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SPECTRAL REPRESENTATION OF A CERTAIN CLASS OF SELF-ADJOINT DIFFERENTIAL OPERATORS AND ITS APPLICATION TO AXISYMMETRIC BOUNDARY VALUE PROBLEMS IN ELASTODYNAMICS

S. C. PAL, M. L. GHOSH and P. K. CHOWDHURI (DARJEELING)

1. Introduction

In this work an integral representation of the Dirac delta function required for solving the axisymmetric boundary value problem has been derived first. This representation is particularly suitable for problems where mixed boundary conditions are encountered. Following FRIEDMANN [1], by contour integration of a suitable Green's function, integral representation of $\delta(R-R_0)$ $(R, R_0 > 1)$ has been derived. This representation has been used to solve a particular type of axisymmetric problem in elastodynamics.

The problem treated is that of a semi-infinite elastic body containing a circular cylindrical cavity, whose axis is perpendicular to the plane surface. The semi-infinite medium is subjected to an axisymmetric concentric torque applied dynamically as a step function in time at the plane surface.

At first LAMB [4] investigated the classical normal loading problem of an elastic half-space. As similar type of problem was investigated by EASON [5], MITRA [6], CHAKRABORTY and DE [7] and many others. They are all point source problems in a homogeneous semi-infinite medium.

The propagation of elastic waves, due to applied boundary tractions, in semi-infinite media containing internal boundaries has as yet not been studied to any large extent.

An earlier and comprehensive survey of the field is given by SCOTT and MIKLOWITZ [8]. Recently this type of work has been done by JOHNSON and PARNES [9].

We have solved the problem of the SH-type of elastic wave propagation in the semi-infinite medium due to a ring source producing SH-waves in the presence of a circular cylindrical cavity (case I). The problem of SH-wave propagation in the presence of rigid circular cylindrical inclusion in the semi-infinite medium due to the ring source has also been treated in the case 2.

2. Integral Representation of a Dirac Delta Function

Consider the operator L with λ as a complex parameter, where

$$L \equiv \frac{d}{dr}\left(r\frac{d}{dr}\right)+\lambda r-\frac{1}{r} \tag{2.1}$$

whose domain, D, is the set of all twice-differentiable functions $u(r)$, $a < r < \infty$ such that

(i) $$r\frac{du}{dr}-u=0 \quad \text{at} \quad r=a>0$$

(ii) the behaviour of u as $r\to\infty$ is that of an outgoing wave.

The solutions of $LG_1=0$ which satisfy (i) are

(2.2) $$G_1=A_1[J_1(\sqrt{\lambda}r)Y_2(\sqrt{\lambda}a)-Y_1(\sqrt{\lambda}r)J_2(\sqrt{\lambda}a)], \quad a<r<r_0,$$

where A_1 is an arbitrary constant and J_n and Y_n are the Bessel functions of the first and second kind, respectively.

Again the function G_2 which will satisfy $LG_2=0$ and the condition (ii) can be written as

(2.3) $$G_2=A_2H_1^{(1)}(\sqrt{\lambda}r) \quad (a<r_0<r<\infty),$$

where A_2 is an arbitrary constant and $H_n^{(1)}$ is the Hankel function of the first kind of order n.

From Eqs. (2.2) and (2.3) the Green's function G satisfying the equation $LG=-\delta(r-r_0)$ and the conditions (i) and (ii) mentioned above is given by (c.f. [1]).

(2.4) $$G(r,r_0;\lambda)=-\frac{\pi H_1^{(1)}(\sqrt{\lambda}r_0)}{2H_2^{(1)}(\sqrt{\lambda}a)}[J_1(\sqrt{\lambda}r)Y_2(\sqrt{\lambda}a)-Y_1(\sqrt{\lambda}r)J_2(\sqrt{\lambda}a)]H(r_0-r)-$$
$$-\frac{\pi H_1^{(1)}(\sqrt{\lambda}r)}{2H_2^{(1)}(\sqrt{\lambda}a)}[J_1(\sqrt{\lambda}r_0)Y_2(\sqrt{\lambda}a)-Y_1(\sqrt{\lambda}r_0)J_2(\sqrt{\lambda}a)]H(r-r_0),$$
$$0<\arg\lambda<2\pi.$$

Now consider

(2.5) $$\frac{1}{2\pi i}\oint G(r,r_0;\lambda)r\,d\lambda,$$

where the contour of integration in the λ-plane is shown in Fig. 1. Since G has a branch point at $\lambda=0$, we introduce a branch eut in the complex λ-plane along the positive real axis and then take the contour as a large circle of radius R_1^2, having the centre at $\lambda=0$, not crossing the branch cut.

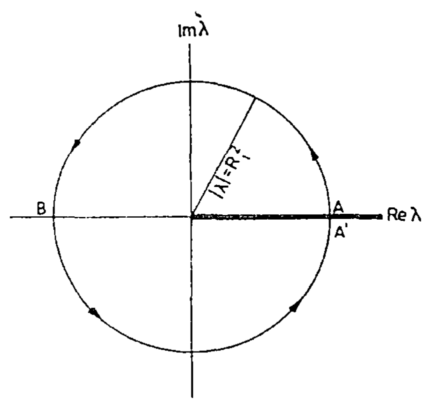

FIG. 1. Circular contour of integration ABA' in the λ-plane.

In terms of Hankel functions Eq. (2.4) can be written as

$$\frac{\pi}{4i}\left[H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)}-H_1^{(1)}(\sqrt{\lambda}r_0)\,H_1^{(2)}(\sqrt{\lambda}r)\right]H(r_0-r)+ \tag{2.6}$$

$$+\frac{\pi}{4i}\left[H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)}-H_1^{(1)}(\sqrt{\lambda}r)H_1^{(2)}(\sqrt{\lambda}r_0)\right]H(r-r_0).$$

For large $|z|$, the asymptotic behaviour of $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ is [2]

$$\begin{aligned} H_n^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}}\exp\left[i\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right], \\ H_n^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}}\exp\left[-i\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right]. \end{aligned} \tag{2.7}$$

Thus, for large values of $|\lambda|$, from the relations (2.7) we obtain

$$\begin{aligned} H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)} &\sim \frac{2}{\sqrt{\lambda r r_0}\,\pi}\exp[i\sqrt{\lambda}(r+r_0-2a)+i\pi], \\ H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(2)}(\sqrt{\lambda}r) &\sim \frac{2}{\pi\sqrt{\lambda r r_0}}\exp[i\sqrt{\lambda}(r_0-r)], \\ H_1^{(1)}(\sqrt{\lambda}r)H_1^{(2)}(\sqrt{\lambda}r_0) &\sim \frac{2}{\pi\sqrt{\lambda r r_0}}\exp[i\sqrt{\lambda}(r-r_0)]. \end{aligned} \tag{2.8}$$

If we put $\lambda = k^2$, then the circle in the λ-plane becomes a semi-circular arc C of radius R_1 in the upper half of the k-plane shown in Fig. 2.

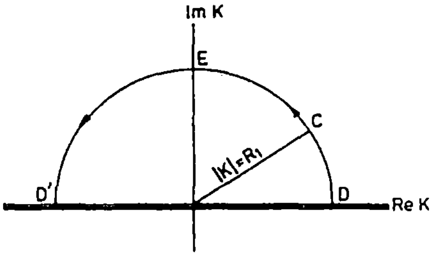

FIG. 2. DED' — the semi-circular path of integration C in the K-plane.

Consequently, for large values of R_1 the integral (2.5) can be written as

$$\frac{1}{2\pi}\sqrt{\frac{r}{r_0}}\int_C [\exp\{ik(r_0-r)\}H(r_0-r)+\exp\{ik(r-r_0)\}H(r-r_0)]dk- \tag{2.9}$$

$$-\frac{1}{2\pi}\int_C\sqrt{\frac{r}{r_0}}\exp\{ik(r+r_0-2a)\}dk =$$

$$= -\frac{1}{2\pi}\sqrt{\frac{r}{r_0}}\int_{-R_1}^{R_1}\exp(ik|r-r_0|)\,dk+\frac{1}{2\pi}\sqrt{\frac{r}{r_0}}\int_{-R_1}^{R_1}\exp\{ik(r+r_0-2a)\}dk =$$

$$= -\frac{1}{\pi}\sqrt{\frac{r}{r_0}}\frac{\sin R_1(r-r_0)}{r-r_0}+\frac{1}{\pi}\sqrt{\frac{r}{r_0}}\frac{\sin R_1(r+r_0-2a)}{r+r_0-2a}.$$

Our object is to show that the integral (2.5) represents $-\delta(r-r_0)$ when $R_1 \to \infty$. To justify the statement, consider a testing function $\phi(r)$, in D which is continuous, has a continuous derivative of order two and vanishes outside a finite interval. Then, from the relations (2.5) and (2.9)

$$\lim_{R_1\to\infty} \int_a^\infty \phi(r) \frac{1}{2\pi i} \oint G(r, r_0;\, \lambda) r\, d\lambda\, dr$$

$$= -\lim_{R_1\to\infty} \frac{1}{\pi} \int_a^\infty \phi(r) \sqrt{\frac{r}{r_0}}\, \frac{\sin R_1(r-r_0)\, dr}{(r-r_0)} +$$

$$+ \lim_{R_1\to\infty} \frac{1}{\pi} \int_a^\infty \phi(r) \sqrt{\frac{r}{r_0}}\, \frac{\sin R_1(r+r_0-2a)\, dr}{(r+r_0-2a)} = -\phi(r_0),$$

where we have used the result of Dirichlet integral and Riemann–Lebesgue Lemma [3].

Therefore

$$\lim_{R_1\to\infty} \frac{1}{2\pi i} \oint G(r, r_0;\, \lambda) r\, d\lambda = -\delta(r-r_0).$$

To obtain an alternative integral representation, which will be useful for our subsequent application in physical problems, we consider the contour Γ (Fig. 3) consisting of the real axis from $k = \varrho$ to $k = R_1$, where $0 < \varrho < R$; a semi-circle C of radius R_1 above the real axis; the real axis again from $-R$ to $-\varrho$; and finally a semi-circle γ of radius ϱ above the real axis with the centre at the origin. We take ϱ small and R_1 large.

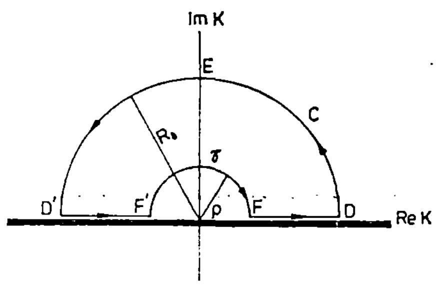

FIG. 3. *FDED'F'F*— the path of integration Γ in the *K*-plane.

The integrand $2G(r, r_0, k^2)\, kr$ has no singularity inside the contour Γ, and so the value of the integral

$$\frac{1}{2\pi i} \int_\Gamma G(r, r_0;\, k^2) 2kr\, dk = 0, \tag{2.10}$$

i.e.

$$\frac{1}{2\pi i} \int_\Gamma G(r, r_0;\, k^2) 2kr\, dk = -\frac{1}{2\pi i} \int_\varrho^{R_1} G(r, r_0;\, u^2) 2ur\, du +$$

$$+ \frac{1}{2\pi i} \int_\varrho^{R_1} G(r, r_0;\, e^{2\pi i} u^2) 2ru\, du - \frac{1}{2\pi} \int_0^\pi G(r, r_0;\, \varrho^2 e^{2i\theta}) 2r\varrho^2 e^{2i\theta} d\theta.$$

The behaviour of $Y_n(z)$ for small values of $|z|$ is described by the formula [2]

$$Y_n(z) \sim -\frac{2^n\Gamma(n)}{\pi z^n}$$

and $J_n(z)$ is bounded for small values of $|z|$ when n is a positive integer. Using these results we conclude

$$|G(r, r_0; \varrho^2 e^{2i\theta})\varrho|$$

is bounded for small values of ϱ. Hence

$$\lim_{\varrho\to 0}\frac{1}{\pi}\int_0^\pi G(r, r_0; \varrho^2 e^{2i\theta})e^{2i\theta}\varrho^2 r\,d\theta = 0.$$

Letting $\varrho \to 0$ and $R_1 \to \infty$ in (2.10), we get

$$\text{(2.11)}\qquad \delta(r-r_0) = -\lim_{R_1\to\infty}\frac{1}{2\pi i}\int_c G(r, r_0; k^2)2kr\,dk =$$

$$= \frac{1}{2\pi i}\int_0^\infty [G(r, r_0; k^2) - G(r, r_0; k^2e^{2i\pi})]2kr\,dk.$$

From Eq. (2.4)

$$G(r, r_0; k^2) - G(r, r_0; k^2e^{2i\pi}) =$$

$$= -\frac{\pi}{2}\left[\frac{J_1(kr_0)+iY_1(kr_0)}{J_2(ka)+iY_2(ka)} - \frac{J_1(kr_0)-iY_1(kr_0)}{J_2(ka)+iY_2(ka)}\right][J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)]\times$$

$$\times H(r_0-r) - \frac{\pi}{2}\left[\frac{J_1(kr)+iY_1(kr)}{J_2(ka)+iY_2(ka)} - \frac{J_1(kr)-iY_1(kr)}{J_2(ka)-iY_2(ka)}\right]\times$$

$$\times [J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)]H(r-r_0) =$$

$$= i\pi\frac{[J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)][J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)]}{J_2^2(ka)+Y_2^2(ka)}.$$

Substituting this expression in Eq. (2.11), we get

$$\delta(r-r_0) = \int_0^\infty \frac{[J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)][J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)]}{J_2^2(ka)+Y_2^2(ka)}rk\,dk.$$

Substituting $r/a = R$, $r_0/a = R_0$ and $ka = \gamma$, Eq. (2.12) can be written as

$$\text{(2.13)}\quad \delta(R-R_0) = \int_0^\infty \frac{[J_1(\gamma R_0)Y_2(\gamma)-Y_1(\gamma R_0)J_2(\gamma)][J_1(\gamma R)Y_2(\gamma)-Y_1(\gamma R)J_2(\gamma)]}{J_2^2(\gamma)+Y_2^2(\gamma)}R\gamma\,d\gamma.$$

Since $\delta(R-R_0)$ is symmetric with respect to R and R_0, then, on the right hand side of Eq. (2.13), R and R_0 can be interchanged. So we write

$$\text{(2.14)}\quad \delta(R-R_0) = R_0\int_0^\infty \frac{\gamma[J_1(\gamma R_0)Y_2(\gamma)-Y_1(\gamma R_0)J_2(\gamma)][J_1(\gamma R)Y_2(\gamma)-Y_1(\gamma R)J_2(\gamma)]}{J_2^2(\gamma)+Y_2^2(\gamma)}\,\alpha\gamma.$$

3. Formulation and General Solution

Case 1. We shall now use the integral representation of the delta function given by Eq. (2.13) to derive the time dependent response of an isotropic linearly elastic half-space containing a cylindrical cavity of radius a due to a ring source. The axis of the cylinder considered as the z-axis, which is perpendicular to the plane surface, is directed downwards (Fig. 4). A torque is applied on the free surface of the half-space over the rim of a concentric circle of radius $r = r_0 (r_0 > a)$ for $t \geqslant 0$.

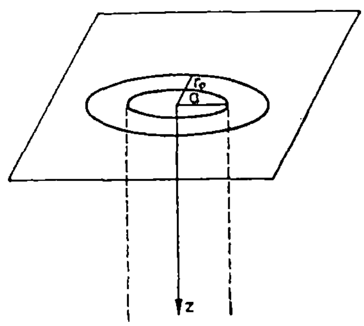

FIG. 4. Geometry of the problem.

Therefore on the cavity surface $r = a$

$$\tau_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = 0 \tag{3.1}$$

and on the plane surface $z = 0$

$$\tau_{\theta z} = \mu \frac{\partial u_\theta}{\partial z} = \delta(r-r_0)H(t) \quad (a < r < \infty, r_0 \gg a), \tag{3.2}$$

where μ is Lame's constant, δ is the Dirac delta function and H is the unit step function.

Now the only non-zero equation of motion is

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = \frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial t^2}, \tag{3.3}$$

where $\beta = \sqrt{\mu/\varrho}$ is the shear wave velocity.

