pi} K_2(v), \qquad (48)$$

the expression (47) becomes

$$\frac{iR_0}{\mu\pi} \int_0^1 \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{ I_1(svR) + K_1(svR) \frac{I_2(sv)}{K_2(sv)} \right\}dv\ -$$

$$-\ \frac{R_0}{\mu\pi} \int_1^{\infty} \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0) \left\{ I_1(svR) + K_1(svR) \frac{I_2(sv)}{K_2(sv)} \right\}dv. \qquad (49)$$

Adding the relations (45) and (49), we obtain

$$\tilde{u}_{eI}(R,0,s) = -\ \frac{2R_0}{\mu\pi} \int_1^{\infty} \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\ \times$$

$$\times \left\{I_1(svR)+K_1(svR)\frac{I_2(sv)}{K_2(sv)} \right\}dv \qquad (50)$$

Similarly, it can be shown that

$$\tilde{u}_{eO}(R,0,s) = -\ \frac{2R_0}{\mu\pi} \int_1^{\infty} \frac{v}{\sqrt{(v^2-1)}} K_1(svR)\ \times$$

$$\times \left\{ I_1(svR_0) + K_1(svR_0) \frac{I_2(sv)}{K_2(sv)} \right\}dv. \qquad (50')$$

Laplace inversion of the relations (50) is now taken to obtain the displacement of points inside the source. Therefore

$$u_{\theta I}(R,0,\tau) = -\frac{1}{2\pi i}\frac{2R_0}{\mu\pi}\int_{Br} e^{\tau s}ds \int_1^{\infty}\frac{v}{\sqrt{(v^2-1)}}\tilde{E}(sv)dv, \tag{51}$$

where

$$\tilde{E}(sv) = K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}. \tag{52}$$

Introducing the change of variable $p = sv$, and changing the order of integration

$$u_{\theta I}(R,0,\tau) = -\frac{2R_0}{\mu\pi}\int_1^{\infty}\frac{1}{\sqrt{(v^2-1)}}dv\left[\frac{1}{2\pi i}\int_{Br} e^{(\tau/v)p}\,\tilde{E}(p)\,dp\right]$$

$$= -\frac{2R_0}{\mu\pi}\int_1^{\infty}\frac{1}{\sqrt{(v^2-1)}}E(\tau/v)\,dv, \tag{53}$$

where $$E(\tau/v) = \mathcal{L}^{-1}\{\tilde{E}(p)\}.$$

We note that $\tilde{E}(p)$ possesses no poles and is analytic for $p > 0$. It has a branch point at the origin and therefore a cut is drawn from the origin along the negative real axis of the complex p-plane in order to make $\tilde{E}(p)$ single valued.

Drawing a large semi-circular contour to the right of the

Bromwich path AB in the complex p-plane, we conclude that $E(\tau/v) = 0$ if the integral

$$\frac{1}{2\pi i} \int_{BC'A} \tilde{E}(p)\, e^{(\tau/v)p}\, dp = 0$$

over the semi-circular arc BC'A (Fig. 6).

Now

$$E(p) = -\frac{1}{2\pi i} \int_{BC'A} \tilde{E}(p) e^{(\tau/v)p}\, dp$$

$$= -\frac{1}{2\pi i} \int_{BC'A} K_1(pR_0) I_1(pR) e^{(\tau/v)p} dp -$$

$$- \frac{1}{2\pi i} \int_{BC'A} K_1(pR_0) K_1(pR) \frac{I_2(p)}{K_2(p)} e^{(\tau/v)p}\, dp. \qquad (54)$$

Since

$$e^{(\tau/v)p} K_1(pR_0) I_1(pR) \sim \frac{1}{2p\sqrt{RR_0}} e^{\left[\frac{\tau}{v} - (R_0 - R)\right] p}$$

and

$$e^{(\tau/v)p} K_1(pR_0) I_1(pR) \frac{I_2(p)}{K_2(p)} \sim \frac{1}{2p\sqrt{RR_0}} e^{\left[\frac{\tau}{v} - (R+R_0-2)\right] p} \quad \text{as } |p| \to \infty$$

then the first integral on the right hand side of Eq.(54) vanishes for $0 < \tau/v < (R_0 - R)$, whereas the second integral vanishes for $0 < \tau/v < (R + R_0 - 2)$.

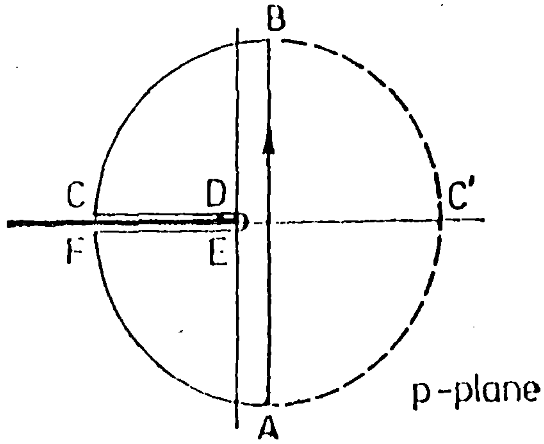

Fig. 6. Laplace inversion contour.

Therefore

$$E(\tau/v) = \begin{cases} 0, & \text{for } 0 < \tau/v < (R_0 - R), \\ E^D(\tau/v), & \text{for } (R_0 - R) < \tau/v < (R + R_0 - 2), \\ E^R(\tau/v), & \text{for } (R + R_0 - 2) < \tau/v. \end{cases} \qquad (55)$$

where

$$E^D(\tau/v) = \mathscr{L}^{-1} [\, K_1(pR_0)\, I_1(pR) \,],$$

$$(56)$$

$$E^R(\tau/v) = \mathscr{L}^{-1} \left[K_1(pR_0)\, I_1(pR) + K_1(pR_0)\, K_1(pR) \frac{I_2(p)}{K_2(p)} \right].$$

For value of τ/v lying in the range $(R_0 - R) < \tau/v < (R + R_0 - 2)$

$$E(\tau/v) = E^D(\tau/v) = \frac{1}{2\pi i} \int_{Br} K_1(pR_0)\, I_1(pR)\, e^{(\tau/v)p}\, dp. \qquad (57)$$

Therefore, putting $\tau/v = (R_0 - R + y)$, where $y > 0$

$$E^D(R_0 - R + y) = \frac{1}{2\pi i} \int_{Br} \left[K_1(pR_0)\, e^{pR_0} \right] \left[I_1(pR)\, e^{-pR} \right] e^{yp}\, dp.$$

From the Laplace inversion table Erdelyi [1954], we find that

$$\mathscr{L}^{-1} \left[K_1(pR_0)\, e^{pR_0} \right] = \frac{H(y)\,(y+R_0)}{R_0 \{y(y+2R_0\}^{1/2}} ,$$

and

$$\mathscr{L}^{-1}\left[I_1(pR)\, e^{-pR} \right] = \frac{[H(y) - H(y-2R)]\,(R-y)}{\pi R\, \{y(2R - y)\}^{1/2}}.$$

So by the convolution theorem

$$E^D(R_0-R+y) = \int_0^y \frac{[H(\eta)-H(\eta-2R)]\; H(y-\eta)(R-\eta)(y-\eta+R_0)}{\pi RR_0\;\; [\eta(2R-\eta)(y-\eta)(y-\eta+2R_0)]^{1/2}}\, d\eta. \qquad (58)$$

For τ/v lying in the range $(R_0-R) < \tau/v < (R+R_0-2)$, τ/v must be less than $(R+R_0)$, i.e. $y < 2R$.

Therefore we can write

$$E^D(R_0-R+y) = \int_0^y \frac{(R-\eta)(y-\eta+R_0)}{\pi RR_0\;\; [\eta(2R-\eta)(y-\eta)(y-\eta+2R_0)]^{1/2}}\, d\eta.$$

So

$$E(\tau/v) = E^D(\tau/v) =$$

$$= \int_0^{\tau/v-(R_0-R)} \frac{(R-\eta)(\tau/v+R-\eta)\;\, d\eta}{\pi RR_0\,[\eta(2R-\eta)(\tau/v-R_0+R-\eta)(\tau/v+R_0+R-\eta)]^{1/2}} \qquad (59)$$

for $(R_0-R) < (\tau/v) < (R+R_0-2)$.

For values of τ/v satisfying the condition $\tau/v > R+R_0-2$,

$$E(\tau/v) = E^{R}(\tau/v) =$$

$$= \frac{1}{2\pi i} \int_{Br} \left[K_1(pR_o)\ I_1(pR) + K_1(pR_o)\ K_1(pR) \frac{I_2(p)}{K_2(p)} \right] e^{(\tau/v)p}\ dp. \quad (60)$$

This integral is equal to the integral along the large semi-circular arc on the left side of the Bromwich path AB plus the integral on the two sides of the branch cut (Fig.6). Since the integral on the large semi-circular arc vanishes, then Eq. (60) becomes

$$E(\tau/v) = \frac{1}{2\pi i} \left[-\int_0^{\infty} \tilde{E}(\eta e^{i\pi}) e^{-(\tau/v)\eta} d\eta + \right.$$

$$\left. + \int_0^{\infty} \tilde{E}(\eta e^{-i\pi}) e^{-(\tau/v)\eta} d\eta \right]. \quad (61)$$

Using the relations

$$I_{\nu}(\eta e^{\pm i\pi}) = e^{\pm i\nu\pi}\ I_{\nu}(\eta),$$

and

$$K_{\nu}(\eta e^{\pm i\pi}) = e^{\mp i\nu\pi}\ K_{\nu}(\eta) \pm i\pi\ I_{\nu}(\eta),$$

we obtain (for $\tau/v > R+R_o-2$)

$$E(\tau/v) = E^{R}(\tau/v) = -\int_{0}^{\infty} \frac{U_2(R,\eta)\, U_2(R_0,\eta)\, e^{-(\tau/v)\eta}}{K_2^2(\eta) + \pi^2 I_2^2(\eta)}\, d\eta, \qquad (62)$$

where $U_2(x,\eta) = K_2(\eta)\, I_1(x,\eta) + I_2(\eta)\, K_1(x,\eta)$.

Substituting these values of $E(\tau/v)$ in Eq. (53), we obtain

$$u_{\theta I}(R,0,\tau) =$$

$$= -\frac{2R_0}{\mu\pi}\left[\left\{H\left(t - \frac{r_0 - r}{\beta}\right) - H\left(t - \frac{r + r_0 - 2a}{\beta}\right)\right\} \int_{1}^{\frac{\tau}{R_0 - R}} \frac{1}{\sqrt{v^2 - 1}}\, E^{D}(\tau/v)\, dv \; + \right.$$

$$\left. + H\left(t - \frac{r + r_0 - 2a}{\beta}\right)\left\{ \int_{\frac{\tau}{R + R_0 - 2}}^{\frac{\tau}{R_0 - R}} \frac{1}{\sqrt{v^2 - 1}}\, E^{D}(\tau/v) dv + \int_{1}^{\frac{\tau}{R + R_0 - 2}} \frac{1}{\sqrt{v^2 - 1}}\, E^{R}(\tau/v) dv \right\}\right], \qquad (63)$$

where the values of $E^{D}(\tau/v)$ and $E^{R}(\tau/v)$ are given in Eqs. (59) and (62), respectively.

Similarly, taking the inverse Laplace transform of Eq. (40′), the displacement $u_{\theta O}(R,0,\tau)$ on the free surface outside the ring source can be derived and it is found that

$$u_{\theta 0}(R,0,\tau) =$$

$$= -\frac{2R_0}{\mu\pi}\left[\left\{H\left(t-\frac{r-r_0}{\beta}\right)-H\left(t-\frac{r+r_0-2a}{\beta}\right)\right\}\int_1^{\frac{\tau}{R-R_0}}\frac{1}{\sqrt{v^2-1}}F^D(\tau/v)\,dv\,+\right.$$

$$\left.+\,H\left(t-\frac{r+r_0-2a}{\beta}\right)\left\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R-R_0}}\frac{1}{\sqrt{v^2-1}}F^D(\tau/v)dv+\int_1^{\frac{\tau}{R+R_0-2}}\frac{1}{\sqrt{v^2-1}}F^R(\tau/v)dv\right\}\right],$$

(63')

where $F^R(\tau/v) = E^R(\tau/v)$, and

$$F^D(\tau/v) = \int_0^{\tau/v-(R-R_0)}\frac{(R_0-\eta)(\tau/v+R_0-\eta)d\eta}{\pi RR_0[\eta(2R_0-\eta)(\tau/v-R+R_0-\eta)(\tau/v+R+R_0-\eta)]^{1/2}}\,.$$

(64)

First, the integrals of Eq. (63) are the displacements due to a direct wave from the ring source before the arrival of the waves reflected from the wall of the cylindrical cavity. The last two integrals together give the displacement after the arrival of the reflected wave.

In order to obtain the response in the vicinity of the SH-wave front, we consider the displacement profile immediately behind the direct outgoing SH-wave. Accordingly, we shall have to study the first integral of Eq. (63′) because it gives the response of the direct SH-wave before the arrival of the reflected wave front.

Let $R_s = R_0 + \tau$ and $R_s^- = R_s - \varepsilon R_0$ where R_s and R_s^- denote points at and immediately behind the SH-wave front, respectively, ε is a small positive quantity.

Then

$$\frac{\tau}{R_s - R_0} = 1 \tag{65}$$

and

$$\frac{\tau}{R_s^- - R_0} = \left[1 + \frac{\varepsilon R_0}{\tau} \right] = q(\tau). \text{ (say)} \tag{65′}$$

Substituting these values in the first integral of Eq. (63′), we obtain

$$u_{\theta 0}(R_s, 0, \tau) = 0,$$

and

$$u_{\theta 0}(R_s^-, 0, \tau) = -\frac{2R_0}{\mu\pi} \int_1^{q(\tau)} \frac{1}{\sqrt{(v-1)}} \left\{ \frac{1}{\sqrt{v+1}} F^D(R^-, R_0, \tau/v) \right\} dv.$$

Therefore, we can write

$$u_{\theta 0}(R_s^-, 0, \tau) = -\frac{2R_0}{\mu\,\pi} \int_1^{q(\tau)} \frac{1}{\sqrt{v-1}} V(v)dv, \tag{66}$$

where V(v) is analytic portion of the integrand. For small value of

ε expanding V(v) by the Taylor's series about the point v = 1 and integrating term by term, we obtain

$$u_{\theta 0}(R_s^-, 0, \tau) \simeq -\frac{4R}{\mu\pi} V(1) \left[\frac{R_0}{\tau}\right]^{1/2} \varepsilon^{1/2} = A\,\varepsilon^{1/2} \text{ (say)}, \qquad (67)$$

where A is a constant.

It therefore follows that the displacement component is continuous i.e. there is no jump in displacement across the direct SH-wave front.

Next, in order to consider the behaviour of response just under the ring source, it should be remembered that the integral representations of transformed displacements given by Eqs. (50) were derived from Eqs. (40) assuming that $R \neq R_0$. For $R = R_0$ the integrals along large quarter circles in the first and fourth quadrants should be reexamined. In this case it is found that though the contributions from the integrals along large circular arcs in the first and fourth quadrants are not separately zero, but the combined sum of the integrals along the large arcs in the first and fourth quadrants of the ζ-plane (Fig. 5a and 5b) vanishes. So the transformed displacements for $R = R_0$ are also given by Eqs. (50). Making $R \rightarrow R_0\pm$, it can easily be shown by help of Eqs. (50) that the displacement has no jump discontinuity across the ring source.

Therefore, in order to derive the nature of the displacement as $R \to R_0$, any one of the relations (63) may be studied. Consider, for example, the displacement at field points outside the source given by (63′). As $R \to R_0$, the upper limit of integration $\tau/(R-R_0) \to \infty$.

Further, as

$$v \to \frac{\tau}{R-R_0} \to \infty,$$

$$\frac{1}{\sqrt{v^2-1}} \to \frac{1}{v} \tag{68}$$

and

$$F^D(\tau/v) \to \frac{1}{2R_0}\,. \tag{68'}$$

Thus, from Eq. (63′)

$$\lim_{R \to R_0} u_{\theta 0}(R,0,\tau) = -\frac{2R_0}{\mu\,\pi}\int_N^{\frac{\tau}{R-R_0}} \frac{1}{v}\,\frac{1}{2R_0}\,dv + \tag{69}$$

+ a finite quantity, where N is large.

The integral is found to contribute a logarithmic singularity to the displacement just on the ring source.

4. FORMULATION AND GENERAL SOLUTION (CASE - 2)

Case. 2. In this case the problem considered is the same in all respects with the first, except that the cavity of the radius a has been replaced by a rigid cylindrical inclusion of the same radius. The cylindrical inclusion being in welded contact with the elastic half-space, there is no relative displacement at the interface. In this case, the condition on the cylindrical boundary is $u_\theta=0$ on r = a. In order to solve this problem, we take the solution in this form:

$$\tilde{u}_\theta(R,Z,s) =$$

$$= \int_0^\infty \left[A_2(\gamma)J_1(\gamma R) + B_2(\gamma)Y_1(\gamma R) \right] e^{-\sqrt{\gamma^2+s^2}\, Z}\, d\gamma, \qquad (70)$$

where $\tilde{u}_\theta(R,Z,s)$ is the Laplace transform of $u_\theta(R,Z,t)$ with respect to t. Now, using the boundary condition

$$\tilde{u}_\theta = 0 \quad \text{on} \quad R = 1,$$

we have

$$B_2(\gamma) = - A_2(\gamma) \frac{J_1(\gamma)}{Y_1(\gamma)} \qquad (71)$$

so $\tilde{u}_\theta$ becomes

$$\tilde{u}_\theta(R,Z,s) =$$

$$= \int_0^\infty A^1(\gamma)\left[J_1(\gamma R)Y_1(\gamma) - J_1(\gamma)Y_1(\gamma R)\right]e^{-\sqrt{\gamma^2+s^2}\,Z}\,d\gamma, \qquad (72)$$

where $A^1(\gamma) = \dfrac{A_2(\gamma)}{Y_1(\gamma)}$.

Therefore, the transformed stress component on the free surface Z = 0 is

$$\tilde{\tau}_{\theta Z}(R,0,s) = -\frac{\mu}{a}\int_0^\infty A^1(\gamma)\sqrt{\gamma^2+s^2}\;C_1(\gamma R)\,d\gamma, \qquad (73)$$

where

$$C_1(\gamma R) = J_1(\gamma R)Y_1(\gamma) - J_1(\gamma)Y_1(\gamma R), \qquad (74)$$

$\tilde{\tau}_{\theta Z}(R,0,s)$ should be equal to $\frac{1}{as}\delta(R - R_0)$. In this case, the required integral representation of the delta function can be obtained from the following expansion formula given by Titchmarsh [1962]:

$$f(r) = \int_0^\infty \frac{\zeta\;[J_1(\zeta r)Y_1(\zeta a) - J_1(\zeta a)Y_1(\zeta r)]}{J_1^2(\zeta a) + Y_1^2(\zeta a)}\,d\zeta \quad \times$$

$$\times \int_a^\infty \xi f(\xi)\;[J_1(\zeta\xi)Y_1(\zeta a) - J_1(\zeta a)Y_1(\zeta\xi)]\,d\xi, \qquad (75)$$

where f(r) is a suitably restricted arbitrary function.

Putting $f(r) = \delta(r-r_0)$,

$f(\xi) = \delta(\xi - r_0)$, where $r_0 > a > 0$,

we get

$$\delta(r-r_0) =$$

$$= r_0\int_0^{\infty} \frac{\zeta\,[J_1(\zeta r)Y_1(\zeta a)-J_1(\zeta a)Y_1(\zeta r)][J_1(\zeta r_0)Y_1(\zeta a)-J_1(\zeta a)Y_1(\zeta r_0)]}{J_1^2(\zeta a) + Y_1^2(\zeta a)}\,d\zeta . \tag{76}$$

Now putting, $\frac{r}{a} = R$, $\frac{r_0}{a} = R_0$, $\zeta a = \gamma$, we have

$$\delta(R-R_0) =$$

$$= R_0\int_0^{\infty} \frac{\gamma\,[J_1(\gamma R)Y_1(\gamma)-J_1(\gamma)Y_1(\gamma R)][J_1(\gamma R_0)Y_1(\gamma)-J_1(\gamma)Y_1(\gamma R_0)]}{J_1^2(\gamma) + Y_1^2(\gamma)}\,d\gamma ,$$

so by the relation (74)

$$\delta(R-R_0) = R_0\int_0^{\infty} \frac{\gamma\, C_1(\gamma R)\, C_1(\gamma R_0)}{J_1^2(\gamma) + Y_1^2(\gamma)}\,d\gamma . \tag{77}$$

This result can also be obtained by the following technique already developed in Section-2 of this paper.

Now, we find the value of $A^1(\gamma)$ as

$$A^1(\gamma) = \frac{R_0}{\mu s} \frac{\gamma\ C_1(\gamma R_0)}{\sqrt{\gamma^2+s^2}} \frac{1}{J_1^2(\gamma)+Y_1^2(\gamma)} . \tag{78}$$

Therefore $\tilde{u}_\theta$ becomes

$$\tilde{u}_\theta(R,0,s) = \frac{R_0}{\mu s} \int_0^\infty \frac{\gamma\ C_1(\gamma R)\ C_1(\gamma R_0)}{\sqrt{\gamma^2+s^2}\ \{J_1^2(\gamma) + Y_1^2(\gamma)\}} d\gamma . \tag{79}$$

Carrying on a similar procedure as followed to obtain the displacement in the case 1, we find that in this case

$$u_{\theta I}(R,0,\tau) =$$

$$= \frac{2R_0}{\mu\pi}\left[\left\{H\left(t - \frac{r_0-r}{\beta}\right) - H\left(t - \frac{r+r_0-2a}{\beta}\right)\right\} \int_1^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}} E^D(\tau/v)\ dv\ +\right.$$

$$\left. + H\left(t - \frac{r+r_0-2a}{\beta}\right)\left\{ \int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}} E^D(\tau/v)dv + \int_1^{\frac{\tau}{R+R_0-2}} \frac{1}{\sqrt{v^2-1}} E_1^R(\tau/v)dv\right\}\right] \tag{80}$$

and

$$u_{\theta O}(R,0,\tau) =$$

$$= \frac{2R_o}{\mu\pi}\left[\left\{H\left(t - \frac{r-r_o}{\beta}\right) - H\left(t - \frac{r+r_o-2a}{\beta}\right)\right\}\int_1^{\frac{\tau}{R-R_o}} \frac{1}{\sqrt{v^2-1}} F^D(\tau/v)dv + \right.$$

$$\left. + H\left(t - \frac{r+r_o-2a}{\beta}\right)\left\{\int_{\frac{\tau}{R+R_o-2}}^{\frac{\tau}{R-R_o}} \frac{1}{\sqrt{v^2-1}} F^D(\tau/v)dv + \int_1^{\frac{\tau}{R-R_o+2}} \frac{1}{\sqrt{v^2-1}} F_1^R(\tau/v)dv\right\}\right], \tag{81}$$

where $E^D(\tau/v)$ and $F^D(\tau/v)$ are respectively given by Eq. (59) and (64) and

$$E_1^R(\tau/v) = F_1^R(\tau/v) = -\int_0^\infty \frac{U_1(R,\eta)\ U_1(R_o,\eta)\ e^{-(\tau/v)\eta}}{K_1^2(\eta) + \pi^2 I_1^2(\eta)} d\eta \tag{82}$$

where

$$U_1(x,\eta) = K_1(\eta)\ I_1(x\eta) - I_1(\eta)\ K_1(x\eta). \tag{83}$$

WAVES IN A SEMI-INFINITE ELASTIC MEDIUM DUE TO AN EXPANDING ELLIPTIC RING SOURCE ON THE FREE SURFACE

1. INTRODUCTION

Since Lamb's original study of the elasitc wave produced by a time-dependent point force acting normally to the surface of an elastic half-space, many authors have elaborated on his work. Aggarwal and Abolw [1967] discussed the exact solution of a class of half-space pulse propagation problems generated by impulsive sources. Gakenheimer and Miklowitz [1969] used a modification of Cagniard's method [1962] to discuss the disturbance created by a moving point load. In case of finite sources, the most widely discussed model is that of a circular ring or disc load. Mitra [1964], Tupholme [1970] and Roy [1975] have studied the various aspects of the same problem. Elastic waves due to uniformly expanding disc or ring loads on the free surface of a semi-infinite medium have been studied extensively by Gakenheimer [1971]. The axisymmetric problem of the determination of the displacement due to a stress discontinuity over a uniformly expanding circular region at a certain depth below the free surface has been studied by Ghosh [1971].

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However exact evaluation of the displacement field for finite source other than the circular model does not seem to have been attempted much in the literature. Burridge and Willis [1969] obtained a solution for radiation from a growing elliptical crack in an anisotropic medium. The problem of an elliptical shear crack growing in prestressed medium has been solved by Richards [1973] by the Cagniard-de Hoop Method. Roy [1981] also attempted the same technique to slove the problem of elastic wave propagation due to prescribed normal stress over an elliptic area on the free surface of an elastic half-space.

In our problem, we have considered the propagation of elastic waves due to an expanding elliptical ring load over the free surface of a semi-infinite medium. The expression for displacement at points on the free surface has been derived in integral form by the application of Cagniard-de Hoop technique for different values of the rate of increase of the major and minor axes of the elliptic ring source. The displacement jumps across the different wave fronts have also been derived.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let an elliptic ring load P acting normal to the surface of an elastic half-space emanating from the origin of co-ordinates expand in such a way that the rates of increase of the major and minor axes of the ellipse are a and b respectively, a and b being constants. Major and minor axes of the ellipse are taken to coincide with the x and y-axes of co-ordinates where as z-axis is taken vertically downwards into the medium (Fig. 1.). Thus we have on z = 0

$$\tau_{zz} = -\frac{P\,\delta\left[t - (x^2a^{-2} + y^2b^{-2})^{1/2}\right]}{(x^2a^{-2} + y^2b^{-2})^{1/2}} \qquad (1)$$

$$\tau_{xz} = \tau_{yz} = 0$$

where P is constant and δ is the Dirac delta function.

The displacement field inside the elastic medium ($z \geq 0$) is given interms of potentials ϕ and ψ as

$$\vec{u} = \nabla\phi + \nabla \times \nabla \times (e_z\psi)$$

where

$$\nabla^2\phi = \frac{1}{c_d^2}\frac{\partial^2\phi}{\partial t^2}, \qquad \nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2} \qquad (2)$$

e_x, e_y, e_z are unit vectors along co-ordinate axes and c_d and c_s are the p- and s-wave velocities of the medium.

In order to obtain solutions of wave equations (2), we introduce Laplace transform with respect to t and denote it by bar and also introduce bilateral Fourier transform with respect to x

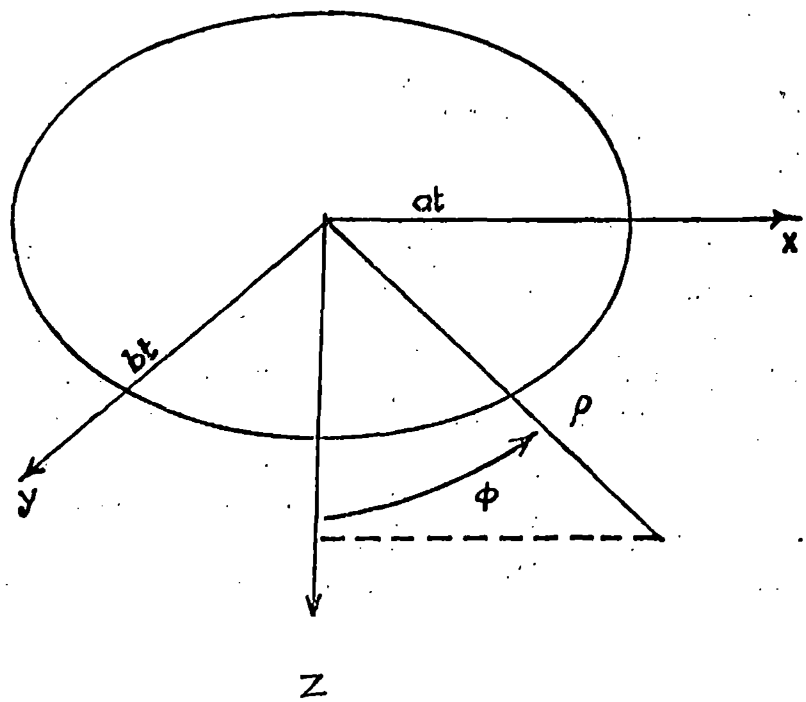


Fig. 1. Geometry of the problem.

and y to supress the time parameter t and the x, y space co-ordinates. Taking Laplace transform with respect to t(-) and also bilateral Fourier transform with respect to x and y ($\cong$), the transformed boundary conditions are

$$\tilde{\tilde{\tau}}_{zz} = - \frac{Pab}{(a^2\xi^2 + b^2\eta^2 + s^2)^{1/2}}, \quad \tilde{\tilde{\tau}}_{xz} = \tilde{\tilde{\tau}}_{yz} = 0 \qquad (3)$$

Then satisfying the transformed boundary conditions (3) and performing the inverse Fourier transform, the Laplace transformed displacement field can be written as

$$\bar{u}_j(x,y,z,s) = \bar{u}_{jd}(x,y,z,s) + \bar{u}_{js}(x,y,z,s) \qquad (4)$$

$$\text{for } j = x,y,z$$

where

$$\bar{u}_{j\alpha_1}(x,y,z,s) =$$

$$= 1/2\pi\mu \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{j\alpha_1}(\xi,\eta,s)\exp[-\zeta_{\alpha_1}z + i(\xi x+\eta y)]d\xi\, d\eta \qquad (5)$$

$$\text{for } \alpha_1 = d,s$$

and

$$F_{xd}(\xi,\eta,s) = - i\xi\zeta_{\upsilon}G, \qquad F_{xs}(\xi,\eta,s) = 2i\xi\zeta_d\zeta_s G,$$

$$F_{yd}(\xi,\eta,s) = - i\eta\zeta_{\upsilon}G, \qquad F_{ys}(\xi,\eta,s) = 2i\eta\,\zeta_d\zeta_s G,$$

$$F_{zd}(\xi,\eta,s) = \zeta_d \zeta_o G, \qquad F_{zs}(\xi,\eta,s) = -2(\xi^2+\eta^2)\,\zeta_d G,$$

$$G = \frac{Pab}{(s^2+r^2)^{1/2}T}, \qquad T = \zeta_o^2 - 4\,\zeta_d \zeta_s(\xi^2+\eta^2)$$

$$r^2 = a^2\xi^2+b^2\eta^2, \tag{6}$$

$$\zeta_d = (\xi^2+\eta^2+k_d^2)^{1/2}, \qquad \zeta_s = (\xi^2+\eta^2+k_s^2)^{1/2},$$

$$\zeta_o = k_s^2 + 2(\xi^2+\eta^2), \qquad k_d = \frac{s}{c_d}, \qquad k_s = \frac{s}{c_s}.$$

Now the De-Hoop transformation,

$$\xi = s/c_d(q\cos\theta - w\sin\theta), \quad \eta = s/c_d(q\sin\theta + w\cos\theta) \tag{7}$$

where $\theta = \tan^{-1} y/x$,

is applied into (5). The Laplace transformed displacement field (5) can be written as

$$\bar{u}_{j\alpha_1}(R,Z,s) = 1/2\pi\mu \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{j\alpha_1}(q,w,s)\exp[-s/c_d(m_\alpha Z - iqR)]\,\frac{s^2}{c_d^2}\,dq\,dw \tag{8}$$

where

$$F_{xd}(q,w,s) = -\frac{i\,Pab\,(q\cos\theta - w\sin\theta)m_o}{s.s/c_d(E_1+O)^{1/2}.N},$$

$$F_{xs}(q,w,s) = \frac{2i\ Pab\ (q \cos\theta - w \sin\theta) m_d m_s}{s.s/c_d (E_1 + O)^{1/2} . N} ,$$

$$F_{yd}(q,w,s) = - \frac{i\ Pab\ (q \sin\theta + w \cos\theta) m_o}{s.s/c_d (E_1 + O)^{1/2} . N.}$$

$$F_{ys}(q,w,s) = \frac{2i\ Pab\ (q \sin\theta + w \cos\theta)\ m_d m_s}{s.s/c_d (E_1 + O)^{1/2} . N.},$$

$$F_{zd}(q,w,s) = \frac{Pab\ m_d m_o}{s.s/c_d (E_1 + O)^{1/2} . N.} ,$$

$$F_{zs}(q,w,s) = - \frac{2\ Pab\ (q^2 + w^2) m_d}{s.s/c_d (E_1 + O)^{1/2} . N.} ,$$

$$m_d = (q^2 + w^2 + 1)^{1/2}, \qquad m_s = (q^2 + w^2 + l^2)^{1/2},$$

$$m_o = l^2 + 2(q^2 + w^2), \qquad N = m_o^2 - 4 m_d m_s (q^2 + w^2),$$

$$E_1 = (1 + q^2 D + w^2 F), \qquad D = \frac{a^2}{c_d^2} \cos^2\theta + \frac{b^2}{c_d^2} \sin^2\theta,$$

$$F = \frac{a^2}{c_d^2} \sin^2\theta + \frac{b^2}{c_d^2} \cos^2\theta, \qquad O = - 2qw \sin\theta \cos\theta (a^2 - b^2)/c_d^2 ,$$

$$l = c_d/c_s \ , \text{ and } R^2 = x^2 + y^2. \tag{9}$$

For mathematical simplicity we confine our attention to the derivation of the displacement field at any point on the xz-plane. Obviously the displacement at any point on any plane through the z-axis can then easily be visualized. Accordingly in order to obtain the displacement at any point on the xz-plane, we put $\theta = 0$ in (8) which then takes the form

$$\bar{u}_{j\alpha_1}(x,z,s) = \frac{Pab}{2\pi\mu c_d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{Re}\left[K_{j\alpha_1}(q,w)\, e^{-\frac{s}{c_d}(m_\alpha z - iqx)} \right] dq\, dw \tag{10}$$

where

$$K_{xd}(q,w) = -\frac{iqm_o}{E^{1/2}.N}, \qquad K_{xs}(q,w) = \frac{2iqm_d m_s}{E^{1/2}.N.},$$

$$K_{yd}(q,w) = -\frac{iwm_o}{E^{1/2}.N}, \qquad K_{ys}(q,w) = \frac{2iwm_d m_s}{E^{1/2}.N.}, \tag{11}$$

$$K_{zd}(q,w) = \frac{m_d m_o}{E^{1/2}.N}, \qquad K_{zs}(q,w) = -\frac{2m_d(q^2 + w^2)}{E^{1/2}.N},$$

and

$$E = (c_d^2 + a^2q^2 + b^2w^2)/c_d^2.$$

3. DILATATIONAL CONTRIBUTION

From (10) $\bar{u}_{zd}$ is converted to the Laplace transform of a known function by mapping $(m_d z - iqx)/c_d$ into t through a contour integration in a complex q-plane.

The singularities of the integrand of $\bar{u}_{zd}$ are branch points at

$$q = S_d^{\pm} = \pm\ i(w^2 + 1)^{1/2}, \qquad q = S_s^{\pm} = \pm\ i(w^2 + 1^2)^{1/2},$$

$$q = S_c^{\pm} = \pm\ i\ \frac{(w^2 b^2 + c_d^2)^{1/2}}{a}, \tag{12}$$

and the poles at (12)

$$q = S_R^{\pm} = \pm\ i(w^2 + \gamma_R^2)^{1/2}.$$

The poles at $q = S_R^{\pm}$ correspond to the zeros of the Rayleigh function N, where $\gamma_R = c_d/c_R$ and c_R is the Rayleigh surface wave speed. The contours of integration in the q-plane are shown in Fig. 2(a,b,c) which also show the positions of singularities lying in the upper half of the q-plane.

Since the positions of the singularities and the transformed contour of integration depend on different values of a and b, three different cases arise for the evaluation of u_{zd}.

(a) Case $a > b > C_d$.

The q-plane for $a > b > C_d$ is shown in Fig. 2(a). The contour $q = q_d^{\pm}$ in the q-plane, is found by solving

$$t = (m_d Z - iqx)/c_d \tag{13}$$

for q, where t is real , we get

$$q = q_d^{\pm} = i\tau \sin\phi \pm (\tau^2 - \tau_{wd}^2)^{1/2} \cos\phi \tag{14}$$

for $\tau > \tau_{wd}$, where

$$\tau_{wd} = (w^2 + 1)^{1/2}, \quad \tau = c_d t/\rho \tag{15}$$

and (ρ,ϕ) are the polar coordinates in the xz-plane as shown in Fig.1. Equations (14) define one branch of a hyperbola with vertex at $q = i(w^2 + 1)^{1/2} x/\rho$, which is parametrically described by the dimensionless time parameter τ as τ varies from τ_{wd} towards infinity.

As shown in Fig. 2(a), the contour of integration has two possible configurations in the q-plane, depending upon ϕ and w.

For the case(1) given by:

$$\text{Case(1)}: \quad \phi < \phi_{da} \quad \text{and} \quad 0 < \omega < \infty$$

or

$$\phi_{da} < \phi < \phi_{ba} \quad \text{and} \quad W_{da} < W < \infty \tag{16}$$

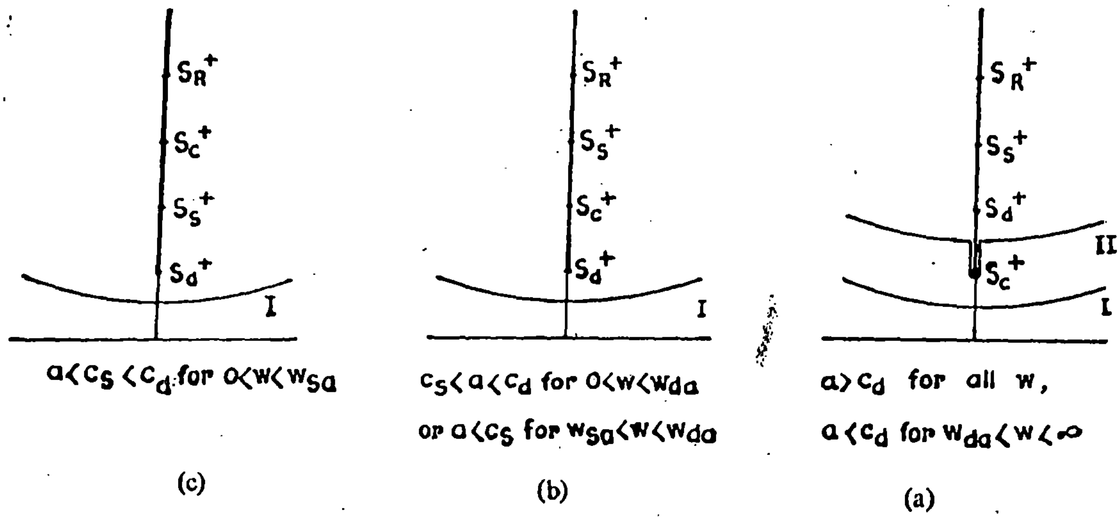


Fig. 2. Cagniard paths of integration in the q-plane.

where $\phi_{da} = \sin^{-1} C_d/a$, $\phi_{ba} = \sin^{-1} b/a$

and

$$W_{da} = \left[\frac{C_d^2 - a^2 \sin^2\phi}{a^2 \sin^2\phi - b^2} \right]^{1/2}, \tag{17}$$

the vertex of the path $q = q_d^{\pm}$ does not lie on the branch cuts and hence the path of integration contour is simply $q = q_d^{\pm}$ and is denoted by I. But for the case (2) given by :

$$\text{Case (2):} \quad \phi_{da} < \phi < \phi_{ba} \quad \text{and} \quad 0 < w < w_{da}$$
$$\text{or} \quad \phi > \phi_{ba} \quad \text{and} \quad 0 < w < \infty \tag{18}$$

the vertex of the path $q = q_d^{\pm}$ lies on the branch cut between the branch points $q = S_c^{+}$ and $q = S_d^{+}$. Hence the integration contour is given by $q = q_d^{\pm}$ for $\tau > \tau_{wd}$ which is denoted by II, plus

$$q = q_{da} = i\tau\sin\phi - i(\tau_{wd}^2 - \tau^2)^{1/2}\cos\phi \tag{19}$$

for $\tau_{wda} < \tau < \tau_{wd}$, where

$$\tau_{wda} = \frac{1}{a}\left[\left\{w^2(a^2 - b^2)+(a^2 - C_d^2)\right\}^{1/2} \cos\phi + (w^2 b^2 + C_d^2)^{1/2}\sin\phi\right]. \tag{20}$$

Transferring the path of integration from the real q-axis to the Cagniard's path we obtain

$$\bar{u}_{zd}(\rho,\phi,s) = \frac{2\,Pab}{\pi\mu C_d}\left[\int_0^{\infty}\int_{t_{vd}}^{\infty} \mathrm{Re}\left[k_{zd}(q_d^+,w)\frac{dq_d^+}{dt}\right]e^{-st}\,dt\,dw\right. +$$

$$+ H(\phi_{ba}-\phi)H(\phi-\phi_{da})\int_0^{w_{da}}\int_{t_{vda}}^{t_{vd}} \mathrm{Re}\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right]e^{-st}\,dtdw +$$

$$\left. + H(\phi-\phi_{ba})\int_0^{\infty}\int_{t_{vda}}^{t_{vd}} \mathrm{Re}\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right]e^{-st}\,dtdw\right] \qquad (21)$$

where $t_{vd} = (\rho/C_d)\tau_{vd}$ and $t_{vda} = (\rho/C_d)\tau_{vda}$. The first term of (21) is the contribution from $q_d^{\pm}$ and the second and third terms are the contributions from q_{da}.

Now interchanging the order of integration in (21) and inverting the Laplace transform, we find that

$$u_{zd}(\rho,\phi,\tau) = \frac{2\,Pab}{\pi\mu C_d}\left[H(\tau-1)\int_0^{T_d}\mathrm{Re}\left[k_{zd}(q_d^+,w)\frac{dq_d^+}{dt}\right]dw\right. +$$

$$+ H(\phi-\phi_{da})H(\phi_{ba}-\phi)H(\tau-\tau_{da})H(\tau'_{da}-\tau)\times$$

$$\times\int_{A'_{da}}^{T_{da}}\mathrm{Re}\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right]dw +$$

$$+ H(\phi - \phi_{ba})H(\tau - \tau_{da}) \times$$

$$\times \int_{A^{o}_{da}}^{T_{da}} \text{Re}\left[k_{zd}(q_{da}, w) \frac{dq_{da}}{dt} \right] dw \Bigg] \tag{22}$$

where

$$\left. \begin{aligned} A'_{da} &= \begin{cases} 0 \text{ for } \tau_{da} < \tau < 1 \\ T_d \text{ for } 1 < \tau < \tau'_{da} \end{cases} \\ A^{o}_{da} &= \begin{cases} 0 \text{ for } \tau_{da} < \tau < 1 \\ T_d \text{ for } \tau > 1 \end{cases} \end{aligned} \right\} \tag{23}$$

$$T_d = (\tau^2 - 1)^{1/2} \tag{24}$$

$$T_{da} = \left[\frac{X_d - \{Y_d - (a^2\cos^2\phi - b^2)Z_d\}^{1/2}}{(a^2\cos^2\phi - b^2)^2} \right]^{1/2} \tag{25}$$

$$X_d = \tau^{o}_{d}\, b^2\sin^2\phi + (a^2 - b^2)\tau_d\cos^2\phi$$

$$Y_d = \tau^{o\,2}_{d}\, b^4 \sin^4\phi + (a^2 - b^2)^2\, \tau^2_d\cos^4\phi +$$

$$+ 2(a^2 - b^2)b^2\tau_d\tau^{o}_{d} \sin^2\phi \cos^2\phi$$

$$Z_d = (\tau_d - 2C^2_d \sin^2\phi)^2 - 4C^2_d(a^2 - C^2_d) \sin^2\phi \cos^2\phi$$

$$\tau_d = a^2\tau^2 + (C^2_d - a^2\cos^2\phi)$$

$$\tau_d^0 = a^2\tau^2 - (C_d^2 - a^2\cos^2\phi) \tag{26}$$

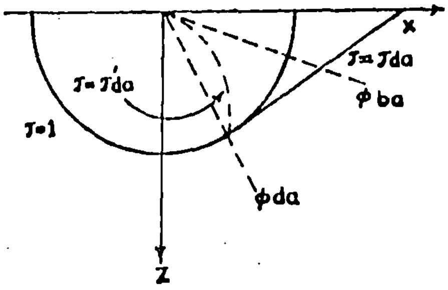
$$\tau_{da} = \frac{1}{a}\left[(a^2 - C_d^2)^{1/2}\cos\phi + C_d \sin\phi \right], \tag{27}$$

$$\tau'_{da} = \left[\frac{C_d^2 - b^2}{a^2 \sin^2\phi - b^2} \right]^{1/2} . \tag{28}$$

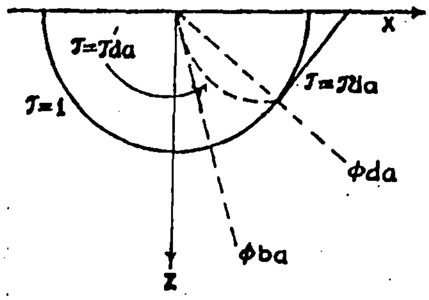
The first term in u_{zd} is due to the dilatational motion behind hemispherical wave front at $\tau = 1$ and the second and third terms are due to the dilatational motion behind the conical wave front at $\tau=\tau_{da}$ for $\phi > \phi_{da}$. These wave fronts are shown in Fig. 3(a), $\tau = \tau'_{da}$ shown in Fig 3(a) by a dashed curve, is not a wave front because it is not a characteristic surface for governing wave equation for the dilatational motion. Similar non characteristic surfaces were found by Gakenheimer and Miklowitz [1969] for a point load travelling on an elastic half-space and also by Aggarwal and Ablow [1967] for the motion of an acoustic half-space due to an expanding surface load. They proved explicitly that their solution was analytic over the surfaces. The same thing can be proved in our case also.

(b) Case $a > c_d > b$

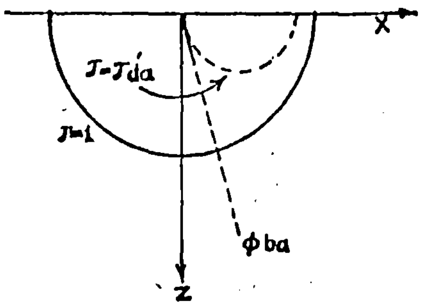
In this case, the path of integration with respect to q transforms to the simple path given by contour I (Fig.2(a)) for all



3 (a) for $\alpha > b > c_d$



3 (b) for $\alpha > c_d > b$



3 (c) for $\alpha < c_d$

FIG. 3. Wave patten for dilatational motion.

w when $\phi < \phi_{ba}$ and also for $0<w<w_{da}$ when $\phi_{ba}<\phi<\phi_{da}$, whereas the path of integration with respect to q transform to the contour II (Fig.2(a)) for $w_{da} < w < \infty$ when $\phi_{ba}<\phi<\phi_{da}$ and also for all w when $\phi > \phi_{da}$. The remaining details of inverting $\bar{u}_{zd}$ for $a > c_d > b$ are exactly the same as for $a > b > c_d$, and one can easily find that

$$u_{zd}(\rho,\phi,\tau) = \frac{2\ Pab}{\pi\mu c_d}\left[H(\tau - 1)\int_0^{\tau_d} Re\left[k_{zd}(q_d^+,w)\frac{dq_d^+}{dt}\right]dw + \right.$$

$$+ H(\phi - \phi_{ba})H(\phi_{da} - \phi)H(\tau - \tau_{da}) \times$$

$$\times \int_{\tau_d}^{\tau_{da}} Re\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right]dw +$$

$$+ H(\phi - \phi_{da})H(\tau - \tau_{da}) \times$$

$$\left. \times \int_{A_{da}^{o}}^{\tau_{da}} Re\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right]dw\right] \qquad (29)$$

where A_{da}^{o} is given by (23). The wave geometry associated with this expression is shown in Fig.3(b).

(c) Case $a < c_d$

For this case the path of integration with respect to q transform to the simple path given by contour I [Figs. 2(b),2(c)] for all w when $\phi<\phi_{ba}$ and also for $0<w<w_{da}$ when $\phi>\phi_{ba}$, whereas the path of integration with respect to q transforms to the contour II [Fig.2 (a)] for $w_{da}<w<\infty$ when $\phi>\phi_{ba}$. Note that in this case the angle ϕ_{da} does not arise. Now proceding as the case $a>b>c_d$ for inverting $\bar{u}_{zd}$ we get

$$u_{zd}(\rho,\phi,\tau) = \frac{2\ Pab}{\pi\mu c_d}\left[H(\tau - 1)\int_0^{T_d} Re\left[k_{zd}(q_d^+,w)\ \frac{dq_d^+}{dt} \right]dw \right. +$$

$$\left. + H(\phi - \phi_{ba})H(\tau - \tau'_{da}) \int_{T_d}^{T_{da}} Re\left[k_{zd}(q_{da},w)\ \frac{dq_{da}}{dt} \right]dw \right]. \quad (30)$$

The wave geometry associated with this expression is shown in Fig.3(c). As expected physically, contribution due to the conical wave front does not exist for this case.

Summary

Combining (22), (29) and (30) one finds that u_{zd} can be written as one expression for all value of a and b.

$$u_{zd}(\rho,\phi,\tau) = \frac{2\ Pab}{\pi\mu c_d}\Bigg[H(\tau - 1)\int_0^{T_d} Re\Big[k_{zd}(q_d^+,w)\ \frac{dq_d^+}{dt}\Big]\ dw\ +$$

$$+\Big[H(\tau - \tau_{da})H(\phi - \phi_{da})\Big\{H(b - c_d)\ +$$

$$+\ H(a-c_d)H(c_d-b)\Big\} + H(\tau - \tau'_{da})H(\phi - \phi_{ba})\Big\{H(a - c_d)\ \times$$

$$\times\ H(c_d - b)H(\phi_{da} - \phi)\ +\ H(c_d - a)\Big\}\Big]\ \times$$

$$\times\int_{A_{da}}^{T_{da}} Re\Big[k_{zd}(q_{da},w)\ \frac{dq_{da}}{dt}\Big]dw\ \Bigg] \qquad (31)$$

where

$$A_{da}\begin{cases} \left.\begin{array}{l} = 0 \text{ for } \tau_{da} < \tau < 1 \\ = T_d \text{ for } 1 < \tau < \tau'_{da} \\ = T_{da} \text{ for } \tau > \tau'_{da} \end{array}\right\} & \text{for } \phi_{da} < \phi < \phi_{ba},\ a > b > c_d \\ \left.\begin{array}{l} = 0 \text{ for } \tau_{da} < \tau < 1 \\ = T_d \text{ for } 1 < \tau \end{array}\right\} & \begin{array}{l}\text{for } \phi > \phi_{ba}, a > b > c_d \\ \text{for } \phi > \phi_{da},\ a > c_d > b\end{array} \\ = T_d \text{ for } \tau > \tau'_{da} & \Big\{\begin{array}{l}\text{for } \phi_{ba} < \phi < \phi_{da}, a > c_d > b \\ \text{for } \phi > \phi_{ba}, a < c_d .\end{array} \end{cases}$$

(32)

4. EQUIVOLUMINAL CONTRIBUTIONS

Inversion of $\bar{u}_{zs}$ is complicated than the inversion of $\bar{u}_{zd}$ because of the appearence of head waves (Von-Schmidt waves) otherwise it is same as $\bar{u}_{zd}$. Here the integration contour has more configurations in the q-plane though the singularities are the same. Here the hyperbola $q = q_s^{\pm}$ arises in a similar way to $q = q_d^{\pm}$, but its vertex can lie on the branch cut between the branch points at $q = S_d^+$ and $q = S_s^+$ and at $q = S_c^+$ and $q = S_s^+$ as well as between $q = S_c^+$ and $q = S_d^+$, depending on the values of w, ϕ, a and b. In this case, the straight line contour lying along the imaginary q-axis is denoted by $q_{s\alpha}$ which is similar to $q_{d\alpha}$ appearing in the dilatational contributions. Now omiting details of inverting $\bar{u}_{zs}$, one can easily find

$$u_{zs}(\rho,\phi,\tau) = \frac{4\ Pab}{\pi\mu c_d}\left[H(\tau - 1)\int_0^{T_s} \mathrm{Re}\left[k_{zs}(q_s^+,w)\frac{dq_s^+}{dt}\right]dw + \right.$$

$$+ [H(\tau - \tau_{s\alpha})H(\phi - \phi_{s\alpha})\{H(b - c_s) + H(c_s - b)H(a - c_s)\} +$$

$$+ H(\tau - \tau'_{s\alpha})H(\phi - \phi_{b\alpha})\ \{H(c_s - b)H(\phi_{s\alpha} - \phi)\ \times$$

$$\times\ H(a - c_s) + H(c_s - a)\}]\ \times$$

$$\times \int_{A_{sa}}^{T_{sa}} \mathrm{Re}\left[k_{zs}(q_{sa},w) \frac{dq_{sa}}{dt} \right] dw +$$

$$+ H(\tau - \tau_{sd})H(\tau'_{sd} - \tau)H(\phi - \phi_{sd}) \times$$

$$\times \int_{A_{sd}}^{T_{sd}} \mathrm{Re}\left[k_{zs}(q_{sa},w) \frac{dq_{sa}}{dt} \right] dw \Bigg] \qquad (33)$$

for $0 \le \rho < \infty$, $0 \le \phi < \pi/2$,

$0 \le \tau < \infty$, $0 \le a < \infty$ and $0 \le b < \infty$, $a > b$

where

$= 0$ for $\tau_{sa} < \tau < 1$ and $= T_s$ for $1 < \tau < \tau'_{sa}$, when: $\phi_{sa} < \phi < \phi_{ba}, a > c_d, a > b > c_s, ac_s > bc_d$; $\phi_{sa} < \phi < \phi_{sd}, a > c_d, a > b > c_s, ac_s < bc_d$; $\phi_{sa} < \phi < \phi_{abs}, c_d > a > b > c_s$

$= 0$ for $\tau_{sa} < \tau < 1$ and $= T_s$ for $\tau > 1$, when: $\phi_{ba} < \phi < \phi_{sd}, a > b > c_d, ac_s > bc_d$; $\phi_{sa} < \phi < \phi_{sd}, a > c_d > c_s > b$

$= 0$ for $\tau_{sa} < \tau < \tau_{sd}$, $= T_{sd}$ for $\tau_{sd} < \tau < \tau'_{sd}$, $= T_s$ for $\tau > \tau'_{sd}$, when: $\phi > \phi_{sd}, a > b > c_d, ac_s > bc_d$; $\phi > \phi_{sd}, a > c_d > c_s > b$

$= 0$ for $\tau_{sa} < \tau < \tau_{sd}$, $= T_{sd}$ for $\tau_{sd} < \tau < \tau'_{sd}$, when: $\phi > \phi_{sd}, a > b > c_d, ac_s < bc_d$

$$
A_{sa}
\begin{cases}
= T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa} & \\[2ex]
= T_s \text{ for } \tau > \tau'_{sa} & \left\{\begin{array}{l} \phi_{ba} < \phi < \phi_{sa},\ a > c_d > c_s > b \\ \phi_{ba} < \phi < \phi_{abs},\ c_d > a > c_s > b \\ \phi_{ba} < \phi < \phi_{abs},\ a < c_s \end{array}\right. \\[2ex]
\left.\begin{array}{l} = T_s \text{ for } \tau'_{sa} < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau > \tau'_{sd} \end{array}\right\} & \begin{array}{l} \phi_{abs} < \phi < \phi_{sa},\ c_d > a > c_s > b \\ \phi > \phi_{abs},\ a < c_s \end{array} \\[2ex]
\left.\begin{array}{l} = 0 \text{ for } \tau_{sa} < \tau < 1 \\ = T_s \text{ for } 1 < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau > \tau'_{sd} \end{array}\right\} & \begin{array}{l} \phi > \phi_{sa},\ c_d > a > c_s > b,\ \alpha > \beta \\ \phi_{sa} < \phi < \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\ \phi > \phi_{ba},\ c_d > a > b > c_s,\ \alpha > \beta \\ \phi_{ba} < \phi < \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \end{array} \\[2ex]
\left.\begin{array}{l} = 0 \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau > \tau'_{sd} \end{array}\right\} & \begin{array}{l} \phi > \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\ \phi > \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\ \phi > \phi_{ba},\ c_d > a > b > c_s,\ \alpha < \gamma \end{array} \\[2ex]
\left.\begin{array}{l} = 0 \text{ for } \tau_{sa} < \tau < 1 \\ = T_s \text{ for } 1 < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa}. \end{array}\right\} & \begin{array}{l} \phi_{abs} < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \alpha > \beta \\ \phi_{abs} < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\ \phi_{abs} < \phi < \phi_x,\ c_d > a > b > c_s,\ \alpha < \gamma \end{array} \\[2ex]
\left.\begin{array}{l} = 0 \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa} \end{array}\right\} & \phi_x < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \alpha < \gamma .
\end{cases}
\qquad (34)
$$

$$
A_{sd}\left\{\begin{array}{lll}
= 0 & \text{for } \tau_{sd}<\tau<1 & \phi>\phi_{sd},\ a>b>c_d \\
 & & \phi>\phi_{sd},\ a>c_d>c_s>b \\
= T_s & \text{for } 1<\tau<\tau'_{sd} & \phi_{sd}<\phi<\phi_{abs},\ c_d>a>c_s>b \\
 & & \phi_{sd}<\phi<\phi_{sa},\ c_d>a>b>c_s \\
 & & \phi_{sd}<\phi<\phi_{abs},\ a<c_s \\
= 0 & \text{for } \tau_{sd}<\tau<1 & \\
= T_s & \text{for } 1<\tau<\tau'_{sa} & \phi_{abs}<\phi<\phi_{sa},\ c_d>a>c_s>b \\
= T_{sa} & \text{for } \tau'_{sa}<\tau<\tau'_{sda} & \phi>\phi_{abs},\ a<c_s \\
= T_s & \text{for } \tau'_{sda}<\tau<\tau'_{sd} & \\
= 0 & \text{for } \tau_{sd}<\tau<\tau_{sa} & \phi>\phi_{sa},\ c_d>a>c_s>b,\ \alpha>\beta \\
 & & \phi_{sa}<\phi<\phi_x,\ c_d>a>c_s>b,\ \beta>\alpha>\gamma' \\
= T_{sa} & \text{for } \tau_{sa}<\tau<\tau'_{sda} & \phi>\phi_{abs},\ c_d>a>b>c_s,\ \alpha>\beta \\
 & & \phi_{abs}<\phi<\phi_x,\ c_d>a>b>c_s,\ \beta>\alpha>\gamma \\
= T_s & \text{for } \tau'_{sda}<\tau<\tau'_{sd} & \phi_{abs}<\phi<\phi_x,\ c_d>a>b>c_s,\ \alpha<\gamma \\
= 0 & \text{for } \tau_{sd}<\tau<\tau_{sa} & \phi>\phi_x,\ c_d>a>c_s>b,\ \beta>\alpha>\gamma' \\
= T_{sa} & \text{for } \tau_{sa}<\tau<\tau'_{sda} & \phi>\phi_x,\ c_d>a>b>c_s,\ \beta>\alpha>\gamma \\
= 0 & \text{for } \tau'_{sda}<\tau<1 & \phi>\phi_x,\ c_d>a>b>c_s,\ \alpha<\gamma \\
= T_s & \text{for } 1<\tau<\tau'_{sd} & \\
= 0 & \text{for } \tau_{sd}<\tau<\tau_{sa} & \\
= T_{sa} & \text{for } \tau_{sa}<\tau<\tau'_{sa} & \phi_{sa}<\phi<\phi_{abs},\ c_d>a>b>c_s \\
= T_s & \text{for } \tau'_{sa}<\tau<\tau'_{sd} &
\end{array}\right. \tag{35}
$$

and also where

$$T_s = (\tau^2 - l^2)^{1/2} \tag{36}$$

$$T_{sa} = \left[\frac{X_s - \{Y_s - (a^2\cos^2\phi - b^2)^2 Z_s\}^{1/2}}{(a^2\cos^2\phi - b^2)^2} \right]^{1/2} \tag{37}$$

$$X_s = \tau_s^0 b^2 \sin^2\phi + (a^2 - b^2)\, \tau_s \cos^2\phi$$

$$Y_s = {\tau_s^0}^2\, b^4 \sin^4\phi + (a^2 - b^2)^2 \tau_s^2 \cos^4\phi +$$

$$+\ 2(a^2 - b^2) b^2 \tau_s \tau_s^0 \sin^2\phi \cos^2\phi$$

$$Z_s = (\tau_s - 2c_d^2 \sin^2\phi)^2 - 4\, l^2\, c_d^2 (a^2 - c_s^2) \sin^2\phi \cos^2\phi$$

$$\tau_s = a^2\tau^2 + l^2(c_s^2 - a^2\cos^2\phi)$$

$$\tau_s^0 = a^2\tau^2 - l^2(c_s^2 - a^2\cos^2\phi) \tag{38}$$

$$T_{sd} = \left[\left\{(\tau - \tau_{sd})\operatorname{cosec}\phi + 1\right\}^2 - 1\right]^{1/2} \tag{39}$$

$$\tau_{sa} = 1/a\left[l(a^2 - c_s^2)^{1/2}\cos\phi + c_d \sin\phi\right] \tag{40}$$

$$\tau_{sd} = \left[(l^2 - 1)^{1/2} \cos\phi + \sin\phi\right] \tag{41}$$

$$\tau'_{sa} = \left[\frac{l^2(b^2 - c_s^2)}{b^2 - a^2\sin^2\phi} \right]^{1/2} \tag{42}$$

$$\tau'_{sd} = (l^2 - 1)^{1/2} \sec\phi \tag{43}$$

$$\tau'_{sda} = \left[(l^2 - 1)^{1/2}\cos\phi + \left(\frac{c_d^2 - b^2}{a^2 - b^2} \right)^{1/2} \sin\phi \right] \tag{44}$$

$$\phi_{sa} = \sin^{-1} c_s/a, \; \phi_{sd} = \sin^{-1} c_s/c_d, \; \phi_{ba} = \sin^{-1} b/a \tag{45}$$

$$\phi_{abs} = \sin^{-1} \left(\frac{c_d^2 - b^2}{l^2(a^2 - b^2)+c_d^2 - a^2} \right)^{1/2} \tag{46}$$

$$\phi_x = \sin^{-1}\left[\frac{(a^2-b^2)^{1/2}\left[l(c_d^2 - b^2)^{1/2} + (l^2 - 1)^{1/2}(c_d^2 - a^2)^{1/2} \right]}{l^2(a^2 - b^2) + c_d^2 - a^2} \right] \tag{47}$$

$$\alpha = \left(\frac{c_d^2 - a^2}{a^2 - b^2} \right)^{1/2}, \qquad \beta = (l^2 - 1)^{1/2},$$

$$\gamma = \frac{b}{a} (l^2 - 1)^{1/2} - \frac{1}{a} (c_d^2 - b^2)^{1/2}, \tag{48}$$

$$\gamma' = \frac{c_s}{a} (l^2 - 1)^{1/2} - \frac{l}{a} \left[\frac{a^2 - c_s^2}{a^2 - b^2} (c_d^2 - b^2) \right]^{1/2}$$

$$q_s^{\pm} = i\, \tau \sin\phi \pm (\tau^2 - \tau_{ws}^2)^{1/2} \cos\phi \tag{49}$$

$$\tau_{vs} = (w^2 + 1^2)^{1/2} \tag{50}$$

$$q_{s\alpha} = i\tau \sin\phi - i(\tau_{vs}^2 - \tau^2)^{1/2} \cos\phi \tag{51}$$

The first term in the expression (33) is the equivoluminal motion behind the hemispherical wave front at $\tau = 1$ and the second is due to the equivoluminal motion behind the conical wave front at $\tau = \tau_{s\alpha}$. The third term in u_{zs} represents the equivoluminal motion due to the head wave fronts at $\tau = \tau_{sd}$. The wave fronts $\tau = \tau_{sd}$ for $\phi > \phi_{sd}$ and $\tau = \tau_{s\alpha}$ are shown in Figs. 4(a-1).

The equations $\tau = \tau'_{s\alpha}$, $\tau = \tau'_{sd}$ and $\tau = \tau'_{sd\alpha}$ are shown in Fig. 4 by dashed curves which are similar to $\tau = \tau'_{d\alpha}$ appearing in the u_{zd}. These dashed curved surfaces are not considered as wave fronts because it can be shown that displacements and their derivatives are continuous across these surfaces.

5. WAVE FRONT EXPANSIONS

The wave forms of the solution given in (31) and (33) are evaluted by approximate estimation of the integrals in the neighbourhood of the first arrival of the different waves. To facilitate this evaluation we put

$$w = [A^2 + (B^2 - A^2)\sin^2\alpha]^{1/2} \tag{52}$$

in the integrals arising in u_{zd} and u_{zs} where A and B are respectively the lower and upper limits of the particular integral in question, and the range of integration with respect to α is from 0 to $\pi/2$.

Now for the first integral of (31), we put $w = T_d \sin\alpha$ and hence for $\tau \to 1+$, we find that for any value of a,

$$w \to 0, \quad q_d^+ \longrightarrow i \sin\phi, \quad \frac{dq_d^+}{dt} \longrightarrow \frac{c_d}{\rho} \frac{\cos\phi}{T_d \cos\alpha},$$

$$m_d \to \cos\phi, \quad m_s \to (1^2 - \sin^2\phi)^{1/2}, \quad m_o \to (1^2 - 2\sin^2\phi),$$

$$E^{1/2} \to \frac{1}{c_d} (c_d^2 - a^2\sin^2\phi)^{1/2}, \text{ for } \phi < \phi_{da} \tag{53}$$

$$\to \frac{i}{c_d} (a^2\sin^2\phi - c_d^2)^{1/2}, \text{ for } \phi > \phi_{da},$$

$$N \to N_1$$

where $N_1 = (1^2 - 2\sin^2\phi)^2 + 4\sin^2\phi \cos\phi(1^2 - \sin^2\phi)^{1/2}$. (54)

Substituting these approximate values in the first integral of (31) one can find, for $\phi < \phi_{da}$

$$[u_z] \to N_{z1} \text{ as } \tau \to 1+ \tag{55}$$

where

$$N_{z1} = \frac{Pabc_d \cos^2\phi(1^2 - 2\sin^2\phi)}{\mu\rho \; (c_d^2 - a^2 \sin^2\phi)^{1/2} . N_1} . \tag{56}$$

Again in the second integral of (31) we put $w = T_{d\alpha}\sin\alpha$ and as $\tau \to 1-$ for $\phi > \phi_{d\alpha}$ we find that

$$q_{d\alpha} \to i\sin\phi - i\cos\phi\, T_{d\alpha}\sin\alpha$$

$$\frac{dq_{d\alpha}}{dt} \to \frac{ic_d}{\rho} \cdot \frac{T_{d\alpha}\sin\alpha\sin\phi + \cos\phi}{(T_{d\alpha}^2\sin^2\alpha + 1 - \tau^2)^{1/2}} \cdot \tag{57}$$

Puting these values in the second integral of (31), we get

$$\int_0^{\pi/2} \mathrm{Re}\left[k_{zd}(i\sin\phi - i\cos\phi\, T_{d\alpha}\sin\alpha,\, T_{d\alpha}\sin\alpha)\frac{ic_d}{\rho} \times \right.$$

$$\left. \times \frac{T_{d\alpha}\sin\alpha\sin\phi + \cos\phi}{(T_{d\alpha}^2\sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{d\alpha}\cos\alpha\, d\alpha \tag{58}$$

$$= \int_0^{\epsilon} \mathrm{Re}\left[k_{zd}(i\sin\phi - i\cos\phi\, T_{d\alpha}\sin\alpha,\, T_{d\alpha}\sin\alpha)\frac{ic_d}{\rho} \times \right.$$

$$\left. \times \frac{T_{d\alpha}\sin\alpha\sin\phi + \cos\phi}{(T_{d\alpha}^2\sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{d\alpha}\cos\alpha\, d\alpha +$$

$$+ \int_{\epsilon}^{\pi/2} \mathrm{Re}\left[k_{zd}(i\sin\phi - i\cos\phi\, T_{d\alpha}\sin\alpha,\, T_{d\alpha}\sin\alpha)\frac{ic_d}{\rho} \times \right.$$

$$\left. \times \frac{T_{d\alpha}\sin\alpha\sin\phi + \cos\phi}{(T_{d\alpha}^2\sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{d\alpha}\cos\alpha\, d\alpha \tag{59}$$

where ϵ is very small.

Since the main contribution to the integral (58) as $\tau \to 1$ arises from the first integral of (59) as $\tau \to 1$, so for the evaluation of (58) as $\tau \to 1$, we consider the approximate value of the integral given by

$$\int_0^{\epsilon} \mathrm{Re}\left[k_{zd}(i \sin\phi - i \cos\phi\, T_{d\alpha} \sin\alpha,\ T_{d\alpha} \sin\alpha) \frac{ic_d}{\rho} \times \right.$$

$$\left. \times \frac{T_{d\alpha} \sin\alpha \sin\phi + \cos\phi}{(T_{d\alpha}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{d\alpha} \cos\alpha \, d\alpha \qquad (60)$$

as $\tau \to 1$.

Since ϵ is very small so α is also small. So for the evaluation of the integral (60) as $\tau \to 1$ we also use the fact that $\alpha \to 0$, from which we get,

$$w \to 0,\ q_{d\alpha} \to i \sin\phi,\ m_d \to \cos\phi,\ m_s \to (1^2 - \sin^2\phi)^{1/2},$$

$$m_0 \to (1^2 - 2 \sin^2\phi), \qquad (61)$$

$$N \to N_1,\ E^{1/2} \to i/c_d(a^2 \sin^2\phi - c_d^2)^{1/2} \text{ for } \phi > \phi_{d\alpha}.$$

Now substituting these approximate values in (60) and integrating we obtain the approximate value of the integral as

$$-\frac{c_d^2\cos^2\phi(l^2 - 2\sin^2\phi)}{\rho(a^2\sin^2\phi - c_d^2)^{1/2}.N_1}\log|\tau - 1| \quad \text{when } \tau \rightarrow 1. \qquad (62)$$

So for $\phi > \phi_{d\alpha}$

$$[u_z] \rightarrow N'_{z4}\ \log|\ \tau - 1\ |\ \text{as } \tau \rightarrow 1 \qquad (63)$$

where

$$N'_{z4} = -\frac{2Pabc_d\ \cos^2\phi(l^2 - 2\ \sin^2\phi)}{\pi\mu\rho\ (a^2\ \sin^2\phi - c_d^2)^{1/2}.N_1}\ . \qquad (64)$$

In order to obtain the value of u_{zd} as $\tau \rightarrow \tau_{d\alpha}$ we put

$$w^2 = A_{d\alpha}^2 + (T_{d\alpha}^2 - A_{d\alpha}^2)\sin^2\alpha.$$

in the second integral of (31).

When $\tau \rightarrow \tau_{d\alpha}+$, we find that

$$w \rightarrow 0$$

$$q_{d\alpha} \rightarrow i\ \frac{c_d}{a}$$

$$dq_{d\alpha}/dt \rightarrow iA'$$

where $$A' = \frac{c_d}{\rho a}\left(\frac{a^2 - c_d^2}{1 - \tau_{d\alpha}^2}\right)^{1/2} \text{ for } a > c_d,$$

$$m_d \rightarrow 1/a(a^2 - c_d^2)^{1/2} \text{ for } a > c_d, \tag{65}$$

$$m_s \rightarrow \frac{1}{a}(a^2 - c_s^2)^{1/2}, \quad m_\alpha \rightarrow \frac{l^2}{a^2}(a^2 - 2c_s^2),$$

$$N \rightarrow N_2$$

where $N_2 = 1/a^4\left[l^4(a^2 - 2c_s^2)^2 + 4lc_d^2(a^2 - c_d^2)^{1/2}(a^2-c_s^2)^{1/2}\right]$

$$E^{1/2} \rightarrow iK^{1/2}(\tau - \tau_{d\alpha})^{1/2}$$

where

$$K = \frac{2a}{c_d}\,\frac{\cos^2\alpha(a^2 - c_d^2)^{1/2}}{\left[(a^2 - c_d^2)^{1/2}\sin\phi - c_d\cos\phi\right]} \qquad \text{for } a > c_d.$$

Using these approximate values in the second integral of (31) we find that for $a > c_d$

$$[u_z] \rightarrow N_{z4} \text{ as } \tau \rightarrow \tau_{d\alpha}+ \tag{66}$$

where

$$N_{z4} = \frac{2Pab}{\pi\mu c_d a^3}\,\frac{l^2(a^2 - c_d^2)^{1/2}(a^2 - 2c_s^2)A'C^{1/2}}{(2KA)^{1/2}N_2} \tag{67}$$

where $$C = 8a^2 c_d \tau_{d\alpha}(a^2-c_d^2)^{1/2}\sin\phi\ \cos\phi$$

$$A = a^2(a^2-b^2)\cos^2\phi\,\tau_{d\alpha}(\tau_{d\alpha}+\tau^o_{d\alpha}) + a^2b^2\sin^2\phi\,\tau_{d\alpha}(\tau_{d\alpha}-\tau^o_{d\alpha})$$

$$\tau^o_{d\alpha} = \frac{1}{a}\left[\ c_d\sin\phi - (a^2-c_d^2)^{1/2}\cos\phi\ \right] \tag{68}$$

It may be noted that conical wave front $\tau = \tau_{d\alpha}$ does not arise for $a < c_d$.

Next when $\phi < \phi_{s\alpha}$, for the evaluation of u_{zs} as $\tau \rightarrow 1$, we put $w = T_s\sin\alpha$ in the first integral of (33). When $\tau \rightarrow 1$, we find that in the above integral

$$w \longrightarrow 0$$

$$q_s^+ \longrightarrow i1\ \sin\phi$$

$$\frac{dq_s^+}{dt} \longrightarrow \frac{c_d}{\rho}\ \frac{1\ \cos\phi}{T_s\cos\alpha}$$

$$(q^2+w^2) \longrightarrow -\ 1^2\sin^2\phi$$

$$m_d \longrightarrow (1-1^2\sin^2\phi)^{1/2}$$

$$m_s \longrightarrow 1\ \cos\phi$$

$$m_0 \rightarrow 1^2(\cos^2\phi - \sin^2\phi)$$

$$E^{1/2} \rightarrow \frac{1}{c_s}(c_s^2 - a^2\sin^2\phi)^{1/2} \quad \text{for } \phi < \phi_{sa}$$

$$\rightarrow \frac{i}{c_s}(a^2\sin^2\phi - c_s^2)^{1/2} \quad \text{for } \phi > \phi_{sa}$$

$$N \rightarrow 1^3 N_s$$

where $N_s = \left[\, 1(\cos^2\phi - \sin^2\phi)^2 + 4\sin^2\phi \cos\phi(1 - 1^2\sin^2\phi)^{1/2} \,\right]$.

Using these approximate values in the first integral of (33) one can find for all values of a and b,

$$[u_z] \rightarrow N_{z2} \text{ for } \phi < \phi_{sa} \text{ as } \tau \rightarrow 1 \tag{70}$$

where

$$N_{z2} = -\frac{2pabc_s}{\mu\rho}\,\frac{\sin^2\phi\cos\phi\,(1-1^2\sin^2\phi)^{1/2}}{(c_s^2-a^2\sin^2\phi)^{1/2}.N_s} \tag{71}$$

For $\phi > \phi_{sa}$, considering approximate evaluation of last two integrals of (33) as $\tau \rightarrow 1$ it can be shown that for the case $a > b > c_d$

$$u_z \rightarrow N'_{z5} \log|\tau - 1| \text{ for } \phi_{sa} < \phi < \phi_{sd} \text{ as } \tau \rightarrow 1 \tag{72}$$

$$u_z \rightarrow N'_{z3} \log|\tau - 1| \text{ for } \phi > \phi_{sd} \text{ as } \tau \rightarrow 1 \tag{73}$$

and for the case $c_d > a > b > c_s$,

$$u_z \longrightarrow N'_{z6} \log|\tau - 1| \text{ for } \phi_{sd} < \phi < \phi_{sa} \text{ as } \tau \longrightarrow 1 \tag{74}$$

$$u_z \longrightarrow N'_{z3} \log|\tau - 1| \text{ for } \phi > \phi_{sa} \text{ as } \tau \longrightarrow 1 \tag{75}$$

and also for the case $c_s > a > b$,

$$u_z \longrightarrow N'_{z6} \log|\tau - 1| \text{ for } \phi > \phi_{sd} \text{ as } \tau \longrightarrow 1 \tag{76}$$

where

$$N'_{z5} = \frac{2pabc_s}{\pi\mu\rho} \frac{\sin^2\phi \cos\phi(1-l^2\sin^2\phi)^{1/2}}{(a^2\sin^2\phi - c_s^2)^{1/2} N_3} \tag{77}$$

$$N'_{z3} = \frac{8pabc_s}{\pi\mu\rho} \frac{\sin^4\phi \cos^2\phi (l^2\sin^2\phi - 1)}{(a^2\sin^2\phi - c_s^2)^{1/2} N_4} \tag{78}$$

$$N'_{z6} = - \frac{2pabc_d}{\pi\mu\rho} \frac{\sin^2\phi \cos\phi (l^2\sin^2\phi - 1)^{1/2}(\cos^2\phi - \sin^2\phi)^2}{(c_s^2 - a^2\sin^2\phi)^{1/2} N_4} \tag{79}$$

$$N_4 = \left[l^2(\cos^2\phi - \sin^2\phi)^4 + 16\sin^4\phi \cos^2\phi(l^2\sin^2\phi - 1)\right] \tag{80}$$

For the approximate evaluation of the displacements at the wave fronts $\tau = \tau_{sa}$ and $\tau = \tau_{sd}$ we follow similar procedure as followed for the evaluation of u_{zd} as $\tau \longrightarrow \tau_{da}$ and we find that

$$[u_z] \longrightarrow N_{z5} \quad \text{as } \tau \longrightarrow \tau_{sa} \text{ for } a > c_d \tag{81}$$

$$[u_z] \longrightarrow N_{z6} \quad \text{as } \tau \longrightarrow \tau_{sa} \text{ for } c_d > a > c_s \tag{82}$$

$$[u_z] \longrightarrow N_{z3}(\tau - \tau_{sd})^{3/2} \quad \text{as } \tau \longrightarrow \tau_{sd} \text{ for } a > c_d \tag{83}$$

$$[u_z] \longrightarrow N_{z7}(\tau - \tau_{sd}) \quad \text{as } \tau \longrightarrow \tau_{sd} \text{ for } a < c_d \tag{84}$$

where

$$N_{z5} = - \frac{4Pbc_d \; A_s' \; [(a^2-c_d^2)D_s]^{1/2}}{\pi\mu a^2 (2K_s B_s A_s)^{1/2}} \tag{85}$$

$$N_{z6} = - \frac{16Pa^2 bc_d^3(c_d^2-a^2) \; A_s' \; [(a^2-c_s^2)D_s]^{1/2}}{\pi\mu(2K_s l^2 A_s)^{1/2} \; [l^6(a^2-2c_s^2)^4 - 16c_d^4(c_d^2-a^2)(a^2-c_s^2)]} \tag{86}$$

$$N_{z3} = - \frac{4Pab}{\pi\mu} A_{sd} \; B_{sd}^2 \; B_{sd}' \; A_{sd}' \left[\frac{2 \; \mathrm{cosec}\phi}{a^2 - c_d^2} \right]^{1/2} \tag{87}$$

$$N_{z7} = \frac{4Pab}{\pi\mu} A_{sd} \; B_{sd}^2 \; A_{sd}' \left[\frac{2 \; \mathrm{cosec}\phi}{c_d^2 - a^2} \right]^{1/2} \tag{88}$$

$$A_s' = \frac{l c_d \; (a^2 - c_s^2)^{1/2}}{\rho \; [\; l(a^2-c_s^2)^{1/2} \sin\phi - c_d \cos\phi \;]} \tag{89}$$

$$D_s = 8a^2 l c_d \tau_{sa} \sin\phi \cos\phi \, (a^2 - c_s^2)^{1/2} \tag{90}$$

$$B_s = \frac{1}{a^4} \left[l^3 (a^2 - 2c_s^2)^2 + 4c_d^2 \left\{ (a^2 - c_d^2)(a^2 - c_s^2) \right\}^{1/2} \right] \tag{91}$$

$$A_s = \left[\tau_{sa} a^2 b^2 (\tau_{sa} - \tau_{sa}^{o}) \sin^2\phi + (a^2 - b^2) a^2 \cos^2\phi (\tau_{sa} + \tau_{sa}^{o}) \right] \tag{92}$$

$$A_{sd} = \frac{\pi}{4} \left[\frac{2(l^2-1)^{1/2}}{(l^2-1)^{1/2} \sin\phi - \cos\phi} \right]^{1/2} \tag{93}$$

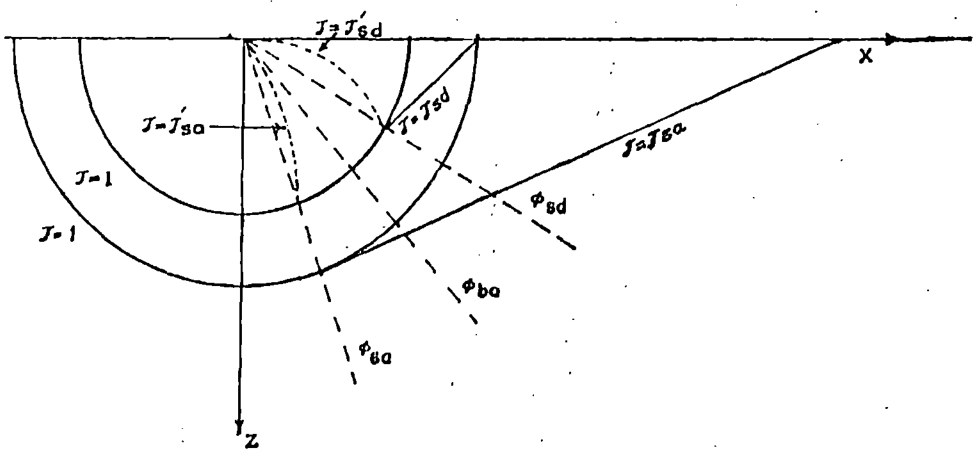
$$B_{sd} = (l^2-2)^{-1} \tag{94}$$

$$B'_{sd} = 4 \, A_{sd} \, (l^2 - 1)^{1/2} \, B_{sd}^2 \tag{95}$$

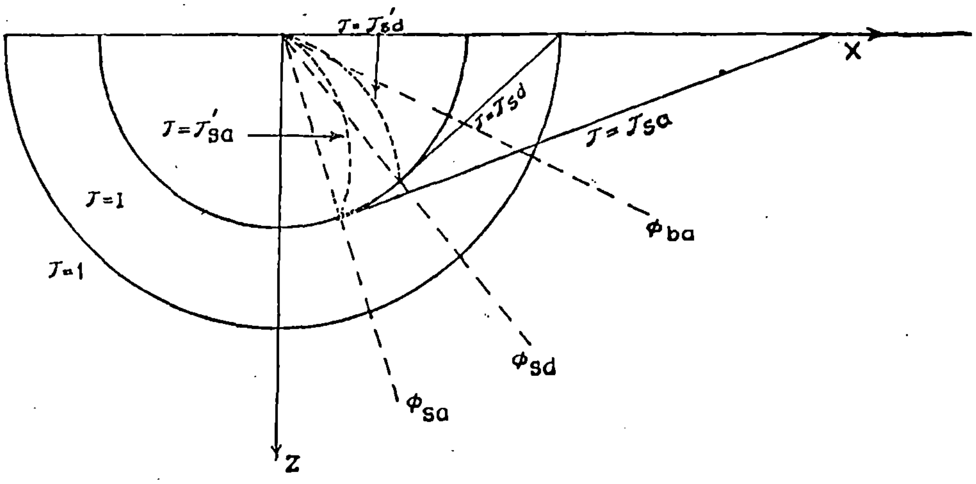
$$A'_{sd} = \frac{c_d}{\rho} (l^2-1)^{1/2} \left[(l^2-1)^{1/2} \sin\phi - \cos\phi \right]^{-1} \tag{96}$$

In these expressions the notations $[u_z]$ stands for the change in u_z across a wave front and N_{z1} etc. are wave front coefficients.