Changing the independent variables (r, z, t) to the no-dimensional variables (R, Z, τ) defined by

$$R = \frac{r}{a}, \quad Z = \frac{z}{a}, \quad \tau = \frac{\beta t}{a}, \quad R_0 = \frac{r_0}{a} \tag{3.4}$$

the above equation reduces to

$$\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R}\frac{\partial u_\theta}{\partial R} + \frac{\partial^2 u_\theta}{\partial Z^2} - \frac{u_\theta}{R^2} = \frac{\partial^2 u_\theta}{\partial \tau^2} \tag{3.5}$$

and boundary conditions become

$$\tau_{r\theta} = \frac{\mu}{a}\left(\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right) = 0 \quad \text{on} \quad R = 1 \tag{3.6}$$

and

$$\tau_{\theta z} = \frac{\mu}{a}\frac{\partial u_\theta}{\partial z} = \frac{1}{a}\delta(R-R_0)H(t) \quad \text{on} \quad Z = 0. \tag{3.7}$$

Now, taking the Laplace transform with respect to nondimensional time (τ) and assuming the homogeneous initial conditions $u_\theta(R, Z, 0) = \frac{\partial u_\theta(R, Z, 0)}{\partial t} = 0$ at $t = 0$ Eq. (3.5) takes the form

$$\frac{\partial^2 \tilde{u}_\theta}{\partial R^2} + \frac{1}{R}\frac{\partial \tilde{u}}{\partial R} + \frac{\partial^2 \tilde{u}_\theta}{\partial Z^2} - \frac{\tilde{u}_\theta}{R^2} = s^2 \tilde{u}_\theta, \tag{3.8}$$

where

$$\tilde{u}_\theta = \int_0^\infty u_\theta e^{-s\tau} d\tau. \tag{3.9}$$

Take solution of Eq. (3.8) in the form

$$\tilde{u}_\theta(R, Z, s) = \int_0^\infty [A_1(\gamma)J_1(\gamma R) + B_1(\gamma)Y_1(\gamma R)]e^{-\sqrt{s^2+\gamma^2}Z} d\gamma, \tag{3.10}$$

where γ is real, J_1 and Y_1 are Bessel functions of the first and second kind respectively.

Using the boundary condition (3.6), we obtain

$$B_1(\gamma) = -A_1(\gamma)\frac{J_2(\gamma)}{Y_2(\gamma)}. \tag{3.11}$$

Substituting the value of $B_1(\gamma)$ an in Eq. (3.10), we have

$$\tilde{u}_\theta(R, Z, s) = \int_0^\infty A(\gamma)[J_1(\gamma R)Y_2(\gamma) - J_2(\gamma)Y_1(\gamma R)]e^{-\sqrt{s^2+\gamma^2}Z} d\gamma, \tag{3.12}$$

where

$$A(\gamma) = \frac{A_1(\gamma)}{Y_2(\gamma)}. \tag{3.13}$$

Therefore the transformed stress component reduces to

$$\tilde{\tau}_{\theta z} = \frac{\mu}{a}\int_0^\infty A(\gamma)\sqrt{(\gamma^2+s^2)}\,C_2(\gamma R)e^{-\sqrt{\gamma^2+s^2}Z} d\gamma, \tag{3.14}$$

where

$$C_2(\gamma R) = J_2(\gamma)Y_1(\gamma R) - Y_2(\gamma)J_1(\gamma R). \tag{3.15}$$

New, using the representation (3.15), Eq. (2.14) becomes

$$\delta(R-R_0) = R_0 \int_0^\infty \frac{\gamma C_2(\gamma R)C_2(\gamma R_0)}{J_2^2(\gamma)+Y_2^2(\gamma)} d\gamma. \tag{3.16}$$

Using Eqs. (3.7), (3.14) and (3.16), the value of $A(\gamma)$ is obtained as

$$A(\gamma) = \frac{R_0}{\mu s}\frac{\gamma C_2(\gamma R_0)}{\sqrt{(s^2-\gamma^2)}\,\{J_2^2(\gamma)+Y_2^2(\gamma)\}}. \tag{3.17}$$

Therefore $\tilde{u}_\theta$ becomes

$$\tilde{u}_\theta(R, Z, s) = -\frac{R_0}{\mu s}\int_0^\infty \frac{\gamma C_2(\gamma R)C_2(\gamma R_0)}{\sqrt{(\gamma^2+s^2)}\,\{J_2^2(\gamma)+Y_2^2(\gamma)\}}\,e^{-\sqrt{\gamma^2+s^2}\,Z}d\gamma. \tag{3.18}$$

On the plane boundary $Z = 0$

$$\tilde{u}_\theta(R, o, s) = -\frac{R_0}{\mu s}\int_0^\infty \frac{\gamma C_2(\gamma R)C_2(\gamma R_0)}{\sqrt{(\gamma^2+s^2)}\,\{J_2^2(\gamma)+Y_2^2(\gamma)\}}\,d\gamma. \tag{3.19}$$

Now, introducing the change of the variable $\gamma = s\zeta$ into the above expression (3.19), we obtain

$$\tilde{u}_\theta(R, o, s) = -\frac{R_0}{\mu}\int_0^\infty \frac{\zeta C_2(s\zeta R)C_2(s\zeta R_0)}{\sqrt{(\zeta^2+1)}\,\{J_2^2(s\zeta)+Y_2^2(s\zeta)\}}\,d\zeta, \tag{3.20}$$

Next, using

$$J_n(s\zeta R) = \frac{H_n^{(1)}(s\zeta R)+H_n^{(2)}(s\zeta R)}{2} \tag{3.21}$$

and

$$Y_n(s\zeta R) = \frac{H_n^{(1)}(s\zeta R)-H_n^{(2)}(s\zeta R)}{2i}, \tag{3.21'}$$

we obtain

$$C_2(s\zeta R) = J_2(s\zeta)Y_1(s\zeta R)-Y_2(s\zeta)J_1(s\zeta R) = \tag{3.22}$$
$$= \frac{1}{2i}[H_1^{(1)}(s\zeta R)H_2^{(2)}(s\zeta) - H_1^{(2)}(s\zeta R)H_2^{(1)}(s\zeta)]$$

and

$$C_2(s\zeta R_0) = \frac{1}{2i}[H_1^{(1)}(s\zeta R_0)H_2^{(2)}(s\zeta)-H_1^{(2)}(s\zeta R_0)H_2^{(1)}(s\zeta)]. \tag{3.22'}$$

Also

$$J_2^2(s\zeta)+Y_2^2(s\zeta) = H_2^{(1)}(s\zeta)H_2^{(2)}(s\zeta). \tag{3.22''}$$

Therefore, Eq. (3.20) becomes

$$\tilde{u}_\theta(R, o, s) = -\frac{R_0}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}}\,F(R, R_0, s\zeta)d\zeta, \tag{3.23}$$

where

$$(R, R_0, s\zeta) = F_1(R, R_0, s\zeta)+F_2(R, R_0, s\zeta) = F_1(R_0, R, s\zeta)+F_2(R_0, R, s\zeta) = \tag{3.24}$$
$$= F(R_0, R, s\zeta)$$

and

$$F_1(\alpha, \beta, s\zeta) = H_1^{(2)}(s\zeta\beta)\left\{H_1^{(1)}(s\zeta\alpha)-H_1^{(2)}(s\zeta\alpha)\frac{H_1^{(1)}(s\zeta)}{H_2^{(2)}(s\zeta)}\right\}, \tag{3.24'}$$

$$F_2(\alpha, \beta, s\zeta) = H_1^{(1)}(s\zeta\beta)\left\{H_1^{(2)}(s\zeta\alpha)-H_1^{(1)}(s\zeta\alpha)\frac{H_2^{(2)}(s\zeta)}{H_2^{(1)}(s\zeta)}\right\}. \tag{3.24''}$$

Using the asymptotic values of the Hankel functions for a large argument, it can be shown that

$$\frac{\zeta}{\sqrt{(\zeta^2+1)}} F_1(R, R_0, s\zeta) \to \frac{2}{\pi s\zeta \sqrt{RR_0}} [e^{-is\zeta(R_0-R)} + e^{-is\zeta(R+R_0-2)}] \tag{3.25}$$

as $|s\zeta| \to \infty$, showing that $\frac{\zeta F_1(R, R_0, s\zeta)}{\sqrt{(\zeta^2+1)}}$ vanishes over a large circular arc in the fourth quadrant of the complex ζ-plane for $R < R_0$.

Also

$$\frac{\zeta F_2(R, R_0, s\zeta)}{\sqrt{(\zeta^2+1)}} \to \frac{2}{\pi s\zeta \sqrt{RR_0}} [e^{is\zeta(R_0-R)} + e^{is\zeta(R+R_0-2)}] \tag{3.25'}$$

showing that $\frac{\zeta F_2(R, R_0, s\zeta)}{\sqrt{\zeta(^2+1)}}$ vanishes over a large circular arc in the first quadrant of the complex ζ-plane for $R < R_0$. Therefore, for $R > R_0$,

$$\frac{\zeta F_2(R_0, R, s\zeta)}{\sqrt{(\zeta^2+1)}} \quad \text{and} \quad \frac{\zeta F_1(R_0, R, s\zeta)}{\sqrt{(\zeta^2+1)}}$$

vanish over large circular arcs in the first and fourth quadrants, respectively, of the complex ζ-plane.

Denoting the responses for field points inside ($R < R_0$) and outside ($R > R_0$) the source by the subscripts I and 0 respectively, we have for points inside the source ($R < R_0$)

$$\bar{u}_{\theta 1}(R, o, s) = -\frac{R_0}{4\mu} \int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} [F_2(R, R_0, s\zeta) + F_1(R, R_0, s\zeta)] d\zeta \tag{3.26}$$

and points outside the source ($R > R_0$)

$$\bar{u}_{\theta 0}(R, o, s) = -\frac{R_0}{4\mu} \int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} [F_2(R_0, R, s\zeta) + F_1(R_0, R, s\zeta)] d\zeta. \tag{3.26'}$$

In order to evaluate

$$-\frac{R_0}{4\mu} \int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_2(R, R_0, s\zeta) d\zeta, \tag{3.27}$$

which is the first part of $\bar{u}_{\theta 1}(R, 0, s)$ we note first that the integrand has branch points at $\zeta = \pm i$ and also has a branch point at the origin of coordinates due to the presence of Hankel functions in the integrand. The integrand has also poles which correspond to the zeros of $H_2^{(1)}(s\zeta)$. From Eq. (3.18) we note that in order that $\bar{u}_\theta(R, Z, s)$ may be finite for large positive values of Z, $(\zeta^2+1)^{1/2}$ should have a positive real part on the path of integration. Accordingly, we draw cuts parallel to the real axis from $+i$ to $-\infty+i$ and from $-i$ to $\infty - i$ to satisfy our requirement. A cut along the negative real axis from the origin is also drawn to make Hankel functions single valued

$$-\frac{R_0}{4\mu} \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_2(R, R_0, s\zeta)$$

is now integrated along the quadrant of a large circle lying in the first quadrant of the complex ζ-plane as shown in Fig. 5a. Since poles of the integrand are outside the path of integration, the integral (3.27) becomes

$$\frac{R_0}{4\mu}\left[\int_0^1 \frac{v}{\sqrt{(1-v^2)}} F_2(R, R_0, isv)dv + \int_1^\infty \frac{v}{i\sqrt{(v^2-1)}} F_2(R, R_0, isv)dv\right]. \tag{3.28}$$

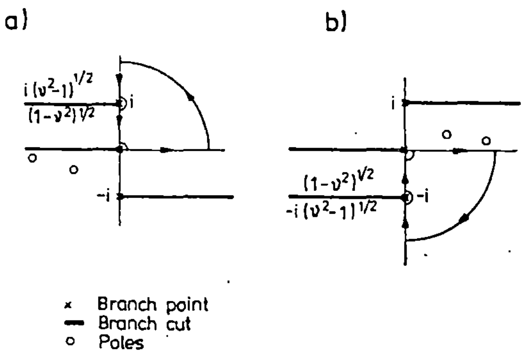

FIG. 5. Integration paths in the complex ζ-plane.

Using the relations

$$\begin{aligned} H_1^{(1)}(iv) &= -\frac{2}{\pi} K_1(v), \\ H_1^{(2)}(iv) &= \frac{2}{\pi} K_1(v) + 2iI_1(v), \\ H_2^{(1)}(iv) &= \frac{2i}{\pi} K_2(v), \\ H_2^{(2)}(iv) &= -2I_2(v) - \frac{2i}{\pi} K_2(v), \end{aligned} \tag{3.29}$$

we have

$$F_2(R, R_0, isv) = -\frac{4i}{\pi} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}. \tag{3.30}$$

Therefore, the expression (3.28) becomes

$$-\frac{iR_0}{\mu\pi}\int_0^1 \frac{v}{\sqrt{(1-v^2)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv - \tag{3.31}$$

$$-\frac{R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv.$$

The second part of $\tilde{u}_{\theta 1}(R, 0, s)$ is equal to

$$-\frac{R_0}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_1(R, R_0 s\zeta)d\zeta \tag{3.32}$$

we draw cuts from $+i$ to $\infty+i$ and from $-i$ to $-\infty-i$ as shown in Fig. (5b). A cut from the origin along the negative real axis is also drawn to make Hankel functions single valued.

Taking a quadrant of a large circular contour in the fourth quadrant (Fig. (5b)) and noting that the poles of F_1 $(R, R_0 s\zeta)$ lie outside the contour, the integral (3.32) takes the form

$$(3.33)\quad \frac{R_0}{4\mu}\left[\int_0^1 \frac{v}{\sqrt{(1-v^2)}} F_1(R, R_0, -isv)\,dv - \int_1^\infty \frac{v}{i\sqrt{(v^2-1)}} F_1(R, R_0, -isv)\,dv\right].$$

Using the relations

$$(3.34)\quad \begin{aligned} H_1^{(1)}(-iv) &= \frac{2}{\pi} K_1(v) - 2iI_1(v),\\ H_1^{(2)}(-iv) &= -\frac{2}{\pi} K_1(v),\\ H_2^{(1)}(-iv) &= -2I_2(v) + \frac{2i}{\pi} K_2(v),\\ H_2^{(2)}(-iv) &= +\frac{2}{i\pi} K_2(v), \end{aligned}$$

the expression (3.33) becomes

$$(3.35)\quad \frac{iR_0}{\mu\pi}\int_0^1 \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv -$$

$$-\frac{R_0}{\mu\pi}\int_0^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv.$$

Adding the relations (3.31) and (3.35), we obtain

$$(3.36)\quad \tilde{u}_{01}(R, o, s) = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv.$$

Similarly, it can be shown that

$$(3.36')\quad \tilde{u}_{00}(R, o, s) = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR)\left\{I_1(svR_0) + K_1(svR_0)\frac{I_2(sv)}{K_2(sv)}\right\}dv.$$

Laplace inversion of the relations (3.36) is now taken to obtain the displacement of points inside the source.
Therefore

$$(3.37)\quad u_{01}(R, o, \tau) = -\frac{1}{2\pi i}\cdot\frac{2R_0}{\mu\pi}\int_{Br} e^{\tau s}ds\int_1^\infty \frac{v}{\sqrt{(v^2-1)}}\tilde{E}(sv)\,dv,$$

where

$$(3.38)\quad \tilde{E}(sv) = K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}.$$

Introducing the change of variable $p = sv$, and changing the order of integration

$$(3.39)\qquad u_{\theta 1}(R, 0, \tau) = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{1}{\sqrt{(v^2-1)}}\, dv\left[\frac{1}{2\pi i}\int_{Br} e^{(\tau/v)p}\tilde{E}(p)\, dp\right] = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{1}{\sqrt{(v^2-1)}}\, E(\tau/v)\, dv,$$

where $E(\tau/v) = \mathscr{L}^{-1}\{\tilde{E}(p)\}$.

We note that $\tilde{E}(p)$ possesses no poles and is analytic for $p > 0$. It has a branch point at the origin and therefore a cut is drawn from the origin along the negative real axis of the complex p-plane in order to make $\tilde{E}(p)$ single valued.

Drawing a large semi-circular contour to the right of the Bromwich path AB in the complex p-plane, we conclude that $E(\tau/v) = 0$ if the integral

$$\frac{1}{2\pi i}\int_{BC'A} \tilde{E}(p)\, e^{(\tau/v)p} dp = 0$$

over the semi-circular arc $BC'A$ (Fig. 6).

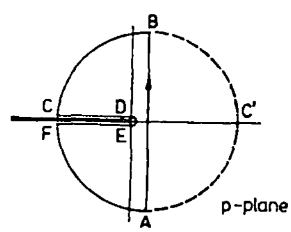

FIG. 6. Laplace inversion contour.

Now

$$(3.40)\qquad E(p) = -\frac{1}{2\pi i}\int_{BC'A} \tilde{E}(p)\, e^{(\tau/v)p} dp =$$

$$= -\frac{1}{2\pi i}\int_{BC'A} K_1(pR_0)I_1(pR)e^{(\tau/v)p}dp - \frac{1}{2\pi i}\int_{BC'A} K_1(pR_0)K_1(pR)\frac{I_2(p)}{K_2(p)}e^{(\tau/v)p}\, dp.$$

Since

$$e^{(\tau/v)p}K_1(pR_0)I_1(pR) \sim \frac{1}{2p\sqrt{RR_0}}\, e^{\left[\frac{\tau}{v}-(R_0-R)\right]p}$$

and

$$e^{(\tau/vp)}K_1(pR_0)I_1(pR)\frac{I_2(p)}{K_2(p)} \sim \frac{1}{2p\sqrt{RR_0}}\, e^{\left[\frac{\tau}{v}-(R+R_0-2)\right]p} \quad \text{as} \quad |p| \to \infty$$

then the first integral on the right hand side of Eq. (3.40) vanishes for $0 < \tau/v < (R_0-R)$, whereas the second integral vanishes for $0 < \tau/v < (R+R_0-2)$.

Therefore

$$E(\tau/v) = \begin{cases} 0, & \text{for} \quad 0 < \tau/v < (R_0 - R), \\ E^D(\tau/v), & \text{for} \quad (R_0 - R) < \tau/v < (R + R_0 - 2), \\ E^R(\tau/v), & \text{for} \quad (R + R_0 - 2) < \tau/v. \end{cases} \tag{3.41}$$

Where

$$\begin{aligned} E^D(\tau/v) &= \mathscr{L}^{-1}[K_1(pR_0)I_1(R)], \\ E^R(\tau/v) &= \mathscr{L}^{-1}\left[K_1(pR_0)I_1(pR) + K_1(pR_0)K_1(pR)\frac{I_2(p)}{K_2(p)}\right]. \end{aligned} \tag{3.42}$$

For value of τ/v lying in the range $(R - R_0) < \tau/v < (R + R_0 - 2)$

$$E(\tau/v) = E^D(\tau/v) = \frac{1}{2\pi i}\int_{Br} K_1(pR_0)I_1(pR)e^{(\tau/v)p}dp. \tag{3.43}$$

Therefore, putting $\tau/v = (R_0 - R + y)$, where $y > 0$

$$E^D(R_0 - R + y) = \frac{1}{2\pi i}\int_{Br} [K_1(pR)e^{pR_0}][I_1(pR)e^{-pR}]e^{yp}dp.$$

From the Laplace inversion table [12], we find that

$$\mathscr{L}^{-1}[K_1(pR_0)e^{pR_0}] = \frac{H(y)(y + R_0)}{R_0\{y(y + 2R_0)\}^{1/2}},$$

and

$$\mathscr{L}^{-1}[I_1(pR)e^{-pR}] = \frac{[H(y) - H(y - 2R)](R - y)}{\pi R\{y(2R - y)\}^{1/2}}.$$

So by the convolution theorem

$$E^D(R_0 - R + y) = \int_0^y \frac{[H(\eta) - H(\eta - 2R)]H(y - \eta)(R - \eta)(y - \eta + R_0)}{\pi R R_0[\eta(2R - \eta)(y - \eta)(y - \eta + 2R_0)]^{1/2}}d\eta. \tag{3.44}$$

For τ/v lying in the range $(R - R_0) < \tau/v < (R + R_0 - 2)$ τ/v must be less than $(R + R_0)$, i.e. $y < 2R$.

Therefore we can write

$$E^D(R_0 - R + y) = \int_0^y \frac{(R - \eta)(y - \eta + R_0)d\eta}{\pi R R_0[\eta(2R - \eta)(y - \eta)(y - \eta + 2R_0)]^{1/2}}.$$

So

$$E(\tau/v) = E^D(\tau/v) = \int_0^{\frac{\tau}{v} - (R_0 - R)} \frac{(R - \eta)(\tau/v + R - \eta)\mathrm{d}\eta}{\pi R R_0[\eta(2R - \eta)(\tau/v - R_0 + R - \eta)(\tau/v + R_0 + R - \eta)]^{1/2}}. \tag{3.45}$$

For values of τ/v satisfying the condition $\tau/v > R + R_0 - 2$,

$$E(\tau/v) = E^R(\tau/v) = \frac{1}{2\pi i}\int_{Br}\left\{K_1(pR_0)I_1(pR) + K_1(pR_0)K_1(pR)\frac{I_2(p)}{K_2(p)}\right\}e^{(\tau/v)p}dp. \tag{3.46}$$

This integral is equal to the integral along the large semi-circular arc on the left side of the Bromwich path AB plus the integral on the two sides of the branch cut (Fig. 6). Since the integral on the large semi-circular arc vanishes, then Eq. (3.46) becomes

(3.47) $$E(\tau/v) = \frac{1}{2\pi i}\Big[-\int_0^\infty \tilde{E}(\eta e^{i\pi})e^{-(\tau/v)\eta}d\eta + \int_0^\infty \tilde{E}(\eta e^{-i\pi})e^{-(\tau/v)\eta}\,d\eta\Big].$$

Using the relations

$$I_\nu(\eta e^{\pm i\pi}) = e^{\pm i\nu\pi}I_\nu(\eta),$$

and

$$K_\nu(\eta e^{\pm i\pi}) = e^{\mp i\nu\pi}K_\nu(\eta) \pm i\pi I_\nu(\eta),$$

we obtain (for $\tau/v > R+R_0-2$)

(3.48) $$E(\tau/v) = E^R(\tau/v) = -\int_0^\infty \frac{U_2(R,\eta)\,U_2(R_0,\eta)\,e^{-(\tau/v)\eta}}{K_2^2(\eta)+\pi^2 I_2^2(\eta)}\,d\eta;$$

where

$$U_2(x,\eta) = K_2(\eta)I_1(x,\eta)+I_2(\eta)K_1(x,\eta).$$

Substituting these values of $E(\tau/v)$ in Eq. (3.39), we obtain

(3.49)

$$u_{\theta 1}(R,0,\tau) = -\frac{2R_0}{\mu\pi}\Bigg[\Big\{H\Big(t-\frac{r_0-r}{\beta}\Big)-H\Big(t-\frac{r+r_0-2a}{\beta}\Big)\Big\}\int_1^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}}E^D(\tau/v)\,dv +$$

$$+H\Big(t-\frac{r+r_0-2a}{\beta}\Big)\Big\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}}E^D(\tau/v)\,dv + \int_1^{\frac{\tau}{R+R_0-2}} \frac{1}{\sqrt{v^2-1}}E^R(\tau/v)\,dv\Big\}\Bigg],$$

where the values of $E^D(\tau/v)$ and $E^R(\tau/v)$ are given in Eqs. (3.45) and (3.48), respectively.

Similarly, taking the inverse Laplace transform of Eq. (3.36′), the displacement $u_{\theta 0}(R,0,\tau)$ on the free surface outside the ring source can be derived and it is found that

(3.49′)

$$U_{\theta 0}(R,o,\tau) = -\frac{2R_0}{\mu\pi}\Bigg[\Big\{H\Big(t-\frac{r-r_0}{\beta}\Big)-H\Big(t-\frac{r+r_0-2a}{\beta}\Big)\Big\}\int_1^{\frac{\tau}{R-R_0}} \frac{1}{\sqrt{(v^2-1)}}F^D(\tau/v)\,dv +$$

$$+H\Big(t-\frac{r+r_0-2a}{\beta}\Big)\Big\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R-R_0}} \frac{1}{\sqrt{v^2-1}}F^D(\tau/v)\,dv + \int_1^{\frac{\tau}{R+R_0-2}} \frac{1}{\sqrt{(v^2-1)}}F^R(\tau/v)\,dv\Big\}\Bigg],$$

where $F^R(\tau/v) = E^R(\tau/v)$, and

(3.50) $$F^D(\tau/v) = \int_0^{\frac{\tau}{v}-(R-R_0)} \frac{(R_0-\eta)(\tau/v+R_0-\eta)\,d\eta}{\pi R R_0\{\eta(2R_0-\eta(\tau/v-R+R_0-\eta)(\tau/v+R+R_0-\eta)\}^{1/2}}.$$

First, the integrals of Eqs. (3.49) are the displacements due to a direct wave from the ring source before the arrival of the waves reflected from the wall of the cylindrical cavity. The last two integrals together give the displacement after the arrival of the reflected wave.

In order to obtain the response in the vicinity of the SH-wave front, we consider the displacement profile immediately behind the direct outgoing SH-wave. Accordingly, we shall have to study the first integral of Eq. (3.49′) because it gives the response of the direct SH-wave before the arrival of the reflected wave front.

Let $R_s = R_0 + \tau$ and $R_s^- = R_s - \varepsilon R_0$ where R_s and R^- denote points at and immediately behind the SH-wave front, respectively, ϵ is a small positive quantity.

Then

$$\frac{\tau}{R_s - R_0} = 1 \tag{3.51}$$

and

$$\frac{\tau}{R_s^- - R_0} = \left(1 + \frac{\varepsilon R_0}{\tau}\right) = q(\tau). \tag{3.51'}$$

Substituting these values in the first integral of Eq. 3.49′, we obtain

$$u_{00}(R_s, o, \tau) = 0,$$

and

$$u_{00}(R_s^-, o, \tau) = -\frac{2R_0}{\mu\pi}\int_1^{q(\tau)} \frac{1}{\sqrt{(v-1)}}\left\{\frac{1}{\sqrt{v+1}} \cdot F^D(R_s^-, R_0, \tau/v)\right\} dv.$$

Therefore, we can write

$$u_{00}(R_s^-, 0, \tau) = -\frac{2R_0}{\mu\pi}\int_1^{q(\tau)} \frac{1}{\sqrt{v-1}} V(v)\, dv, \tag{3.52}$$

where $V(v)$ is an analytic portion of the integrand. For small values of ε expanding $V(v)$ by the Taylor's series about the point $v = 1$ and integrating term by term, we obtain

$$u_{00}(R_s^-, o, \tau) \simeq -\frac{4R}{\mu\pi} V(1)\left(\frac{R_0}{\tau}\right)^{1/2} \varepsilon^{1/2} = A\varepsilon^{1/2} \quad \text{(say)}, \tag{3.53}$$

where A is a constant.

It therefore follows that the displacement component is continuous i.e. there is no jump in displacement across the direct SH-wave front.

Next, in order to consider the behaviour of response just under the ring source, it should be remembered that the integral representations of transformed displacements given by Eqs. (3.36) were derived from Eqs. (3.26) assuming that $R \neq R_0$. For $R = R_0$ the integrals along large quarter circles in the first and fourth quadrants should be reexamined. In this case it is found that though the contributions from the integrals along large circular arcs in the first and fourth quadrants are not separately zero, but the combined sum of the integrals along the large arcs in the first and fourth quadrants of the ζ-plane (Fig. 5a and 5b) vanishes. So the transformed displacements for $R = R_0$ are also given by Eqs. (3.36). Making $R \to R_0 \pm$, it can easily by shown by help of Eqs. (3.36) that the displacement has no jump discontinuity accross the ring source.

Therefore, in order to derive the nature of the displacement as $R \to R_0$, any one of the relations (3.49) may be studied. Consider, for example, the displacement at field points outside the source given by (3.49'). As $R \to R_0$, the upper limit of integration $\tau/(R-R_0) \to \infty$.

Further, as

(3.54)
$$v \to \frac{\tau}{R-R_0} \to \infty, \qquad \frac{1}{\sqrt{(v^2-1)}} \to \frac{1}{v}$$

and

(3.54')
$$F^D(\tau/v) \to \frac{1}{2R_0}.$$

Thus, from Eq. (3.49')

(3.55)
$$\lim_{R \to R_0} u_{\theta 0}(R, o, \tau) = -\frac{2R_0}{\mu\pi} \int_N^{\frac{\tau}{R-R_0}} \frac{1}{v} \cdot \frac{1}{2R_0} dv + \text{a finite quantity},$$

where N is large.

The integral is found to contribute a logarithmic singularity to the displacement just on the ring source.

Case 2. In this case the problem considered is the same in all respects with the first, except that the cavity of the radius a has been replaced by a rigid cylindrical inclusion of the same radius. The cylindrical inclusion being in welded contact with the elastic half-space, there is no relative displacement at the interface. In this case, the condition on the cylindrical boundary is $u_\theta = 0$ on $r = a$.

In order to solve this problem, we take the solution in this form:

(3.56)
$$\bar{u}_\theta(R, Z, s) = \int_0^\infty [A_2(\gamma) J_1(\gamma R) + B_2(\gamma) Y_1(\gamma R)] e^{-\sqrt{\gamma^2+s^2} Z} d\gamma,$$

where $\bar{u}_\theta(R, Z, s)$ is the Laplace transform of $u_\theta(R, Z, t)$ with respect to t. Now, using the boundary condition

$$\bar{u}_\theta = 0 \quad \text{on} \quad R = 1,$$

we have

(3.57)
$$B_2(\gamma) = -A_2(\gamma) \frac{J_1(\gamma)}{Y_1(\gamma)}$$

so $\bar{u}_\theta$ becomes

(3.58)
$$\bar{u}_\theta(R, Z, s) = \int_0^\infty A^1(\gamma) [J_1(\gamma R) Y_1(\gamma) - J_1(\gamma) Y_1(\gamma R)] e^{-\sqrt{\gamma^2+s^2} Z} d\gamma,$$

where

$$A^1(\gamma) = \frac{A_2(\gamma)}{Y_1(\gamma)}.$$

Therefore, the transformed stress component on the free surface $Z = 0$ is

(3.59) $$\bar{\tau}_{\theta z}(R, o, s) = -\frac{\mu}{a}\int_0^\infty A^1(\gamma)\sqrt{\gamma^2+s^2}\, C_1(\gamma R)\, d\gamma,$$

where

(3.60) $$C_1(\gamma R) = J_1(\gamma R)\, Y_1(\gamma) - J_1(\gamma)\, Y_1(\gamma R),$$

$\tau_{\theta z}(R, 0, s)$ should be equal to $\frac{1}{as}\,\delta\,(R-R_0)$. In this case, the required integral representation of the delta function can be obtained from the following expansion formula given by Titchmarsh [11]:

(3.61) $$f(r) = \int_0^\infty \frac{\zeta[J_1(\zeta r)\, Y_1(\zeta a) - J_1(\zeta a)\, Y_1(\zeta r)]}{J_1^2(\zeta a)+Y_1^2(\zeta a)}\, d\zeta \int_a^\infty \xi f(\xi)\,[J_1(\zeta\xi)\, Y_1(\zeta a) - J_1(\zeta a)\, Y_1(\zeta\xi)]\, d\xi,$$

where $f(r)$ is a suitably restricted arbitrary function.

Putting

$$f(r) = \delta(r-r_0),$$
$$f(\xi) = \delta(\xi-r_0), \quad \text{where} \quad r_0 > a > 0,$$

we get

(3.62) $$\delta(r-r_0) = r_0\int_0^\infty \frac{\zeta[J_1(\zeta r)\, Y_1(\zeta a) - J_1(\zeta a)\, Y_1(\zeta r)]\,[J_1(\zeta r_0)\, Y_1(\zeta a) - J_1(\zeta a)\, Y_1(\zeta r_0)]}{J_1^2(\zeta a)+Y_1^2(\zeta a)}\, d\zeta.$$

Now putting, $\frac{r}{a} = R$, $\frac{r_0}{a} = R_0$, $\zeta a = \gamma$, we have

$$\delta(R-R_0) = R_0\int_0^\infty \frac{\gamma[J_1(\gamma R)\, Y_1(\gamma) - J_1(\gamma)\, Y_1(\gamma R)]\,[J_1(\gamma R_0)\, Y_1(\gamma) - J_1(\gamma)\, Y_1(\gamma R_0)]}{J_1^2(\gamma)+Y_1^2(\gamma)}\, d\gamma,$$

so by the relation (3.60)

(3.63) $$\delta(R-R_0) = R_0\int_0^\infty \frac{\gamma C_1(\gamma R)\, C_1(\gamma R_0)}{J_1^2(\gamma)+Y_1^2(\gamma)}\, d\gamma.$$

This result can also be obtained by the following technique already developed in Sect. 2 of this paper.