It may also be noted that if we put a = b in this problem, it reduces to the problem of uniformly expanding circular ring source and in that case our derived results coincide with the results given in the paper of Gakenheimer [1971].

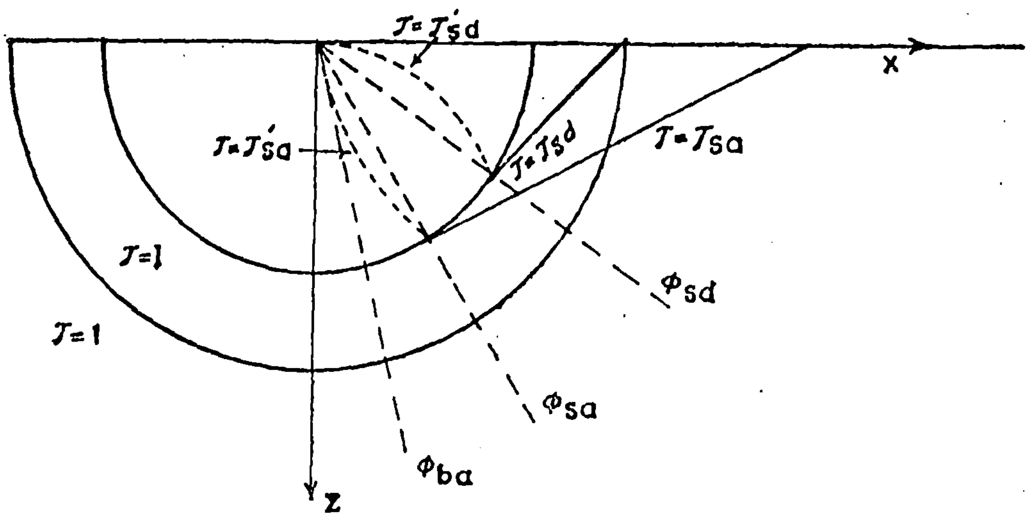


4 (a) for $a > c_d$, $a > b > c_s$, $a\,c_s > b\,c_d$.

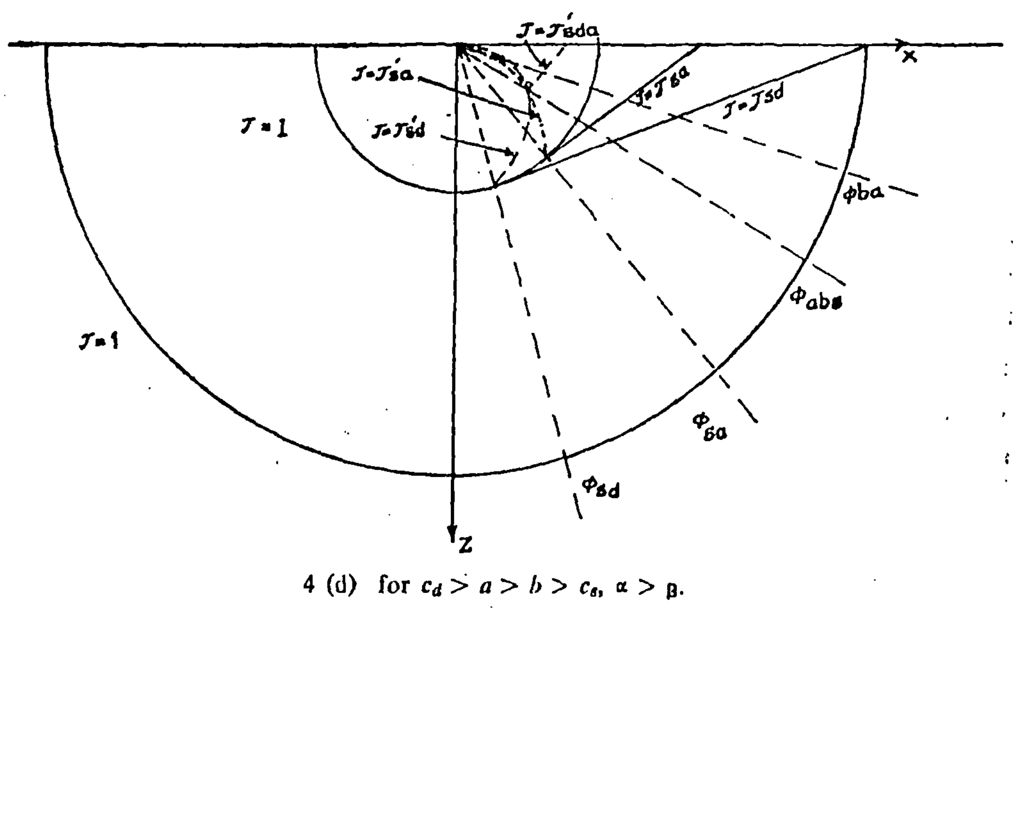


4 (b) for $a > c_d$, $a > b > c_s$, $a\,c_s < bc_d$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

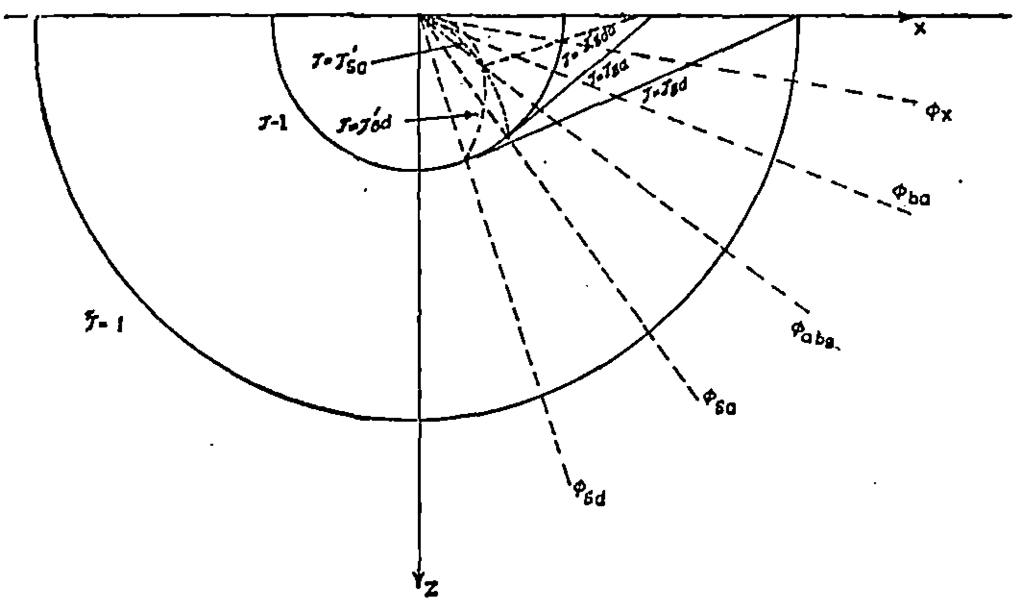


4 (c) for $a > c_d > c_s > b$.

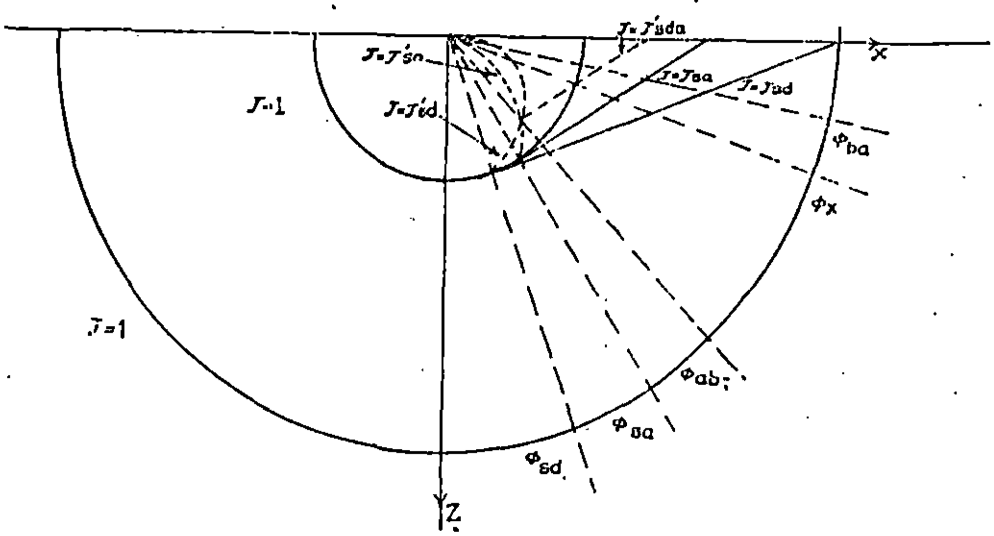


4 (d) for $c_d > a > b > c_s$, $\alpha > \beta$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

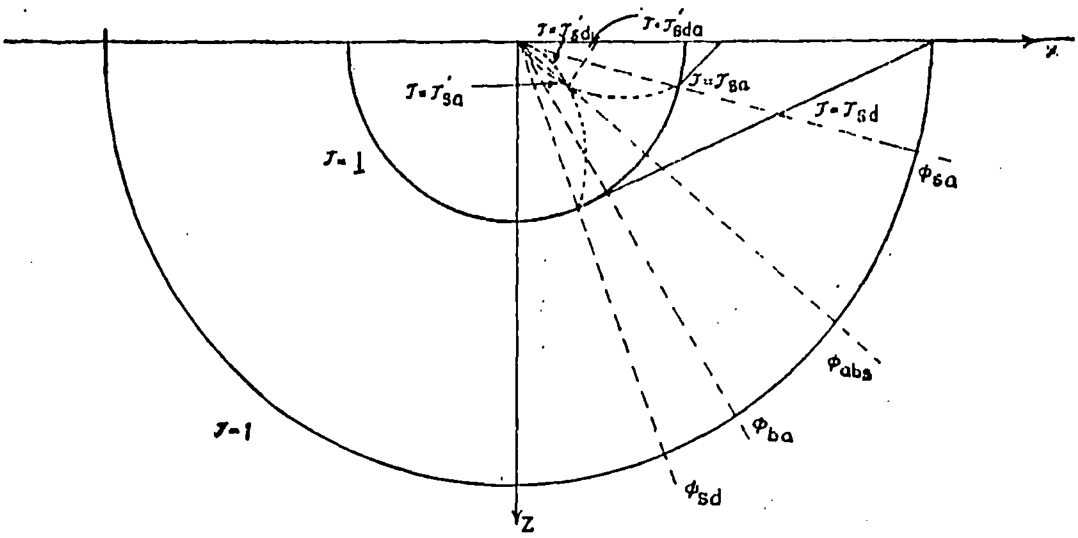


4 (e) for $c_d > a > b > c_s$, $\beta > \alpha > \gamma$.

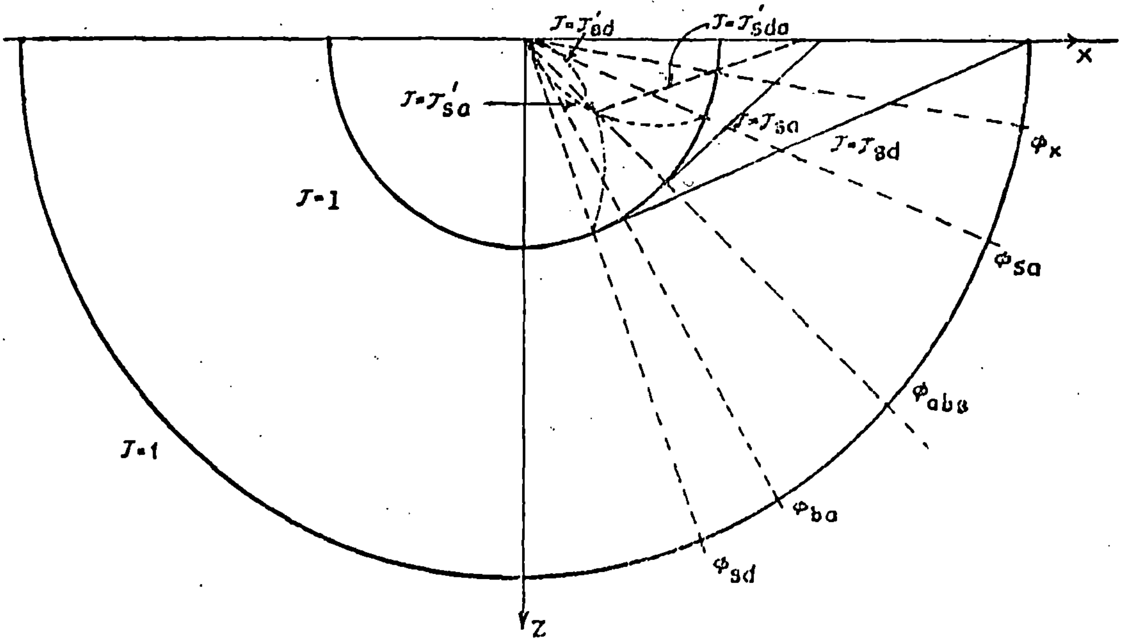


4 (f) for $c_d > a > b > c_s$, $\alpha < \gamma$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

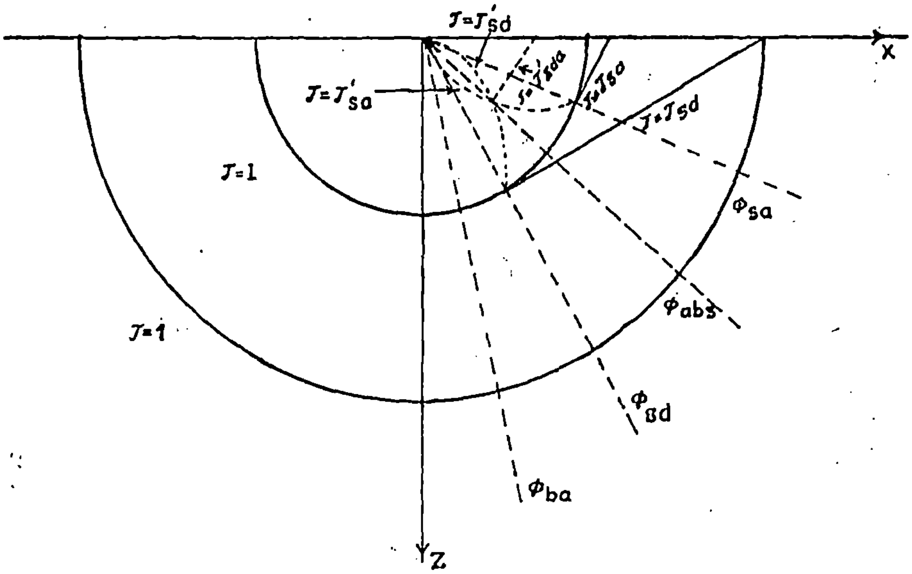


4 (g) for $c_d > a > c_s > b$, $\alpha > \beta$, $ac_s < bc_d$.

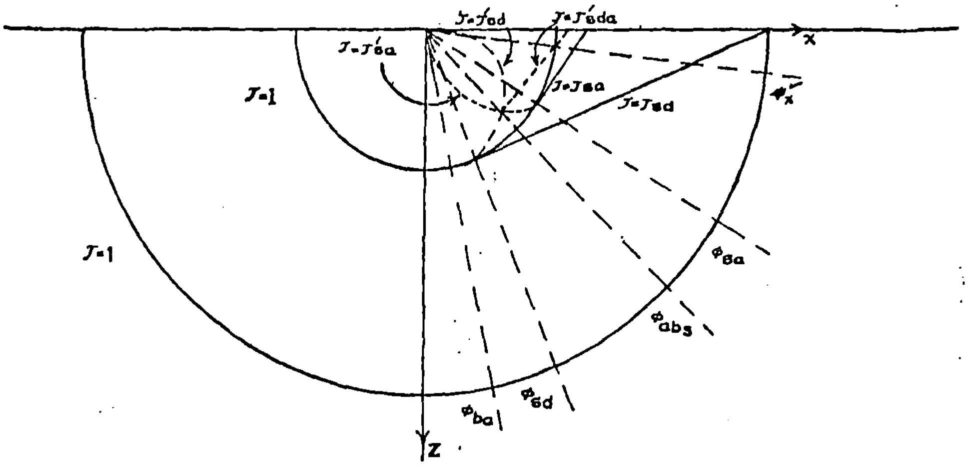


4 (h) for $c_d > a > c_s > b$, $\beta > \alpha > \gamma'$, $ac_s < bc_d$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

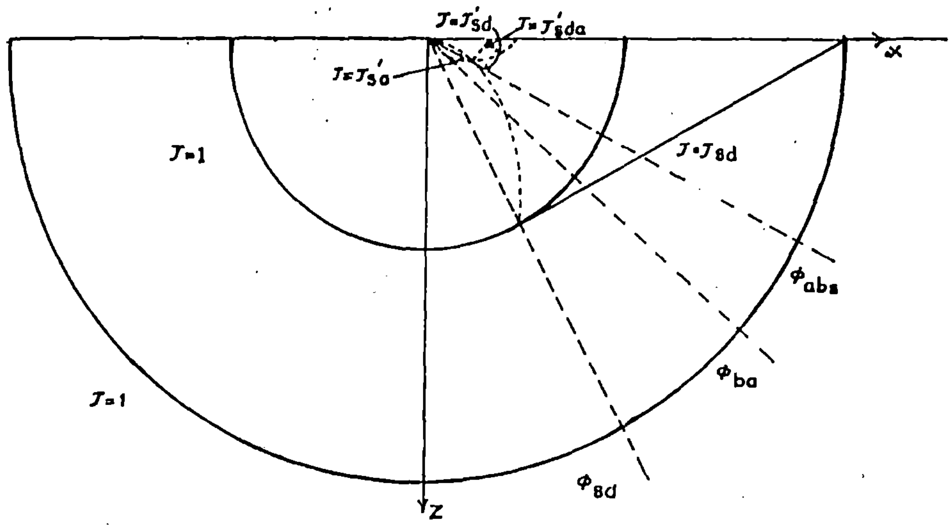


4 (i) for $c_d > a > c_s > b$, $\alpha > \beta$, $a\,c_s > b\,c_d$.

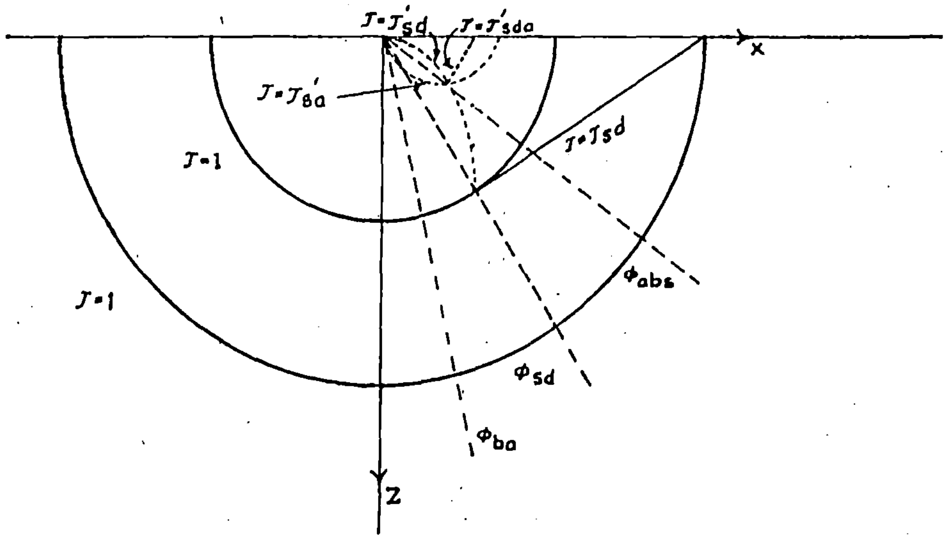


4 (j) for $c_d > a > c_s > b$, $\beta > \alpha > \gamma'$, $ac_t > bc_d$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.



4 (k) for $a < c_s$, $ac_s < bc_d$.



4 (l) for $a < c_s$, $ac_s > b c_d$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

CHAPTER - II

CRACK PROBLEMS IN ELASTODYNAMICS

HIGH FREQUENCY SCATTERING OF ANTIPLANE SHEAR WAVES BY AN INTERFACE CRACK

1. INTRODUCTION.

Scattering of elastic waves by a crack of finite length at the interface of two dissimilar elastic materials is important in view of its application in Geophysics and in Mechanical engineering problems. The extensive use of composite materials in modern technology has created interest in the wave propagation problems in layered media with interfacial discontinuities. The diffraction of Love waves by a crack of finite width at the interface of a layered half space was studied by Neerhoff [1979]. Kuo [1984] carried out numerical and analytical studies of transient response of an interfacial crack between two dissimilar orthotropic half spaces. Following the method of Mal [1970], Srivastava et al. [1980] also considered the low frequency aspect of the interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half space.

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But high frequency solution of the diffraction of elastic waves by a crack of finite size is interesting in view of the fact that transient solution close to the wave front can be represented by an integral of high frequency component of the solution. Green's function method together with a function-theoretic technique based upon an extended Wiener-Hopf argument has been developed by Keogh [1985 a], [1985 b] for solving the problem of high frequency scattering of elastic waves by a Griffith crack situated in an infinite homogeneous elastic medium.

In the present paper, we have derived the high frequency solution of the diffraction of SH-wave when it interacts with a Griffith crack located at the interface of two bonded dissimilar elastic half spaces. To solve the problem, following the method of Chang [1971], the problem has been formulated as an extended Wiener-Hopf equation and the asymptotic solutions for high frequencies or for wavelengths short compared to the length of the crack have been derived. Expressions for the dynamic stress intensity factor and the crack opening displacement have been obtained and the results have been illustrated graphically for two pairs of different types of material.

2. FORMULATION OF THE PROBLEM

Let (x,y,z) be a rectangular Cartesian coordinates. Let an open crack of finite length 2L be located at the interface of two bonded dissimilar elastic semi-infinite solids lying parallel to x-axis. The x-axis is taken along the interface, y-axis vertically upwards into the medium and z-axis is perpendicular to the plane of the paper. (μ_1,ρ_1) and (μ_2,ρ_2) are coefficients of rigidity and density respectively of the upper and lower semi-infinite medium. The crack is subjected to a normally incoming antiplane shear wave originating at $y = -\infty$.

We are interested in finding the high frequency solution of the diffraction problem i.e. the solution when the length of the crack is large compared to the wave length of the incident wave.

Accordingly we shall have to solve the problem when the crack is subject to the following boundary conditions:

$$\sigma_{yz}^{(1)}(x,0+) = \sigma_{yz}^{(2)}(x,0-) = -P_S - P_0 e^{-i\omega t}, \quad |x| < L \qquad (1)$$

$$\sigma_{yz}^{(1)}(x,0+) = \sigma_{yz}^{(2)}(x,0-), \quad |x| > L \qquad (2)$$

$$w_1(x,0+) = w_2(x,0-), \quad |x| > L \qquad (3)$$

where ω is the circular frequency and P_S is the static pressure.

Assume

$$w_1(x,y,t) = W_1(x,y)\, e^{-i\omega t} \quad (4)$$

$$w_2(x,y,t) = W_2(x,y)\, e^{-i\omega t} \quad (5)$$

where W_1 and W_2 satisfy the following two wave equations

$$\nabla^2 W_1(x,y) + k_1^2 W_1(x,y) = 0 \quad (6)$$

$$\nabla^2 W_2(x,y) + k_2^2 W_2(x,y) = 0 \quad (7)$$

with $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

The shear wave number k_1 and k_2 are related to the two shear wave velocities C_1 and C_2 of medium (1) and (2) respectively by

$$k_1 = \omega/C_1 \quad (8) \qquad k_2 = \omega/C_2 \quad (9)$$

Without any loss of generality we assume that $k_2 > k_1$.

Let $$\sigma_{yz}^{(1)}(x,y,t) = \tau_{yz}^{(1)}(x,y)\, e^{-i\omega t} \quad (10)$$

$$\sigma_{yz}^{(2)}(x,y,t) = \tau_{yz}^{(2)}(x,y)\, e^{-i\omega t} \quad (11)$$

In the boundary condition (1), P_S is the static pessure assumed to be sufficiently large so that crack faces do not come in contact during vibration. Since we are interested in the dynamic part of the stress distribution, so the boundary conditions (1), (2) and

(3) may be written as

$$\tau_{yz}^{(1)}(x,0+) = \tau_{yz}^{(2)}(x,0-) = -P_o, \quad |x| < L \tag{12}$$

$$\tau_{yz}^{(1)}(x,0+) = \tau_{yz}^{(2)}(x,0-), \quad |x| > L \tag{13}$$

and

$$W_1(x,0+) = W_2(x,0-), \quad |x| > L \tag{14}$$

that is

$$\mu_1 \frac{\partial W_1}{\partial y} = \mu_2 \frac{\partial W_2}{\partial y} = - P_o, \quad |x|<L, \quad y = 0 \tag{15}$$

$$\mu_1 \frac{\partial W_1}{\partial y} = \mu_2 \frac{\partial W_2}{\partial y}, \quad |x|>L, \quad y = 0 \tag{16}$$

and $$W_1(x,0+) = W_2(x,0-), \quad |x|>L \tag{17}$$

In order to obtain solutions of wave equations (6) and (7) we introduce Fourier transform defined by

$$\bar{W}(\alpha,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(x,y)\, e^{i\alpha x}\, dx \tag{18}$$

Thus we obtain the transformed wave equations as

$$\frac{d^2 \bar{W}_1}{dy^2} - (\alpha^2-k_1^2)\bar{W}_1 = 0 \tag{19}$$

$$\frac{d^2 \bar{W}_2}{dy^2} - (\alpha^2 - k_2^2)\bar{W}_2 = 0 \qquad (20)$$

The solutions of (19) and (20), bounded as y tends to infinity, are

$$\bar{W}_1(\alpha,y) = A_1(\alpha)\, e^{-\gamma_1 y} \quad , \quad y \geq 0 \qquad (21)$$

$$\bar{W}_2(\alpha,y) = A_2(\alpha)\, e^{\gamma_2 y} \quad , \quad y \leq 0 \qquad (22)$$

where $\gamma_1 = (\alpha^2 - k_1^2)^{1/2}$ (23) $\gamma_2 = (\alpha^2 - k_2^2)^{1/2}$ (24)

Introducing for a complex α

$$G_+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_L^\infty \tau_{yz}^{(1)}(x,0)\, e^{i\alpha(x-L)}\, dx \qquad (25)$$

$$G_-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-L} \tau_{yz}^{(1)}(x,0)\, e^{i\alpha(x+L)}\, dx \qquad (26)$$

and

$$G_1(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} \tau_{yz}^{(1)}(x,0)\, e^{i\alpha x}\, dx \qquad (27)$$

the transformed stress at the interface y = 0 can be written as

$$\bar{\tau}_{yz}^{(1)}(\alpha,0) = G_+(\alpha)\, e^{i\alpha L} + G_1(\alpha) + G_-(\alpha)\, e^{-i\alpha L} \qquad (28)$$

Using the boundary condition (12) we note that

$$G_1(\alpha) = -\frac{P_o}{\sqrt{2\pi}\ i\alpha}\left[e^{i\alpha L} - e^{-i\alpha L}\right] \tag{29}$$

Further using the fact that

$$\bar{\tau}_{yz}^{(1)}(\alpha,0) = -\mu_1\gamma_1 A_1(\alpha) \tag{30}$$

we obtain from (28)

$$-\mu_1\gamma_1 A_1(\alpha) = G_+(\alpha)e^{i\alpha L} + G_-(\alpha)e^{-i\alpha L} - \frac{P_o}{\sqrt{2\pi}\ i\alpha}\left[e^{i\alpha L} - e^{-i\alpha L}\right] \tag{31}$$

Since from (12) and (13) stress τ_{yz} is continuous at all points of the interface, so we obtain

$$A_2(\alpha) = -\frac{\mu_1\gamma_1}{\mu_2\gamma_2} A_1(\alpha). \tag{32}$$

So (21) and (22) take the forms

$$\bar{W}_1(\alpha,y) = A_1(\alpha)\ e^{-\gamma_1 y}, \quad y \geq 0 \tag{33}$$

$$\bar{W}_2(\alpha,y) = -\frac{\mu_1\gamma_1}{\mu_2\gamma_2} A_1(\alpha)\ e^{\gamma_2 y}, \quad y \leq 0 \tag{34}$$

Now $$\bar{W}_1(\alpha,0+) - \bar{W}_2(\alpha,0-) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} \Big[W_1(x,0+) - W_2(x,0-) \Big] e^{i\alpha x}\, dx$$

$$= B(\alpha), \qquad \text{(say)} \tag{35}$$

wich is the measure of the discontinuity of displacement along the surface of the crack. From (35) we get

$$A_1(\alpha) = \frac{\mu_2\gamma_2 B(\alpha)}{\mu_1\gamma_1 + \mu_2\gamma_2} \tag{36}$$

Eliminating $A_1(\alpha)$ from (31) and (36) we obtain an extended Wiener-Hopf equation, namely

$$G_+(\alpha)\, e^{i\alpha L} + G_-(\alpha)\, e^{-i\alpha L} + B(\alpha)K(\alpha) = \frac{P_0}{\sqrt{2\pi}\; i\alpha}\left[e^{i\alpha L} - e^{-i\alpha L}\right] \tag{37}$$

where

$$K(\alpha) = \frac{\mu_1\mu_2\gamma_1\gamma_2}{\mu_1\gamma_1 + \mu_2\gamma_2} = \frac{\mu_1\mu_2(\alpha^2-k_1^{\,2})^{1/2}}{(\mu_1+\mu_2)}\, R(\alpha) \tag{38}$$

$$R(\alpha) = \frac{(\mu_1+\mu_2)\;(\alpha^2-k_2^{\,2})^{1/2}}{\mu_1(\alpha^2-k_1^{\,2})^{1/2} + \mu_2(\alpha^2-k_2^{\,2})^{1/2}} \tag{39}$$

In order to solve the Wiener-Hopf equation given by (37) we assume that the branch points $\alpha = k_1$ and k_2 of $K(\alpha)$ possess a small imaginary part such that

$$k_1 = k_1 + ik_1' \quad \text{and} \quad k_2 = k_2 + ik_2'$$

where k_1' and k_2' are infinitesimally small positive quantities which would ultimately be made to tend to zero.

Now we write $K(\alpha) = K_+(\alpha)K_-(\alpha)$ where $K_+(\alpha)$ is analytic in the upper half plane $\mathrm{Im}\,\alpha > -k_2'$ whereas $K_-(\alpha)$ is analytic in the lower half plane given by $\mathrm{Im}\,\alpha < k_2'$. Since $\tau_{yz}(x,0)$ decreases exponentially as $x \longrightarrow \pm\infty$, $G_+(\alpha)$ and $G_-(\alpha)$ have the same common region of regularity as $K_+(\alpha)$ and $K_-(\alpha)$.

Now (37) can easily be expressed as two integral equations relating $G_+(\alpha)$, $G_-(\alpha)$ and $B(\alpha)$ as follows:

$$\frac{G_+(\alpha)}{K_+(\alpha)} - \frac{P_0}{\sqrt{2\pi}\; i\alpha}\left[\frac{1}{K_+(\alpha)} - \frac{1}{K_+(0)}\right] +$$

$$+ \frac{1}{2\pi i}\int_{C_+} \frac{e^{-2isL}}{(s-\alpha)K_+(s)}\left[G_-(s) + \frac{P_0}{\sqrt{2\pi}\; is}\right] ds$$

$$= - B(\alpha)\, K_-(\alpha)\, e^{-i\alpha L} + \frac{P_0}{\sqrt{2\pi}\; i\alpha\, K_+(0)} -$$

$$- \frac{1}{2\pi i}\int_{C_-} \frac{e^{-2isL}}{(s-\alpha)K_+(s)}\left[G_-(s) + \frac{P_0}{\sqrt{2\pi}\; is}\right] ds \qquad (40)$$

and

$$\frac{G_-(\alpha)}{K_-(\alpha)} + \frac{P_o}{\sqrt{2\pi}\ i\alpha\ K_-(\alpha)} + \frac{1}{2\pi i}\int_{C_-} \frac{e^{2isL}}{(s-\alpha)\ K_-(s)}\left[G_+(s) - \frac{P_o}{\sqrt{2\pi}\ is}\right]ds$$

$$= -B(\alpha)K_+(\alpha)\ e^{i\alpha L} - \frac{1}{2\pi i}\int_{C_+} \frac{e^{2isL}}{(s-\alpha)\ K_-(s)}\left[G_+(s) - \frac{P_o}{\sqrt{2\pi}\ is}\right]ds \quad (41)$$

where C_+ and C_- are the straight contours below the pole at s = 0 and situated within the common region of regularity of $G_+(s)$, $G_-(s)$, $K_+(s)$, and $K_-(s)$ as shown in Fig. 1.

In (40), the left-hand side is analytic in the upper half plane whereas the right-hand side is analytic in the lower-half plane and both of them are equal in the common region of analyticity of these two functions. So by analytic continuation, both sides of (40) are analytic in the whole of the s-plane. Now since

$$\tau_{yz} \sim (x \mp L)^{-1/2} \qquad \text{as } x \longrightarrow \pm L$$

so,

$$G_{\pm}(\alpha) \sim \alpha^{-1/2} \qquad \text{as } |\alpha| \longrightarrow \infty$$

and also

$$K_{\pm}(\alpha) \sim \alpha^{1/2} \qquad \text{as } |\alpha| \longrightarrow \infty$$

so it follows that

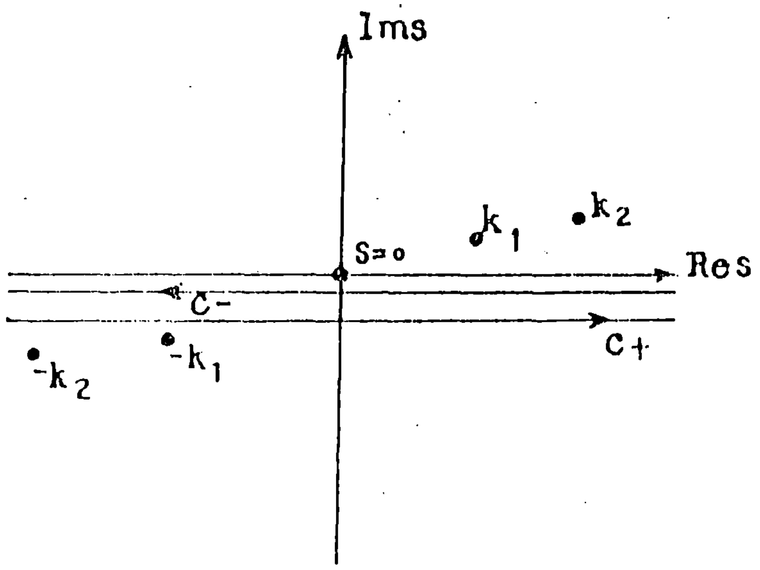

Fig. 1. Path of integration in the complex s-plane.

$$\frac{G_{\pm}(\alpha)}{K_{\pm}(\alpha)} \sim \alpha^{-1} \quad \text{as } |\alpha| \longrightarrow \infty.$$

Therefore by Liouville's theorem, both sides of (40) are equal to zero. Equation (41) can be treated similarly.

Therefore from (40) and (41) we obtain the system of integral equations given by

$$\left[G_{+}(\alpha) - \frac{P_0}{\sqrt{2\pi}\, i\alpha} \right] \frac{1}{K_{+}(\alpha)} + \frac{P_0}{\sqrt{2\pi}\, i\alpha\, K_{+}(0)} +$$

$$+ \frac{1}{2\pi i} \int_{C_+} \frac{e^{-2isL}}{(s-\alpha)\, K_{+}(s)} \left[G_{-}(s) + \frac{P_0}{\sqrt{2\pi}\, is} \right] ds = 0 \qquad (42)$$

and

$$\left[G_{-}(\alpha) + \frac{P_0}{\sqrt{2\pi}\, i\alpha} \right] \frac{1}{K_{-}(\alpha)} +$$

$$+ \frac{1}{2\pi i} \int_{C_-} \frac{e^{2isL}}{(s-\alpha)\, K_{-}(s)} \left[G_{+}(s) - \frac{P_0}{\sqrt{2\pi}\, is} \right] ds = 0 \qquad (43)$$

Since $\tau_{yz}^{(1)}(x,0)$ is an even function of x, so from (25) and (26) it can be shown that $G_{+}(-\alpha) = G_{-}(\alpha)$ and it has been shown in the Appendix that $K_{+}(-\alpha) = iK_{-}(\alpha)$. Using these results and replacing α by $-\alpha$ and s by -s in (42) it can easily be shown that equations (42) and (43) are identical. So $G_{+}(\alpha)$ and $G_{-}(\alpha)$ are to be determined from any one of the integral equation (42) or (43).

3. HIGH FREQUENCY SOLUTION OF THE INTEGRAL EQUATION

To solve the integral equation (43) in the case when normalized wave number $k_1 L \gg 1$, the integration along the path C_- in (43) is replaced by the integration round the circular contour C_0 round the pole at $s = 0$ and by the integration along the contours C_{k_1} and C_{k_2} round the branch cuts through the branch points k_1 and k_2 of the function $K_-(s)$ as shown in Fig. 2.

Thus equation (43) takes the form

$$\left[G_-(\alpha) + \frac{P_0}{\sqrt{2\pi}\; i\alpha} \right] - \frac{P_0\; K_-(\alpha)}{\sqrt{2\pi}\; i\alpha\; K_-(0)} +$$

$$+ \frac{K_-(\alpha)}{2\pi i} \int_{C_{k_1}+C_{k_2}} \frac{e^{2isL}}{(s-\alpha)K_-(s)} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\; is} \right] ds = 0 \qquad (44)$$

Now

$$\int_{C_{k_1}} \frac{e^{2isL}}{(s-\alpha)K_-(s)} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\; is} \right] ds$$

$$= \frac{1}{\mu_1} \int_{C_{k_1}} \frac{e^{2isL}\; K_+(s)}{(s-\alpha)(s^2-k_1^2)^{1/2}} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\; is} \right] ds$$

which can easily be evaluated when $k_1 L \gg 1$ and is found to be equal to

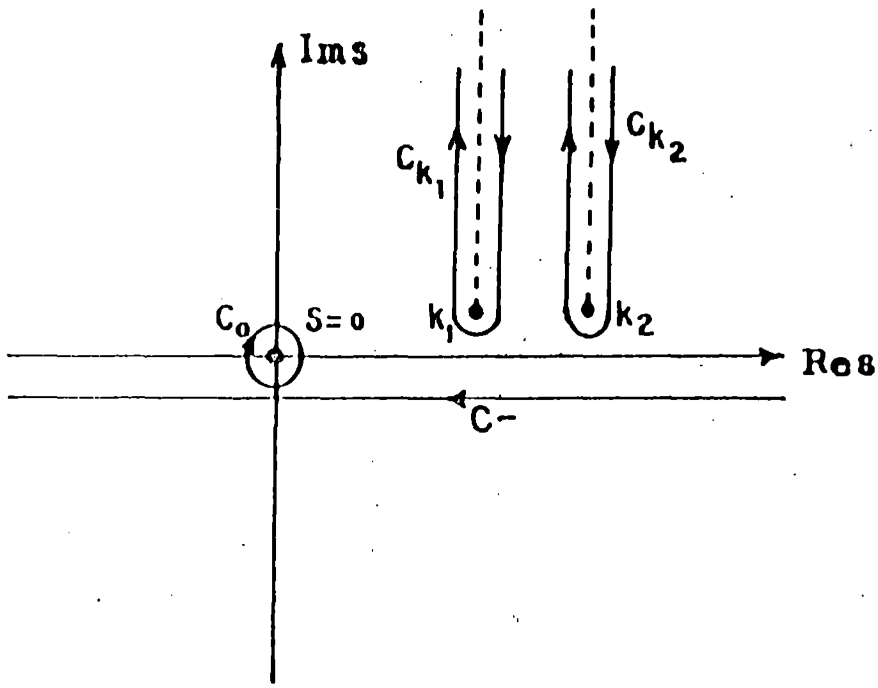

FIG. 2. Path of integration C_0, C_{k_1}, C_{k_2}.

$$-\frac{1}{\mu_1}\sqrt{\frac{\pi}{k_1 L}}\ \frac{e^{2ik_1L}\ K_+(k_1)\ e^{i\pi/4}}{(k_1-\alpha)}\left[G_+(k_1) - \frac{P_0}{\sqrt{2\pi}\ ik_1}\right] \qquad (45)$$

Similarly for $k_1 L >> 1$

$$\int_{C_{k_2}} \frac{e^{2isL}}{(s-\alpha)\ K_-(s)}\left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\ is}\right] ds$$

$$= -\frac{1}{\mu_2}\sqrt{\frac{\pi}{k_2 L}}\ \frac{e^{2ik_2L}\ K_+(k_2)\ e^{i\pi/4}}{(k_2-\alpha)}\left[G_+(k_2) - \frac{P_0}{\sqrt{2\pi}\ ik_2}\right] \qquad (46)$$

Using the results (45) and (46) and also the relations $G_+(-\alpha) = G_-(\alpha)$ and $K_-(-\alpha) = -iK_+(\alpha)$, we obtain from (44)

$$F_+(-\alpha) + \frac{A(k_1)F_+(k_1)e^{2ik_1L}}{\mu_1(k_1-\alpha)\sqrt{k_1 L}} + \frac{A(k_2)F_+(k_2)e^{2ik_2L}}{\mu_2(k_2-\alpha)\sqrt{k_2 L}} = C(\alpha). \qquad (47)$$

where

$$F_+(\xi) = \frac{1}{K_-(-\xi)}\left[G_+(\xi) - \frac{P_0}{\sqrt{2\pi}\ i\xi}\right] \qquad (48)$$

$$A(\xi) = \frac{[K_+(\xi)]^2\ e^{i\pi/4}}{2\sqrt{\pi}} \qquad (49)$$

and

$$C(\xi) = \frac{P_0}{\sqrt{2\pi}\ iK_-(0)\xi} \qquad (50)$$

Substituting $\alpha = -k_1$ and $\alpha = -k_2$ in (47), we obtain respectively the equations

$$\left[1 + \frac{A(k_1)e^{2ik_1L}}{2\mu_1 k_1 \sqrt{k_1 L}} \right] F_+(k_1) + \frac{A(k_2)F_+(k_2)e^{2ik_2L}}{\mu_2 (k_1+k_2)\sqrt{k_2 L}} = -C(k_1) \quad (51)$$

and

$$\frac{A(k_1)e^{2ik_1L}}{\mu_1 (k_1+k_2)\sqrt{k_1 L}} F_+(k_1) + \left[1 + \frac{A(k_2)e^{2ik_2L}}{2\mu_2 k_2 \sqrt{k_2 L}} \right] F_+(k_2) = -C(k_2)$$

(52)

Now solving (51) and (52) we get

$$F_+(k_1) = C(k_1) \left[\frac{A(k_2)\ (k_1-k_2)e^{2ik_2L}}{2\mu_2 k_2 (k_1+k_2)\sqrt{k_2 L}} - 1 \right] U(k_1,k_2) \quad (53)$$

and

$$F_+(k_2) = C(k_2) \left[\frac{A(k_1)\ (k_2-k_1)e^{2ik_1L}}{2\mu_1 k_1 (k_1+k_2)\sqrt{k_1 L}} - 1 \right] U(k_1,k_2) \quad (54)$$

where

$$U(k_1,k_2) = \left[1 + \frac{A(k_1)\ e^{2ik_1L}}{2\mu_1 k_1 \sqrt{k_1 L}} + \frac{A(k_2)\ e^{2ik_2L}}{2\mu_2 k_2 \sqrt{k_2 L}} + \right.$$

$$\left. + \frac{A(k_1)A(k_2)(k_1-k_2)^2 e^{2i(k_1+k_2)L}}{4\mu_1\mu_2 k_1 k_2 (k_1+k_2)^2 \sqrt{Lk_1}\ \sqrt{Lk_2}} \right]^{-1} \quad (55)$$

Now expanding $U(k_1,k_2)$ and neglecting higher order terms of $(k_1L)^{-1/2}$ and $(k_2L)^{-1/2}$ and using (47) we get

$$
\begin{aligned}
G_-(\alpha) &= - C(\alpha)\, K_-(0) + C(\alpha)\, K_-(\alpha) + \\
&+ \frac{K_-(\alpha)A(k_1)e^{2ik_1L}.C(k_1)}{\mu_1(k_1-\alpha)\sqrt{k_1L}}\left[1-\frac{A(k_1)e^{2ik_1L}}{2\mu_1k_1\sqrt{k_1L}}-\frac{A(k_2)k_1e^{2ik_2L}}{\mu_2k_2\sqrt{k_2L}\,(k_1+k_2)}\right]+ \\
&+ \frac{K_-(\alpha)A(k_2)e^{2ik_2L}.C(k_2)}{\mu_2(k_2-\alpha)\sqrt{k_2L}}\left[1-\frac{A(k_1)k_2e^{2ik_1L}}{\mu_1k_1\sqrt{k_1L}\,(k_1+k_2)}-\frac{A(k_2)e^{2ik_2L}}{2\mu_2k_2\sqrt{k_2L}}\right]
\end{aligned}
\tag{56}
$$

Now replacing α by $-\alpha$ and using $C(-\alpha) = - C(\alpha)$. We have

$$
\begin{aligned}
G_+(\alpha) &= C(\alpha)\, K_-(0) - C(\alpha)K_-(-\alpha) + \\
&+ \frac{K_-(-\alpha)A(k_1)e^{2ik_1L}.C(k_1)}{\mu_1(k_1+\alpha)\sqrt{k_1L}}\left[1-\frac{A(k_1)e^{2ik_1L}}{2\mu_1k_1\sqrt{k_1L}}-\frac{A(k_2)k_1e^{2ik_2L}}{\mu_2k_2\sqrt{k_2L}\,(k_1+k_2)}\right]+ \\
&+ \frac{K_-(-\alpha)A(k_2)e^{2ik_2L}.C(k_2)}{\mu_2(k_2+\alpha)\sqrt{k_2L}}\left[1-\frac{A(k_1)k_2e^{2ik_1L}}{\mu_1k_1\sqrt{k_1L}\,(k_1+k_2)}-\frac{A(k_2)e^{2ik_2L}}{2\mu_2k_2\sqrt{k_2L}}\right].
\end{aligned}
\tag{57}
$$

4. STRESS INTENSITY FACTOR AND CRACK OPENING DISPLACEMENT NEAR THE CRACK TIPS

Now as $\alpha \longrightarrow \infty$

$$K_-(-\alpha) = - iK_+(\alpha) = - i(\alpha+k_1)^{1/2}\left[\frac{\mu_1\mu_2}{\mu_1+\mu_2}\right]^{1/2} \simeq - i\alpha^{1/2}\left[\frac{\mu_1\mu_2}{\mu_1+\mu_2}\right]^{1/2}$$

$$\frac{K_-(-\alpha)}{\alpha + k_1} \approx - i\alpha^{-1/2}\left[\frac{\mu_1\mu_2}{\mu_1+\mu_2}\right]^{1/2}$$

$$\frac{K_-(-\alpha)}{\alpha+k_2} \approx -i\alpha^{-1/2}\left[\frac{\mu_1\mu_2}{\mu_1+\mu_2}\right]^{1/2}$$

So as $\alpha \longrightarrow \infty$ we get from (56) and (57)

$$G_+(\alpha) \approx S\,\alpha^{-1/2} + \frac{P_o}{\sqrt{2\pi}\; i\alpha}$$

and

$$G_-(\alpha) \approx - iS\,\alpha^{-1/2} - \frac{P_o}{\sqrt{2\pi}\; i\alpha} \tag{58}$$

where

$$S = \frac{P_0}{\sqrt{2\pi}\, K_-(0)} \left[1 - \frac{A(k_1)\, e^{2ik_1L}}{\mu_1 k_1 \sqrt{k_1 L}} + \frac{A(k_2)\, e^{2ik_2L}}{\mu_2 k_2 \sqrt{k_2 L}} + \right.$$

$$+ \frac{1}{2} \left(\frac{A^2(k_1)\, e^{4ik_1L}}{\mu_1^2\, k_1^2\, k_1\, L} + \frac{A^2(k_2)\, e^{4ik_2L}}{\mu_2^2\, k_2^2\, k_2\, L} \right) +$$

$$\left. + \frac{A(k_1)\, A(k_2)\, e^{2i(k_1+k_2)L}}{\mu_1\, k_1\, \mu_2\, k_2 \sqrt{k_1 L\, k_2 L}} \right] \left(\frac{\mu_1\, \mu_2}{\mu_1 + \mu_2} \right)^{1/2} \tag{59}$$

Now from equation (37) using (58) and also the fact that

$$K(\alpha) \longrightarrow \pm\, \alpha\, \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \quad \text{as } \alpha \longrightarrow \pm\, \infty \tag{60}$$

we get

$$B(\alpha) = \frac{\pm\, S}{\alpha\sqrt{\alpha}} \left[ie^{-i\alpha L} - e^{i\alpha L} \right] \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \quad \text{as } \alpha \longrightarrow \pm\, \infty \tag{61}$$

Taking inverse Fourier-Transform of (35) and using the results of Fresnel integrals viz.

$$\int_0^\infty \frac{\begin{matrix}\sin\\ \cos\end{matrix}(x+L)\alpha}{\sqrt{\alpha}}\, d\alpha = \left(\frac{\pi}{2(x+L)} \right)^{1/2} \tag{62}$$

We get the displacement jump across the surface of the crack as

$$\Delta W = W_1(x,0+) - W_2(x,0-) = 2S_1(1-i)\sqrt{(L-x)} \quad \text{for } x \to L-0 \qquad (63)$$

and

$$\Delta W = W_1(x,0+) - W_2(x,0-) = 2S_1(1-i)\sqrt{(x+L)} \quad \text{for } x \to -L+0 \qquad (64)$$

where $$S_1 = \frac{(\mu_1+\mu_2)}{\mu_1\ \mu_2} S \qquad (65)$$

Next in order to find the value of τ_{xy} near about the crack tip we use (61) in (36) and (32) and to obtain

$$A_j(\alpha) = \frac{(-1)^{j+1} S}{\mu_j\ \alpha\sqrt{\alpha}}\left[ie^{-i\alpha L} - e^{i\alpha L}\right], \ (j = 1,2) \text{ as } \alpha \to \infty \qquad (66)$$

and

$$A_j(\alpha) = \frac{(-1)^{j+1} S}{\mu_j\ \alpha\sqrt{-\alpha}}\left[e^{-i\alpha L} - ie^{i\alpha L}\right], \ (j = 1,2) \text{ as } \alpha \to -\infty \qquad (67)$$

Now

$$\tau_{yz}(x,y) = \mu_j \frac{\partial W_j(x,y)}{\partial y}, \quad j=1,2$$

$$= \mu_j \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_j(\alpha) \exp\left\{- \gamma_j|y| - i\alpha x\right\} d\alpha \right] \qquad (68)$$

Substituting the values of $A_j(\alpha)$ as $|\alpha| \to \infty$, we can write the stress near about the crack tip as

$$\tau_{yz}(x,y) = \frac{S}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-\alpha|y|}}{\sqrt{\alpha}} \left[e^{i\alpha(x+L)} - ie^{i\alpha(x-L)} - ie^{-i\alpha(x+L)} + e^{-i\alpha(x-L)} \right] d\alpha$$

$$= \frac{S(1-i)}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-\alpha|y|}}{\sqrt{\alpha}} \left[\cos\alpha(x+L) - \sin\alpha(x+L) + \cos\alpha(x-L) + \sin\alpha(x-L) \right] d\alpha$$

$$= S\,(1-i) \left[\frac{1}{\sqrt{r_2}} \sin\frac{\phi_2}{2} + \frac{1}{\sqrt{r_1}} \cos\frac{\phi_1}{2} \right] \qquad (69)$$

near about the crack tips, where

$$r_1 = \left[(x-L)^2 + y^2 \right]^{1/2}, \quad \phi_1 = \sin^{-1}\frac{|y|}{r_1} \qquad (70)$$

$$r_2 = \left[(x+L)^2 + y^2 \right]^{1/2}, \quad \phi_2 = \sin^{-1}\frac{|y|}{r_2} \qquad (71)$$

Therefore at the interface (y = 0) we obtain

$$\tau_{yz} \longrightarrow \frac{S(1-i)}{\sqrt{(x-L)}} \quad \text{as } x \longrightarrow L+0 \qquad (72)$$

and

$$\tau_{yz} \longrightarrow \frac{S(1-i)}{\sqrt{-(x+L)}} \quad \text{as } x \longrightarrow -L-0 \qquad (73)$$

Now the stress intensity factor is defined by

$$K = \frac{|(1-i)\ S|\sqrt{2\pi k_1}}{P_0} \qquad (74)$$

The absolute value of the complex stress intensity factor defined by (74) has been plotted against k_1L in Fig.3 for values of $k_1L > 1$ for the following two sets of materials, given by

First Set:

Steel	$\rho_1 = 7.6\ gm/cm^3$	$\mu_1 = 8.32 \times 10^{11}\ dyne/cm^2$
Aluminium	$\rho_2 = 2.7\ gm/cm^3$	$\mu_2 = 2.63 \times 10^{11}\ dyne/cm^2$

Second Set:

Wrought iron	$\rho_1 = 7.8\ gm/cm^3$	$\mu_1 = 7.7 \times 10^{11}\ dyne/cm^2$
Copper	$\rho_2 = 8.96\ gm/cm^3$	$\mu_2 = 4.5 \times 10^{11}\ dyne/cm^2$.

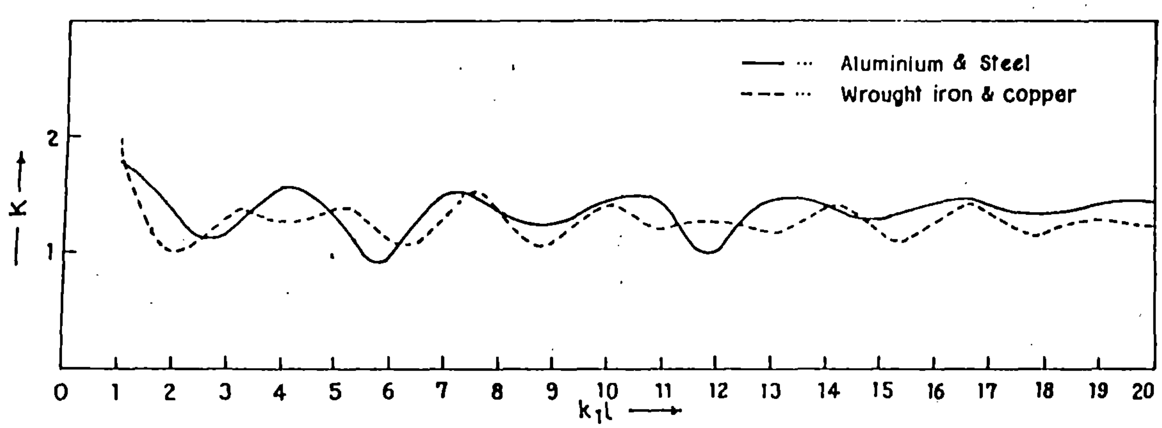

FIG. 3. Stress intensity factor K versus dimensionless frequency $k_1\, l$.

5. CRACK OPENING DISPLACEMENT AT POINTS AWAY FROM THE CRACK TIPS

Next in order to obtain the displacement jump for the large values of $k_1(L-x)$ and $k_1(L+x)$ we write $G_+(\alpha)$ and $G_-(\alpha)$ from (57) and (56) respectively as

$$G_+(\alpha) = \frac{p}{\alpha} - \frac{QK_-(-\alpha)}{\alpha} + \frac{R(k_1,k_2)\ K_-(-\alpha)}{k_1+\alpha} + \frac{R(k_2,k_1)\ K_-(-\alpha)}{k_2+\alpha} \qquad (75)$$

and

$$G_-(\alpha) = -\frac{P}{\alpha} + \frac{QK_-(\alpha)}{\alpha} + \frac{R(k_1,k_2)\ K_-(\alpha)}{k_1-\alpha} + \frac{R(k_2,k_1)\ K_-(\alpha)}{k_2-\alpha} \qquad (76)$$

where
$$P = \frac{P_0}{\sqrt{2\pi}\ i} \qquad (77)$$

$$Q = \frac{P_0}{\sqrt{2\pi}\ iK_-(0)} = \frac{P}{K_-(0)} \qquad (78)$$

and

$$R(k_m,k_n) = \frac{QA(k_m)\ e^{2ik_mL}}{\mu_m k_m \sqrt{Lk_m}}\left[1 - \frac{e^{2ik_mL}\ A(k_m)}{\sqrt{Lk_m}\ 2\mu_m k_m} - \frac{e^{2ik_nL}\ A(k_n)k_m}{\sqrt{Lk_n}\ \mu_n k_n(k_m+k_n)}\right] \qquad (79)$$

where $m = 1$ when $n = 2$

and $m = 2$ when $n = 1$.

Again using $K_-(-\alpha) = -iK_+(\alpha)$ we get from (37)

$$B(\alpha) = -\frac{Qie^{i\alpha L}}{\alpha\, K_-(\alpha)} + \frac{iR(k_1,k_2)\, e^{i\alpha L}}{(k_1+\alpha)\, K_-(\alpha)} + \frac{iR(k_2,k_1)\, e^{i\alpha L}}{(k_2+\alpha)\, K_-(\alpha)} -$$

$$- \frac{Q\, e^{-i\alpha L}}{\alpha\, K_+(\alpha)} - \frac{R(k_1,k_2)\, e^{-i\alpha L}}{(k_1-\alpha)\, K_+(\alpha)} - \frac{R(k_2,k_1)\, e^{-i\alpha L}}{(k_2-\alpha)\, K_+(\alpha)} \qquad (80)$$

From (35) we get the displacement jump across the surface of the crack as

$$W_1(x,0+) - W_2(x,0-) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(\alpha)\, e^{-i\alpha x}\, d\alpha. \qquad (81)$$

Now substituting the expression of $B(\alpha)$ from (80) in (81) and approximately evaluating the integrals arising in (81) term by term for large values of $k_1(L-x)$, $k_2(L-x)$, $k_1(L+x)$ and $k_2(L+x)$ and neglecting terms of order higher than $(k_1L)^{-3/2}$ and $(k_2L)^{-3/2}$, we obtain finally the crack opening displacement across the cracked-surface in the following form:

$$\Delta W = W_1(x,0+) - W_2(x,0-) = 2\pi\ Qi\ K_+(0) \left(\frac{1}{\mu_1 k_1} + \frac{1}{\mu_2 k_2} \right) +$$

$$+ \sqrt{2}\ Q\ e^{-i\pi/4} \left[\left(\frac{e^{ik_1(L-x)}}{\sqrt{k_1(L-x)}} + \frac{e^{ik_1(L+x)}}{\sqrt{k_1(L+x)}} \right) \times \right.$$

$$\times \left\{ R_1 + \frac{R_1\ R_{11}\ e^{2ik_1L}}{\sqrt{2k_1L}} + \frac{R_2\ R_{21}\ e^{2ik_2L}}{\sqrt{2k_2L}} + \frac{R_1(R_{11})^2\ e^{4ik_1L}}{\sqrt{2k_1L}\ \sqrt{2k_1L}} + \right.$$

$$\left. + \frac{R_2R_{22}R_{21}e^{4ik_2L}}{\sqrt{2k_2L}\ \sqrt{2k_2L}} + \frac{R_1R_{12}R_{21}e^{2i(k_1+k_2)L}}{\sqrt{2k_1L}\ \sqrt{2k_2L}} + \frac{R_2R_{21}R_{11}e^{2i(k_1+k_2)L}}{\sqrt{2k_1L}\ \sqrt{2k_2L}} \right\} +$$

$$+ \left(\frac{e^{ik_2(L-x)}}{\sqrt{k_2(L-x)}} + \frac{e^{ik_2(L+x)}}{\sqrt{k_2(L+x)}} \right) \times$$

$$\times \left\{ R_2 + \frac{R_2R_{22}e^{2ik_2L}}{\sqrt{2k_2L}} + \frac{R_1R_{12}e^{2ik_1L}}{\sqrt{2k_1L}} + \frac{R_2(R_{22})^2\ e^{4ik_2L}}{\sqrt{2k_2L}\ \sqrt{2k_2L}} + \right.$$

$$\left.\left. + \frac{R_1R_{11}R_{12}e^{4ik_1L}}{\sqrt{2k_1L}\ \sqrt{2k_1L}} + \frac{R_2R_{21}R_{12}e^{2i(k_1+k_2)L}}{\sqrt{2k_1L}\ \sqrt{2k_2L}} + \frac{R_1R_{12}R_{22}e^{2i(k_1+k_2)L}}{\sqrt{2k_1L}\ \sqrt{2k_2L}} \right\} \right] \tag{82}$$

where

$$R_1 = \frac{K_+(k_1)}{\sqrt{2}\,\mu_1 k_1} \qquad R_2 = \frac{K_+(k_2)}{\sqrt{2}\,\mu_2 k_2}$$

$$R_{11} = \frac{D\,[K_+(k_1)]^2}{\mu_1\ (k_1+k_1)} \qquad R_{22} = \frac{D\,[K_+(k_2)]^2}{\mu_2\ (k_2+k_2)}$$

$$R_{21} = \frac{D\,K_+(k_1)\ K_+(k_2)}{\mu_1\ (k_1+k_2)} \qquad R_{12} = \frac{D\,K_+(k_1)\ K_+(k_2)}{\mu_2\ (k_1+k_2)}$$

$$D = (-1)\,\frac{e^{i\pi/4}}{\sqrt{2\pi}}\,. \tag{83}$$

Expressions in (63) and (64) give the displacement jump nearabout the crack tips where as the displacement jump at points away from the crack tips are given by (82).

From these two results we can obtain the crack opening displacement at any point of the crack surface $-L < x < L$, $y = 0$.

Here also normalized crack opening displacement has been plotted against normalized distance x/L from the centre of the crack for two different sets of materials in Fig. 4. It is interesting to note that oscillatory nature of the crack opening displacement increases with the increase of frequencies as a result of the interference of waves inside the crack. Further we note that amplitude of the crack opening displacement decreases with the increase of frequency.

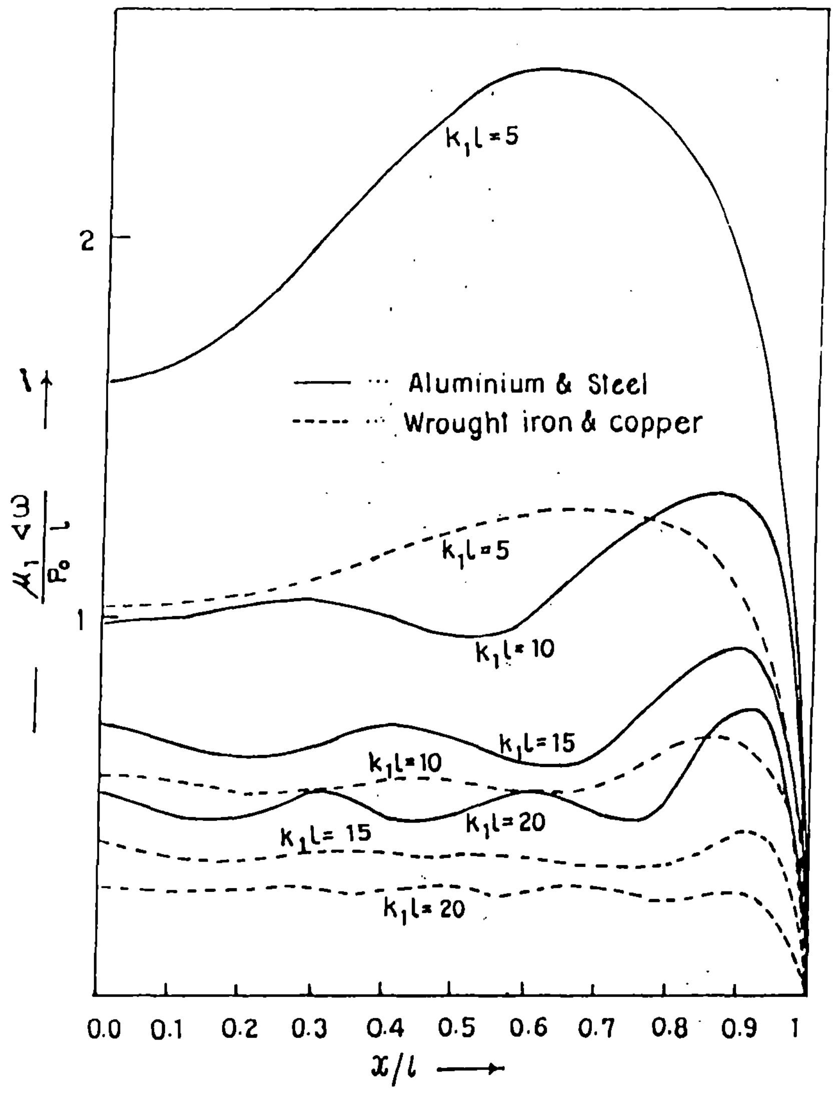

FIG. 4. Normalized crack opening displacement versus normalized distance x/l from the centre of the crack.

Appendix

$$K(\alpha) = \frac{\mu_1 \mu_2 (\alpha^2 - k_1^2)^{1/2}}{(\mu_1 + \mu_2)} R(\alpha)$$

where $$R(\alpha) = \frac{(\mu_1 + \mu_2)(\alpha^2 - k_2^2)^{1/2}}{\mu_1 (\alpha^2 - k_1^2)^{1/2} + \mu_2 (\alpha^2 - k_2^2)^{1/2}}$$

put $$m = \frac{\mu_2}{\mu_1}$$

Therefore

$$K(\alpha) = \frac{\mu_2 (\alpha^2 - k_1^2)^{1/2}}{1 + m} R(\alpha) \tag{A1}$$

where

$$R(\alpha) = \frac{(1+m)\ (\alpha^2 - k_2^2)^{1/2}}{(\alpha^2 - k_1^2)^{1/2} + m(\alpha^2 - k_2^2)^{1/2}} \quad \rightarrow 1 \quad \text{as } |\alpha| \rightarrow \infty$$

Now $$R_+(\alpha)\ R_-(\alpha) = \left[\frac{m}{1+m} + \frac{(\alpha^2 - k_1^2)^{1/2}}{(m+1)(\alpha^2 - k_2^2)^{1/2}} \right]^{-1}$$

Therefore

$\log R_+(\alpha) + \log R_-(\alpha) =$

$$= \text{Log} \left[\frac{m}{1+m} + \frac{(\alpha^2 - k_1^2)^{1/2}}{(m+1)(\alpha^2 - k_2^2)^{1/2}} \right]^{-1} = \log R(\alpha)$$

Therefore

$$\log R_+(\alpha) = \frac{1}{2\pi i} \int_{C_L} \frac{\log R(z)}{(z-\alpha)} dz = \frac{1}{2\pi i} \int_{-ic-\infty}^{-ic+\infty} \frac{\log R(z)}{(z-\alpha)} dz$$

where the path of integration C_L is shown in Fig. 5.

Putting z = -z and using the fact that R(z) = R(-z), we get

$$\log R_+(\alpha) = -\frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{\log R(z)}{(z+\alpha)} dz$$

$$= -\frac{1}{2\pi i} \int_{C_1} \frac{\log R(z)}{(z+\alpha)} dz$$

where C_1 is the contour round the branch points k_1 and k_2 as shown in Fig. 6.

So,

$$\log R_+(\alpha) = \frac{1}{2\pi i} \int_{C_1} \frac{\log \left[\frac{m}{m+1} + \frac{(z^2-k_1^2)^{1/2}}{(m+1)(z^2-k_2^2)^{1/2}} \right]}{(z+\alpha)} dz$$

$$= \frac{1}{2\pi i} \int_{k_1}^{k_2} \frac{\log \left[1 + \frac{i(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}} \right]}{(z+\alpha)} dz \quad -$$

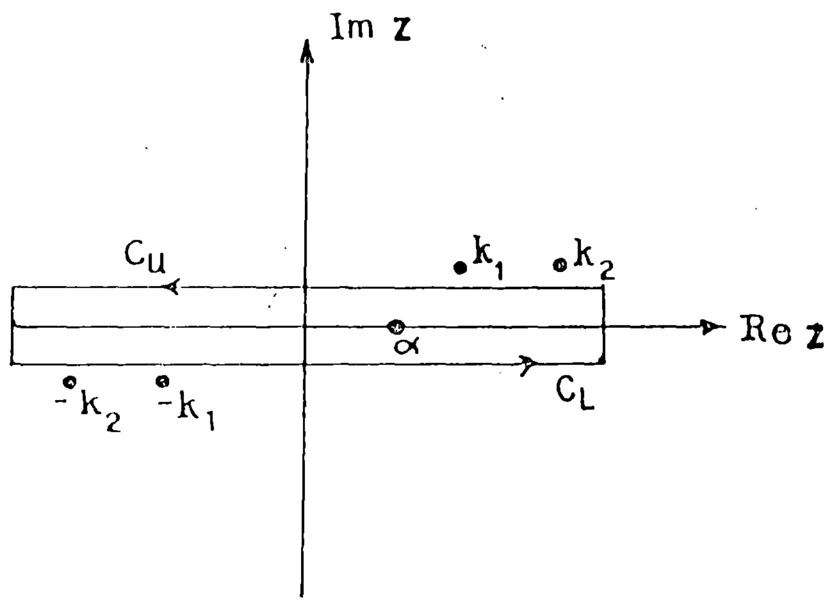

Fig. 5. Complex z-plane.

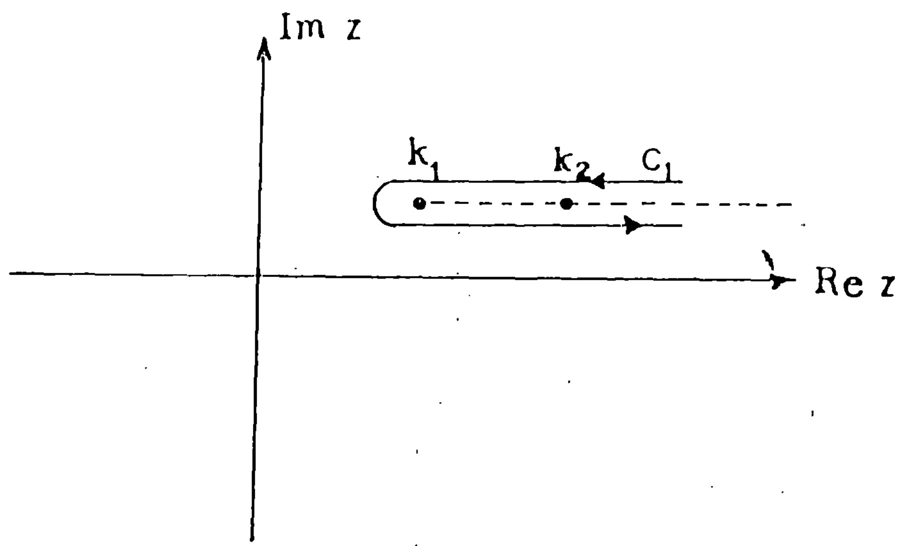

Fig. 6. Path of integration round the branch points.

$$-\frac{1}{2\pi i}\int_{k_1}^{k_2}\frac{\log\left[1-\dfrac{i(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\,dz$$

$$=\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\,dz$$

Therefore $$R_+(\alpha)=\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\,dz\right]$$

Similarly $$R_-(\alpha)=\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right]$$

Therefore from (A1) we can write

$$K_+(\alpha) = \frac{\sqrt{\mu_2}\ (\alpha+k_1)^{1/2}}{\sqrt{(1+m)}} \exp\left[\frac{1}{\pi}\int_{k_1}^{k_2} \frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\,dz\right] \tag{A2}$$

and

$$K_-(\alpha) = \frac{\sqrt{\mu_2}\ (\alpha-k_1)^{1/2}}{\sqrt{(1+m)}} \exp\left[\frac{1}{\pi}\int_{k_1}^{k_2} \frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right] \tag{A3}$$

Hence from (A2) and (A3) we get

$$K_+(-\alpha) = \frac{\sqrt{\mu_2}\, i(\alpha-k_1)^{1/2}}{\sqrt{(1+m)}} \exp\left[\frac{1}{\pi}\int_{k_1}^{k_2} \frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right]$$

$$= iK_-(\alpha)$$

i.e. $$K_+(-\alpha) = iK_-(\alpha) \tag{A4}$$

---x---

HIGH FREQUENCY SCATTERING OF PLANE HORIZONTAL SHEAR WAVES BY A GRIFFITH CRACK PROPAGATING ALONG THE BIMATERIAL INTERFACE

1. INTRODUCTION

Scattering of elastic waves by a stationary or a moving crack of finite length at the interface of two dissimilar elastic materials is important in view of its application in fracture mechanics as well as in seismology. Recently, Takei, Shindo and Atsumi [1982] considered the problem of diffraction of transient horizontal shear waves by a finite crack lying on a bimaterial interface. The method of solution was extended by Ueda, Shindo and Atsumi [1983] to solve the problem of torsional impact response of a penny shaped interface crack. Srivastava et al [1980] also considered the low frequency aspect of the interaction of an antiplane shear wave by a Griffith crack at the interface of two bonded dissimilar elastic half spaces.

In the case of cracks of finite size, travelling at a constant velocity, loads, for mathematical simplicity, are usually assumed to be independent of time. However, in practice, structures

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are often required to sustain oscillating loads where the dynamic disturbances propagate through the elastic medium in the form of stress waves. The problem of diffraction of plane harmonic polarized shear wave by a half plane crack extended under antiplane strain was first studied by Jahanshahi [1967]. Later Chen and Sih [1973] considered the interaction of stress waves with a semi-infinite running crack under either the plane strain or the generalized plane stress condition. Sih and Loeber [1970] and Chen and Sih [1975] also considered the problem of scattering of plane harmonic waves by a running crack of finite length. In both the cases the problem was reduced to a system of simultaneous Fredholm integral equations which were solved numerically.

In the present paper, we have investigated the high frequency solution of the problem of diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface. The high frequency solution of the diffraction of elastic waves by a crack of finite size is important in view of the fact that transient solution close to the wave front can be represented by an integral of the high frequency component of the solution. In order to solve the problem, following the method of Chang [1971], the problem has been formulated as an extended Wiener-Hopf equation and the asymptotic solutions for high frequencies or for wave lengths which are short compared to the length of the crack have been derived. Expressions for the dynamic stress intensity factor at the crack tip and the crack opening displacement have been derived. The dynamic stress intensity factor for high frequencies has been illustrated graphically for two pairs of different types of materials for different crack velocities and angles of incidence.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let a plane crack of width 2L move at a constant velocity V at the interface of two bonded dissimilar elastic semi-infinite media due to the incidence of the plane horizontal SH-wave

$$W_i = A \exp[-\{k_1(X \cos\theta_1 + Y \sin\theta_1) + \Omega T\}] \quad (1)$$

in the medium. The crack lies on the bimaterial interface along Y=0 with respect to the fixed rectangular co-ordinate system (X,Y,Z) as shown in Fig.1.

We assume that the displacement and stress fields W_j, τ_{yz_j} (j=1,2) are

$$W_j = W_j(X,Y,T) \quad (2)$$

$$\tau_{yz_j} = \mu_j \frac{\partial W_j(X,Y)}{\partial Y}, \quad (3)$$

in which subscripts j=1,2 refer to the upper and lower half planes,

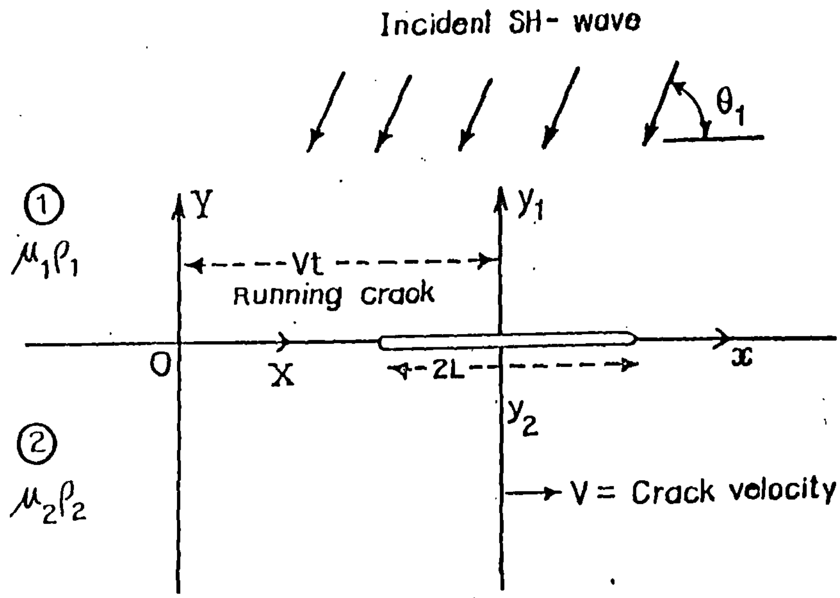

Fig. 1. Running interface crack.

respectively, T denotes time and μ_j is shear modulus of elasticity. The displacement W_j is governed by the classical wave equation

$$\frac{\partial^2 W_j}{\partial X^2} + \frac{\partial^2 W_j}{\partial Y^2} = \frac{1}{c_j^2} \frac{\partial^2 W_j}{\partial T^2} , \qquad (j=1,2) \tag{4}$$

where $c_j=(\mu_j/\rho_j)^{1/2}$ is shear wave velocity and ρ_j is the density of the material. Without any loss of generality, we further assume that $c_1 > c_2$.

Due to the incident wave given by (1), reflected and transmitted waves in the absence of the crack may be written in the form

$$W_r = B \exp [-i\{k_1(X \cos\theta_1 - Y \sin\theta_1) + \Omega T \}] \tag{5}$$

and

$$W_T = C \exp [-i\{k_2(X \cos\theta_2 + Y \sin\theta_2) + \Omega T \}], \tag{6}$$

where

$$B = \frac{k_1 \sin\theta_1 - mk_2 \sin\theta_2}{k_1 \sin\theta_1 + mk_2 \sin\theta_2} A \tag{7}$$

$$C = \frac{2k_1 \sin\theta_1}{k_1 \sin\theta_1 + mk_2 \sin\theta_2} A \tag{8}$$

$$m = \mu_2/\mu_1 \qquad \text{and} \qquad k_1 \cos\theta_1 = k_2 \cos\theta_2 \tag{9}$$

A,B,C are incident, reflected and transmitted wave amplitude, k_j is the wave number, $\Omega = k_j c_j$ is the circular frequency and θ_1, θ_2 are the angles of incidence and refraction, respectively.

A set of moving co-ordinates (x, y_j, z, t) attached to the centre of the crack moving at a constant velocity V is introduced in accordance with

$$x = X - Vt, \qquad y_j = s_j Y, \qquad z = Z, \qquad t = T \tag{10}$$

where $s_j = (1-M_j^2)^{1/2}$ and $M_j = V/c_j$ is the Mach number.

In terms of the translating co-ordinates x, y_j, equation (4) becomes

$$\frac{\partial^2 W_j}{\partial x^2} + \frac{\partial^2 W_j}{\partial y_j^2} + \frac{1}{c_j^2 s_j^2}\frac{\partial}{\partial t}\left[2M_j c_j \frac{\partial W_j}{\partial x} - \frac{\partial W_j}{\partial t}\right] = 0 \tag{11}$$

In the moving system (x,y,z,t) equations (1),(5) and (6) take the form

$$e^{-i\omega t}\begin{bmatrix} W_i \\ W_r \\ W_T \end{bmatrix} = \begin{bmatrix} A \exp[-i\{k_1(x\cos\theta_1 + \frac{y_1}{s_1}\sin\theta_1) + \omega t\}] \\ B \exp[-i\{k_1(x\cos\theta_1 - \frac{y_1}{s_1}\sin\theta_1) + \omega t\}] \\ C \exp[-i\{k_2(x\cos\theta_2 + \frac{y_2}{s_2}\sin\theta_2) + \omega t\}] \end{bmatrix}, \tag{12}$$

where $\omega = \Omega\alpha$ and $\alpha = (1+M_1\cos\theta_1) = (1+M_2\cos\theta_2)$.