Now, we find the value of $A^1(\gamma)$ as

(3.64) $$A^1(\gamma) = \frac{R_0}{\mu s}\, \frac{\gamma C_1(\gamma R_0)}{\sqrt{\gamma^2+s^2}}\, \frac{1}{J_1^2(\gamma)+Y_1^2(\gamma)}.$$

Therefore $\bar{u}_\theta$ becomes

(3.65) $$\bar{u}_\theta(R, o, s) = \frac{R_0}{\mu s}\int_0^\infty \frac{\gamma C_1(\gamma R)\, C_1(\gamma R_0)}{\sqrt{\gamma^2+s^2}\,\{J_1^2(\gamma)+Y_1^2(\gamma)}\, d\gamma.$$

Carrying on a similar procedure as followed to obtain the displacement in the case 1, we find that in this case

(3.66)
$$u_{\theta 1}(R, o, \tau) = \frac{2R_0}{\mu\pi}\Bigg[\Bigg\{H\left(t-\frac{r_0-r}{\beta}\right)-H\left(t-\frac{r+r_0-2a}{\beta}\right)\Bigg\}\int_1^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}}\, E^D(\tau/v)\,dv+$$
$$+H\left(t-\frac{r+r_0-2a}{\beta}\right)\Bigg\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}}\, E^D(\tau/v)\,dv+\int_1^{\frac{\tau}{R+R_0-2}} \frac{1}{\sqrt{v^2-1}}\, E(\tau/v)\,dv\Bigg\}\Bigg]$$

and

(3.67)
$$u_{\theta 0}(R, o, \tau) = \frac{2R_0}{\mu\pi}\Bigg[\Bigg\{H\left(t-\frac{r-r_0}{\beta}\right)-H\left(t-\frac{r+r_0-2a}{\beta}\right)\Bigg\}\int_1^{\frac{\tau}{R-R_0}} \frac{1}{\sqrt{v^2-1}}\, F^D(\tau/v)\,dv+$$
$$+H\left(t-\frac{r+r_0-2a}{\beta}\right)\Bigg\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R-R_0}} \frac{1}{\sqrt{v^2-1}}\, F^D(\tau/v)\,dv+\int_1^{\frac{\tau}{R-R_0+2}} \frac{1}{\sqrt{v^2-1}}\, F_1^R(\tau/v)\,dv\Bigg\}\Bigg],$$

where $E^D(\tau/v)$ and $F^D(\tau/v)$ are (respectively) given by Eqs. (3.45) and (3.50) and

(3.68)
$$E_1^R(\tau/v) = F_1^R(\tau/v) = -\int_0^{\infty} \frac{U_1(R, \eta)\, U_1(R_0, \eta)\, e^{-\left(\frac{\tau}{v}\right)\eta}}{K_1^2(\eta)+\pi^2 I_1^2(\eta)}\, d\eta$$

where

(3.69)
$$U_1(x, \eta) = K_1(\eta) I_1(x\eta) - I_1(\eta) K_1(x\eta).$$

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Streszczenie

SPEKTRALNA REPREZENTACJA PEWNEJ KLASY SAMOSPRZĘŻONYCH OPERATORÓW RÓŻNICZKOWYCH I JEJ ZASTOSOWANIE DO OSIOWO-SYMETRYCZNYCH ZAGADNIEŃ BRZEGOWYCH W ELASTODYNAMICE

Praca jest próbą znalezienia zamkniętej postaci osiowo-symetrycznej dynamicznej funkcji Greena typu SH dla izotropowej jednorodnej liniowej półprzestrzeni sprężystej, zawierającej cylindryczny otwór kołowy prostopadły do brzegu półprzestrzeni. Rozważono dwa przypadki: pierwszy odpowiada swobodnemu od obciążeń brzegowi cylindrycznemu oraz nagle przyłożonemu osiowo-symetrycznemu obciążeniu stycznemu, które jest skupione na konturze pewnego koła w płaszczyźnie brzegu półprzestrzeni; drugi odpowiada utwierdzonemu brzegowi otworu oraz obciążeniu takiemu jak w przypadku pierwszym. Stosując pewną całkową reprezentację celowo-symetrycznego obciążenia dla rozważanego ciała oraz technikę transformacji Laplace'a, podano zamkniętą postać funkcji Greena tylko na brzegu półprzestrzeni. Przeprowadzono też analizę jakościową tej postaci w otoczeniu pewnego kołowego frontu falowego.

Резюме

СПЕКТРАЛЬНОЕ ПРЕДСТАВЛЕНИЕ НЕКОТОРОГО КЛАССА САМОСОПРЯЖЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ И ЕГО ПРИМЕНЕНИЕ К ОСЕСИММЕТРИЧНЫМ КРАЕВЫМ ЗАДАЧАМ В ЭЛАСТОДИНАМИКЕ

Работа является попыткой нахождения замкнутого вида осесимметричной динамической функции Грина типа SH для изотропнового однородного линейного упругого полупространства, содержавшего цилиндрическое круговое отверстие перпедикулярное к границы полупространства. Рассмотрены два случая: первый отвечает свободному от нагрузок краю цилиндрического отверстия и внезапно приложенной осесимметричной касательной нагрузке, которая сосредоточена на контуре некоторого круга в плоскости границы полупространства, второй отвечает закрепленному краю отверстия и нагрузке такой как в первом случае.

Применяя некоторое интегральное представление осесимметричной нагрузки для рассматриваемого тела и технику преобразования Лапласа, приведен замкнутый вид функции Грина только на границе полупространства. Проведен тоже качественный анализ этого вида в окрестности некоторого кругового волнового фронта.

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WAVES IN A SEMI-INFINITE ELASTIC MEDIUM DUE TO AN EXPANDING ELLIPTIC RING SOURCE ON THE FREE SURFACE

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An elliptic ring load emanating from the origin of co-ordinates at $t = 0$ is assumed to expand on the free-surface of an elastic half-space. The rates of increase of the major and minor axes of the ellipse are assumed to be equal to a and b respectively. The displacement at points on the free-surface has been derived in integral form by Cagniard-de Hoop technique. Displacement jumps across different wave fronts have also been derived.

1. INTRODUCTION

Since Lamb's original study of the elastic wave produced by a time-dependent point force acting normally to the surface of an elastic half-space, many authors have elaborated on his work. Aggarwal and Ablow[1] discussed the exact solution of a class of half-space pulse propagation problems generated by impulsive sources. Gakenheimer and Miklowitz[4] used a modification of Cagniard's method[3] to discuss the disturbance created by a moving point load. In case of finite sources, the most widely discussed model is that of a circular ring or disc load. Mitra[7], Tupholme[11] and Roy[9] have studied the various aspects of the same problem. Elastic waves due to uniformly expanding disc or ring loads on the free surface of a semi-infinite medium have been studied extensively by Gakenheimer[5]. The axisymmetric problem of the determination of the displacement due to a stress discontinuity over a uniformly expanding circular region at a certain depth below the free surface has been studied by Ghosh[6].

However exact evaluation of the displacement field for finite source other than the circular model does not seem to have been attempted much in the literature. Burridge and Willis[2] obtained a solution for radiation from a growing elliptical crack in an anisotropic medium. The problem of an elliptical shear crack growing in prestressed medium has been solved by Richards[8] by the Cagniard-de Hoop Method. Roy[10] also attempted the same technique to solve the problem of elastic wave propagation due to prescribed normal stress over an elliptic area on the free surface of an elastic half-space.

In our problem, we have considered the propagation of elastic waves due to an expanding elliptical ring load over the free surface of a semi-infinite medium. The expression for displacement at points on the free surface has been derived in integral from by the application of Cagniard-de Hoop technique for different values of the rate of increase of the major and minor axes of the elliptic ring source. The displacement jumps across the different wave fronts have also been derived.

2. Formulation of the Problem and its Solution

Let an elliptic ring load P acting normal to the surface of an elastic half-space emanating from the origin of co-ordinates expand in such a way that the rates of increase of the major and minor axes of the ellipse are a and b respectively, a and b being constants. Major and minor axes of the ellipse are taken to coincide with the x and y-axes of co-ordinates where as z-axis is taken vertically downwards into the medium (Fig.1).

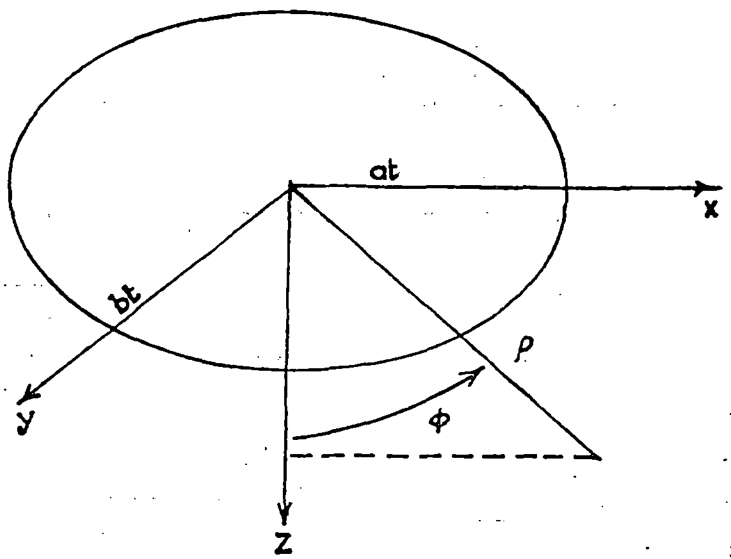


Fig. 1. Geometry of the Problem.

Thus we have on $z = 0$

$$\tau_{zz} = -\frac{P\,\delta\, t - (x^2 a^{-2} + y^2 b^{-2})^{1/2}}{(x^2 a^{-2} + y^2 b^{-2})^{1/2}} \qquad \ldots(1)$$

$$\tau_{xz} = \tau_{yz} = 0$$

where P is constant and δ is the Dirac delta function.

The displacement field inside the elastic medium $(z > 0)$ is given interms of potentials ϕ and ψ as

$$u = \nabla\phi + \nabla \times \nabla \times (e_z\,\psi)$$

where

$$\nabla^2 \phi = \frac{1}{c_d^2} \frac{\partial^2 \phi}{\partial t^2}, \nabla^2 \psi = \frac{1}{c_s^2} \frac{\partial^2 \psi}{\partial t^2} \qquad \text{...(2)}$$

e_x, e_y, e_z are unit vectors along co-ordinate axes and c_d and c_s are the p — and s-wave velocities of the medium.

In order to obtain solutions of wave equations (2), we introduce Laplace transform with respect to t and denote it by bar and also introduce bilateral Fourier transform with respect to x and y to supress the time parameter t and the x, y space co-ordinates. Taking Laplace transform with respect to t (—) and also bilateral Fourier transform with respect to x and y ($\cong$), the transformed boundary conditions are

$$\overset{\cong}{\tau}_{zz} = -\frac{Pab}{(a^2 \xi^2 + b^2 \eta^2 + s^2)^{1/2}}, \overset{\cong}{\tau}_{xz} = \overset{\cong}{\tau}_{yz} = 0. \qquad \text{...(3)}$$

Then satisfying the transformed boundary conditions (3) and performing the inverse Fourier transform, the Laplace transformed displacement field can be written as

$$\bar{u}_j (x, y, z, s) = \bar{u}_{jd} (x, y, z, s) + \bar{u}_{js} (x, y, z, s) \qquad \text{...(4)}$$

for $j = x, y, z$

where

$$\bar{u}_{j\alpha_1} (x, y, z, s) = 1/2\pi\mu \int_{-\infty}^{\infty} \int_{\infty}^{\infty} F_{j\alpha_1} (\xi, \eta, s) \exp [\zeta_{\alpha_1} z + i (\xi x + \eta y)] \, d\xi \, d\eta \qquad \text{...(5)}$$

for $\alpha_1 = d, s$

and

$$\left.\begin{aligned}
&F_{xd} (\xi, \eta, s) = - i \xi \zeta_0 G, \quad F_{xs} (\xi, \eta, s) = 2i\xi \zeta_d \zeta_s G, \\
&F_{yd} (\xi, \eta, s) = - i\eta \zeta_0 G, \quad F_{ys} (\xi, \eta, s) = 2i\eta \zeta_d \zeta_s G, \\
&F_{zd} (\xi, \eta, s) = \zeta_d \zeta_0 G, \; F_{zs} (\xi, \eta, s) = - 2 (\xi^2 + \eta^2) \zeta_d G, \\
&G = \frac{Pab}{(s^2 + r^2)^{1/2} T}, \; T = \zeta_0^2 - 4 \zeta_d \zeta_s (\xi^2 + \eta^2) \\
&r^2 = a^2 \xi^2 + b^2 \eta^2, \\
&\zeta_d = (\xi^2 + \eta^2 + k_d^2)^{1/2}, \; \zeta_s = (\xi^2 + \eta^2 + k_s^2)^{1/2}, \\
&\zeta_0 = k_s^2 + 2 (\xi^2 + \eta^2), \quad k_d = \frac{s}{c_d}, k_s = \frac{s}{c_s}.
\end{aligned}\right\} \qquad \text{...(6)}$$

Now the De-Hoop transformation,

$$\xi = s/c_d (q \cos \theta - w \sin \theta), \eta = s/c_d (q \sin \theta + w \cos \theta) \qquad \text{...(7)}$$

where $\theta = \tan^{-1} y/x$.

is applied into (5). The Laplace transformed displacement field (5) can be written as

$$\bar{u}_{j\alpha 1} (R, Z, s) = 1/2\pi\mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{j\alpha 1} (q, w, s) \exp [- s/c_d (m_\alpha Z - iqR)]$$

$$\times \frac{s^2}{c_d^2} dq\, dw \qquad \text{...(8)}$$

where

$$F_{xd} (q, w, s) = - \frac{i\, Pab\, (q \cos \theta - w \sin \theta)\, m_0}{s.\, s/c_d (E_1 + 0)^{1/2} .N},$$

$$F_{xs} (q, w, s) = \frac{2i\ Pab\, (q \cos \theta - w \sin \theta)\, m_d\ m_s}{s.\, s/c_d (E_1 + 0)^{1/2}.\ N.},$$

$$F_{yd} (q, w, s) = - \frac{i\ Pab\, (q \sin \theta + w \cos \theta)\, m_0}{s.\ s/c_d\, (E_1 + 0)^{1/2}.\ N.}$$

$$F_{ys} (q, w, s) = \frac{2i\ Pab\, (q \sin \theta + w \cos \theta)\, m_d\, m_s}{s.\ s/c_d\ (E_1 + 0)^{1/2}\ .\ N.},$$

$$F_{zd} (q, w, s) = \frac{Pab\ m_d\, m_0}{s.\, s/c_d\, (E_1 + 0)^{1/2}\ .N.},$$

$$F_{zs} (q, w, s) = \frac{- 2\ Pab\ (q^2 + w^3)\, m_d}{s.\ s/c_d\, (E_1 + 0)^{1/2}\ .N.},$$

$$m_d = (q^2 + w^2 + 1)^{1/2}, \qquad m_s = (q^2 + w^2 + l^2)^{1/2}$$

$$m_0 = l^2 + 2\, (q^2 + w^2), \qquad N = m_0^2 - 4m_d m_s\, (q^2 + w^2),$$

$$E_1 = (1 + q^2\, D + w^2 F), \qquad D = \frac{a^2}{c_d^2} \cos^2 \theta + \frac{b^2}{c_d^2} \sin^2 \theta,$$

$$F = \frac{a^2}{c_d^2} \sin^2 \theta + \frac{b^2}{c_d^2} \cos^2 \theta,\ 0 = - 2qw \sin \theta \cos \theta\ (a^2 - b^2)/c_d^2,$$

$$l = c_d/c_s \text{ and } R^2 = x^2 + y^2. \qquad \text{...(9)}$$

For mathematical simplicity we confine our attention to the derivation of the displacement field at any point on the xz-plane. Obviously the displacement at any point on any plane through the z-axis can then easily be visualized. Accordingly in order to obtain the displacement at any point on the xz-plane, we put $\theta = 0$ in (8) which then takes the form

$$\bar{u}_{j\alpha_1}(x, z, s) = \frac{Pab}{2\pi\mu c_d}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Re}\left[K_{j\alpha_1}(q,w)e^{-\frac{s}{c_d}(m_\alpha z - iqx)}\right] dq\, dw \quad \ldots(10)$$

where

$$\left.\begin{aligned} K_{xd}(qw) &= -\frac{iqm_0}{E^{1/2}.N},\quad K_{xs}(q, w) = \frac{2iqm_d m_s}{E^{1/2}.N.}, \\ K_{yd}(q, w) &= -\frac{iwm_0}{E^{1/2}.N},\quad K_{ys}(q, w) = \frac{2iwm_d m_s}{E^{1/2}.N.}, \\ K_{zd}(q, w) &= \frac{m_d m_0}{E^{1/2}.N},\quad K_{zs}(q, w) = -\frac{2m_d(q^2 + w^2)}{E^{1/2}.N}, \end{aligned}\right\} \quad \ldots(11)$$

and

$$E = 1/c_d^2\,(c_d^2 + a^2 q^2 + b^2 w^2).$$

3. Dilatational Contribution

From (10) $\bar{u}_{zd}$ is converted to the Laplace transform of a known function by mapping $1/c_d\,(m_d z - iqx)$ into t through a contour integration in a complex q-plane.

The singularities of the integrand of $\bar{u}_{zd}$ are branch points at

$$q = S_d^{\pm} = \pm\, i\,(w^2 + 1)^{1/2},\quad q = S_s^{\pm} = \pm\, i\,(w^2 + l^2)^{1/2},$$

$$q = S_c^{\pm} = \pm\, i\,\frac{(w^2 b^2 + c_d^2)^{1/2}}{a},$$

and the poles at

$$q = S_R^{\pm} = \pm\, i(w^2 + \gamma_R^2)^{1/2}. \quad \ldots(12)$$

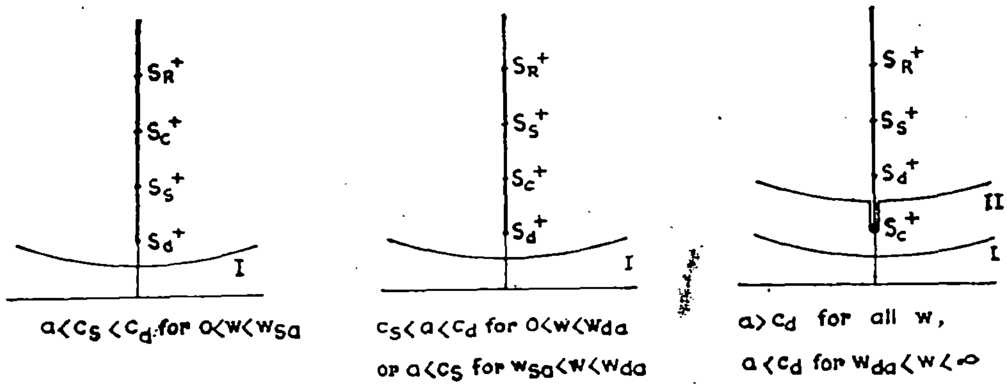
The poles at $q = S_R^{\pm}$ correspond to the zeros of the Rayleigh function N, where $\gamma_R = c_d/c_R$ and c_R is the Rayleigh surface wave speed. The contours of integration in the q-plane are shown in Fig. 2 (a, b, c) which also show the positions of singularities lying in the upper half of the q-plane.

Since the positions of the singularities and the transformed cottour of integration depend on different values of a and b, three different cases arise for the evaluation of u_{zd}.

(a) Case $a > b > C_d$.

The q-plane for $a > b > C_d$ is shown in Fig. 2 (a). The contour $q = q_d^{\pm}$ in the q-plane, is found by solving

$$t = 1/C_d\,(m_d Z - iqx) \quad \ldots(13)$$



(c) (b) (a)

FIG. 2. Cagniard paths of integration in the q-plane.

for q, where t is real, we get

$$q = q_d^{\pm} = i \sin \phi \pm (\tau^2 - \tau_{wd}^2)^{1/2} \cos \phi \qquad .. (14)$$

for

$$\tau > \tau_{wd}, \text{ where } \tau_{wd} = (w^2 + 1)^{1/2}, \ \tau = c_d t/\rho \qquad .. (15)$$

and (ρ, ϕ) are the polar coordinates in the xz-plane as shown in Fig. 1. Equations (14) define one branch of a hyperbola with vertex at $q = i\,(w^2 + 1)^{1/2}\ x/\rho$, which is parametrically described by the dimensionless time parameter τ as τ varies from τ_{wd} towards infinity.

As shown in Fig. 2 (a), the contour of integration has two possible configurations in the q-plane, depending upon ϕ and w.

For the case (1) given by :

Case (1) : $\phi < \phi_{da}$ and $0 < \omega < \infty$

or

$$\phi_{da} < \phi < \phi_{ba} \text{ and } w_{da} < w < \infty \qquad ...(16)$$

where $\phi_{da} = \sin^{-1} C_d/a$, $\phi_{ba} = \sin^{-1} b/a$

and

$$w_{da} = \left(\frac{C_d^2 - a^2 \sin^2 \phi}{a^2 \sin^2 \phi - b^2} \right)^{1,2} \qquad ...(17)$$

the vertex of the path $= q_d^{\pm}$ does not lie on the branch cuts and hence the path of integration contour is simply $q = q_d^{\pm}$ and is denoted by I.

But for the case (2) given by :

Case (2): $\phi_{da} < \phi < \phi_{ba}$ and $0 < w < w_{da}$

or $\phi > \phi_{ba}$ and $0 < w < \infty$ ...(18)

the vertex of the path $q = q_d^{\pm}$ lies on the branch cut between the branch points $q = S_c^{+}$ and $q = S_d^{+}$. Hence the integration contour is given by $q = q_d^{\pm}$ for $\tau > \tau_{wd}$ which is denoted by II, plus $q = q_{da} = i\,\tau\,\sin\phi - i\,(\tau_{dw}^2 - \tau^2)^{1/2}\cos\phi$ ...(19)

for $\tau_{wda} < \tau < \tau_{wd}$, where

$$\tau_{wda} = \frac{1}{a}\left[\left\{w^2(a^2 - b^2) + (a^2 - C_d^2)\right\}^{1/2} \times \cos\phi + (w^2 b^2 + C_d^2)^{1/2}\sin\phi\right]. \qquad ...(20)$$

Transferring the path of integration from the real q-axis to the Cagniard's–path we obtain

$$\bar{u}_{zd}(\rho, \phi, s) = \frac{2\,Pab}{\pi\mu C_d}\left[\int_0^{\infty}\int_{t_{wd}}^{\infty} \mathrm{Re}\left[k_{zd}(q_d^+, w)\frac{dq_d^+}{dt}\right]e^{-st}\,dt\,dw\right.$$

$$+ H(\phi_{ba} - \phi)\,H(\phi - \phi_{da})\int_0^{w_{da}}\int_{t_{wda}}^{t_{wd}} \mathrm{Re}\left[k_{zd}(q_{da}, w)\frac{dq_{da}}{dt}\right]e^{-st}\,dt\,dw$$

$$\left. + H(\phi - \phi_{ba})\int_0^{\infty}\int_{t_{wda}}^{t_{wd}} \mathrm{Re}\left[k_{zd}(q_{da}, w)\frac{dq_{da}}{dt}\right]e^{-st}\,dt\,dw\right] \qquad ...(21)$$

where $t_{wd} = \rho/C_d\;\tau_{wd}$ and $t_{wda} = \rho/C_d\;\tau_{wda}$. The first term of (21) is the contribution from $q_d^{\pm}$ and the second and third terms are the contributions from q_{da}.

Now interchanging the order of integration in (21) and inverting the Laplace transform, we find that

$$u_{zd}(\rho, \phi, \tau) = \frac{2\,Pab}{\pi\mu C_d}\left[H(\tau - 1)\int_0^{T_d} \mathrm{Re}\left[k_{zd}(q_d^+, w)\frac{dq_d^+}{dt}\right]dw\right.$$

$$+ H(\phi - \phi_{da})\,H(\phi_{ba} - \phi)\,H(\tau - \tau_{da})\,H(\tau'_{da} - \tau)$$

(*equation continued on p. 655*)

$$\times \int_{A'_{da}}^{T_{da}} \text{Re}\left[k_{zd}\,(q_{da}, w)\frac{dq_{da}}{dt}\right] dw$$

$$+ H(\phi - \phi_{b_a})\, H(\tau - \tau_{da})$$

$$\times \int_{A^0_{da}}^{T_{da}} R_e\left[k_{zd}\,(q_{da}, w)\,\frac{dq_{da}}{dt}\right] dw \Bigg] \qquad \ldots(22)$$

where

$$\left.\begin{aligned} A^0_{da} &= \begin{cases} 0 & \text{for } \tau_{da} < \tau < 1 \\ T_d & \text{for } 1 < \tau < \tau'_{da} \end{cases} \\ A^0_0 &= \begin{cases} 0 & \text{for } \tau_{aa} < \tau < 1 \\ T_d & \text{for } \tau > 1 \end{cases} \end{aligned}\right\} \qquad \ldots(23)$$

$$T_d = (\tau^2 - 1)^{1/2} \qquad \ldots(24)$$

$$T_{da} = \left[\frac{X_d - \{Y_d - (a^2\cos^2\phi - b^2)\, Z_d\}^{1/2}}{(a^2\cos^2\phi - b^2)^2}\right]^{1/2} \qquad \ldots(25)$$

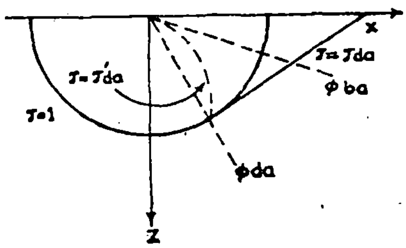
$$\left.\begin{aligned} X_d &= \tau^0_d\, b^2\sin^2\phi + (a^2 - b^2)\,\tau_d\cos^2\phi \\ Y_d &= \tau^{0^2}_d\, b^4\sin^4\phi + (a^2 - b^2)^2\,\tau^2_d\cos^4\phi + 2\,(a^2 + b^2)b^2\,\tau_d. \\ &\quad \times \tau^0_d\sin^2\phi\cos^2\phi \\ Z_d &= (\tau_d - 2C^2_d\sin^2\phi)^2 - 4C^2_d\,(a^2 - C^2_d)\sin^2\phi\cos^2\phi \\ \tau_d &= a^2\tau^2 + (C^2_d - a^2\cos^2\phi) \\ \tau^0_d &= a^2\tau^2 - (C^2_d - a^2\cos^2\phi) \end{aligned}\right\} \qquad (26)$$

$$\tau_{da} = \frac{1}{a}\left[(a^2 - C^2_d)^{1/2}\cos\phi + C_d\sin\phi\right], \qquad (27)$$

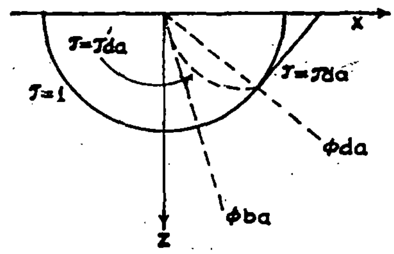
$$\tau'_{da} = \left[\frac{C^2_d - b^2}{a^2\sin^2\phi - b^2}\right]^{1/2}. \qquad (28)$$

~~...(26)~~

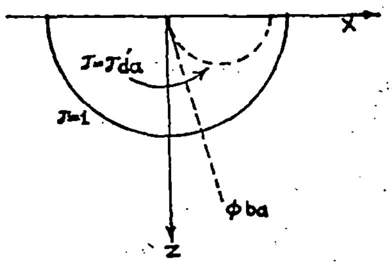
The first term in u_{zd} is due to the dilatational motion behind hemispherical wave front at $\tau = 1$ and the second and third terms are due to the dilational motion behind the conical wave front at $\tau = \tau_{da}$ for $\phi > \phi_{da}$. These wave fronts are shown in Fig. 3 (*a*),



3 (a) for $\alpha > b > c_d$



3 (b) for $\alpha > c_d > b$



3 (c) for $\alpha < c_d$

FIG. 3. Wave patten for dilatational motion.

$\tau = \tau'_{da}$ shown in Fig 3 (a) by a dashed curve, is not a wave front because it is not a characteristic surface for governing wave equation for the dilatational motion. Similar non characteristic surfaces were found by Gakenheimer and Miklowitz[4] for a point load travelling on an elastic half space and also by Aggarwal and Ablow[1] for the motion of an acoustic half-space due to an expanding surface load. They prove explicitly that their solution was analytic over the surfaces. The same thing can be proved in our case also.

(b) *Case* $a > c_d > b$

In this case, the path of integration with respect to q transforms to the simple path given by contour I (Fig. 2 (a)) for all w when $\phi < \phi_{ba}$ and also for $0 < w < w_{da}$ when $\phi_{ba} < \phi < \phi_{da}$, whereas the path of integration with respect to q transform to the contour II (Fig. 2 (a)) for $w_{da} < w < \infty$ when $\phi_{ba} < \phi < \phi_{da}$ and also for all w when $\phi > \phi_{da}$. The remaining details of inverting $\bar{u}_{zd}$ for $a > c_d > b$ are exactly the same as for $a > b > c_d$, and one can easily find that

$$u_{zd}(\rho, \phi, \tau) = \frac{2\ Pab}{\pi\mu\ c_d}\left[H(\tau - 1)\int_0^{\tau_d} R_e\left[k_{zd}(q_d^+, w)\frac{dq_d^+}{dt}\right]dw\right.$$

(*equation continued on p. 657*)

$$+ H(\phi - \phi_{ba}) H(\phi_{da} - \phi) H(\tau - \tau'_{da})$$

$$\times \int_{T_d}^{T_{da}} \text{Re}\left[k_{zd}(q_{da}, w) \frac{dq_{da}}{dt}\right] dw$$

$$+ H(\phi - \phi_{aa}) H(\tau - \tau_{da})$$

$$\times \int_{A^0_{da}}^{T_{da}} \text{Re}\left[k_{zd}(q_{da}, w) \frac{dq_{da}}{dt}\right] dw \Bigg] \quad \text{...(27)}$$

where A^0_{da} is given by (23).

The wave geometry associated with this expression is shown in Fig. 3 (b).

(c) *Case* $a < c_d$

For this case the path of integration with respect to q transform to the simple path given by contour I [Figs. 2(b), 2 (c)] for all w when $\phi < \phi_{ba}$ and also for $0 < w < w_{da}$ when $\phi > \phi_{ba}$, whereas the path of integration with respect to q transforms to the contour II [Fig. 2 (a)] for $w_{da} < w < \infty$ when $\phi > \phi_{ba}$. Note that in this case the angle ϕ_{da} does not arise. Now proceeding as the case $a > b > c_d$ for inverting $\bar{u}_{zd}$ we get

$$u_{zd}(\rho, \phi, \tau) = \frac{2\,Pab}{\pi\mu c_d}\Bigg[H(\tau - 1)\int_0^{T_d} R_e\left[k_{zd}(q_d^+, w) \frac{dq_d^+}{dt}\right] dw$$

$$+ H(\phi - \phi_{ba}) H(\tau - \tau'_{da})$$

$$\times \int_{T_d}^{T_{da}} R_e\left[k_{zd}(q_{da}, w) \frac{dq_{da}}{dt}\right] dw\Bigg]. \quad \text{...(30)}$$

The wave geometry associated with this expression is shown in Fig. 3 (c). As expected physically, contribution due to the conical wave front does not exist for this case.

Summary

Combining (22), (29) and (30) one finds that u_{zd} can be written as one expression for all values of a and b.

$$u_{zd}(\rho, \phi, \tau) = \frac{2\,Pab}{\pi\mu c_d}\Big[H(\tau - 1)\int_0^{T_d} R_e\Big[k_{zd}(q_d^+, w)\,\frac{dq_d^+}{dt}\Big]dw$$

$$+ [H(\tau - \tau_{da})\,H(\phi - \phi_{ba})\{H(b - c_d)$$

$$+ H(a - c_d)\,H(c_d - b)\} + H(\tau - \tau'_{da})\,H(\phi - \phi_{ba})\{H(a - c_d)$$

$$\times H(c_d - b)\,H(\phi_{da} - \phi) + H(c_d - a)\}]$$

$$\times \int_{A_{da}}^{T_{da}} \mathrm{Re}\Big[k_{zd}(q_{da}, w)\,\frac{dq_{da}}{dt}\Big]dw\Big] \qquad \ldots(31)$$

where

$$A_{da} = \begin{cases} \left.\begin{array}{l} 0 \text{ for } \tau_{da} < \tau < 1 \\ T_d \text{ for } 1 < \tau < \tau'_{da} \\ T_{da} \text{ for } \tau > \tau'_{da} \end{array}\right\} \text{for } \phi_{da} < \phi < \phi_{ba},\ a > b > c_d \\ \left.\begin{array}{l} 0 \text{ for } \tau_{da} < \tau < 1 \\ T_d \text{ for } 1 < \tau \end{array}\right\} \begin{array}{l} \text{for } \phi > \phi_{ba},\ a > b > c_d \\ \text{for } \phi > \phi_{da},\ a > c_d > b \end{array} \\ T_d \text{ for } \tau > \tau'_{da} \quad \left\{\begin{array}{l} \text{for } \phi_{ba} < \phi < \phi_{da},\ a > c_d > b \\ \text{for } \phi > \phi_{ba},\ a < c_d. \end{array}\right. \end{cases} \qquad \ldots(32)$$