In view of the equation (12) we take the solution of (11) as

$$W_j(x,y_j)e^{-i\omega t} = w_j(x,y_j)\exp[i(M_j\lambda_j x - \omega t)]. \tag{13}$$

Substitution of equation (13) into equation (11) yields the

Helmholtz equation governing w_j :

$$\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y_j^2} + \lambda_j^2 w_j = 0 , \qquad (j = 1,2) \qquad (14)$$

where $\lambda_j = \frac{k_j \alpha}{s_j^2}$

Applying Fourier transform, equation (14) can be solved and the result is

$$w_1(x,y_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B_1(\xi)\exp[-i\xi x-(\xi^2-\lambda_1^2)^{1/2} y_1]d\xi , \qquad y_1>0 \qquad (15)$$

$$w_2(x,y_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B_2(\xi)\exp[-i\xi x+(\xi^2-\lambda_2^2)^{1/2} y_2]d\xi , \qquad y_2<0 \qquad (16)$$

From (13),(15) and (16) we obtain the displacement components due to scattered field as

$$W_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_1(\xi)\exp[-i\xi x - \nu_1 y_1]d\xi , \qquad y_1>0 \qquad (17)$$

and

$$W_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_2(\xi)\exp[-i\xi x + \nu_2 y_2]d\xi , \qquad y_2<0, \qquad (18)$$

where

$$\nu_j = [(\xi + \lambda_j M_j)^2 - \lambda_j^2]^{1/2}, \qquad j=1,2 \qquad (19)$$

$A_1(\xi)$ and $A_2(\xi)$ are the unknown quantities to be determined from the following boundary conditions:

$$\mu_1 s_1 \frac{\partial W_1}{\partial y_1} = \mu_2 s_2 \frac{\partial W_2}{\partial y_2} \quad , \text{ for all } x, \quad y=0 \tag{20}$$

$$W_1 = W_2 \ , \qquad |x|>L, \ y=0 \tag{21}$$

$$\frac{\partial W_1}{\partial y_1} + \frac{\partial W_i}{\partial y_1} + \frac{\partial W_r}{\partial y_1} = 0, \qquad |x|<L, \qquad y=0+ \tag{22}$$

From the boundary condition (22) we obtain

$$\frac{\partial W_1}{\partial y_1} = A_1 \exp[-ik_1 x \cos\theta_1], \quad |x|<L, \quad y=0, \tag{23}$$

where $$A_1 = \frac{i(A-B)k_1 \sin\theta_1}{s_1} \tag{24}$$

Using (17), the above equation can be written as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} A_1(\xi)\nu_1 \exp[-i\xi x]d\xi = - A_1\exp[-ik_1 x \cos\theta_1], \qquad -L<x<L$$

$$= P(x), \quad x>L \quad \text{(say)}$$

$$= Q(x), \quad x<-L \quad \text{(say)}$$

Therefore,

$$A_1(\xi)\nu_1 = \exp[i\xi L]\ G_+(\xi) + \exp[-i\xi L]\ G_-(\xi) -$$

$$- \frac{A_1}{i(\xi-\xi_0)} \left[\exp\{i(\xi-\xi_0)L\} - \exp\{-i(\xi-\xi_0)L\} \right], \tag{25}$$

where

$$G_+(\xi) = \int_{L}^{\infty} P(x)\ \exp[i\xi(x-L)]dx \tag{26}$$

$$G_-(\xi) = \int_{-\infty}^{-L} Q(x)\ \exp[i\xi(x+L)]dx \tag{27}$$

$$\xi_0 = k_1 \cos\theta_1 . \tag{28}$$

From the boundary condition (20) we obtain

$$A_2(\xi) = - \frac{M\ \nu_1 A_1(\xi)}{\nu_2} \tag{29}$$

where $$M = \frac{\mu_1 s_1}{\mu_2 s_2}. \tag{30}$$

Next using the boundary condition (21), we obtain

$$A_1(\xi) - A_2(\xi) = \int_{-\infty}^{\infty} (W_1 - W_2)\ \exp[i\xi x]dx$$

$$= \int_{-L}^{L} P_1(x)\ \exp[i\xi x]dx$$

$$= N(\xi) \qquad \text{(say)}, \tag{31}$$

which is the measure of the discontinuity of displacement along the surface of the crack. Now with the aid of (29) and (31), we find

$$A_1(\xi) = \frac{\nu_2 N(\xi)}{\nu_2 + M\nu_1} \tag{32}$$

Eliminating $A_1(\xi)$ from (25) and (32) we obtain an extended Wiener-Hopf equation, namely

$$\exp[\,i\xi L\,]G_+(\xi) + \exp[-i\xi L\,]G_-(\xi) - N(\xi)K(\xi)$$

$$= \frac{A_1}{i(\xi-\xi_0)}\left[\exp\{i(\xi-\xi_0)L\} - \exp\{-i(\xi-\xi_0)L\}\right], \tag{33}$$

where $$K(\xi) = \frac{\nu_1\nu_2}{\nu_2 + M\,\nu_1} = \frac{\nu_1}{1+M}R(\xi) \tag{34}$$

$$R(\xi) = \frac{(1+M)\nu_2}{\nu_2 + M\,\nu_1} \tag{35}$$

In order to solve the Wiener-Hopf equation given by (33) we assume that branch points $\xi=\lambda_1(1-M_1)$, $\lambda_2(1-M_2)$, $-\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ of $K(\xi)$ possess small imaginary parts, which would ultimately be made to tend to zero.

Now we write $K(\xi) = K_+(\xi)K_-(\xi)$, where $K_+(\xi)$ is analytic in the upper-half plane $\mathrm{Im}\,\xi > \mathrm{Im}\,[-\lambda_1(1+M_1)]$, whereas $K_-(\xi)$ is analytic in the lower-half plane given by $\mathrm{Im}\,\xi < \mathrm{Im}\,[\,\lambda_1(1-M_1)]$. The expressions of $K_+(\xi)$ and $K_-(\xi)$ are derived in the Appendix. Since $\frac{\partial W_1}{\partial y_1}$ decreases exponentially as $x \rightarrow \pm\infty$, $G_+(\xi)$ and $G_-(\xi)$ have the same common region of regularity as $K_+(\xi)$ and $K_-(\xi)$.

Now equation (33) can easily be expressed as two integral equations involving $G_+(\xi)$, $G_-(\xi)$ and $N(\xi)$ as follows:

$$\frac{G_+(\xi)}{K_+(\xi)} - \frac{A_1 e^{-i\xi_0 L}}{i(\xi-\xi_0)}\left[\frac{1}{K_+(\xi)} - \frac{1}{K_+(\xi_0)}\right] +$$

$$+ \frac{1}{2\pi i}\int_{C_+} \frac{e^{-2isL}}{(s-\xi)K_+(s)}\left[G_-(s) + \frac{A_1 e^{i\xi_0 L}}{i(s-\xi_0)}\right]ds$$

$$= N(\xi)K_-(\xi)e^{-i\xi L} + \frac{A_1 e^{-i\xi_0 L}}{i(\xi-\xi_0)K_+(\xi_0)} -$$

$$- \frac{1}{2\pi i}\int_{C_-} \frac{e^{-2isL}}{(s-\xi)K_+(s)}\left[G_-(s) + \frac{A_1 e^{i\xi_0 L}}{i(s-\xi_0)}\right]ds, \qquad (36)$$

where c_+ and c_- are the straight contours below the pole at $\xi = \xi_0$ and situated within the common region of regularity of $G_+(\xi)$, $G_-(\xi)$, $K_+(\xi)$ and $K_-(\xi)$ as shown in Fig.2.

The left hand side of (36) is analytic in the upper-half plane whereas the right hand side is analytic in the lower-half plane and both of them are equal in common region of analyticity of these two functions. Therefore, by analytic continuation, both sides of (36) are analytic in the whole of the s-plane. Next, by Liouville's theorem, it can be shown that both sides of (36) are equal to zero. Thus we obtain

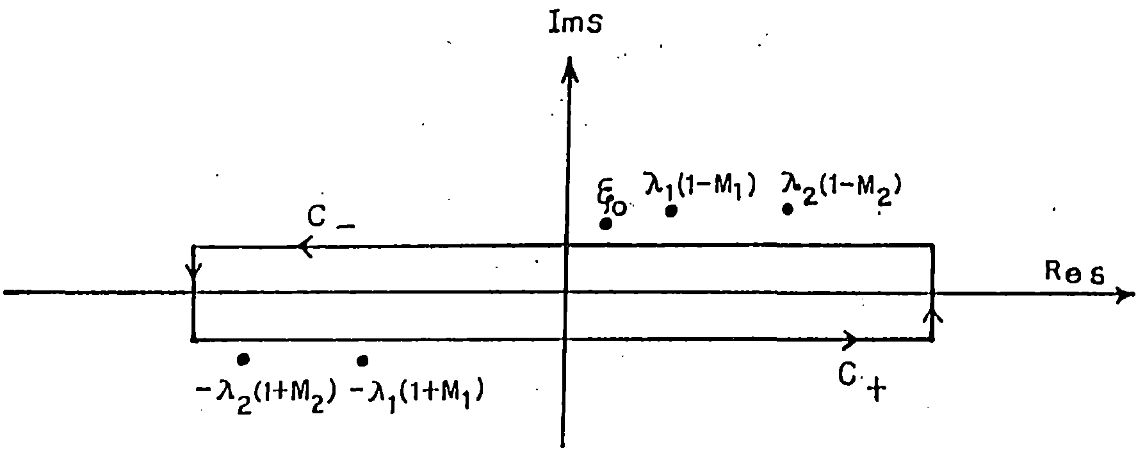

Fig. 2. Path of integration in the complex s-plane.

$$\frac{1}{K_+(\xi)}\left[G_+(\xi) - \frac{A_1 e^{-i\xi_0 L}}{i(\xi-\xi_0)}\right] + \frac{A_1 e^{-i\xi_0 L}}{i(\xi-\xi_0)K_+(\xi_0)} +$$

$$+ \frac{1}{2\pi i}\int_{C_+} \frac{e^{2isL}}{(s-\xi)K_+(s)}\left[G_-(s) + \frac{A_1 e^{i\xi_0 L}}{i(s-\xi_0)}\right]ds = 0 \qquad (37)$$

similarly, we also obtain

$$\frac{1}{K_-(\xi)}\left[G_-(\xi) + \frac{A_1 e^{i\xi_0 L}}{i(\xi-\xi_0)}\right] +$$

$$+ \frac{1}{2\pi i}\int_{C_-} \frac{e^{2isL}}{(s-\xi)K_-(s)}\left[G_+(s) - \frac{A_1 e^{-i\xi_0 L}}{i(s-\xi_0)}\right]ds = 0 \qquad (38)$$

3. HIGH FREQUENCY SOLUTION OF THE INTEGRAL EQUATIONS

In order to obtain $G_+(\xi)$ and $G_-(\xi)$ from the integral equations (37) and (38) in case when the normalized wave number $\lambda_1(1+M_1)L>>1$, the integration along the path c_+ in (37) is replaced by the integration along the loops $L_{-\lambda_1}$ and $L_{-\lambda_2}$ round the branch points $-\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ of $K_+(s)$, respectively. Also, the integration along the path c_- in (38) is replaced by the integration round the circular contour L_0, round the pole $s=\xi_0$ and by the integrations along the loops L_{λ_1} and L_{λ_2} round the branch cuts through the branch points $\lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ of the function $K_-(s)$ as shown in Fig. 3.

Finally evaluating the integrals along the straight line paths round the branch points for large values of frequency, we obtain two equations given by

$$F_{\pm}(\xi)+C_{\pm}(\xi)+\sum_{j=1}^{2}\frac{\sigma_j e^{2i\lambda_j(1\pm M_j)}\,A_{\mp}[\mp\lambda_j(1\pm M_j)]\;F_{\mp}[\mp\lambda_j(1\pm M_j)]}{2\{\lambda_j(1\pm M_j)-\xi\}\;(\lambda_j L)^{1/2}} = 0, \quad (39)$$

where $\sigma_1=1$ and $\sigma_2=M$, and

$$F_{\pm}(\xi) = \frac{1}{K_{\pm}(\xi)}\left[G_{\pm}(\xi) \mp \frac{A_1 e^{\mp i\xi_0 L}}{i(\xi-\xi_0)}\right]$$

$$A_{\pm}(\xi) = \frac{i e^{i\pi/4}}{\pi^{1/2}}\,[K_{\pm}(\xi)]^2$$

$$C_{\pm}(\xi) = \frac{A_1 e^{\mp i\xi_0 L}}{i(\xi-\xi_0)K_{\pm}(\xi_0)} \quad (40)$$

Now substituting $\xi = \lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ and $\xi = -\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ in (39) a system of linear equations of $F_+[\lambda_1(1-M_1)]$, $F_+[\lambda_2(1-M_2)]$, $F_-[-\lambda_1(1+M_1)]$ and $F_-[-\lambda_2(1+M_2)]$ are obtained. Now solving them and neglecting higher order terms of $(\lambda_1 L)^{-1/2}$ and $(\lambda_2 L)^{-1/2}$ we obtain, finally, after some algebraic manipulation:

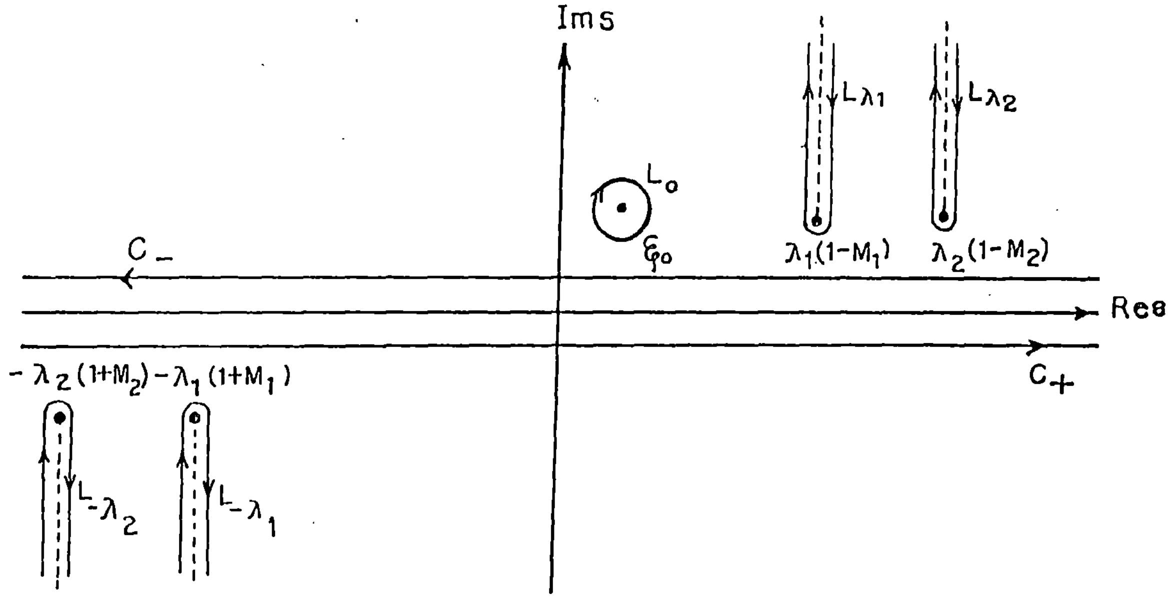

Fig. 3. Path of integration L_0, L_{λ_1}, L_{λ_2} and $L_{-\lambda_1}$, $L_{-\lambda_2}$.

$$F_{\pm}[\pm\lambda_k(1\mp M_k)] = - C_{\pm}[\pm\lambda_k(1\mp M_k)] \times$$

$$\times \left[1-\sum_{j=1}^{2} \frac{\sigma_j e^{2i\lambda_j(1\mp M_k)L} A_{\mp}[\mp\lambda_j(1\pm M_j)]\, C_{\mp}[\mp\lambda_j(1\pm M_j)]}{2(\lambda_j L)^{1/2}\{\lambda_j(1\pm M_j)+\lambda_k(1\mp M_k)\}C_{\pm}[\pm\lambda_k(1\mp M_k)]} \right], \quad k=1,2 \quad (41)$$

Now using (39) we get from (41)

$$G_{\pm}(\xi) = \pm \frac{A_1 e^{\mp i\xi_o L}}{i(\xi-\xi_o)} \mp \frac{A_1 e^{\mp i\xi_o L} K_{\pm}(\xi)}{i(\xi-\xi_o)K_{\pm}(\xi_o)} +$$

$$+ \sum_{k=1}^{2}\left[\frac{\sigma_k e^{2i\lambda_k(1\pm M_k)L} A_{\mp}[\mp\lambda_k(1\pm M_k)]\, C_{\mp}[\mp\lambda_k(1\pm M_k)]\, K_{\pm}(\xi)}{2(\lambda_k L)^{1/2}\{\lambda_k(1\pm M_k)\pm\xi\}} \times \right.$$

$$\left. \times \left(1-\sum_{j=1}^{2} \frac{\sigma_j e^{2i\lambda_j(1\mp M_j)L} A_{\pm}[\pm\lambda_j(1\mp M_j)]\, C_{\pm}[\pm\lambda_j(1\mp M_j)]}{2(\lambda_j L)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\}C_{\mp}[\mp\lambda_k(1\pm M_k)]}\right)\right] \quad (42)$$

4. CRACK OPENING DISPLACEMENT AT POINTS AWAY FROM THE CRACK TIPS

In order to obtain the displacement jump for the large values of $\lambda_1(L-x)$, $\lambda_2(L-x)$, $\lambda_1(L+x)$ and $\lambda_2(L+x)$, we can write $G_+(\xi)$ and $G_-(\xi)$ from (42) as

$$G_{\pm}(\xi) = \pm \frac{P_{\pm}}{\xi-\xi_o} \mp \frac{Q_{\pm}K_{\pm}(\xi)}{\xi-\xi_o} + \sum_{k=1}^{2} \frac{K_{\pm}(\xi)\, R_{\pm}^{(k)}}{\{\lambda_k(1\pm M_k)\pm\xi\}}, \quad (43)$$

where

$$P_{\pm} = \frac{A_1 e^{\mp i\xi_0 L}}{i}, \tag{44}$$

$$Q_{\pm} = \frac{A_1 e^{\mp i\xi_0 L}}{iK_{\pm}(\xi_0)} = \frac{P_{\pm}}{K_{\pm}(\xi_0)} \tag{45}$$

$$R_{\pm}^{(k)} = \frac{\sigma_k e^{2i\lambda_k(1\pm M_k)L} A_{\mp}[\mp\lambda_k(1\pm M_k)]\ C_{\mp}[\mp\lambda_k(1\pm M_k)]}{2(\lambda_k L)^{1/2}} \times$$

$$\times \left[1-\sum_{j=1}^{2} \frac{\sigma_j e^{2i\lambda_j(1\mp M_j)L} A_{\pm}[\pm\lambda_j(1\mp M_j)]\ C_{\pm}[\pm\lambda_j(1\mp M_j)]}{2(\lambda_j L)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\}\ C_{\mp}[\mp\lambda_k(1\pm M_k)]}\right] \tag{46}$$

Now we obtain from (33)

$$N(\xi) = -\frac{Q_+ e^{i\xi L}}{(\xi-\xi_0)K_-(\xi)} + \frac{R_+^{(1)} e^{i\xi L}}{\{\xi+\lambda_1(1+M_1)\}K_-(\xi)} + \frac{R_+^{(2)} e^{i\xi L}}{\{\xi+\lambda_2(1+M_2)\}K_-(\xi)} +$$

$$+ \frac{Q_- e^{-i\xi L}}{(\xi-\xi_0)K_+(\xi)} - \frac{R_-^{(1)} e^{-i\xi L}}{\{\xi-\lambda_1(1-M_1)\}K_+(\xi)} - \frac{R_-^{(2)} e^{-i\xi L}}{\{\xi-\lambda_2(1-M_2)\}K_+(\xi)} . \tag{47}$$

From (31) we obtain the displacement jump across the surface of the crack as

$$W_1(x,0+) - W_2(x,0-) = \frac{1}{2\pi}\int_{-\infty}^{\infty} N(\xi)e^{-i\xi x}\, d\xi \tag{48}$$

Substituting the expression of $N(\xi)$ from (47) in (48) and approximately evaluating the integrals arising in (48) term by term for large values of $\lambda_1(L-x)$, $\lambda_2(L-x)$, $\lambda_1(L+x)$, and $\lambda_2(L+x)$, and neglecting terms of order higher than $(\lambda_1 L)^{-3/2}$ and $(\lambda_2 L)^{-3/2}$, we finally obtain the crack opening displacement across the cracked surface at points away from the crack tips in the following form:

$$\Delta W = W_1(x,0+) - W_2(x,0-)$$

$$= - iQ_+K_+(\xi_0)e^{i\xi_0(L-x)} \times$$

$$\times \left[\frac{1}{\{(\xi_0+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}} + \frac{M}{\{(\xi_0+\lambda_2 M_2)^2-\lambda_2^2\}^{1/2}} \right] -$$

$$- \frac{e^{-i\pi/4}}{\pi^{1/2}} \left[T_+ - T_- \right], \qquad (49)$$

where

$$T_\pm = \sum_{k=1}^{2} \frac{\sigma_k e^{i\lambda_k(1\mp M_k)(L\mp x)}}{\{\lambda_k(L\mp x)\}^{1/2}} \left[\frac{Q_\pm K_\pm[\pm\lambda_k(1\mp M_k)]}{2^{1/2}[\lambda_k(1\mp M_k)\mp\xi_0]} - \right.$$

$$- \sum_{j=1}^{2} \frac{\sigma_j A_\mp[\mp\lambda_j(1\pm M_j)]\, K_\pm[\pm\lambda_k(1\mp M_k)]}{2(2\lambda_j L)^{1/2}\{\lambda_k(1\mp M_k)+\lambda_j(1\pm M_j)\}} \left(\frac{Q_\mp e^{2i\lambda_j(1\pm M_j)L}}{\{\lambda_j(1\pm M_j)\pm\xi_0\}} - \right.$$

$$\left.\left. - \sum_{r=1}^{2} \frac{\sigma_r A_\pm[\pm\lambda_r(1\mp M_r)]\, Q_\pm\, e^{2i[\lambda_r(1\mp M_r)+\lambda_j(1\pm M_j)]L}}{2(\lambda_r L)^{1/2}\{\lambda_r(1\mp M_r)+\lambda_j(1\pm M_j)\}\{\lambda_r(1\mp M_r)\mp\xi_0\}} \right) \right]. \qquad (50)$$

5. STRESS INTENSITY FACTOR AND CRACK OPENING DISPLACEMENT NEAR THE CRACK TIPS

Now considering the behaviour of ξ at infinity we obtain from (42)

$$G_{\pm}(\xi) \simeq \pm \frac{A_1 e^{\mp i\xi_o L}}{i(\xi-\xi_o)} + S_{\pm}\xi^{-1/2} \quad \text{as } \xi \longrightarrow \infty, \tag{51}$$

where
$$S_{\pm} = \frac{1}{(1+M)^{1/2}} \left[\mp \frac{A_1 e^{\mp i\xi_o L}}{iK_{\pm}(\xi_o)} \pm \right.$$

$$\pm \sum_{k=1}^{2} \frac{\sigma_k e^{2i\lambda_k(1\pm M_k)L} A_{\mp}[\mp\lambda_k(1\pm M_k)]\, C_{\mp}[\mp\lambda_k(1\pm M_k)]}{2(\lambda_k L)^{1/2}} \times$$

$$\left. \times \left\{ 1 - \sum_{j=1}^{2} \frac{\sigma_j e^{2i\lambda_j(1\mp M_j)L} A_{\pm}[\pm\lambda_j(1\mp M_j)]\, C_{\pm}[\pm\lambda_j(1\mp M_j)]}{2(\lambda_j L)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\} C_{\mp}[\mp\lambda_k(1\pm M_k)]} \right\} \right]. \tag{52}$$

Now, from equation (33), using (51) and also the fact that

$$K(\xi) \longrightarrow \pm \frac{\xi}{1+M} \quad \text{as } \xi \longrightarrow \pm\infty, \tag{53}$$

we obtain

$$N(\xi) = \frac{1+M}{\pm\xi(\xi)^{1/2}} \left[S_+ e^{i\xi L} + S_- e^{-i\xi L} \right] \quad \text{as } \xi \longrightarrow \pm\infty. \tag{54}$$

Taking inverse Fourier transform of (31) and using the results of Fresnel integrals, viz.

$$\int_0^\infty \frac{\begin{matrix}\sin\\ \cos\end{matrix}(x+L)\alpha}{(\alpha)^{1/2}}\, d\alpha = \left[\frac{\pi}{2(x+L)} \right]^{1/2}, \tag{55}$$

we obtain the displacement jump across the surface of the crack as

$$\Delta W = W_1(x,0+) - W_2(x,0-)$$

$$= -(1+M)(1+i)S_- \left[\frac{2(x+L)}{\pi} \right]^{1/2} \quad \text{for } x \longrightarrow -L+0 \tag{56}$$

$$= -(1+M)(1-i)S_+ \left[\frac{2(L-x)}{\pi} \right]^{1/2} \quad \text{for } x \longrightarrow L-0. \tag{57}$$

Expressions (56) and (57) give the displacement jump near to the crack tips, whereas the displacement jump away from the crack tips is given by (49).

Next, in order to find the value of τ_{yz} near to the crack tip we use (54) in (32) and (29) and obtain

$$A_j(\xi) = \frac{(-1)^{j+1}\sigma_j}{\xi(\xi)^{1/2}} \left[S_+ e^{i\xi L} + S_- e^{-i\xi L} \right], \quad j=1,2 \quad \text{as } \xi \longrightarrow \infty \tag{58}$$

$$A_j(\xi) = \frac{i(-1)^{j+1}\sigma_j}{\xi(-\xi)^{1/2}} \left[S_+ e^{i\xi L} - S_- e^{-i\xi L} \right], \quad j=1,2 \quad \text{as } \xi \longrightarrow -\infty \tag{59}$$

Now $$\tau_{yz}(x,y_j) = \mu_j \frac{\partial w_j(x,y_j)}{\partial y} = \mu_j s_j \frac{\partial w_j(x,y_j)}{\partial y_j}$$

$$= \frac{\mu_j s_j}{2\pi} \frac{\partial}{\partial y_j} \left[\int_{-\infty}^{\infty} A_j(\xi)\, e^{-i\xi x-\nu_j|y_j|}\, d\xi \right]. \quad (60)$$

Now substituting the values of $A_j(\xi)$ as $|\xi| \longrightarrow \infty$ in (60) and integrating, we obtain the stress near to the crack tip as

$$\tau_{yz}(x,y_1) = -\frac{\mu_1 s_1}{(2\pi)^{1/2}} \left[(1-i)S_+ \frac{\cos(\psi_1/2)}{r_1^{1/2}} + (1+i)S_- \frac{\sin(\psi_2/2)}{r_2^{1/2}} \right] \quad (61)$$

and

$$\tau_{yz}(x,y_2) = -\frac{\mu_1 s_1}{(2\pi)^{1/2}} \left[(1-i)S_+ \frac{\cos(\phi_1/2)}{d_1^{1/2}} + (1+i)S_- \frac{\cos(\phi_2/2)}{d_2^{1/2}} \right], \quad (62)$$

where $$r_1 = \{(x-L)^2+y_1^2\}^{1/2}, \quad \psi_1 = \sin^{-1}\frac{|y_1|}{r_1}$$

$$r_2 = \{(x+L)^2+y_1^2\}^{1/2}, \quad \psi_2 = \sin^{-1}\frac{|y_1|}{r_2} \quad (63)$$

$$d_1 = \{(x-L)^2+y_2^2\}^{1/2}, \quad \phi_1 = \sin^{-1}\frac{|y_2|}{d_1}$$

$$d_2 = \{(x+L)^2+y_2^2\}^{1/2}, \quad \phi_2 = \sin^{-1}\frac{|y_2|}{d_2}.$$

Therefore at the interface (y=0) near to the right-hand crack vertex, we obtain

$$\tau_{yz} \longrightarrow -\frac{\mu_1 s_1 (1-i) S_+}{\{2\pi(x-L)\}^{1/2}} \qquad \text{as } x \longrightarrow L+0. \tag{64}$$

Now the normalized dynamic stress intensity factor K at the crack tip x = L is defined by

$$K = \left| \frac{[2\pi k_1 (x-L)]^{1/2} \tau_{yz}}{\mu_1 A_1} \right| = s_1 \left| \frac{(1-i) S_+ (k_1)^{1/2}}{A_1} \right| \quad \text{for } x \longrightarrow L+0, \tag{65}$$

where A_1 is given by (24).

The absolute values of the complex stress intensity factor defined by (65) has been plotted against k_1L in Fig.4 for values $k_1L>1$ for different values of the Mach number M_2 and the angle of incidence for the following sets of materials:

first set:

Steel	$\rho_1 = 7.6$ gm/cm^3,	$\mu_1 = 8.32 \times 10^{11}$ dyne/cm^2
Aluminium	$\rho_2 = 2.7$ gm/cm^3,	$\mu_2 = 2.63 \times 10^{11}$ dyne/cm^2

second set:

Wrought iron	$\rho_1 = 7.8$ gm/cm^3,	$\mu_1 = 7.7 \times 10^{11}$ dyne/cm^2
Copper	$\rho_2 = 8.96$ gm/cm^3,	$\mu_2 = 4.5 \times 10^{11}$ dyne/cm^2

As the Mach number $M_2 \longrightarrow 0$ the stress intensity factor K tends to the value of the stress intensity factor corresponding to the stationary crack. The problem for $\theta_1 = \pi/2$ and $M_2=0.0$ was solved earlier by Pal and Ghosh [1990]. The graph of stress intensity factor vs K_1L corresponding to $\theta_1=\pi/2$ and $M_2=0.0$ as given in Fig.4a is found to coincide exactly with that given by Pal and Ghosh [1990]. It is interesting to note that for both pairs of materials, as M_2 increases, the peaks of the curves of stress intensity factors decrease in magnitude and occur at lower values of K_1L. Further, it may be noted that for any fixed value of M_2 the stress intensity factor decreases with the decrease in the value of the angle of incidence.

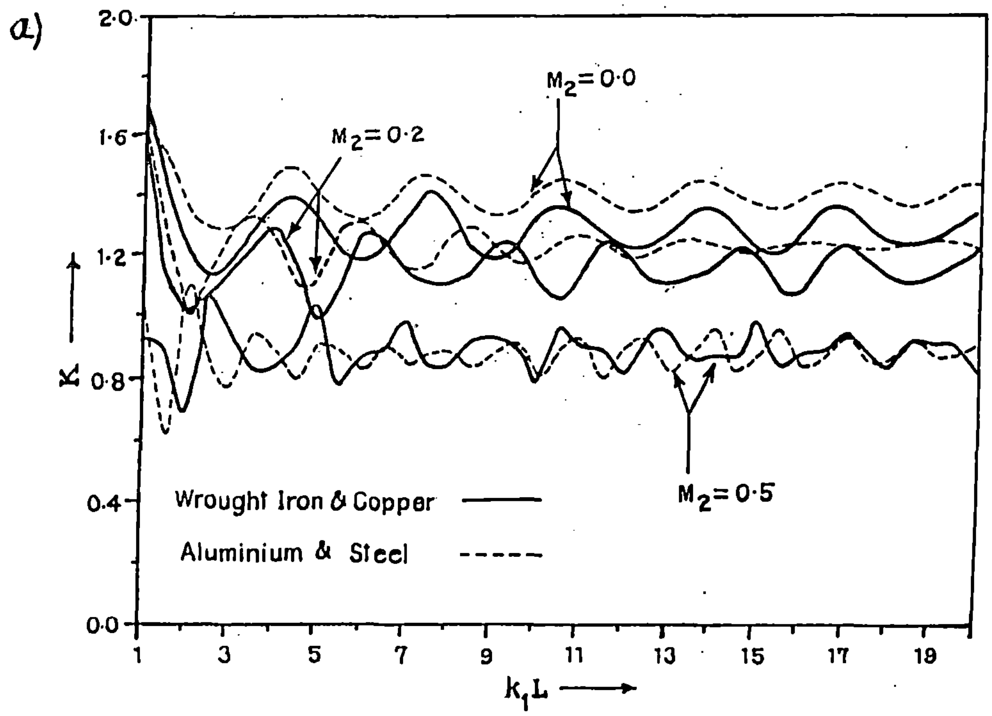

Fig.4(a). Stress intensity factor K versus dimensionless k_1L for $\theta_1 = \pi/2$.

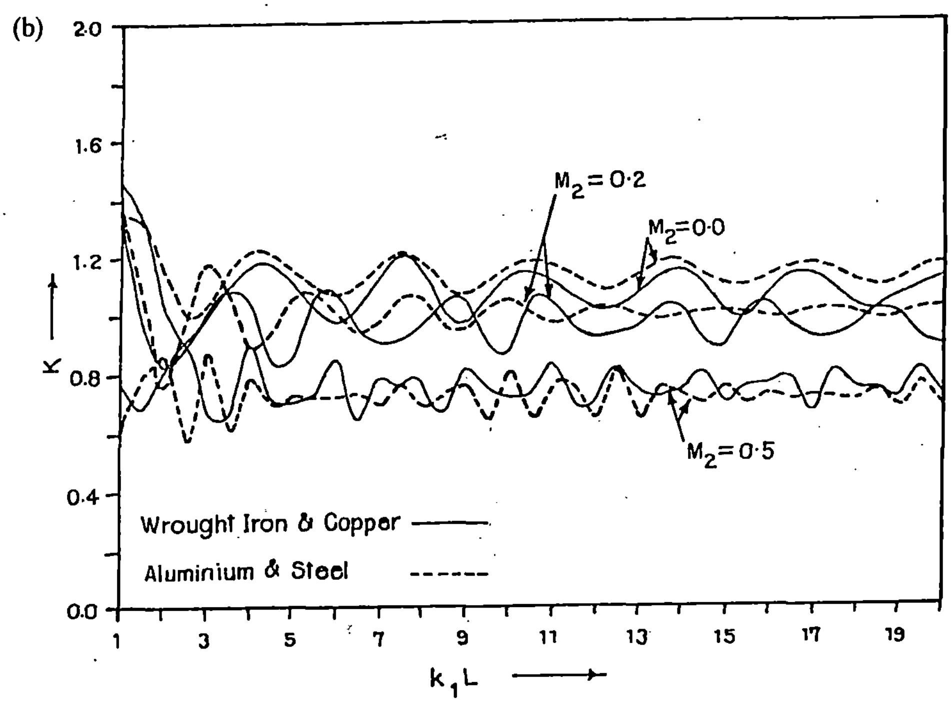

Fig.4(b). Stress intensity factor K versus dimensionless k_1L for $\theta_1 = \pi/3$.

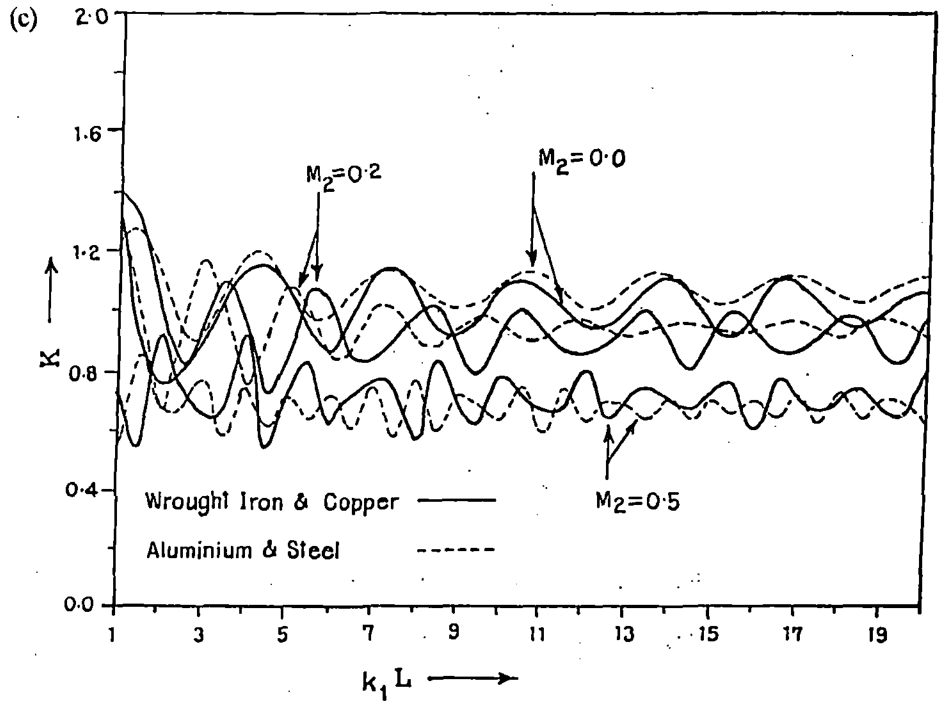

Fig.4(c). Stress intensity factor K versus dimensionless k_1L for $\theta_1 = \pi/4$.

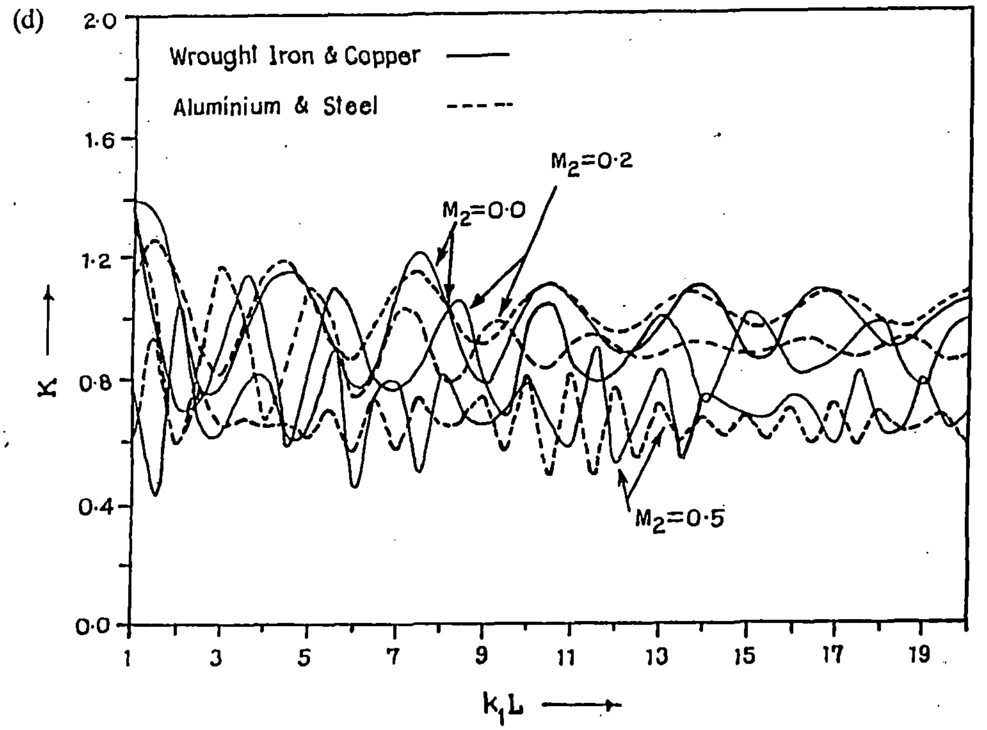

Fig.4(d). Stress intensity factor K versus dimensionless k_1L for $\theta_1 = \pi/6$.

APPENDIX

$$K(\xi) = \frac{\{(\xi+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{1+M} R(\xi) \tag{A1}$$

where $$M = \frac{\mu_1 s_1}{\mu_2 s_2}$$

and $$R(\xi) = \frac{(1+M)\{(\xi+\lambda_2 M_2)^2-\lambda_2^2\}^{1/2}}{M\{(\xi+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}+\{(\xi+\lambda_2 M_2)^2-\lambda_2^2\}^{1/2}} \longrightarrow 1$$

as $|\xi| \longrightarrow \infty$

Now $$R_+(\xi)R_-(\xi) = \frac{1}{\dfrac{1}{1+M} + \dfrac{M\{(\xi+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{(\xi+\lambda_2 M_2)^2-\lambda_2^2\}^{1/2}}}$$

Taking log on both sides

$$\log R(\xi) = \log R_+(\xi) + \log R_-(\xi) = \frac{1}{2\pi i} \int_{c_L+c_U} \frac{\log R(\eta)}{\eta-\xi} d\eta$$

where the paths of integration c_L and c_U are shown in Fig.A1.

Therefore $$\log R_+(\xi) = \frac{1}{2\pi i} \int_{c_L} \frac{\log R(\eta)}{\eta-\xi} d\eta$$

$$\log R_-(\xi) = \frac{1}{2\pi i}\int_{c_U} \frac{\log R(\eta)}{\eta-\xi}\,d\eta$$

or $$\log R_+(\xi) = \frac{1}{2\pi i}\int_{-ic-\infty}^{-ic+\infty} \frac{\log R(\eta)}{\eta-\xi}\,d\eta$$

Putting $\eta = -\eta$

$$\log R_+(\xi) = \frac{1}{2\pi i}\int_{ic+\infty}^{ic-\infty} \frac{\log R(-\eta)}{\eta+\xi}\,d\eta$$

$$\log R_-(\xi) = \frac{1}{2\pi i}\int_{ic+\infty}^{ic-\infty} \frac{\log R(\eta)}{\eta-\xi}\,d\eta,$$

therefore

$$\log R_-(\xi) = \frac{1}{2\pi i}\int_{c_1} \frac{1}{(\eta-\xi)} \log\left[\frac{1}{\dfrac{1}{1+M} + \dfrac{M\{(\eta+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{(\eta+\lambda_2 M_2)^2-\lambda_2^2\}^{1/2}}}\right]d\eta$$

where c_1 is the contour round the branch points $\lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ as shown in Fig. A2.

Therefore

$$\log R_-(\xi) =$$

$$= \frac{1}{2\pi i}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)} \frac{1}{(\eta-\xi)}\left[\log\left(\frac{1}{1+M} + \frac{iM\{(\eta+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{\lambda_2^2-(\eta+\lambda_2 M_2)^2\}^{1/2}}\right) -\right.$$

$$\left. - \log\left(\frac{1}{1+M} - \frac{iM\{(\eta+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{\lambda_2^2-(\eta+\lambda_2 M_2)^2\}^{1/2}}\right)\right]d\eta$$

$$= \frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)} \frac{1}{(\eta-\xi)} \tan^{-1}\left[\frac{M\{(\eta+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{\{\lambda_2^2-(\eta+\lambda_2 M_2)^2\}^{1/2}}\right]d\eta,$$

and therefore

$$R_-(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)} \frac{1}{(\eta-\xi)} \tan^{-1}\left(\frac{M\,[(\eta+\lambda_1 M_1)^2-\lambda_1^2\,]^{1/2}}{[\,\lambda_2^2-(\eta+\lambda_2 M_2)^2\,]^{1/2}}\right)d\eta\right].$$

Similarly

$$R_+(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1(1+M_1)}^{\lambda_2(1+M_2)} \frac{1}{(\eta+\xi)} \tan^{-1}\left(\frac{M\,[(\eta-\lambda_1 M_1)^2-\lambda_1^2\,]^{1/2}}{[\,\lambda_2^2-(\eta-\lambda_2 M_2)^2\,]^{1/2}}\right)d\eta\right].$$

Therefore from (A1) we can write

$$K_+(\xi) = \left[\frac{\xi+\lambda_1(1+M_1)}{(1+M)}\right]^{1/2} \times$$

$$\times \exp\left[\frac{1}{\pi}\int_{\lambda_1(1+M_1)}^{\lambda_2(1+M_2)} \frac{1}{(\eta+\xi)}\tan^{-1}\left(\frac{M\,[(\eta-\lambda_1 M_1)^2-\lambda_1^2\,]^{1/2}}{[\,\lambda_2^2-(\eta-\lambda_2 M_2)^2\,]^{1/2}}\right)d\eta\right] \quad \text{(A2)}$$

and

$$K_-(\xi) = \left[\frac{\xi-\lambda_1(1-M_1)}{(1+M)}\right]^{1/2} \times$$

$$\times \exp\left[\frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)} \frac{1}{(\eta-\xi)}\tan^{-1}\left(\frac{M\,[(\eta+\lambda_1 M_1)^2-\lambda_1^2\,]^{1/2}}{[\,\lambda_2^2-(\eta+\lambda_2 M_2)^2\,]^{1/2}}\right)d\eta\right]. \quad \text{(A3)}$$

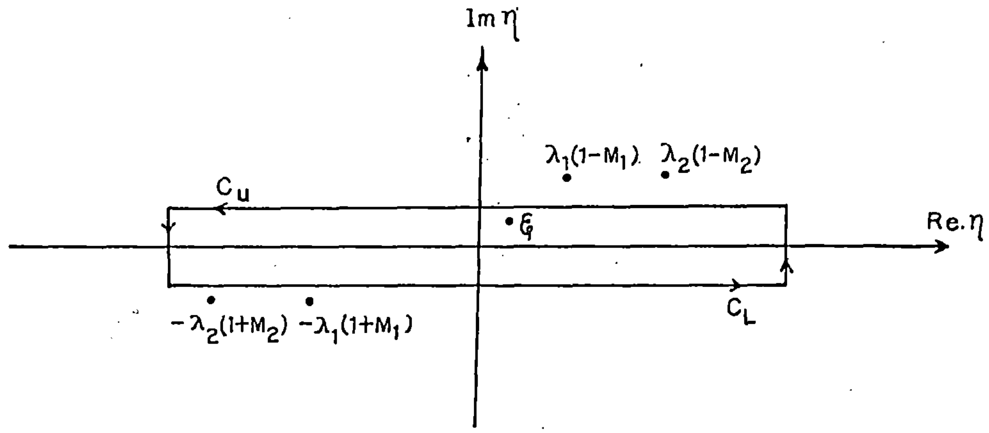

Fig. A1. Complex η-plane.

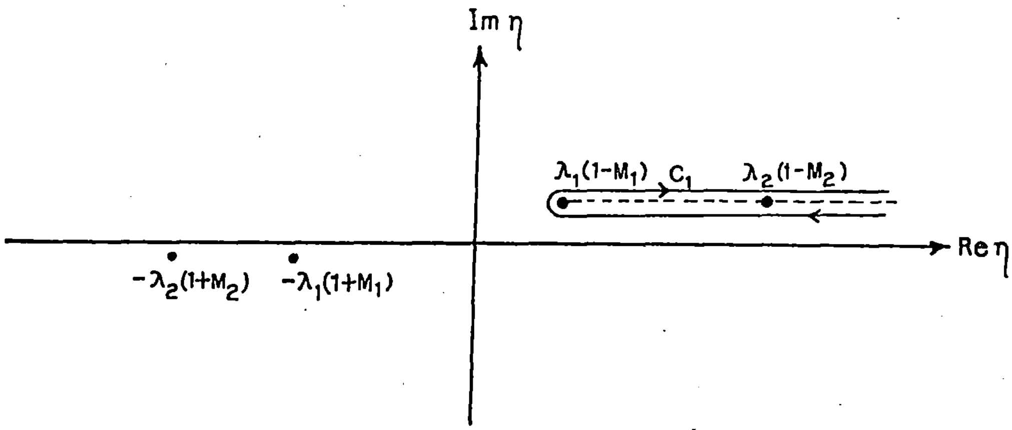

Fig. A2. Path of integration round the branch points.

CHAPTER - III

DIFFRACTION PROBLEMS IN ELASTODYNAMICS

FORCED VERTICAL VIBRATION OF FOUR RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

1. INTRODUCTION

The problem of the effect of vibrating source in different forms on the surface of an elastic medium have aroused attention in view of their application in seismology and geophysics. Reissner [1937], and later Millar and Pursey [1954], treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. Analytical treatment of the dynamical response of footings and solid-structure interaction are usually available in the literature only for circular and elliptical footings, and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, and also in the study of the vibration of dams and large structures subjected to earthquakes. The problem of circular punch has been solved analytically by Awojobi and Grootenhuis [1965], Robertson [1966], Gladwell [1968] and others. Roy [1986] considered the dynamic

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response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. Karasudhi, Keer and Lee [1968] obtained a low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium. Wickham [1977] worked out in detail the problem of forced two-dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium. Recently, Mandal and Ghosh [1992] treated the problem of forced vertical vibration of two rigid strips on a semi-infinite elastic medium.

To improve the dynamic models of buildings and other structures, it will be fruitful to have analytic results for foundations of a more complicated nature. In what follows, the problem of vertical vibration of four rigid strips in smooth contact with a semi-infinite elastic medium has been considered. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by running wheels on a railway track. The resulting mixed boundary value problem has been reduced to the solution of quadruple integral equations, which have further been reduced to the solution of integral-differential equations. Finally, an iterative solution valid for low frequency has been obtained.

From the solution of the integral equations, the stress just below the strips and also the vertical displacement at points outside the strips on the free surface have been found. The effects of stress intensity factors at the edges of the strips and vertical displacement outside the strips have been shown by means of graphs.

2. FORMULATION OF THE PROBLEM

Consider the normal vibration of frequency ω of four rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty<X<\infty$, $Y\geq 0$, $-\infty<Z<\infty$. It is assumed that the motion is forced by prescribed displacement distribution $v_0 e^{-i\omega t}$ normal to the four infinite strips located in the region $d_1\leq|X|\leq d_2$, $d_3\leq|X|\leq d$, $Y=0$, $|Z|<\infty$, where v_0 is a constant.

Normalizing all the lengths with respect to d and putting

$$\frac{X}{d} = x, \quad \frac{Y}{d} = y, \quad \frac{Z}{d} = z, \quad \frac{d_1}{d} = a, \quad \frac{d_2}{d} = b, \quad \frac{d_3}{d} = c,$$

one finds that the rigid strips are defined by $a\leq|x|\leq b$, $c \leq|x|\leq 1$, $y=0$, $|z|<\infty$ (fig.1). With the time factor $e^{-i\omega t}$ suppressed throughout the analysis, the displacement components can be written as

$$u(x,y) = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y} \; ; \quad v(x,y) = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x} \; ; \quad w(x,y) = 0 \qquad (1)$$

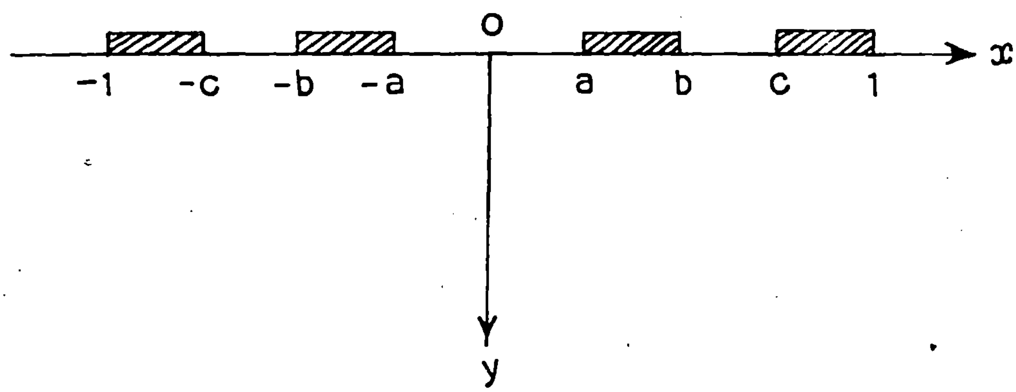

Fig. 1. Geometry of the problem.

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + m_1^2\phi = 0 .$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + m_2^2\psi = 0 \tag{2}$$

in which $m_1^2 = \dfrac{\omega^2 d^2}{c_1^2}$ and $m_2^2 = \dfrac{\omega^2 d^2}{c_2^2}$.

In terms of ϕ and ψ the stress components are

$$\tau_{xy} = \mu \left\{ 2 \frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} \right\}$$

$$\tau_{yy} = - \mu \left\{\left(m_2^2 + 2 \frac{\partial^2}{\partial x^2} \right)\phi - 2 \frac{\partial^2\psi}{\partial x\partial y} \right\} \tag{3}$$

$$\tau_{yz} = 0$$

The boundary conditions are

$$v(x,0) = v_0 \quad , \quad x \in I_2 , I_4 \tag{4}$$

$$\tau_{yy}(x,0) = 0 \quad , \quad x \in I_1 , I_3 , I_5 \tag{5}$$

$$\tau_{xy}(x,0) = 0 \quad , \quad -\infty < x < \infty \tag{6}$$

where $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,1)$, $I_5 = (1,\infty)$.

The solution of the Helmholtz equation (2) can be written as

$$\phi = 2\int_0^\infty A(\xi)\cos\xi x\, e^{-\gamma_1 y}\, d\xi$$

$$\psi = 2\int_0^\infty B(\xi)\sin\xi x\, e^{-\gamma_2 y}\, d\xi \qquad (7)$$

where

$$\gamma_j = \begin{cases} (\xi^2 - m_j^2)^{1/2}, & |\xi| \geq m_j \\ -i(m_j^2 - \xi^2)^{1/2}, & |\xi| \leq m_j \end{cases}, \qquad j = 1,2$$

and $A(\xi)$ and $B(\xi)$ are unknown functions, to be determined from the boundary conditions.

By using the boundary condition (6) it can be shown that

$$B(\xi) = \frac{2\gamma_1\xi}{\xi^2 + \gamma_2^2} A(\xi) \qquad (8)$$

Now the displacement component v and stress τ_{yy} become

$$v(x,y) = 2\int_0^\infty \left[\frac{2\xi^2}{2\xi^2 - m_2^2} e^{-\gamma_2 y} - e^{-\gamma_1 y} \right] A(\xi)\cos\xi x\, d\xi \qquad (9)$$

$$\tau_{yy}(x,y) = -2\mu\int_0^\infty \left[(m_2^2 - 2\xi^2)e^{-\gamma_1 y} + \frac{2\xi^2\gamma_1\gamma_2}{2\xi^2 - m_2^2} e^{-\gamma_2 y} \right] A(\xi)\cos\xi x\, d\xi \qquad (10)$$

From the boundary conditions (4) and (5) we get the following set of integral equations in $P(\xi)$:

$$\int_0^{\infty} \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} P(\xi)\cos\xi x \, d\xi = \frac{1}{2} v_0 , \quad x \in I_2 , I_4 \tag{11}$$

and

$$\int_0^{\infty} P(\xi)\cos\xi x \, d\xi = 0 , \qquad x \in I_1 , I_2 , I_5 \tag{12}$$

where

$$P(\xi) = \frac{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2}{(2\xi^2 - m_2^2)} A(\xi) .$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (11) and (12) in the form

$$P(\xi) = \int_a^b t f(t^2)\cos\xi t \, dt + \int_c^1 u g(u^2)\cos\xi u \, du \tag{13}$$

where $f(t^2)$ and $g(u^2)$ are unknown functions to be determined.

By the choice of $P(\xi)$ given by (13) the relation (12) is satisfied automatically and the equation (11) becomes

$$\int_a^b tf(t^2)dt \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \cos\xi x \cos\xi t \, d\xi \; +$$

$$+ \int_c^1 ug(u^2)du \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \cos\xi x \cos\xi u \, d\xi = \frac{v_0}{2} ,$$

$$x \in I_2 \, , \, I_4 \qquad (14)$$

using the relation

$$\frac{\sin\xi x \sin\xi t}{\xi^2} = \int_0^x \int_0^t \frac{wvJ_0(\xi w)\, J_0(\xi v)\, dvdw}{(x^2-w^2)^{1/2}\, (t^2-v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{d}{dx} \int_a^b t\, f(t^2)\, dt \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{wv\, L_1(v,w)\, dvdw}{(x^2-w^2)^{1/2}\, (t^2-v^2)^{1/2}} \; +$$

$$+ \frac{d}{dx} \int_c^1 u\, g(u^2)du \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{wv\, L_1(v,w)\, dvdw}{(x^2-w^2)^{1/2}\, (u^2-v^2)^{1/2}}$$

$$= \frac{v_0}{2} \, , \; x \in I_2, \; I_4 \qquad (15)$$

where

$$L_1(v,w) = \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} J_0(\xi w) J_0(\xi v) d\xi \qquad (16)$$

By a simple contour integration technique used by Ghosh and Ghosh (1985), $L_1(v,w)$ can be written as

$$L_1(v,w) = -i\tau^2 \int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2 \; H_0^{(1)}(m_1\eta w)\; J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)} d\eta \; -$$

$$- 4i\tau^2 \int_0^{\tau} \frac{\eta^2(\eta^2-1)(\tau^2-\eta^2)^{1/2} \; H_0^{(1)}(m_1\eta w)\; J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)} d\eta \; +$$

$$+ \pi i\tau^2 \left[\frac{(\eta^2-1)^{1/2} H_0^{(1)}(m_1\eta w)\; J_0(m_1\eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0} , \quad w>v$$

$$= \frac{-i\tau^2}{16(1-\tau^2)} \left[\sum_{j=0}^{2} P_j \int_0^1 \frac{(1-\eta^2)^{1/2} H_0^{(1)}(m_1\eta w) J_0(m_1\eta v)}{\eta^2-\tau_j^2} d\eta \; + \right.$$

$$\left. + \sum_{j=0}^{2} S_j \int_0^{\tau} \frac{(\tau^2-\eta^2)^{1/2} H_0^{(1)}(m_1\eta w) J_0(m_1\eta v)}{\eta^2-\tau_j^2} d\eta \right] +$$

$$+ \pi i\tau^2 \left[\frac{(\eta^2-1)^{1/2} H_0^{(1)}(m_1\eta w)\; J_0(m_1\eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0} , \quad w>v \qquad (17)$$

where $\tau = \frac{m_2}{m_1} = \frac{c_1}{c_2}$, $Q_0(\eta) = (2\eta^2-\tau^2)^2 - 4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$ and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta) = 0$. τ_1, τ_2 are the roots of the equation

$$(2\eta^2-\tau^2)^2 + 4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2} = 0.$$

$Q_0'(\eta)$ denotes the derivative of $Q_0(\eta)$ with respect to η and

$$P_j = \frac{(2\tau_j^2 - \tau^2)}{\prod_i (\tau_j^2 - \tau_i^2)} ,$$

$$S_j = \frac{4\tau_j^2 (\tau_j^2 - 1)}{\prod_i (\tau_j^2 - \tau_i^2)} , \quad i,j = 0,1,2 \text{ and } i \neq j.$$

The corresponding expression for $L_1(v,w)$ for $w < v$ follows from equation (17) by interchanging w and v. For a Poisson ratio $\sigma = \frac{1}{4}$, the values of τ, τ_0, τ_1, and τ_2 are given by

$$\tau^2 = \frac{2(1-\sigma)}{(1-2\sigma)} = 3, \quad \tau_0^2 = \frac{3}{(0.9194)^2}, \quad \tau_1^2 = \frac{3}{(2+2\sqrt{3})} \text{ and } \tau_2^2 = \frac{3}{4} .$$

Hence, in this case $\tau_2 < \tau_1 < 1 < \tau < \tau_0$.

By using the series expansions of J_0 and $H_0^{(1)}$ and evaluating the integrals arising in equation (17), we obtain, after some algebraic manipulation,

$$L_1(v,w) = \frac{2}{\pi}\tau^2\left[\left(\gamma + \log\frac{m_1 w}{2} - \frac{\pi i}{2}\right)M + N - \frac{P}{4}(w^2+v^2)m_1^2\log m_1\right] + O(m_1^2) \qquad w > v.$$

$$= \frac{2}{\pi}\tau^2\left[\left(\gamma + \log\frac{m_1 v}{2} - \frac{\pi i}{2}\right)M + N - \frac{P}{4}(w^2+v^2)m_1^2\log m_1\right] + O(m_1^2) \qquad w < v. \quad (18)$$

where $\gamma = 0.5772157\ldots$ is Euler's constant,

$$M = -\frac{\pi}{4(1-\tau^2)} \quad (19)$$

$$N = \frac{\pi}{32(1-\tau^2)}\left[4\log\frac{4}{\tau} + \sum_{j=1}^{2} P_j\frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j} - \right.$$

$$- P_0\frac{\sqrt{(\tau_0^2-1)}}{\tau_0}\log\left\{\tau_0 + \sqrt{(\tau_0^2-1)}\right\} +$$

$$+ \sum_{j=1}^{2} S_j\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j} -$$

$$\left. - S_0\frac{\sqrt{(\tau_0^2-\tau^2)}}{\tau_0}\log\left\{\frac{\tau_0 + \sqrt{(\tau_0^2-\tau^2)}}{\tau}\right\}\right], \quad (20)$$

$$P = \frac{\pi}{32(1-\tau^2)}\left[\sum_{j=0}^{2} P_j\left(\frac{1}{2} - \tau_j^2\right) + \sum_{j=0}^{2} S_j\left(\frac{\tau^2}{2} - \tau_j^2\right)\right]. \quad (21)$$

Next, differentiating both sides of the relation (14) with respect to x, we obtain

$$\int_a^b tf(t^2)dt \int_0^{\infty} \frac{\gamma_1 m_2^2 \xi}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \sin\xi x \sin\xi t \, d\xi +$$

$$+ \int_c^1 ug(u^2)du \int_0^{\infty} \frac{\gamma_1 m_2^2 \xi}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \sin\xi x \sin\xi u \, d\xi = 0,$$

$$x \in I_2 , I_4$$

Following similar procedure as done for deriving equation (15), we get

$$x \int_a^b \frac{tf(t^2)}{x^2-t^2} dt + x \int_c^1 \frac{ug(u^2)}{x^2-u^2} du$$

$$= \int_a^b t\, f(t^2)\, dt \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{wv\, L_2(v,w)\, dvdw}{(x^2-w^2)^{1/2}\, (t^2-v^2)^{1/2}} +$$

$$+ \int_c^1 u\, g(u^2)du \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{wv\, L_2(v,w)\, dwdv}{(x^2-w^2)^{1/2}\, (u^2-v^2)^{1/2}} , \quad x \in I_2, I_4 \qquad (22)$$

where $$L_2(v,w) = \int_0^{\infty} \left[\xi - \frac{2\gamma_1\xi^2(m_1^2-m_2^2)}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2}\right] J_0(\xi w)\, J_0(\xi v)d\xi \qquad (23)$$

For small values of m_1 and m_2 such that $m_1=O(m_2)$, one can use the contour integration technique mentioned above and obtain

$$L_2(v,w) = 2im_1^2(1-\tau^2)\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\eta^2\; H_0^{(1)}(m_1\eta w)\; J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4\;(\eta^2-1)(\tau^2-\eta^2)}\,d\eta$$

$$+ 4im_1^2(1-\tau^2)\int_0^\tau -\frac{2\eta^4(\eta^2-1)(\tau^2-\eta^2)^{1/2}\; H_0^{(1)}(m_1\eta w)\; J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta\; -$$

$$- 2\pi im_1^2(1-\tau^2)\left[\frac{\eta^2(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)\; J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0},\quad w>v \tag{24}$$

By a process similar to the one which led to equation (18), equation (24) can be written as

$$L_2(v,w) = -\frac{4P}{\pi}(1-\tau^2)\; m_1^2\;\log m_1 + O(m_1^2) \tag{25}$$

where P is given by equation (21).

Now examining the relation (15) and (18) we assume the expressions of the functions $f(t^2)$ and $g(u^2)$ as

$$f(t^2) = f_0(t^2) + f_1(t^2)\; m_1^2\;\log m_1 + O(m_1^2)$$

$$g(u^2) = g_0(u^2) + g_1(u^2)\; m_1^2\;\log m_1 + O(m_1^2). \tag{26}$$

Putting the above expressions of $f(t^2)$ and $g(u^2)$ and the value of $L_2(v,w)$ given by (25) in equation (22) and equating the coefficients of like powers of m_1 we obtain

$$\int_a^b \frac{tf_0(t^2)}{x^2-t^2}\,dt + \int_c^1 \frac{ug_0(u^2)}{x^2-u^2}\,du = 0 , \quad x \in I_2 , I_4 \qquad (27)$$

and

$$\int_a^b \frac{tf_1(t^2)}{x^2-t^2}\,dt + \int_c^1 \frac{ug_1(u^2)}{x^2-u^2}\,du =$$

$$= -\frac{4}{\pi} P(1-\tau^2)\left[\int_a^b tf_0(t^2)dt + \int_c^1 ug_0(u^2)\,du\right], \quad x \in I_2, I_4. \qquad (28)$$

Following Srivastava and Lowengrub (1970) the solutions of the above integral equations (27) can be obtained as

$$f_0(t^2) = D_1 \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{c^2-t^2}{1-t^2}\right)^{1/2} \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} -$$

$$- D_2 \left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} \frac{1}{\sqrt{(1-t^2)(c^2-t^2)}} , \quad t \in I_2 \qquad (29)$$

and

$$g_0(u^2) = D_1 \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{u^2-c^2}{1-u^2}\right)^{1/2} \frac{1}{\sqrt{(u^2-a^2)(u^2-b^2)}} +$$

$$+ D_2 \left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2} \frac{1}{\sqrt{(u^2-c^2)(1-u^2)}} , \quad u \in I_4 \qquad (30)$$

where D_1 and D_2 are constants which can be calculated as follows:

We substitute the value of $L_1(v,w)$ from (18) as well as the expansions of $f(t^2)$ and $g(u^2)$ obtained from (26), (29) and (30) upto $O(m_1^2 \log m_1)$ in the equation (15). When the coefficients of like powers of m_1 from both sides of the resulting equation are equated and we get after some algebraic manipulation, the following

$$D_1 = \frac{\pi v_0}{4\tau^2} \frac{(X_2-X_1)}{(X_1X_4-X_2X_3)} \quad ; \quad D_2 = \frac{\pi v_0}{4\tau^2} \frac{(X_1-X_3)}{(X_1X_4-X_2X_3)} \qquad (31)$$

where

$$X_1 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left[\left\{\left(\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2}\right)M + N\right\}(J_1+J_3) + \right.$$

$$\left. + \frac{1}{2}MJ_1 \log(b^2-a^2) + MJ_5\right] \qquad (32)$$

$$X_2 = \left\{\left[\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2}\right]M + N\right\}(J_4 - J_2) - \frac{1}{2} MJ_2 \log(b^2 - a^2) + MJ_6 \quad (33)$$

$$X_3 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left[\left\{\left[\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2}\right]M + N\right\}(J_1 + J_3) + \right.$$

$$\left. + \frac{1}{2} MJ_3 \log(1-c^2) + MJ_7\right] \quad (34)$$

$$X_4 = \left\{\left[\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2}\right]M + N\right\}(J_4 - J_2) + \frac{1}{2} MJ_4 \log(1-c^2) - MJ_8 \quad (35)$$

$$J_1 = \int_a^b \left(\frac{c^2-t^2}{1-t^2}\right)^{1/2} \frac{t\,dt}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$

$$J_2 = \int_a^b \left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} \frac{t\,dt}{\sqrt{(1-t^2)(c^2-t^2)}}$$

$$J_3 = \int_c^1 \left(\frac{u^2-c^2}{1-u^2}\right)^{1/2} \frac{u\,du}{\sqrt{(u^2-a^2)(u^2-b^2)}}$$

$$J_4 = \int_c^1 \left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2} \frac{u\,du}{\sqrt{(u^2-c^2)(1-u^2)}}$$

$$J_5 = \int_c^1 \frac{u\log\left(\sqrt{u^2-b^2}+\sqrt{u^2-a^2}\right)}{\sqrt{(u^2-a^2)(u^2-b^2)}}\left[\frac{u^2-c^2}{1-u^2}\right]^{1/2} du$$

$$J_6 = \int_c^1 \frac{u\log\left(\sqrt{u^2-b^2}+\sqrt{u^2-a^2}\right)}{\sqrt{(1-u^2)(u^2-c^2)}}\left[\frac{u^2-a^2}{u^2-b^2}\right]^{1/2} du$$

$$J_7 = \int_a^b \frac{t\log\left(\sqrt{c^2-t^2}+\sqrt{1-t^2}\right)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\left[\frac{c^2-t^2}{1-t^2}\right]^{1/2} dt$$

$$J_8 = \int_a^b \frac{t\log\left(\sqrt{c^2-t^2}+\sqrt{1-t^2}\right)}{\sqrt{(1-t^2)(c^2-t^2)}}\left[\frac{t^2-a^2}{b^2-t^2}\right]^{1/2} dt.$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ on the plane y=0 can be found from the relations (10), (13), (26), (29) and (30) as

$$\tau_{yy}(x,0) = \frac{\pi\mu x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\left[D_1 \left[\frac{1-a^2}{c^2-a^2}\right]^{1/2} \left[\frac{c^2-x^2}{1-x^2}\right]^{1/2} - \right.$$

$$- D_2 \frac{(x^2-a^2)}{\sqrt{(1-x^2)(c^2-x^2)}} \Big] + O(m_1^2 \log m_1) , \quad x \in I_2$$

$$= \frac{\pi\mu x}{\sqrt{(x^2-c^2)(1-x^2)}} \left[D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \frac{(x^2-c^2)}{\sqrt{(x^2-a^2)(x^2-b^2)}} + \right.$$

$$\left. + D_2 \left(\frac{x^2-a^2}{x^2-b^2} \right)^{1/2} \right] + O(m_1^2 \log m_1), \quad x \in I_4 \qquad (36)$$

Defining the stress intensity factors at the edges of the strips by the relations

$$K_a = \underset{x\to a+}{\mathrm{Lt}} \left| \frac{\tau_{yy}(x,0)\sqrt{x-a}}{\pi \mu v_0} \right| ; \qquad K_b = \underset{x\to b-}{\mathrm{Lt}} \left| \frac{\tau_{yy}(x,0)\sqrt{b-x}}{\pi \mu v_0} \right|$$

$$K_c = \underset{x\to c+}{\mathrm{Lt}} \left| \frac{\tau_{yy}(x,0)\sqrt{x-c}}{\pi \mu v_0} \right| ; \qquad K_1 = \underset{x\to 1-}{\mathrm{Lt}} \left| \frac{\tau_{yy}(x,0)\sqrt{1-x}}{\pi \mu v_0} \right|$$

We get

$$K_a = \left| \frac{\sqrt{a}\, D_1/v_0}{\sqrt{2(b^2-a^2)}} \right| \qquad (37)$$

$$K_b = \left| \frac{\gamma\sqrt{b}}{\sqrt{2(b^2-a^2)}} \left\{ \frac{D_1}{v_0}\left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-b^2}{1-b^2} \right)^{1/2} - \frac{D_2}{v_0} \frac{(b^2-a^2)}{\sqrt{(1-b^2)(c^2-b^2)}} \right\} \right| \tag{38}$$

$$K_c = \left| \frac{\gamma\sqrt{c}}{\sqrt{2(1-c^2)}} \frac{D_2}{v_0} \left(\frac{c^2-a^2}{c^2-b^2} \right)^{1/2} \right| \tag{39}$$

$$K_1 = \left| \frac{1}{\sqrt{2(1-c^2)}} \left\{ \frac{(1-c^2)\, D_1}{\sqrt{(c^2-a^2)(1-b^2)}} + \left(\frac{1-a^2}{1-b^2} \right)^{1/2} D_2 \right\} \right| \tag{40}$$

The vertical displacement v(x,y) on the plane y=0 can be obtained from equations (9), (13), (26), (29), and (30) as

$$v(x,o) = \frac{4\tau^2}{\pi} \Bigg[\left\{ \left[\gamma + \log m_1 - \frac{\pi i}{2} \right] M + N \right\} \left\{ D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} (J_1+J_3) + \right.$$

$$\left. + D_2(J_4-J_2) \right\} + \frac{M}{2} \left\{ (J_9+J_{11}) \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} D_1 + D_2(J_{12}-J_{10}) \right\} \Bigg]$$

$$x \in I_1,\ I_3,\ I_5 \tag{41}$$

where

$$J_9 = \int_a^b \frac{t \log|t^2-x^2|}{\sqrt{(t^2-a^2)(b^2-t^2)}} \left(\frac{c^2-t^2}{1-t^2} \right)^{1/2} dt$$

$$J_{10} = \int_a^b \frac{t \log|t^2-x^2|}{\sqrt{(1-t^2)(c^2-t^2)}} \left[\frac{t^2-a^2}{b^2-t^2} \right]^{1/2} dt$$

$$J_{11} = \int_c^1 \frac{u \log|u^2-x^2|}{\sqrt{(u^2-a^2)(u^2-b^2)}} \left[\frac{u^2-c^2}{1-u^2} \right]^{1/2} du$$

$$J_{12} = \int_c^1 \frac{u \log|u^2-x^2|}{\sqrt{(u^2-c^2)(1-u^2)}} \left[\frac{u^2-a^2}{u^2-b^2} \right]^{1/2} du.$$

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_a, K_b, K_c and K_1 at the edges of the strips and vertical displacement $|v(x,0)/v_0|$ near about the rigid strips have been plotted against dimensionless frequency m_1 and distance x respectively for a Poisson solid ($\tau^2=3$).

It is found that whatever the lengths of the strips are, SIFs at the four edges of the strips increase with increase in the value of m_1($0.1 \leq m_1 \leq 0.6$).

From the graphs, it may be noted further that with a decrease in the length of the inner strip, which might be induced either by increasing 'a' or by decreasing 'b', the SIFs gradually increase (fig.2 - fig.9).

Also, a decrease in the value of the length of the outer strip, which might be induced by increasing the value of c, causes an increase in the values of the SIFs (fig.10 - fig.13), from which an interesting conclusion might be drawn: i.e, that the presence of the outer strip suppresses the SIFs at both the edges of the inner strip and the presence of the inner strip suppresses the SIFs at both the edges of the outer strip.

The vertical displacement has been plotted for different strip lengths. It is found from fig.14 - fig.16 that with the increase in the value of strip lengths, the displacement increases.

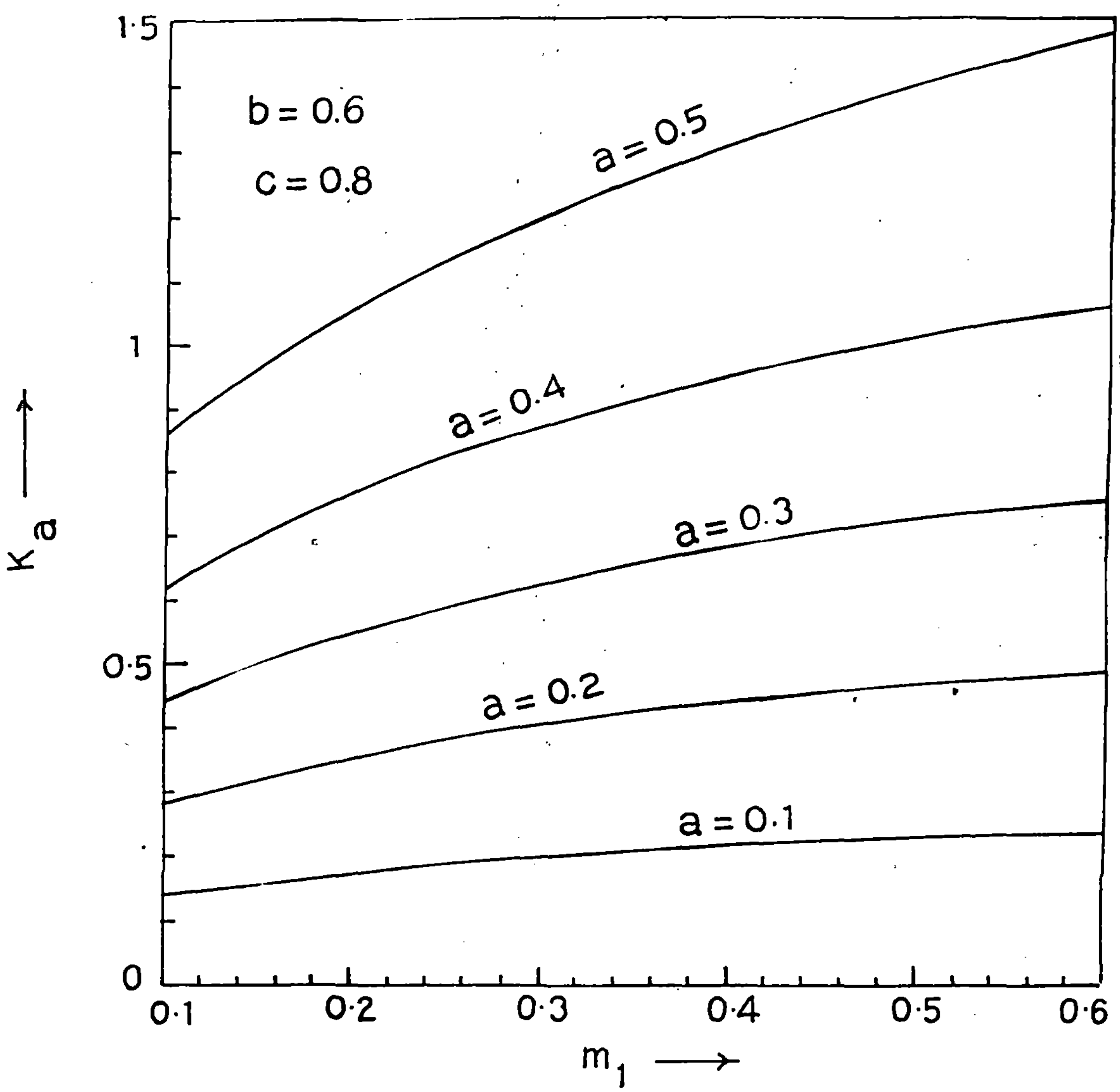

Fig. 2. Stress intensity factor K_a versus dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

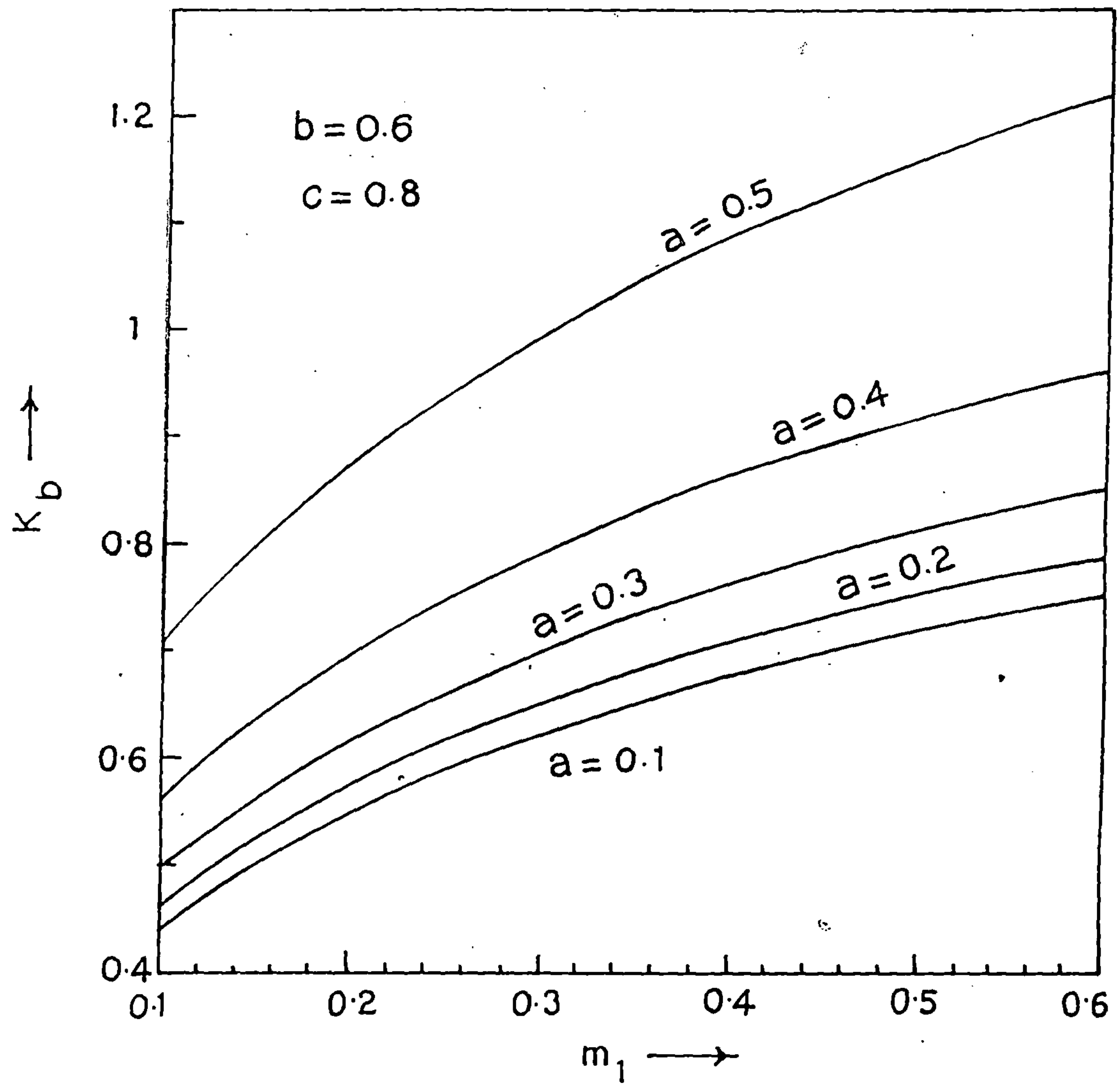

Fig. 3. Stress intensity factor K_b versus dimensionless frequency m_1 for b = 0.6, c = 0.8 and for different values of a.

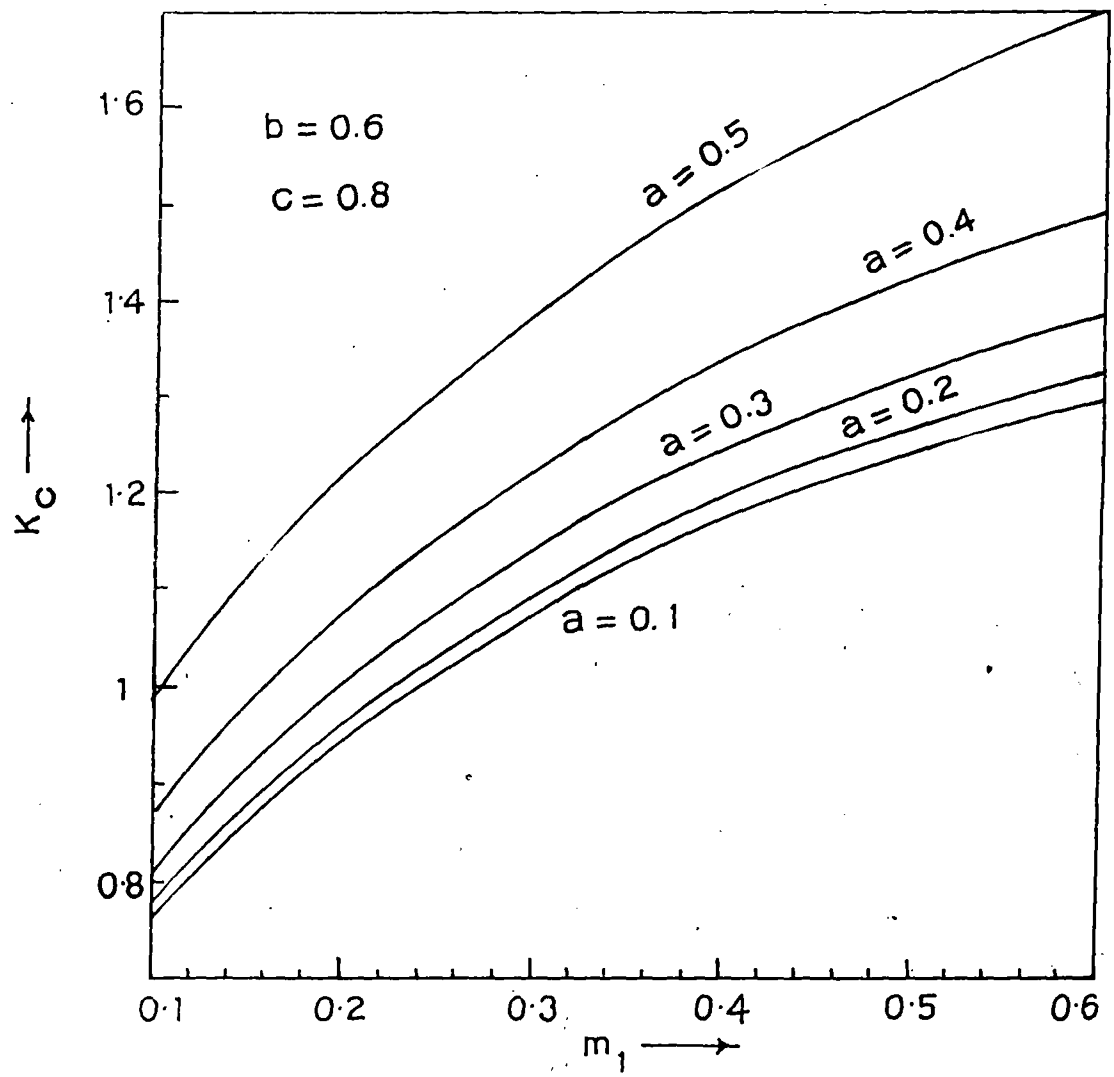

Fig. 4. Stress intensity factor K_c versus dimensionless frequency m_1 for b = 0.6, c = 0.8 and for different values of a.