4. Equivoluminal Contributions

Inversion of $\bar{u}_{zs}$ is complicated than the inversion of $\bar{u}_{zd}$ because of the appearence of head waves (Von-Schmidt waves) otherwise it is same as $\bar{u}_{zd}$. Here the integration contour has more configurations in the q-plane though the singularities are the same. Here the hyperbola $q = q_s^{\pm}$ arises in a similar way to $q = q_d^{\pm}$, but its vertex can lie on the branch cut between the branch points at $q = S_d^+$ and $q = S_s^+$ and at $q = S_c^+$ and $q = S_s^+$ as well as between $q = S_c^+$ and $q = S_d^+$, depending on the values of w, ϕ, a and b. In this case, the straight line contour lying along the imaginary q-axis is denoted by q_{sa} which is similar to q_{da} appearing in the dilatational contributions. Now ommiting details of inverting $\bar{u}_{zs}$, one can easily find

$$u_{zs}(\rho, \phi, \tau) = \frac{4\,Pab}{\pi\mu c_d}\Big[H(\tau - l)\int_0^{T_s} R_e\Big[k_{zs}(q_s^+, w)\,\frac{dq_s^+}{dt}\Big]dw$$

(*equation continued on p.* 659)

$$+ [H(\tau-\tau_{sa}) H(\phi-\phi_{sa}) \{H(b-c_s)+H(c_s-b) H(a-c_s)\}$$

$$+ H(\tau - \tau'_{sa}) H(\phi - \phi_{ba}) \{H(c_s - b) H(\phi_{sa} - \phi)$$

$$\times H(a - c_s) + H(c_s - a)\}]$$

$$\times \int_{A_{sa}}^{T_{sa}} R_e \left[k_{z_s}(q_{sa}, w) \frac{dq_{sa}}{dt} \right] dw$$

$$+ H(\tau - \tau_{sd}) H(\tau'_{sd} - \tau) H(\phi - \phi_{sd})$$

$$\times \int_{A_{sd}}^{T_{sd}} R_e \left[k_{z_s}(q_{sa}, w) \frac{dq_{sa}}{dt} \right] dw \Big] \quad \text{...(33)}$$

for $0 \leqslant \rho < \infty, 0 \leqslant \phi < \pi/2$,

$0 \leqslant \tau < \infty, 0 \leqslant a < \infty$ and

$0 \leqslant b < \infty, a > b$

where

$= 0$ for $\tau_{sa} < \tau < l$
$= T_s$ for $l < \tau < \tau'_{sa}$
$\left\{ \begin{array}{l} \phi_{sa} < \phi < \phi_{ba}, a > c_d, a > b > c_s, ac_s > bc_d \\ \phi_{sa} < \phi < \phi_{sd}, a > c_d, a > b > c_s, ac_s < bc_d \\ \phi_{sa} < \phi < \phi_{abs}, c_d > a > b > c_s \end{array} \right.$

$= 0$ for $\tau_{sa} < \tau < l$
$= T_s$ for $\tau > l$
$\left\} \begin{array}{l} \phi_{ba} < \phi < \phi_{sd}, a > b > c_d, ac_s > bc_d \\ \phi_{sa} < \phi < \phi_{sd}, a > c_d > c_s > b \end{array} \right.$

$= 0$ for $\tau_{sa} \leqslant \tau < \tau_{sd}$
$= T_{sd}$ for $\tau_{sd} < \tau < \tau'_{sd}$
$= T_s$ for $\tau > \tau'_{sd}$
$\left\} \begin{array}{l} \phi > \phi_{sd}, a > b > c_d, ac_s > bc_a \\ \phi > \phi_{sd}, a > c_d > c_s > b \end{array} \right.$

$= 0$ for $\tau_{sa} < \tau < \tau_{sd}$
$= T_{sd}$ for $\tau_{sd} < \tau < \tau'_{sd}$
$= T_s$ for $\tau'_{sd} < \tau < \tau'_{sa}$
$\left\} \; \phi > \phi_{sd}, a > b > c_d, ac_s < bc_d \right.$

$= T_s$ for $\tau > \tau'_{sa}$
$\left\{ \begin{array}{l} \phi_{ba} < \phi < \phi_{sa}, a > c_d > c_s > b \\ \phi_{ba} < \phi < \phi_{abs}, c_d > a > c_s > b \\ \phi_{ba} < \phi < \phi_{abs}, a < c_s \end{array} \right.$

$$
A_{sa}\left\{\begin{array}{ll}
\left.\begin{array}{l}
= T_s \text{ for } \tau'_{sa} < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau > \tau'_{sd}
\end{array}\right\} &
\begin{array}{l}
\phi_{abs} < \phi < \phi_{sa},\ c_d > a > c_s > b \\
\phi > \phi_{abs},\ a < c_s
\end{array} \\
\begin{array}{l}
= 0 \text{ for } \tau_{sa} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau > \tau'_{sd}
\end{array} &
\left\{\begin{array}{l}
\phi > \phi_{sa},\ c_d > a > c_s > b,\ \alpha > \beta \\
\phi_{sa} < \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\
\phi > \phi_{ba},\ c_d > a > b > c_s,\ \alpha > \beta \\
\phi_{ba} < \phi < \phi_x,\ c_d > a > b > c_s\ \beta > \alpha > \gamma
\end{array}\right. \\
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau > \tau'_{sd}
\end{array}\right\} &
\begin{array}{l}
\phi > \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\
\phi > \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\
\phi > \phi_{ba},\ c_d > a > b > c_s,\ \alpha < \gamma
\end{array} \\
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sa} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa}
\end{array}\right\} &
\begin{array}{l}
\phi_{abs} < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \alpha > \beta \\
\phi_{abs} < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\
\phi_{abs} < \phi < \phi_x,\ c_d > a > b > c_s,\ \alpha < \gamma
\end{array} \\
\left.\begin{array}{l}
= 0 \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\
= T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\
= T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa}
\end{array}\right\} &
\phi_x < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \alpha < \gamma.
\end{array}\right.
$$

$$
\left\{\begin{array}{l}
= 0 \text{ for } \tau_{sd} < \tau < l \\
= T_s \text{ for } l < \tau < \tau'_{sd}
\end{array}\right. \qquad
\left.\right\}\begin{array}{l}
\phi > \phi_{sd},\ a > b > c_d \\
\phi > \phi_{sd},\ a > c_d > c_s > b \\
\phi_{sd} < \phi < \phi_{abs},\ c_d > a > c_s > b \\
\phi_{sd} < \phi < \phi_{sa},\ c_d > a > b > c_s \\
\phi_{sd} < \phi < \phi_{abs},\ a < c_s
\end{array}
$$

$$A_{sd}\begin{cases} = 0 \text{ for } \tau_{sd} < \tau < l \\ = T_s \text{ for } l < \tau < \tau'_{sa} \\ = T_{sa} \text{ for } \tau'_{sa} < \tau < \tau'_{sda} \\ = T_s \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \end{cases} \quad \begin{matrix} \phi_{abs} < \phi < \phi_{sa},\ c_d > a > c_s > b \\ \phi > \phi_{abs},\ a < c_s \end{matrix}$$

$$\begin{cases} = 0 \text{ for } \tau_{sd} < \tau < \tau_{sa} \\ = T_{sa} \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\ = T_s \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \end{cases} \quad \begin{matrix} \phi > \phi_{sa},\ c_d > a > c_s > b,\ \alpha > \beta \\ \phi_{sa} < \phi < \phi_x,\ c_d > a > c_s > b, \beta > \alpha > \gamma' \\ \phi > \phi_{abs},\ c_d > a > b > c_s,\ \alpha > \beta \\ \phi_{abs} < \phi < \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\ \phi_{abs} < \phi < \phi_x,\ c_d > a > b > c_s,\ \alpha < \gamma \end{matrix}$$

$$\begin{cases} = 0 \text{ for } \tau_{sd} < \tau < \tau_{sa} \\ = T_{sa} \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\ = 0 \text{ for } \tau'_{sda} < \tau < l \\ = T_s \text{ for } l < \tau < \tau'_{sd} \end{cases} \quad \begin{matrix} \phi > \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\ \phi > \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\ \phi > \phi_x,\ c_d > a > b > c_s,\ \alpha < \gamma \end{matrix}$$

$$\begin{cases} = 0 \text{ for } \tau_{sd} < \tau < \tau_{sa} \\ = T_{sa} \text{ for } \tau_{sa} < \tau < \tau'_{sa} \\ = T_s \text{ for } \tau'_{sa} < \tau < \tau'_{sd} \end{cases} \quad \phi_{sa} < \phi < \phi_{abs},\ c_d > a > b > c_s \qquad \ldots(35)$$

and also where

$$T_s = (\tau^2 - l^2)^{1/2} \qquad \ldots(36)$$

$$T_{sa} = \left[\frac{X_s - \{Y_s - (a^2 \cos^2 \phi - b^2)^2 Z_s\}^{1/2}}{(a^2 \cos^2 \phi - b^2)^2} \right]^{1/2} \qquad \ldots(37)$$

$$\left.\begin{aligned} X_s &= \tau_s^0\, b^2 \sin^2 \phi + (a^2 - b^2)\, \tau_s \cos^2 \phi \\ Y_s &= \tau_s^{0\,2}\, b^4 \sin^4 \phi + (a^2 - b^2)^2 \tau_s^2 \cos^4 \phi \\ &\quad + 2\,(a^2 - b^2) b^2\, \tau_s\, \tau_s^0 \sin^2 \phi \cos^2 \phi \\ Z_s &= (\tau_s - 2c_d^2 \sin^2 \phi)^2 - 4\, l^2\, c_d^2\, (a^2 - c_s^2) \sin^2 \phi \cos^2 \phi \\ \tau_s &= a^2 \tau^2 + l^2\, (c_s^2 - a^2 \cos^2 \phi) \\ \tau_s^0 &= a^2 \tau^2 - l^2\, (c_s^2 - a^2 \cos^2 \phi) \end{aligned}\right\} \qquad \ldots(38)$$

$$T_{sd} = [\{(\tau - \tau_{sd}) \operatorname{cosec} \phi + 1\}^2 - 1]^{1/2} \quad \text{...(39)}$$

$$\tau_{sa} = 1/a\, [(l\, a^2 - c_s^2)^{1/2} \cos \phi + c_d \sin \phi] \quad \text{...(40)}$$

$$\tau_{sd} = [(l^2 - 1)^{1/2} \cos \phi + \sin \phi] \quad \text{...(41)}$$

$$\tau'_{sa} = \left[\frac{l^2 (b^2 - c_s^2)}{b^2 - a^2 \sin^2 \phi} \right]^{1/2} \quad \text{...(42)}$$

$$\tau'_{sd} = (l^2 - 1)^{1/2} \text{ see } \phi \quad \text{...(43)}$$

$$\tau'_{sda} = \left[(l^2 - 1)^{1/2} \cos \phi + \left(\frac{c_d^2 - b^2}{a^2 - b^2} \right)^{1/2} \sin \phi \right] \quad \text{...(44)}$$

$$\phi_{sa} = \sin^{-1} c_s/a,\ \phi_{sd} = \sin^{-1} c_s/c_d,\ \phi_{ba} = \sin^{-1} b/a \quad \text{.. (45)}$$

$$\phi_{abs} = \sin^{-1} \left(\frac{c_d^2 - b^2}{l^2 (a^2 - b^2) + c_d^2 - a^2} \right)^{1/2} \quad \text{...(46)}$$

$$\phi_x = \sin^{-1} \left[\frac{(a^2 - b^2)^{1/2} \left[l (c_d^2 - b^2)^{1/2} + (l^2 - 1)^{1/2} (c_d^2 - a^2)^{1/2} \right]}{l^2 (a^2 - b^2) + c_d^2 - a^2} \right] \quad \text{...(47)}$$

$$\alpha = \left(\frac{c_d^2 - a^2}{a^2 - b^2} \right)^{1/2},\ \beta = (l^2 - 1)^{1/2},\ \gamma = b/a\,(l^2 - 1)^{1/2}$$

$$- \frac{1}{a} (c_d^2 - b^2)^{1/2},\ \gamma' = \frac{c_s}{a} (l^2 - 1)^{1/2}$$

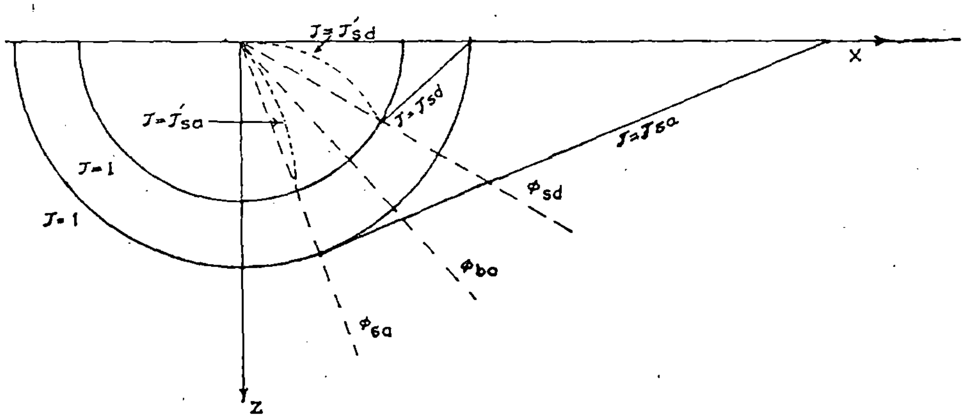
$$- 1/a \left[\frac{a^2 - c_s^2}{a^2 - b^2} (c_d^2 - b^2) \right]^{1/2} \quad \text{...(48)}$$

$$q_s^{\pm} = i \tau \sin \phi \pm (\tau^2 - \tau_{ws}^2)^{1/2} \cos \phi \quad \text{...(49)}$$

$$\tau_{ws} = (w^2 + l^2)^{1/2} \quad \text{...(50)}$$

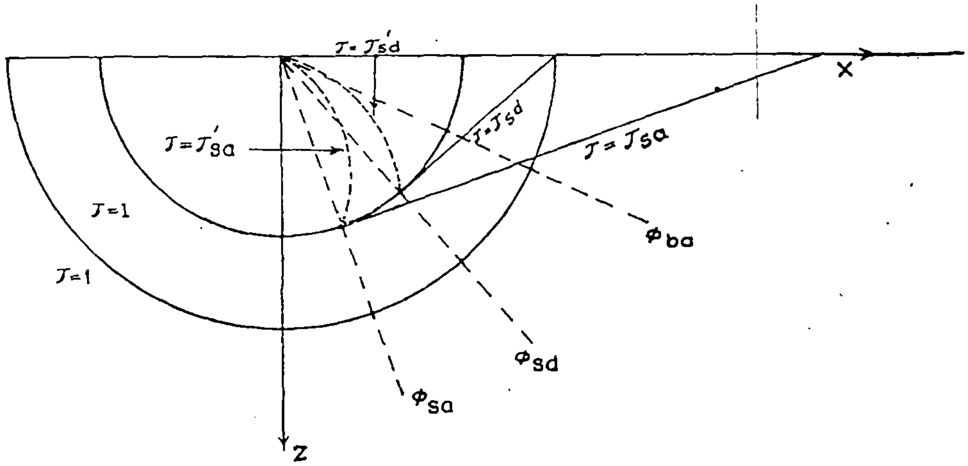
$$q_{sa} = i \tau \sin \phi - i (\tau_{ws}^2 - \tau^2)^{1/2} \cos \phi. \quad \text{...(51)}$$

The first term in the expression (33) is the equivoluminal motion behind the hemispherical wave front at $\tau = l$ and the second is due to the equivoluminal motion behind the conical wave front at $\tau = \tau_{sa}$. The third term in u_{rs} represents the equivoluminal motion due to the head wave fronts at $\tau = \tau_{sd}$. The wave fronts $\tau = \tau_{sd}$ for $\phi > \phi_{sd}$ and $\tau = \tau_{sa}$ are shown in Figs. 4(a—l).



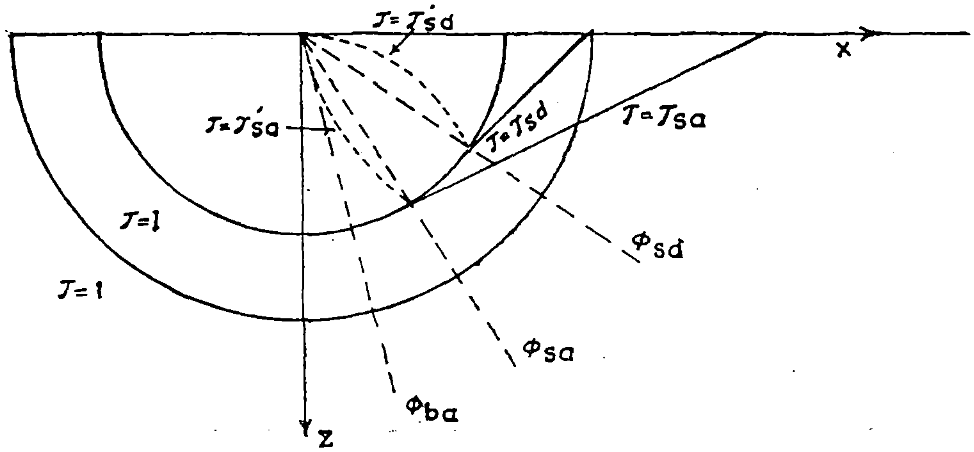
FIGS. 4(a—l). Wave pattern for equivoluminal and head wave motion.