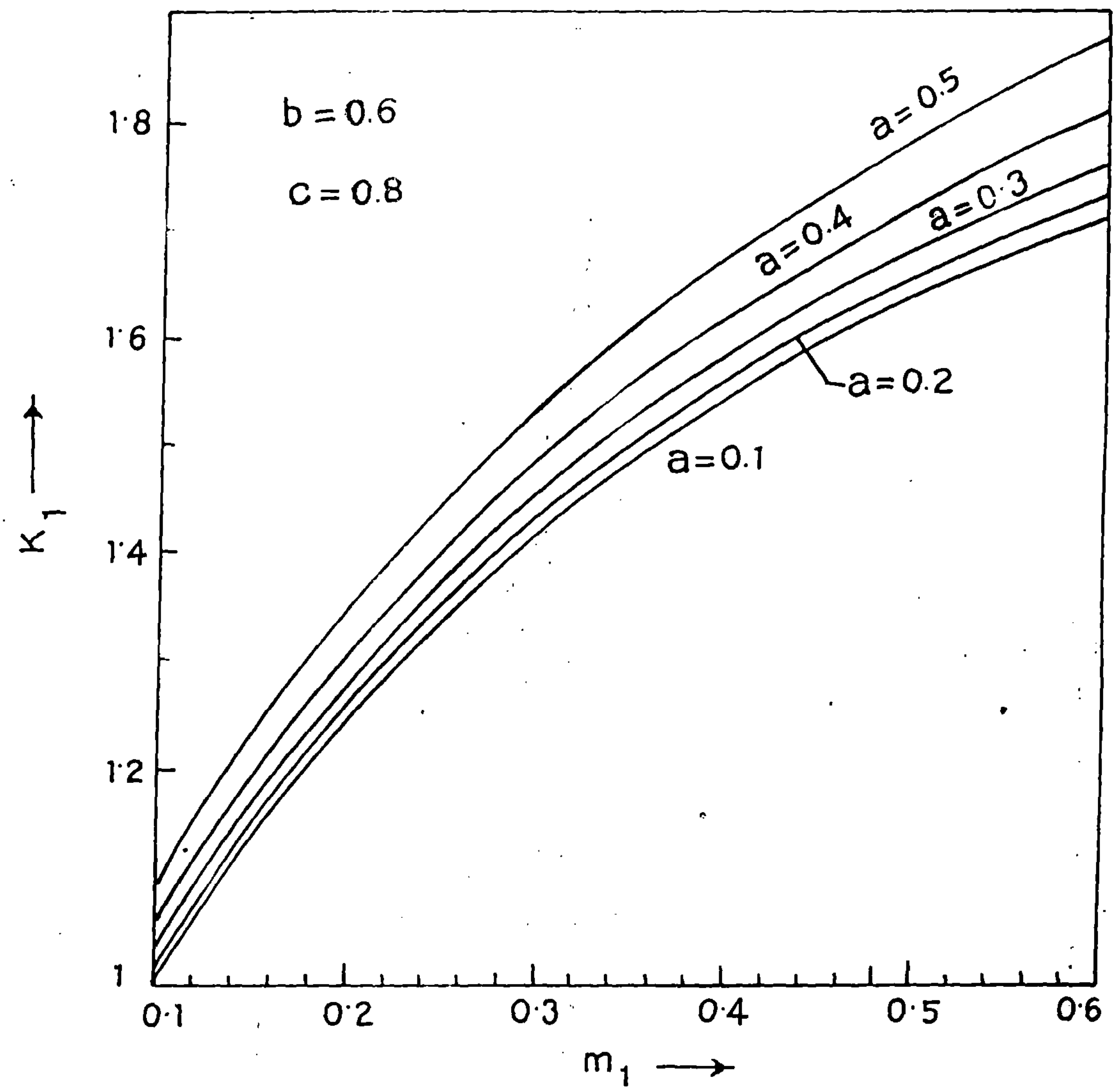

Fig. 5. Stress intensity factor K_1 versus dimensionless frequency m_1 for b = 0.6, c = 0.8 and for different values of a.

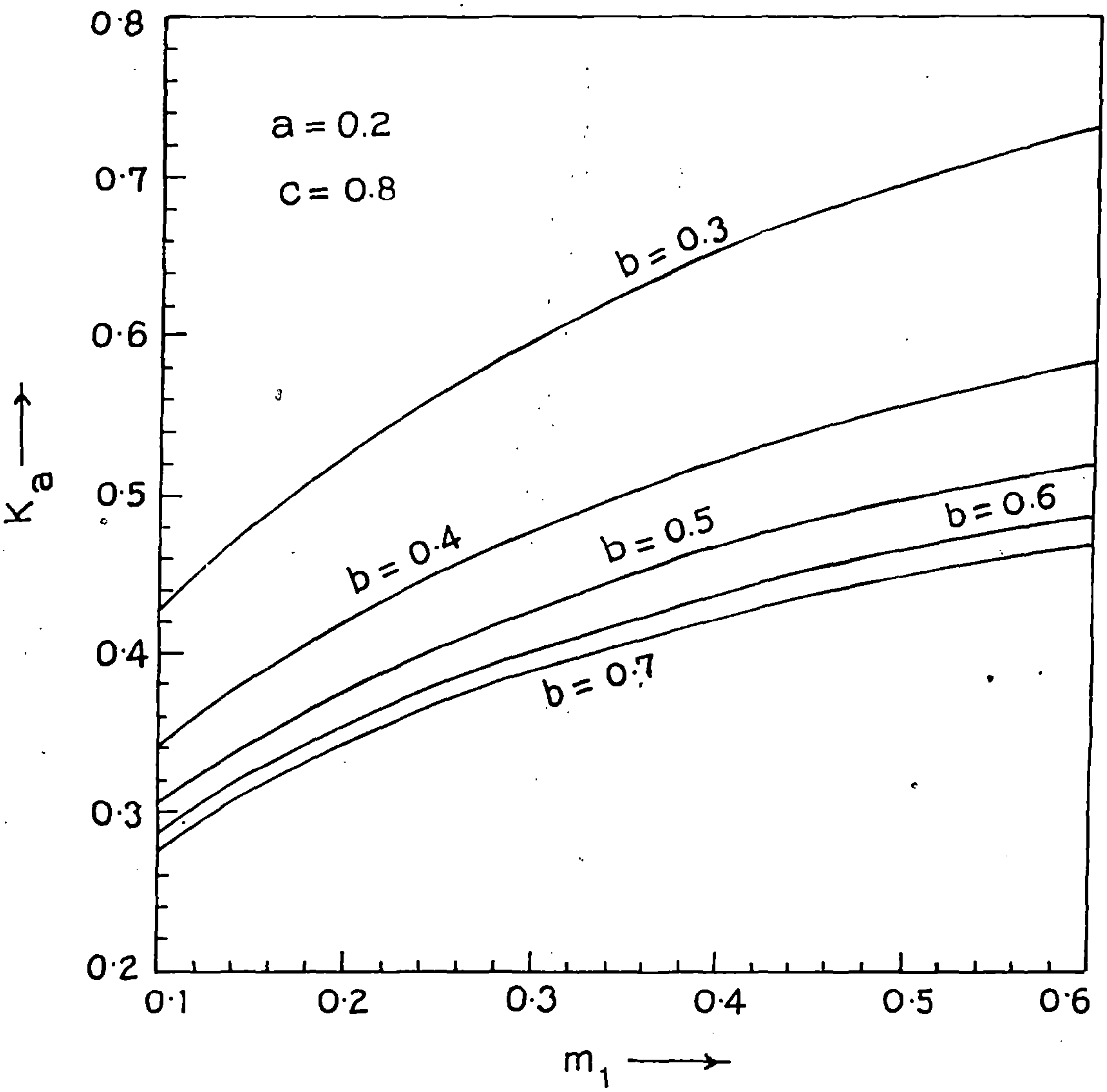

Fig. 6. Stress intensity factor K_a versus dimensionless frequency m_1 for a = 0.2, c = 0.8 and for different values of b.

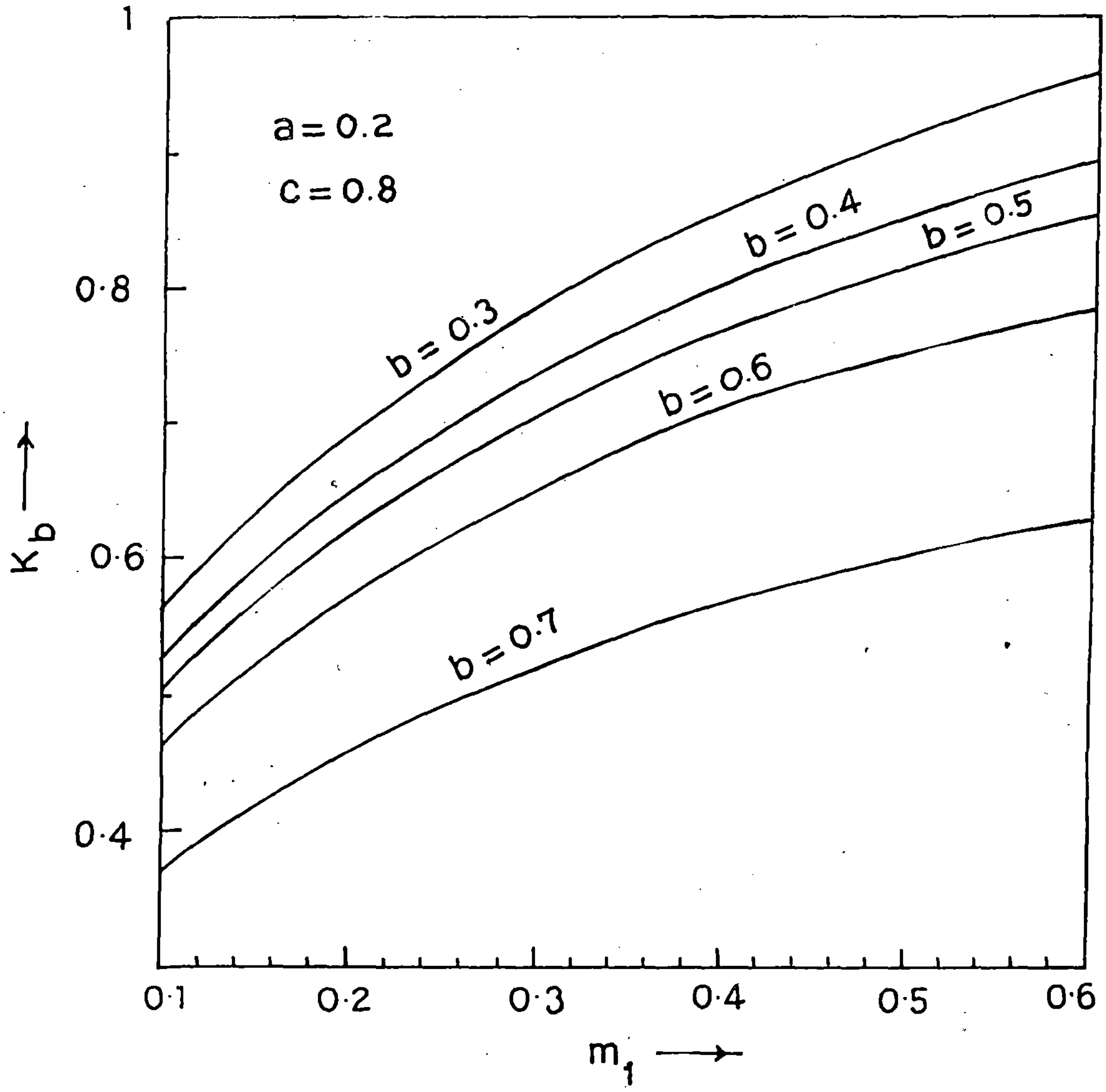

Fig. 7. Stress intensity factor K_b versus dimensionless frequency m_1 for a = 0.2, c = 0.8 and for different values of b.

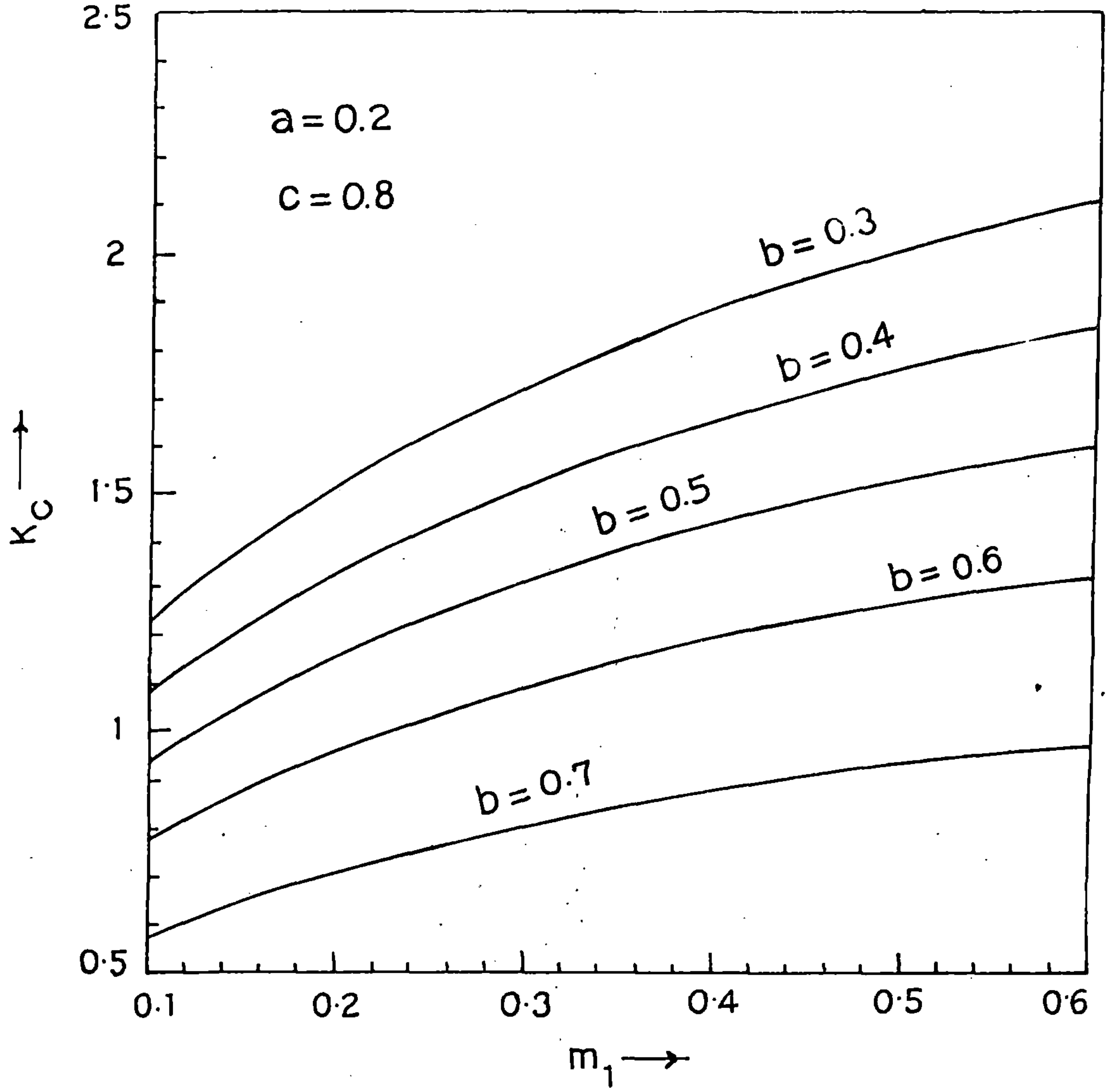

Fig. 8. Stress intensity factor K_c versus dimensionless frequency m_1 for a = 0.2, c = 0.8 and for different values of b.

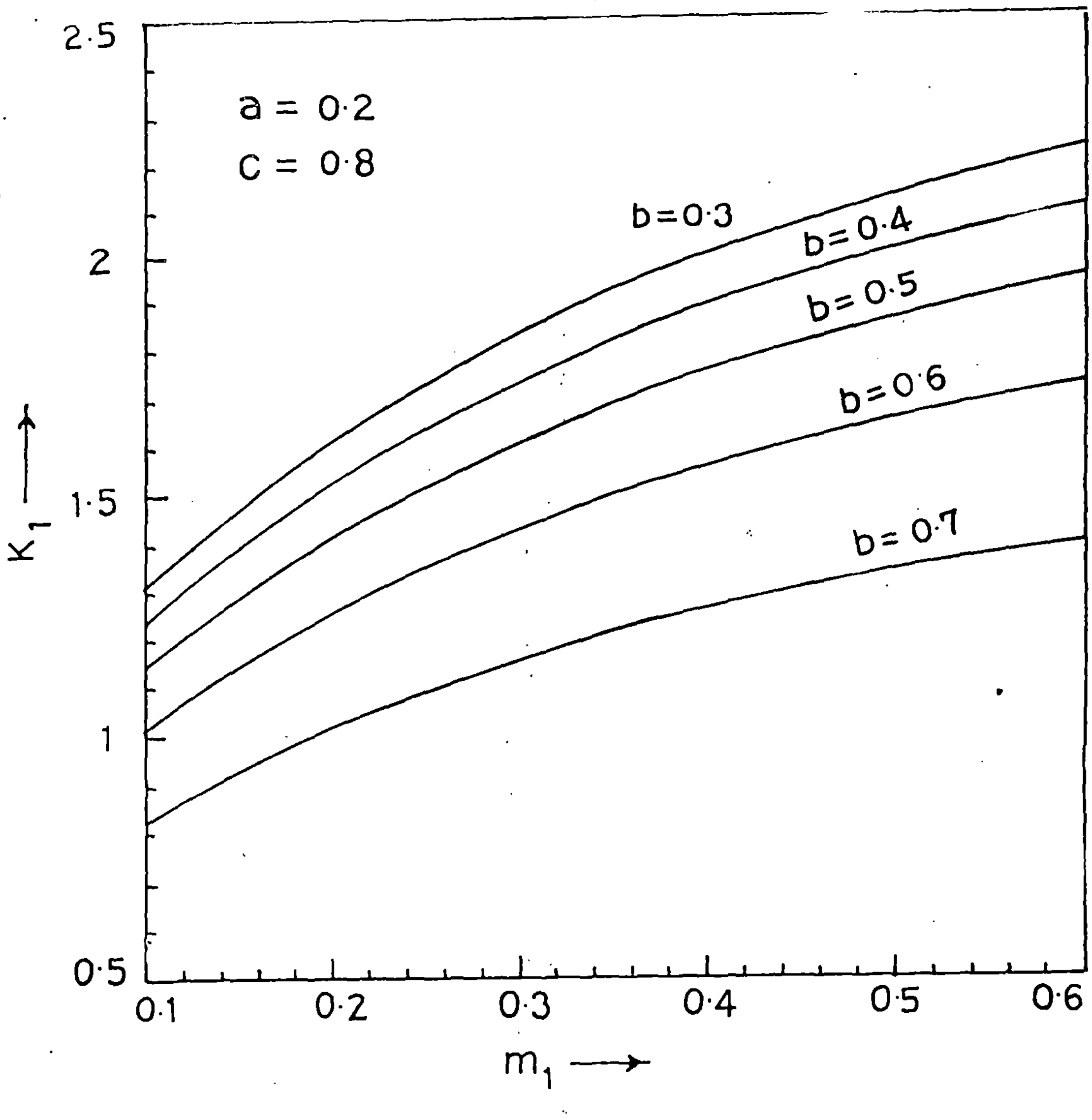

Fig. 9. Stress intensity factor K_1 versus dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

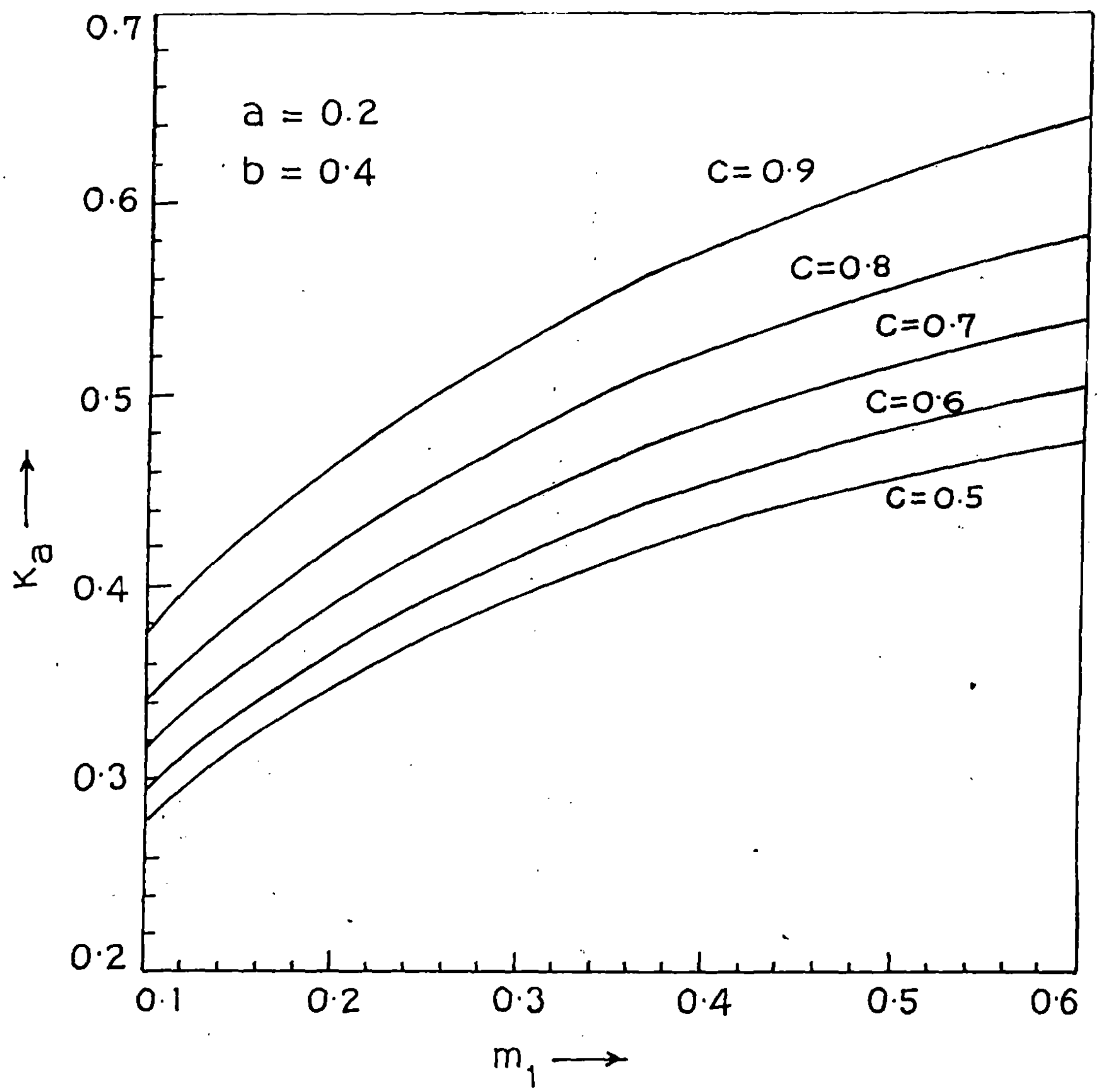

Fig. 10. Stress intensity factor K_a versus dimensionless frequency m_1 for a = 0.2, b = 0.4 and for different values of c.

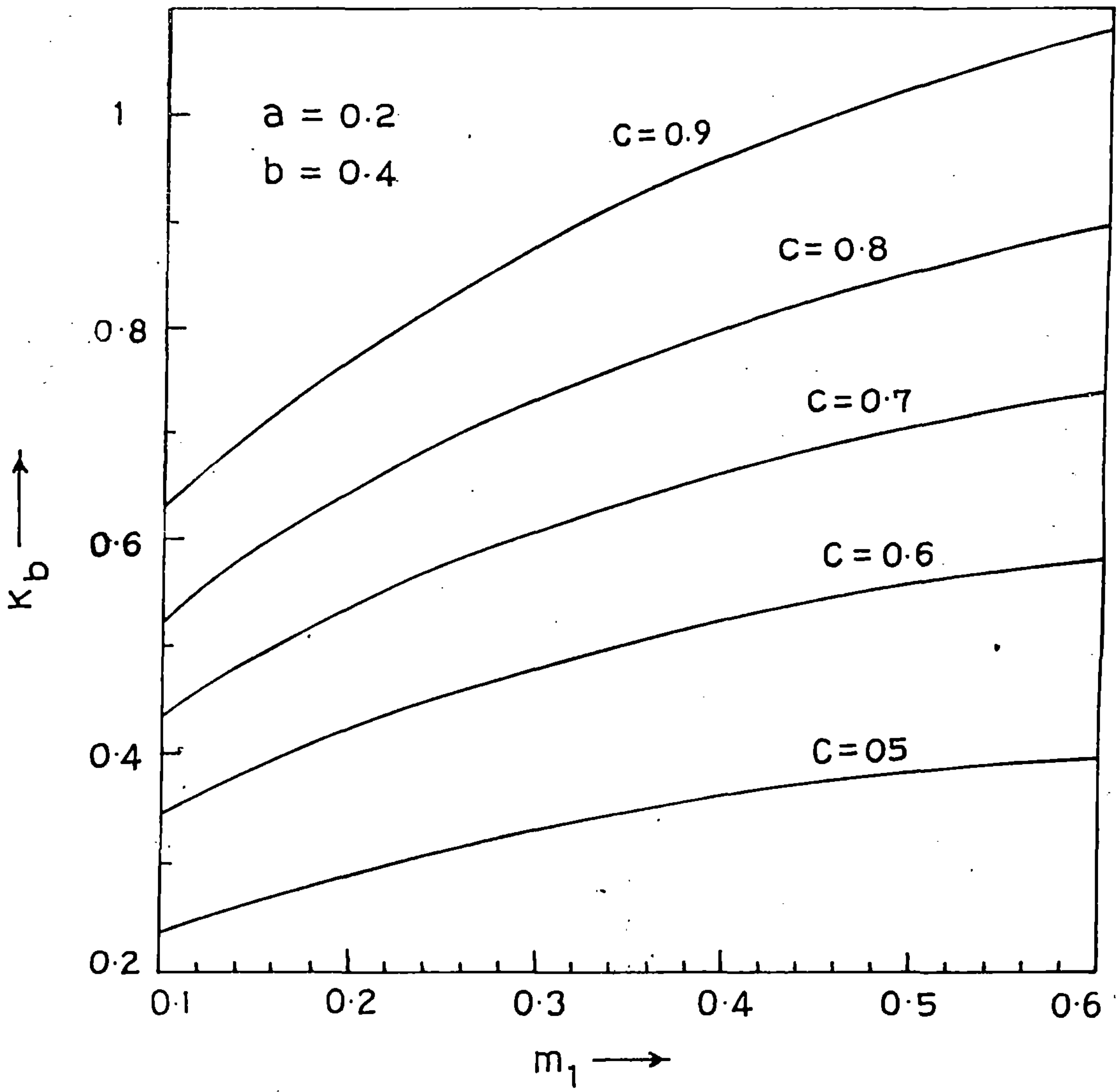

Fig. 11. Stress intensity factor K_b versus dimensionless frequency m_1 for a = 0.2, b = 0.4 and for different values of c.

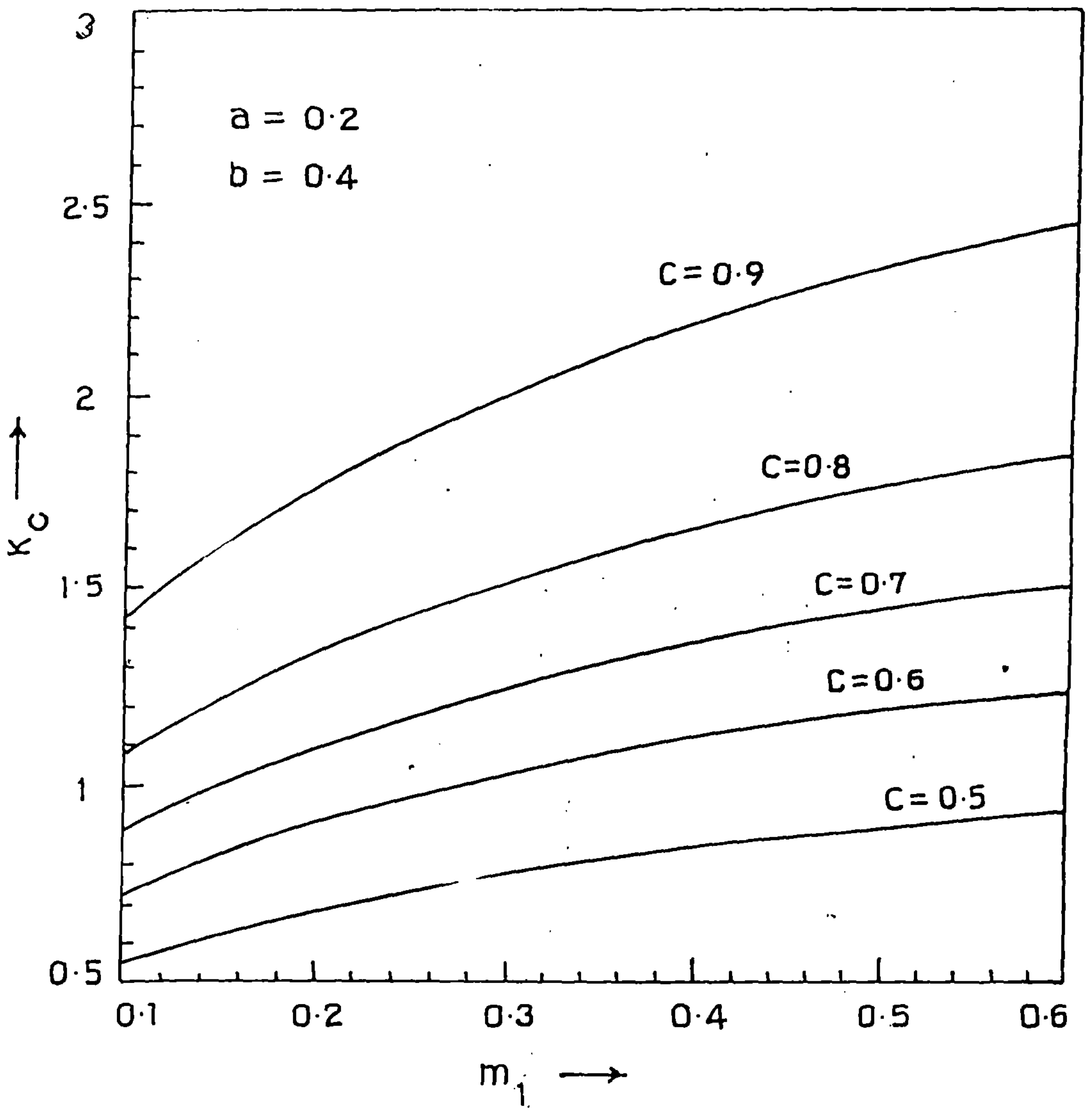

Fig. 12. Stress intensity factor K_c versus dimensionless frequency m_1 for a = 0.2, b = 0.4 and for different values of c.

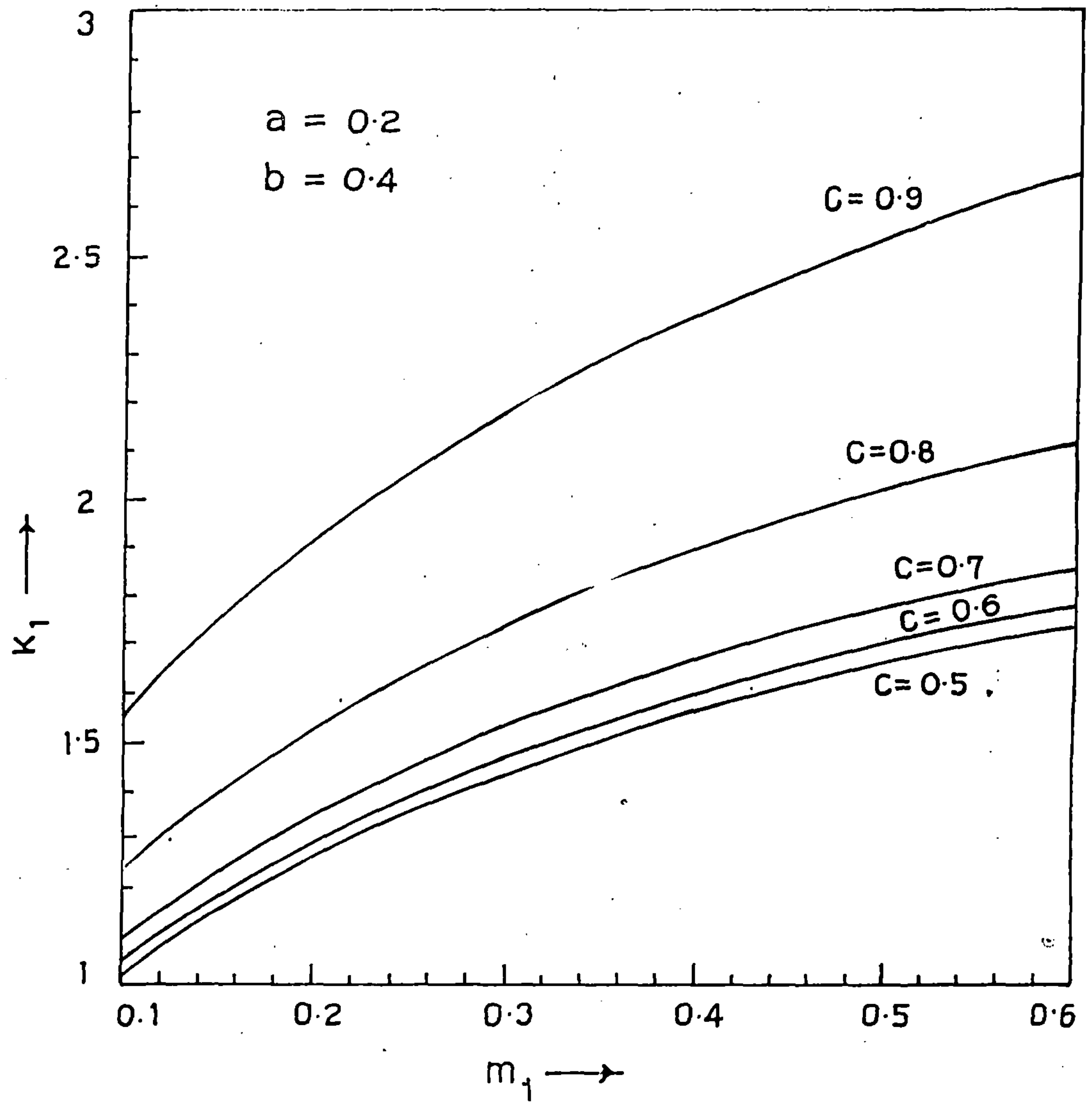

Fig. 13. Stress intensity factor K_1 versus dimensionless frequency m_1 for a = 0.2, b = 0.4 and for different values of c.

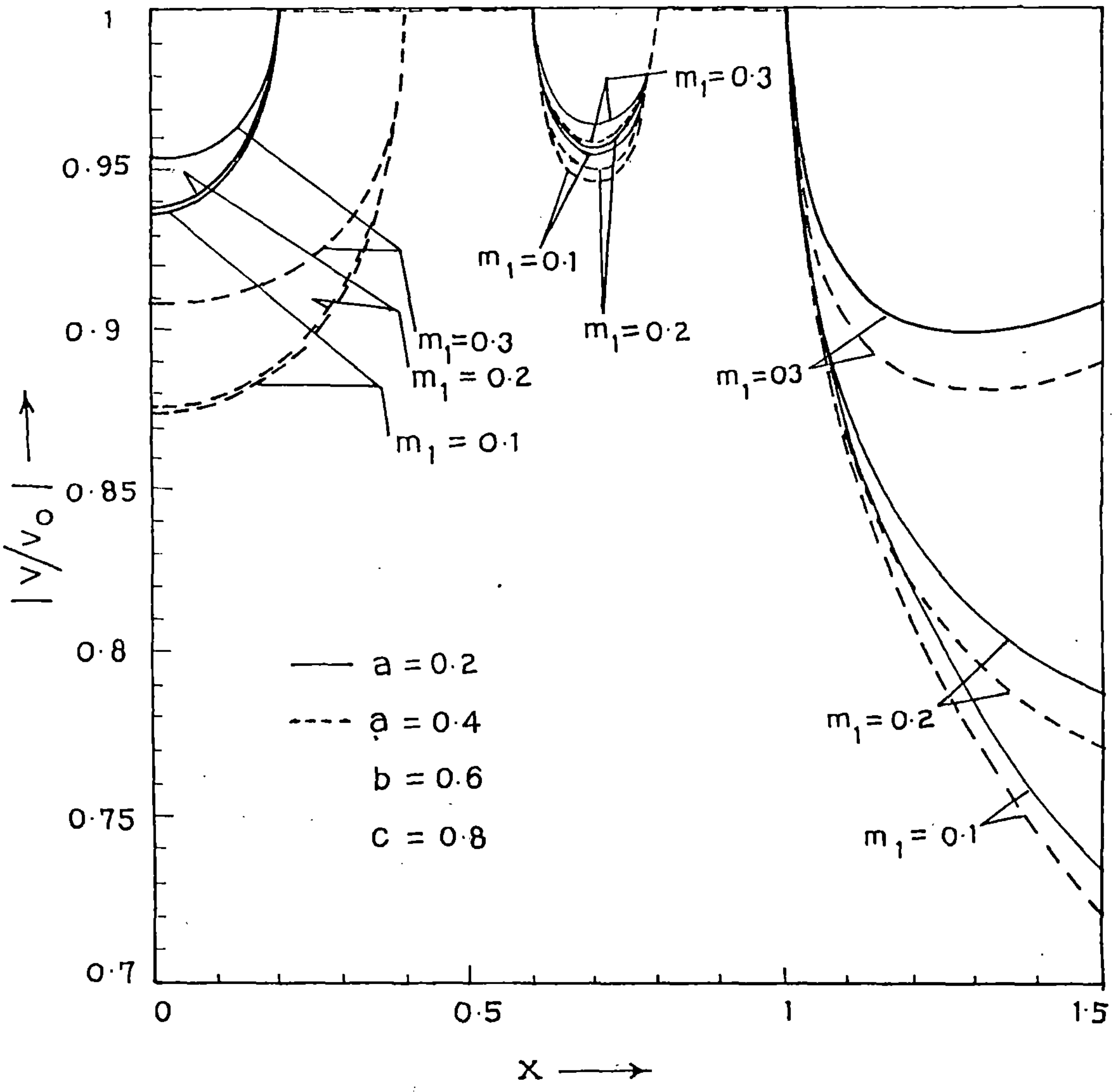

Fig. 14. Vertical displacement $| v(x,0)/v_0 |$ versus dimensionless distance x for b = 0.6, c = 0.8, a = 0.2, 0.4 and for m_1 = 0.1, 0.2, 0.3.

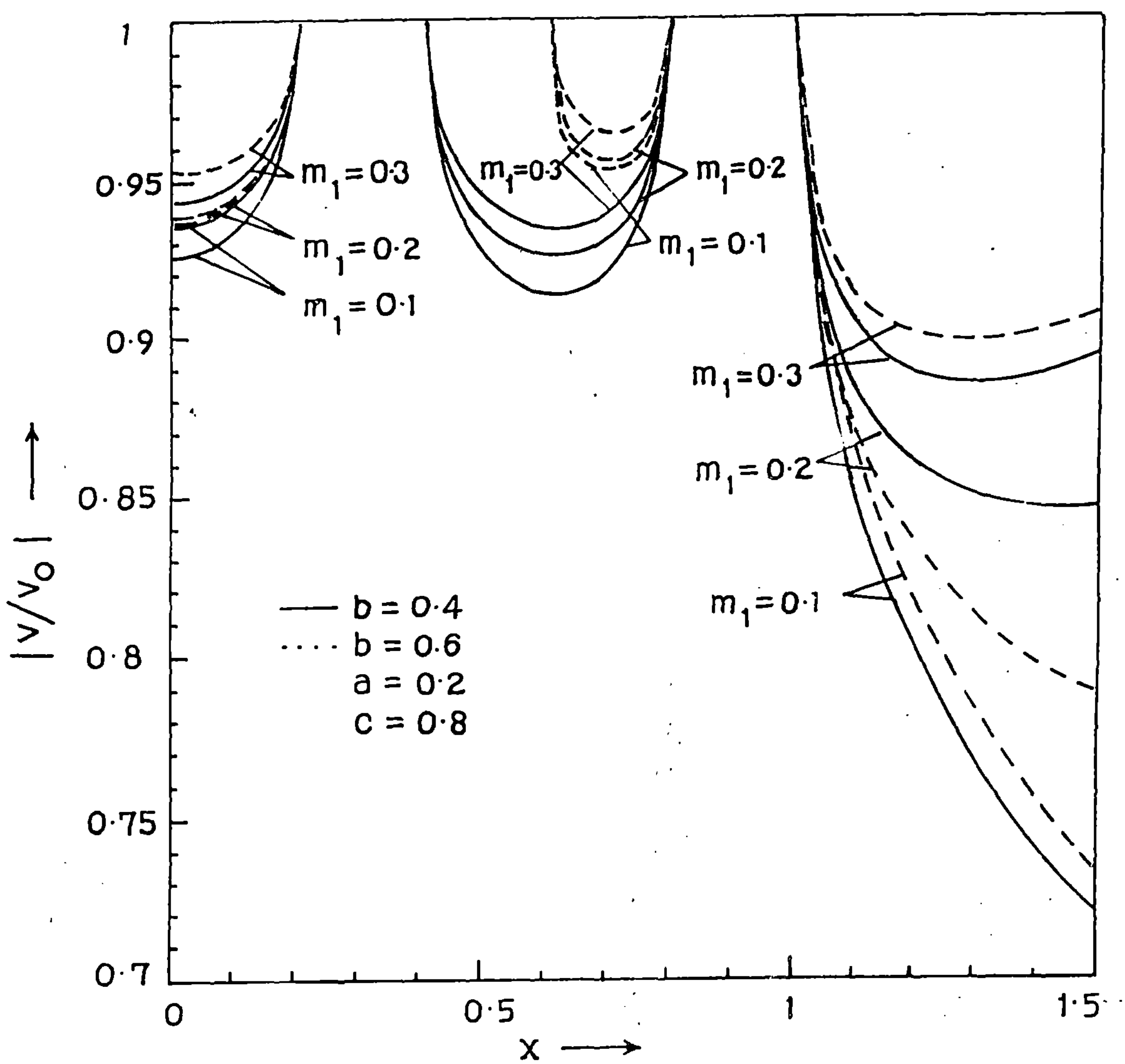

Fig. 15. Vertical displacement $| v(x,0)/v_0 |$ versus dimensionless distance x for a = 0.2, c = 0.8, b = 0.4, 0.6 and for m_1 = 0.1, 0.2, 0.3.

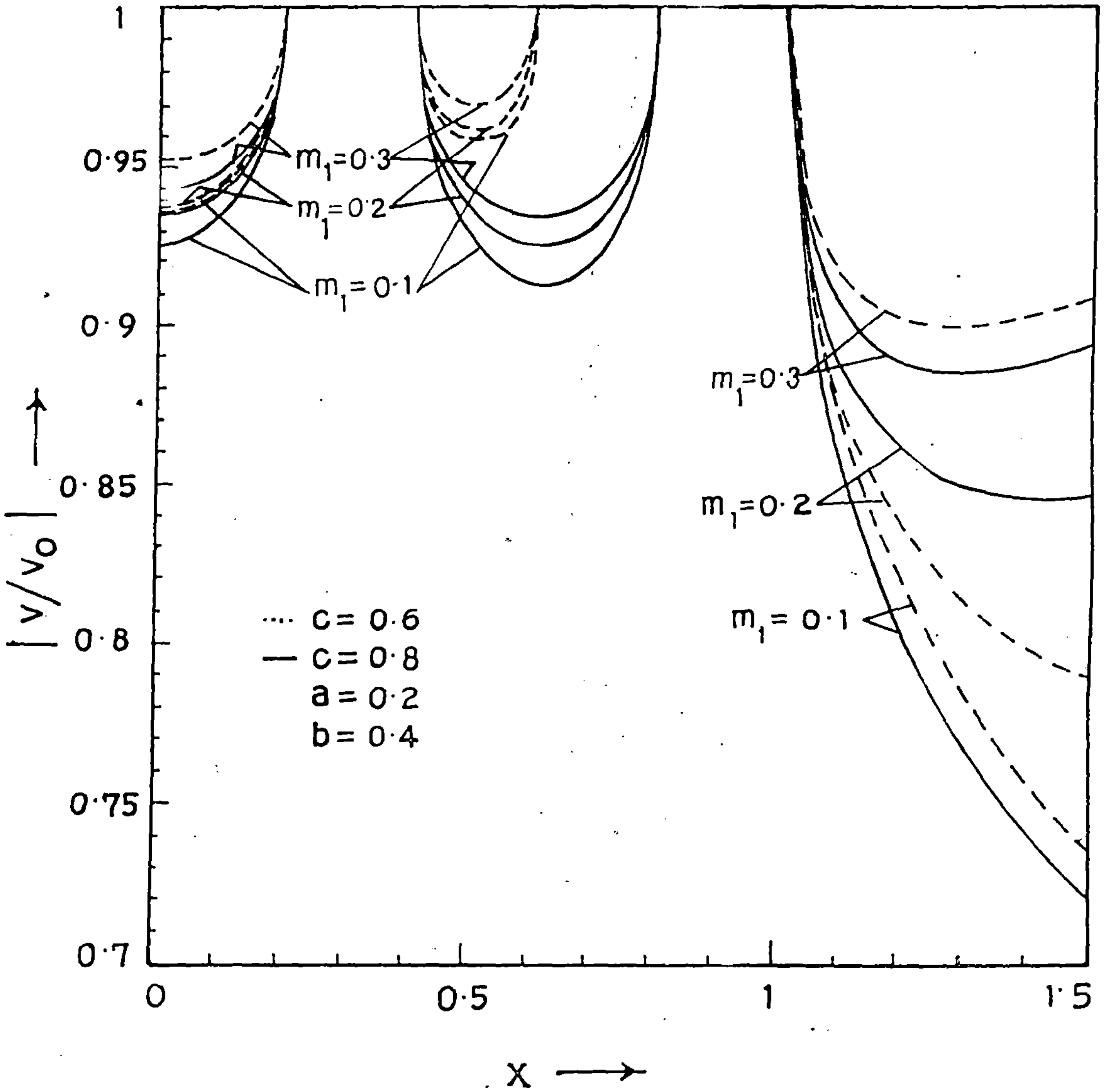

Fig. 16. Vertical displacement $| v(x,0)/v_0 |$ versus dimensionless distance x for a = 0.2, b = 0.4, c = 0.6, 0.8 and for m_1 = 0.1, 0.2, 0.3.

DIFFRACTION OF ELASTIC WAVES BY FOUR RIGID STRIPS EMBEDDED IN AN INFINITE ORTHOTROPIC MEDIUM

1. INTRODUCTION

In recent years, the study of the problems involving cracks or inclusions in composite and anisotropic materials has gained much importance. The problems of diffraction of elastic waves by cracks or inclusions have aroused attention in the field of fracture mechanics in view of their application in Seismology and Geophysics. Studies of a single Griffith crack as well as two parallel and coplanar Griffith cracks have been made by Mal [1970], Jain and Kanwal [1972] and Itou [1980]. The corresponding problems of diffraction by a single and two parallel rigid strips have been solved by Wickham [1977], Jain and Kanwal [1972] and Mandal and Ghosh [1992] respectively. In most of the cases the problems were solved by the integral equation technique, but the solutions of interesting problems involving the scattering of elastic waves by more than two coplanar Griffith cracks or strips are still lacking. The problem involving single Griffith crack in orthotropic medium was investigated by Kassir and Bandyopadhya [1983], Shindo et al

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[1986] and De and Patra [1990]. Shindo et al [1991] have investigated the impact response of symmetric edge cracks in an orthotropic strip. Mandal and Ghosh [1994] considered the problem of interaction of elastic waves with a periodic array of coplanar Griffith cracks in an orthotropic elastic medium. The problem of scattering of elastic waves by a circular crack in transversely isotropic medium was investigated by Kundu and Bostrom [1991].

In our case, we have considered the two-dimensional problems of diffraction of elastic waves by four coplanar parallel rigid strips embedded in an infinite orthotropic medium. The five part mixed boundary value problem was reduced to the solution of a set of integral equations. Following the technique developed by Srivastava and Lowengrub [1970], the integral equations were solved. The normal stress under the strips and displacement outside the strips were derived in closed analytical form. To display the influence of the material orthotropy numerical values of stress intensity factors at the edges of the strips and vertical displacement have been plotted against dimensionless frequency and distance respectively for several orthotropic materials. This type of problem is important in view of their application in detecting the presence of inhomogeneities embedded in material structure and in seismology while studing the scattering of elastic waves by inhomogeneities like rigid hard rocks inside the earth.

2. FORMULATION OF THE PROBLEM

Consider the diffraction of normally incident longitudinal wave by four coplanar and parallel rigid strips embedded in an infinite orthotropic elastic medium and the strips occupy the region $d_1 \leq |x_1| \leq d_2$, $d_3 \leq |x_1| \leq d$, $x_2=0$, $|x_3|<\infty$. Let E_i, μ_{ij} and ν_{ij} $(i,j=1,2,3)$ denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the x_1, x_2, x_3 directions which coincide with the axes of material orthotropy. Normalizing all lengths with respect to 'd' and putting $x_1/d=x$, $x_2/d=y$, $x_3/d=z$, $d_1/d=a$, $d_2/d=b$, $d_3/d=c$, the rigid strips are defined by $a \leq |x| \leq b$, $c \leq |x| \leq 1$, $y=0$, $|z|<\infty$ (Fig.1).

Let a time harmonic wave given by $u_i=0$ and $v_i=v_0\exp[\,i(ky-\omega t)]$ where $k=\omega d/c_s \sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$ and v_0 is a constant, travelling in the direction of positive y-axis be incident normally on the strips. The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy} / \mu_{12} = c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y}$$

$$\tau_{xy} / \mu_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (1)$$

where c_{ij} $(i,j = 1,2)$ are nondimensional parameters related to the elastic constants by the relations

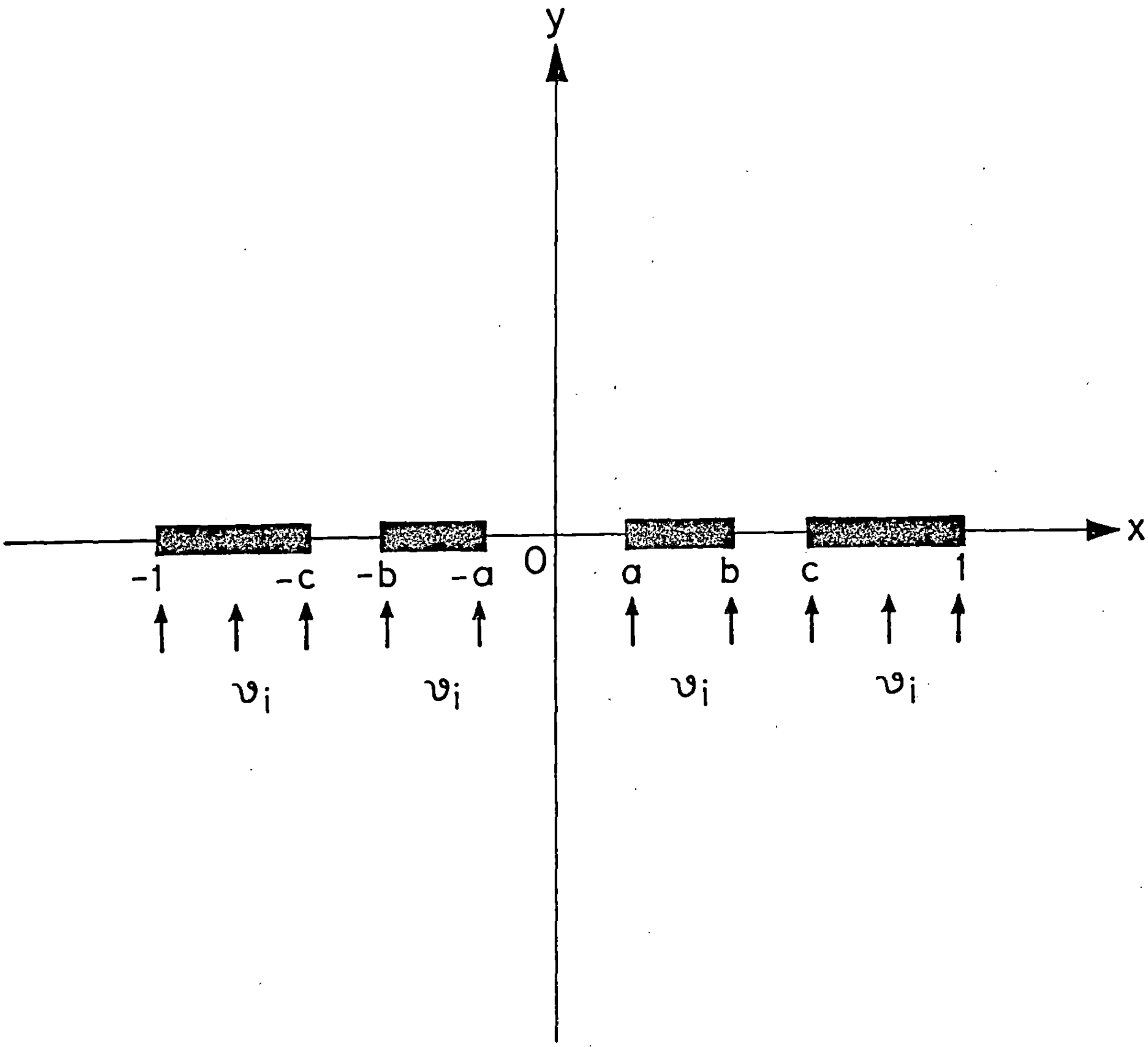

Fig. 1. Geometry of the strips and incident field.

$$c_{11} = E_1 / \mu_{12} (1 - \nu_{12}^2 E_2/E_1)$$

$$c_{22} = E_2 / \mu_{12} (1 - \nu_{12}^2 E_2/E_1) = c_{11}E_2/E_1 \qquad (2)$$

$$c_{12} = \nu_{12}E_2 / \mu_{12} (1 - \nu_{12}^2 E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11}$$

The constants E_i and ν_{ij} satisfy the Maxwell's relation

$$\nu_{ij} / E_i = \nu_{ji} / E_j .$$

The equations of motion for orthotropic material, interms of displacements are

$$c_{11} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1 + c_{12}) \frac{\partial^2 v}{\partial x \partial y} = \frac{d^2}{c_s^2} \frac{\partial^2 u}{\partial t^2}$$

$$(3)$$

$$c_{22} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1 + c_{12}) \frac{\partial^2 u}{\partial x \partial y} = \frac{d^2}{c_s^2} \frac{\partial^2 v}{\partial t^2}$$

where u, v are the displacement components of the scattered field (Fig.2).

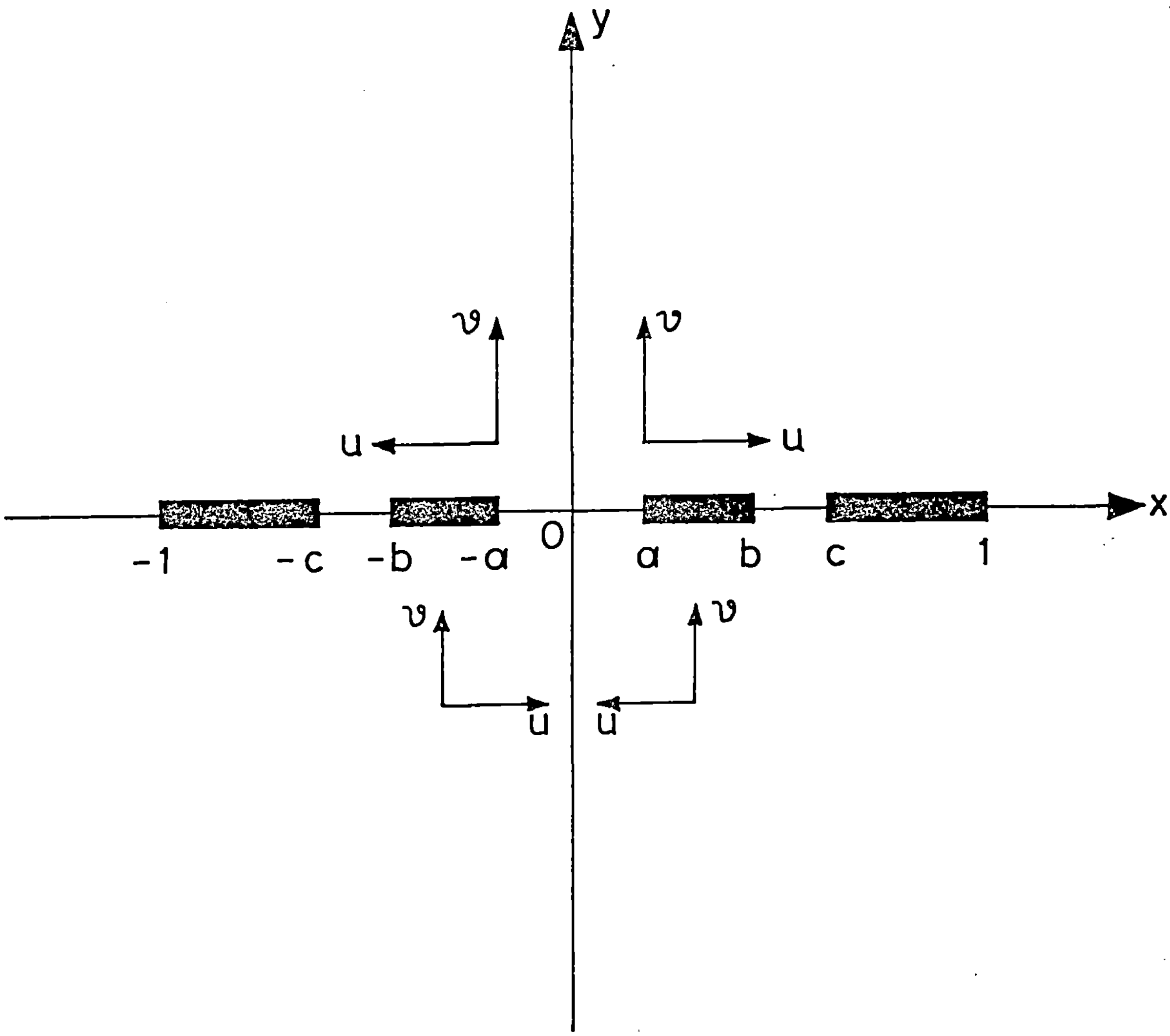

Fig. 2. Displacement components of the scattered field.

The boundary conditions are

(i) $u(x,y,t) = 0$, $v(x,y,t) + v_i(x,y,t) = 0$ across $y=0$ on the surface of the strips.

(ii) u and v are continuous across $y=0$ for $|x| < \infty$.

(iii) τ_{yy}, τ_{xy} are continuous across $y=0$ outside the strips.

Further, the scattered field should satisfy the radiation condition at infinity. Substituting $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ our problem reduces to the solution of the equations

$$c_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1 + c_{12})\frac{\partial^2 v}{\partial x \partial y} + \frac{d^2\omega^2}{c_s^2}u = 0$$

and

$$c_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1 + c_{12})\frac{\partial^2 u}{\partial x \partial y} + \frac{d^2\omega^2}{c_s^2}v = 0 \qquad (4)$$

Boundary conditions on u and v suggest that u and v are odd and even functions of y respectively. Accordingly, equations (4) are to be solved subject to the boundary conditions

$$v(x,0) = -v_0 \, , \qquad x \in I_2 \, , I_4 \tag{5}$$

$$\tau_{yy}(x,0) = 0 \, , \qquad x \in I_1 \, , \, I_3 \, , \, I_5 \tag{6}$$

$$u(x,0) = 0 \, , \qquad |x| < \infty \tag{7}$$

with $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,1)$, $I_5 = (1,\infty)$.

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of equations (4) are taken as

$$u(x,y) = \pm \frac{2}{\pi} \int_0^\infty \left[A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|) \right] \sin\xi x \, d\xi \, , \quad y \gtrless 0 \tag{8}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)\right] \cos\xi x \, d\xi \, , \tag{9}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1 + c_{12})\gamma_i} \, , \qquad i = 1,2 \, , \qquad k_s^2 = \frac{d^2\omega^2}{c_s^2} \tag{10}$$

and $A_i(\xi)$ $(i = 1,2)$ are the unknowns to be solved, γ_1^2 and γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0 \tag{11}$$

From the boundary condition (7) it is found that

$$A_2(\xi) = - A_1(\xi).$$

Therefore displacements u, v and stresses τ_{yy}, τ_{xy} finally can be written as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin\xi x\, d\xi, \quad y>0 \qquad (12)$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1\exp(-\gamma_1|y|) - \alpha_2\exp(-\gamma_2|y|)\right]A_1(\xi)\cos\xi x\, d\xi \qquad (13)$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \right.$$

$$\left. - \left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos\xi x\, d\xi, \quad y>0 \qquad (14)$$

$$\tau_{xy}/\mu_{12} = -\frac{2}{\pi}\int_0^\infty \left[(\gamma_1 + \alpha_1)\exp(-\gamma_1|y|) - \right.$$

$$\left. - (\gamma_2 + \alpha_2)\exp(-\gamma_2|y|)\right]A_1(\xi)\sin\xi x\, d\xi \qquad (15)$$

Next putting

$$A(\xi) = \frac{\alpha_1\gamma_1 - \alpha_2\gamma_2}{\xi}A_1(\xi)$$

the boundary conditions (5) and (6) lead to the following integral equations in $A(\xi)$:

$$\int_0^{\infty} \left[\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right] A(\xi) \cos\xi x \, d\xi = -\frac{\pi}{2} v_0 , \qquad x \in I_2 , I_4 \tag{16}$$

and

$$\int_0^{\infty} A(\xi) \cos\xi x \, d\xi = 0 , \qquad x \in I_1 , I_3 , I_5 \tag{17}$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (16) and (17) in the form

$$A(\xi) = \int_a^b tf(t^2)\cos\xi t \, dt + \int_c^1 ug(u^2)\cos\xi u \, du \tag{18}$$

where $f(t^2)$ and $g(u^2)$ are unknown functions to be determined.

By the choice of $A(\xi)$ given by (18) the relation (17) is satisfied automatically and the equation (16) becomes

$$\int_a^b tf(t^2)dt \int_0^{\infty} \left[\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right] \cos\xi x \cos\xi t \, d\xi \quad +$$

$$+ \int_c^1 ug(u^2)du \int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) \cos\xi x \cos\xi u \, d\xi$$

$$= -\frac{\pi}{2} v_0 , \quad x \in I_2, I_4 \tag{19}$$

Using the relation

$$\frac{\sin\xi x \sin\xi t}{\xi^2} = \int_0^x \int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)dvdw}{(x^2-w^2)^{1/2} (t^2-v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{d}{dx} \int_a^b tf(t^2)dt \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{vwL_1(v,w) \, dw \, dv}{(x^2-w^2)^{1/2} (t^2-v^2)^{1/2}} +$$

$$+ \frac{d}{dx} \int_c^1 ug(u^2)du \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{vwL_1(v,w) \, dw \, dv}{(x^2-w^2)^{1/2} (u^2-v^2)^{1/2}}$$

$$= -\frac{\pi}{2} v_0 , \quad x \in I_2, I_4 \tag{20}$$

where $$L_1(v,w) = \int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) J_0(\xi w) \, J_0(\xi v) \, d\xi . \tag{21}$$

By a contour integration technique (Mandal and Ghosh [1994]) the infinite integral in $L_1(v,w)$ can be converted to the following finite integrals

$$L_1(v,w) = -i\left[\int_0^{1/\sqrt{c}_{11}} \frac{c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,d\eta\right. -$$

$$\left. - \int_{1/\sqrt{c}_{11}}^{1} \frac{c_{11}\eta^2-1+\bar{\gamma}_2'^2}{\bar{\gamma}_2'^2(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,d\eta\right], \quad w>v \qquad (22)$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{R_1 - (R_1^2 - 4R_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{R_1 + (R_1^2 - 4R_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_1' = \left[\frac{1}{2}\left\{-R_1 + (R_1^2 + 4R_3)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2' = \left[\frac{1}{2}\left\{R_1 + (R_1^2 + 4R_3)^{1/2}\right\}\right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}}\left\{(c_{12}^2 + 2c_{12} - c_{12}c_{22})\,\eta^2 + (1 + c_{22})\right\}$$

$$R_2 = \frac{c_{11}}{c_{22}}(1-\eta^2)\left[\frac{1}{c_{11}} - \eta^2\right]$$

$$R_3 = \frac{c_{11}}{c_{22}}(1-\eta^2)\left[\eta^2 - \frac{1}{c_{11}}\right] \qquad (23)$$

The corresponding expression of $L_1(v,w)$ for $w<v$ follows from (22) by interchanging w and v.

Substituting the series expansion of $J_0()$ and $H_0^{(1)}()$ for small k_s, in (22) we find after some algebraic manipulation

$$L_1(v,w) = \frac{2}{\pi}\left[\left(\gamma+\log(k_s w/2) - \frac{\pi i}{2}\right)M + N - \frac{(w^2+v^2)}{4} R k_s^2 \log k_s\right] + O(k_s^2) \quad , \; w>v$$

$$= \frac{2}{\pi}\left[\left(\gamma+\log(k_s v/2) - \frac{\pi i}{2}\right)M + N - \frac{(w^2+v^2)}{4} R k_s^2 \log k_s\right] + O(k_s^2) \quad , \; v>w \qquad (24)$$

where $\gamma = 0.5772157.......$ is Euler's constant,

$$M = \int_0^{1/\sqrt{c}_{11}} \frac{c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\,d\eta - \int_{1/\sqrt{c}_{11}}^{1} \frac{c_{11}\eta^2-1+\bar{\gamma}_2'^2}{\bar{\gamma}_2'^2(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\,d\eta \qquad (25)$$

$$N = \int_0^{1/\sqrt{c}_{11}} \frac{c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\log\eta\,d\eta - \int_{1/\sqrt{c}_{11}}^{1} \frac{c_{11}\eta^2-1+\bar{\gamma}_2'^2}{\bar{\gamma}_2'^2(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\log\eta\,d\eta \qquad (26)$$

and

$$R = \int_0^{1/\sqrt{c}_{11}} \frac{\eta^2(c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2)}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\,d\eta - \int_{1/\sqrt{c}_{11}}^{1} \frac{\eta^2(c_{11}\eta^2-1+\bar{\gamma}_2'^2)}{\bar{\gamma}_2'^2(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\,d\eta \qquad (27)$$

Now differentiating both sides of the relation (19) with respect to x we obtain

$$\int_a^b tf(t^2)dt \int_0^\infty \xi \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) \sin\xi x \cos\xi t \, d\xi \quad +$$

$$+ \int_c^1 ug(u^2)du \int_0^\infty \xi \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) \sin\xi x \cos\xi u \, d\xi = 0 \ , \ x \in I_2, I_4$$

Following similar procedure as done for deriving equation (20), we obtain

$$x\int_a^b \frac{tf(t^2)}{(x^2-t^2)} dt + x\int_c^1 \frac{ug(u^2)}{(x^2-u^2)} du$$

$$= \int_a^b tf(t^2)dt \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{vwL_2(v,w) \, dw \, dv}{(x^2-w^2)^{1/2} \ (t^2-v^2)^{1/2}} +$$

$$+ \int_c^1 ug(u^2)du \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{vwL_2(v,w) \, dw \, dv}{(x^2-w^2)^{1/2} \ (u^2-v^2)^{1/2}}$$

$$= 0, \qquad x \in I_2, I_4 \tag{28}$$

where

$$L_2(v,w) = \int_0^\infty \left[\xi - \frac{\xi^2}{\theta} \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) \right] J_0(\xi w) \ J_0(\xi v) d\xi \tag{29}$$

$$\ominus = \frac{c_{11} + N_1 N_2}{N_1 + N_2} \tag{30}$$

$$N_1^2 = \frac{1}{2c_{22}} \left[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}} \right] \tag{31}$$

and

$$N_2^2 = \frac{1}{2c_{22}} \left[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}} \right].$$

We use the contour integration technique mentioned earlier and get from (29)

$$L_2(v,w) = \frac{ik_s^2}{\ominus} \left[\int_0^{1/\sqrt{c}_{11}} \frac{\eta^2 (c_{11}\eta^2 - 1 - \bar{\gamma}_1 \bar{\gamma}_2)}{\bar{\gamma}_1 \bar{\gamma}_2 (\bar{\gamma}_1 + \bar{\gamma}_2)} J_0(k_s \eta v) H_0^{(1)}(k_s \eta w)\, d\eta \right. -$$

$$\left. - \int_{1/\sqrt{c}_{11}}^{1} \frac{\eta^2 (c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2)}{\bar{\gamma}_2' \, (\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)} J_0(k_s \eta v) H_0^{(1)}(k_s \eta w)\, d\eta \right], \quad w > v \tag{32}$$

By the process similar to the one which led to the equation (24), (32) for small values of k_s can be written as

$$L_2(v,w) = -\frac{2}{\pi} P\, k_s^2 \log k_s + O(k_s^2) \tag{33}$$

where $P = \frac{1}{\ominus} R$ and R is given by (27).

Now, let us consider

$$f(t^2) = f_0(t^2) + k_s^2 \log(k_s)\, f_1(t^2) + O(k_s^2)$$

and $$g(u^2) = g_0(u^2) + k_s^2 \log(k_s)\, g_1(u^2) + O(k_s^2) \qquad (34)$$

Putting the above expressions of $f(t^2)$, $g(u^2)$ and the value of $L_2(v,w)$ given by (33) in the equation (28) and equating the coefficients of like powers of k_s we obtain,

$$\int_a^b \frac{tf_0(t^2)}{(x^2 - t^2)} dt + \int_c^1 \frac{ug_0(u^2)}{(x^2 - u^2)} du = 0 , \quad x \in I_2 , I_4 \qquad (35)$$

and $$\int_a^b \frac{tf_1(t^2)}{(x^2 - t^2)} dt + \int_c^1 \frac{ug_1(u^2)}{(x^2 - u^2)} du$$

$$= - \frac{2P}{\pi} \left[\int_a^b tf_0(t^2)dt + \int_c^1 ug_0(u^2)du \right], \quad x \in I_2, I_4 \qquad (36)$$

Following Srivastava and Lowengrub [1970] the solutions of the above integral equation (35) can be obtained as

$$f_0(t^2) = D_1 \left[\frac{1-a^2}{c^2-a^2}\right]^{1/2} \left[\frac{c^2-t^2}{1-t^2}\right]^{1/2} \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} -$$

$$- D_2 \left[\frac{t^2-a^2}{b^2-t^2}\right]^{1/2} \frac{1}{\sqrt{(1-t^2)(c^2-t^2)}}, \quad x \in I_2 \qquad (37)$$

and $$g_0(u^2) = D_1 \left[\frac{1-a^2}{c^2-a^2}\right]^{1/2} \left[\frac{u^2-c^2}{1-u^2}\right]^{1/2} \frac{1}{\sqrt{(u^2-a^2)(u^2-b^2)}} +$$

$$+ D_2 \left[\frac{u^2-a^2}{u^2-b^2}\right]^{1/2} \frac{1}{\sqrt{(1-u^2)(u^2-c^2)}}, \quad x \in I_4 \qquad (38)$$

where D_1 and D_2 are constants which can be calculaed as follows.

We substitute the value of $L_1(v,w)$ from (24) as well as the expansion of $f(t^2)$ and $g(u^2)$ obtained from (34), (37) and (38) up to $O(k_s^2 \log k_s)$ in the equation (20). When the coefficients of like powers of k_s from both sides of the resulting equation are equated we get after some manipulation, the following results:

$$D_1 = - v_0 \frac{\pi^2}{4} \frac{(X_4-X_2)}{(X_1X_4-X_2X_3)} \quad ; \qquad D_2 = - v_0 \frac{\pi^2}{4} \frac{(X_3-X_1)}{(X_2X_3-X_1X_4)} \qquad (39)$$

Where

$$X_1 = \left(\frac{1 - a^2}{c^2 - a^2} \right)^{1/2} \left[\left\{ \left(\gamma + \log(k_s/2) - \frac{\pi i}{2} \right) M + N \right\} (J_1 + J_3) + \right.$$

$$\left. + \frac{1}{2} M J_1 \log(b^2 - a^2) + M J_5 \right] \qquad (40)$$

$$X_2 = \left\{ \left(\gamma + \log(k_s/2) - \frac{\pi i}{2} \right) M + N \right\} (J_4 - J_2) -$$

$$- \frac{1}{2} M J_2 \log(b^2 - a^2) + M J_6 \qquad (41)$$

$$X_3 = \left(\frac{1 - a^2}{c^2 - a^2} \right)^{1/2} \left[\left\{ \left(\gamma + \log(k_s/2) - \frac{\pi i}{2} \right) M + N \right\} (J_1 + J_3) + \right.$$

$$\left. + \frac{1}{2} M J_3 \log(1 - c^2) + M J_7 \right] \qquad (42)$$

$$X_4 = \left\{ \left(\gamma + \log(k_s/2) - \frac{\pi i}{2} \right) M + N \right\} (J_4 - J_2) +$$

$$+ \frac{1}{2} M J_4 \log(1 - c^2) - M J_8 \qquad (43)$$

$$J_1 = \int_a^b \left(\frac{c^2 - t^2}{1 - t^2} \right)^{1/2} \frac{t\,dt}{\sqrt{(t^2 - a^2)(b^2 - t^2)}}$$

$$J_2 = \int_a^b \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \frac{tdt}{\sqrt{(1-t^2)(c^2-t^2)}}$$

$$J_3 = \int_c^1 \left(\frac{u^2-c^2}{1-u^2} \right)^{1/2} \frac{udu}{\sqrt{(u^2-a^2)(u^2-b^2)}}$$

$$J_4 = \int_c^1 \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \frac{udu}{\sqrt{(1-u^2)(u^2-c^2)}}$$

$$J_5 = \int_c^1 \left(\frac{u^2-c^2}{1-u^2} \right)^{1/2} \frac{u\log\left(\sqrt{u^2-b^2} + \sqrt{u^2-a^2}\right)}{\sqrt{(u^2-a^2)(u^2-b^2)}} du$$

$$J_6 = \int_c^1 \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \frac{u\log\left(\sqrt{u^2-b^2} + \sqrt{u^2-a^2}\right)}{\sqrt{(1-u^2)(u^2-c^2)}} du.$$

$$J_7 = \int_a^b \left(\frac{c^2-t^2}{1-t^2} \right)^{1/2} \frac{t\log\left(\sqrt{c^2-t^2} + \sqrt{1-t^2}\right)}{\sqrt{(t^2-a^2)(b^2-t^2)}} dt$$

$$J_8 = \int_a^b \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \frac{t\log\left(\sqrt{c^2-t^2} + \sqrt{1-t^2}\right)}{\sqrt{(1-t^2)(c^2-t^2)}} dt$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ on the plane $y = 0$ can be found from the relations (14), (18), (34),(37) and (38) as

$$\tau_{yy}(x,0) = -\frac{\mu_{12}c_{22}x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\left\{ D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-x^2}{1-x^2}\right)^{1/2} - \frac{D_2(x^2-a^2)}{\sqrt{(1-x^2)(c^2-x^2)}}\right\} + O(k_s^2\log k_s)\ , \quad x \in I_2$$

$$= -\frac{\mu_{12}c_{22}x}{\sqrt{(x^2-c^2)(1-x^2)}}\left\{ D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\frac{(x^2-c^2)}{\sqrt{(x^2-a^2)(x^2-b^2)}} + D_2\left(\frac{x^2-a^2}{x^2-b^2}\right)^{1/2}\right\} + O(k_s^2\log k_s)\ , \quad x \in I_4 \qquad (44)$$

Defining the stress intensity factors at the edges of the strips by the relations

$$K_a = \underset{x\to a+}{\mathrm{Lt}}\left|\frac{\tau_{yy}(x,0)\sqrt{(x-a)}}{v_0\mu_{12}}\right|$$

$$K_b = \underset{x \to b-}{\mathrm{Lt}} \left| \frac{\tau_{yy}(x,0)\sqrt{(b-x)}}{v_0 \mu_{12}} \right|$$

$$K_c = \underset{x \to c+}{\mathrm{Lt}} \left| \frac{\tau_{yy}(x,0)\sqrt{(x-c)}}{v_0 \mu_{12}} \right|$$

$$K_1 = \underset{x \to 1-}{\mathrm{Lt}} \left| \frac{\tau_{yy}(x,0)\sqrt{(1-x)}}{v_0 \mu_{12}} \right|$$

we get

$$K_a = \left| \frac{c_{22}\sqrt{a}\ D_1}{\sqrt{2(b^2-a^2)}} \right| \qquad (45)$$

$$K_b = \left| \frac{c_{22}\sqrt{b}}{\sqrt{2(b^2-a^2)}} \left\{ D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-b^2}{1-b^2} \right)^{1/2} - \frac{D_2(b^2-a^2)}{\sqrt{(1-b^2)(c^2-b^2)}} \right\} \right| \qquad (46)$$

$$K_c = \left| \frac{c_{22}\sqrt{c}}{\sqrt{2(1-c^2)}}\ D_2 \left(\frac{c^2-a^2}{c^2-b^2} \right)^{1/2} \right| \qquad (47)$$

$$K_1 = \left| \frac{c_{22}}{\sqrt{2(1-c^2)}} \left\{ \frac{D_1(1-c^2)}{\sqrt{(1-b^2)(c^2-a^2)}} + D_2 \left(\frac{1-a^2}{1-b^2} \right)^{1/2} \right\} \right| \qquad (48)$$

The vertical displacement $v(x,y)$ on the plane $y = 0$ can be obtained from equations (13), (18), (34), (37) and (38) as

$$v(x,0) = \frac{4}{\pi^2}\Bigg[\left\{ \left[\gamma + \log(k_s) - \frac{\pi i}{2} \right] M + N \right\} \times$$

$$\times \left\{ D_1 \left[\frac{1-a^2}{c^2-a^2} \right]^{1/2} (J_1 + J_2) + D_2 (J_4 - J_2) \right\} +$$

$$+ \frac{M}{2} \left\{ D_1 \left[\frac{1-a^2}{c^2-a^2} \right]^{1/2} (J_9 + J_{11}) + D_2 (J_{12} - J_{10}) \right\} \Bigg] ,$$

$$x \in I_1 , I_3 , I_5 \qquad (49)$$

where

$$J_9 = \int_a^b \left[\frac{c^2-t^2}{1-t^2} \right]^{1/2} \frac{t \log|t^2-x^2|}{\sqrt{(t^2-a^2)(b^2-t^2)}} \, dt$$

$$J_{10} = \int_a^b \left[\frac{t^2-a^2}{b^2-t^2} \right]^{1/2} \frac{t \log|t^2-x^2|}{\sqrt{(1-t^2)(c^2-t^2)}} \, dt$$

$$J_{11} = \int_c^1 \left[\frac{u^2-c^2}{1-u^2} \right]^{1/2} \frac{u \log|u^2-x^2|}{\sqrt{(u^2-a^2)(u^2-b^2)}} \, du$$

$$J_{12} = \int_c^1 \left[\frac{u^2-a^2}{u^2-b^2} \right]^{1/2} \frac{u \log|u^2-x^2|}{\sqrt{(u^2-c^2)(1-u^2)}} \, du$$

In order to obtain the solution of the problem corresponding to two rigid strips taking $b \longrightarrow c$ we find from (37) and (38) that in this particular case

$$f_0(t^2) = g_0(t^2) = D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \frac{1}{\sqrt{(t^2-a^2)(1-t^2)}} -$$

$$- D_2 \left(\frac{t^2-a^2}{1-t^2} \right)^{1/2} \frac{1}{b^2 - t^2}, \quad a \leq t \leq 1.$$

It can further be shown that $X_1 = X_3$ so that

$$D_2 = 0 \text{ and } D_1 = - \frac{V_0 \pi^2}{4X_1},$$

where

$$X_1 = \frac{\pi}{2} \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left[\left\{ \gamma + \log(k_s/2) - \frac{\pi i}{2} + \log(1-a^2)^{1/2} \right\} M + N \right]$$

It can easily be shown that in the isotropic case this result is identical with result given by Jain and Kanwal [1972].

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_a, K_b, K_c and K_1 given by (45) - (48) at the edges of the strips and vertical displacement $|v(x,0)/v_0|$ near about the rigid strips have been plotted against dimensionless frequency k_s and distance x respectively for three different types of orthotropic materials whose engineering constants have been listed in table 1.

TABLE - 1. ENGINEERING ELASTIC CONSTANTS

	E_1(Pa)	E_2(Pa)	μ_{12}(Pa)	γ_{12}
Type I	Modulite II Graphite-Epoxi Composite :			
	15.3×10^9	$1{\cdot}58.0\times10^9$	5.52×10^9	0.033
Type II	E-Type Glass-Epoxi Composite :			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless Steel-Aluminium Composite :			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

It is found that whatever the lengths of the strips are, SIFs at the four edges of the strips increase with increase in the value of

k_s ($0.1 \le k_s \le 0.6$). From the graphs, it may be noted further that with a decrease in the length of the inner strip, which might be induced either by increasing 'a' or by decreasing 'b', the SIF K_a at the innermost edge gradually decreases, wheareas the SIFs at the other edges show just the opposite behavior (Fig.3 - Fig.4).

Also, a decrease in the value of the length of the outer strip, which might be induced by increasing the value of 'c', causes an increase in the values of the SIFs (Fig.5) from which an interesting conclusion might be drawn : i.e., the presence of the inner strip suppresses the SIFs at both edges of the outer strip and the presence of the outer strip suppresses the SIFs at the edges of the inner strip.

The SIF K_a has been plotted (Fig. 6) for different orthotropic materials to show the effect of material orthotropy. Similar effect are being seen for other SIFs.

The vertical displacement has been plotted for different strip lengths. It is found from Fig.7 - Fig.9 that with the increase in the value of strip length, the displacement increases.

For a fixed material the variation of displacement with frequency is found to be insignificant.

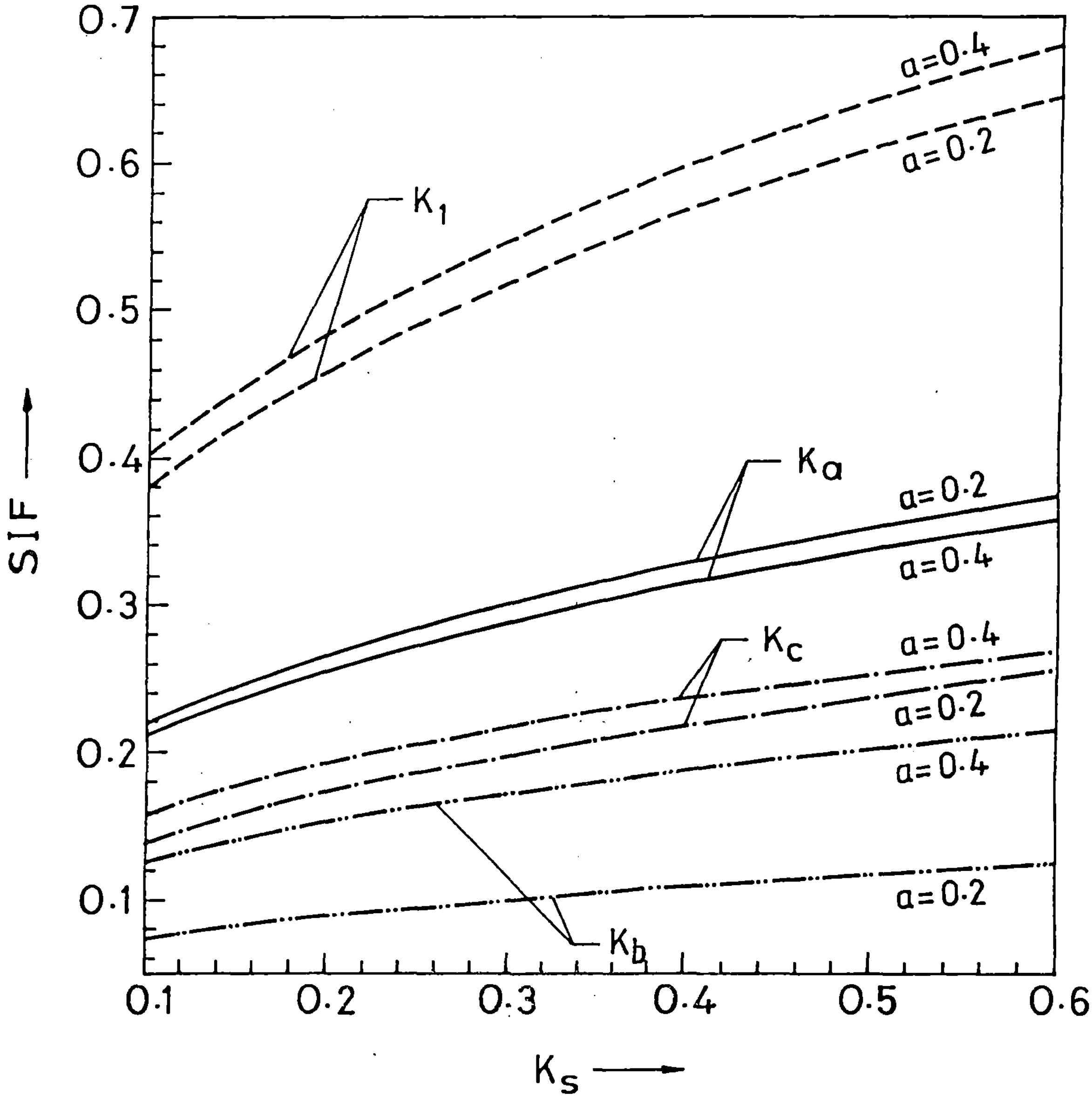

Fig. 3. Stress intensity factors vs. frequency k_s for generalized plane stress. (for material of type III).

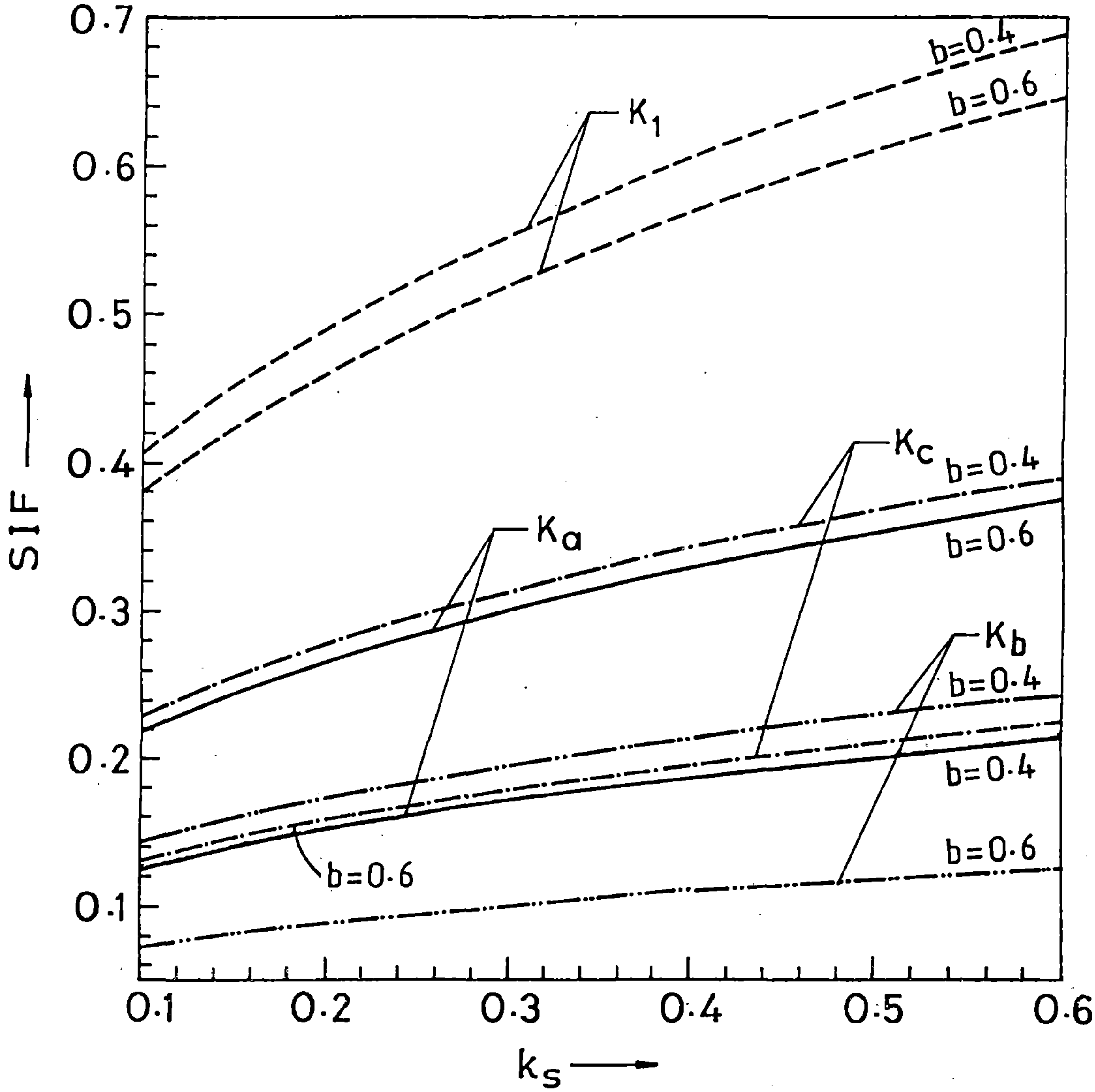

Fig. 4. Stress intensity factors vs. frequency k_s for generalized plane stress.

(for material of type III).

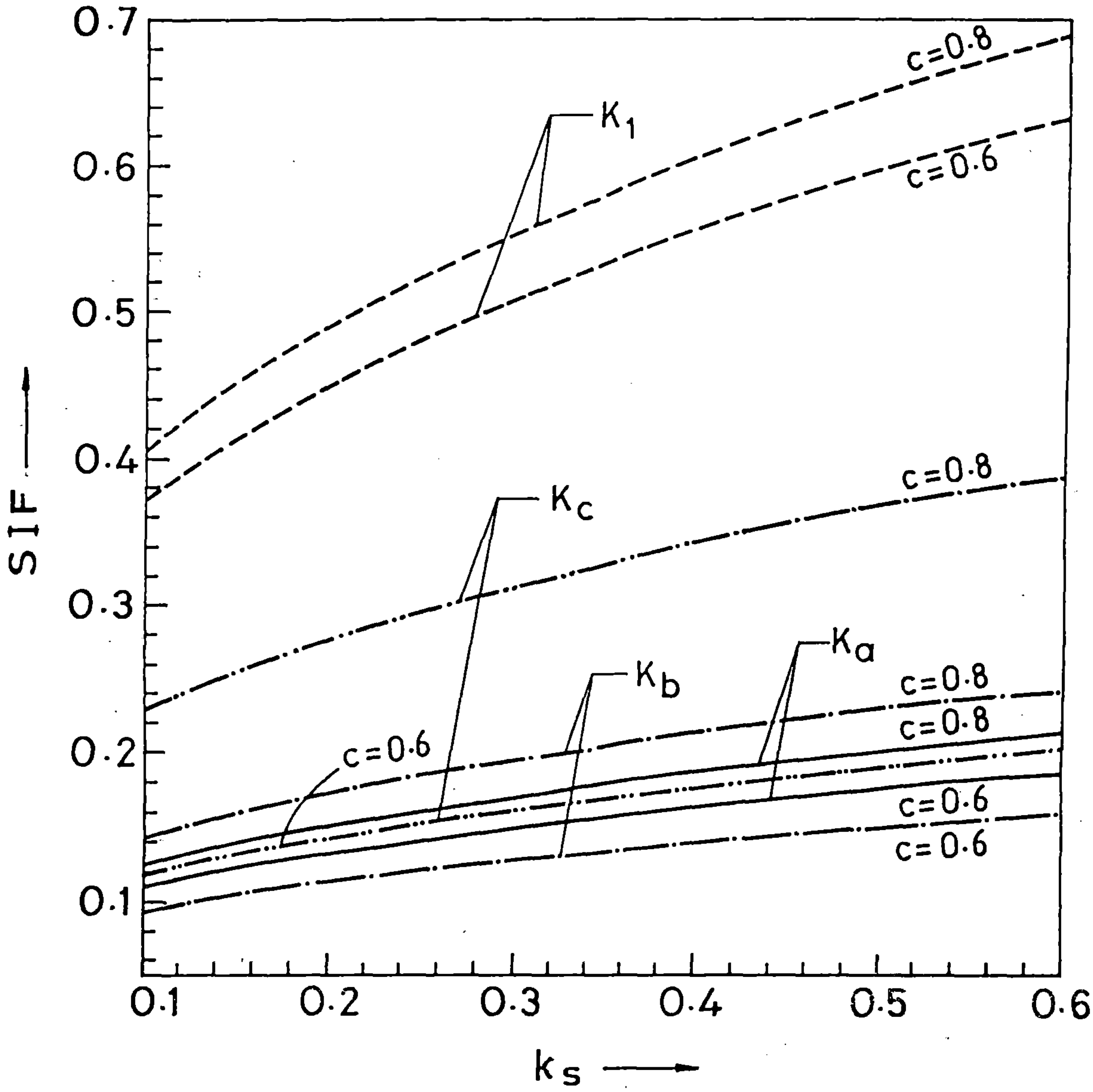

Fig. 5. Stress intensity factors vs. frequency k_s for generalized plane stress. (for material of type III).

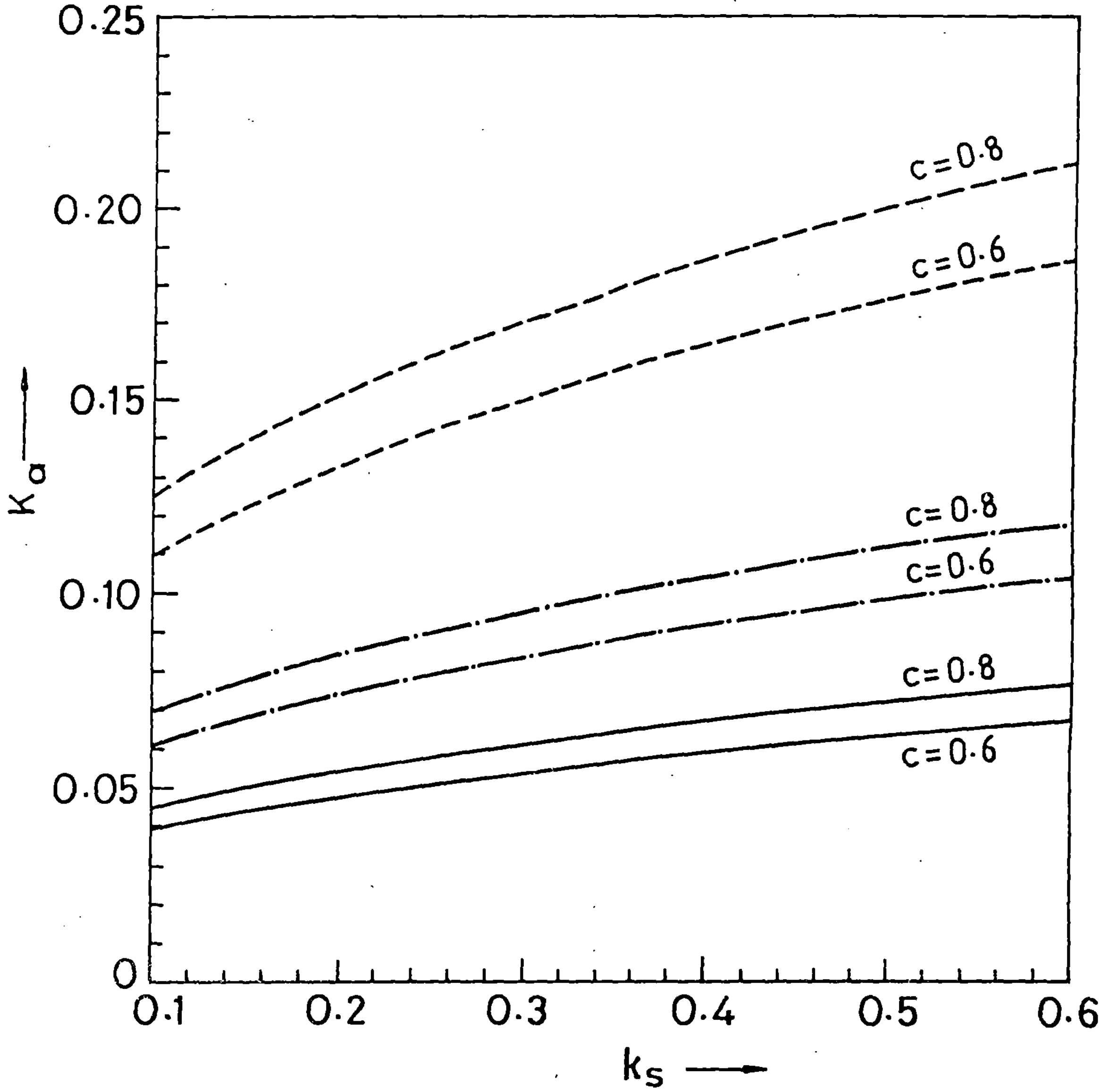

Fig. 6. Stress intensity factor K_a vs. frequency k_s for generalized plane stress.

(——— Type I, -.-.-. Type II, ----- Type III).

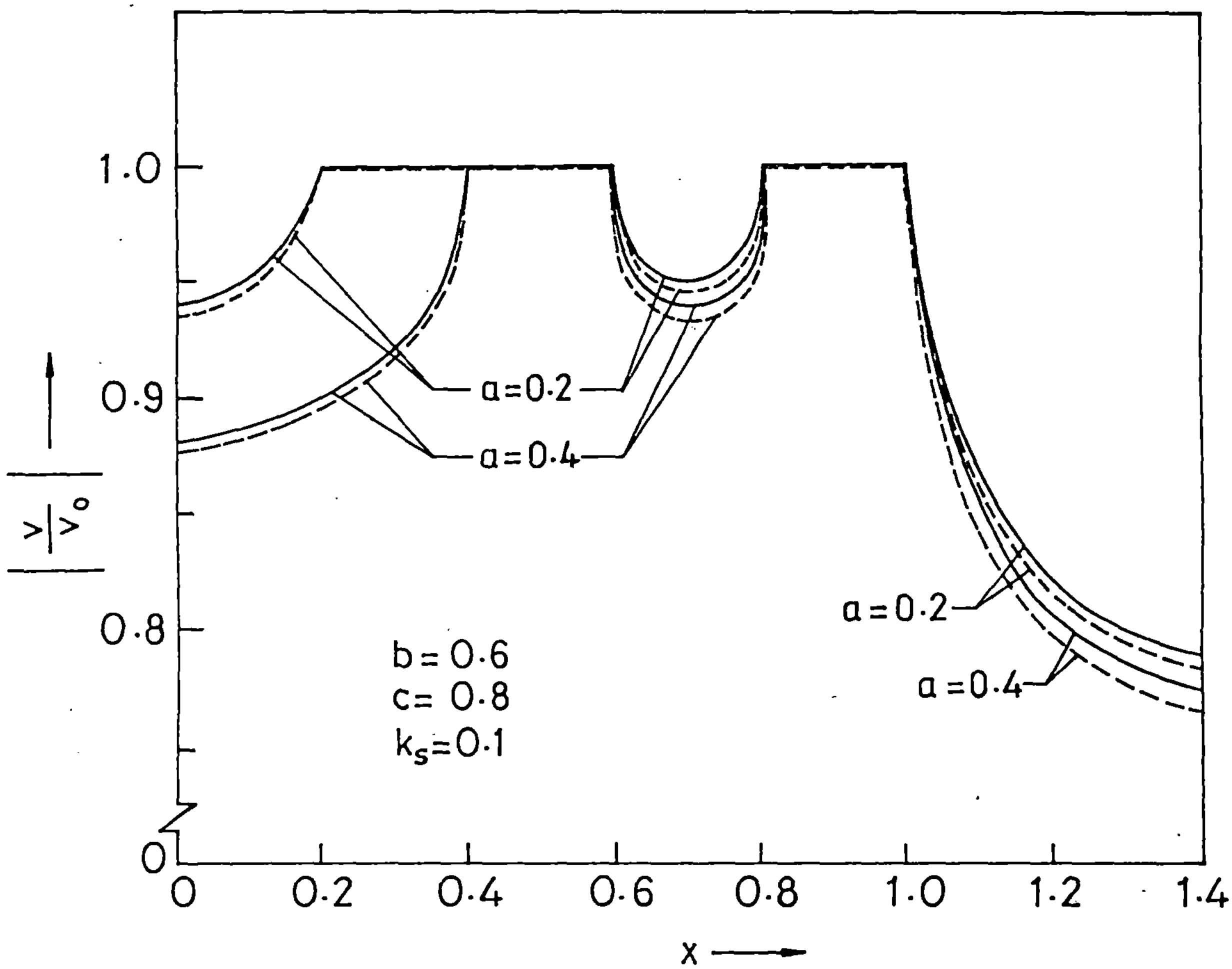

Fig. 7. Vertical displacement $|v/v_0|$ vs. distance x for generalized plane stress.

(——— Type I, ----- Type II).

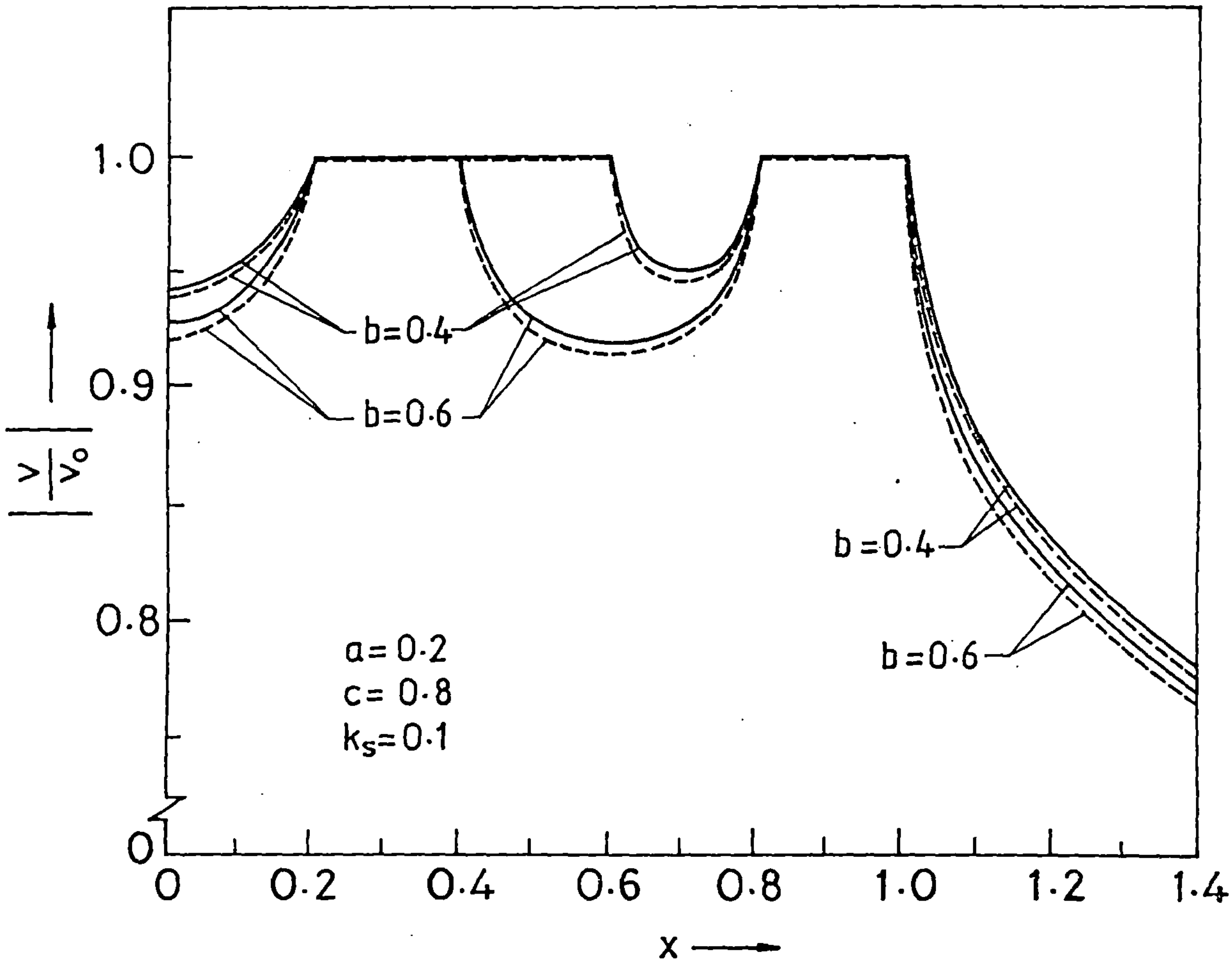

Fig. 8. Vertical displacement $|v/v_0|$ vs. distance x for generalized plane stress.

(——— Type I, ----- Type II).

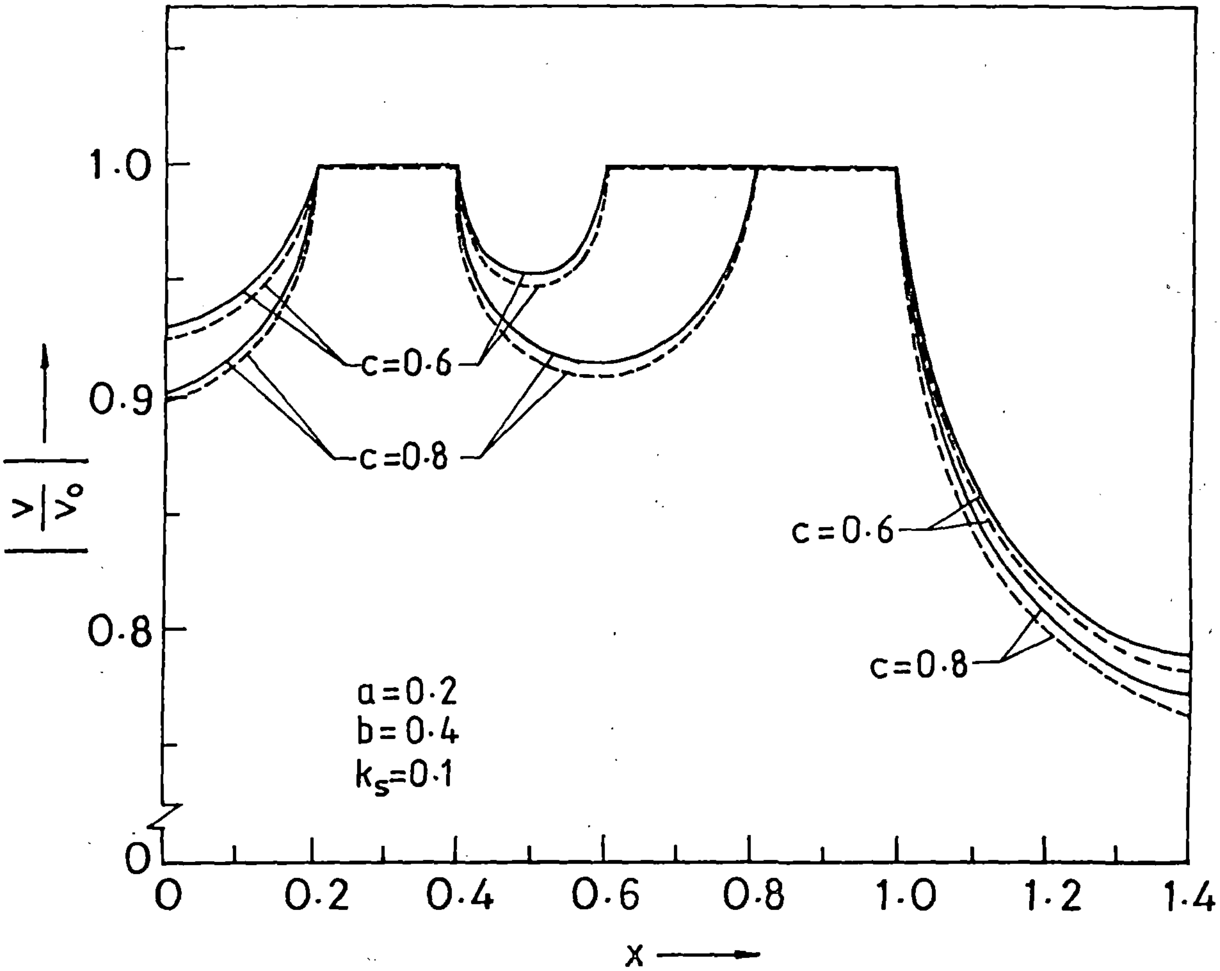

Fig. 9. Vertical displacement $|v/v_0|$ vs. distance x for generalized plane stress.

(——— Type I, ----- Type II).

BIBLIOGRAPHY

1. ACHENBACH, J. D. [1972] :

   Dynamic effect in brittle fracture., Mechanics Today, V1; ed S, Nemat-Nasser, Pergamon Press, New York.

2. ACHENBACH, J. D. [1975] :

   Wave propagation in elastic solids., North-Holand Publishing Company., Amsterdam - Oxford, American Elsevier Publishing Company, INC., New York.

3. ACHENBACH, J. D. [1976] :

   Wave propagation, elastodynamic stress singularities and fracture., Theoritical and Applied Mechnics, ed. W. T. Koiter, Proc. 14th IUTM Congress., North-Holland Publishing Co. Amsterdam.

4. AGGARWAL, H. R. and ABLOW, C. M. [1967] :

   Solution to a class of three-dimensional pulse propagation problems in an elastic half-space., Int. J. Engng. Sci., V5, P663.

5. AKI, K. and RICHARDS, P. G. [1980] :
Quantitative seismology ; theory and methods, Volume I & II., W. H. Freeman and Company., SAN FRANCISCO.

6. ANG, D. D. and KNOPOFF, L. [1964] :
Diffraction of scalar elastic waves by a finite crack., Proc. Nat. Acd. Sci., V51, P593.

7. ATKINSON, C. and LIST, R. D. [1978] :
Steady state crack propagation into media with spatially varying elastic properties., Int. J. Engng. Sci., V16, P717.

8. AWOJOBI, O. A. and GROOTENHUIS, P. [1965] :
Vibration of rigid bodies on semi-infinite elastic media., Proc. Roy. Soc. London., V A287, P27.

9. BAKER, B. R. [1962] :
Dynamic stress created by a moving crack.,J. Appl. Mech., V15, P449.

10. BOSTROM, A. [1987] :
Elastic wave scattering from an infinite crack : Antiplane strain., J. Appl. Mech., V54, P503.

11. BULLEN, K. E. [1963] :
An introduction to the theory of seismology., University Press, Cambridge.

12. BURRIDGE, R. and WILLIS, J.R. [1969] :

The self-similar problem of the expanding elliptical crack in an anisotropic solid.,Proc. Camb. Phil. Soc., V 66, P443.

13. CAGNIARD, L. [1962] :

Reflection and refraction of progressive seismic waves, Translated by Flinn, E. A. and Dix, C.H., McGraw-Hill Book Co. Inc., New York.

14. CHAKRABORTY, S. K. and DE, T. K. [1971] :

On stress and displacement due to a moving line load over the plane boundary of a hetrogeneous elastic half-space., Pure Appl. Geophys., V85, P214.

15. CHANG, SHIH-JUNG. [1971] :

Diffraction of plane dilatational waves by a finite crack., Quart. J. Mech. Appl. Math., V24, P424.

16. CHAO, C. C. [1960] :

Dynamic response of an elastic half-space to tangential surface loadings.,J. Appl. Mech. E., V27, P559.

17. CHEN, E. P. and SIH, G. C. [1973] :

Running crack in an incident wave field., Int. J. Solids Structures., V9, P897.

18. CHEN, E. P. and SIH, G. C. [1975] :

Scattering of plane waves by a propagating crack., J. Appl. Mech., V42, P705.

19. CHEN, E. P. [1978] :

Sudden appearence of a crack in a stretched finite strip., J. Appl. Mech., V45, P277.

20. DE, J. and PATRA, B. [1990] :

Moving Griffith crack in an orthotropic strip., Int. J. Engng. Sci., V28, P809.

21. DUFFY, DEAN G. [1994] :

Transform method for Solving Partial Differential Equations., ERC Press., London.

22. EASON, G. [1964] :

On the torsional impulsive loading of an elastic half-space., Quart. J. Mech. Appl. Math., V17, P279.

23. EASON, G. [1966] :

The displacement produced in an elastic half-space by a suddenly applied surface force., J. Inst. Math. Applic., V2, P299.

24. ERDELYI, A (Ed), [1954] :

Tables of integral transforms, V 1, Bateman Manuscript Project, McGraw-Hill.

25. ERINGEN, A. C. and SUHUBI, E. S. [1975] :

Elastodynamics, Vol. II, Linear theory, Academic Press, New York.

26. EWING, W. M., JARDETZKY, W. S. and PRESS, F. [1957] :

Elastic waves in layered media., McGraw-Hill Book Company, New York.

27. FREUND, L. B. [1975] :

Dynamic crack propagation. The Mathematics of Fracture, ed. F. Erdogan, ASME AMD, V19, American Society of Mechanical Engineers, New York.

28. FREUND, L. B. [1976] :

The analysis of elastodynamic crack tip stress fields., Mechanics Today, ed. S Nemat-Nasser, V111, Pargamon Press, New York.

29. FRIEDMANN, B. [1966] :

Principles and techniques of applied Mathematics, John Wiley and Sons.

30. GAKENHEIMER, D.C. and MIKLOWITZ, J. [1969] :

Transient exitation of an elastic half-space by a point load travelling on the surface., J. Appl. Mech., V36, Trans. ASME 91E, P505.

31. GAKENHEIMER, D.C. [1971] :

Response of an elastic half-space to expanding surface loads., J. Appl. Mech., Trans. ASME , V38, P99.

32. GHOSH, M. L. [1971] :
The axisymmetric problems of a normal stress discontinuity in a semi-infinite elastic medium., Appl. Sci. Res. V24, P149.

33. GHOSH, M. [1980/81] :
Displacement produced in an elastic half-space by the impulsive torsional motion of a circular ring source., Pageoph, V119, P102.

34. GHOSH, M. and GHOSH, M. L. [1985] :
Harmonic rocking of a rigid strip on a semi-infinite elastic medium., Indian J. Pure Appl. Math., V16, P938.

35. GLADWELL, G. M. L. [1968] :
Forced tangential and rotating vibration of a rigid circular disc on a semi-infinite solid., Int. J. Engng. Sci., V6, P591.

36. ITOU, S. [1978] :
Dynamic stress concentration around two coplnar Griffith cracks in an infinite elastic medium., J. Appl. Mech., V4, P803.

37. ITOU, S. [1979] :

Three dimensional problem of running crack., Int. J. Engng. Sci., V17, P59.

38. ITOU, S. [1980 a] :

Transient analysis of stress waves around two coplanar Griffith cracks under impact load., Engng. Frac. Mech., V13, P349.

39. ITOU, S. [1980 b] :

Diffraction of an antiplane shear wave by two coplanar Griffith cracks in an infinite elastic medium., Int. J. Solids Structures., V16, P1147.

40. JAHANSHAHI, A. [1967] :

A diffraction problem and crack propagation., J. Appl. Mech., V34, P100.

41. JAIN, D. L. and KANWAL, R. P. [1972 a] :

Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium., Int. J. Solids Structures., V8, P961.

42. JAIN, D. L. and KANWAL, R. P. [1972 b] :

Diffraction of elastic waves by two coplanar and parallel rigid strips., Int. J. Engng. Sci., V10, P925.

43. JOHNSON, E. R. and PARNES, R. [1977] :

Propagating of axisymmetric waves in an elastic half-space containing a cylindrical inclusion. Part-I., Quart. J. Mech. Appl. Math., V30, P235.

44. JOHNSON, E. R. and PARNES, R. [1977] :

Propagating of axisymmetric waves in an elastic half-space containing a cylindrical inclusion. Part-II., Quart. J. Mech. Appl. Math., V30, P255.

45. KANNINEN, M. F. [1978] :

A critical appraisal of solution techniques in dynamic fracture mechanics., Numerical Methods in Fracture Mechanics, ed. A. R. Luxmoore and D. R. J. Owen, University College of Swansea, U.K.

46. KARASUDHI, P., KEER, L. M. and LEE, S. L. [1968] :

Vibratory motion of a body on an elastic half-plane., J. Appl. Mech., V35, P697.

47. KASSIR, M. K. and BANDYOPADHYAY, K. K. [1983] :

Impact response of cracked orthotropic medium., J. Appl. Mech. V50, P630.

48. KASSIR, M. K. and TSE, S. [1983] :

Moving Griffith crack in an orthotropic material., Int. J. Engng. Sci., V21, P315.

49. KELLER, J. B. [1958] :

"A geometrical theory of diffraction" in calculus of variations and its applications. Vol. VIII MaGraw-Hill Publishing Company, New York.

50. KEOGH, P. S. [1985 a] :

High-frequency scattering by a Griffiyh crack I : A crack Green's function., Quart. J. Mech. Appl. Math., V38, P185.

51. KEOGH, P. S. [1985 b] :

High-frequency scattering by a Griffith crack II : Incident plane and cylindrical waves., Quart. J. Mech. Appl. Math., V38, P205.

52. KNOPOFF, L. and GILBERT, F. [1959] :

First motion methods in theoretical seismology., J. Accoust. Soc. Am., V31, P1161.

53. KRENK, S. and SCHMIDT, H. [1982] :

Elastic wave scattering by a circular crack., Phil. Trans. R. Soc. Lond., V A308, P167.

54. KRYLOV, V. I. and KANTOROVICH, L. V. [1957] :

Approximate methods of higher analysis., Interscience, New York.

55. KUNDU, T. and BOSTROM, A. [1991] :

Axisymmetric scattering of plane longitudinal wave by a circular crack in a transversely isotropic solid., J. Appl. Mech., V58, P695.

56. KUO, A. Y. [1984] :

Transient stress intensity factors of an interfacial crack between two dissimilar anisotropic half-spaces., J. Appl. Mech., V51, P71.

57. LAMB, H. [1904] :

On the propagation of tremors over the surface of an elastic solid., Phil. Trans. R. Soc. London., V A203, P1.

58. LANG, H. A. [1961] :

Surface displacements in an elastic half-space., Z. Angew. Math. Mech., V41, P141.

59. LEBDEB, N. N. [1965] :

Special functions and their applications., Prentice-Hall Inc.

60. LOEBER, J. F. AND SIH, G. C. [1967] :

Diffraction of antiplane shear waves by finite crack., J. Accous. Soc. Am., V44, P90.

61. MAITI, N. C. [1978] :

Transient disturbance produced in an elastic half-space by impulsive shearing traction distributed over a circular region., Pageoph, V116, P198.

62. MAL, A. K. [1970] :

Interaction of elastic waves with a Griffith crack., Int. J. Engng. Sci., V8, P763.

63. MAL, A. K. [1970] :

Interaction of elastic waves with a penny-shaped crack., Int. J. Engng. Sci., V8, P381.

64. MAL, A. K. [1972] :

A note on the low frequency diffraction of elastic waves by a Griffith crack., Int. J. Engng. Sci., V10, P609.

65. MANDAL, S. C. and GHOSH, M. L. [1992] :

Forced vertical vibration of two rigid strips on a semi-infinite elastic solid., J. Sound and Vibration., V158, P169.

66. MANDAL, S. C. and GHOSH, M. L. [1994] :

Interaction of elastic waves with a periodic array of coplanar Griffith cracks in an orthotropic elastic medium., Int. J. Engng. Sci., V32, P167.

67. MANDAL, S. C., PAL, S. C. and GHOSH, M. L. [1996 a] :

Diffraction of elastic waves by four rigid strips embedded in an infinite orthotropic medium., Journal of Engineering Mathematics., (IN PRESS).

68. MANDAL, S. C., PAL, S. C. and GHOSH, M. L. [1996 b] :
Forced vertical vibration of four rigid strips on a semi-infinite elastic solid., Int. J. Solids and Structures., (IN PRESS).

69. MATCZYNSKI, M. [1973] :
Motion of a crack in antiplane strain of an elastic strip., Archives of Mechanics., V25, P823.

70. MAUE, A. W. [1954] :
Die entspannuangswelle bei plotzlichen einschnitt eines gespannten elastischen korpers, ZAMM, V34, P1.

71. MILLER, G. F. and PURSEY, H. [1964] :
The field and radiation impedence of mechanical radiators on the surface of semi-infinite isotropic solid., Proc. Roy. Soc. London., V A223, P521.

72. MILLER, M. K. and GUY, W. T. [1966] :
Numerical inversion of the Laplace transform by use of Jacobi polynomials. SIAM J. Numerical Analysis., V3, P624.

73. MITRA, M. [1964] :
Disturbance produced in an elastic half-space by impulsive normal pressure., Proc. Camb. Phil. Soc., V60, P683.

74. NEERHOFF, F. L. [1979] :

Diffraction of Love waves by astress-free crack of a finite width in the plane interface of a layered composite., Appl. Sci. Res., V35, P265.

75. NISHIDA, Y., SHINDO, Y. and ATSUMI, A. [1984] :

Diffraction of horizontal shear waves by a moving interface crack., Acta Mechanica., V54, P23.

76. NOBLE, B. [1958] :

Method based on Wiener-Hopf technique for the solution of partial differential equations., Pergammon Press. New York.

77. NOBLE, B. [1962] :

Electromagnetic waves., ed. R. E. Langer., University of Wisconsin Press.

78. NOBLE, E. B. [1963] :

The solution of Bessel function dual integral equations by a multiplying-factor method. Proc. Camb. Phil. Soc., V59, P351.

79. PAL, S. C., GHOSH, M. L. and CHOWDHURI, P. K. [1985] :

Spectral representation of a certain class of self-adjoint differential operators and its application to axisymmetric boundary value problems in elastodynamics., J. Tech. Phys., V26, 1, P97.

80. PAL, S. C. and GHOSH, M. L. [1987] :

Waves in a semi-infinite elastic medium due to an expanding elliptic ring source on the free surface., Indian J. Pure Appl. Math., V18(7), P648.

81. PAL, S. C. and GHOSH, M. L. [1990] :

High fequency scattering of antiplane shear waves by an interface crack., Indian J. Pure Appl. Math., V21(12), P1107.

82. PAL, S. C. and GHOSH, M. L. [1993] :

High frequency scattering of plane horizontal shear waves by a Griffith crack propagating along the bimaterial interface., Engineering Fracture Mechanics., V45, No1, P107.

83. PEKERIS, C. L. [1955] :

The seismic surface pulse., Proc. Nat. Acad. Sci. V41, P469.

84. PEKERIS, C. L. and LONGMAN, I. M. [1958] :

The motion of the surface of a uniform elastic half-space produced by a buried torque-pulse., Geophysical J. Roy. Ast. Soc., V1, P146.

85. PILANT, WALTER L. [1979] :

Elastic waves in the earth., Elsevier Scientific Publishing Company, Amsterdam-Oxford-New York.

86. REISSNER, E. [1937] :

Freie and erzwungene torsionsschwingungen den elastischen halbraumes., Ingenien Archiv., V2, P229.

87. RICHARDS, PAUL G. [1973] :

The dynamic field of a growing plane elliptical shear crack., Int. J. Solids Structures.,V9, P843.

88. ROACH, G. F. [1982] :

Green's Functions., Second edition., Cambridge University Press., London.

89. ROBERTSON, I. A. [1966] :

Forced vertical vibration of a rigid circular disc on a semi-infinite elastic solid., Proc. Camb. Phil. Soc., V62, P547.

90. ROY, A. [1975] :

First motion response of an elastic half-space to normal pressure over a circular area., Int. J. Engng. Sci., V13, P641.

91. ROY, A. [1981] :

Response of an elastic half-space to normal pressure over an elliptic area., Int. J. Engng. Sci., V19, P129.

92. ROY, A. [1986] :
Dynamic response of elliptic footing., Int. J. Solids Structures., V22, P293.

93. SARKAR, J., GHOSH, M. L. and MANDAL, S. C. [1991] :
Scattering of antiplane shear wave by a propagating crack at the interface of two dissimilar elastic media., Proc. Indian Acad. Sci. (Math. Sci.)., V101, P183.

94. SAUTER, F. [1950] :
Der elastische halbraum bei einer mechanischen beeinflussung seiner oberflache., ZAMM, V30, P203.

95. SCOTT, R. A. and MIKLOWITZ, J. [1964] :
Transient compressional waves in an infinite elastic plate with a circular cylindrical cavity., J. Appl. Mech., V , P627.

96. SHINDO, Y., NOZAKI, H. and HIGAKI, H. [1986] :
Impact response of a finite crack in an orthotropic strip., Acta Mech., V62, P87.

97. SHINDO, Y., HIGAKI, H. and NOZAKI, H. [1991] :
Impact response of symmetric edge crack in an orthotropic strip., JSME, Int. J., V34, P7.

98. SIH, G. C. [1968] :

Some elastodynamic problems of cracks., iNT.J. Frac. Mech., V1, P51.

99. SIH, G. C. and LOEBER, J. F. [1970] :

Interaction of horizontal shear waves with a running crack., J. Appl. Mech., V37, P324.

100. SIH, G. C. and CHEN, E. P. [1980] :

Normal and shear impact of layered composite with a crack : Dynamic stress intensification., J. Appl. Mech., V47, P351.

101. SRIVASTAVA, K. N. and LOWENGRUB, M. [1970] :

Finite Hilbert transform technique for triple integral equations with trigonometrical kernels., Proc. Roy. Soc. Edn. LXVIII, P309.

102. SRIVASTAVA, K.N., PALAIYA, R.M. & KARAULIA, D.S. [1980] :

Interaction of antiplane shear waves by a Griffith crack at interface of two bonded dissimilar elastic half-spaces., Int. J. Fracture., V16, P349.

103. SRIVASTAVA, K.N., GUPTA, O.P. and PALAIYA, R.M. [1981] :

Interaction of elastic waves with a Griffith crack situated in an infinitely long strip., ZAMM.,V61, P583.

104. SRIVASTAVA, K.N., PALAIYA, R.M. & KARAULIA, D.S. [1983] : Interaction of shear waves with a Griffith crack situated in an infinitely long elastic strip., Int J. Fracture., V21, P39.

105. TAKEI, M., SHINDO. Y. and ATSUMI, A. [1982] : Diffraction of transient horizontal shear waves by a finite crack at the interface of two bonded dissimilar elastic solids., Engng. Fracture Mech., V. 16, P799.

106. THIRUVENKATACHAR, V. R. [1955] : Proc. 1st Cong. Theor. Appl. Math., P181.

107. TITCHMARSH, E. C. [1962] : Eigenfunction expansions, Part I, Oxford University Press, London.

108. TRANTER, C. J. [1968] : Bessel functions with some physical applications., The English University Press Ltd., London.

109. TUPHOLME, G. E. [1970] : Pulse generation in an elastic half-space by the uniform normal loading of a circular surface segment. Int. J. Engng. Sci., V8, P617.

110. UEDA, S., SHINDO, Y. and ATSUMI, A. [1983] : Torsional impact response of a penny-shaped crack

lying on a bimaterial interface., Engng. Fracture Mech., V18, P1058.

111. WHITTAKER, E. T. and WATSON, G. N. [1963] :
A course of modern analysis., The Cambridge University Press.

112. WICKHAM, G. R. [1977] :
The forced two dimensional oscillations of a rigid strip in smooth contact with a semi-infinite elastic solid., Proc. Camb. Phil. Soc., V81, P291.

113. YOFFE, E. H. [1951] :
The moving Griffith crack., Phil. Mag., V42, P739.

----x----

Journal of Technical Physics, J. Tech. Phys., 26, 1, 97–115, 1985.
Polish Academy of Sciences, Institute of Fundamental Technological Research, Warszawa.

SPECTRAL REPRESENTATION OF A CERTAIN CLASS OF SELF-ADJOINT DIFFERENTIAL OPERATORS AND ITS APPLICATION TO AXISYMMETRIC BOUNDARY VALUE PROBLEMS IN ELASTODYNAMICS

S. C. PAL, M. L. GHOSH and P. K. CHOWDHURI (DARJEELING)

1. Introduction

In this work an integral representation of the Dirac delta function required for solving the axisymmetric boundary value problem has been derived first. This representation is particularly suitable for problems where mixed boundary conditions are encountered. Following FRIEDMANN [1], by contour integration of a suitable Green's function, integral representation of $\delta(R-R_0)$ $(R, R_0 > 1)$ has been derived. This representation has been used to solve a particular type of axisymmetric problem in elastodynamics.

The problem treated is that of a semi-infinite elastic body containing a circular cylindrical cavity, whose axis is perpendicular to the plane surface. The semi-infinite medium is subjected to an axisymmetric concentric torque applied dynamically as a step function in time at the plane surface.

At first LAMB [4] investigated the classical normal loading problem of an elastic half-space. As similar type of problem was investigated by EASON [5], MITRA [6], CHAKRABORTY and DE [7] and many others. They are all point source problems in a homogeneous semi-infinite medium.

The propagation of elastic waves, due to applied boundary tractions, in semi-infinite media containing internal boundaries has as yet not been studied to any large extent.

An earlier and comprehensive survey of the field is given by SCOTT and MIKLOWITZ [8]. Recently this type of work has been done by JOHNSON and PARNES [9].

We have solved the problem of the SH-type of elastic wave propagation in the semi-infinite medium due to a ring source producing SH-waves in the presence of a circular cylindrical cavity (case I). The problem of SH-wave propagation in the presence of rigid circular cylindrical inclusion in the semi-infinite medium due to the ring source has also been treated in the case 2.

2. Integral Representation of a Dirac Delta Function

Consider the operator L with λ as a complex parameter, where

$$L \equiv \frac{d}{dr}\left(r\frac{d}{dr}\right)+\lambda r-\frac{1}{r} \tag{2.1}$$

whose domain, D, is the set of all twice-differentiable functions $u(r)$, $a < r < \infty$ such that

(i) $$r\frac{du}{dr}-u=0 \quad \text{at} \quad r=a>0$$

(ii) the behaviour of u as $r\to\infty$ is that of an outgoing wave.

The solutions of $LG_1=0$ which satisfy (i) are

(2.2) $$G_1=A_1[J_1(\sqrt{\lambda}r)Y_2(\sqrt{\lambda}a)-Y_1(\sqrt{\lambda}r)J_2(\sqrt{\lambda}a)], \quad a<r<r_0,$$

where A_1 is an arbitrary constant and J_n and Y_n are the Bessel functions of the first and second kind, respectively.

Again the function G_2 which will satisfy $LG_2=0$ and the condition (ii) can be written as

(2.3) $$G_2=A_2H_1^{(1)}(\sqrt{\lambda}r) \quad (a<r_0<r<\infty),$$

where A_2 is an arbitrary constant and $H_n^{(1)}$ is the Hankel function of the first kind of order n.

From Eqs. (2.2) and (2.3) the Green's function G satisfying the equation $LG=-\delta(r-r_0)$ and the conditions (i) and (ii) mentioned above is given by (c.f. [1]).

(2.4) $$G(r,r_0;\lambda)=-\frac{\pi H_1^{(1)}(\sqrt{\lambda}r_0)}{2H_2^{(1)}(\sqrt{\lambda}a)}[J_1(\sqrt{\lambda}r)Y_2(\sqrt{\lambda}a)-Y_1(\sqrt{\lambda}r)J_2(\sqrt{\lambda}a)]H(r_0-r)-$$
$$-\frac{\pi H_1^{(1)}(\sqrt{\lambda}r)}{2H_2^{(1)}(\sqrt{\lambda}a)}[J_1(\sqrt{\lambda}r_0)Y_2(\sqrt{\lambda}a)-Y_1(\sqrt{\lambda}r_0)J_2(\sqrt{\lambda}a)]H(r-r_0),$$
$$0<\arg\lambda<2\pi.$$

Now consider

(2.5) $$\frac{1}{2\pi i}\oint G(r,r_0;\lambda)r\,d\lambda,$$

where the contour of integration in the λ-plane is shown in Fig. 1. Since G has a branch point at $\lambda=0$, we introduce a branch cut in the complex λ-plane along the positive real axis and then take the contour as a large circle of radius R_1^2, having the centre at $\lambda=0$, not crossing the branch cut.

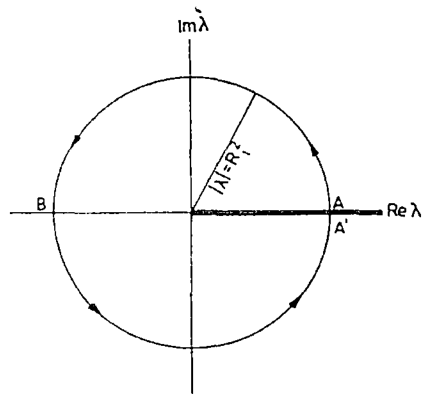

FIG. 1. Circular contour of integration ABA' in the λ-plane.

In terms of Hankel functions Eq. (2.4) can be written as

$$(2.6)\qquad \frac{\pi}{4i}\left[H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)}-H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(2)}(\sqrt{\lambda}r)\right]H(r_0-r)+$$

$$+\frac{\pi}{4i}\left[H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)}-H_1^{(1)}(\sqrt{\lambda}r)H_1^{(2)}(\sqrt{\lambda}r_0)\right]H(r-r_0).$$

For large $|z|$, the asymptotic behaviour of $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ is [2]

$$(2.7)\qquad \begin{aligned} H_n^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}}\exp\left[i\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right],\\ H_n^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}}\exp\left[-i\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right]. \end{aligned}$$

Thus, for large values of $|\lambda|$, from the relations (2.7) we obtain

$$(2.8)\qquad \begin{aligned} H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)} &\sim \frac{2}{\sqrt{\lambda r r_0}\,\pi}\exp[i\sqrt{\lambda}(r+r_0-2a)+i\pi],\\ H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(2)}(\sqrt{\lambda}r) &\sim \frac{2}{\pi\sqrt{\lambda r r_0}}\exp[i\sqrt{\lambda}(r_0-r)],\\ H_1^{(1)}(\sqrt{\lambda}r)H_1^{(2)}(\sqrt{\lambda}r_0) &\sim \frac{2}{\pi\sqrt{\lambda r r_0}}\exp[i\sqrt{\lambda}(r-r_0)]. \end{aligned}$$

If we put $\lambda = k^2$, then the circle in the λ-plane becomes a semi-circular arc C of radius R_1 in the upper half of the k-plane shown in Fig. 2.

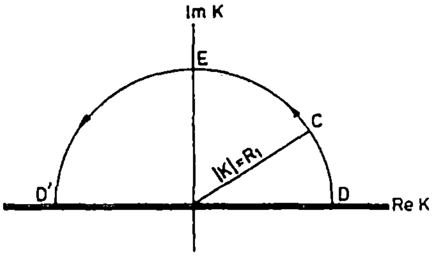

FIG. 2. DED' — the semi-circular path of integration C in the K-plane.

Consequently, for large values of R_1 the integral (2.5) can be written as

$$(2.9)\qquad \frac{1}{2\pi}\sqrt{\frac{r}{r_0}}\int_C[\exp\{ik(r_0-r)\}H(r_0-r)+\exp\{ik(r-r_0)\}H(r-r_0)]dk-$$

$$-\frac{1}{2\pi}\int_C\sqrt{\frac{r}{r_0}}\exp\{ik(r+r_0-2a)\}dk=$$

$$=-\frac{1}{2\pi}\sqrt{\frac{r}{r_0}}\int_{-R_1}^{R_1}\exp(ik|r-r_0|)\,dk+\frac{1}{2\pi}\sqrt{\frac{r}{r_0}}\int_{-R_1}^{R_1}\exp\{ik(r+r_0-2a)\}dk=$$

$$=-\frac{1}{\pi}\sqrt{\frac{r}{r_0}}\frac{\sin R_1(r-r_0)}{r-r_0}+\frac{1}{\pi}\sqrt{\frac{r}{r_0}}\frac{\sin R_1(r+r_0-2a)}{r+r_0-2a}.$$

Our object is to show that the integral (2.5) represents $-\delta(r-r_0)$ when $R_1 \to \infty$. To justify the statement, consider a testing function $\phi(r)$, in D which is continuous, has a continuous derivative of order two and vanishes outside a finite interval. Then, from the relations (2.5) and (2.9)

$$\lim_{R_1\to\infty} \int_a^\infty \phi(r) \frac{1}{2\pi i} \oint G(r, r_0;\, \lambda) r\, d\lambda\, dr$$

$$= -\lim_{R_1\to\infty} \frac{1}{\pi} \int_a^\infty \phi(r) \sqrt{\frac{r}{r_0}}\, \frac{\sin R_1(r-r_0)\,dr}{(r-r_0)} +$$

$$+ \lim_{R_1\to\infty} \frac{1}{\pi} \int_a^\infty \phi(r) \sqrt{\frac{r}{r_0}}\, \frac{\sin R_1(r+r_0-2a)\,dr}{(r+r_0-2a)} = -\phi(r_0),$$

where we have used the result of Dirichlet integral and Riemann–Lebesgue Lemma [3].

Therefore

$$\lim_{R_1\to\infty} \frac{1}{2\pi i} \oint G(r, r_0;\, \lambda) r\, d\lambda = -\delta(r-r_0).$$

To obtain an alternative integral representation, which will be useful for our subsequent application in physical problems, we consider the contour Γ (Fig. 3) consisting of the real axis from $k = \varrho$ to $k = R_1$, where $0 < \varrho < R$; a semi-circle C of radius R_1 above the real axis; the real axis again from $-R$ to $-\varrho$; and finally a semi-circle γ of radius ϱ above the real axis with the centre at the origin. We take ϱ small and R_1 large.

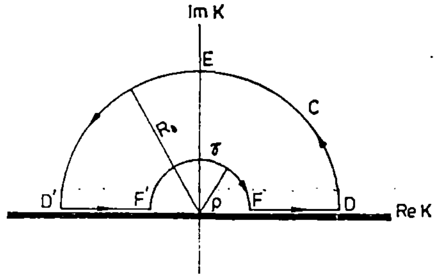

FIG. 3. $FDED'F'F$ — the path of integration Γ in the K-plane.

The integrand $2G(r, r_0, k^2)\, kr$ has no singularity inside the contour Γ, and so the value of the integral

$$\frac{1}{2\pi i} \int_\Gamma G(r, r_0;\, k^2) 2kr\, dk = 0, \tag{2.10}$$

i.e.

$$\frac{1}{2\pi i} \int_\Gamma G(r, r_0;\, k^2) 2kr\, dk = -\frac{1}{2\pi i} \int_\varrho^{R_1} G(r, r_0;\, u^2) 2ur\, du +$$

$$+ \frac{1}{2\pi i} \int_\varrho^{R_1} G(r, r_0;\, e^{2\pi i} u^2) 2ru\, du - \frac{1}{2\pi} \int_0^\pi G(r, r_0;\, \varrho^2 e^{2i\theta}) 2r\varrho^2 e^{2i\theta} d\theta.$$

The behaviour of $Y_n(z)$ for small values of $|z|$ is described by the formula [2]

$$Y_n(z) \sim -\frac{2^n\Gamma(n)}{\pi z^n}$$

and $J_n(z)$ is bounded for small values of $|z|$ when n is a positive integer. Using these results we conclude

$$|G(r, r_0; \varrho^2 e^{2i\theta})\varrho|$$

is bounded for small values of ϱ. Hence

$$\lim_{\varrho\to 0}\frac{1}{\pi}\int_0^{\pi} G(r, r_0; \varrho^2 e^{2i\theta})e^{2i\theta}\varrho^2 r\,d\theta = 0.$$

Letting $\varrho \to 0$ and $R_1 \to \infty$ in (2.10), we get

$$(2.11)\qquad \delta(r-r_0) = -\lim_{R_1\to\infty}\frac{1}{2\pi i}\int_c G(r, r_0; k^2)2kr\,dk =$$

$$= \frac{1}{2\pi i}\int_0^{\infty}[G(r, r_0; k^2)-G(r, r_0; k^2 e^{2i\pi})]2kr\,dk.$$

From Eq. (2.4)

$$G(r, r_0; k^2)-G(r, r_0; k^2e^{2i\pi}) =$$

$$= -\frac{\pi}{2}\left[\frac{J_1(kr_0)+iY_1(kr_0)}{J_2(ka)+iY_2(ka)}-\frac{J_1(kr_0)-iY_1(kr_0)}{J_2(ka)+iY_2(ka)}\right][J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)]\times$$

$$\times H(r_0-r)-\frac{\pi}{2}\left[\frac{J_1(kr)+iY_1(kr)}{J_2(ka)+iY_2(ka)}-\frac{J_1(kr)-iY_1(kr)}{J_2(ka)-iY_2(ka)}\right]\times$$

$$\times [J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)]H(r-r_0) =$$

$$= i\pi\frac{[J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)][J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)]}{J_2^2(ka)+Y_2^2(ka)}.$$

Substituting this expression in Eq. (2.11), we get

$$\delta(r-r_0) = \int_0^{\infty}\frac{[J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)][J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)]}{J_2^2(ka)+Y_2^2(ka)}rk\,dk.$$

Substituting $r/a = R$, $r_0/a = R_0$ and $ka = \gamma$, Eq. (2.12) can be written as

$$(2.13)\quad \delta(R-R_0) = \int_0^{\infty}\frac{[J_1(\gamma R_0)Y_2(\gamma)-Y_1(\gamma R_0)J_2(\gamma)][J_1(\gamma R)Y_2(\gamma)-Y_1(\gamma R)J_2(\gamma)]}{J_2^2(\gamma)+Y_2^2(\gamma)}R\gamma\,d\gamma.$$

Since $\delta(R-R_0)$ is symmetric with respect to R and R_0, then, on the right hand side of Eq. (2.13), R and R_0 can be interchanged. So we write

$$(2.14)\quad \delta(R-R_0) = R_0\int_0^{\infty}\frac{\gamma[J_1(\gamma R_0)Y_2(\gamma)-Y_1(\gamma R_0)J_2(\gamma)][J_1(\gamma R)Y_2(\gamma)-Y_1(\gamma R)J_2(\gamma)]}{J_2^2(\gamma)+Y_2^2(\gamma)}\,\alpha\gamma.$$

3. Formulation and General Solution

Case 1. We shall now use the integral representation of the delta function given by Eq. (2.13) to derive the time dependent response of an isotropic linearly elastic half-space containing a cylindrical cavity of radius a due to a ring source. The axis of the cylinder considered as the z-axis, which is perpendicular to the plane surface, is directed downwards (Fig. 4). A torque is applied on the free surface of the half-space over the rim of a concentric circle of radius $r = r_0(r_0 > a)$ for $t \geqslant 0$.

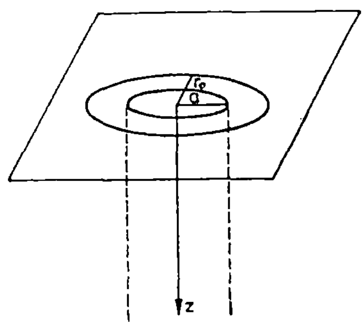

FIG. 4. Geometry of the problem.

Therefore on the cavity surface $r = a$

$$\tau_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = 0 \tag{3.1}$$

and on the plane surface $z = 0$

$$\tau_{\theta z} = \mu\frac{\partial u_\theta}{\partial z} = \delta(r-r_0)H(t) \qquad (a < r < \infty, r_0 \gg a), \tag{3.2}$$

where μ is Lame's constant, δ is the Dirac delta function and H is the unit step function.

Now the only non-zero equation of motion is

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = \frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial t^2}, \tag{3.3}$$

where $\beta = \sqrt{\mu/\varrho}$ is the shear wave velocity.

Changing the independent variables (r, z, t) to the no-dimensional variables (R, Z, τ) defined by

$$R = \frac{r}{a}, \qquad Z = \frac{z}{a}, \qquad \tau = \frac{\beta t}{a}, \qquad R_0 = \frac{r_0}{a} \tag{3.4}$$

the above equation reduces to

$$\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R}\frac{\partial u_\theta}{\partial R} + \frac{\partial^2 u_\theta}{\partial Z^2} - \frac{u_\theta}{R^2} = \frac{\partial^2 u_\theta}{\partial \tau^2} \tag{3.5}$$

and boundary conditions become

$$\tau_{r\theta} = \frac{\mu}{a}\left(\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right) = 0 \quad \text{on} \quad R = 1 \tag{3.6}$$

and

(3.7) $$\tau_{\theta z} = \frac{\mu}{a}\frac{\partial u_\theta}{\partial z} = \frac{1}{a}\delta(R-R_0)H(t) \quad \text{on} \quad Z = 0.$$

Now, taking the Laplace transform with respect to nondimensional time (τ) and assuming the homogeneous initial conditions $u_\theta(R, Z, 0) = \frac{\partial u_\theta(R, Z, 0)}{\partial t} = 0$ at $t = 0$ Eq. (3.5) takes the form

(3.8) $$\frac{\partial^2 \tilde{u}_\theta}{\partial R^2} + \frac{1}{R}\frac{\partial \tilde{u}}{\partial R} + \frac{\partial^2 \tilde{u}_0}{\partial Z^2} - \frac{\tilde{u}_\theta}{R^2} = s^2\tilde{u}_0,$$

where

(3.9) $$\tilde{u}_\theta = \int_0^\infty u_\theta e^{-s\tau} d\tau.$$

Take solution of Eq. (3.8) in the form

(3.10) $$\tilde{u}_\theta(R, Z, s) = \int_0^\infty [A_1(\gamma)J_1(\gamma R) + B_1(\gamma)Y_1(\gamma R)]e^{-\sqrt{s^2+\gamma^2}Z} d\gamma,$$

where γ is real, J_1 and Y_1 are Bessel functions of the first and second kind respectively.

Using the boundary condition (3.6), we obtain

(3.11) $$B_1(\gamma) = -A_1(\gamma)\frac{J_2(\gamma)}{Y_2(\gamma)}.$$

Substituting the value of $B_1(\gamma)$ an in Eq. (3.10), we have

(3.12) $$\tilde{u}_\theta(R, Z, s) = \int_0^\infty A(\gamma)[J_1(\gamma R)Y_2(\gamma) - J_2(\gamma)Y_1(\gamma R)]e^{-\sqrt{s^2+\gamma^2}Z} d\gamma,$$

where

(3.13) $$A(\gamma) = \frac{A_1(\gamma)}{Y_2(\gamma)}.$$

Therefore the transformed stress component reduces to

(3.14) $$\tilde{\tau}_{\theta z} = \frac{\mu}{a}\int_0^\infty A(\gamma)\sqrt{(\gamma^2+s^2)}\,C_2(\gamma R)e^{-\sqrt{\gamma^2+s^2}Z} d\gamma,$$

where

(3.15) $$C_2(\gamma R) = J_2(\gamma)Y_1(\gamma R) - Y_2(\gamma)J_1(\gamma R).$$

New, using the representation (3.15), Eq. (2.14) becomes

(3.16) $$\delta(R-R_0) = R_0\int_0^\infty \frac{\gamma C_2(\gamma R)C_2(\gamma R_0)}{J_2^2(\gamma)+Y_2^2(\gamma)} d\gamma.$$

Using Eqs. (3.7), (3.14) and (3.16), the value of $A(\gamma)$ is obtained as

(3.17) $$A(\gamma) = \frac{R_0}{\mu s}\frac{\gamma C_2(\gamma R_0)}{\sqrt{(s^2-\gamma^2)}\,\{J_2^2(\gamma)+Y_2^2(\gamma)\}}.$$

Therefore $\tilde{u}_\theta$ becomes

$$\tilde{u}_\theta(R, Z, s) = -\frac{R_0}{\mu s}\int_0^\infty \frac{\gamma C_2(\gamma R) C_2(\gamma R_0)}{\sqrt{(\gamma^2+s^2)}\,\{J_2^2(\gamma)+Y_2^2(\gamma)\}}\, e^{-\sqrt{\gamma^2+s^2}\,Z} d\gamma. \tag{3.18}$$

On the plane boundary $Z = 0$

$$\tilde{u}_\theta(R, o, s) = -\frac{R_0}{\mu s}\int_0^\infty \frac{\gamma C_2(\gamma R) C_2(\gamma R_0)}{\sqrt{(\gamma^2+s^2)}\,\{J_2^2(\gamma)+Y_2^2(\gamma)\}} d\gamma. \tag{3.19}$$

Now, introducing the change of the variable $\gamma = s\zeta$ into the above expression (3.19), we obtain

$$\tilde{u}_\theta(R, o, s) = -\frac{R_0}{\mu}\int_0^\infty \frac{\zeta C_2(s\zeta R) C_2(s\zeta R_0)}{\sqrt{(\zeta^2+1)}\,\{J_2^2(s\zeta)+Y_2^2(s\zeta)\}} d\zeta, \tag{3.20}$$

Next, using

$$J_n(s\zeta R) = \frac{H_n^{(1)}(s\zeta R)+H_n^{(2)}(s\zeta R)}{2} \tag{3.21}$$

and

$$Y_n(s\zeta R) = \frac{H_n^{(1)}(s\zeta R)-H_n^{(2)}(s\zeta R)}{2i}, \tag{3.21'}$$

we obtain

$$C_2(s\zeta R) = J_2(s\zeta) Y_1(s\zeta R) - Y_2(s\zeta) J_1(s\zeta R) = \tag{3.22}$$
$$= \frac{1}{2i}[H_1^{(1)}(s\zeta R) H_2^{(2)}(s\zeta) - H_1^{(2)}(s\zeta R) H_2^{(1)}(s\zeta)]$$

and

$$C_2(s\zeta R_0) = \frac{1}{2i}[H_1^{(1)}(s\zeta R_0) H_2^{(2)}(s\zeta) - H_1^{(2)}(s\zeta R_0) H_2^{(1)}(s\zeta)]. \tag{3.22'}$$

Also

$$J_2^2(s\zeta)+Y_2^2(s\zeta) = H_2^{(1)}(s\zeta) H_2^{(2)}(s\zeta). \tag{3.22''}$$

Therefore, Eq. (3.20) becomes

$$\tilde{u}_\theta(R, o, s) = -\frac{R_0}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} F(R, R_0, s\zeta) d\zeta, \tag{3.23}$$

where

$$(R, R_0, s\zeta) = F_1(R, R_0, s\zeta)+F_2(R, R_0, s\zeta) = F_1(R_0, R, s\zeta)+F_2(R_0, R, s\zeta) = \tag{3.24}$$
$$= F(R_0, R, s\zeta)$$

and

$$F_1(\alpha, \beta, s\zeta) = H_1^{(2)}(s\zeta\beta)\left\{H_1^{(1)}(s\zeta\alpha) - H_1^{(2)}(s\zeta\alpha)\frac{H_2^{(1)}(s\zeta)}{H_2^{(2)}(s\zeta)}\right\}, \tag{3.24'}$$

$$F_2(\alpha, \beta, s\zeta) = H_1^{(1)}(s\zeta\beta)\left\{H_1^{(2)}(s\zeta\alpha) - H_1^{(1)}(s\zeta\alpha)\frac{H_2^{(2)}(s\zeta)}{H_2^{(1)}(s\zeta)}\right\}. \tag{3.24''}$$

Using the asymptotic values of the Hankel functions for a large argument, it can be shown that

$$\text{(3.25)} \qquad \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_1(R, R_0, s\zeta) \to \frac{2}{\pi s\zeta \sqrt{RR_0}} [e^{-is\zeta(R_0-R)} + e^{-is\zeta(R+R_0-2)}]$$

as $|s\zeta| \to \infty$, showing that $\dfrac{\zeta F_1(R, R_0, s\zeta)}{\sqrt{(\zeta^2+1)}}$ vanishes over a large circular arc in the fourth quadrant of the complex ζ-plane for $R < R_0$.

Also

$$\text{(3.25')} \qquad \frac{\zeta F_2(R, R_0, s\zeta)}{\sqrt{(\zeta^2+1)}} \to \frac{2}{\pi s\zeta \sqrt{RR_0}} [e^{is\zeta(R_0-R)} + e^{is\zeta(R+R_0-2)}]$$

showing that $\dfrac{\zeta F_2(R, R_0, s\zeta)}{\sqrt{\zeta(^2+1)}}$ vanishes over a large circular arc in the first quadrant of the complex ζ-plane for $R < R_0$. Therefore, for $R > R_0$,

$$\frac{\zeta F_2(R_0, R, s\zeta)}{\sqrt{(\zeta^2+1)}} \quad \text{and} \quad \frac{\zeta F_1(R_0, R, s\zeta)}{\sqrt{(\zeta^2+1)}}$$

vanish over large circular arcs in the first and fourth quadrants, respectively ,of the complex ζ-plane.

Denoting the responses for field points inside ($R < R_0$) and outside ($R > R_0$) the source by the subscripts I and 0 respectively, we have for points inside the source ($R < R_0$)

$$\text{(3.26)} \qquad \bar{u}_{\theta I}(R, o, s) = -\frac{R_0}{4\mu} \int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} [F_2(R, R_0, s\zeta) + F_1(R, R_0, s\zeta)] d\zeta$$

and points outside the source ($R > R_0$)

$$\text{(3.26')} \qquad \bar{u}_{\theta 0}(R, o, s) = -\frac{R_0}{4\mu} \int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} [F_2(R_0, R, s\zeta) + F_1(R_0, R, s\zeta)] d\zeta .$$

In order to evaluate

$$\text{(3.27)} \qquad -\frac{R_0}{4\mu} \int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_2(R, R_0, s\zeta) d\zeta ,$$

which is the first part of $\bar{u}_{\theta I}(R, 0, s)$ we note first that the integrand has branch points at $\zeta = \pm i$ and also has a branch point at the origin of coordinates due to the presence of Hankel functions in the integrand. The integrand has also poles which correspond to the zeros of $H_2^{(1)}(s\zeta)$. From Eq. (3.18) we note that in order that $\bar{u}_\theta(R, Z, s)$ may be finite for large positive values of Z,$(\zeta^2+1)^{1/2}$ should have a positive real part on the path of integration. Accordingly, we draw cuts parallel to the real axis from $+i$ to $-\infty+i$ and from $-i$ to $\infty - i$ to satisfy our requirement. A cut along the negative real axis from the origin is also drawn to make Hankel functions single valued

$$-\frac{R_0}{4\mu} \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_2(R, R_0, s\zeta)$$

is now integrated along the quadrant of a large circle lying in the first quadrant of the complex ζ-plane as shown in Fig. 5a. Since poles of the integrand are outside the path of integration, the integral (3.27) becomes

$$\frac{R_0}{4\mu}\left[\int_0^1 \frac{v}{\sqrt{(1-v^2)}} F_2(R, R_0, isv)dv + \int_1^\infty \frac{v}{i\sqrt{(v^2-1)}} F_2(R, R_0, isv)dv\right]. \tag{3.28}$$

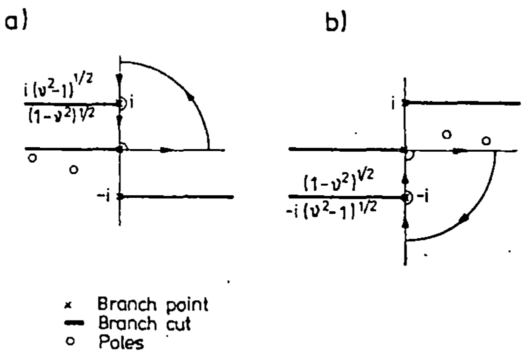

FIG. 5. Integration paths in the complex ζ-plane.

Using the relations

$$\begin{aligned} H_1^{(1)}(iv) &= -\frac{2}{\pi} K_1(v), \\ H_1^{(2)}(iv) &= \frac{2}{\pi} K_1(v) + 2iI_1(v), \\ H_2^{(1)}(iv) &= \frac{2i}{\pi} K_2(v), \\ H_2^{(2)}(iv) &= -2I_2(v) - \frac{2i}{\pi} K_2(v), \end{aligned} \tag{3.29}$$

we have

$$F_2(R, R_0, isv) = -\frac{4i}{\pi} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}. \tag{3.30}$$

Therefore, the expression (3.28) becomes

$$-\frac{iR_0}{\mu\pi}\int_0^1 \frac{v}{\sqrt{(1-v^2)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv - \tag{3.31}$$

$$-\frac{R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv.$$

The second part of $\tilde{u}_{\theta 1}(R, 0, s)$ is equal to

$$-\frac{R_0}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_1(R, R_0 s\zeta)d\zeta \tag{3.32}$$

we draw cuts from $+i$ to $\infty+i$ and from $-i$ to $-\infty-i$ as shown in Fig. (5b). A cut from the origin along the negative real axis is also drawn to make Hankel functions single valued.

Taking a quadrant of a large circular contour in the fourth quadrant (Fig. (5b)) and noting that the poles of F_1 $(R, R_0 s\zeta)$ lie outside the contour, the integral (3.32) takes the form

$$(3.33)\quad \frac{R_0}{4\mu}\left[\int_0^1 \frac{v}{\sqrt{(1-v^2)}} F_1(R, R_0, -isv)\,dv - \int_1^\infty \frac{v}{i\sqrt{(v^2-1)}} F_1(R, R_0, -isv)\,dv\right].$$

Using the relations

$$(3.34)\quad \begin{aligned} H_1^{(1)}(-iv) &= \frac{2}{\pi} K_1(v) - 2iI_1(v), \\ H_1^{(2)}(-iv) &= -\frac{2}{\pi} K_1(v), \\ H_2^{(1)}(-iv) &= -2I_2(v) + \frac{2i}{\pi} K_2(v), \\ H_2^{(2)}(-iv) &= +\frac{2}{i\pi} K_2(v), \end{aligned}$$

the expression (3.33) becomes

$$(3.35)\quad \frac{iR_0}{\mu\pi}\int_0^1 \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv -$$

$$-\frac{R_0}{\mu\pi}\int_0^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv.$$

Adding the relations (3.31) and (3.35), we obtain

$$(3.36)\quad \bar{u}_{01}(R, o, s) = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-p)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv.$$

Similarly, it can be shown that

$$(3.36')\quad \bar{u}_{00}(R, o, s) = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR)\left\{I_1(svR_0) + K_1(svR_0)\frac{I_2(sv)}{K_2(sv)}\right\}dv.$$

Laplace inversion of the relations (3.36) is now taken to obtain the displacement of points inside the source.
Therefore

$$(3.37)\quad u_{01}(R, o, \tau) = -\frac{1}{2\pi i}\cdot\frac{2R_0}{\mu\pi}\int_{Br} e^{\tau s}ds \int_1^\infty \frac{v}{\sqrt{(v^2-1)}} \bar{E}(sv)\,dv,$$

where

$$(3.38)\quad \bar{E}(sv) = K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}.$$

Introducing the change of variable $p = sv$, and changing the order of integration

$$(3.39) \quad u_{01}(R, 0, \tau) = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{1}{\sqrt{(v^2-1)}}\,dv\left[\frac{1}{2\pi i}\int_{Br} e^{(\tau/v)p}\tilde{E}(p)\,dp\right] = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{1}{\sqrt{(v^2-1)}}E(\tau/v)\,dv,$$

where $E(\tau/v) = \mathscr{L}^{-1}\{\tilde{E}(p)\}$.

We note that $\tilde{E}(p)$ possesses no poles and is analytic for $p > 0$. It has a branch point at the origin and therefore a cut is drawn from the origin along the negative real axis of the complex p-plane in order to make $\tilde{E}(p)$ single valued.

Drawing a large semi-circular contour to the right of the Bromwich path AB in the complex p-plane, we conclude that $E(\tau/v) = 0$ if the integral

$$\frac{1}{2\pi i}\int_{BC'A} \tilde{E}(p)e^{(\tau/v)p}dp = 0$$

over the semi-circular arc $BC'A$ (Fig. 6).

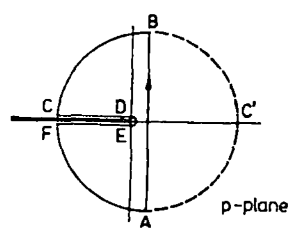

FIG. 6. Laplace inversion contour.

Now

$$(3.40) \quad E(p) = -\frac{1}{2\pi i}\int_{BC'A}\tilde{E}(p)e^{(\tau/v)p}dp =$$

$$= -\frac{1}{2\pi i}\int_{BC'A} K_1(pR_0)I_1(pR)e^{(\tau/v)p}dp - \frac{1}{2\pi i}\int_{BC'A} K_1(pR_0)K_1(pR)\frac{I_2(p)}{K_2(p)}e^{(\tau/v)p}dp.$$

Since

$$e^{(\tau/v)p}K_1(pR_0)I_1(pR) \sim \frac{1}{2p\sqrt{RR_0}}e^{\left[\frac{\tau}{v}-(R_0-R)\right]p}$$

and

$$e^{(\tau/vp)}K_1(pR_0)I_1(pR)\frac{I_2(p)}{K_2(p)} \sim \frac{1}{2p\sqrt{RR_0}}e^{\left[\frac{\tau}{v}-(R+R_0-2)\right]p} \quad \text{as} \quad |p| \to \infty$$

then the first integral on the right hand side of Eq. (3.40) vanishes for $0 < \tau/v < (R_0-R)$, whereas the second integral vanishes for $0 < \tau/v < (R+R_0-2)$.

Therefore

(3.41)
$$E(\tau/v) = \begin{cases} 0, & \text{for} \quad 0 < \tau/v < (R_0-R), \\ E^D(\tau/v), & \text{for} \quad (R_0-R) < \tau/v < (R+R_0-2), \\ E^R(\tau/v), & \text{for} \quad (R+R_0-2) < \tau/v. \end{cases}$$

Where

(3.42)
$$E^D(\tau/v) = \mathcal{L}^{-1}[K_1(pR_0)I_1(R)],$$
$$E^R(\tau/v) = \mathcal{L}^{-1}\left[K_1(pR_0)I_1(pR)+K_1(pR_0)K_1(pR)\frac{I_2(p)}{K_2(p)}\right].$$

For value of τ/v lying in the range $(R-R_0) < \tau/v < (R+R_0-2)$

(3.43)
$$E(\tau/v) = E^D(\tau/v) = \frac{1}{2\pi i}\int_{Br} K_1(pR_0)I_1(pR)e^{(\tau/v)p}dp.$$

Therefore, putting $\tau/v = (R_0-R+y)$, where $y > 0$

$$E^D(R_0-R+y) = \frac{1}{2\pi i}\int_{Br} [K_1(pR)e^{pR_0}][I_1(pR)e^{-pR}]e^{yp}dp.$$

From the Laplace inversion table [12], we find that

$$\mathcal{L}^{-1}[K_1(pR_0)e^{pR_0}] = \frac{H(y)(y+R_0)}{R_0\{y(y+2R_0)\}^{1/2}},$$

and

$$\mathcal{L}^{-1}[I_1(pR)e^{-pR}] = \frac{[H(y)-H(y-2R)](R-y)}{\pi R\{y(2R-y)\}^{1/2}}.$$

So by the convolution theorem

(3.44)
$$E^D(R_0-R+y) = \int_0^y \frac{[H(\eta)-H(\eta-2R)]H(y-\eta)(R-\eta)(y-\eta+R_0)}{\pi RR_0[\eta(2R-\eta)(y-\eta)(y-\eta+2R_0)]^{1/2}}d\eta.$$

For τ/v lying in the range $(R-R_0) < \tau/v < (R+R_0-2)$ τ/v must be less than $(R+R_0)$, i.e. $y < 2R$.