4 (a) for $a > c_d$, $a > b > c_s$, $a c_s > b c_d$.

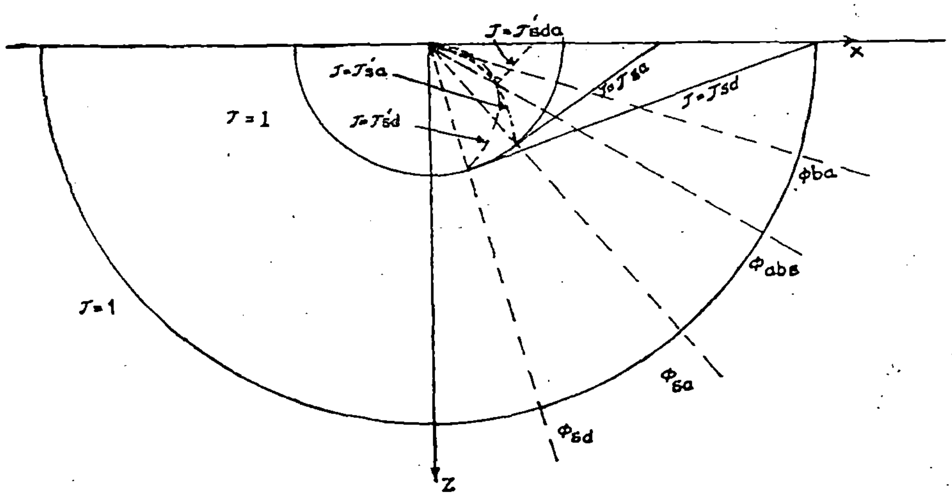


4 (b) for $a > c_d$, $a > b > c_s$, $a c_s < b c_d$.

The equations $\tau = \tau'_{sa}$, $\tau = \tau'_{sd}$ and $\tau = \tau'_{sda}$ are shown in Fig. 4 by dashed curve which are similar to $\tau = \tau'_{da}$ appearing in the u_{zd}. These dashed curved surfaces are not considered as wave fronts because it can be shown that displacements and their derivatives are continuous across these surfaces.



4 (c) for $a > c_d > c_s > b$.



4 (d) for $c_d > a > b > c_s, \alpha > \beta$.

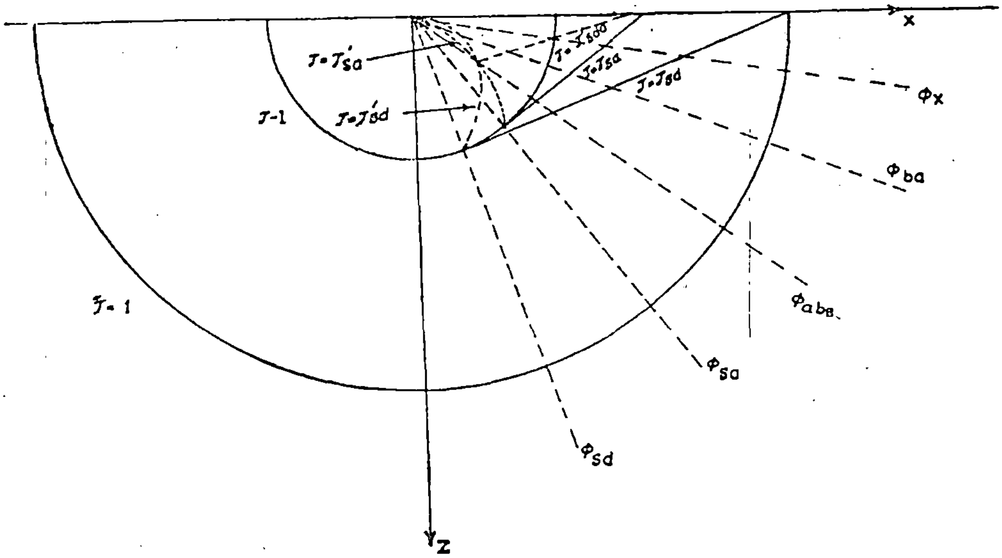
5. Wave Front Expansions

The wave forms of the solution given in (31) and (33) are evaluted by approximate estimation of the integrals in the neighbourhood of the first arrival of the different waves. To facilitate this evaluation we put

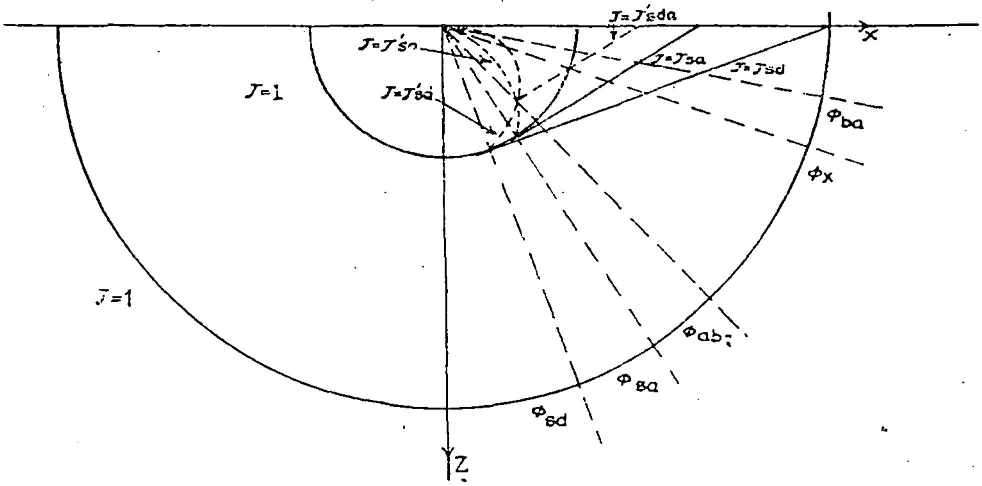
$$w = [A^2 + (B^2 - A^2) \sin^2 \alpha]^{1/2} \qquad ...(52)$$

in the integrals arising in u_{xd} and u_{xs} where A and B are respectively the lower and upper limits of the particular integral in question, and the range of integration with respect to α is form 0 to $\pi/2$.

Now for the first integral of (31), we put $w = T_d \sin \alpha$ and hence for $\tau \to 1+$, we find that for any value of a.

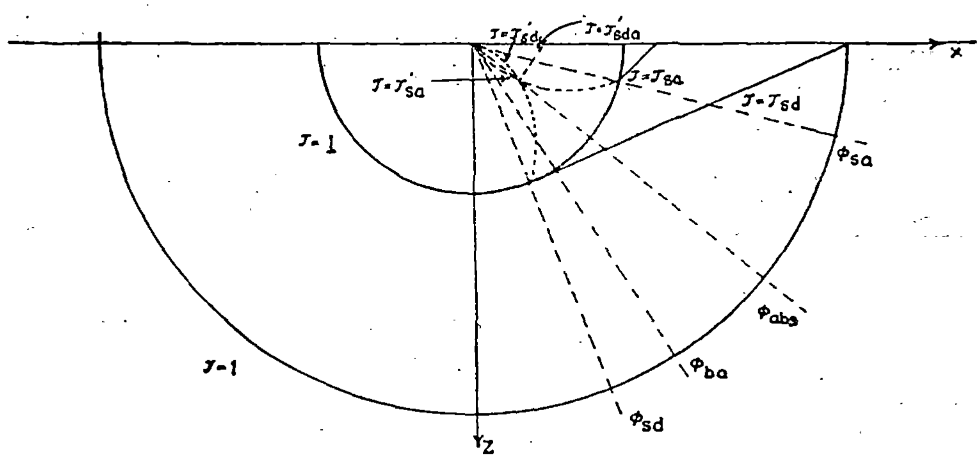


4 (e) for $c_d > a > b \geqslant c_s$, $\beta > \alpha > \gamma$.

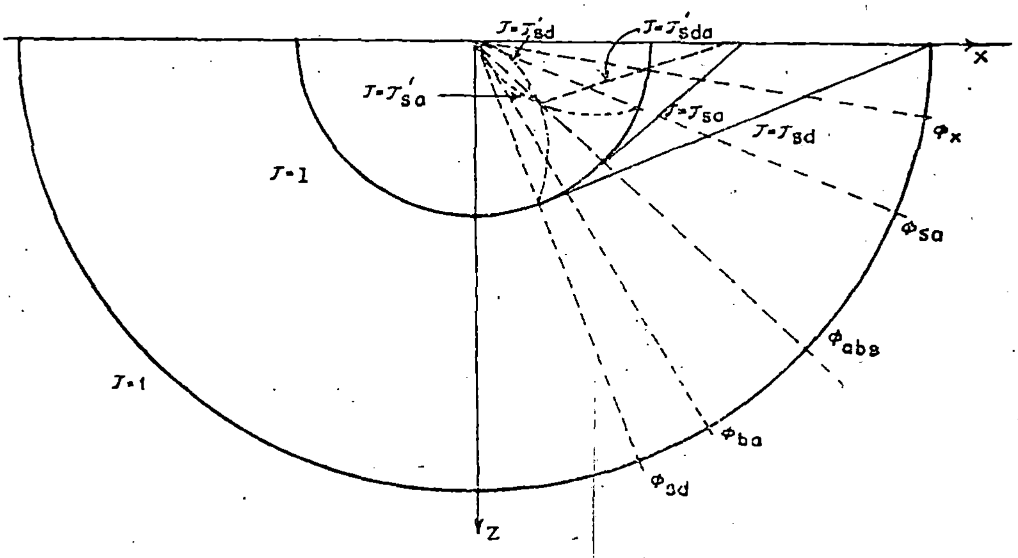


4 (f) for $c_d > a > b > c_s$, $\alpha < \gamma$.

$$\left.\begin{aligned}
&w \to 0,\ q_d^+ \to i \sin\phi,\ \frac{dq_d^+}{dt} \to \frac{c_d}{\rho_s}\frac{\cos\phi}{T_d \cos\alpha},\\
&m_d \to \cos\phi,\ m_s \to (l^2 - \sin^2\phi)^{1/2},\ m_0 \to (l^2 - 2\sin^2\phi),\\
&E^{1/2} \to \frac{1}{c_d}\,(c_d^2 - a^2\sin^2\phi)^{1/2},\ \text{for } \phi < \phi_{da}\\
&\qquad \to \frac{i}{c_d}\,(a^2\sin^2\phi - c_d^2)^{1/2},\ \text{for } \phi > \phi_{da},\\
&N \to N_1
\end{aligned}\right\} \quad \ldots(53)$$



4 (g) for $c_d > a > c_s > b$, $\alpha > \beta$, $ac_s < bc_d$.



4 (h) for $c_d > a > c_s > b$, $\beta > \alpha > \gamma'$, $ac_s < bc_d$.

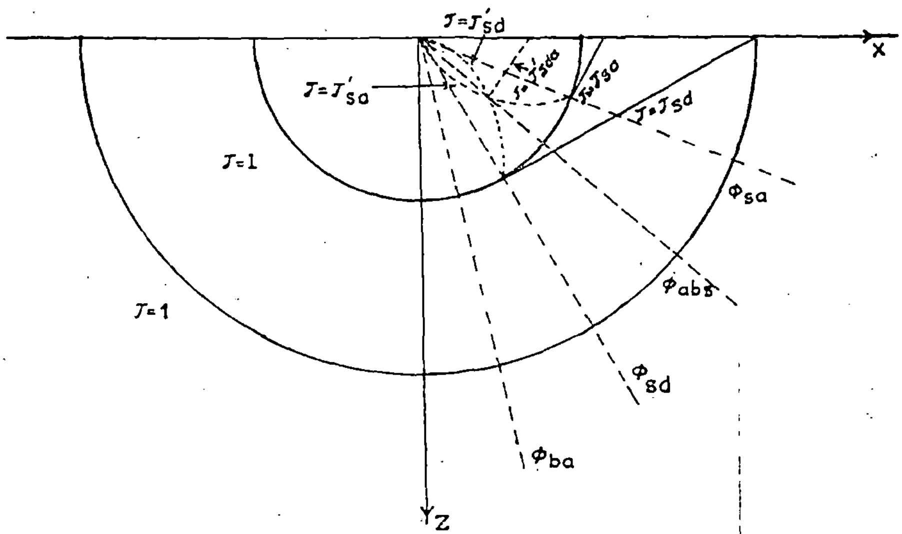
where $N_1 = (l^2 - 2\sin^2\phi)^2 + 4\sin^2\phi\cos\phi\,(l^2 - \sin^2\phi)^{1/2}$. ...(54)

Substituting these approximate values in the first integral of (31) one can find, for $\phi < \phi_{da}$

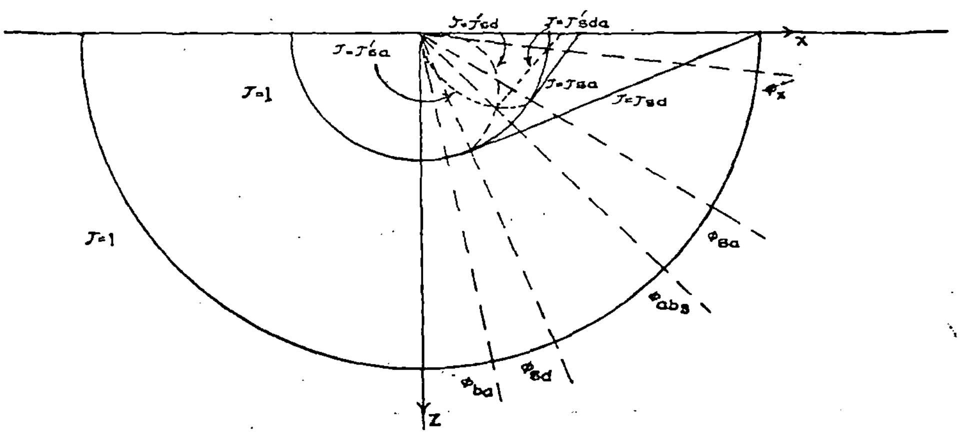
$$[u_z] \to N_{z1} \text{ as } \tau \to 1+ \qquad ...(55)$$

where

$$N_{z1} = \frac{Pabc_d}{\mu\rho}\,\frac{\cos^2\phi\,(l^2 - 2\sin^2\phi)}{(c_d^2 - a^2\sin\phi)^{1/2}.\,N_1}. \qquad ..(56)$$



4 (i) for $c_d > a > c_s > b$, $\alpha > \beta$, $a\,c_s > b\,c_d$.



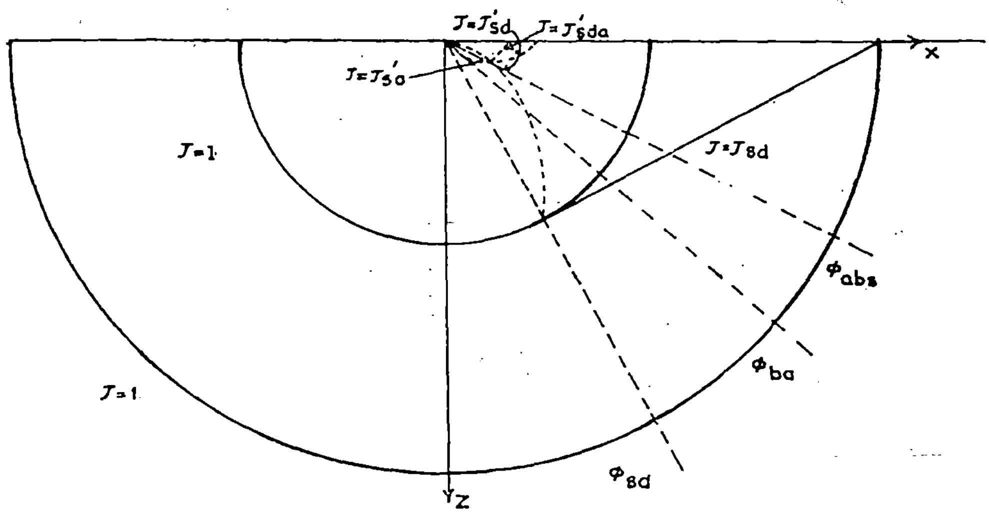
4 (j) for $c_d > a > c_s > b$, $\beta > \alpha > \gamma'$, $ac_i > bc_d$.