Therefore we can write

$$E^D(R_0-R+y) = \int_0^y \frac{(R-\eta)(y-\eta+R_0)d\eta}{\pi RR_0[\eta(2R-\eta)(y-\eta)(y-\eta+2R_0)]^{1/2}}.$$

So

(3.45)
$$E(\tau/v) = E^D(\tau/v) = \int_0^{\frac{\tau}{v}-(R_0-R)} \frac{(R-\eta)(\tau/v+R-\eta)\mathrm{d}\eta}{\pi RR_0[\eta(2R-\eta)(\tau/v-R_0+R-\eta)(\tau/v+R_0+R-\eta)]^{1/2}}.$$

For values of τ/v satisfying the condition $\tau/v > R+R_0-2$,

(3.46)
$$E(\tau/v) = E^R(\tau/v) = \frac{1}{2\pi i}\int_{Br}\left\{K_1(pR_0)I_1(pR)+K_1(pR_0)K_1(pR)\frac{I_2(p)}{K_2(p)}\right\}e^{(\tau/v)p}dp.$$

This integral is equal to the integral along the large semi-circular arc on the left side of the Bromwich path AB plus the integral on the two sides of the branch cut (Fig. 6). Since the integral on the large semi-circular arc vanishes, then Eq. (3.46) becomes

$$E(\tau/v) = \frac{1}{2\pi i}\left[-\int_0^\infty \tilde{E}(\eta e^{i\pi})e^{-(\tau/v)\eta}d\eta + \int_0^\infty \tilde{E}(\eta e^{-i\pi})e^{-(\tau/v)\eta}\,d\eta\right]. \tag{3.47}$$

Using the relations

$$I_\nu(\eta e^{\pm i\pi}) = e^{\pm i\nu\pi}I_\nu(\eta),$$

and

$$K_\nu(\eta e^{\pm i\pi}) = e^{\mp i\nu\pi}K_\nu(\eta) \pm i\pi I_\nu(\eta),$$

we obtain (for $\tau/v > R+R_0-2$)

$$E(\tau/v) = E^R(\tau/v) = -\int_0^\infty \frac{U_2(R,\eta)\,U_2(R_0,\eta)\,e^{-(\tau/v)\eta}}{K_2^2(\eta)+\pi^2 I_2^2(\eta)}\,d\eta; \tag{3.48}$$

where

$$U_2(x,\eta) = K_2(\eta)I_1(x,\eta)+I_2(\eta)K_1(x,\eta).$$

Substituting these values of $E(\tau/v)$ in Eq. (3.39), we obtain

(3.49)

$$u_{\theta 1}(R,0,\tau) = -\frac{2R_0}{\mu\pi}\Bigg[\Big\{H\Big(t-\frac{r_0-r}{\beta}\Big)-H\Big(t-\frac{r+r_0-2a}{\beta}\Big)\Big\}\int_1^{\frac{\tau}{R_0-R}}\frac{1}{\sqrt{v^2-1}}E^D(\tau/v)\,dv +$$

$$+H\Big(t-\frac{r+r_0-2a}{\beta}\Big)\Big\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R_0-R}}\frac{1}{\sqrt{v^2-1}}E^D(\tau/v)\,dv+\int_1^{\frac{\tau}{R+R_0-2}}\frac{1}{\sqrt{v^2-1}}E^R(\tau/v)\,dv\Big\}\Bigg],$$

where the values of $E^D(\tau/v)$ and $E^R(\tau/v)$ are given in Eqs. (3.45) and (3.48), respectively.

Similarly, taking the inverse Laplace transform of Eq. (3.36′), the displacement $u_{\theta 0}(R,0,\tau)$ on the free surface outside the ring source can be derived and it is found that

(3.49′)

$$U_{\theta 0}(R,o,\tau) = -\frac{2R_0}{\mu\pi}\Bigg[\Big\{H\Big(t-\frac{r-r_0}{\beta}\Big)-H\Big(t-\frac{r+r_0-2a}{\beta}\Big)\Big\}\int_1^{\frac{\tau}{R-R_0}}\frac{1}{\sqrt{(v^2-1)}}F^D(\tau/v)\,dv+$$

$$+H\Big(t-\frac{r+r_0-2a}{\beta}\Big)\Big\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R-R_0}}\frac{1}{\sqrt{v^2-1}}F^D(\tau/v)\,dv+\int_1^{\frac{\tau}{R+R_0-2}}\frac{1}{\sqrt{(v^2-1)}}F^R(\tau/v)\,dv\Big\}\Bigg],$$

where $F^R(\tau/v) = E^R(\tau/v)$, and.

$$F^D(\tau/v) = \int_0^{\frac{\tau}{v}-(R-R_0)} \frac{(R_0-\eta)(\tau/v+R_0-\eta)d\eta}{\pi R R_0\{\eta(2R_0-\eta(\tau/v-R+R_0-\eta)(\tau/v+R+R_0-\eta)\}^{1/2}}. \tag{3.50}$$

First, the integrals of Eqs. (3.49) are the displacements due to a direct wave from the ring source before the arrival of the waves reflected from the wall of the cylindrical cavity. The last two integrals together give the displacement after the arrival of the reflected wave.

In order to obtain the response in the vicinity of the SH-wave front, we consider the displacement profile immediately behind the direct outgoing SH-wave. Accordingly, we shall have to study the first integral of Eq. (3.49′) because it gives the response of the direct SH-wave before the arrival of the reflected wave front.

Let $R_s = R_0 + \tau$ and $R_s^- = R_s - \varepsilon R_0$ where R_s and R^- denote points at and immediately behind the SH-wave front, respectively, ϵ is a small positive quantity.

Then

$$\frac{\tau}{R_s - R_0} = 1 \tag{3.51}$$

and

$$\frac{\tau}{R_s^- - R_0} = \left(1 + \frac{\varepsilon R_0}{\tau}\right) = q(\tau). \tag{3.51'}$$

Substituting these values in the first integral of Eq. 3.49′, we obtain

$$u_{00}(R_s, o, \tau) = 0,$$

and

$$u_{00}(R_s^-, o, \tau) = -\frac{2R_0}{\mu\pi} \int_1^{q(\tau)} \frac{1}{\sqrt{(v-1)}} \left\{\frac{1}{\sqrt{v+1}} \cdot F^D(R_s^-, R_0, \tau/v)\right\} dv.$$

Therefore, we can write

$$u_{00}(R_s^-, 0, \tau) = -\frac{2R_0}{\mu\pi} \int_1^{q(\tau)} \frac{1}{\sqrt{v-1}} V(v)\, dv, \tag{3.52}$$

where $V(v)$ is an analytic portion of the integrand. For small values of ε expanding $V(v)$ by the Taylor's series about the point $v = 1$ and integrating term by term, we obtain

$$u_{00}(R_s^-, o, \tau) \simeq -\frac{4R}{\mu\pi} V(1) \left(\frac{R_0}{\tau}\right)^{1/2} \varepsilon^{1/2} = A\varepsilon^{1/2} \quad \text{(say)}, \tag{3.53}$$

where A is a constant.

It therefore follows that the displacement component is continuous i.e. there is no jump in displacement across the direct SH-wave front.

Next, in order to consider the behaviour of response just under the ring source, it should be remembered that the integral representations of transformed displacements given by Eqs. (3.36) were derived from Eqs. (3.26) assuming that $R \neq R_0$. For $R = R_0$ the integrals along large quarter circles in the first and fourth quadrants should be reexamined. In this case it is found that though the contributions from the integrals along large circular arcs in the first and fourth quadrants are not separately zero, but the combined sum of the integrals along the large arcs in the first and fourth quadrants of the ζ-plane (Fig. 5a and 5b) vanishes. So the transformed displacements for $R = R_0$ are also given by Eqs. (3.36). Making $R \to R_0 \pm$, it can easily by shown by help of Eqs. (3.36) that the displacement has no jump discontinuity accross the ring source.

Therefore, in order to derive the nature of the displacement as $R \to R_0$, any one of the relations (3.49) may be studied. Consider, for example, the displacement at field points outside the source given by (3.49′). As $R \to R_0$, the upper limit of integration $\tau/(R-R_0) \to \infty$.

Further, as

(3.54)
$$v \to \frac{\tau}{R-R_0} \to \infty,$$
$$\frac{1}{\sqrt{(v^2-1)}} \to \frac{1}{v}$$

and

(3.54′)
$$F^D(\tau/v) \to \frac{1}{2R_0}.$$

Thus, from Eq. (3.49′)

(3.55)
$$\lim_{R\to R_0} u_{\theta 0}(R, o, \tau) = -\frac{2R_0}{\mu\pi} \int_N^{\frac{\tau}{R-R_0}} \frac{1}{v} \cdot \frac{1}{2R_0} dv + \text{a finite quantity},$$

where N is large.

The integral is found to contribute a logarithmic singularity to the displacement just on the ring source.

Case 2. In this case the problem considered is the same in all respects with the first, except that the cavity of the radius a has been replaced by a rigid cylindrical inclusion of the same radius. The cylindrical inclusion being in welded contact with the elastic half-space, there is no relative displacement at the interface. In this case, the condition on the cylindrical boundary is $u_\theta = 0$ on $r = a$.

In order to solve this problem, we take the solution in this form:

(3.56)
$$\bar{u}_\theta(R, Z, s) = \int_0^\infty [A_2(\gamma) J_1(\gamma R) + B_2(\gamma) Y_1(\gamma R)] e^{-\sqrt{\gamma^2+s^2}\, Z} d\gamma,$$

where $\bar{u}_\theta(R, Z, s)$ is the Laplace transform of $u_\theta(R, Z, t)$ with respect to t. Now, using the boundary condition

$$\bar{u}_\theta = 0 \quad \text{on} \quad R = 1,$$

we have

(3.57)
$$B_2(\gamma) = -A_2(\gamma) \frac{J_1(\gamma)}{Y_1(\gamma)}$$

so $\bar{u}_\theta$ becomes

(3.58)
$$\bar{u}_\theta(R, Z, s) = \int_0^\infty A^1(\gamma) [J_1(\gamma R) Y_1(\gamma) - J_1(\gamma) Y_1(\gamma R)] e^{-\sqrt{\gamma^2+s^2}\, Z} d\gamma,$$

where

$$A^1(\gamma) = \frac{A_2(\gamma)}{Y_1(\gamma)}.$$

Therefore, the transformed stress component on the free surface $Z = 0$ is

(3.59) $$\bar{\tau}_{\theta z}(R, o, s) = -\frac{\mu}{a} \int_0^\infty A^1(\gamma)\sqrt{\gamma^2+s^2}\, C_1(\gamma R)\, d\gamma,$$

where

(3.60) $$C_1(\gamma R) = J_1(\gamma R)\, Y_1(\gamma) - J_1(\gamma)\, Y_1(\gamma R),$$

$\tau_{\theta z}(R, 0, s)$ should be equal to $\frac{1}{as}\,\delta\,(R-R_0)$. In this case, the required integral representation of the delta function can be obtained from the following expansion formula given by Titchmarsh [11]:

(3.61) $$f(r) = \int_0^\infty \frac{\zeta[J_1(\zeta r)\, Y_1(\zeta a) - J_1(\zeta a)\, Y_1(\zeta r)]}{J_1^2(\zeta a) + Y_1^2(\zeta a)}\, d\zeta \int_a^\infty \xi f(\xi)\, [J_1(\zeta\xi)\, Y_1(\zeta a) - J_1(\zeta a)\, Y_1(\zeta\xi)]\, d\xi,$$

where $f(r)$ is a suitably restricted arbitrary function.

Putting

$$f(r) = \delta(r - r_0),$$
$$f(\xi) = \delta(\xi - r_0), \quad \text{where} \quad r_0 > a > 0,$$

we get

(3.62) $$\delta(r - r_0) = r_0 \int_0^\infty \frac{\zeta[J_1(\zeta r)\, Y_1(\zeta a) - J_1(\zeta a)\, Y_1(\zeta r)]\, [J_1(\zeta r_0)\, Y_1(\zeta a) - J_1(\zeta a)\, Y_1(\zeta r_0)]}{J_1^2(\zeta a) + Y_1^2(\zeta a)}\, d\zeta.$$

Now putting, $\frac{r}{a} = R$, $\frac{r_0}{a} = R_0$, $\zeta a = \gamma$, we have

$$\delta(R - R_0) = R_0 \int_0^\infty \frac{\gamma[J_1(\gamma R)\, Y_1(\gamma) - J_1(\gamma)\, Y_1(\gamma R)]\, [J_1(\gamma R_0)\, Y_1(\gamma) - J_1(\gamma)\, Y_1(\gamma R_0)]}{J_1^2(\gamma) + Y_1^2(\gamma)}\, d\gamma,$$

so by the relation (3.60)

(3.63) $$\delta(R - R_0) = R_0 \int_0^\infty \frac{\gamma C_1(\gamma R)\, C_1(\gamma R_0)}{J_1^2(\gamma) + Y_1^2(\gamma)}\, d\gamma.$$

This result can also be obtained by the following technique already developed in Sect. 2 of this paper.

Now, we find the value of $A^1(\gamma)$ as

(3.64) $$A^1(\gamma) = \frac{R_0}{\mu s} \frac{\gamma C_1(\gamma R_0)}{\sqrt{\gamma^2+s^2}} \frac{1}{J_1^2(\gamma) + Y_1^2(\gamma)}.$$

Therefore $\bar{u}_\theta$ becomes

(3.65) $$\bar{u}_\theta(R, o, s) = \frac{R_0}{\mu s} \int_0^\infty \frac{\gamma C_1(\gamma R)\, C_1(\gamma R_0)}{\sqrt{\gamma^2+s^2}\,\{J_1^2(\gamma) + Y_1^2(\gamma)}\, d\gamma.$$

Carrying on a similar procedure as followed to obtain the displacement in the case 1, we find that in this case

(3.66)

$$u_{\theta 1}(R, o, \tau) = \frac{2R_0}{\mu\pi}\Bigg[\Bigg\{H\left(t-\frac{r_0-r}{\beta}\right)-H\left(t-\frac{r+r_0-2a}{\beta}\right)\Bigg\}\int_1^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}}E^D(\tau/v)\,dv+$$

$$+H\left(t-\frac{r+r_0-2a}{\beta}\right)\Bigg\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}}E^D(\tau/v)\,dv+\int_1^{\frac{\tau}{R+R_0-2}} \frac{1}{\sqrt{v^2-1}}E(\tau/v)\,dv\Bigg\}\Bigg]$$

and

(3.67)

$$u_{\theta 0}(R, o, \tau) = \frac{2R_0}{\mu\pi}\Bigg[\Bigg\{H\left(t-\frac{r-r_0}{\beta}\right)-H\left(t-\frac{r+r_0-2a}{\beta}\right)\Bigg\}\int_1^{\frac{\tau}{R-R_0}} \frac{1}{\sqrt{v^2-1}}F^D(\tau/v)\,dv+$$

$$+H\left(t-\frac{r+r_0-2a}{\beta}\right)\Bigg\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R-R_0}} \frac{1}{\sqrt{v^2-1}}F^D(\tau/v)\,dv+\int_1^{\frac{\tau}{R-R_0+2}} \frac{1}{\sqrt{v^2-1}}F_1^R(\tau/v)\,dv\Bigg\}\Bigg],$$

where $E^D(\tau/v)$ and $F^D(\tau/v)$ are (respectively) given by Eqs. (3.45) and (3.50) and

(3.68)
$$E_1^R(\tau/v) = F_1^R(\tau/v) = -\int_0^\infty \frac{U_1(R,\eta)\,U_1(R_0,\eta)\,e^{-\left(\frac{\tau}{v}\right)\eta}}{K_1^2(\eta)+\pi^2 I_1^2(\eta)}\,d\eta$$

where

(3.69)
$$U_1(x,\eta) = K_1(\eta)I_1(x\eta)-I_1(\eta)K_1(x\eta).$$

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References

1. B. Friedmann, *Principles and techniques of applied mathematics*, John Wiley and Sons, 1966.
2. N. N. Lebedev, *Special functions and their applications*, Prentice-Hall Inc., 1965.
3. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, The Cambridge University Press. 1963.
4. H. Lamb, Phil. Trans. R. Soc. London, A **203**, P.1, 1904.
5. G. Eason, Quat. J. Mech. Appl. Math., **17** Part 3, pp. 279, 1964.
6. M. Mitra, Proc. Camb. Phil. Soc., **60**, pp. 683, 1964.
7. S. K. Chakraborty and T. K. De, Pure Appl. Geophys., **85**, pp. 214, 1971.

8. J. Miklowitz and R. A. Scott, J. Appl. Mech., p. 627, 1964.
9. E. R. Johnson, R. Parnes, Quart. J. Mech. Appl. Math., **30**, p. 235, 1977.
10. E. R. Johnson, R. Parnes, Quart. J. Mech. Appl. Math., **30**, p. 255, 1977.
11. E. C. Titchmarsh, *Eigenfunction expansions, Part I*, Oxford University Press, London 1962.
12. A. Erdelyi [Ed], *Tables of integral transforms*, vol. 1, Bateman Manuseript Project, Mc Graw Hill, 1954.

Streszczenie

SPEKTRALNA REPREZENTACJA PEWNEJ KLASY SAMOSPRZĘŻONYCH OPERATORÓW RÓŻNICZKOWYCH I JEJ ZASTOSOWANIE DO OSIOWO-SYMETRYCZNYCH ZAGADNIEŃ BRZEGOWYCH W ELASTODYNAMICE

Praca jest próbą znalezienia zamkniętej postaci osiowo-symetrycznej dynamicznej funkcji Greena typu SH dla izotropowej jednorodnej liniowej półprzestrzeni sprężystej, zawierającej cylindryczny otwór kołowy prostopadły do brzegu półprzestrzeni. Rozważono dwa przypadki: pierwszy odpowiada swobodnemu od obciążeń brzegowi cylindrycznemu oraz nagle przyłożonemu osiowo-symetrycznemu obciążeniu stycznemu, które jest skupione na konturze pewnego koła w płaszczyźnie brzegu półprzestrzeni; drugi odpowiada utwierdzonemu brzegowi otworu oraz obciążeniu takiemu jak w przypadku pierwszym. Stosując pewną całkową reprezentację celowo-symetrycznego obciążenia dla rozważanego ciała oraz technikę transformacji Laplace'a, podano zamkniętą postać funkcji Greena tylko na brzegu półprzestrzeni. Przeprowadzono też analizę jakościową tej postaci w otoczeniu pewnego kołowego frontu falowego.

Резюме

СПЕКТРАЛЬНОЕ ПРЕДСТАВЛЕНИЕ НЕКОТОРОГО КЛАССА САМОСОПРЯЖЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ И ЕГО ПРИМЕНЕНИЕ К ОСЕСИММЕТРИЧНЫМ КРАЕВЫМ ЗАДАЧАМ В ЭЛАСТОДИНАМИКЕ

Работа является попыткой нахождения замкнутого вида осесимметричной динамической функции Грина типа SH для изотропнового однородного линейного упругого полупространства, содержавшего цилиндрическое круговое отверстие перпедикулярное к границы полупространства. Рассмотрены два случая: первый отвечает свободному от нагрузок краю цилиндрического отверствия и внезапно приложенной осесимметричной касательной нагрузке, которая сосредоточена на контуре некоторого круга в плоскости границы полупространства, второй отвечает закрепленному краю отверстия и нагрузке такой как в первом случае.

Применяя некоторое интегральное представление осесимметричной нагрузки для рассматриваемого тела и технику преобразования Лапласа, приведен замкнутый вид функции Грина только на границе полупространства. Проведен тоже качественный анализ этого вида в окрестности некоторого кругового волнового фронта.

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WAVES IN A SEMI-INFINITE ELASTIC MEDIUM DUE TO AN EXPANDING ELLIPTIC RING SOURCE ON THE FREE SURFACE

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An elliptic ring load emanating from the origin of co-ordinates at $t = 0$ is assumed to expand on the free-surface of an elastic half-space. The rates of increase of the major and minor axes of the ellipse are assumed to be equal to a and b respectively. The displacement at points on the free-surface has been derived in integral form by Cagniard-de Hoop technique. Displacement jumps across different wave fronts have also been derived.

1. INTRODUCTION

Since Lamb's original study of the elastic wave produced by a time-dependent point force acting normally to the surface of an elastic half-space, many authors have elaborated on his work. Aggarwal and Ablow[1] discussed the exact solution of a class of half-space pulse propagation problems generated by impulsive sources. Gakenheimer and Miklowitz[4] used a modification of Cagniard's method[3] to discuss the disturbance created by a moving point load. In case of finite sources, the most widely discussed model is that of a circular ring or disc load. Mitra[7], Tupholme[11] and Roy[9] have studied the various aspects of the same problem. Elastic waves due to uniformly expanding disc or ring loads on the free surface of a semi-infinite medium have been studied extensively by Gakenheimer[5]. The axisymmetric problem of the determination of the displacement due to a stress discontinuity over a uniformly expanding circular region at a certain depth below the free surface has been studied by Ghosh[6].

However exact evaluation of the displacement field for finite source other than the circular model does not seem to have been attempted much in the literature. Burridge and Willis[2] obtained a solution for radiation from a growing elliptical crack in an anisotropic medium. The problem of an elliptical shear crack growing in prestressed medium has been solved by Richards[8] by the Cagniard-de Hoop Method. Roy[10] also attempted the same technique to solve the problem of elastic wave propagation due to prescribed normal stress over an elliptic area on the free surface of an elastic half-space.

In our problem, we have considered the propagation of elastic waves due to an expanding elliptical ring load over the free surface of a semi-infinite medium. The expression for displacement at points on the free surface has been derived in integral from by the application of Cagniard-de Hoop technique for different values of the rate of increase of the major and minor axes of the elliptic ring source. The displacement jumps across the different wave fronts have also been derived.

2. Formulation of the Problem and its Solution

Let an elliptic ring load P acting normal to the surface of an elastic half-space emanating from the origin of co-ordinates expand in such a way that the rates of increase of the major and minor axes of the ellipse are a and b respectively, a and b being constants. Major and minor axes of the ellipse are taken to coincide with the x and y-axes of co-ordinates where as z-axis is taken vertically downwards into the medium (Fig.1).

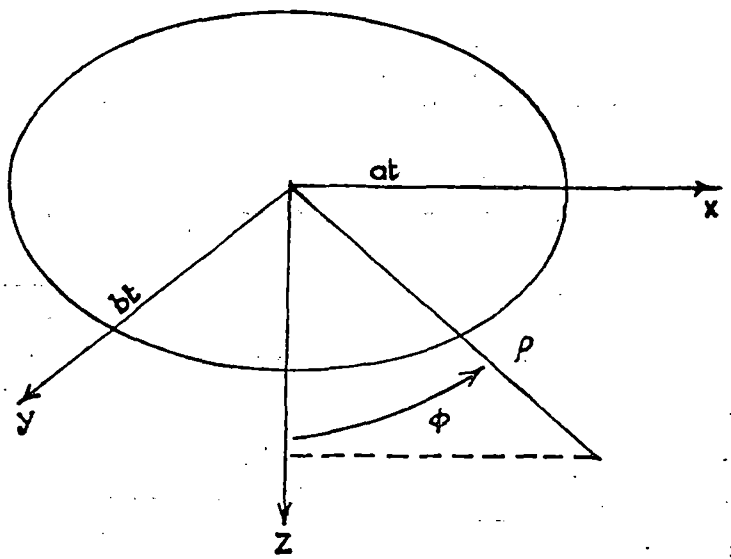


Fig. 1. Geometry of the Problem.

Thus we have on $z = 0$

$$\tau_{zz} = -\frac{P\,\delta\, t - (x^2 a^{-2} + y^2 b^{-2})^{1/2}}{(x^2 a^{-2} + y^2 b^{-2})^{1/2}} \qquad \ldots(1)$$

$$\tau_{xz} = \tau_{yz} = 0$$

where P is constant and δ is the Dirac delta function.

The displacement field inside the elastic medium $(z > 0)$ is given interms of potentials ϕ and ψ as

$$u = \nabla\phi + \nabla \times \nabla \times (e_z \psi)$$

where

$$\nabla^2 \phi = \frac{1}{c_d^2} \frac{\partial^2 \phi}{\partial t^2}, \nabla^2 \psi = \frac{1}{c_s^2} \frac{\partial^2 \psi}{\partial t^2} \quad ...(2)$$

e_x, e_y, e_z are unit vectors along co-ordinate axes and c_d and c_s are the p — and s-wave velocities of the medium.

In order to obtain solutions of wave equations (2), we introduce Laplace transform with respect to t and denote it by bar and also introduce bilateral Fourier transform with respect to x and y to supress the time parameter t and the x, y space co-ordinates. Taking Laplace transform with respect to t (—) and also bilateral Fourier transform with respect to x and y ($\cong$), the transformed boundary conditions are

$$\overset{\cong}{\tau}_{zz} = -\frac{Pab}{(a^2 \xi^2 + b^2 \eta^2 + s^2)^{1/2}}, \overset{\cong}{\tau}_{xz} = \overset{\cong}{\tau}_{yz} = 0. \quad ...(3)$$

Then satisfying the transformed boundary conditions (3) and performing the inverse Fourier transform, the Laplace transformed displacement field can be written as

$$\bar{u}_j (x, y, z, s) = \bar{u}_{jd} (x, y, z, s) + \bar{u}_{js} (x, y, z, s) \quad ...(4)$$

for $j = x, y, z$

where

$$\bar{u}_{j\alpha_1} (x, y, z, s) = 1/2\pi\mu \int_{-\infty}^{\infty} \int_{\infty}^{\infty} F_{j\alpha_1} (\xi, \eta, s) \exp [\zeta_{\alpha_1} z + i (\xi x + \eta y)] \, d\xi \, d\eta \quad ...(5)$$

for $\alpha_1 = d, s$

and

$$\left.\begin{aligned}
&F_{xd} (\xi, \eta, s) = - i \xi \zeta_0 G, \quad F_{xs} (\xi, \eta, s) = 2i\xi \zeta_d \zeta_s G, \\
&F_{yd} (\xi, \eta, s) = - i\eta \zeta_0 G, \quad F_{yz} (\xi, \eta, s) = 2i\eta \zeta_d \zeta_s G, \\
&F_{zd} (\xi, \eta, s) = \zeta_d \zeta_0 G, \; F_{zs} (\xi, \eta, s) = - 2 (\xi^2 + \eta^2) \zeta_d G, \\
&G = \frac{Pab}{(s^2 + r^2)^{1/2} T}, \; T = \zeta_0^2 - 4 \zeta_d \zeta_s (\xi^2 + \eta^2) \\
&r^2 = a^2 \xi^2 + b^2 \eta^2, \\
&\zeta_d = (\xi^2 + \eta^2 + k_d^2)^{1/2}, \; \zeta_s = (\xi^2 + \eta^2 + k_s^2)^{1/2}, \\
&\zeta_0 = k_s^2 + 2 (\xi^2 + \eta^2), \quad k_d = \frac{s}{c_d}, k_s = \frac{s}{c_s}.
\end{aligned}\right\} \quad ...(6)$$

Now the De-Hoop transformation,

$$\xi = s/c_d (q \cos \theta - w \sin \theta), \eta = s/c_d (q \sin \theta + w \cos \theta) \quad ...(7)$$

where $\theta = \tan^{-1} y/x$.

is applied into (5). The Laplace transformed displacement field (5) can be written as

$$\bar{u}_{j\alpha 1} (R, Z, s) = 1/2\pi\mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{j\alpha 1} (q, w, s) \exp [- s/c_d (m_\alpha Z - iqR)]$$

$$\times \frac{s^2}{c_d^2} dq\, dw \qquad ...(8)$$

where

$$F_{xd} (q, w, s) = - \frac{i\, Pab\, (q \cos\theta - w \sin\theta)\, m_0}{s. s/c_d (E_1 + 0)^{1/2} .N},$$

$$F_{xs} (q, w, s) = \frac{2i\ Pab\, (q \cos\theta - w \sin\theta)\, m_d\ m_s}{s. s/c_d (E_1 + 0)^{1/2}. N.},$$

$$F_{yd} (q, w, s) = - \frac{i\, Pab\, (q \sin\theta + w \cos\theta)\, m_0}{s. s/c_d (E_1 + 0)^{1/2}. N.}$$

$$F_{ys} (q, w, s) = \frac{2i\ Pab\, (q \sin\theta + w \cos\theta)\, m_d\, m_s}{s. s/c_d\ (E_1 + 0)^{1/2} . N.},$$

$$F_{zd} (q, w, s) = \frac{Pab\ m_d\, m_0}{s. s/c_d (E_1 + 0)^{1/2} .N.},$$

$$F_{zs} (q, w, s) = \frac{- 2\, Pab\, (q^2 + w^3)\, m_d}{s. s/c_d (E_1 + 0)^{1/2} .N.},$$

$$m_d = (q^2 + w^2 + 1)^{1/2}, \qquad m_s = (q^2 + w^2 + l^2)^{1/2},$$

$$m_0 = l^2 + 2\,(q^2 + w^2), \qquad N = m_0^2 - 4 m_d m_s\, (q^2 + w^2),$$

$$E_1 = (1 + q^2 D + w^2 F), \qquad D = \frac{a^2}{c_d^2} \cos^2\theta + \frac{b^2}{c_d^2} \sin^2\theta,$$

$$F = \frac{a^2}{c_d^2} \sin^2\theta + \frac{b^2}{c_d^2} \cos^2\theta,\ 0 = - 2qw \sin\theta \cos\theta\ (a^2 - b^2)/c_d^2,$$

$$l = c_d/c_s \text{ and } R^2 = x^2 + y^2. \qquad ...(9)$$

For mathematical simplicity we confine our attention to the derivation of the displacement field at any point on the xz-plane. Obviously the displacement at any point on any plane through the z-axis can then easily be visualized. Accordingly in order to obtain the displacement at any point on the xz-plane, we put $\theta = 0$ in (8) which then takes the form

$$\bar{u}_{j\alpha_1}(x, z, s) = \frac{Pab}{2\pi\mu c_d}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Re}\left[K_{j\alpha_1}(q,w)e^{-\frac{s}{c_d}(m_\alpha z - iqx)}\right] dq\,dw \quad \text{...(10)}$$

where

$$\left.\begin{aligned} K_{xd}(qw) &= -\frac{iqm_0}{E^{1/2}.N}, \quad K_{xs}(q, w) = \frac{2iqm_dm_s}{E^{1/2}.N.}, \\ K_{yd}(q, w) &= -\frac{iwm_0}{E^{1/2}.N}, \quad K_{ys}(q, w) = \frac{2iwm_dm_s}{E^{1/2}.N.}, \\ K_{zd}(q, w) &= \frac{m_dm_0}{E^{1/2}.N}, \quad K_{zs}(q, w) = -\frac{2m_d(q^2 + w^2)}{E^{1/2}.N}, \end{aligned}\right\} \quad \text{...(11)}$$

and

$$E = 1/c_d^2\,(c_d^2 + a^2q^2 + b^2w^2).$$

3. Dilatational Contribution

From (10) $\bar{u}_{zd}$ is converted to the Laplace transform of a known function by mapping $1/c_d\,(m_dz - iqx)$ into t through a contour integration in a complex q-plane.

The singularities of the integrand of $\bar{u}_{zd}$ are branch points at

$$q = S_d^{\pm} = \pm\, i\,(w^2 + 1)^{1/2}, \quad q = S_s^{\pm} = \pm\, i\,(w^2 + l^2)^{1/2},$$

$$q = S_c^{\pm} = \pm\, i\,\frac{(w^2b^2 + c_d^2)^{1/2}}{a},$$

and the poles at

$$q = S_R^{\pm} = \pm\, i(w^2 + \gamma_R^2)^{1/2}. \quad \text{...(12)}$$

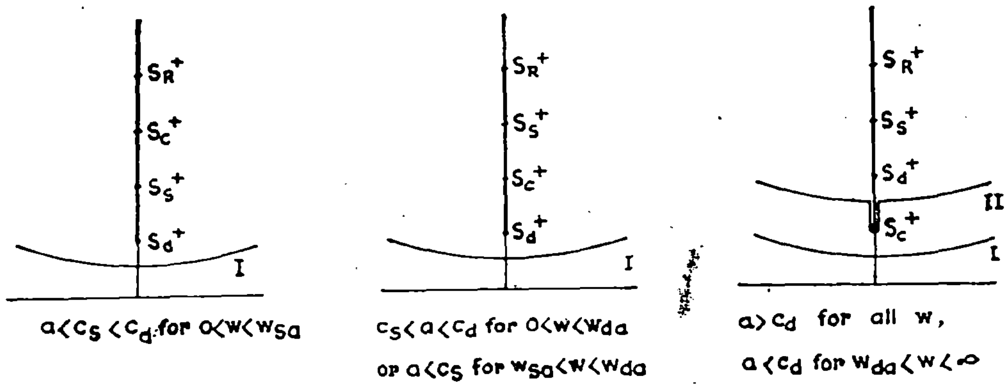
The poles at $q = S_R^{\pm}$ correspond to the zeros of the Rayleigh function N, where $\gamma_R = c_d/c_R$ and c_R is the Rayleigh surface wave speed. The contours of integration in the q-plane are shown in Fig. 2 (a, b, c) which also show the positions of singularities lying in the upper half of the q-plane.

Since the positions of the singularities and the transformed cottour of integration depend on different values of a and b, three different cases arise for the evaluation of u_{zd}.

(a) Case $a > b > C_d$.

The q-plane for $a > b > C_d$ is shown in Fig. 2 (a). The contour $q = q_d^{\pm}$ in the q-plane, is found by solving

$$t = 1/C_d\,(m_dZ - iqx) \quad \text{.. (13)}$$



(c) (b) (a)

FIG. 2. Cagniard paths of integration in the q-plane.

for q, where t is real, we get

$$q = q_d^{\pm} = i \sin \phi \pm (\tau^2 - \tau_{wd}^2)^{1/2} \cos \phi \qquad .. (14)$$

for

$$\tau > \tau_{wd}, \text{ where } \tau_{wd} = (w^2 + 1)^{1/2}, \ \tau = c_d t/\rho \qquad .. (15)$$

and (ρ, ϕ) are the polar coordinates in the xz-plane as shown in Fig. 1. Equations (14) define one branch of a hyperbola with vertex at $q = i\,(w^2 + 1)^{1/2}\ x/\rho$, which is parametrically described by the dimensionless time parameter τ as τ varies from τ_{wd} towards infinity.

As shown in Fig. 2 (a), the contour of integration has two possible configurations in the q-plane, depending upon ϕ and w.

For the case (1) given by :

Case (1) : $\phi < \phi_{da}$ and $0 < \omega < \infty$

or

$$\phi_{da} < \phi < \phi_{ba} \text{ and } w_{da} < w < \infty \qquad ...(16)$$

where $\phi_{da} = \sin^{-1} C_d/a, \ \phi_{ba} = \sin^{-1} b/a$

and

$$w_{da} = \left(\frac{C_d^2 - a^2 \sin^2 \phi}{a^2 \sin^2 \phi - b^2} \right)^{1,2} \qquad ...(17)$$

the vertex of the path $= q_d^{\pm}$ does not lie on the branch cuts and hence the path of integration contour is simply $q = q_d^{\pm}$ and is denoted by I.

But for the case (2) given by :

Case (2): $\phi_{da} < \phi < \phi_{ba}$ and $0 < w < w_{da}$

or $\phi > \phi_{ba}$ and $0 < w < \infty$ ...(18)

the vertex of the path $q = q_d^{\pm}$ lies on the branch cut between the branch points $q = S_c^+$ and $q = S_d^+$. Hence the integration contour is given by $q = q_d^{\pm}$ for $\tau > \tau_{wd}$ which is denoted by II, plus $q = q_{da} = i\,\tau\,\sin\phi - i\,(\tau_{dw}^2 - \tau^2)^{1/2}\cos\phi$ ...(19)

for $\tau_{wda} < \tau < \tau_{wd}$, where

$$\tau_{wda} = \frac{1}{a}\left[\left\{w^2(a^2 - b^2) + (a^2 - C_d^2)\right\}^{1/2}\right.$$

$$\left.\times\cos\phi + (w^2 b^2 + C_d^2)^{1/2}\sin\phi\right]. \qquad ...(20)$$

Transferring the path of integration from the real q-axis to the Cagniard's–path we obtain

$$\bar{u}_{zd}(\rho, \phi, s) = \frac{2\,Pab}{\pi\mu C_d}\left[\int_0^{\infty}\int_{t_{wd}}^{\infty}\mathrm{Re}\left[k_{zd}(q_d^+, w)\,\frac{dq_d^+}{dt}\right]e^{-st}\,dt\,dw\right.$$

$$+ H(\phi_{ba} - \phi)\,H(\phi - \phi_{da})\int_0^{w_{da}}\int_{t_{wda}}^{t_{wd}}\mathrm{Re}\left[k_{zd}(q_{da}, w)\,\frac{dq_{da}}{dt}\right]e^{-st}\,dt\,dw$$

$$\left.+ H(\phi - \phi_{ba})\int_0^{\infty}\int_{t_{wda}}^{t_{wd}}\mathrm{Re}\left[k_{zd}(q_{da}, w)\,\frac{dq_{da}}{dt}\right]e^{-st}\,dt\,dw\right] \qquad ...(21)$$

where $t_{wd} = \rho/C_d\;\tau_{wd}$ and $t_{wda} = \rho/C_d\;\tau_{wda}$. The first term of (21) is the contribution from $q_d^{\pm}$ and the second and third terms are the contributions from q_{da}.

Now interchanging the order of integration in (21) and inverting the Laplace transform, we find that

$$u_{zd}(\rho, \phi, \tau) = \frac{2\,Pab}{\pi\mu C_d}\left[H(\tau - 1)\int_0^{T_d}\mathrm{Re}\left[k_{zd}(q_d^+, w)\,\frac{dq_d^+}{dt}\right]dw\right.$$

$$+ H(\phi - \phi_{da})\,H(\phi_{ba} - \phi)\,H(\tau - \tau_{da})\,H(\tau'_{da} - \tau)$$

(*equation continued on p. 655*)

$$\times \int_{A'_{da}}^{T_{da}} \operatorname{Re}\left[k_{zd}(q_{da}, w)\frac{dq_{da}}{dt}\right] dw$$

$$+ H(\phi - \phi_{b_a})\, H(\tau - \tau_{da})$$

$$\times \int_{A^0_{da}}^{T_{da}} R_e\left[k_{zd}(q_{da}, w)\,\frac{dq_{da}}{dt}\right] dw \Bigg] \qquad \ldots(22)$$

where

$$\left.\begin{aligned} A^0_{da} &= \begin{cases} 0 & \text{for } \tau_{da} < \tau < 1 \\ T_d & \text{for } 1 < \tau < \tau'_{da} \end{cases} \\ A^0_0 &= \begin{cases} 0 & \text{for } \tau_{aa} < \tau < 1 \\ T_d & \text{for } \tau > 1 \end{cases} \end{aligned}\right\} \qquad \ldots(23)$$

$$T_d = (\tau^2 - 1)^{1/2} \qquad \ldots(24)$$

$$T_{da} = \left[\frac{X_d - \{Y_d - (a^2\cos^2\phi - b^2)\, Z_d\}^{1/2}}{(a^2\cos^2\phi - b^2)^2}\right]^{1/2} \qquad \ldots(25)$$

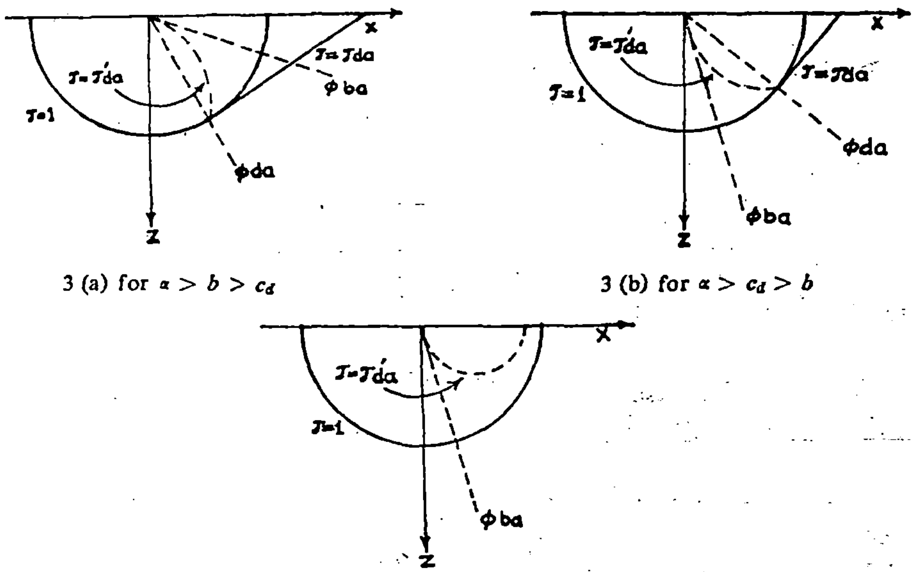
$$\left.\begin{aligned} X_d &= \tau^0_d\, b^2\sin^2\phi + (a^2 - b^2)\,\tau_d\cos^2\phi \\ Y_d &= \tau^{0^2}_d\, b^4\sin^4\phi + (a^2 - b^2)^2\,\tau^2_d\cos^4\phi + 2\,(a^2 + b^2)b^2\,\tau_d. \\ &\quad \times \tau^0_d\sin^2\phi\cos^2\phi \\ Z_d &= (\tau_d - 2C^2_d\sin^2\phi)^2 - 4C^2_d\,(a^2 - C^2_d)\sin^2\phi\cos^2\phi \\ \tau_d &= a^2\tau^2 + (C^2_d - a^2\cos^2\phi) \\ \tau^0_d &= a^2\tau^2 - (C^2_d - a^2\cos^2\phi) \end{aligned}\right\} \qquad (26)$$

$$\tau_{da} = \frac{1}{a}\left[(a^2 - C^2_d)^{1/2}\cos\phi + C_d\sin\phi\right], \qquad (27)$$

$$\tau'_{da} = \left[\frac{C^2_d - b^2}{a^2\sin^2\phi - b^2}\right]^{1/2}. \qquad (28)$$

...~~(26)~~

The first term in u_{zd} is due to the dilatational motion behind hemispherical wave front at $\tau = 1$ and the second and third terms are due to the dilational motion behind the conical wave front at $\tau = \tau_{da}$ for $\phi > \phi_{da}$. These wave fronts are shown in Fig. 3 (*a*),



3 (a) for $\alpha > b > c_d$ 3 (b) for $\alpha > c_d > b$

3 (c) for $\alpha < c_d$

FIG. 3. Wave patten for dilatational motion.

$\tau = \tau'_{da}$ shown in Fig 3 (a) by a dashed curve, is not a wave front because it is not a characteristic surface for governing wave equation for the dilatational motion. Similar non characteristic surfaces were found by Gakenheimer and Miklowitz[4] for a point load travelling on an elastic half space and also by Aggarwal and Ablow[1] for the motion of an acoustic half-space due to an expanding surface load. They prove explicitly that their solution was analytic over the surfaces. The same thing can be proved in our case also.

(b) *Case $a > c_d > b$*

In this case, the path of integration with respect to q transforms to the simple path given by contour I (Fig. 2 (a)) for all w when $\phi < \phi_{ba}$ and also for $0 < w < w_{da}$ when $\phi_{ba} < \phi < \phi_{da}$, whereas the path of integration with respect to q transform to the contour II (Fig. 2 (a)) for $w_{da} < w < \infty$ when $\phi_{ba} < \phi < \phi_{da}$ and also for all w when $\phi > \phi_{da}$. The remaining details of inverting $\bar{u}_{zd}$ for $a > c_d > b$ are exactly the same as for $a > b > c_d$, and one can easily find that

$$u_{zd}(\rho, \phi, \tau) = \frac{2\,Pab}{\pi\mu\, c_d}\left[H(\tau - 1)\int_0^{\tau_d} R_e\left[k_{zd}(q_d^+, w)\frac{dq_d^+}{dt}\right]dw\right.$$

(*equation continued on p. 657*)

$$+ H(\phi - \phi_{ba}) H(\phi_{da} - \phi) H(\tau - \tau'_{da})$$

$$\times \int_{T_d}^{T_{da}} \mathrm{Re}\left[k_{zd}(q_{da}, w) \frac{dq_{da}}{dt}\right] dw$$

$$+ H(\phi - \phi_{aa}) H(\tau - \tau_{da})$$

$$\times \int_{A^0_{da}}^{T_{da}} \mathrm{Re}\left[k_{zd}(q_{da}, w) \frac{dq_{da}}{dt}\right] dw \Bigg] \quad \ldots(27)$$

where A^0_{da} is given by (23).

The wave geometry associated with this expression is shown in Fig. 3 (b).

(c) *Case $a < c_d$*

For this case the path of integration with respect to q transform to the simple path given by contour I [Figs. 2(b), 2 (c)] for all w when $\phi < \phi_{ba}$ and also for $0 < w < w_{da}$ when $\phi > \phi_{ba}$, whereas the path of integration with respect to q transforms to the contour II [Fig. 2 (a)] for $w_{da} < w < \infty$ when $\phi > \phi_{ba}$. Note that in this case the angle ϕ_{da} does not arise. Now proceeding as the case $a > b > c_d$ for inverting $\bar{u}_{zd}$ we get

$$u_{zd}(\rho, \phi, \tau) = \frac{2\,Pab}{\pi\mu c_d}\Bigg[H(\tau - 1)\int_0^{T_d} R_e\left[k_{zd}(q_d^+, w)\frac{dq_d^+}{dt}\right] dw$$

$$+ H(\phi - \phi_{ba}) H(\tau - \tau'_{da})$$

$$\times \int_{T_d}^{T_{da}} R_e\left[k_{zd}(q_{da}, w)\frac{dq_{da}}{dt}\right] dw\Bigg]. \quad \ldots(30)$$

The wave geometry associated with this expression is shown in Fig. 3 (c). As expected physically, contribution due to the conical wave front does not exist for this case.

Summary

Combining (22), (29) and (30) one finds that u_{zd} can be written as one expression for all values of a and b.

$$u_{zd}(\rho, \phi, \tau) = \frac{2\,Pab}{\pi\mu c_d}\Bigg[H(\tau - 1)\int_0^{T_d} R_e\left[k_{zd}(q_d^+, w)\,\frac{dq_d^+}{dt}\right]dw$$

$$+ [H(\tau - \tau_{da})\,H(\phi - \phi_{ba})\,\{H(b - c_d)$$

$$+ H(a-c_d)\,H(c_d-b)\} + H(\tau - \tau'_{da})\,H(\phi-\phi_{ba})\,\{H(a-c_d)$$

$$\times H(c_d - b)\,H(\phi_{da} - \phi) + H(c_d - a)\}]$$

$$\times \int_{A_{da}}^{T_{da}} \mathrm{Re}\left[k_{zd}(q_{da}, w)\,\frac{dq_{da}}{dt}\right]dw\Bigg] \qquad \ldots(31)$$

where

$$A_{da} = \begin{cases} \left.\begin{array}{l} 0 \text{ for } \tau_{da} < \tau < 1 \\ T_d \text{ for } 1 < \tau < \tau'_{da} \\ T_{da} \text{ for } \tau > \tau'_{da} \end{array}\right\} \text{for } \phi_{da} < \phi < \phi_{ba},\ a > b > c_d \\ \left.\begin{array}{l} 0 \text{ for } \tau_{da} < \tau < 1 \\ T_d \text{ for } 1 < \tau \end{array}\right\} \begin{array}{l} \text{for } \phi > \phi_{ba},\ a > b > c_d \\ \text{for } \phi > \phi_{da},\ a > c_d > b \end{array} \\ T_d \text{ for } \tau > \tau'_{da} \quad \left\{\begin{array}{l} \text{for } \phi_{ba} < \phi < \phi_{da},\ a > c_d > b \\ \text{for } \phi > \phi_{ba},\ a < c_d. \end{array}\right. \end{cases} \qquad \ldots(32)$$

4. Equivoluminal Contributions

Inversion of $\bar{u}_{zs}$ is complicated than the inversion of $\bar{u}_{zd}$ because of the appearence of head waves (Von-Schmidt waves) otherwise it is same as $\bar{u}_{zd}$. Here the integration contour has more configurations in the q-plane though the singularities are the same. Here the hyperbola $q = q_s^{\pm}$ arises in a similar way to $q = q_d^{\pm}$, but its vertex can lie on the branch cut between the branch points at $q = S_d^+$ and $q = S_s^+$ and at $q = S_c^+$ and $q = S_s^+$ as well as between $q = S_c^+$ and $q = S_d^+$, depending on the values of w, ϕ, a and b. In this case, the straight line contour lying along the imaginary q-axis is denoted by q_{sa} which is similar to q_{da} appearing in the dilatational contributions. Now omiting details of inverting $\bar{u}_{zs}$, one can easily find

$$u_{zs}(\rho, \phi, \tau) = \frac{4\,Pab}{\pi\mu c_d}\left[H(\tau - l)\int_0^{T_s} R_e\left[k_{zs}(q_s^+, w)\,\frac{dq_s^+}{dt}\right]dw\right.$$

(*equation continued on p.* 659)

$$+ [H(\tau-\tau_{sa}) H(\phi-\phi_{sa}) \{H(b-c_s)+H(c_s-b) H(a-c_s)\}$$

$$+ H(\tau - \tau'_{sa}) H(\phi - \phi_{ba}) \{H(c_s - b) H(\phi_{sa} - \phi)$$

$$\times H(a - c_s) + H(c_s - a)\}]$$

$$\times \int_{A_{sa}}^{T_{sa}} R_e \left[k_{z_s}(q_{sa}, w) \frac{dq_{sa}}{dt} \right] dw$$

$$+ H(\tau - \tau_{sd}) H(\tau'_{sd} - \tau) H(\phi - \phi_{sd})$$

$$\times \int_{A_{sd}}^{T_{sd}} R_e \left[k_{z_s}(q_{sa}, w) \frac{dq_{sa}}{dt} \right] dw \Big] \qquad \ldots(33)$$

for $0 \leqslant \rho < \infty, 0 \leqslant \phi < \pi/2,$

$0 \leqslant \tau < \infty, 0 \leqslant a < \infty$ and

$0 \leqslant b < \infty, a > b$

where

$$\begin{cases}
= 0 \text{ for } \tau_{sa} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sa}
\end{cases} \quad \begin{cases} \phi_{sa} < \phi < \phi_{ba}, a > c_d, a>b>c_s, ac_s>bc_d \\ \phi_{sa} < \phi<\phi_{sd}, a>c_d, a>b>c_s, ac_s<bc_d \\ \phi_{sa}<\phi<\phi_{abs}, c_d>a>b>c_s \end{cases}$$

$$\begin{cases}
= 0 \text{ for } \tau_{sa} < \tau < l \\
= T_s \text{ for } \tau > l
\end{cases} \quad \begin{cases} \phi_{ba} < \phi < \phi_{sd}, a>b>c_d, ac_s>bc_d \\ \phi_{sa}<\phi<\phi_{sd}, a>c_d>c_s>b \end{cases}$$

$$\begin{cases}
= 0 \text{ for } \tau_{sa} \leqslant \tau<\tau_{sd} \\
= T_{sd} \text{ for } \tau_{sd} < \tau< \tau'_{sd} \\
= T_s \text{ for } \tau > \tau'_{sd}
\end{cases} \quad \begin{cases} \phi >\phi_{sd}, a>b>c_d, ac_s>bc_a \\ \phi >\phi_{sd}, a > c_d > c_s >b \end{cases}$$

$$\begin{cases}
= 0 \text{ for } \tau_{sa} < \tau < \tau_{sd} \\
= T_{sd} \text{ for } \tau_{sd} < \tau<\tau'_{sd} \\
= T_s \text{ for } \tau'_{sd} <\tau < \tau'_{sa}
\end{cases} \quad \phi > \phi_{sd}, a > b > c_d, ac_s < bc_d$$

$$= T_s \text{ for } \tau > \tau'_{sa} \quad \begin{cases} \phi_{ba} <\phi < \phi_{sa}, a > c_d > c_s > b \\ \phi_{ba} < \phi < \phi_{abs}, c_d > a > c_s > b \\ \phi_{ba} < \phi < \phi_{abs}, a < c_s \end{cases}$$

$$
A_{sa}\left\{
\begin{array}{l}
\left.\begin{array}{l}
= T_s \text{ for } \tau'_{sa} < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau > \tau'_{sd}
\end{array}\right\}
\begin{array}{l}
\phi_{abs} < \phi < \phi_{sa},\ c_d > a > c_s > b \\
\phi > \phi_{abs},\ a < c_s
\end{array} \\[2ex]
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sa} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau > \tau'_{sd}
\end{array}\right\}
\begin{array}{l}
\phi > \phi_{sa},\ c_d > a > c_s > b,\ \alpha > \beta \\
\phi_{sa} < \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\
\phi > \phi_{ba},\ c_d > a > b > c_s,\ \alpha > \beta \\
\phi_{ba} < \phi < \phi_x,\ c_d > a > b > c_s\ \beta > \alpha > \gamma
\end{array} \\[2ex]
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau > \tau'_{sd}
\end{array}\right\}
\begin{array}{l}
\phi > \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\
\phi > \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\
\phi > \phi_{ba},\ c_d > a > b > c_s,\ \alpha < \gamma
\end{array} \\[2ex]
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sa} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa}
\end{array}\right\}
\begin{array}{l}
\phi_{abs} < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \alpha > \beta \\
\phi_{abs} < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\
\phi_{abs} < \phi < \phi_x,\ c_d > a > b > c_s,\ \alpha < \gamma
\end{array} \\[2ex]
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa}
\end{array}\right\}
\phi_x < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \alpha < \gamma.
\end{array}\right.
$$

$$
\left\{
\begin{array}{l}
= 0 \text{ for } \tau_{sd} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sd}
\end{array}\right.
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\end{array}}\right\}
\begin{array}{l}
\phi > \phi_{sd},\ a > b > c_d \\
\phi > \phi_{sd},\ a > c_d > c_s > b \\
\phi_{sd} < \phi < \phi_{abs},\ c_d > a > c_s > b \\
\phi_{sd} < \phi < \phi_{sa},\ c_d > a > b > c_s \\
\phi_{sd} < \phi < \phi_{abs},\ a < c_s
\end{array}
$$

$$
A_{sd}\left\{
\begin{array}{ll}
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sd} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sa} \\
= T_{sa} \text{ for } \tau'_{sa} < \tau < \tau'_{sda} \\
= T_s \text{ for } \tau'_{sda} < \tau < \tau'_{sd}
\end{array}\right\} &
\begin{array}{l}
\phi_{ab_s} < \phi < \phi_{sa},\ c_d > a > c_s > b \\
\phi > \phi_{ab_s},\ a < c_s
\end{array} \\
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sd} < \tau < \tau_{sa} \\
= T_{sa} \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\
= T_s \text{ for } \tau'_{sda} < \tau < \tau'_{sd}
\end{array}\right\} &
\begin{array}{l}
\phi > \phi_{sa},\ c_d > a > c_s > b,\ \alpha > \beta \\
\phi_{sa} < \phi < \phi_x,\ c_d > a > c_s > b, \beta > \alpha > \gamma' \\
\phi > \phi_{ab_s},\ c_d > a > b > c_s,\ \alpha > \beta \\
\phi_{ab_s} < \phi < \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\
\phi_{ab_s} < \phi < \phi_x,\ c_d > a > b > c_s,\ \alpha < \gamma
\end{array} \\
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sd} < \tau < \tau_{sa} \\
= T_{sa} \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\
= 0 \text{ for } \tau'_{sda} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sd}
\end{array}\right\} &
\begin{array}{l}
\phi > \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\
\phi > \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\
\phi > \phi_x,\ c_d > a > b > c_s,\ \alpha < \gamma
\end{array} \\
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sd} < \tau < \tau_{sa} \\
= T_{sa} \text{ for } \tau_{sa} < \tau < \tau'_{sa} \\
= T_s \text{ for } \tau'_{sa} < \tau < \tau'_{sd}
\end{array}\right\} &
\phi_{sa} < \phi < \phi_{ab_s},\ c_d > a > b > c_s
\end{array}\right. \qquad \ldots(35)
$$

and also where

$$T_s = (\tau^2 - l^2)^{1/2} \qquad \ldots(36)$$

$$T_{sa} = \left[\frac{X_s - \{Y_s - (a^2 \cos^2 \phi - b^2)^2 Z_s\}^{1/2}}{(a^2 \cos^2 \phi - b^2)^2} \right]^{1/2} \qquad \ldots(37)$$

$$
\left.\begin{array}{l}
X_s = \tau_s^0\, b^2 \sin^2 \phi + (a^2 - b^2)\, \tau_s \cos^2 \phi \\
Y_s = \tau_s^{0^2}\, b^4 \sin^4 \phi + (a^2 - b^2)^2 \tau_s^2 \cos^4 \phi \\
\qquad + 2\,(a^2 - b^2) b^2\, \tau_s\, \tau_s^0 \sin^2 \phi \cos^2 \phi \\
Z_s = (\tau_s - 2 c_d^2 \sin^2 \phi)^2 - 4\, l^2\, c_d^2\, (a^2 - c_s^2) \sin^2 \phi \cos^2 \phi \\
\tau_s = a^2 \tau^2 + l^2\, (c_s^2 - a^2 \cos^2 \phi) \\
\tau_s^0 = a^2 \tau^2 - l^2\, (c_s^2 - a^2 \cos^2 \phi)
\end{array}\right\} \qquad \ldots(38)
$$

$$T_{sd} = [\{(\tau - \tau_{sd}) \operatorname{cosec} \phi + 1\}^2 - 1]^{1/2} \qquad ...(39)$$

$$\tau_{sa} = 1/a\,[(l\,a^2 - c_s^2)^{1/2} \cos \phi + c_d \sin \phi] \qquad ...(40)$$

$$\tau_{sd} = [(l^2 - 1)^{1/2} \cos \phi + \sin \phi] \qquad ...(41)$$

$$\tau'_{sa} = \left[\frac{l^2 (b^2 - c_s^2)}{b^2 - a^2 \sin^2\phi} \right]^{1/2} \qquad ...(42)$$

$$\tau'_{sd} = (l^2 - 1)^{1/2} \sec \phi \qquad ...(43)$$

$$\tau'_{sda} = \left[(l^2 - 1)^{1/2} \cos \phi + \left(\frac{c_d^2 - b^2}{a^2 - b^2} \right)^{1/2} \sin \phi \right] \qquad ...(44)$$

$$\phi_{sa} = \sin^{-1} c_s/a,\ \phi_{sd} = \sin^{-1} c_s/c_d,\ \phi_{ba} = \sin^{-1} b/a \qquad ...(45)$$

$$\phi_{abs} = \sin^{-1} \left(\frac{c_d^2 - b^2}{l^2 (a^2 - b^2) + c_d^2 - a^2} \right)^{1/2} \qquad ...(46)$$

$$\phi_x = \sin^{-1} \left[\frac{(a^2 - b^2)^{1/2} \left[l (c_d^2 - b^2)^{1/2} + (l^2 - 1)^{1/2} (c_d^2 - a^2)^{1/2} \right]}{l^2 (a^2 - b^2) + c_d^2 - a^2} \right] \qquad ...(47)$$

$$\alpha = \left(\frac{c_d^2 - a^2}{a^2 - b^2} \right)^{1/2},\ \beta = (l^2 - 1)^{1/2},\ \gamma = b/a\,(l^2 - 1)^{1/2}$$

$$- \frac{1}{a} (c_d^2 - b^2)^{1/2},\ \gamma' = \frac{c_s}{a} (l^2 - 1)^{1/2}$$

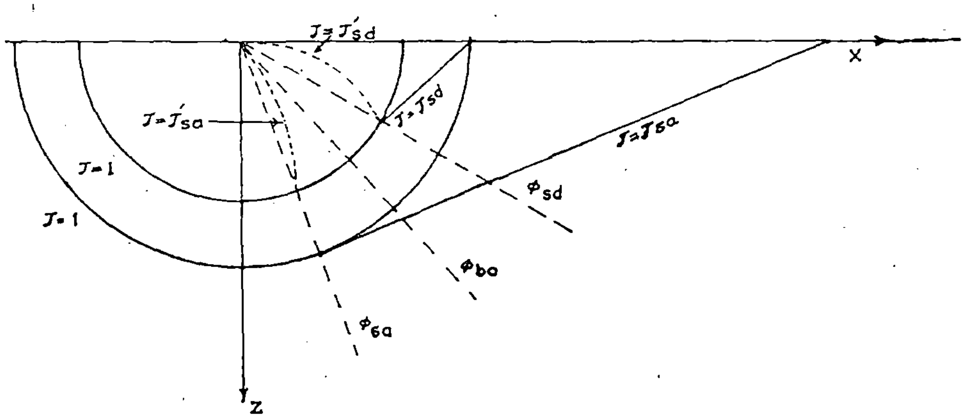
$$- 1/a \left[\frac{a^2 - c_s^2}{a^2 - b^2} (c_d^2 - b^2) \right]^{1/2} \qquad ...(48)$$

$$q_s^{\pm} = i\,\tau \sin \phi \pm (\tau^2 - \tau_{ws}^2)^{1/2} \cos \phi \qquad ...(49)$$

$$\tau_{ws} = (w^2 + l^2)^{1/2} \qquad ...(50)$$

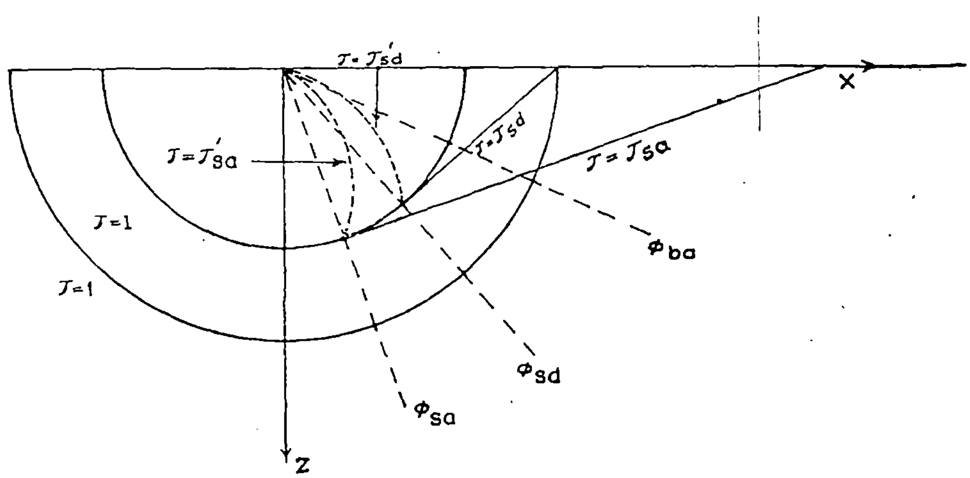
$$q_{sa} = i\,\tau \sin \phi - i\,(\tau_{ws}^2 - \tau^2)^{1/2} \cos \phi. \qquad ...(51)$$

The first term in the expression (33) is the equivoluminal motion behind the hemispherical wave front at $\tau = l$ and the second is due to the equivoluminal motion behind the conical wave front at $\tau = \tau_{sa}$. The third term in u_{rs} represents the equivoluminal motion due to the head wave fronts at $\tau = \tau_{sd}$. The wave fronts $\tau = \tau_{sd}$ for $\phi > \phi_{sd}$ and $\tau = \tau_{sa}$ are shown in Figs. 4(a—l).



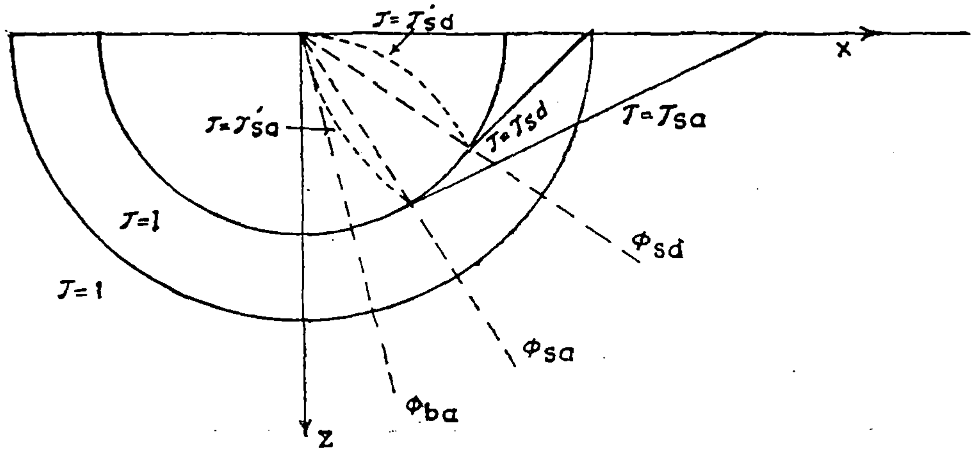
FIGS. 4(a—l). Wave pattern for equivoluminal and head wave motion.

4 (a) for $a > c_d$, $a > b > c_s$, $a\,c_s > b\,c_d$.

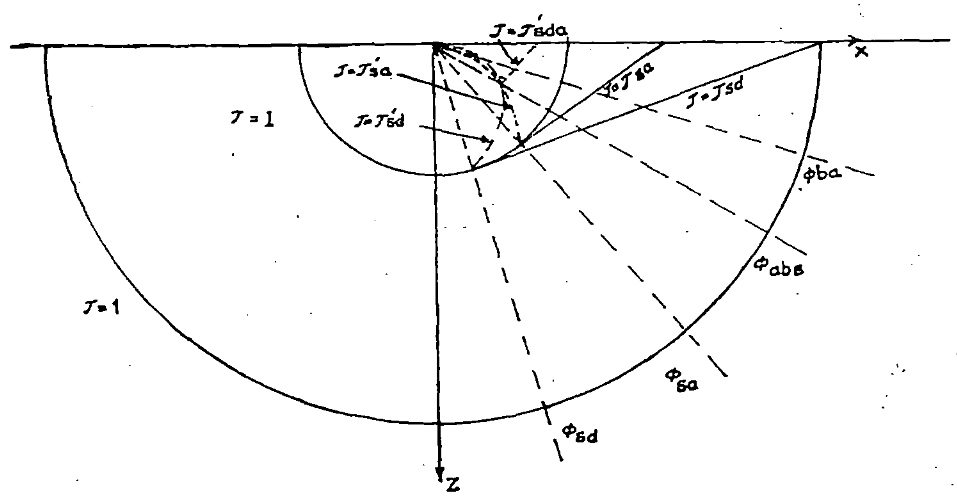


4 (b) for $a > c_d$, $a > b > c_s$, $a\,c_s < b c_d$.

The equations $\tau = \tau'_{sa}$, $\tau = \tau'_{sd}$ and $\tau = \tau'_{sda}$ are shown in Fig. 4 by dashed curve which are similar to $\tau = \tau'_{da}$ appearing in the u_{zd}. These dashed curved surfaces are not considered as wave fronts because it can be shown that displacements and their derivatives are continuous across these surfaces.



4 (c) for $a > c_d > c_s > b$.



4 (d) for $c_d > a > b > c_s$, $\alpha > \beta$.

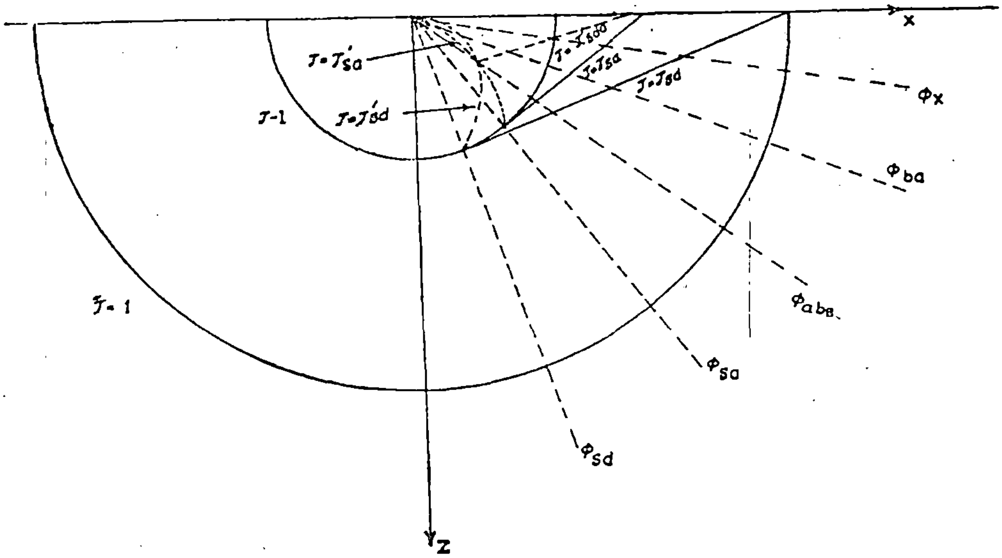
5. Wave Front Expansions

The wave forms of the solution given in (31) and (33) are evaluted by approximate estimation of the integrals in the neighbourhood of the first arrival of the different waves. To facilitate this evaluation we put

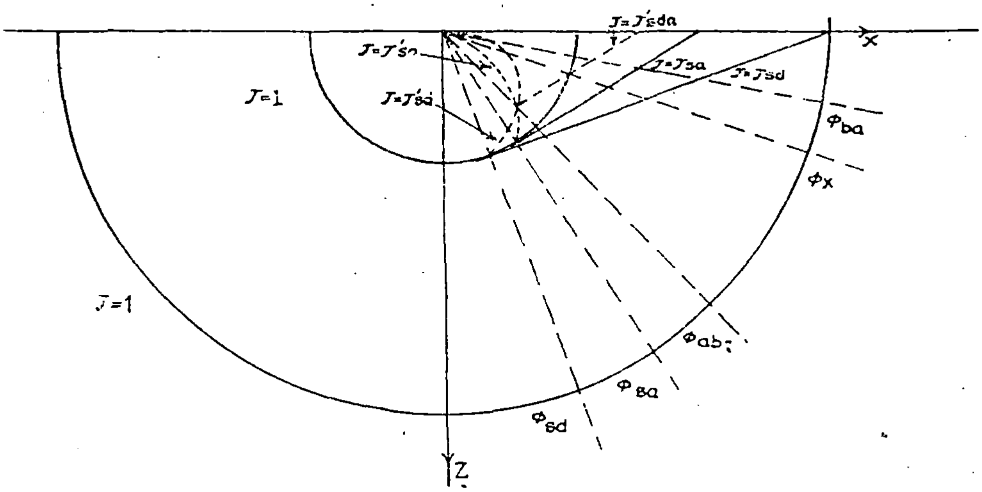
$$w = [A^2 + (B^2 - A^2) \sin^2 \alpha]^{1/2} \qquad \ldots(52)$$

in the integrals arising in u_{zd} and u_{zs} where A and B are respectively the lower and upper limits of the particular integral in question, and the range of integration with respect to α is form 0 to $\pi/2$.

Now for the first integral of (31), we put $w = T_d \sin \alpha$ and hence for $\tau \to 1+$, we find that for any value of a.

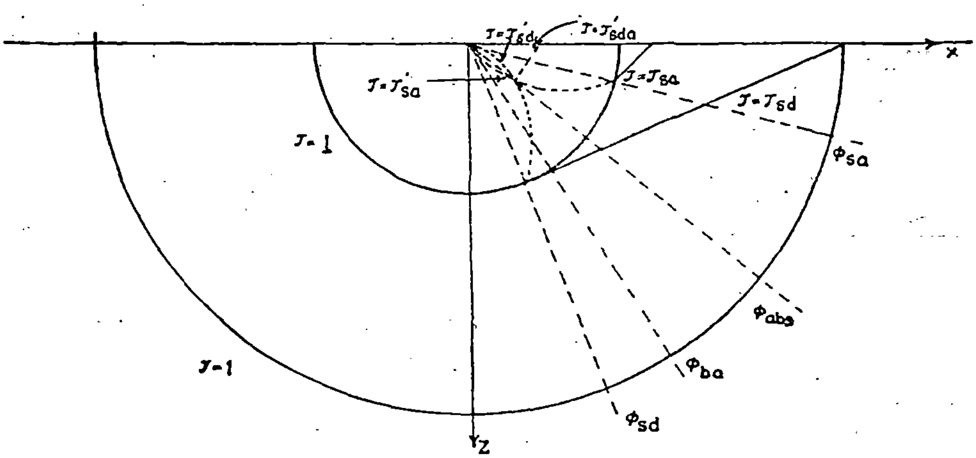


4 (e) for $c_d > a > b \geqslant c_s$, $\beta > \alpha > \gamma$.

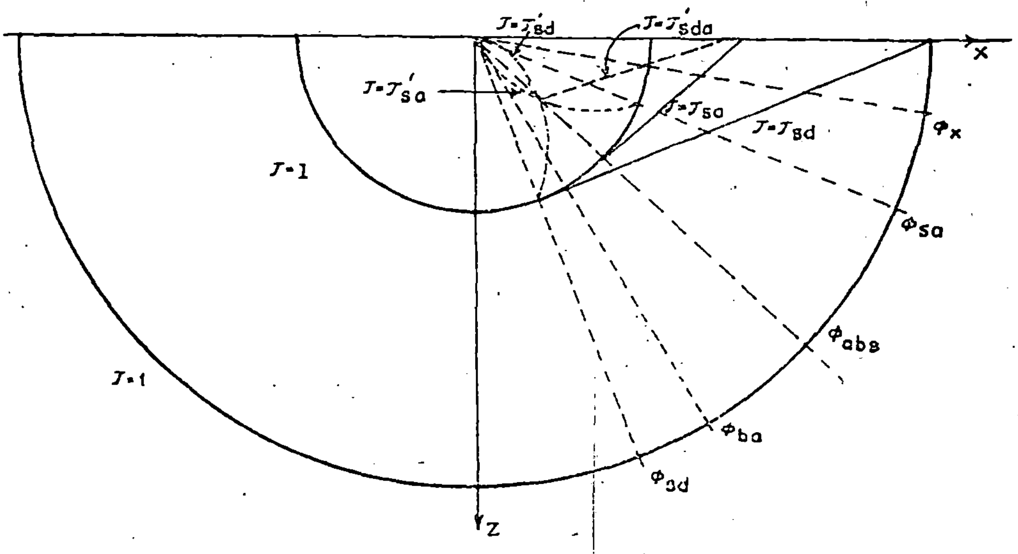


4 (f) for $c_d > a > b > c_s$, $\alpha < \gamma$.

$$\left.\begin{aligned}
&w \to 0,\ q_d^+ \to i \sin \phi,\ \frac{dq_d^+}{dt} \to \frac{c_d}{\rho_s} \frac{\cos \phi}{T_d \cos \alpha},\\
&m_d \to \cos \phi,\ m_s \to (l^2 - \sin^2 \phi)^{1/2},\ m_0 \to (l^2 - 2\sin^2 \phi),\\
&E^{1/2} \to \frac{1}{c_d} (c_d^2 - a^2 \sin^2 \phi)^{1/2},\ \text{for } \phi < \phi_{da}\\
&\qquad \to \frac{i}{c_d} (a^2 \sin^2 \phi - c_d^2)^{1/2},\ \text{for } \phi > \phi_{da},\\
&N \to N_1
\end{aligned}\right\} \qquad \ldots(53)$$



4 (g) for $c_d > a > c_s > b$, $\alpha > \beta$, $ac_s < bc_d$.



4 (h) for $c_d > a > c_s > b$, $\beta > \alpha > \gamma'$, $ac_s < bc_d$.

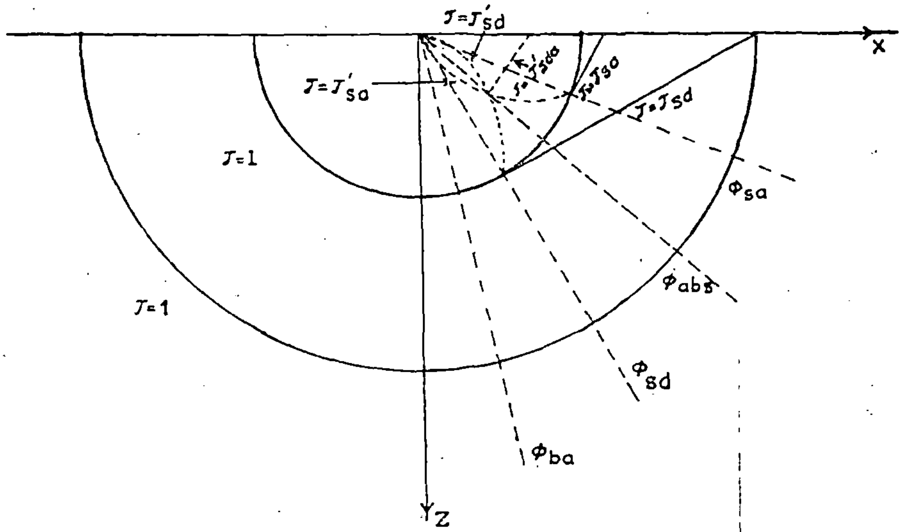
where $N_1 = (l^2 - 2\sin^2\phi)^2 + 4\sin^2\phi\cos\phi\,(l^2 - \sin^2\phi)^{1/2}$. ...(54)

Substituting these approximate values in the first integral of (31) one can find, for $\phi < \phi_{da}$

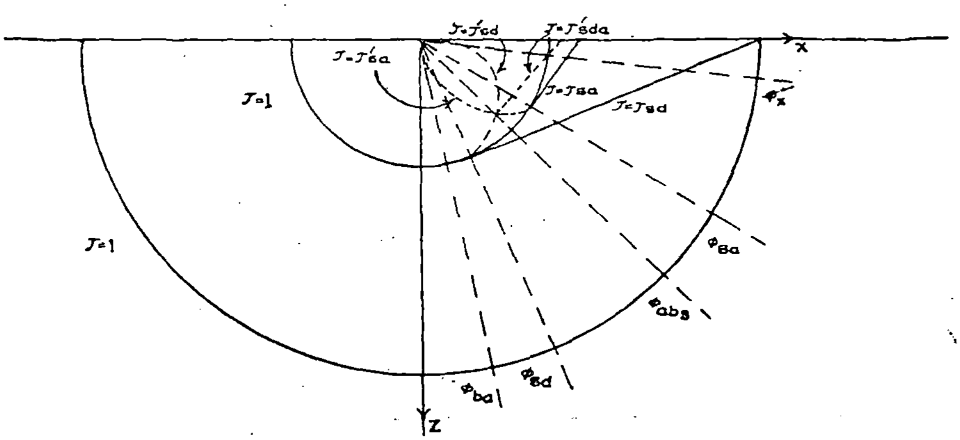
$$[u_z] \rightarrow N_{z1} \text{ as } \tau \rightarrow 1+ \qquad \text{...(55)}$$

where

$$N_{z1} = \frac{Pabc_d}{\mu\rho}\frac{\cos^2\phi\,(l^2 - 2\sin^2\phi)}{(c_d^2 - a^2\sin\phi)^{1/2}.\,N_1}. \qquad \text{..(56)}$$



4 (i) for $c_d > a > c_s > b,\ \alpha > \beta,\ a\,c_s > b\,c_d$.



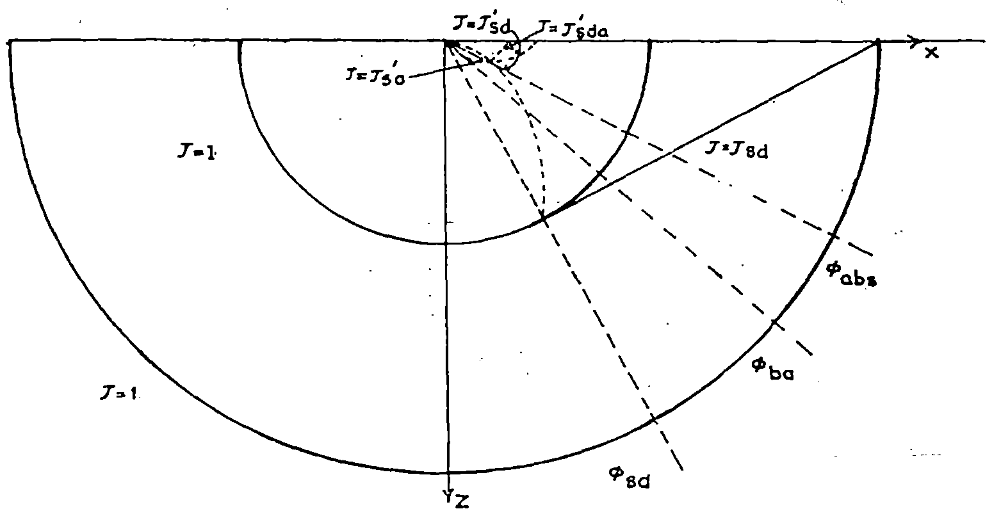
4 (j) for $c_d > a > c_s > b,\ \beta > \alpha > \gamma',\ ac_t > bc_d$.

Again in tne second integral of (31) we put $w = T_{da} \sin\alpha$ and as $\tau \to 1-$ for $\phi > \phi_{da}$ we find that

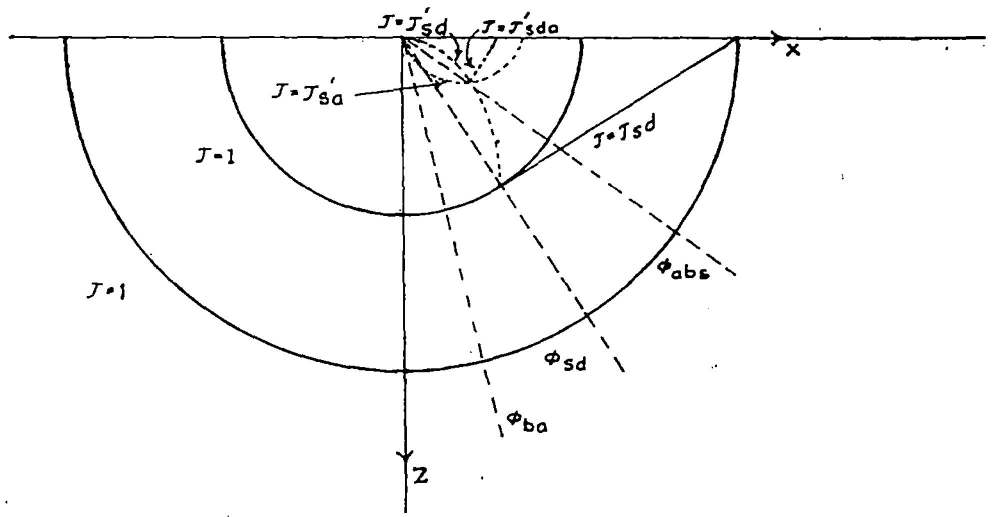
$$q_{da} \to i \sin\phi - i \cos\phi\ T_{da} \sin\alpha$$

$$\frac{dq_{da}}{dt} \to \frac{ic_d}{\rho} \cdot \frac{T_{da} \sin\alpha \sin\phi + \cos\phi}{(T_{da}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} . \qquad \text{.. (57)}$$

Puting these values in the second integral of (31), we get



4 (k) for $a < c_e$, $ac_s < bc_d$.



4 (l) for $a < c_s$, $ac_s > b\,c_d$.

$$\int_0^{\pi/2} R_e \left[k_{zd} \left(i \sin\phi - i \cos\phi\, T_{da} \sin\alpha,\ T_{da} \sin\alpha\right) \frac{ic_d}{\rho}, \right.$$

$$\left. \times \frac{T_{da} \sin\alpha \sin\phi + \cos\phi}{(T_{da}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{da} \cos\alpha\, d\alpha \qquad \ldots(58)$$

$$= \int_0^{\epsilon} R_e \left[k_{zd} \left(i \sin\phi - i \cos\phi\, T_{da} \sin\alpha, T_{da} \sin\alpha\right) \frac{ic_d}{\rho} \right.$$

(equation continued on p. 669)

$$\times \frac{T_{da} \sin \alpha \sin \phi + \cos \phi}{(T_{da}^2 \sin^2 \alpha + 1 - \tau^2)^{1/2}} \Bigg] T_{da} \cos \alpha \, d\alpha$$

$$+ \int_{\epsilon}^{\pi/2} R_e \Bigg[k_{zd} (i \sin \phi - i \cos \phi \, T_{da} \sin \alpha, T_{da} \sin \alpha) \frac{ic_d}{\rho}$$

$$\times \frac{T_{da} \sin \alpha \sin \phi + \cos \phi}{(T_{da}^2 \sin^2 \alpha + 1 - \tau^2)^{1/2}} \Bigg] T_{da} \cos \alpha \, d\alpha \qquad \text{.. (59)}$$

where ϵ is very small.

Since the main contribution to the integral (58) as $\tau \to 1$ arises from the first integral of (59) as $\tau \to 1$, so for the evaluation of (58) as $\tau \to 1$, we consider the approximate value of the integral given by

$$\int_0^{\epsilon} R_e \Bigg[k_{zd} (i \sin \phi - i \cos \phi \, T_{da} \sin \alpha, T_{da} \sin \alpha) \frac{ic_d}{\rho}$$

$$\times \frac{T_{da} \sin \alpha \sin \phi + \cos \phi}{(T_{da}^2 \sin^2 \alpha + 1 - \tau^2)} \Bigg] T_{da} \cos \alpha \, d\alpha \qquad \text{...(60)}$$

as $\tau \to 1$.

Since ϵ is very small so α is also small. So for the evaluation of the integral (60) as $\tau \to 1$ we also use the fact that $\alpha \to 0$, from which we get,

$$w \to 0,\ q_{da} \to i \sin \phi,\ m_d \to \cos \phi,\ m_s \to (l^2 - \sin^2 \phi)^{1/2},$$

$$m_0 \to (l^2 - 2 \sin^2 \phi), \qquad \text{...(61)}$$

$$N \to N_1,\ E^{1/2} \to i/c_d \, (a^2 \sin^2 \phi - c_d^2)^{1/2} \ \text{for}\ \phi > \phi_{da}.$$

Now substituting these approximate values in (60) and integrating we obtain the approximate value of the integral as

$$- \frac{c_d^2 \cos^2 \phi \, (l^2 - 2 \sin^2 \phi)}{\rho \ (a^2 \sin^2 \phi - c_d^2)^{1/2} . N_1} \log | \tau - 1 | \ \text{when}\ \tau \to 1. \qquad \text{...(62)}$$

So for $\phi > \phi_{da}$

$$[u_z] \to N'_{z4} \log | \tau - 1 | \ \text{as}\ \tau \to 1 \qquad \text{...(63)}$$

where

$$N'_{z4} = -\frac{2Pabc_d}{\pi\mu\rho}\frac{\cos^2\phi\,(l^2 - 2\sin^2\phi)}{(a^2\sin^2\phi - c_d^2)^{1/2}.\,N_1}. \qquad ...(64)$$

In order to obtain the value of u_{zd} as $\tau \to \tau_{da}$ we put in the second integral of (31).

$$w^2 = A_{da}^2 + (T_{da}^2 - A_{da}^2)\sin^2\alpha.$$

When $\tau \to \tau_{da}+$, we find that

$$\left.\begin{aligned}
& w \to 0 \\
& q_{da} \to i\frac{c_d}{a} \\
& dq_{da}/dt \to iA' \\
& \text{where } A' = \frac{c_d}{\rho a}\left(\frac{a^2 - c_d^2}{1-\tau_{da}^2}\right)^{1/2} \text{ for } a > c_d, \\
& m_d \to 1/a\,(a^2 - c_d^2)^{1/2} \text{ for } a > c_d, \\
& m_s \to \frac{l}{a}(a^2 - c_s^2)^{1/2},\quad m_0 \to \frac{l^2}{a^2}(a^2 - 2c_s^2), \\
& N \to N_2
\end{aligned}\right\} \qquad ...(65)$$

$$\text{where } N_2 = l/a^4\left[l^4(a^2 - 2c_s^2)^2 + 4\,l\,c_d^2\,(a^2 - c_d^2)^{1/2}\right]$$

$$E^{1/2} \to iK^{1/2}(\tau - \tau_{da})^{1/2}$$

where

$$K = \frac{2a}{c_d}\frac{\cos^2\alpha\,(a^2 - c_d^2)^{1/2}}{\left\{(a^2 - c_d^2)^{1/2}\sin\phi - c_d\cos\phi\right\}} \quad \text{for } a > c_d.$$

Using these approximate values in the second integral of (31) we find that for $a > c_d$

$$[u_z] \to N_{z4} \text{ as } \tau \to \tau_{da}+ \qquad ...(66)$$

where

$$N_{z4} = \frac{2Pab}{\pi\mu c_d a^3}\frac{l^2(a^2 - c_d^2)^{1/2}(a^2 - 2c_s^2)\,A'\,C^{1/2}}{(2KA)^{1/2}.\,N_2} \qquad ...(67)$$

where $C = 8a^2 c_d \tau_{da} (a^2 - c_d^2)^{1/2} \sin\phi \cos\phi$

$$A = a^2 (a^2 - b^2) \cos^2\phi \, \tau_{da} (\tau_{da} + \tau^0_{da}) + a^2 b^2 \sin^2\phi \, \tau_{da} (\tau_{da} - \tau^0_{da})$$

$$\tau^0_{da} = 1/a \left[c_d \sin\phi - (a^2 - c_d^2)^{1/2} \cos\phi \right]. \qquad \ldots(68)$$

It may be noted that conical wave front $\tau = \tau_{da}$ does not arise for $a < c_d$.

Next when $\phi < \phi_{sa}$, for the evaluation of u_{zs} as $\tau \to l$, we put $w = T_s \sin\alpha$ in the first integral of (33). When $\tau \to l$, we find that in the above integral

$$w \to 0$$

$$q^+_s \to il \sin\phi$$

$$\frac{dq^+_s}{dt} \to \frac{c_d}{\rho} \frac{l \cos\phi}{T_s \cos\alpha}$$

$$(q^2 + w^2) \to l^2 \sin^2\phi$$

$$m_d \to (1 - l^2 \sin^2\phi)^{1/2}$$

$$m_s \to l \cos\phi$$

$$m_0 \to l^2 (\cos^2\phi - \sin^2\phi)$$

$$E^{1/2} \to 1/c_s \, (c_s^2 - a^2 \sin^2\phi)^{1/2} \text{ for } \phi < \phi_{sa}$$

$$\to i/c_s \, (a^2 \sin^2\phi - c_s^2)^{1/2} \text{ for } \phi > \phi_{sa}$$

$$N \to l^3 N_3$$

where $N_3 = [l (\cos^2\phi - \sin^2\phi)^2 + 4 \sin^2\phi \cos\phi \, (1 - l^2 \sin^2\phi)^{1/2}]$.

Using these approximate values in the first integral of (33) one can find for all values of a and b.

$$[u_z] \to N_{z_2} \text{ for } \phi < \phi_{sa} \text{ as } \tau \to l$$

where

$$N_{z_2} = -\frac{2Pabc_s}{\mu\rho} \frac{\sin^2\phi \cos\phi \, (1 - l^2 \sin^2\phi)^{1/2}}{\left(c_s^2 - a^2 \sin^2\phi\right)^{1/2} . N_3}. \qquad \ldots(71)$$

For $\phi > \phi_{sa}$, considering approximate evalution of last two integrals of (33) as $\tau \to l$ it can be shown that for the case $a > b > c_d$

$$u_z \to N'_{z5} \log|\tau - l| \text{ for } \phi_{sa} < \phi < \phi_{sd} \text{ as } \tau \to l \quad \ldots(72)$$

$$u_z \to N'_{z3} \log|\tau - l| \text{ for } \phi > \phi_{sd} \text{ as } \tau \to l \quad \ldots(73)$$

and for the case $c_d > a > b > c_s$,

$$u_z \to N'_{z6} \log|\tau - l| \text{ for } \phi_{sd} < \phi < \phi_{sa} \text{ as } \tau \to l \quad \ldots(74)$$

$$u_z \to N'_{z3} \log|\tau - l| \text{ for } \phi > \phi_{sa} \text{ as } \tau \to l \quad \ldots(75)$$

and also for the case $c_s > a > b$,

$$u_z \to N'_{z6} \log|\tau - l| \text{ for } \phi > \phi_{sd} \text{ as } \tau \to 1 \quad \ldots(76)$$

where

$$N'_{z5} = \frac{2Pabc_s}{\pi\mu\rho} \frac{\sin^2\phi\cos\phi(1 - l^2\sin^2\phi)^{1/2}}{(a^2\sin^2\phi - c_s^2)^{1/2}.\ N_3} \quad \ldots(77)$$

$$N'_{z3} = \frac{8Pabc_s}{\pi\mu\rho} \frac{\sin^4\phi\cos^2\phi\,(l^2\sin^2\phi - 1)}{(a^2\sin^2\phi - c_s^2)^{1/2}.N_4} \quad \ldots(78)$$

$$N'_{z6} = -\frac{2Pabc_d}{\pi\mu\rho} \frac{\sin^2\phi\cos^2\phi(l^2\sin^2\phi - 1)^{1/2}(\cos^2\phi - \sin^2\phi)^2}{(c_s^2 - a^2\sin^2\phi)^{1/2}.\ N_4} \quad \ldots(79)$$

$$N_4 = [l^2(\cos^2\phi - \sin^2\phi)^4 + 16\sin^4\phi\cos^2\phi(l^2\sin^2\phi - 1)]. \quad \ldots(80)$$

For the approximate evaluation of the displacements at the wave fronts $\tau = \tau_{sa}$ and $\tau = \tau_{sd}$ we follow similar procedure as followed for the evaluation of u_{zd} as $\tau \to \tau_{da}$ and we find that

$$[u_z] \to N_{z5} \text{ as } \tau \to \tau_{sa} \text{ for } a > c_d \quad \ldots(81)$$

$$[u_z] \to N_{z6} \text{ as } \tau \to \tau_{sa} \text{ for } c_d > a > c_s \quad \ldots(82)$$

$$[u_z] \to N_{z3}(\tau - \tau_{sd})^{3/2} \text{ as } \tau \to \tau_{sd} \text{ for } a > c_d \quad \ldots(83)$$

$$[u_z] \to N_{z7}(\tau - \tau_{zd}) \text{ as } \tau \to \tau_{sd} \text{ for } a < c_d \quad \ldots(84)$$

where

$$N_{z5} = -\frac{4Pb\,c_d\,A'_s\sqrt{(a^2 - c_d^2)D_s}}{\pi\,\mu\,a^2(2\,K_s\,B_s\,A_s)^{1/2}}$$

$$N_{z6} = -\frac{16\, Pa^2\, bc_d^3\, (c_d^2 - a^2)\, A'_s \sqrt{(a^2 - c_s^2) D_s}}{\pi\mu\, (2K_s\, l^2\, A_s)^{1/2}\, [l^6 (a^2 - 2c_s^2)^4 - 16 c_d^4\, (c_d^2 - a^2)(a^2 - c_s^2)]} \quad \ldots(86)$$

$$N_{z3} = -\frac{4Pab}{\pi\mu} A_{sd}\, B_{sd}^2 B'_{sd}\, A'_{sd} \left(\frac{2 \operatorname{cosec} \phi}{a^2 - c_d^2} \right)^{1/2} \quad \ldots(87)$$

$$N_{z7} = \frac{4Pab}{\pi\mu}\, A_{sd}\, B_{sd}^2\, A'_{sd} \left(\frac{2 \operatorname{cosec} \phi}{c_d^2 - a^2} \right)^2 \quad \ldots(88)$$

$$A'_s = \frac{l c_d\, (a^2 - c_s^2)^{1/2}}{\rho\, [l\, (a^2 - c_s^2)^{1/2} \sin\phi - c_d \cos\phi]} \quad \ldots(89)$$

$$D_s = 8\, a^2\, l c_d\, \tau_{sa} \sin\phi \cos\phi\, (a^2 - c_s^2)^{1/2} \quad \ldots(90)$$

$$B_s = \frac{l}{a^4} [l^3\, (a^2 - 2c_s^2)^2 + 4c_d^2 \sqrt{(a^2 - c_d^2)(a^2 - c_s^2)}] \quad \ldots(91)$$

$$A_s = [\tau_{sa}\, a^2\, b^2\, (\tau_{sa} - \tau_{sa}^0)\, \sin^2\phi + (a^2 - b^2)\, a^2 \cos^2\phi\, (\tau_{sa} + \tau_{sa}^0)] \quad \ldots(92)$$

$$A_{sd} = \frac{\pi}{4} \left[\frac{2\, (l^2 - 1)^{1/2}}{(l^2 - 1)^{1/2} \sin\phi - \cos\phi} \right]^{1/2} \quad \ldots(93)$$

$$B_{sd} = (l^2 - 2)^{-1} \quad \ldots(94)$$

$$B'_{sd} = 4\, A_{sd}\, (l^2 - 1)^{1/2}\, B_{sd}^2 \quad \ldots(26)$$

$$A'_{sd} = \frac{c_d}{\rho}\, (l^2 - 1)^{1/2}\, [(l^2 - 1)^{1/2} \sin\phi - \cos\phi]^{-1} \quad \ldots(96)$$

In these expressions the notations $[u_s]$ stands for the change in u_s across a wave front and N_{z1} etc. are wave front coefficients.

It may also be noted that if we put $a = b$ in this problem, it reduces to the problem of uniformly expanding circular ring source and in that case our derived results coincide with the results given in the paper of Gakenheimer[5].

REFERENCES

1. H. R. Aggarwal, and C. M. Ablow, *Int. J. Engng. Sci.* 5 (1967), 663–79.
2. R. Burridge, and J. R. Willis, *Proc. Camb. Phil. Soc.* 66 (1969), 443–50.

3. L. Cagniard, *Reflection and Refraction of Progressive Seismic Waves*, Translated by Flinn E. A., and Dix C. H., McGraw-Hill Book Co., Inc. New York, 1962.
4. D. C. Gakenheimer, and J. Miklowitz, *J. Appl. Mech.* 36 (1969) *Trans. ASME* 91 E, 505-15.
5. D. C. Gakenheimer, *J. Appl. Mech.* 38 (1971), *Trans. ASME* 93E, 99–110.
6. M. L. Ghosh, *Appl. Sci. Res.* 24 (1971), 149-67.
7. M. Mitra, *Proc. Camb. Phil. Soc.* 60 (1964), 683–94.
8. Paul G. Richard, *Int. J. Solids Structures*, 9 (1973), 843-61.
9. A. Roy, *Int. J. Engng. Sci.* 13 (1975), 641-48.
10. A. Roy, *Int. J. Engng. Sci.* 19 (1981), 129-36.
11. G. E. Tupholme, *Int. J. Engng. Sci.* 8 (1970), 617-39.

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HIGH FREQUENCY SCATTERING OF ANTIPLANE SHEAR WAVES BY AN INTERFACE CRACK

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The problem of diffraction of normally incident antiplane shear wave by a crack of finite length situated at the interface of two bonded dissimilar elastic half spaces has been studied. The problem is reduced to the solution of a Wiener-Hopf problem. The expressions for the stress intensity factor and the crack opening displacement have been derived for the case of wave-lengths short compared to the length of the crack. The numerical results for two different pairs of samples have been presented graphically.

1. INTRODUCTION

Scattering of elastic waves by a crack of finite length at the interface of two dissimilar elastic materials is important in view of its application in Geophysics and in Mechanical engineering problems. The extensive use of composite materials in modern technology has created interest in the wave propagation problems in layered media with interfacial discontinuities. The diffraction of Love waves by a crack of finite width at the interface of a layered half space was studied by Neerhoff[5]. Kuo[6] carried out numerical and analytical studies of transient response of an interfacial crack between two dissimilar orthotropic half spaces. Following the method of Mal[7], Srivastava et al.[1] also considered the low frequency aspect of the interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half space.

But high frequency solution of the diffraction of elastic waves by a crack of finite size is interesting in view of the fact that transient solution close to the wave front can be represented by an integral of the high frequency component of the solution. Green's function method together with a function-theoretic technique based upon an extended Wiener-Hopf argument has been developed by Keogh[3,4] for solving the problem of high frequency scattering of elastic waves by a Griffith crack situated in an infinite homogeneous elastic medium.

In the present paper, we have derived the high frequency solution of the diffraction of SH-wave when it interacts with a Griffith crack located at the interface of two bonded dissimilar elastic half spaces. To solve the problem, following the method of Chang[2], the problem has been formulated as an extended Wiener-Hopf equation and the asymptotic solutions for high frequencies or for wavelengths short compared to the length of the crack have been derived. Expressions for the dynamic stress intensity factor and the crack opening displacement have been obtained and the results have been illustrated graphically for two pairs of different types of material.

2. Formulation of the Problem

Let (x, y, z) be a rectangular Cartesian coordinates. Let an open crack of finite length 2l be located at the interface of two bonded dissimilar elastic semi-infinite solids lying parallel to x-axis. The x-axis is taken along the interface, y-axis vertically upwards into the medium and z-axis is perpendicular to the plane of the paper. (μ_1, ρ_1) and (μ_2, ρ_2) are coefficients of rigidity and density respectively of the upper and lower semi-infinite medium. The crack is subjected to a normally incoming antiplane shear wave originating at $y = -\infty$.

We are interested in finding the high frequency solution of the diffraction problem i.e. the solution when the length of the crack is large compared to the wave length of the incident wave.

Accordingly we shall have to solve the problem when the crack is subject to the following boundary conditions:

$$\sigma_{yz}^{(1)}(x, 0+) = \sigma_{yz}^{(2)}(x, 0-) = -P_s - P_0\, e^{-\omega t}; \ |x| < l \qquad \ldots(1)$$

$$\sigma_{yz}^{(1)}(x, 0+) = \sigma_{yz}^{(2)}(x, 0-), \ |x| > l \qquad \ldots(2)$$

$$w_1(x, 0^+) = w_2(x, 0^{-1}), \ |x| > l \qquad \ldots(3)$$

where ω is the circular frequency and P_s is the static pressure.
Assume

$$w_1(x, y, t) = W_1(x, y)\, e^{-i\omega t} \qquad \ldots(4)$$

$$w_2(x, y, t) = W_2(x, y)\, e^{-i\omega t} \qquad \ldots(5)$$

where W_1 and W_2 satisfy the following two wave equations

$$\nabla^2 W_1(x, y) + k_1^2 W_1(x, y) = 0 \qquad \ldots(6)$$

$$\nabla^2 W_2(x, y) + k_2^2 W_2(x, y) = 0 \qquad \ldots(7)$$

with $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

The shear wave numbers k_1 and k_2 are related to the two shear wave velocities C_1 and C_2 of medium (1) and (2) respectively by

$$k_1 = \omega / C_1 \qquad \ldots(8)$$

$$k_2 = \omega / C_2 \qquad \ldots(9)$$

Without any loss of generality we assume that $k_2 > k_1$.

Let $$\sigma_{yz}^{(1)}(x, y, t) = \tau_{yz}^{(1)}(x, y)\, e^{-i\omega t} \quad ...(10)$$

$$\sigma_{yz}^{(2)}(x, y, t) = \tau_{yz}^{(2)}(x, y)\, e^{-i\omega t}. \quad ...(11)$$

In the boundary condition (1), P_s is the static pressure assumed to be sufficiently large so that crack faces do not come in contact during vibration. Since we are interested in the dynamic part of the stress distribution, so the boundary conditions (1), (2) and (3) may be written as

$$\tau_{yz}^{(1)}(x, 0^+) = \tau_{yz}^{(2)}(x, 0^-) = -P_0,\ |x| < l \quad ...(12)$$

$$\tau_{yz}^{(1)}(x, 0^+) = \tau_{yz}^{(2)}(x, 0^-),\quad |x| > l \quad ...(13)$$

and $$W_1(x, 0^+) = W_2(x, 0^-),\ |x| > l \quad ...(14)$$

that is

$$\mu_1 \frac{\partial W_1}{\partial y} = \mu_2 \frac{\partial W_2}{\partial y} = -P0,\ |x| < l,\ y = 0 \quad ...(15)$$

$$\mu_1 \frac{\partial W_1}{\partial y} = \mu_2 \frac{\partial W_2}{\partial y},\ |x| > l,\ y = 0 \quad ...(16)$$

and $$W_1(x, 0^+) = W_2(x, 0^-),\ |x| > l \quad ...(17)$$

In order to obtain solutions of wave equations (6) and (7) we introduce Fourier transform defined by

$$\overline{W}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(x, y)\, e^{i\alpha x}\, dx. \quad ...(18)$$

Thus we obtain the transformed wave equations as

$$\frac{d^2\overline{W}_1}{dy^2} - (\alpha^2 - k_1^2)\,\overline{W}_1 = 0 \quad ...(19)$$

$$\frac{d^2\overline{W}_2}{dy^2} - (\alpha^2 - k_2^2)\,\overline{W}_2 = 0. \quad ...(20)$$

The solutions of (19) and (20), bounded as y tends to infinity, are

$$\overline{W}_1(\alpha, y) = A_1(\alpha)\, e^{-\gamma_1 y},\ y \geq 0 \quad ...(21)$$

$$\overline{W}_2(\alpha, y) = A_2(\alpha)\, e^{\gamma_2 y},\ y \leq 0 \quad ...(22)$$

where

$$\gamma_1 = (\alpha^2 - k_1^2)^{1/2} \quad ...(23)$$

$$\gamma_2 = (\alpha^2 - k_2^2)^{1/2}. \quad ...(24)$$

Introducing for a complex α

$$G_+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_l^\infty \tau_{yz}^{(1)}(x, 0)\, e^{i\alpha(x-l)}\, dx \qquad ...(25)$$

$$G_-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-l} \tau_{yz}^{(1)}(x, 0)\, e^{i\alpha(x+l)}\, dx \qquad ...(26)$$

and

$$G_1(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-l}^{l} \tau_{yz}^{(1)}(x, 0)\, e^{i\alpha x}\, dx \qquad ...(27)$$

the transformed stress at the interface $y = 0$ can be written as

$$\bar{\tau}_{yz}^{(1)}(\alpha, 0) = G_+(\alpha)\, e^{i\alpha l} + G_1(\alpha) + G_-(\alpha)\, e^{-i\alpha l}. \qquad ...(28)$$

Using the boundary condition (12) we note that

$$G_1(\alpha) = \frac{-P_0}{\sqrt{2\pi}\, i\alpha} \left[e^{i\alpha l} - e^{-i\alpha l} \right]. \qquad ...(29)$$

Further using the fact that

$$\bar{\tau}_{yz}^{(1)}(\alpha, 0) = -\mu_1 \gamma_1 A_1(\alpha) \qquad ...(30)$$

we obtain from (28)

$$-\mu_1\gamma_1 A_1(\alpha) = G_+(\alpha)\, e^{i\alpha l} + G_-(\alpha)\, e^{-i\alpha l} - \frac{P0}{\sqrt{2\pi}\, i\alpha}\left[e^{i\alpha l} - e^{-i\alpha l}\right]. \qquad ...(31)$$

Since from (12) and (13) stress τ_{yz} is continuous at all points of the interface so we obtain

$$A_2(\alpha) = -\frac{\mu_1 \gamma_1}{\mu_2 \gamma_2} A_1(\alpha) \qquad ...(32)$$

so (21) and (22) take the forms

$$\bar{W}_1(\alpha, y) = A_1(\alpha)\, e^{-\gamma_1 y},\ y \geq 0 \qquad ...(33)$$

$$\bar{W}_2(\alpha, y) = -\frac{\mu_1 \gamma_1}{\mu_2 \gamma_2} A_1(\alpha)\, e^{\gamma_2 y},\ y \leq 0. \qquad ...(34)$$

Now $\bar{W}_1(\alpha, 0^+) - \bar{W}_2(\alpha, 0^-) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-l}^{l} \left[W_1(x, 0^+) - W_2(x, 0^-) \right] e^{i\alpha x}\, dx$

$$= B(\alpha) \quad \text{(say)} \qquad ...(35)$$

which is the measure of the discontinuity of displacement along the surface of the crack. From (35) we get

$$A_1(\alpha) = \frac{\mu_2\gamma_2 B(\alpha)}{\mu_1\gamma_1 + \mu_2\gamma_2}. \qquad \ldots(36)$$

Eliminating $A_1(\alpha)$ from (31) and (36) we obtain an extended Wiener-Hopf equation, namely

$$G_+(\alpha)\, e^{i\alpha l} + G_-(\alpha)\, e^{-i\alpha l} + B(\alpha)K(\alpha) = \frac{P_0}{\sqrt{2\pi}\, i\alpha}\left[e^{i\alpha l} - e^{-i\alpha l}\right] \qquad \ldots(37)$$

where

$$K(\alpha) = \frac{\mu_1\mu_2\gamma_1\gamma_2}{\mu_1\gamma_1 + \mu_2\gamma_2} = \frac{\mu_1\mu_2\,(\alpha^2 - k_1^2)^{1/2}}{(\mu_1 + \mu_2)}.\, R(\alpha) \qquad \ldots(38)$$

$$R(\alpha) = \frac{(\mu_1 + \mu_2)\,(\alpha^2 - k_2^2)^{1/2}}{\mu_1\,(\alpha^2 - k_1^2)^{1/2} + \mu_2\,(\alpha^2 - k_2^2)^{1/2}}. \qquad \ldots(39)$$

In order to solve the Wiener-Hopf equation given by (37) we assume that the branch points $\alpha = k_1$ and k_2 of $K(\alpha)$ possess a small imaginary part such that

$$k_1 = k_1 + i\,k_1' \quad \text{and} \quad k_2 = k_2 + i\,k_2'$$

where k_1' and k_2' are infinitesimally small positive quantities which would ultimately be made to tend to zero.

Now we write $K(\alpha) = K_+(\alpha)\,K_-(\alpha)$ where $K_+(\alpha)$ is analytic in the upper half plane $\text{Im}\,\alpha > -k_2'$ whereas $K_-(\alpha)$ is analytic in the lower half plane given by $\text{Im}\,\alpha < k_2'$. Since $\tau_{yz}(x, 0)$ decreases exponentially as $x \to \pm\infty$, $G_+(\alpha)$ and $G_-(\alpha)$ have the same common region of regularity as $K_+(\alpha)$ and $K_-(\alpha)$.

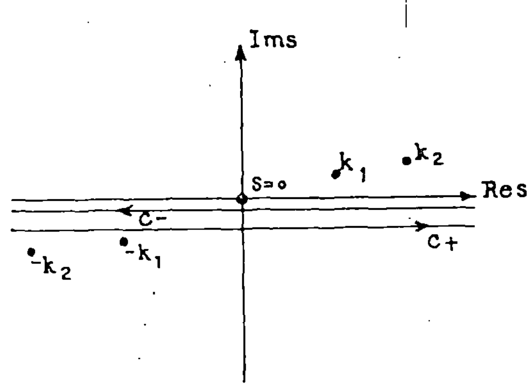

FIG. 1. Path of integration in the complex s-plane.

Now (37) can easily be expressed as two integral equations relating $G_+(\alpha)$, $G_-(\alpha)$ and $B(\alpha)$ as follows:

$$\frac{G_+(\alpha)}{K_+(\alpha)} - \frac{P_0}{\sqrt{2\pi}\, i\alpha}\left[\frac{1}{K_+(\alpha)} - \frac{1}{K_+(0)}\right]$$

$$+ \frac{1}{2\pi i}\int_{C_+} \frac{e^{-2isl}}{(s-\alpha)\, K_+(s)}\left[G_-(s) + \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

$$= - B(\alpha)\, K_-(\alpha)\, e^{-i\alpha l} + \frac{P_0}{\sqrt{2\pi}\, i\alpha\, K_+(0)} - \frac{1}{2\pi i}\cdot\int_{C_-}$$

$$\frac{e^{-2isl}}{(s-\alpha)\, K_+(s)}\left[G_-(s) + \frac{P_0}{\sqrt{2\pi}\, is}\right] ds \qquad \ldots(40)$$

and

$$\frac{G_-(\alpha)}{K_-(\alpha)} + \frac{P_0}{\sqrt{2\pi}\, i\alpha\, K_-(\alpha)} + \frac{1}{2\pi i}\int_{C_-} \frac{e^{2isl}}{(s-\alpha)\, K_-(s)}$$

$$\left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

$$= - B(\alpha)\, K_+(\alpha)\, e^{i\alpha l} - \frac{1}{2\pi i}\int_{C_+} \frac{e^{2isl}}{(s-\alpha)\, K_-(s)}\left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds \qquad \ldots(41)$$

where C_+ and C_- are the straight contours below the pole at $s = 0$ and situated within the common region of regularity of $G_+(s)$, $G_-(s)$, $K_+(s)$ and $K_-(s)$ as shown in Fig. 1.

In (40), the left-hand side is analytic in the upper half plane whereas the right-hand side is analytic in the lower-half plane and both of them are equal in the common region of analyticity of these two functions. So by analytic continuation, both sides of (40) are analytic in the whole of the s-plane. Now since

$$\tau_{yz} \sim (x \mp l)^{-1/2} \qquad \text{as } x \to \pm l$$

so $\quad G_\pm(\alpha) \sim \alpha^{-1/2} \qquad \text{as } |\alpha| \to \infty$

and also $\quad K_\pm(\alpha) \sim \alpha^{1/2} \qquad \text{as } |\alpha| \to \infty$

so it follows that

$$\frac{G_\pm(\alpha)}{K_\pm(\alpha)} \sim \alpha^{-1} \qquad \text{as } |\alpha| \to \infty.$$

Therefore by Liouville's theorem, both sides of (40) are equal to zero. Equation (41) can be treated similarly.

Therefore from (40) and (41) we obtain the system of integral equations given by

$$\left[G_+ (\alpha) - \frac{P_0}{\sqrt{2\pi}\, i\alpha}\right] \frac{1}{K_+ (\alpha)} + \frac{P_0}{\sqrt{2\pi}\, i\alpha\, K_+ (0)}$$

$$+ \frac{1}{2\pi i} \int_{C_+} \frac{e^{-2isl}}{(s-\alpha)\, K_+ (s)} \left[G_- (s) + \frac{P_0}{\sqrt{2\pi}\, is}\right] ds = 0 \quad \text{...(42)}$$

and

$$= \left[G_- (\alpha) + \frac{P_0}{\sqrt{2\pi}\, i\alpha}\right] \frac{1}{K_- (\alpha)} + \frac{1}{2\pi i} \int_{C_-} \frac{e^{2isl}}{(s-\alpha)\, K_- (s)}$$

$$\left[G_+ (s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds = 0. \quad \text{...(43)}$$

Since $\tau_{yz}^{(1)} (x, 0)$ is an even function of x, so from (25) and (26) it can be shown that $G_+(-\alpha) = G_- (\alpha)$ and it has been shown in the appendix that $K_+ (-\alpha) = iK_- (\alpha)$. Using these results and replacing α by $-\alpha$ and s by $-s$ in (42) it can easily be shown that equations (42) and (43) are identical. So $G_+ (\alpha)$ and $G_- (\alpha)$ are to be determined from any one of the integral equation (42) or (43).

3. High Frequency Solution of the Integral Equation

To solve the integral equation (43) in the case when normalized wave number $k_1 l \gg 1$, the integration along the path C_- in (43) is replaced by the integration round the circular contour C_0 round the pole at $s = 0$ and by the integration along the contours C_{k_1} and C_{k_2} round the branch cuts through the branch points k_1 and k_2 of the function $K_- (s)$ as shown in Fig. 2.

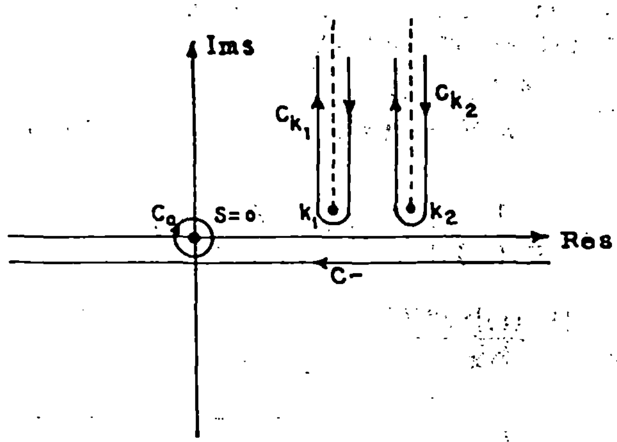

FIG. 2. Path of integration C_0, C_{k_1}, C_{k_2}.

Thus eqn. (43) takes the form

$$\left[G_-(\alpha) + \frac{P_0}{\sqrt{2\pi}\, i\alpha}\right] - \frac{P_0 K_-(\alpha)}{\sqrt{2\pi}\, i\alpha K_-(0)}$$

$$+ \frac{K_-(\alpha)}{2\pi i} \int_{C_{k_1}+C_{k_1}} \frac{\exp(2isl)}{(s-\alpha) K_-(s)} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds = 0. \qquad \text{...(44)}$$

Now

$$\int_{C_{k_1}} \frac{\exp(2isl)}{(s-\alpha) K_-(s)} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

$$= \frac{1}{\mu_1} \int_{C_{k_1}} \frac{e^{2isl} K_+(s)}{(s-\alpha)(s^2-k_1^2)^{1/2}} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

which can easily be evaluated when $k_1 l \gg 1$ and is found to be equal to

$$-\frac{1}{\mu_1}\sqrt{\frac{\pi}{k_1 l}}\, \frac{\exp(2ik_1 l)\, K_+(k_1)\, e^{i\pi/4}}{(k_1-\alpha)} \left[G_+(k_1) - \frac{P_0}{\sqrt{2\pi}\, ik_1}\right]. \qquad \text{...(45)}$$

Similarly for $k_1 l \gg 1$

$$\int_{C_{k_2}} \frac{\exp(2isl)}{(s-\alpha) K_-(s)} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

$$= -\frac{1}{\mu_2}\sqrt{\frac{\pi}{k_2 l}} \cdot \frac{\exp(2ik_2 l)\, K_+(k_2)\, e^{i\pi/4}}{(k_2-\alpha)} \left[G_+(k_2) - \frac{P_0}{\sqrt{2\pi}\, ik_2}\right]. \qquad \text{...(46)}$$

Using the results (45) and (46) and also the relations $G_+(-\alpha) = G_-(\alpha)$ and $K_-(\alpha) = -iK_+(\alpha)$, we obtain from (44)

$$F_+(-\alpha) + \frac{A(k_1)\, F_+(k_1)\, e^{2ik_1 l}}{\mu_1 (k_1-\alpha)\sqrt{k_1 l}} + \frac{A(k_2)\, F_+(k_2)\, e^{2ik_2 l}}{\mu_2 (k_2-\alpha)\sqrt{k_2 l}} = C(\alpha). \qquad \text{...(47)}$$

where

$$F_+(\xi) = \frac{1}{K_-(-\xi)}\left[G_+(\xi) - \frac{P_0}{\sqrt{2\pi}\, i\xi}\right] \qquad \text{...(48)}$$

$$A(\xi) = \frac{[K_+(\xi)]^2\, e^{i\pi/4}}{2\sqrt{\pi}} \qquad \text{...(49)}$$

and

$$C(\xi) = \frac{P_0}{\sqrt{2\pi}\, iK_-(0)\xi} \qquad \text{...(50)}$$

Substituting $\alpha = -k_1$ and $\alpha = -k_2$ in (47) we obtain respectively the equations

$$\left[1 + \frac{A(k_1)\, e^{2ik_1 l}}{2\mu_1 k_1 \sqrt{k_1 l}}\right] F_+(k_1) + \frac{A(k_2)\, F_+(k_2)\, e^{2ik_2 l}}{\mu_2 (k_1 + k_2) \sqrt{k_2 l}} = -C(k_1) \qquad \ldots(51)$$

and

$$\frac{A(k_1)\, e^{2ik_1 l}}{\mu_1 (k_1 + k_2)\sqrt{k_1 l}} F_+(k_1) + \left[1 + \frac{A(k_2)\, e^{2ik_2 l}}{2\mu_2 k_2 \sqrt{k_2 l}}\right] F_+(k_2) = -C(k_2). \qquad \ldots(52)$$

Now solving (51) and (52) we get

$$F_+(k_1) = C(k_1)\left[\frac{A(k_2)\,(k_1 - k_2)\, e^{2ik_2 l}}{2\mu_2 k_2 (k_1 + k_2)\sqrt{k_2 l}} - 1\right] L(k_1, k_2) \qquad \ldots(53)$$

and

$$F_+(k_2) = C(k_2)\left[\frac{A(k_1)\,(k_2 - k_1)\, e^{2ik_1 l}}{2\mu_1 k_1 (k_1 + k_2)\sqrt{k_1 l}} - 1\right] L(k_1, k_2) \qquad \ldots(54)$$

where

$$L(k_1, k_2) = \left[1 + \frac{A(k_1)\, e^{2ik_1 l}}{2\mu_1 k_1 \sqrt{k_1 l}} + \frac{A(k_2)\, e^{2ik_2 l}}{2\mu_2 k_2 \sqrt{k_2 l}} + \frac{A(k_1)A(k_2)\,(k_1 - k_2)^2\, e^{2i(k_1+k_2)l}}{4\mu_1 \mu_2 k_1 k_2 (k_1 + k_2)^2 \sqrt{lk_1}\sqrt{lk_2}}\right]^{-1}. \qquad \ldots(55)$$

Now expanding $L(k_1, k_2)$ and neglecting higher order terms of $1/\sqrt{k_1 l}$ and $1/\sqrt{k_2 l}$ and using (47) we get

$$\begin{aligned} G_-(\alpha) &= -C(\alpha) K_-(0) + C(\alpha) K_-(\alpha) \\ &+ \frac{K_-(\alpha) A(k_1)\, e^{2ik_1 l} \cdot C(k_1)}{\mu_1 (k_1 - \alpha)\sqrt{k_1 l}} \left[1 - \frac{A(k_1)\, e^{2ik_1 l}}{2\mu_1 k_1 \sqrt{k_1 l}} - \frac{A(k_2)\, k_1\, e^{2ik_2 l}}{\mu_2 k_2 \sqrt{k_2 l}\,(k_1 + k_2)}\right] \\ &+ \frac{K_-(\alpha) A(k_2)\, e^{2ik_2 l} \cdot C(k_2)}{\mu_2 (k_2 - \alpha)\sqrt{k_2 l}} \left[1 - \frac{A(k_1)\, k_2\, e^{2ik_1 l}}{\mu_1 k_1 \sqrt{k_1 l}\,(k_1 + k_2)} - \frac{A(k_2)\, e^{2ik_2 l}}{2\mu_2 k_2 \sqrt{k_2 l}}\right] \end{aligned} \qquad \ldots(56)$$

Now replacing α by $-\alpha$ and using $C(-\alpha) = -C(\alpha)$. We have

$$\begin{aligned} G_+(\alpha) &= C(\alpha) K_-(0) - C(\alpha) K_-(-\alpha). \\ &+ \frac{K_-(-\alpha) A(k_1)\, e^{2ik_1 l} \cdot C(k_1)}{\mu_1 (k_1 + \alpha)\sqrt{k_1 l}} \left[1 - \frac{A(k_1)\, e^{2ik_1 l}}{2\mu_1 k_1 \sqrt{k_1 l}} - \frac{A(k_2)\, k_1\, e^{2ik_1 l}}{\mu_2 k_2 \sqrt{k_2 l}\,(k_1 + k_2)}\right] \\ &+ \frac{K_-(-\alpha) A(k_2)\, e^{2ik_2 l} \cdot C(k_2)}{\mu_2 (k_2 + \alpha)\sqrt{k_2 l}} \left[1 - \frac{A(k_1)\, k_2\, e^{2ik_1 l}}{\mu_1 k_1 \sqrt{k_1 l}\,(k_1 + k_2)} - \frac{A(k_2)\, e^{2ik_2 l}}{2\mu_2 k_2 \sqrt{k_2 l}}\right]. \end{aligned} \qquad \ldots(57)$$

4. Stress Intensity Factor and Crack Opening Displacement near the Crack Tips

Now as $\alpha \to \infty$

$$\therefore \quad K_-(-\alpha) = -iK_+(\alpha) = -i\,(\alpha+k_1)^{1/2}\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}} \simeq -i\alpha^{1/2}\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}$$

$$\frac{K_-(-\alpha)}{\alpha+k_1} \approx -i\alpha^{-1/2}\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}$$

$$\frac{K_-(-\alpha)}{\alpha+k_2} \approx -i\alpha^{-1/2}\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}.$$

So as $\alpha \to \infty$ we get from (56) and (57)

$$\left.\begin{aligned} G_+(\alpha) &\approx S\,\alpha^{-1/2} + \frac{P_0}{\sqrt{2\pi}\,i\alpha} \\ \text{and} \quad & \\ G_-(\alpha) &\approx -iS\alpha^{-1/2} - \frac{P_0}{\sqrt{2\pi}\,i\alpha} \end{aligned}\right\} \qquad \ldots(58)$$

where

$$S = \frac{P_0}{\sqrt{2\pi}\,K_-(0)}\left[1 - \frac{A(k_1)\,e^{2ik_1l}}{\mu_1k_1\sqrt{k_1l}} + \frac{A(k_2)\,e^{2ik_2l}}{\mu_2k_2\sqrt{k_2l}} + \right.$$

$$\left. + \frac{1}{2}\left(\frac{A^2(k_1)\,e^{4ik_1l}}{\mu_1^2\,k_1^2\,k_1l} + \frac{A_2^2(k_2)\,e^{4ik_2l}}{\mu_2^2\,k_2^2\,k_2l}\right) + \frac{A(k_1)\,A(k_2)\,e^{2i(k_1+k_2)l}}{\mu_1\,k_1\,\mu_2\,k_2\sqrt{k_1l.k_2l}}\right]$$

$$\times\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}. \qquad \ldots(59)$$

Now from eqn. (37) using (58) and also the fact that

$$K(\alpha) \to \pm\,\alpha.\,\frac{\mu_1\mu_2}{\mu_1+\mu_2} \quad \text{as } \alpha \to \pm\infty \qquad \ldots(60)$$

we get

$$B(\alpha) = \frac{\pm S}{\alpha\sqrt{\alpha}}\left[ie^{-i\alpha l} - e^{i\alpha l}\right]\frac{\mu_1+\mu_2}{\mu_1\mu_2} \qquad \ldots(61)$$

$$\text{as } \alpha \to \pm\infty.$$

Taking inverse Fourier-Transform of (35) and using the results of Fresnel integrals viz.

$$\int_0^\infty \frac{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(x+l)\alpha}{\sqrt{\alpha}}\,d\alpha = \sqrt{\frac{\pi}{2(x+l)}}. \qquad \ldots(62)$$

We get the displacement jump across the surface of the crack as

$$\Delta W = W_1 (x, 0+) - W_2 (x, 0 -) = 2S_1 (1 - i) \sqrt{(l - x)} \quad \text{for } x \to l - 0 \qquad ...(63)$$

and

$$\Delta W = W_1 (x, 0+) - W_2 (x, 0 -) = 2S_1 (1 - i) \sqrt{(x+l)} \quad \text{for } x \to - l + 0 \qquad ...(64)$$

where

$$S_1 = \frac{(\mu_1 + \mu_2)}{\mu_1 \mu_2} \cdot S. \qquad ...(65)$$

Next inorder to find the value of τ_{xy} near about the crack tip we use (61) in (36) and (32) and to obtain

$$A_j (\alpha) = \frac{(-1)^{j+1} \cdot S}{\mu_j \alpha \sqrt{\alpha}} \left[i e^{-i\alpha l} - e^{i\alpha l} \right], \quad (j = 1,2) \quad \text{as } \alpha \to \infty \qquad ...(66)$$

and

$$A_j (\alpha) = \frac{(-1)^{j+1} \cdot S}{\mu_j \alpha \sqrt{-\alpha}} \left[e^{-i\alpha l} - i e^{i\alpha l} \right], \quad (j = 1,2) \quad \text{as } \alpha \to -\infty. \qquad ...(67)$$

Now

$$\tau_{yz} (x, y) = \mu_j \frac{\partial W_j (x, y)}{\partial y}, \quad j = 1,2$$

$$= \mu_j \frac{\partial}{\partial y} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_j (\alpha) \exp^{(-\gamma_j |y| - i\alpha x)} d\alpha \right]. \qquad ...(68)$$

Substituting the values of $A_j (\alpha)$ as $|\alpha| \to \infty$, we can write the stress near about the crack tip as

$$\tau_{yz} (x, y) = \frac{S}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-\alpha|y|}}{\sqrt{\alpha}} \left[e^{i\alpha(x+l)} - i e^{i\alpha(x-l)} - i e^{-i\alpha(x+l)} + e^{-i\alpha(x-l)} \right] d\alpha$$

$$= \frac{S(1-i)}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-\alpha|y|}}{\sqrt{\alpha}} \left[\cos \alpha (x+l) - \sin \alpha (x+l) + \cos \alpha (x - l) + \sin \alpha (x - l) \right] d\alpha$$

$$= S (1 - i) \left[\frac{1}{\sqrt{r_2}} \sin \frac{\phi_2}{2} + \frac{1}{\sqrt{r_1}} \cos \frac{\phi_1}{2} \right] \qquad ...(69)$$

near about the crack tips, where

$$r_1 = \left[(x - l)^2 + y^2 \right]^{1/2}, \ \phi_1 = \sin^{-1} \frac{|y|}{r_1} \qquad ...(70)$$

$$r_2 = \left[(x + l)^2 + y^2 \right]^{1/2}, \ \phi_2 = \sin^{-1} \frac{|y|}{r_2}. \qquad ...(71)$$

Therefore at the interface ($y = 0$) we obtain

$$\tau_{yz} \to \frac{S\,(1-i)}{\sqrt{x-l}} \quad \text{as} \quad x \to l+0 \qquad \ldots(72)$$

and
$$\tau_{yz} \to \frac{S\,(1-i)}{\sqrt{-(x+l)}} \quad \text{as} \quad x \to -l-0. \qquad \ldots(73)$$

Now the stress intensity factor is defined by

$$K = \frac{|(1-i)\,S|\sqrt{2\pi k_1}}{P_0}. \qquad \ldots(74)$$

The absolute value of the complex stress intensity factor defined by (74) has been plotted against $k_1\, l$ in Fig. 3 for values of $k_1\, l > 1$ for the following two sets of materials, given by

First Set:	Steel	$\rho_1 = 7.6$ gm/cm^3	$\mu_1 = 8.32\times10^{11}$ dyne/cm^2
	Aluminium	$\rho_2 = 2.7$ gm/cm^3	$\mu_2 = 2.63\times10^{11}$ dyne/cm^2
Second Set:	Wrought iron	$\rho_1 = 7.8$ gm/cm^3	$\mu_1 = 7.7\times10^{11}$ dyne/cm^2
	Copper	$\rho_2 = 8.96$ gm/cm^3	$\mu_2 = 4.5\times10^{11}$ dyne/cm^2.

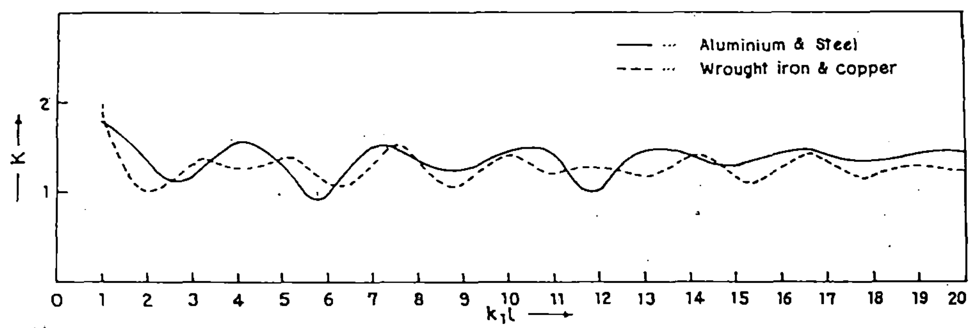

FIG. 3. Stress intensity factor K versus dimensionless frequency $k_1\, l$.

5. Crack Opening Displacement at Points Away from the Crack Tips

Next in order to obtain the displacement jump for the large values of $k_1\,(l-x)$ and $k_1\,(l+x)$ we write $G_+(\alpha)$ and $G_-(\alpha)$ from (57) and (56) respectively as

$$G_+(\alpha) = \frac{P}{\alpha} - \frac{QK_-(-\alpha)}{\alpha} + \frac{R(k_1, k_2)\,K_-(-\alpha)}{k_1+\alpha} + \frac{R(k_2, k_1)\,K_-(-\alpha)}{k_2+\alpha} \qquad \ldots(75)$$

and
$$G_-(\alpha) = \frac{-P}{\alpha} + \frac{QK_-(\alpha)}{\alpha} + \frac{R(k_1, k_2)\,K_-(\alpha)}{k_1-\alpha} + \frac{R(k_2, k_1)\,K_-(\alpha)}{k_2-\alpha} \qquad \ldots(76)$$

where $$P = \frac{P_0}{\sqrt{2\pi}.\, i} \qquad \ldots(77)$$

$$Q = \frac{P_0}{\sqrt{2\pi}\, iK_-(0)} = \frac{P}{K_-(0)} \qquad \ldots(78)$$

and $$R(k_m, k_n) = \frac{QA(k_m)\cdot e^{2ik_m l}}{\mu_m\, k_m\, \sqrt{lk_m}}\left[1 - \frac{e^{2ik_m l}\cdot A(k_m)}{\sqrt{lk_m}\, 2\mu_m k_m} - \frac{e^{2ik_n l}\cdot A(k_n)k_m}{\sqrt{lk_n}\ \mu_n k_n\ (k_m+k_n)}\right], \qquad \ldots(79)$$

where $m = 1$ when $n = 2$

and $m = 2$ when $n = 1$.

Again using $K_-(-\alpha) = -iK_+(\alpha)$ we get from (37)

$$B(\alpha) = -\frac{Qi\, e^{i\alpha l}}{\alpha\, K_-(\alpha)} + \frac{iR\,(k_1, k_2)\, e^{i\alpha l}}{(k_1+\alpha)\, K_-(\alpha)} + \frac{iR(k_2, k_1)\, e^{i\alpha l}}{(k_2+\alpha)\, K_-(\alpha)}$$

$$- \frac{Q\, e^{-i\alpha l}}{\alpha\, K_+(\alpha)} - \frac{R\,(k_1, k_2)\, e^{-i\alpha l}}{(k_1-\alpha)\, K_+(\alpha)} - \frac{R(k_2, k_1)\, e^{-i\alpha l}}{(k_2-\alpha)\, K_+(\alpha)}. \qquad \ldots(80)$$

From (35) we get the displacement jump across the surface of the crack as

$$W_1\,(x, 0^+) - W_2\,(x, 0^-) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B\,(\alpha)\, e^{-i\alpha x}\, d\alpha. \qquad \ldots(81)$$

Now substituting the expression of $B(\alpha)$ from (80) in (81) and approximately evaluating the integrals arising in (81) term by term for large values of $k_1\ (l-x)$, $k_2(l-x)$, $k_1\ (l+x)$ and $k_2\ (l+x)$ and neglecting terms of order higher than $(k_1 l)^{-3/2}$ and $(k_2 l)^{-3/2}$ we obtain finally the crack opening displacement across the cracked-surface in the following form:

$$\Delta W = W_1\,(x, 0^+) - W_2\,(x, 0^-) = 2\pi\, Qi\, K_+\,(0)\left(\frac{1}{\mu_1 k_1} + \frac{1}{\mu_2 k_2}\right)$$

$$+ \sqrt{2}\, Qe^{-i\pi/4}\left[\left(\frac{e^{ik_1(l-x)}}{\sqrt{k_1\,(l-x)}} + \frac{e^{ik_1(l+x)}}{\sqrt{k_1\,(l+x)}}\right)\right.$$

$$\times\left(R_1 + \frac{R_1\, R_{11}\, e^{2ik_1 l}}{\sqrt{2k_1\, l}} + \frac{R_2\, R_{21}\, e^{2ik_2 l}}{\sqrt{2k_2\, l}} + \frac{R_1\,(R_{11})^2\, e^{4ik_1 l}}{\sqrt{2k_1\, l}\,\sqrt{2k_1\, l}}\right.$$

$$+ \frac{R_2\, R_{22}\, R_{21}\, e^{4ik_2 l}}{\sqrt{2k_2\, l}\,\sqrt{2k_2\, l}} + \frac{R_1\, R_{12}\, R_{21}\, e^{2i(k_1+k_2)\, l}}{\sqrt{2k_1\, l}\,\sqrt{2k_2\, l}} + \frac{R_2\, R_{21}\, R_{11}\, e^{2i(k_1+k_2)\, l}}{\sqrt{2k_1\, l}\ \sqrt{2k_2\, l}}$$

$$+ \left(\frac{e^{ik_2\,(l-x)}}{\sqrt{k_2\,(l-x)}} + \frac{e^{ik_2\,(l+x)}}{\sqrt{k_2\,(l+x)}}\right)$$

$$\times \left(R_2 + \frac{R_2 R_{22} e^{2ik_2 l}}{\sqrt{2k_2 l}} + \frac{R_1 R_{12} e^{2ik_1 l}}{\sqrt{2k_1 l}} + \frac{R_2 (R_{22})^2 e^{4ik_2 l}}{\sqrt{2k_2 l}\ \sqrt{2k_2 l}}\right.$$

$$+ \frac{R_1 R_{11} R_{12} e^{4ik_1 l}}{\sqrt{2k_1 l}\ \sqrt{2k_1 l}} + \frac{R_2 R_{21} R_{12} e^{2i(k_1+k_2) l}}{\sqrt{2k_1 l}\ \sqrt{2k_2 l}}$$

$$\left. + \frac{R_1 R_{12} R_{22} e^{2i(k_1+k_2) l}}{\sqrt{2k_1 l}\ \sqrt{2k_2 l}}\right)\Bigg] \qquad \ldots(82)$$

where

$$R_1 = \frac{K_+ (k_1)}{\sqrt{2}\ \mu_1 k_1} \qquad R_2 = \frac{K_+ (k_2)}{\sqrt{2}\ \mu_2 k_2}$$

$$R_{11} = \frac{D[K_+ (k_1)]^2}{\mu_1 (k+k_1)} \qquad R_{22} = \frac{D[K_+ (k_2)]^2}{\mu_2 (k_2+k_2)} \qquad \ldots(83)$$

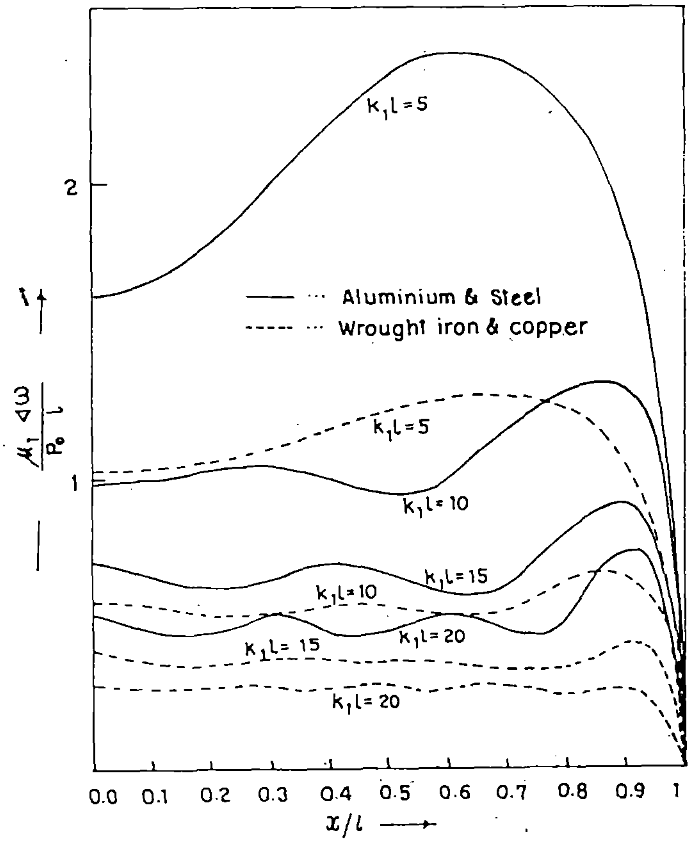

FIG. 4. Normalized crack opening displacement versus normalized distance x/l from the centre of the crack.

$$R_{21} = \frac{D.K_+(K_1)\,K_+\,(k_2)}{\mu_1\,(k_1+k_2)} \qquad R_{12} = \frac{DK_+\,(K_1)\,K_+(k_2)}{\mu_2\,(k_1+k_2)}$$

$$D = (-1)\,\frac{e^{i\pi/4}}{\sqrt{2\pi}}.$$

Expressions in (63) and (64) give the displacement jump nearabout the crack tips where as the displacement jump at points away from the crack tips are given by (82).

From these two results we can obtain the crack opening displacement at any point of the crack surface $-l < x < l, y = 0$.

Here also normalized crack opening displacement has been plotted against normalized distance x/l from the centre of the crack for two different sets of materials in Fig. 4. It is interesting to note that oscillatory nature of the crack opening displacement increases with the increase of frequencies as a result of the interference of waves inside the crack. Further we note that amplitude of the crack opening displacement decreases with the increase of frequency.

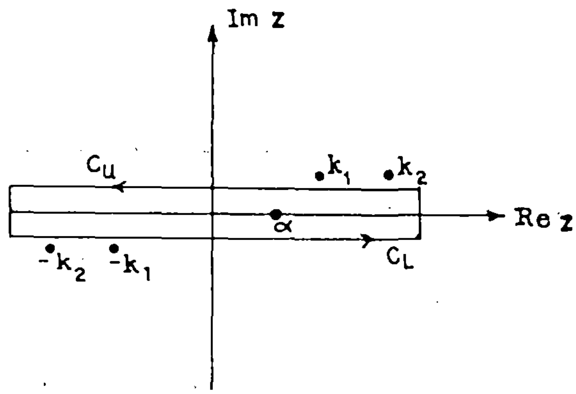

FIG. 5. Complex z-plane.

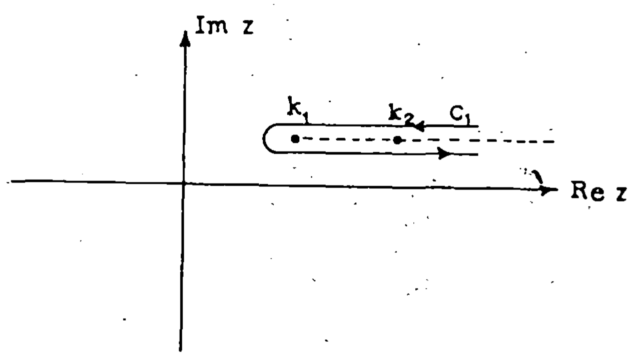

FIG. 6. Path of integration round the branch points.

References

1. K. N. Srivastava, R. M. Palaiya and D. S. Karaulia, *Int. J. Fract.* **16** (1980), 349-58.
2. S. J. Chang, *Quart. J. Mech. Appl. Math.* **24** (1971), 424-43.
3. P. S. Keogh, *Quart. J. Mech. Appl. Math.* **38** (1985), 185-204.
4. P. S. Keogh, *Quart. J. Mech. Appl. Math.* **38** (1985), 205-32.
5. F. L. Neerhoff, *Appl. Sci. Res.*, 35 (1979), 265-315.
6. A. Y. Kuo, *J. Appl. Mech.* **51** (1984), 71-76.
7. A. K. Mal, *Int. J. Engng. Sci.* **8** (1970), 763-76.

Appendix A

$$K(\alpha) = \frac{\mu_1\mu_2\,(\alpha^2 - k_1^2)^{1/2}}{(\mu_1 + \mu_2)}\,R(\alpha)$$

where

$$R(\alpha) = \frac{(\mu_1 + \mu_2)\,(\alpha^2 - k_2^2)^{1/2}}{\mu_1\,(\alpha^2 - k_1^2)^{1/2} + \mu_2\,(\alpha^2 - k_2^2)^{1/2}}.$$

Put $m = \dfrac{\mu_2}{\mu_1}$.

Therefore

$$K(\alpha) = \frac{\mu_2\,(\alpha^2 - k_1^2)^{1/2}}{1+m}\,R(\alpha) \qquad \text{...(A1)}$$

where

$$R(\alpha) = \frac{(1 + m)\,(\alpha^2 - k_2^2)^{1/2}}{(\alpha^2 - k_1^2)^{1/2} + m(\alpha^2 - k_2^2)^{1/2}} \to 1 \text{ as } |\alpha| \to \infty.$$

Now

$$R_+(\alpha)R_-(\alpha) = \frac{1}{\dfrac{m}{1+m} + \dfrac{(\alpha^2 - k_1^2)^{1/2}}{(m+1)\,(\alpha^2 - k_2^2)^{1/2}}}$$

Therefore

$$\log R_+(\alpha) + \log R_-(\alpha) = \log \frac{1}{\dfrac{m}{1+m} + \dfrac{(\alpha^2 - k_1^2)^{1/2}}{(m+1)\,(\alpha^2 - k_2^2)^{1/2}}} = \log R(\alpha)$$

$$\therefore \quad \log R_+(\alpha) = \frac{1}{2\pi i} \int_{C_L} \frac{\log R(z)}{(z-\alpha)}\, dz$$

$$= \frac{1}{2\pi i} \int_{-ic-\infty}^{-ic+\infty} \frac{\log R(z)}{(z-\alpha)}\, dz$$

where the path of integration C_L is shown in Fig. 5.
Putting $z = -z$ and using the fact that $R(z) = R(-z)$, we get

$$\log R_+(\alpha) = -\frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{\log R(z)}{(z+\alpha)}\, dz$$

$$= -\frac{1}{2\pi i} \int_{C_1} \frac{\log R(z)}{(z+\alpha)}\, dz$$

where C_1 is the contour round the branch points k_1 and k_2 as shown in Fig. 6.

So,

$$\log R_+(\alpha) = \frac{1}{2\pi i} \int_{C_1} \frac{\log\left[\dfrac{m}{m+1} + \dfrac{(z^2-k_1^2)^{1/2}}{(m+1)(z^2-k_2^2)^{1/2}}\right]}{(z+\alpha)}\, dz$$

$$= \frac{1}{2\pi i} \int_{k_1}^{k_2} \frac{\log\left[1 + \dfrac{i(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}$$

$$- \frac{1}{2\pi i} \int_{k_1}^{k_2} \frac{\log\left[1 - \dfrac{i(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\, dz$$

$$= \frac{1}{\pi} \int_{k_1}^{k_2} \frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\, dz$$

$$\therefore R_+(\alpha) = \exp\left[\frac{1}{\pi} \int_{k_1}^{k_2} \frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\, dz\right]$$

Similarly

$$R_{-}(\alpha) = \exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right]$$

Therefore from (A1) we can write

$$K_{+}(\alpha) = \frac{\sqrt{\mu_2}(\alpha+k_1)^{1/2}}{\sqrt{(1+m)}}\cdot\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\,dz\right] \quad \text{...(A2)}$$

and

$$K_{-}(\alpha) = \frac{\sqrt{\mu_2}(\alpha-k_1)^{1/2}}{\sqrt{(1+m)}}\cdot\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right]. \quad \text{...(A3)}$$

Hence from (A2) and (A3) we get

$$K_{+}(-\alpha) = \frac{\sqrt{\mu_2}\,i(\alpha-k_1)^{1/2}}{\sqrt{(1+m)}}\cdot\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right]$$

$$= iK_{-}(\alpha)$$

i.e. $K_{+}(-\alpha) = iK_{-}(\alpha)$ (A4)

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HIGH FREQUENCY SCATTERING OF PLANE HORIZONTAL SHEAR WAVES BY A GRIFFITH CRACK PROPAGATING ALONG THE BIMATERIAL INTERFACE

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Abstract—The problem of diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface is studied. In order to obtain a high frequency solution, the problem is formulated as an extended Wiener–Hopf problem. The expressions for the dynamic stress intensity factor at the crack tip and the crack opening displacement are derived for the case of wave lengths which are short compared to the length of the crack. The dynamic stress intensity factor for high frequencies is illustrated graphically for two pairs of different types of material for different crack velocities and angles of incidence.

1. INTRODUCTION

SCATTERING of elastic waves by a stationary or a moving crack of finite length at the interface of two dissimilar elastic materials is important in view of its application in fracture mechanics as well as in seismology. Recently, Takei *et al.* [1] considered the problem of diffraction of transient horizontal shear waves by a finite crack lying on a bimaterial interface. The method of solution was extended by Ueda *et al.* [2] to solve the problem of torsional impact response of a penny shaped interface crack. Srivastava *et al.* [3] also considered the low frequency aspect of the interaction of an antiplane shear wave by a Griffith crack at the interface of two bonded dissimilar elastic half spaces.

In the case of cracks of finite size, travelling at a constant velocity, loads, for mathematical simplicity, are usually assumed to be independent of time. However, in practice, structures are often required to sustain oscillating loads where the dynamic disturbances propagate through the elastic medium in the form of stress waves. The problem of diffraction of a plane harmonic polarized shear wave by a half plane crack extended under antiplane strain was first studied by Jahanshahi [4]. Later Chen and Sih [5] considered the interaction of stress waves with a semi-infinite running crack under either the plane strain or the generalized plane stress condition. Sih and Loeber [6] and Chen and Sih [7] also considered the problem of scattering of plane harmonic waves by a running crack of finite length. In both the cases the problem was reduced to a system of simultaneous Fredholm integral equations which were solved numerically.

In the present paper, we have investigated the high frequency solution of the problem of diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface. The high frequency solution of the diffraction of elastic waves by a crack of finite size is important in view of the fact that the transient solution close to the wave front can be represented by an integral of the high frequency component of the solution. In order to solve the problem, following the method of Chang [8], the problem has been formulated as an extended Wiener–Hopf equation and the asymptotic solutions for high frequencies or for wave lengths which are short compared to the length of the crack have been derived. Expressions for the dynamic stress intensity factor at the crack tip and the crack opening displacement have been derived. The dynamic stress intensity factor for high frequencies has been illustrated graphically for two pairs of different types of materials for different crack velocities and angles of incidence.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let a plane crack of width $2L$ move at a constant velocity V at the interface of two bonded dissimilar elastic semi-infinite media due to the incidence of the plane horizontal SH-wave

$$W_i = A\exp[-\{k_1(X\cos\theta_1 + Y\sin\theta_1) + \Omega T\}] \tag{1}$$

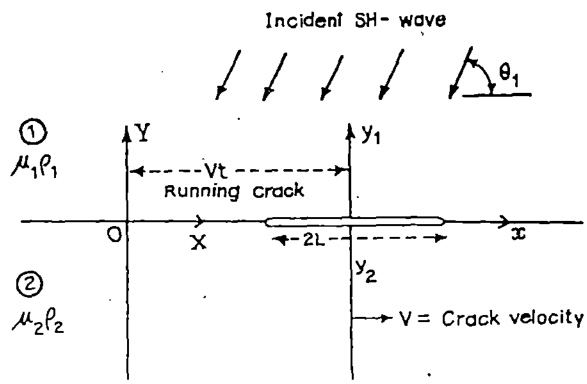

Fig. 1. Running interface crack.

in the medium. The crack lies on the bimaterial interface along $Y = 0$ with respect to the fixed rectangular co-ordinate system (X, Y, Z) as shown in Fig. 1.

We assume that the displacement and stress fields W_j, τ_{YZ_j} $(j = 1, 2)$ are

$$W_j = W_j(X, Y, T) \tag{2}$$

$$\tau_{YZ_j} = \mu_j \frac{\partial W_j(X, Y)}{\partial Y}, \tag{3}$$

in which subscripts $j = 1, 2$ refer to the upper and lower half planes, respectively, T denotes time and μ_j is the shear modulus of elasticity. The displacement W_j is governed by the classical wave equation

$$\frac{\partial^2 W_j}{\partial X^2} + \frac{\partial^2 W_j}{\partial Y^2} = \frac{1}{c_j^2}\frac{\partial^2 W}{\partial T^2}, (j = 1, 2), \tag{4}$$

where $c_j = (\mu_j/\rho_j)^{1/2}$ is the shear wave velocity and ρ_j is the density of the material. Without any loss of generality, we further assume that $c_1 > c_2$.

Due to the incident wave given by (1), reflected and transmitted waves in the absence of the crack may be written in the form

$$W_r = B \exp[-i\{k_1(X\cos\theta_1 - Y\sin\theta_1) + \Omega T\}] \tag{5}$$

and

$$W_T = C \exp[-i\{k_2(X\cos\theta_2 + Y\sin\theta_2) + \Omega T\}], \tag{6}$$

where

$$B = \frac{k_1\sin\theta_1 - mk_2\sin\theta_2}{k_1\sin\theta_1 + mk_2\sin\theta_2} A \tag{7}$$

$$C = \frac{2k_1\sin\theta_1}{k_1\sin\theta_1 + mk_2\sin\theta_2} A \tag{8}$$

$$m = \mu_2/\mu_1 \quad \text{and} \quad k_1\cos\theta_1 = k_2\cos\theta_2. \tag{9}$$

A, B, C are incident, reflected and transmitted wave amplitude, k_j is the wave number, $\Omega = k_j c_j$ is the circular frequency and θ_1, θ_2 are the angles of incidence and refraction, respectively.

A set of moving co-ordinates (x, y_j, z, t) attached to the centre of the crack moving at a constant velocity V is introduced in accordance with

$$x = X - Vt, \quad y_j = s_j Y, \quad z = Z, \quad t = T, \tag{10}$$

where $s_j = (1 - M_j^2)^{1/2}$ and $M_j = V/c_j$ is the Mach number.

In terms of the translating co-ordinates x, y_j, eq. (4) becomes

$$\frac{\partial^2 W_j}{\partial x^2}+\frac{\partial^2 W_j}{\partial y_j^2}+\frac{1}{c_j^2 s_j^2}\frac{\partial}{\partial t}\left[2M_j c_j\frac{\partial W_j}{\partial x}-\frac{\partial W_j}{\partial t}\right]=0. \tag{11}$$

In the moving system (x, y, z, t) eqs (1), (5) and (6) take the form

$$e^{-i\omega t}\begin{bmatrix} W_i \\ W_r \\ W_T \end{bmatrix} = \begin{bmatrix} A\exp\left[-i\left\{k_1\left(x\cos\theta_1+\frac{y_1}{s_1}\sin\theta_1\right)+\omega t\right\}\right] \\ B\exp\left[-i\left\{k_1\left(x\cos\theta_1-\frac{y_1}{s_1}\sin\theta_1\right)+\omega t\right\}\right] \\ C\exp\left[-i\left\{k_2\left(x\cos\theta_2+\frac{y_2}{s_2}\sin\theta_2\right)+\omega t\right\}\right] \end{bmatrix}, \tag{12}$$

where $\omega = \Omega\alpha$ and $\alpha = (1 + M_1\cos\theta_1) = (1 + M_2\cos\theta_2)$.

In view of eq. (12) we take the solution of (11) as

$$W_j(x, y_j)\,e^{-i\omega t} = w_j(x, y_j)\exp[i(M_j\lambda_j x - \omega t)]. \tag{13}$$

Substitution of eq. (13) into eq. (11) yields the Helmholtz equation governing w_j:

$$\frac{\partial^2 w_j}{\partial x^2}+\frac{\partial^2 w_j}{\partial y_j^2}+\lambda_j^2 w_j=0 \quad (j=1,2), \tag{14}$$

where

$$\lambda_j=\frac{k_j\alpha}{s_j^2}.$$

Applying Fourier transform, eq. (14) can be solved and the result is

$$w_1(x, y_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}B_1(\xi)\exp[-i\xi x-(\xi^2-\lambda_1^2)^{1/2}y_1]\,d\xi, \quad y_1>0 \tag{15}$$

$$w_2(x, y_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}B_2(\xi)\exp[-i\xi x+(\xi^2-\lambda_2^2)^{1/2}y_2]\,d\xi, \quad y_2<0. \tag{16}$$

From (13), (15) and (16) we obtain the displacement components due to scattered field as

$$W_1=\frac{1}{2\pi}\int_{-\infty}^{\infty}A_1(\xi)\exp[-i\xi x-v_1 y_1]\,d\xi, \quad y_1>0 \tag{17}$$

$$W_2=\frac{1}{2\pi}\int_{-\infty}^{\infty}A_2(\xi)\exp[-i\xi x+v_2 y_2]\,d\xi, \quad y_2<0, \tag{18}$$

where

$$v_j=[(\xi+\lambda_j M_j)^2-\lambda_j^2]^{1/2}, \quad j=1,2. \tag{19}$$

$A_1(\xi)$ and $A_2(\xi)$ are the unknown quantities to be determined from the following boundary conditions:

$$\mu_1 s_1\frac{\partial W_1}{\partial y_1}=\mu_2 s_2\frac{\partial W_2}{\partial y_2}, \quad \text{for all } x,\ y=0 \tag{20}$$

$$W_1=W_2, \quad |x|>L, \quad y=0 \tag{21}$$

$$\frac{\partial W_1}{\partial y_1}+\frac{\partial W_i}{\partial y_1}+\frac{\partial W_r}{\partial y_1}=0, \quad |x|<L, \quad y=0+. \tag{22}$$

From the boundary condition (22) we obtain

$$\frac{\partial W_1}{\partial y_1} = A_1 \exp[-ik_1 x\cos\theta_1], \quad |x| < L, \qquad y = 0, \tag{23}$$

where

$$A_1 = \frac{i(A-B)k_1 \sin\theta_1}{s_1}. \tag{24}$$

Using (17), the above equation can be written as

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)\nu_1 \exp[-i\xi x]\,\mathrm{d}\xi &= -A_1 \exp[-ik_1 x\cos\theta_1], \quad -L < x < L\\ &= P(x), \quad x > L \quad \text{(say)}\\ &= Q(x), \quad x < -L \quad \text{(say)}.\end{aligned}$$

Therefore

$$A_1(\xi)\nu_1 = \exp[i\xi L]G_+(\xi) + \exp[-i\xi L]G_-(\xi) - \frac{A_1}{i(\xi-\xi_0)}[\exp\{i(\xi-\xi_0)L\} - \exp\{-i(\xi-\xi_0)L\}], \tag{25}$$

where

$$G_+(\xi) = \int_L^{\infty} P(x)\exp[i\xi(x-L)]\,\mathrm{d}x \tag{26}$$

$$G_-(\xi) = \int_{-\infty}^{-L} Q(x)\exp[i\xi(x+L)]\,\mathrm{d}x \tag{27}$$

$$\xi_0 = k_1 \cos\theta_1. \tag{28}$$

From the boundary condition (20) we obtain

$$A_2(\xi) = -\frac{M\nu_1 A_1(\xi)}{\nu_2}, \tag{29}$$

where

$$M = \frac{\mu_1 s_1}{\mu_2 s_2}. \tag{30}$$

Next using the boundary condition (21), we obtain

$$\begin{aligned}A_1(\xi) - A_2(\xi) &= \int_{-\infty}^{\infty} (W_1 - W_2)\exp[i\xi x]\,\mathrm{d}x\\ &= \int_{-L}^{L} P_1(x)\exp[i\xi x]\,\mathrm{d}x\\ &= N(\xi) \quad \text{(say)},\end{aligned} \tag{31}$$

which is the measure of the discontinuity of displacement along the surface of the crack. Now with the aid of (29) and (31), we find

$$A_1(\xi) = \frac{\nu_2 N(\xi)}{\nu_2 + M\nu_1}. \tag{32}$$

Eliminating $A_1(\xi)$ from (25) and (32) we obtain an extended Wiener–Hopf equation, namely

$$\begin{aligned}&\exp[i\xi L]G_+(\xi) + \exp[-i\xi L]G_-(\xi) - N(\xi)K(\xi)\\ &\qquad = \frac{A_1}{i(\xi-\xi_0)}[\exp\{i(\xi-\xi_0)L\} - \exp\{-i(\xi-\xi_0)L\}],\end{aligned} \tag{33}$$

where

$$K(\xi) = \frac{\nu_1 \nu_2}{\nu_2 + M\nu_1} = \frac{\nu_1}{1+M} R(\xi) \tag{34}$$

$$R(\xi) = \frac{(1+M)\nu_2}{\nu_2 + M\nu_1}. \tag{35}$$

In order to solve the Wiener–Hopf equation given by (33) we assume that branch points $\xi = \lambda_1(1-M_1)$, $\lambda_2(1-M_2)$, $-\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ of $K(\xi)$ possess small imaginary parts, which would ultimately be made to tend to zero.

Now we write $K(\xi) = K_+(\xi)K_-(\xi)$, where $K_+(\xi)$ is analytic in the upper-half plane $\mathrm{Im}\,\xi > \mathrm{Im}[-\lambda_1(1+M_1)]$, whereas $K_-(\xi)$ is analytic in the lower-half plane given by $\mathrm{Im}\,\xi < \mathrm{Im}[\lambda_1(1-M_1)]$. The expressions of $K_+(\xi)$ and $K_-(\xi)$ are derived in the Appendix. Since $\partial W_1/\partial y_1$ decreases exponentially as $x \to \pm\infty$, $G_+(\xi)$ and $G_-(\xi)$ have the same common region of regularity as $K_+(\xi)$ and $K_-(\xi)$.

Now eq. (33) can easily be expressed as two integral equations involving $G_+(\xi)$, $G_-(\xi)$ and $N(\xi)$ as follows:

$$\frac{G_+(\xi)}{K_+(\xi)} - \frac{A_1 e^{-i\xi_0 L}}{i(\xi-\xi_0)}\left[\frac{1}{K_+(\xi)} - \frac{1}{K_+(\xi_0)}\right] + \frac{1}{2\pi i}\int_{c_+} \frac{e^{-2isL}}{(s-\xi)K_+(s)}\left[G_-(s) + \frac{A_1 e^{i\xi_0 L}}{i(s-\xi_0)}\right] ds$$

$$= N(\xi)K_-(\xi)e^{-i\xi L} + \frac{A_1 e^{-i\xi_0 L}}{i(\xi-\xi_0)K_+(\xi_0)} - \frac{1}{2\pi i}\int_{c_-} \frac{e^{-2isL}}{(s-\xi)K_+(s)}\left[G_-(s) + \frac{A_1 e^{i\xi_0 L}}{i(s-\xi_0)}\right] ds, \tag{36}$$

where c_+ and c_- are the straight contours below the pole at $\xi = \xi_0$ and situated within the common region of regularity of $G_+(\xi)$, $G_-(\xi)$, $K_+(\xi)$ and $K_-(\xi)$ as shown in Fig. 2.

The left hand side of (36) is analytic in the upper-half plane whereas the right hand side is analytic in the lower-half plane and both of them are equal in the common region of analyticity of these two functions. Therefore, by analytic continuation, both sides of (36) are analytic in the whole of the s-plane. Next, by Liouville's theorem, it can be shown that both sides of (36) are equal to zero. Thus we obtain

$$\frac{1}{K_+(\xi)}\left[G_+(\xi) - \frac{A_1 e^{-i\xi_0 L}}{i(\xi-\xi_0)}\right] + \frac{A_1 e^{-i\xi_0 L}}{i(\xi-\xi_0)K_+(\xi_0)}$$

$$+ \frac{1}{2\pi i}\int_{c_+} \frac{e^{2isL}}{(s-\xi)K_+(s)}\left[G_-(s) + \frac{A_1 e^{i\xi_0 L}}{i(s-\xi_0)}\right] ds = 0. \tag{37}$$

Similarly, we also obtain

$$\frac{1}{K_-(\xi)}\left[G_-(\xi) + \frac{A_1 e^{i\xi_0 L}}{i(\xi-\xi_0)}\right] + \frac{1}{2\pi i}\int_{c_-} \frac{e^{2isL}}{(s-\xi)K_-(s)}\left[G_+(s) - \frac{A_1 e^{-i\xi_0 L}}{i(s-\xi_0)}\right] ds = 0. \tag{38}$$

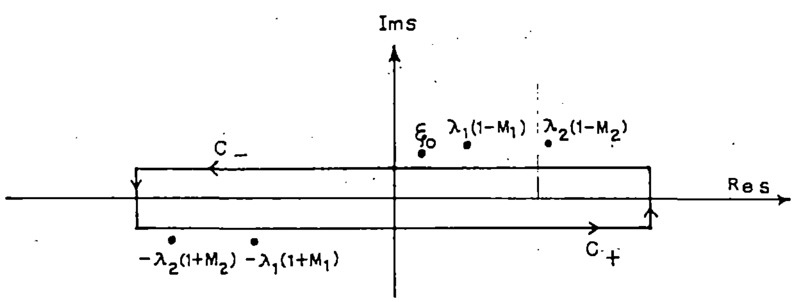

Fig. 2. Path of integration in the complex s-plane.

3. HIGH FREQUENCY SOLUTION OF THE INTEGRAL EQUATIONS

In order to obtain $G_+(\xi)$ and $G_-(\xi)$ from the integral equations (37) and (38) in the case when the normalized wave number $\lambda_1(1+M_1)L \gg 1$, the integration along the path c_+ in (37) is replaced by the integration along the loops $L_{-\lambda_1}$ and $L_{-\lambda_2}$ round the branch points $-\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ of $K_+(s)$, respectively. Also, the integration along the path c_- in (38) is replaced by the integration round the circular contour L_0, round the pole $s=\xi_0$ and by the integrations along the loops L_{λ_1} and L_{λ_2} round the branch cuts through the branch points $\lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ of the function $K_-(s)$ as shown in Fig. 3.

Finally evaluating the integrals along the straight line paths round the branch points for large values of frequency, we obtain two equations given by

$$F_\pm(\xi)+C_\pm(\xi)+\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\pm M_j)}A_\mp[\mp\lambda_j(1\pm M_j)]F_\mp[\mp\lambda_j(1\pm M_j)]}{2\{\lambda_j(1\pm M_j)-\xi\}(\lambda_jL)^{1/2}}=0, \tag{39}$$

where $\sigma_1=1$ and $\sigma_2=M$, and

$$F_\pm(\xi)=\frac{1}{K_\pm(\xi)}\left[G_\pm(\xi)\mp\frac{A_1\,e^{\mp i\xi_0L}}{i(\xi-\xi_0)}\right]$$

$$A_\pm(\xi)=\frac{i\,e^{i\pi/4}}{\pi^{1/2}}[K_\pm(\xi)]^2$$

$$C_\pm(\xi)=\frac{A_1\,e^{\mp i\xi_0L}}{i(\xi-\xi_0)K_\pm(\xi_0)}. \tag{40}$$

Now substituting $\xi=\lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ and $\xi=-\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ in (39) a system of linear equations of $F_+[\lambda_1(1-M_1)]$, $F_+[\lambda_2(1-M_2)]$, $F_-[-\lambda_1(1+M_1)]$ and $F_-[-\lambda_2(1+M_2)]$ are obtained. Now solving them and neglecting higher order terms of $(\lambda_1L)^{-1/2}$ and $(\lambda_2L)^{-1/2}$ we obtain, finally, after some algebraic manipulation:

$$F_\pm[\pm\lambda_k(1\mp M_k)]=-C_\pm[\pm\lambda_k(1\mp M_k)]$$
$$\times\left[1-\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\mp M_k)L}A_\mp[\mp\lambda_j(1\pm M_j)]C_\mp[\mp\lambda_j(1\pm M_j)]}{2(\lambda_jL)^{1/2}\{\lambda_j(1\pm M_j)+\lambda_k(1\mp M_k)\}C_\pm[\pm\lambda_k(1\mp M_k)]}\right],\quad k=1,2. \tag{41}$$

Now using (39) we obtain from (41)

$$G_\pm(\xi)=\pm\frac{A_1\,e^{\mp i\xi_0L}}{i(\xi-\xi_0)}\mp\frac{A_1\,e^{\mp i\xi_0L}K_\pm(\xi)}{i(\xi-\xi_0)K_\pm(\xi_0)}$$
$$+\sum_{k=1}^{2}\left[\frac{\sigma_k\,e^{2i\lambda_k(1\pm M_k)L}A_\mp[\mp\lambda_k(1\pm M_k)]C_\mp[\mp\lambda_k(1\pm M_k)]K_\pm(\xi)}{2(\lambda_kL)^{1/2}\{\lambda_k(1\pm M_k)\pm\xi\}}\right.$$
$$\left.\times\left(1-\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\mp M_j)L}A_\pm[\pm\lambda_j(1\mp M_j)]C_\pm[\pm\lambda_j(1\mp M_j)]}{2(\lambda_jL)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\}C_\mp[\mp\lambda_k(1\pm M_k)]}\right)\right]. \tag{42}$$

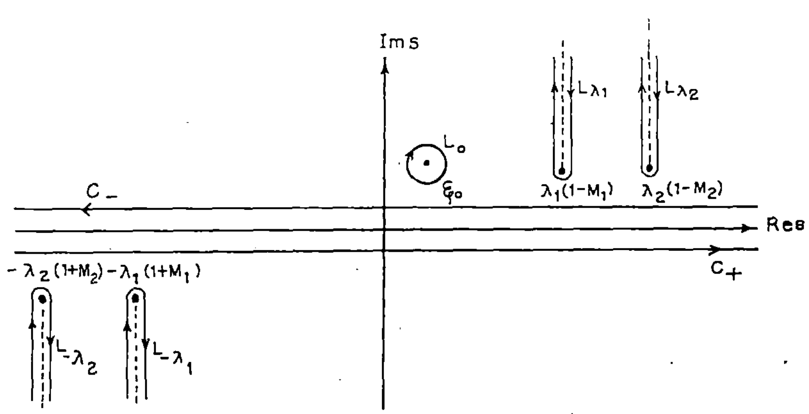

Fig. 3. Path of integration L_0, L_{λ_1}, L_{λ_2} and $L_{-\lambda_1}$, $L_{-\lambda_2}$.

4. CRACK OPENING DISPLACEMENT AT POINTS AWAY FROM THE CRACK TIPS

In order to obtain the displacement jump for the large values of $\lambda_1(L-x)$, $\lambda_2(L-x)$, $\lambda_1(L+x)$ and $\lambda_2(L+x)$, we can write $G_+(\xi)$ and $G_-(\xi)$ from (42) as

$$G_\pm(\xi) = \pm\frac{P_\pm}{\xi-\xi_0} \mp \frac{Q_\pm K_\pm(\xi)}{\xi-\xi_0} + \sum_{k=1}^{2}\frac{K_\pm(\xi)R_\pm^{(k)}}{\{\lambda_k(1\pm M_k)\pm\xi\}}, \tag{43}$$

where

$$P_\pm = \frac{A_1\,e^{\mp i\xi_0 L}}{i} \tag{44}$$

$$Q_\pm = \frac{A_1\,e^{\mp i\xi_0 L}}{iK_\pm(\xi_0)} = \frac{P_\pm}{K_\pm(\xi_0)} \tag{45}$$

$$R_\pm^{(k)} = \frac{\sigma_k\,e^{2i\lambda_k(1\pm M_k)L}A_\mp[\mp\lambda_k(1\pm M_k)]C_\mp[\mp\lambda_k(1\pm M_k)]}{2(\lambda_k L)^{1/2}} \times\left(1-\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\mp M_j)L}A_\pm[\pm\lambda_j(1\mp M_j)]C_\pm[\pm\lambda_j(1\mp M_j)]}{2(\lambda_j L)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\}C_\mp[\mp\lambda_k(1\pm M_k)]}\right). \tag{46}$$

Now we obtain from (33)

$$N(\xi) = -\frac{Q_+\,e^{i\xi L}}{(\xi-\xi_0)K_-(\xi)} + \frac{R_+^{(1)}\,e^{i\xi L}}{\{\xi+\lambda_1(1+M_1)\}K_-(\xi)} + \frac{R_+^{(2)}\,e^{i\xi L}}{\{\xi+\lambda_2(1+M_2)\}K_-(\xi)} + \frac{Q_-\,e^{-i\xi L}}{(\xi-\xi_0)K_+(\xi)} - \frac{R_-^{(1)}\,e^{-i\xi L}}{\{\xi-\lambda_1(1-M_1)\}K_+(\xi)} - \frac{R_-^{(2)}\,e^{-i\xi L}}{\{\xi-\lambda_2(1-M_2)\}K_+(\xi)}. \tag{47}$$

From (31) we obtain the displacement jump across the surface of the crack as

$$W_1(x,0+) - W_2(x,0-) = \frac{1}{2\pi}\int_{-\infty}^{\infty} N(\xi)\,e^{-i\xi x}\,d\xi. \tag{48}$$

Substituting the expression of $N(\xi)$ from (47) in (48) and approximately evaluating the integrals arising in (48) term by term for large values of $\lambda_1(L-x)$, $\lambda_2(L-x)$, $\lambda_1(L+x)$, and $\lambda_2(L+x)$, and neglecting terms of order higher than $(\lambda_1 L)^{-3/2}$ and $(\lambda_2 L)^{-3/2}$, we finally obtain the crack opening displacement across the cracked surface at points away from the crack tips in the following form:

$$\Delta W = W_1(x,0+) - W_2(x,0-) = -iQ_+K_+(\xi_0)\,e^{i\xi_0(L-x)} \times\left[\frac{1}{\{(\xi_0+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}} + \frac{M}{\{(\xi_0+\lambda_2 M_2)^2-\lambda_2^2\}^{1/2}}\right] - \frac{e^{-i\pi/4}}{\pi^{1/2}}[T_+ - T_-], \tag{49}$$

where

$$T_\pm = \sum_{k=1}^{2}\frac{\sigma_k\,e^{i\lambda_k(1\mp M_k)(L\mp x)}}{\{\lambda_k(L\mp x)\}^{1/2}}\left[\frac{Q_\pm K_\pm[\pm\lambda_k(1\mp M_k)]}{2^{1/2}[\lambda_k(1\mp M_k)\mp\xi_0]} - \sum_{j=1}^{2}\frac{\sigma_j A_\mp[\mp\lambda_j(1\pm M_j)]K_\pm[\pm\lambda_k(1\mp M_k)]}{2(2\lambda_j L)^{1/2}\{\lambda_k(1\mp M_k)+\lambda_j(1\pm M_j)\}}\left(\frac{Q_\mp\,e^{2i\lambda_j(1\pm M_j)L}}{\{\lambda_j(1\pm M_j)\pm\xi_0\}} - \sum_{r=1}^{2}\frac{\sigma_r A_\pm[\pm\lambda_r(1\mp M_r)]Q_\pm\,e^{2i[\lambda_r(1\mp M_r)+\lambda_j(1\pm M_j)]L}}{2(\lambda_r L)^{1/2}\{\lambda_r(1\mp M_r)+\lambda_j(1\pm M_j)\}\{\lambda_r(1\mp M_r)\mp\xi_0\}}\right)\right]. \tag{50}$$

5. STRESS INTENSITY FACTOR AND CRACK OPENING DISPLACEMENT NEAR THE CRACK TIPS

Now considering the behaviour of ξ at infinity we obtain from (42)

$$G_\pm(\xi) \simeq \pm\frac{A_1\,e^{\mp i\xi_0 L}}{i(\xi-\xi_0)} + S_\pm\,\xi^{-1/2} \quad \text{as } \xi\to\infty, \tag{51}$$

where

$$S_{\pm}=\frac{1}{(1+M)^{1/2}}\left[\mp\frac{A_1\,e^{\mp i\xi_0 L}}{iK_{\pm}(\xi_0)}\pm\sum_{k=1}^{2}\frac{\sigma_k\,e^{2i\lambda_k(1\pm M_k)L}A_{\mp}[\mp\lambda_k(1\pm M_k)]C_{\mp}[\mp\lambda_k(1\pm M_k)]}{2(\lambda_k L)^{1/2}}\right.$$

$$\left.\times\left(1-\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\mp M_j)L}A_{\pm}[\pm\lambda_j(1\mp M_j)]C_{\pm}[\pm\lambda_j(1\mp M_j)]}{2(\lambda_j L)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\}C_{\mp}[\mp\lambda_k(1\pm M_k)]}\right)\right]. \tag{52}$$

Now, from eq. (33), using (51) and also the fact that

$$K(\xi)\to\pm\frac{\xi}{1+M}\quad\text{as }\xi\to\pm\infty, \tag{53}$$

we obtain

$$N(\xi)=\frac{1+M}{\pm\xi(\xi)^{1/2}}[S_+\,e^{i\xi L}+S_-\,e^{-i\xi L}]\quad\text{as }\xi\to\pm\infty. \tag{54}$$

Taking the inverse Fourier transform of (31) and using the results of Fresnel integrals, *viz.*

$$\int_0^{\infty}\frac{\begin{matrix}\sin\\ \cos\end{matrix}\,(x+L)\alpha}{(\alpha)^{1/2}}\,d\alpha=\left[\frac{\pi}{2(x+L)}\right]^{1/2}, \tag{55}$$

we obtain the displacement jump across the surface of the crack as

$$\Delta W=W_1(x,0+)-W_2(x,0-)=-(1+M)(1+i)S_-\left[\frac{2(x+L)}{\pi}\right]^{1/2}\quad\text{for }x\to-L+0 \tag{56}$$

$$=-(1+M)(1-i)S_+\left[\frac{2(L-x)}{\pi}\right]^{1/2}\quad\text{for }x\to L-0. \tag{57}$$

Expressions (56) and (57) give the displacement jump near to the crack tips, whereas the displacement jump away from the crack tips is given by (49).

Next, in order to find the value of τ_{yz} near to the crack tip we use (54) in (32) and (29) and obtain

$$A_j(\xi)=\frac{(-1)^{j+1}\sigma_j}{\xi(\xi)^{1/2}}[S_+\,e^{i\xi L}+S_-\,e^{-i\xi L}],\quad j=1,2\quad\text{as }\xi\to\infty \tag{58}$$

$$A_j(\xi)=\frac{i(-1)^{j+1}\sigma_j}{\xi(-\xi)^{1/2}}[S_+\,e^{i\xi L}-S_-\,e^{-i\xi L}],\quad j=1,2\quad\text{as }\xi\to-\infty. \tag{59}$$

Now

$$\tau_{yz}(x,y_j)=\mu_j\frac{\partial W_j(x,y_j)}{\partial y}=\mu_j s_j\frac{\partial W_j(x,y_j)}{\partial y_j}=\frac{\mu_j s_j}{2\pi}\frac{\partial}{\partial y_j}\left[\int_{-\infty}^{\infty}A_j(\xi)\,e^{-i\xi x-\nu_j|y_j|}\,d\xi\right]. \tag{60}$$

Now substituting the values of $A_j(\xi)$ as $|\xi|\to\infty$ in (60) and integrating, we obtain the stress near to the crack tip as

$$\tau_{yz}(x,y_1)=-\frac{\mu_1 s_1}{(2\pi)^{1/2}}\left[(1-i)S_+\frac{\cos(\psi_1/2)}{r_1^{1/2}}+(1+i)S_-\frac{\sin(\psi_2/2)}{r_2^{1/2}}\right] \tag{61}$$

and

$$\tau_{yz}(x,y_2)=-\frac{\mu_1 s_1}{(2\pi)^{1/2}}\left[(1-i)S_+\frac{\cos(\phi_1/2)}{d_1^{1/2}}+(1+i)S_-\frac{\cos(\phi_2/2)}{d_2^{1/2}}\right], \tag{62}$$

where

$$r_1 = \{(x-L)^2 + y_1^2\}^{1/2}, \quad \psi_1 = \sin^{-1}\frac{|y_1|}{r_1}$$

$$r_2 = \{(x+L)^2 + y_1^2\}^{1/2}, \quad \psi_2 = \sin^{-1}\frac{|y_1|}{r_2}$$

$$d_1 = \{(x-L)^2 + y_2^2\}^{1/2}, \quad \phi_1 = \sin^{-1}\frac{|y_2|}{d_1}$$

$$d_2 = \{(x+L)^2 + y_2^2\}^{1/2}, \quad \phi_2 = \sin^{-1}\frac{|y_2|}{d_2}. \tag{63}$$

Therefore at the interface ($y = 0$) near to the right-hand crack vertex, we obtain

$$\tau_{yz} \to -\frac{\mu_1 s_1 (1-i) S_+}{\{2\pi(x-L)\}^{1/2}} \quad \text{as } x \to L + 0. \tag{64}$$

Now the normalized dynamic stress intensity factor K at the crack tip $x = L$ is defined by

$$K = \left|\frac{[2\pi k_1 (x-L)]^{1/2}\tau_{yz}}{\mu_1 A_1}\right| = s_1\left|\frac{(1-i)S_+(k_1)^{1/2}}{A_1}\right| \quad \text{for } x \to L+0, \tag{65}$$

where A_1 is given by (24).

The absolute values of the complex stress intensity factor defined by (65) have been plotted against $k_1 L$ in Fig. 4 for values $k_1 L > 1$ for different values of the Mach number M_2 and the angle of incidence for the following sets of materials:

first set:	steel	$\rho_1 = 7.6$ gm/cm^3,	$\mu_1 = 8.32 \times 10^{11}$ dyne/cm^2
	aluminium	$\rho_2 = 2.7$ gm/cm^3,	$\mu_2 = 2.63 \times 10^{11}$ dyne/cm^2
second set:	wrought iron	$\rho_1 = 7.8$ gm/cm^3,	$\mu_1 = 7.7 \times 10^{11}$ dyne/cm^2
	copper	$\rho_2 = 8.96$ gm/cm^3,	$\mu_2 = 4.5 \times 10^{11}$ dyne/cm^2.

As the Mach number $M_2 \to 0$ the stress intensity factor K tends to the value of the stress intensity factor corresponding to the stationary crack. The problem for $\theta_1 = \pi/2$ and $M_2 = 0.0$ was solved earlier by Pal and Ghosh [9]. The graph of stress intensity factor vs $k_1 L$ corresponding to $\theta_1 = \pi/2$ and $M_2 = 0.0$ as given in Fig. 4a is found to coincide exactly with that given by Pal and

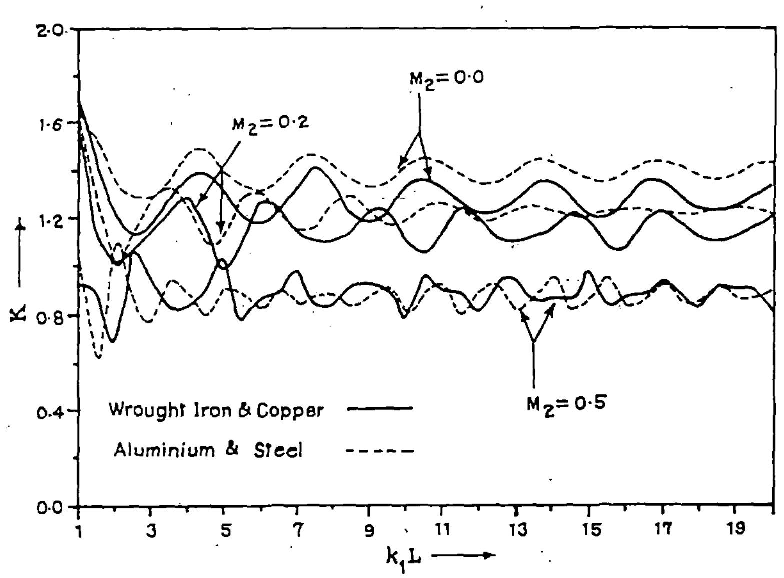

Fig. 4(a) (*caption overleaf*)

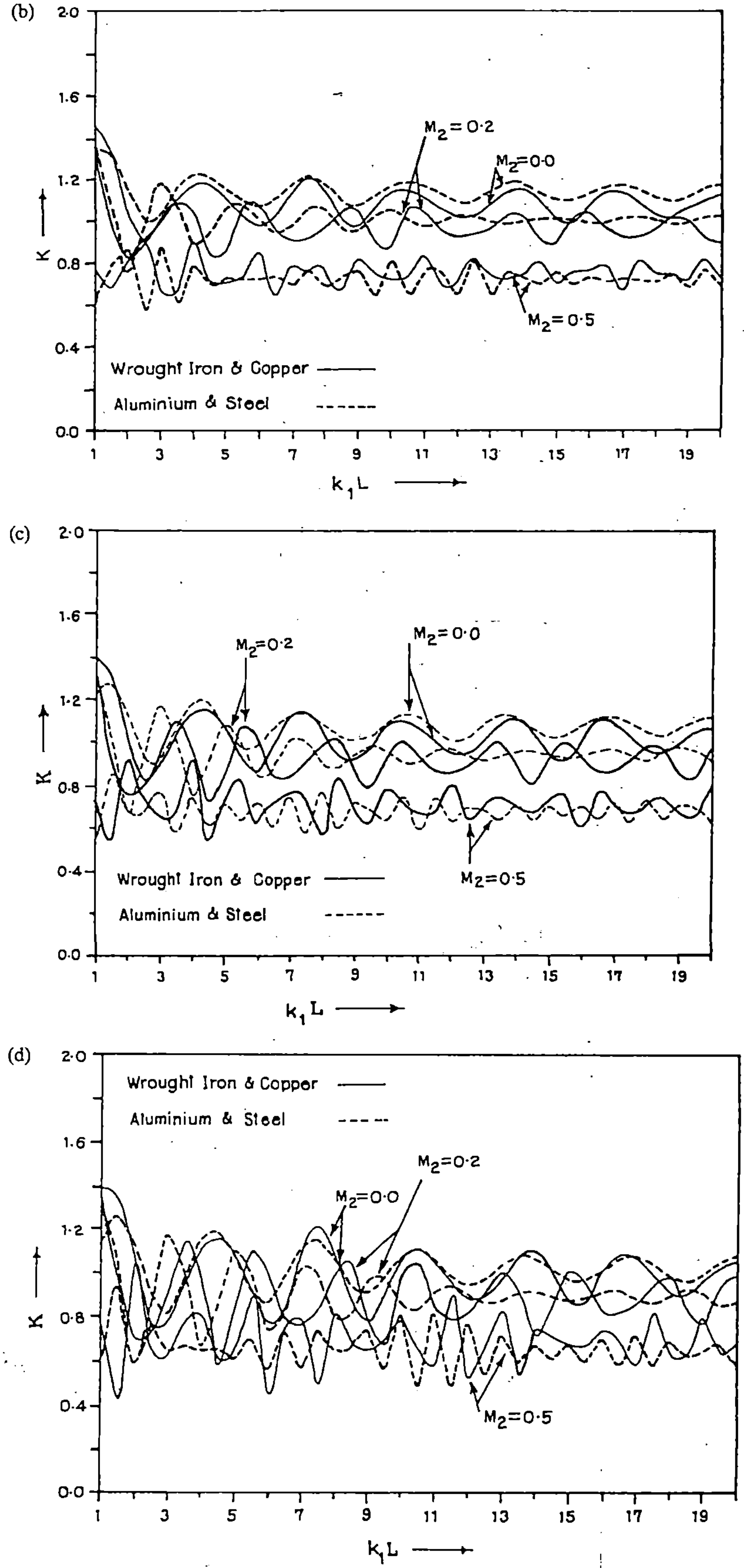

Fig. 4. Stress intensity factor K versus dimensionless $k_1 L$. (a) $\theta_1 = \pi/2$. (b) $\theta_1 = \pi/3$. (c) $\theta_1 = \pi/4$. (d) $\theta_1 = \pi/6$.

Ghosh [9]. It is interesting to note that for both pairs of materials, as M_2 increases, the peaks of the curves of stress intensity factors decrease in magnitude and occur at lower values of $k_1 L$. Further, it may be noted that for any fixed value of M_2 the stress intensity factor decreases with the decrease in the value of the angle of incidence.

REFERENCES

[1] M. Takei, Y. Shindo and A. Atsumi, Diffraction of transient horizontal shear waves by a finite crack at the interface of two bonded dissimilar elastic solids. *Engng Fracture Mech.* **16,** 799–807 (1982).
[2] S. Ueda, Y. Shindo and A. Atsumi, Torsional impact response of a penny-shaped crack lying on a bimaterial interface. *Engng Fracture Mech.* **18,** 1058–1066 (1983).
[3] K. N. Srivastava, R. M. Palaiya and D. S. Karaulia, Interaction of antiplane shear waves by a Griffith crack at interface of two bonded dissimilar elastic half-spaces. *Int. J. Fracture* **16,** 349–358 (1980).
[4] A. Jahanshahi, A diffraction problem and crack propagation. *J. appl. Mech.* **34,** 100–103 (1967).
[5] E. P. Chen and G. C. Sih, Running crack in an incident wave field. *Int. J. Solids Structures* **9,** 897–919 (1973).
[6] G. C. Sih and J. F. Loeber, Interaction of horizontal shear waves with a running crack. *J. appl. Mech.* **37,** 324–330 (1970).
[7] E. P. Chen and G. C. Sih, Scattering of plane waves by a propagating crack. *J. appl. Mech.* **42,** 705–711 (1975).
[8] S. J. Chang, Diffraction of plane dilatational waves by a finite crack. *Q. Jl Mech. appl. Math.* **24,** 424–443 (1971).
[9] S. C. Pal and M. L. Ghosh, High frequency scattering of antiplane shear waves by an interface crack. *Ind. Jl Pure appl. Math.* **21,** 1107–1124 (1990).

APPENDIX

$$K(\xi) = \frac{\{(\xi + \lambda_1 M_1)^2 - \lambda_1^2\}^{1/2}}{1 + M} R(\xi), \tag{A1}$$

where

$$M = \frac{\mu_1 s_1}{\mu_2 s_2}$$

$$R(\xi) = \frac{(1 + M)\{(\xi + \lambda_2 M_2)^2 - \lambda_2^2\}^{1/2}}{M\{(\xi + \lambda_1 M_1)^2 - \lambda_1^2\}^{1/2} + \{(\xi + \lambda_2 M_2)^2 - \lambda_2^2\}^{1/2}} \to 1 \quad \text{as } |\xi| \to \infty.$$

Now

$$R_+(\xi) R_-(\xi) = \frac{1}{\dfrac{1}{1+M} + \dfrac{M\{(\xi + \lambda_1 M_1)^2 - \lambda_1^2\}^{1/2}}{(1+M)\{(\xi + \lambda_2 M_2)^2 - \lambda_2^2\}^{1/2}}}.$$

Taking logs on both sides

$$\log R(\xi) = \log R_+(\xi) + \log R_-(\xi) = \frac{1}{2\pi i} \int_{c_L + c_U} \frac{\log R(\eta)}{\eta - \xi} \, d\eta,$$

where the paths of integration c_L and c_U are as shown in Fig. A1. Therefore

$$\log R_+(\xi) = \frac{1}{2\pi i} \int_{c_L} \frac{\log R(\eta)}{\eta - \xi} \, d\eta$$

$$\log R_-(\xi) = \frac{1}{2\pi i} \int_{c_U} \frac{\log R(\eta)}{\eta - \xi} \, d\eta$$

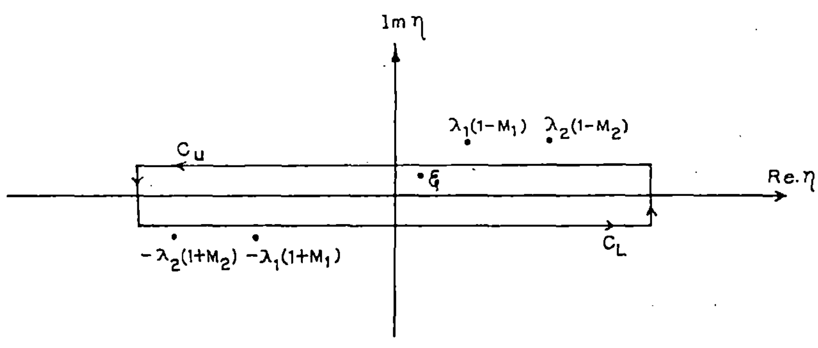

Fig. A1. Complex η-plane.

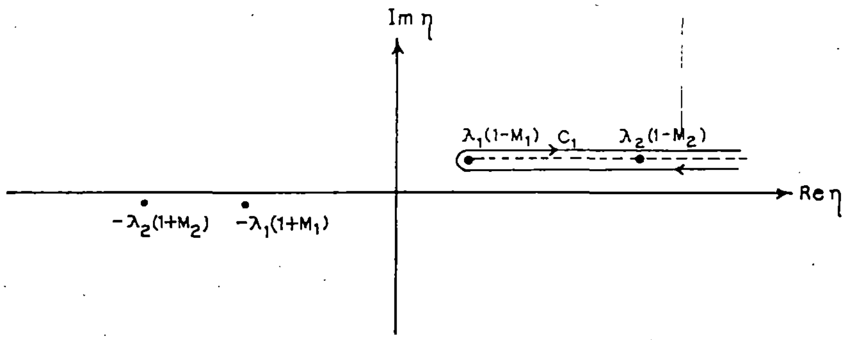

Fig. A2. Path of integration round the branch points.

or

$$\log R_+(\xi)=\frac{1}{2\pi i}\int_{-i\kappa-\infty}^{-i\kappa+\infty}\frac{\log R(\eta)}{\eta-\xi}\,d\eta.$$

Putting $\eta=-\eta$

$$\log R_+(\xi)=\frac{1}{2\pi i}\int_{i\kappa+\infty}^{i\kappa-\infty}\frac{\log R(-\eta)}{\eta+\xi}\,d\eta$$

$$\log R_-(\xi)=\frac{1}{2\pi i}\int_{i\kappa+\infty}^{i\kappa-\infty}\frac{\log R(\eta)}{\eta-\xi}\,d\eta,$$

therefore

$$\log R_-(\xi)=\frac{1}{2\pi i}\int_{c_1}\frac{1}{(\eta-\xi)}\log\left[\frac{1}{\dfrac{1}{1+M}+\dfrac{M\{(\eta+\lambda_1M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{(\eta+\lambda_2M_2)^2-\lambda_2^2\}^{1/2}}}\right]d\eta,$$

where c_1 is the contour round the branch points $\lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ as shown in Fig. A2. Therefore

$$\log R_-(\xi)=\frac{1}{2\pi i}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)}\frac{1}{(\eta-\xi)}\left[\log\left(\frac{1}{1+M}+\frac{iM\{(\eta+\lambda_1M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{\lambda_2^2-(\eta+\lambda_2M_2)^2\}^{1/2}}\right)\right.$$
$$\left.-\log\left(\frac{1}{1+M}-\frac{iM\{(\eta+\lambda_1M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{\lambda_2^2-(\eta+\lambda_2M_2)^2\}^{1/2}}\right)\right]d\eta$$
$$=\frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)}\frac{1}{(\eta-\xi)}\tan^{-1}\left[\frac{M\{(\eta+\lambda_1M_1)^2-\lambda_1^2\}^{1/2}}{\{\lambda_2^2-(\eta+\lambda_2M_2)^2\}^{1/2}}\right]d\eta,$$

and therefore

$$R_-(\xi)=\exp\left[\frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)}\frac{1}{(\eta-\xi)}\tan^{-1}\left(\frac{M[(\eta+\lambda_1M_1)^2-\lambda_1^2]^{1/2}}{[\lambda_2^2-(\eta+\lambda_2M_2)^2]^{1/2}}\right)d\eta\right].$$

Similarly

$$R_+(\xi)=\exp\left[\frac{1}{\pi}\int_{\lambda_1(1+M_1)}^{\lambda_2(1+M_2)}\frac{1}{(\eta+\xi)}\tan^{-1}\left(\frac{M[(\eta-\lambda_1M_1)^2-\lambda_1^2]^{1/2}}{[\lambda_2^2-(\eta-\lambda_2M_2)^2]^{1/2}}\right)d\eta\right].$$

Therefore from (A1) we can write

$$K_+(\xi)=\left[\frac{\xi+\lambda_1(1+M_1)}{(1+M)}\right]^{1/2}\exp\left[\frac{1}{\pi}\int_{\lambda_1(1+M_1)}^{\lambda_2(1+M_2)}\frac{1}{(\eta+\xi)}\tan^{-1}\left(\frac{M[(\eta-\lambda_1M_1)^2-\lambda_1^2]^{1/2}}{[\lambda_2^2-(\eta-\lambda_2M_2)^2]^{1/2}}\right)d\eta\right] \tag{A2}$$

and

$$K_-(\xi)=\left[\frac{\xi-\lambda_1(1-M_1)}{(1+M)}\right]^{1/2}\exp\left[\frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)}\frac{1}{(\eta-\xi)}\tan^{-1}\left(\frac{M[(\eta+\lambda_1M_1)^2-\lambda_1^2]^{1/2}}{[\lambda_2^2-(\eta+\lambda_2M_2)^2]^{1/2}}\right)d\eta\right]. \tag{A3}$$

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FORCED VERTICAL VIBRATION OF FOUR RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

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Abstract—In this paper, the problem of two-dimensional oscillations of four rigid strips, situated on a homogeneous isotropic semi-infinite elastic solid and forced by a specified normal component of the displacement has been considered. The mixed boundary value problem of determining the unknown stress distribution just below the strips and vertical displacement outside the strips has been converted to the determination of the solution of quadruple integral equations by the use of Fourier transform. An iterative solution of these integral equations valid for low frequency has been found by the application of the finite Hilbert transform. The normal stress just below the strips and the vertical displacement away from the strips have been obtained. Finally, graphs are presented which illustrate the salient features of the displacement and stress intensity factors at the edges of the strips.

1. INTRODUCTION

The problem of the effect of vibrating source in different forms on the surface of an elastic medium have aroused attention in view of their application in seismology and geophysics. Reissner (1937), and later Millar and Pursey (1954), treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. Analytical treatment of the dynamical response of footings and solid-structure interaction are usually available in the literature only for circular and elliptical footings, and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, and also in the study of the vibration of dams and large structures subjected to earthquakes. The problem of circular punch has been solved analytically by Awojobi and Grootenhuis (1965), Robertson (1966), Gladwell (1968) and others. Roy (1986) considered the dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. Karasudhi *et al.* (1968) obtained a low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium. Wickham (1977) worked out in detail the problem of forced two-dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium. Recently, Mandal and Ghosh (1992) treated the problem of forced vertical vibration of two rigid strips on a semi-infinite elastic medium.

To improve the dynamic models of buildings and other structures, it will be fruitful to have analytic results for foundations of a more complicated nature. In what follows, the problem of vertical vibration of four rigid strips in smooth contact with a semi-infinite elastic medium has been considered. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by running wheels on a railway track. The resulting mixed boundary value problem has been reduced to the solution of

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quadruple integral equations, which have further been reduced to the solution of integral-differential equations. Finally, an iterative solution valid for low frequency has been obtained.

From the solution of the integral equations, the stress just below the strips and also the vertical displacement at points outside the strips on the free surface have been found. The effects of stress intensity factors at the edges of the strips and vertical displacement outside the strips have been shown by means of graphs.

2. FORMULATION OF THE PROBLEM

Consider the normal vibration of frequency ω of four rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty < X < \infty$, $Y \geqslant 0$, $-\infty < Z < \infty$. It is assumed that the motion is forced by prescribed displacement distribution $v_0 e^{-i\omega t}$ normal to the four infinite strips located in the region $d_1 \leqslant |X| \leqslant d_2$, $d_3 \leqslant |X| \leqslant d$, $Y = 0$, $|Z| < \infty$, where v_0 is a constant.

Normalizing all the lengths with respect to d and putting $X/d = x$, $Y/d = y$, $Z/d = z$, $d_1/d = a$, $d_2/d = b$, $d_3/d = c$, one finds that the rigid strips are defined by $a \leqslant |x| \leqslant b$, $c \leqslant |x| \leqslant 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

With the time factor $e^{-i\omega t}$ suppressed throughout the analysis, the displacement components can be written as

$$u(x,y) = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}; \quad v(x,y) = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x}; \quad w(x,y) = 0 \tag{1}$$

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + m_1^2\phi &= 0 \\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + m_2^2\psi &= 0 \end{aligned} \tag{2}$$

in which

$$m_1^2 = \frac{\omega^2 d^2}{c_1^2} \quad \text{and} \quad m_2^2 = \frac{\omega^2 d^2}{c_2^2}.$$

In terms of ϕ and ψ the stress components are

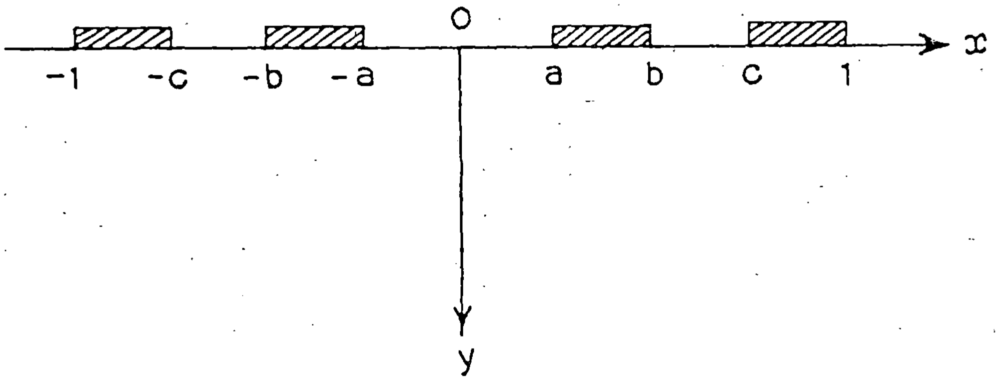

Fig. 1. Geometry of the problem.

$$\tau_{xy} = \mu\left\{2\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right\}$$

$$\tau_{yy} = -\mu\left\{\left(m_2^2+2\frac{\partial^2}{\partial x^2}\right)\phi-2\frac{\partial^2\psi}{\partial x\partial y}\right\}$$

$$\tau_{yz} = 0. \tag{3}$$

The boundary conditions are

$$v(x,0) = v_0, \quad x\in I_2, I_4 \tag{4}$$

$$\tau_{yy}(x,0) = 0, \quad x\in I_1, I_3, I_5 \tag{5}$$

$$\tau_{xy}(x,0) = 0, \quad -\infty < x < \infty \tag{6}$$

where $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,1)$, $I_5 = (1,\infty)$. The solution of the Helmholtz equation (2) can be written as

$$\phi = 2\int_0^\infty A(\xi)\cos\xi x e^{-\gamma_1 y}\,d\xi$$

$$\psi = 2\int_0^\infty B(\xi)\sin\xi x e^{-\gamma_2 y}\,d\xi \tag{7}$$

where

$$\gamma_j = \begin{cases}(\xi^2-m_j^2)^{1/2}, & |\xi|\geqslant m_j\\ -i(m_j^2-\xi^2)^{1/2}, & |\xi|\leqslant m_j\end{cases}, \quad j = 1,2$$

and $A(\xi)$ and $B(\xi)$ are unknown functions to be determined from the boundary conditions.

By using the boundary condition (6), it can be shown that

$$B(\xi) = \frac{2\gamma_1\xi}{\xi^2+\gamma_2^2}A(\xi). \tag{8}$$

Now the displacement component v and stress τ_{yy} become

$$v(x,y) = 2\int_0^\infty\left[\frac{2\xi^2}{2\xi^2-m_2^2}e^{-\gamma_2 y}-e^{-\gamma_1 y}\right]A(\xi)\cos\xi x\,d\xi \tag{9}$$

$$\tau_{yy}(x,y) = -2\mu\int_0^\infty\left[(m_2^2-2\xi^2)e^{-\gamma_1 y}+\frac{2\xi^2\gamma_1\gamma_2}{2\xi^2-m_2^2}e^{-\gamma_2 y}\right]A(\xi)\cos\xi x\,d\xi. \tag{10}$$

From the boundary conditions (4) and (5) we get the following set of integral equations in $P(\xi)$:

$$\int_0^\infty\frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}P(\xi)\cos\xi x\,d\xi = \frac{1}{2}v_0, \quad x\in I_2, I_4 \tag{11}$$

and

$$\int_0^\infty P(\xi)\cos\xi x\,\mathrm{d}\xi = 0, \quad x\in I_1, I_2, I_5 \tag{12}$$

where

$$P(\xi) = \frac{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2}{(2\xi^2-m_2^2)}A(\xi).$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (11) and (12) in the form

$$P(\xi) = \int_a^b tf(t^2)\cos\xi t\,\mathrm{d}t + \int_c^1 ug(u^2)\cos\xi u\,\mathrm{d}u \tag{13}$$

where $f(t^2)$ and $g(u^2)$ are unknown functions to be determined.

By the choice of $P(\xi)$ given by eqn (13) the relation (12) is satisfied automatically and eqn (11) becomes

$$\int_a^b tf(t^2)\,\mathrm{d}t\int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\cos\xi x\cos\xi t\,\mathrm{d}\xi$$
$$+\int_c^1 ug(u^2)\,\mathrm{d}u\int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\cos\xi x\cos\xi u\,\mathrm{d}\xi = \frac{v_0}{2}, \quad x\in I_2, I_4 \tag{14}$$

using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b tf(t^2)\,\mathrm{d}t\frac{\partial}{\partial t}\int_0^x\int_0^t \frac{wvL_1(v,w)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} +$$
$$+\frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 ug(u^2)\,\mathrm{d}u\frac{\partial}{\partial u}\int_0^x\int_0^u \frac{wvL_1(v,w)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}} = \frac{v_0}{2}, \quad x\in I_2, I_4 \tag{15}$$

where

$$L_1(v,w) = \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}J_0(\xi w)J_0(\xi v)\,\mathrm{d}\xi. \tag{16}$$

By a simple contour integration technique used by Ghosh and Ghosh (1985), $L_1(v,w)$ can be written as

$$L_1(v,w) = -\mathrm{i}\tau^2\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\mathrm{d}\eta$$
$$-4\mathrm{i}\tau^2\int_0^1 \frac{\eta^2(\eta^2-1)(\tau^2-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\mathrm{d}\eta$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v$$

$$=\frac{-i\tau^2}{16(1-\tau^2)}\left[\sum_{j=0}^{2}P_j\int_0^1\frac{(1-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{\eta^2-\tau_j^2}\,\mathrm{d}\eta\right.$$

$$\left.+\sum_{j=0}^{2}S_j\int_0^\tau\frac{(\tau^2-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{\eta^2-\tau_j^2}\,\mathrm{d}\eta\right]$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v \tag{17}$$

where

$$\tau=\frac{m_2}{m_1}=\frac{c_1}{c_2}, \quad Q_0(\eta)=(2\eta^2-\tau^2)^2-4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$$

and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta)=0$. τ_1, τ_2 are the roots of the equation

$$(2\eta^2-\tau^2)^2+4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}=0.$$

$Q_0'(\eta)$ denotes the derivative of $Q_0(\eta)$ with respect to η and

$$P_j=\frac{(2\tau_j^2-\tau^2)}{\prod_i(\tau_j^2-\tau_i^2)}, \quad S_j=\frac{4\tau_j^2(\tau_j^2-1)}{\prod_i(\tau_j^2-\tau_i^2)}, \quad i,j=0,1,2 \quad \text{and} \quad i\neq j.$$

The corresponding expression for $L_1(v,w)$ for $w<v$ follows from eqn (17) by interchanging w and v. For a Poisson ratio $\sigma=\frac{1}{4}$, the values of τ, τ_0, τ_1 and τ_2 are given by

$$\tau^2=\frac{2(1-\sigma)}{(1-2\sigma)}=3, \quad \tau_0^2=\frac{3}{(0.9194)^2}, \quad \tau_1^2=\frac{3}{(2+2\sqrt{3})} \quad \text{and} \quad \tau_2^2=\frac{3}{4}.$$

Hence, in this case $\tau_2<\tau_1<1<\tau<\tau_0$.

By using the series expansions of J_0 and $H_0^{(1)}$, and evaluating the integrals arising in eqn (17), we obtain, after some algebraic manipulation,

$$L_1(v,w)=\frac{2}{\pi}\tau^2\left[\left(\gamma+\log\frac{m_1 w}{2}-\frac{\pi i}{2}\right)M+N-\frac{P}{4}(w^2+v^2)m_1^2\log m_1\right]+O(m_1^2) \quad w>v.$$

$$=\frac{2}{\pi}\tau^2\left[\left(\gamma+\log\frac{m_1 v}{2}-\frac{\pi i}{2}\right)M+N-\frac{P}{4}(w^2+v^2)m_1^2\log m_1\right]+O(m_1^2) \quad w<v, \tag{18}$$

where $\gamma=0.5772157\ldots$ is Euler's constant,

$$M\equiv-\frac{\pi}{4(1-\tau^2)} \tag{19}$$

$$N = \frac{\pi}{32(1-\tau^2)}\left[4\log\frac{4}{\tau} + \sum_{j=1}^{2} P_j \frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j} - P_0\frac{\sqrt{(\tau_0^2-1)}}{\tau_0}\right.$$

$$\log\{\tau_0 + \sqrt{(\tau_0^2-1)}\} + \sum_{j=1}^{2} S_j \frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j}$$

$$\left. - S_0\frac{\sqrt{(\tau_0^2-\tau^2)}}{\tau_0}\log\left\{\frac{\tau_0+\sqrt{(\tau_0^2-\tau^2)}}{\tau}\right\}\right], \quad (20)$$

$$P = \frac{\pi}{32(1-\tau^2)}\left[\sum_{j=0}^{2} P_j\left(\frac{1}{2}-\tau_j^2\right) + \sum_{j=0}^{2} S_j\left(\frac{\tau^2}{2}-\tau_j^2\right)\right]. \quad (21)$$

Next, differentiating both sides of relation (14) with respect to x, we obtain

$$\int_a^b tf(t^2)\,dt \int_0^\infty \frac{\gamma_1 m_2^2 \xi}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\sin\xi x \sin\xi t\, d\xi$$

$$+\int_c^1 ug(u^2)\,du \int_0^\infty \frac{\gamma_1 m_2^2 \xi}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\sin\xi x\sin\xi u\, d\xi = 0, \quad x\in I_2, I_4.$$

Following a similar procedure as for deriving eqn (15), we get

$$x\int_a^b \frac{tf(t^2)}{x^2-t^2}dt + x\int_c^1 \frac{ug(u^2)}{x^2-u^2}du = \int_a^b tf(t^2)\,dt\frac{\partial}{\partial t}\int_0^x\int_0^t \frac{wvL_2(v,w)\,dv\,dw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

$$+\int_c^1 ug(u^2)\,du\frac{\partial}{\partial u}\int_0^x\int_0^u \frac{wvL_2(v,w)\,dw\,dv}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}}, \quad x\in I_2, I_4 \quad (22)$$

where

$$L_2(v,w) = \int_0^\infty\left[\xi - \frac{2\gamma_1\xi^2(m_1^2-m_2^2)}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\right]J_0(\xi w)J_0(\xi v)\,d\xi. \quad (23)$$

For small values of m_1 and m_2 such that $m_1 = O(m_2)$, one can use the contour integration technique mentioned above and obtain

$$L_2(v,w) = 2im_1^2(1-\tau^2)\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\eta^2 H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}d\eta$$

$$+4im_1^2(1-\tau^2)\int_0^\tau -\frac{2\eta^4(\eta^2-1)(\tau^2-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}d\eta$$

$$-2\pi im_1^2(1-\tau^2)\left[\frac{\eta^2(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v \quad (24)$$

By a process similar to the one which led to eqn (18), eqn (24) can be written as

$$L_2(v,w) = -\frac{4P}{\pi}(1-\tau^2)m_1^2\log m_1 + O(m_1^2) \quad (25)$$

where P is given by eqn (21).

Now examining relations (15) and (18), we assume the expressions of the functions $f(t^2)$ and $g(u^2)$ as

$$f(t^2) = f_0(t^2) + f_1(t^2) m_1^2 \log m_1 + O(m_1^2)$$
$$g(u^2) = g_0(u^2) + g_1(u^2) m_1^2 \log m_1 + O(m_1^2). \tag{26}$$

Putting the above expressions of $f(t^2)$ and $g(u^2)$, and the value of $L_2(v, w)$ given by eqn (25) in eqn, (22) and equating the coefficients of like powers of m_1 we obtain

$$\int_a^b \frac{t f_0(t^2)}{x^2 - t^2} \mathrm{d}t + \int_c^1 \frac{u g_0(u^2)}{x^2 - u^2} \mathrm{d}u = 0, \quad x \in I_2, I_4 \tag{27}$$

and

$$\int_a^b \frac{t f_1(t^2)}{x^2 - t^2} \mathrm{d}t + \int_c^1 \frac{u g_1(u^2)}{x^2 - u^2} \mathrm{d}u = -\frac{4}{\pi} P(1-\tau^2) \left[\int_a^b t f_0(t^2)\, \mathrm{d}t + \int_c^1 u g_0(u^2)\, \mathrm{d}u \right], \quad x \in I_2, I_4. \tag{28}$$

Following Srivastava and Lowengrub (1970), the solutions of the above integral equations (27) can be obtained as

$$f_0(t^2) = D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-t^2}{1-t^2} \right)^{1/2} \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$
$$- D_2 \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \frac{1}{\sqrt{(1-t^2)(c^2-t^2)}}, \quad t \in I_2 \tag{29}$$

and

$$g_0(u^2) = D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{u^2-c^2}{1-u^2} \right)^{1/2} \frac{1}{\sqrt{(u^2-a^2)(u^2-b^2)}}$$
$$+ D_2 \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \frac{1}{\sqrt{(u^2-c^2)(1-u^2)}}, \quad u \in I_4 \tag{30}$$

where D_1 and D_2 are constants which can be calculated as follows:

We substitute the value of $L_1(v, w)$ from eqn (18), as well as the expansions of $f(t^2)$ and $g(u^2)$ obtained from eqns (26), (29) and (30) up to $O(m_1^2 \log m_1)$ in eqn (15). When the coefficients of like powers of m_1 from both sides of the resulting equation are equated, after some algebraic manipulation we get the following

$$D_1 = \frac{\pi v_0}{4\tau^2} \frac{(X_2 - X_1)}{(X_1 X_4 - X_2 X_3)}; \quad D_2 = \frac{\pi v_0}{4\tau^2} \frac{(X_1 - X_3)}{(X_1 X_4 - X_2 X_3)} \tag{31}$$

where

$$X_1 = \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left[\left\{ \left(\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2} \right) M + N \right\} (J_1 + J_3) + \frac{1}{2} M J_1 \log(b^2 - a^2) + M J_5 \right] \tag{32}$$

$$X_2 = \left\{ \left(\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2} \right) M + N \right\} (J_4 - J_2) - \frac{1}{2} M J_2 \log(b^2 - a^2) + M J_6 \tag{33}$$

$$X_3 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left[\left\{\left(\gamma+\log\frac{m_1}{2}-\frac{\pi i}{2}\right)M+N\right\}(J_1+J_3)+\frac{1}{2}MJ_3\log(1-c^2)+MJ_7\right] \tag{34}$$

$$X_4 = \left\{\left(\gamma+\log\frac{m_1}{2}-\frac{\pi i}{2}\right)M+N\right\}(J_4-J_2)+\frac{1}{2}MJ_4\log(1-c^2)-MJ_8$$

$$J_1 = \int_a^b\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}\frac{t\,dt}{\sqrt{(t^2-a^2)(b^2-t^2)}};\quad J_2 = \int_a^b\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\frac{t\,dt}{\sqrt{(1-t^2)(c^2-t^2)}}$$

$$J_3 = \int_c^1\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}\frac{u\,du}{\sqrt{(u^2-a^2)(u^2-b^2)}};\quad J_4 = \int_c^1\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\frac{u\,du}{\sqrt{(u^2-c^2)(1-u^2)}}$$

$$J_5 = \int_c^1\frac{u\log(\sqrt{u^2-b^2}+\sqrt{u^2-a^2})}{\sqrt{(u^2-a^2)(u^2-b^2)}}\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}du$$

$$J_6 = \int_c^1\frac{u\log(\sqrt{u^2-b^2}+\sqrt{u^2-a^2})}{\sqrt{(1-u^2)(u^2-c^2)}}\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}du$$

$$J_7 = \int_a^b\frac{t\log(\sqrt{c^2-t^2}+\sqrt{1-t^2})}{\sqrt{(t^2-a^2)(b^2-t^2)}}\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}dt$$

$$J_8 = \int_a^b\frac{t\log(\sqrt{c^2-t^2}+\sqrt{1-t^2})}{\sqrt{(1-t^2)(c^2-t^2)}}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}dt. \tag{35}$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ on the plane $y=0$ can be found from the relations (10), (13), (26), (29) and (30) as

$$\begin{aligned}\tau_{yy}(x,0) &= \frac{\pi\mu x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\left[D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-x^2}{1-x^2}\right)^{1/2}\right.\\&\quad\left.-D_2\frac{(x^2-a^2)}{\sqrt{(1-x^2)(c^2-x^2)}}\right]+O(m_1^2\log m_1),\quad x\in I_2\\&= \frac{\pi\mu x}{\sqrt{(x^2-c^2)(1-x^2)}}\left[D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\frac{(x^2-c^2)}{\sqrt{(x^2-a^2)(x^2-b^2)}}\right.\\&\quad\left.+D_2\left(\frac{x^2-a^2}{x^2-b^2}\right)^{1/2}\right]+O(m_1^2\log m_1),\quad x\in I_4.\end{aligned} \tag{36}$$

Defining the stress intensity factors at the edges of the strips by the relations

$$K_a = \mathop{Lt}_{x\to a^+}\left|\frac{\tau_{yy}(x,0)\sqrt{x-a}}{\pi\mu v_0}\right|;\quad K_b = \mathop{Lt}_{x\to b}\left|\frac{\tau_{yy}(x,0)\sqrt{b-x}}{\pi\mu v_0}\right|$$

$$K_c = \mathop{Lt}_{x\to c^+}\left|\frac{\tau_{yy}(x,0)\sqrt{x-c}}{\pi\mu v_0}\right|;\quad K_1 = \mathop{Lt}_{x\to 1}\left|\frac{\tau_{yy}(x,0)\sqrt{1-x}}{\pi\mu v_0}\right|.$$

We get

$$K_a = \left| \frac{\sqrt{a} D_1 / v_0}{\sqrt{2(b^2 - a^2)}} \right| \tag{37}$$

$$K_b = \left| \frac{\sqrt{b}}{\sqrt{2(b^2 - a^2)}} \left\{ \frac{D_1}{v_0} \left(\frac{1 - a^2}{c^2 - a^2} \right)^{1/2} \left(\frac{c^2 - b^2}{1 - b^2} \right)^{1/2} - \frac{D_2}{v_0} \frac{(b^2 - a^2)}{\sqrt{(1 - b^2)(c^2 - b^2)}} \right\} \right| \tag{38}$$

$$K_c = \left| \frac{\sqrt{c}}{\sqrt{2(1 - c^2)}} \frac{D_2}{v_0} \left(\frac{c^2 - a^2}{c^2 - b^2} \right)^{1/2} \right| \tag{39}$$

$$K_1 = \left| \frac{1}{\sqrt{2(1 - c^2)}} \left\{ \frac{(1 - c^2) D_1}{\sqrt{(c^2 - a^2)(1 - b^2)}} + \left(\frac{1 - a^2}{1 - b^2} \right)^{1/2} D_2 \right\} \right|. \tag{40}$$

The vertical displacement $v(x, y)$ on the plane $y = 0$ can be obtained from eqns (9), (13), (26), (29) and (30) as

$$v(x, 0) = \frac{4\tau^2}{\pi} \left[\left\{ \left(\gamma + \log m_1 - \frac{\pi i}{2} \right) M + N \right\} \left\{ D_1 \left(\frac{1 - a^2}{c^2 - a^2} \right)^{1/2} (J_1 + J_3) \right. \right.$$

$$\left. \left. + D_2 (J_4 - J_2) \right\} + \frac{M}{2} \left\{ (J_9 + J_{11}) \left(\frac{1 - a^2}{c^2 - a^2} \right)^{1/2} D_1 + D_2 (J_{12} - J_{10}) \right\} \right] \quad x \in I_1, I_3, I_5 \tag{41}$$

where

$$J_9 = \int_a^b \frac{t \log |t^2 - x^2|}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \left(\frac{c^2 - t^2}{1 - t^2} \right)^{1/2} \mathrm{d}t$$

$$J_{10} = \int_a^b \frac{t \log |t^2 - x^2|}{\sqrt{(1 - t^2)(c^2 - t^2)}} \left(\frac{t^2 - a^2}{b^2 - t^2} \right)^{1/2} \mathrm{d}t$$

$$J_{11} = \int_c^1 \frac{u \log |u^2 - x^2|}{\sqrt{(u^2 - a^2)(u^2 - b^2)}} \left(\frac{u^2 - c^2}{1 - u^2} \right)^{1/2} \mathrm{d}u$$

$$J_{12} = \int_c^1 \frac{u \log |u^2 - x^2|}{\sqrt{(u^2 - c^2)(1 - u^2)}} \left(\frac{u^2 - a^2}{u^2 - b^2} \right)^{1/2} \mathrm{d}u.$$

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_a, K_b, K_c and K_1 at the edges of the strips and vertical displacement $|v(x, 0)/v_0|$ near the rigid strips have been plotted against dimensionless frequency m_1 and distance x, respectively, for a Poisson solid ($\tau^2 = 3$).

It is found that whatever the lengths of the strips are, SIFs at the four edges of the strips increase with an increase in the value of m_1 ($0.1 \leqslant m_1 \leqslant 0.6$).

From the graphs, it may be further noted that with a decrease in the length of the inner strip, which might be induced either by increasing "a" or by decreasing "b" the SIFs gradually increase (Figs 2–9).

Also, a decrease in the value of the length of the outer strip, which might be induced by increasing the value of c, causes an increase in the values of the SIFs (Figs 10–13), from which an interesting conclusion might be drawn: i.e. that the presence of the outer strip suppresses the SIFs at both the edges of the inner strip and the presence of the inner strip suppresses the SIFs at both the edges of the outer strip.

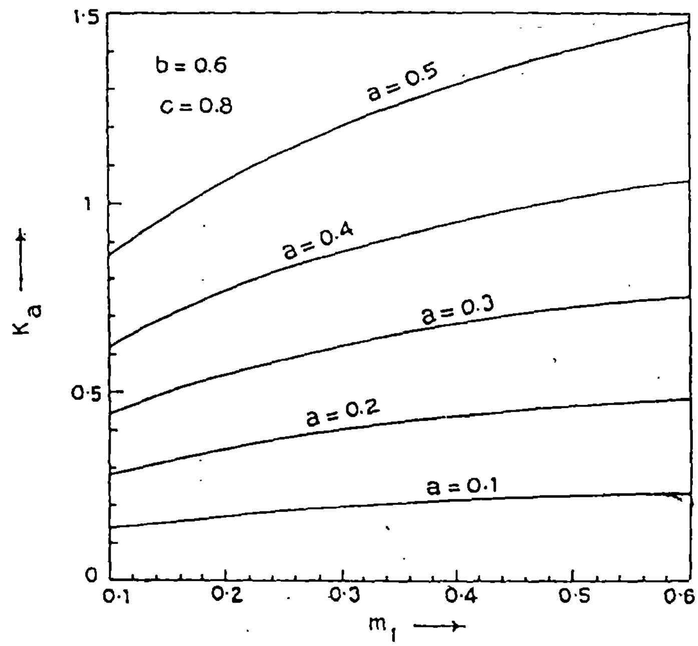

Fig. 2. Stress intensity factor K_a vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

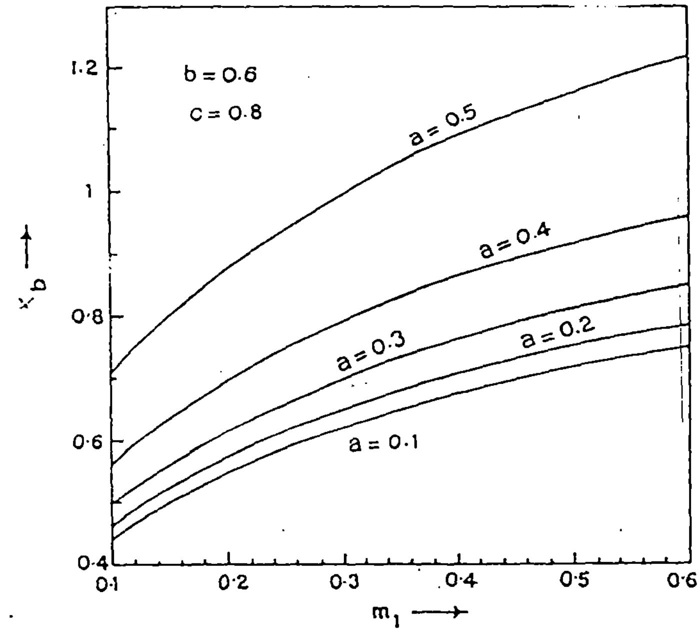

Fig. 3. Stress intensity factor K_b vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

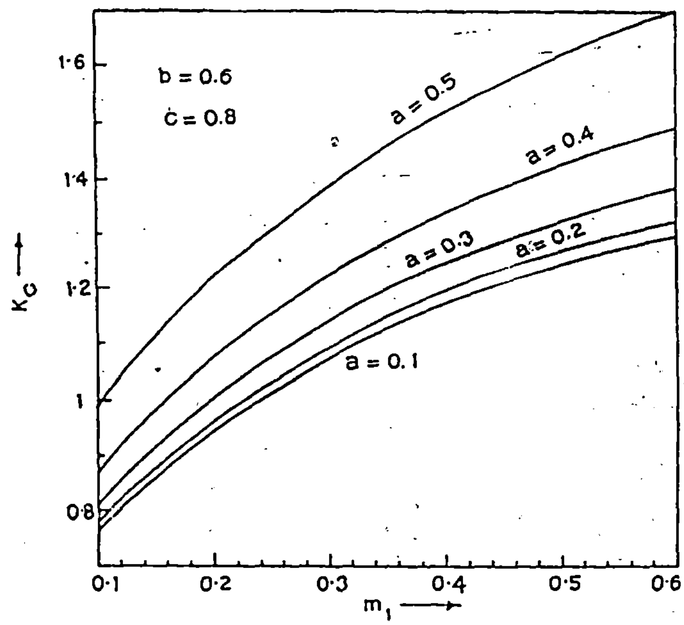

Fig. 4. Stress intensity factor K_c vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

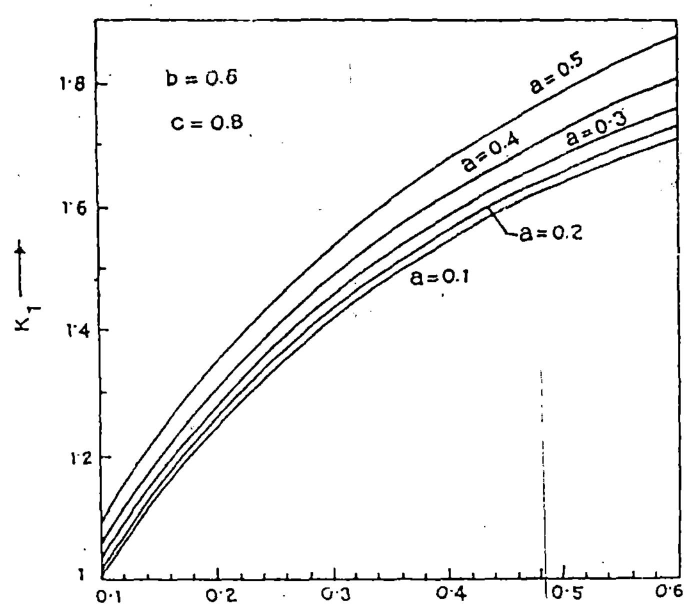

Fig. 5. Stress intensity factor K_1 vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

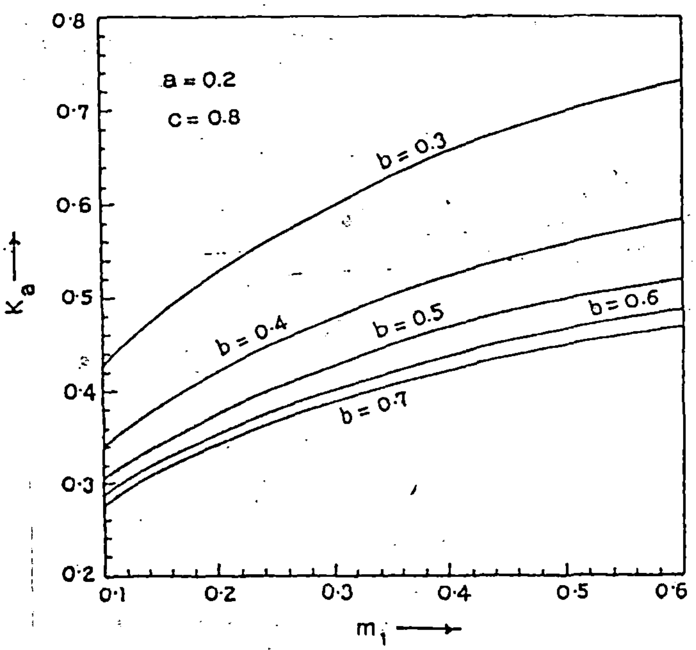

Fig. 6. Stress intensity factor K_a vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

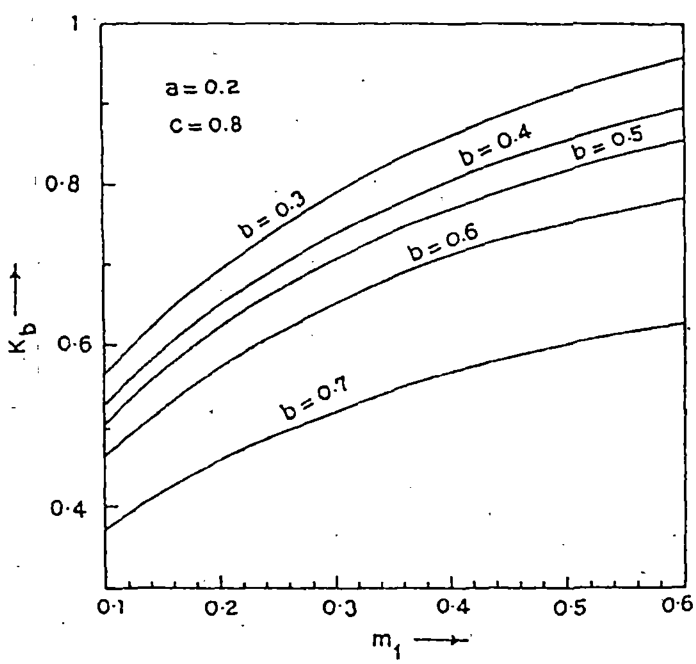

Fig. 7. Stress intensity factor K_b vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

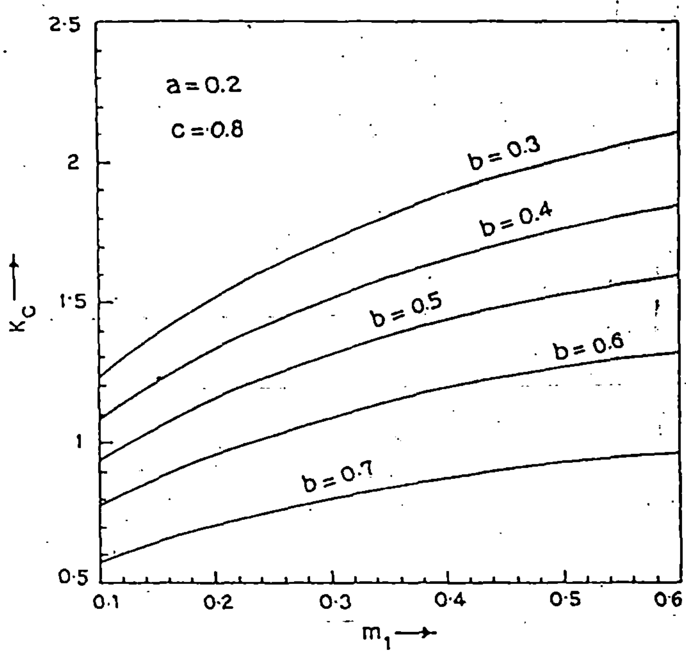

Fig. 8. Stress intensity factor K_c vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

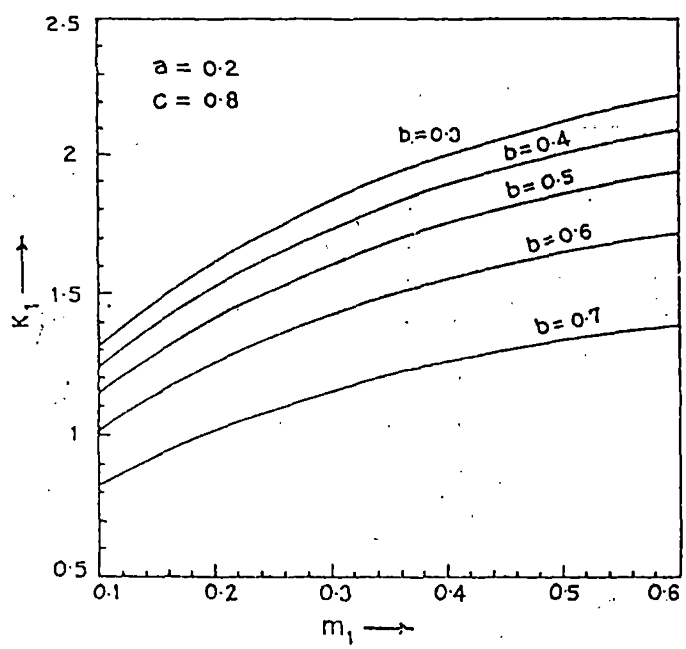

Fig. 9. Stress intensity factor K_1 vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

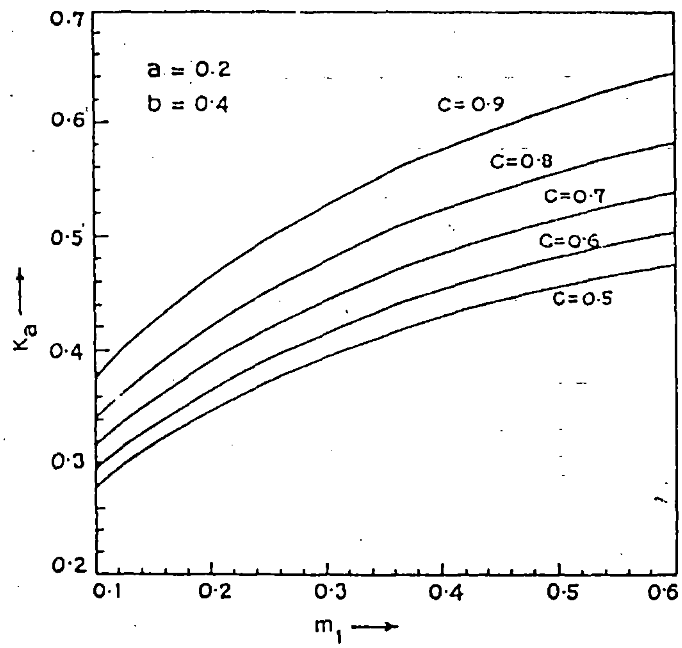

Fig. 10. Stress intensity factor K_a vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

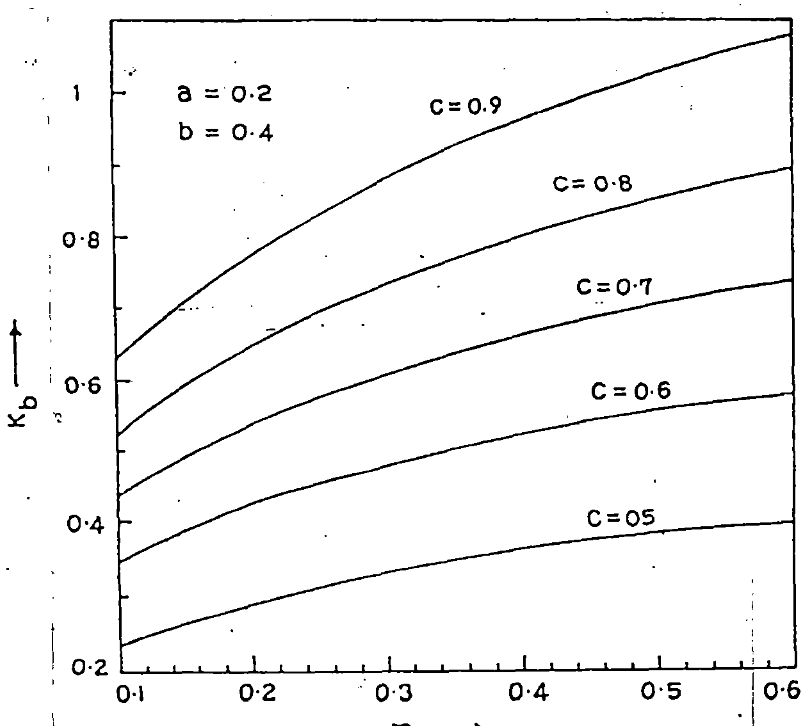

Fig. 11. Stress intensity factor K_b vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

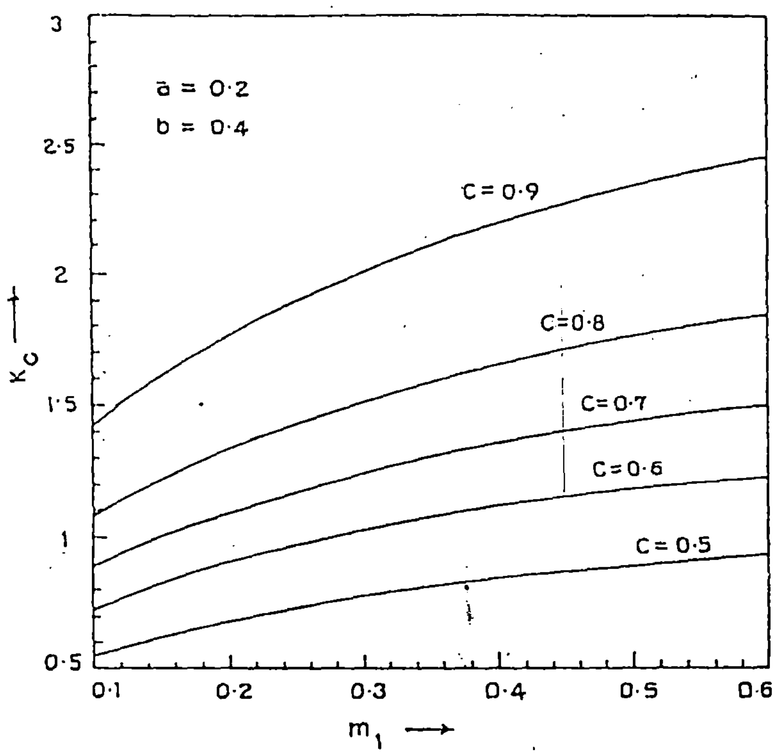

Fig. 12. Stress intensity factor K_c vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

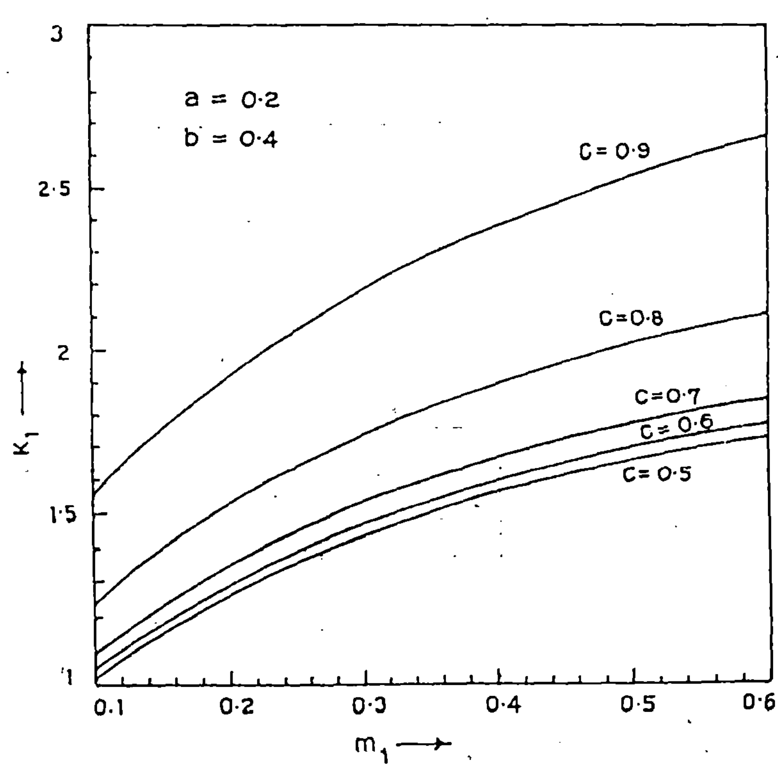

Fig. 13. Stress intensity factor K_1 vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

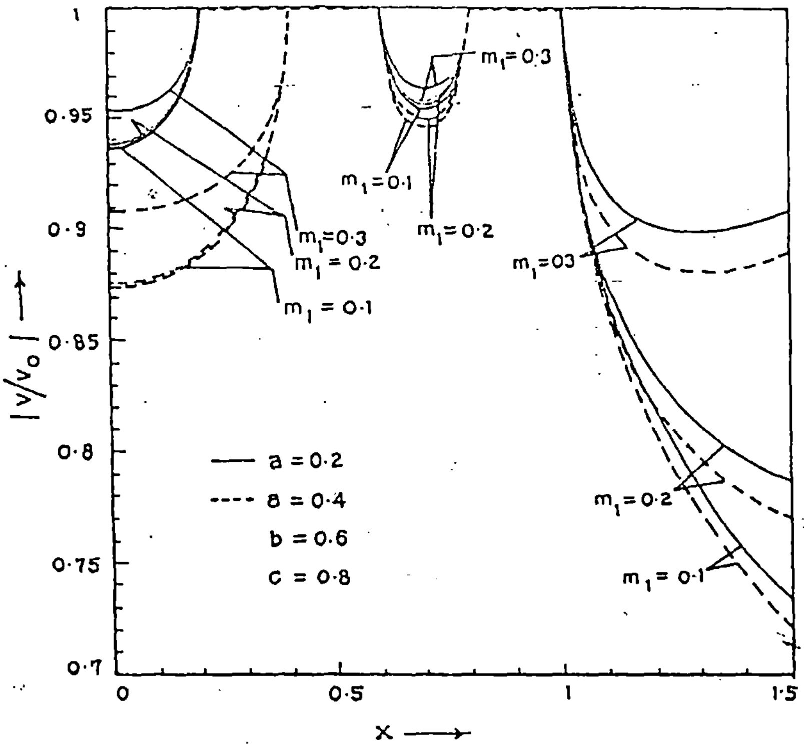

Fig. 14. Vertical displacement $|v(x, 0)/v_0|$ vs dimensionless distance x for $b = 0.6$, $c = 0.8$, $a = 0.2$, 0.4 and for $m_1 = 0.1, 0.2, 0.3$.

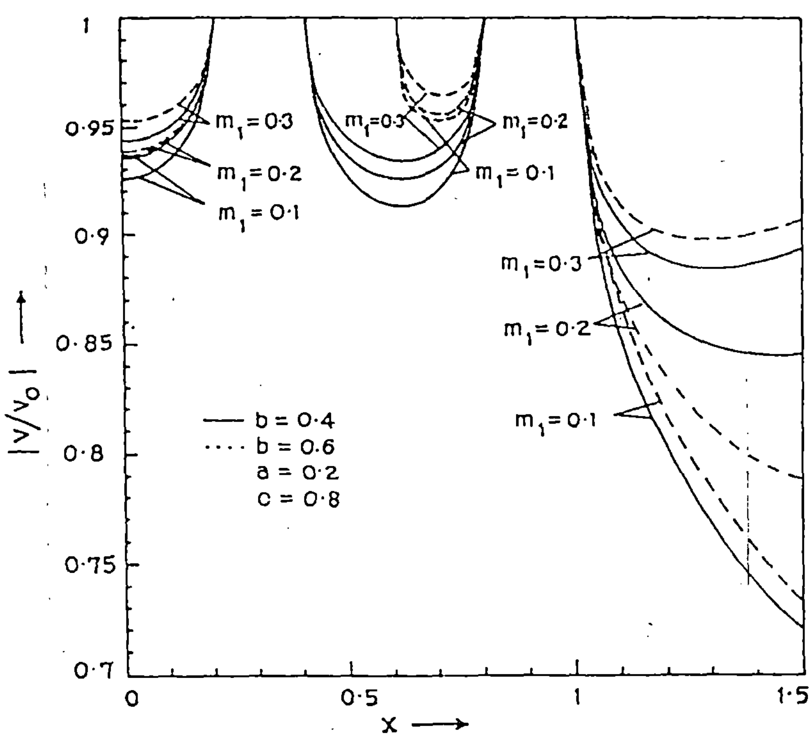

Fig. 15. Vertical displacement $|v(x, 0)/v_0|$ vs dimensionless distance x for $a = 0.2$, $c = 0.8$, $b = 0.4$, 0.6 and for $m_1 = 0.1, 0.2, 0.3$.

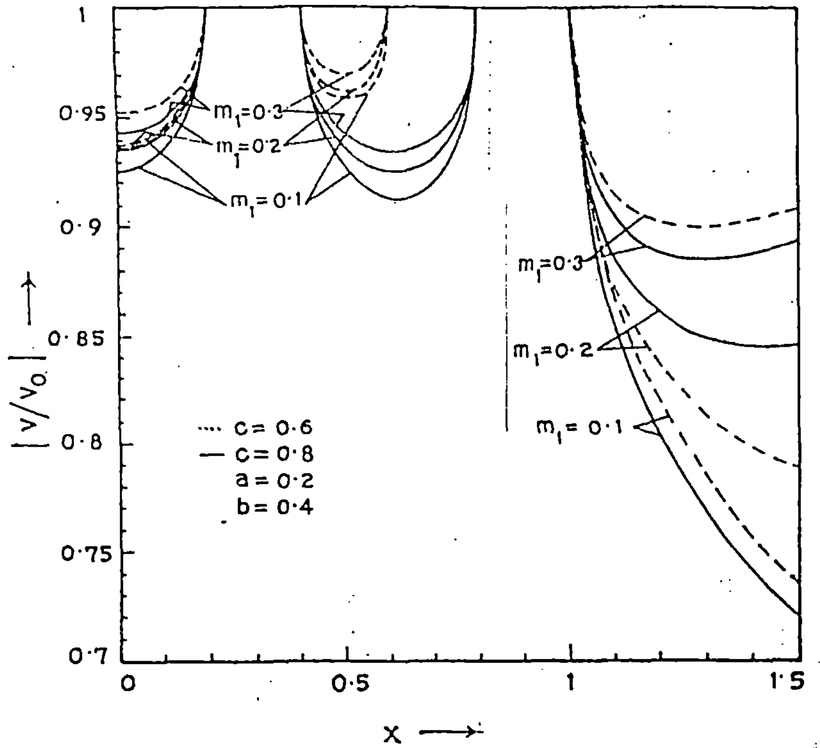

Fig. 16. Vertical displacement $|v(x,0)v_0|$ vs dimensionless distance x for $a = 0.2$, $b = 0.4$, $c = 0.6$, 0.8 and for $m_1 = 0.1, 0.2, 0.3$.

The vertical displacement has been plotted for different strip lengths. It is found from Figs 14–16 that with an increase in value of strip lengths, the displacement increases.

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REFERENCES

Awojobi, O. A. and Grootenhuis, P. (1965). Vibration of rigid bodies on semi-infinite elastic media. *Proc. Roy. Soc. London* A287, 27–63.

Ghosh, M. and Ghosh, M. L. (1985). Harmonic rocking of a rigid strip on a semi-infinite elastic medium. *Indn J. Pure Appl. Math.* 16, 938–955.

Gladwell, G. M. L. (1968). Forced tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid. *Int. J. Engng Sci.* 6, 591–607.

Karasudhi, P., Keer, L. M. and Lee, S. L. (1968). Vibratory motion of a body on an elastic half plane. *J. Appl. Mech.* 4, 697–705.

Mandal, S. C. and Ghosh, M. L. (1992). Forced vertical vibration of two rigid strips on a semi-infinite elastic solid. *J. Sound and Vibration* 158, 169–179.

Miller, G. F. and Pursey, H. (1954). The field and radiation impendence of mechanical radiators on the surface of semi-infinite isotropic solid. *Proc. Roy. Soc. London* A223, 521–541.

Reissner, E. (1937). Freie and erzwungene Torsionsschwingungen den elastischen Halbraumes. *Ingenien Archiv.* 2, 229–445.

Robertson, I. A. (1966). Forced vertical vibration of a rigid circular disc on a semi-infinite elastic solid. *Proc. Camb. Phil. Soc.* 62, 547–553.

Roy, A. (1986). Dynamic response of elliptic footing. *Int. J. Solids* 22, 293–309.

Srivastava, K. N. and Lowengrub, M. (1970). Finite Hilbert transform technique for triple integral equations with trigonometrical kernels. *Proc. Roy. Soc. Edn.* LXVIII 309–321.

Wickham, G. R. (1977). The forced two dimensional oscillations of a rigid strip in a smooth contact with a semi-infinite elastic solid. *Proc. Camb. Phil. Soc.* 81, 291–311.