Again in tne second integral of (31) we put $w = T_{da} \sin \alpha$ and as $\tau \to 1-$ for $\phi > \phi_{da}$ we find that

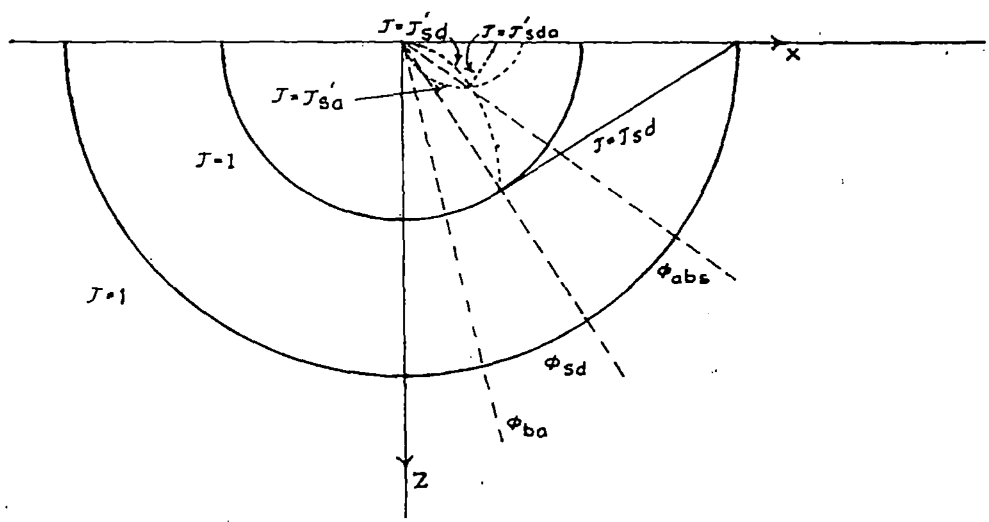
$$q_{da} \to i \sin \phi - i \cos \phi \; T_{da} \sin \alpha$$

$$\frac{dq_{da}}{dt} \to \frac{ic_d}{\rho} \cdot \frac{T_{da} \sin \alpha \sin \phi + \cos \phi}{(T_{da}^2 \sin^2 \alpha + 1 - \tau^2)^{1/2}} . \qquad .. (57)$$

Puting these values in the second integral of (31), we get



4 (k) for $a < c_e$, $ac_s < bc_d$.



4 (l) for $a < c_s$, $ac_s > b\,c_d$.

$$\int_0^{\pi/2} R_e \left[k_{zd} \left(i \sin\phi - i \cos\phi \, T_{da} \sin\alpha, \, T_{da} \sin\alpha \right) \frac{ic_d}{\rho}, \right.$$

$$\left. \times \frac{T_{da} \sin\alpha \sin\phi + \cos\phi}{(T_{da}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{da} \cos\alpha \, d\alpha \qquad \ldots(58)$$

$$= \int_0^{\epsilon} R_e \left[k_{zd} \left(i \sin\phi - i \cos\phi \, T_{da} \sin\alpha, T_{da} \sin\alpha \right) \frac{ic_d}{\rho} \right.$$

(equation continued on p. 669)

$$\times \frac{T_{da} \sin \alpha \sin \phi + \cos \phi}{(T_{da}^2 \sin^2 \alpha + 1 - \tau^2)^{1/2}} \Big] T_{da} \cos \alpha \, d\alpha$$

$$+ \int_{\epsilon}^{\pi/2} R_e \Big[k_{zd} (i \sin \phi - i \cos \phi \, T_{da} \sin \alpha, T_{da} \sin \alpha) \frac{ic_d}{\rho}$$

$$\times \frac{T_{da} \sin \alpha \sin \phi + \cos \phi}{(T_{da}^2 \sin^2 \alpha + 1 - \tau^2)^{1/2}} \Big] T_{da} \cos \alpha \, d\alpha \qquad .. (59)$$

where ϵ is very small.

Since the main contribution to the integral (58) as $\tau \to 1$ arises from the first integral of (59) as $\tau \to 1$, so for the evaluation of (58) as $\tau \to 1$, we consider the approximate value of the integral given by

$$\int_0^{\epsilon} R_e \Big[k_{zd} (i \sin \phi - i \cos \phi \, T_{da} \sin \alpha, T_{da} \sin \alpha) \frac{ic_d}{\rho}$$

$$\times \frac{T_{da} \sin \alpha \sin \phi + \cos \phi}{(T_{da}^2 \sin^2 \alpha + 1 - \tau^2)} \Big] T_{da} \cos \alpha \, d\alpha \qquad ...(60)$$

as $\tau \to 1$.

Since ϵ is very small so α is also small. So for the evaluation of the integral (60) as $\tau \to 1$ we also use the fact that $\alpha \to 0$, from which we get,

$$w \to 0, \; q_{da} \to i \sin \phi, \; m_d \to \cos \phi, \; m_s \to (l^2 - \sin^2 \phi)^{1/2},$$

$$m_0 \to (l^2 - 2 \sin^2 \phi), \qquad ...(61)$$

$$N \to N_1, \; E^{1/2} \to i/c_d \, (a^2 \sin^2 \phi - c_d^2)^{1/2} \text{ for } \phi > \phi_{da}.$$

Now substituting these approximate values in (60) and integrating we obtain the approximate value of the integral as

$$- \frac{c_d^2 \cos^2 \phi \, (l^2 - 2 \sin^2 \phi)}{\rho \, (a^2 \sin^2 \phi - c_d^2)^{1/2} . N_1} \log | \tau - 1 | \text{ when } \tau \to 1. \qquad ...(62)$$

So for $\phi > \phi_{da}$

$$[u_z] \to N'_{z4} \log | \tau - 1 | \text{ as } \tau \to 1 \qquad ...(63)$$

where

$$N'_{z4} = -\frac{2Pabc_d}{\pi\mu\rho}\frac{\cos^2\phi\,(l^2 - 2\sin^2\phi)}{(a^2\sin^2\phi - c_d^2)^{1/2}.\,N_1}. \quad ...(64)$$

In order to obtain the value of u_{zd} as $\tau \to \tau_{da}$ we put in the second integral of (31).

$$w^2 = A_{da}^2 + (T_{da}^2 - A_{da}^2)\sin^2\alpha.$$

When $\tau \to \tau_{da} +$, we find that

$$w \to 0$$

$$q_{da} \to i\frac{c_d}{a}$$

$$dq_{da}/dt \to iA'$$

$$\left.\begin{aligned}
&\text{where } A' = \frac{c_d}{\rho^a}\left(\frac{a^2 - c_d^2}{1 - \tau_{da}^2}\right)^{1/2} \text{ for } a > c_d,\\
&m_d \to l/a\,(a^2 - c_d^2)^{1/2} \text{ for } a > c_d,\\
&m_s \to \frac{l}{a}(a^2 - c_s^2)^{1/2},\quad m_0 \to \frac{l^2}{a^2}(a^2 - 2c_s^2),\\
&N \to N_2
\end{aligned}\right\} \quad ...(65)$$

$$\text{where } N_2 = l/a^4\left[l^4(a^2 - 2c_s^2)^2 + 4\,l\,c_d^2\,(a^2 - c_d^2)^{1/2}\right]$$

$$E^{1/2} \to iK^{1/2}(\tau - \tau_{da})^{1/2}$$

where

$$K = \frac{2a}{c_d}\frac{\cos^2\alpha\,(a^2 - c_d^2)^{1/2}}{\left\{(a^2 - c_d^2)^{1/2}\sin\phi - c_d\cos\phi\right\}} \text{ for } a > c_d.$$

Using these approximate values in the second integral of (31) we find that for $a > c_d$

$$[u_z] \to N_{z4} \text{ as } \tau \to \tau_{da} + \quad ...(66)$$

where

$$N_{z4} = \frac{2Pab}{\pi\mu c_d a^3}\frac{l^2(a^2 - c_d^2)^{1/2}(a^2 - 2c_s^2)\,A'\,C^{1/2}}{(2KA)^{1/2}.\,N_2} \quad ...(67)$$

where $C = 8a^2 c_d \tau_{da} (a^2 - c_d^2)^{1/2} \sin\phi \cos\phi$

$A = a^2 (a^2 - b^2) \cos^2\phi \, \tau_{da} (\tau_{da} + \tau^0_{da}) + a^2 b^2 \sin^2\phi \, \tau_{da} (\tau_{da} - \tau^0_{da})$

$$\tau^0_{da} = 1/a \left[c_d \sin\phi - (a^2 - c_d^2)^{1/2} \cos\phi \right]. \qquad ...(68)$$

It may be noted that conical wave front $\tau = \tau_{da}$ does not arise for $a < c_d$.

Next when $\phi < \phi_{sa}$, for the evaluation of u_{zs} as $\tau \to l$, we put $w = T_s \sin\alpha$ in the first integral of (33). When $\tau \to l$, we find that in the above integral

$w \to 0$

$q^+_s \to il \sin\phi$

$$\frac{dq^+_s}{dt} \to \frac{c_d}{\rho} \frac{l \cos\phi}{T_s \cos\alpha}$$

$(q^2 + w^2) \to l^2 \sin^2\phi$

$m_d \to (1 - l^2 \sin^2\phi)^{1/2}$

$m_s \to l \cos\phi$

$m_0 \to l^2 (\cos^2\phi - \sin^2\phi)$

$E^{1/2} \to 1/c_s \, (c_s^2 - a^2 \sin^2\phi)^{1/2}$ for $\phi < \phi_{sa}$

$\to i/c_s \, (a^2 \sin^2\phi - c_s^2)^{1/2}$ for $\phi > \phi_{sa}$

$N \to l^3 N_3$

where $N_3 = [l (\cos^2\phi - \sin^2\phi)^2 + 4 \sin^2\phi \cos\phi \, (1 - l^2 \sin^2\phi)^{1/2}]$.

Using these approximate values in the first integral of (33) one can find for all values of a and b.

$[u_z] \to N_{z_2}$ for $\phi < \phi_{sa}$ as $\tau \to l$

where

$$N_{z_2} = -\frac{2Pabc_s}{\mu\rho} \frac{\sin^2\phi \cos\phi \, (1 - l^2 \sin^2\phi)^{1/2}}{\left(c_s^2 - a^2 \sin^2\phi \right)^{1/2} . N_3} . \qquad ...(71)$$

For $\phi > \phi_{sa}$, considering approximate evalution of last two integrals of (33) as $\tau \to l$ it can be shown that for the case $a > b > c_d$

$$u_z \to N'_{z5} \log | \tau - l | \text{ for } \phi_{sa} < \phi < \phi_{sd} \text{ as } \tau \to l \quad \text{...(72)}$$

$$u_z \to N'_{z3} \log | \tau - l | \text{ for } \phi > \phi_{sd} \text{ as } \tau \to l \quad \text{...(73)}$$

and for the case $c_d > a > b > c_s$,

$$u_z \to N'_{z6} \log | \tau - l | \text{ for } \phi_{sd} < \phi < \phi_{sa} \text{ as } \tau \to l \quad \text{...(74)}$$

$$u_z \to N'_{z3} \log | \tau - l | \text{ for } \phi > \phi_{sa} \text{ as } \tau \to l \quad \text{...(75)}$$

and also for the case $c_s > a > b$,

$$u_z \to N'_{z6} \log | \tau - l | \text{ for } \phi > \phi_{sd} \text{ as } \tau \to 1 \quad \text{...(76)}$$

where

$$N'_{z5} = \frac{2Pabc_s}{\pi\mu\rho} \frac{\sin^2\phi \cos\phi\,(1 - l^2 \sin^2\phi)^{1/2}}{(a^2 \sin^2\phi - c_s^2)^{1/2}.\ N_3} \quad \text{...(77)}$$

$$N'_{z3} = \frac{8Pabc_s}{\pi\mu\rho} \frac{\sin^4\phi \cos^2\phi\,(l^2 \sin^2\phi - 1)}{(a^2 \sin^2\phi - c_s^2)^{1/2}.N_4} \quad \text{...(78)}$$

$$N'_{z6} = -\frac{2Pabc_d}{\pi\mu\rho} \frac{\sin^2\phi \cos^2\phi\,(l^2 \sin^2\phi - 1)^{1/2} (\cos^2\phi - \sin^2\phi)^2}{(c_s^2 - a^2 \sin^2\phi)^{1/2}.\ N_4} \quad \text{...(79)}$$

$$N_4 = [l^2 (\cos^2\phi - \sin^2\phi)^4 + 16 \sin^4\phi \cos^2\phi\,(l^2 \sin^2\phi - 1)]. \quad \text{...(80)}$$

For the approximate evaluation of the displacements at the wave fronts $\tau = \tau_{sa}$ and $\tau = \tau_{sd}$ we follow similar procedure as followed for the evaluation of u_{zd} as $\tau \to \tau_{da}$ and we find that

$$[u_z] \to N_{z5} \text{ as } \tau \to \tau_{sa} \text{ for } a > c_d \quad \text{...(81)}$$

$$[u_z] \to N_{z6} \text{ as } \tau \to \tau_{sa} \text{ for } c_d > a > c_s \quad \text{..(82)}$$

$$[u_z] \to N_{z3} (\tau - \tau_{sd})^{3/2} \text{ as } \tau \to \tau_{sd} \text{ for } a > c_d \quad \text{...(83)}$$

$$[u_z] \to N_{z7} (\tau - \tau_{zd}) \text{ as } \tau \to \tau_{sd} \text{ for } a < c_d \quad \text{...(84)}$$

where

$$N_{z5} = -\frac{4Pb\,c_d\,A'_s \sqrt{(a^2 - c_d^2)D_s}}{\pi\,\mu\,a^2 (2\,K_s\,B_s\,A_s)^{1/2}}$$

$$N_{z6} = -\frac{16\, Pa^2\, bc_d^3\, (c_d^2 - a^2)\, A'_s \sqrt{(a^2 - c_s^2) D_s}}{\pi\mu\, (2K_s\, l^2\, A_s)^{1/2}\, [l^6\, (a^2 - 2c_s^2)^4 - 16 c_d^4\, (c_d^2 - a^2)(a^2 - c_s^2)]} \quad ...(86)$$

$$N_{z3} = -\frac{4Pab}{\pi\mu} A_{sd}\, B_{sd}^2 B'_{sd}\, A'_{sd} \left(\frac{2 \operatorname{cosec} \phi}{a^2 - c_d^2} \right)^{1/2} \quad ...(87)$$

$$N_{z7} = \frac{4Pab}{\pi\mu}\, A_{sd}\, B_{sd}^2\, A'_{sd} \left(\frac{2 \operatorname{cosec} \phi}{c_d^2 - a^2} \right)^2 \quad ...(88)$$

$$A'_s = \frac{l c_d\, (a^2 - c_s^2)^{1/2}}{\rho\, [l\, (a^2 - c_s^2)^{1/2} \sin \phi - c_d \cos \phi]} \quad ...(89)$$

$$D_s = 8\, a^2\, l c_d\, \tau_{sa} \sin \phi \cos \phi\, (a^2 - c_s^2)^{1/2} \quad ...(90)$$

$$B_s = \frac{l}{a^4}\, [l^3\, (a^2 - 2c_s^2)^2 + 4c_d^2 \sqrt{(a^2 - c_d^2)(a^2 - c_s^2)}] \quad ...(91)$$

$$A_s = [\tau_{sa}\, a^2\, b^2\, (\tau_{sa} - \tau_{sa}^0) \sin^2 \phi + (a^2 - b^2)\, a^2 \cos^2 \phi\, (\tau_{sa} + \tau_{sa}^0)] \quad ...(92)$$

$$A_{sd} = \frac{\pi}{4} \left[\frac{2\, (l^2 - 1)^{1/2}}{(l^2 - 1)^{1/2} \sin \phi - \cos \phi} \right]^{1/2} \quad ...(93)$$

$$B_{sd} = (l^2 - 2)^{-1} \quad ...(94)$$

$$B'_{sd} = 4\, A_{sd}\, (l^2 - 1)^{1/2}\, B_{sd}^2 \quad ...(26)$$

$$A'_{sd} = \frac{c_d}{\rho}\, (l^2 - 1)^{1/2}\, [(l^2 - 1)^{1/2} \sin \phi - \cos\phi]^{-1} \quad ...(96)$$

In these expressions the notations $[u_z]$ stands for the change in u_z across a wave front and N_{z1} etc. are wave front coefficients.

It may also be noted that if we put $a = b$ in this problem, it reduces to the problem of uniformly expanding circular ring source and in that case our derived results coincide with the results given in the paper of Gakenheimer[5].

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HIGH FREQUENCY SCATTERING OF ANTIPLANE SHEAR WAVES BY AN INTERFACE CRACK

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The problem of diffraction of normally incident antiplane shear wave by a crack of finite length situated at the interface of two bonded dissimilar elastic half spaces has been studied. The problem is reduced to the solution of a Wiener-Hopf problem. The expressions for the stress intensity factor and the crack opening displacement have been derived for the case of wave-lengths short compared to the length of the crack. The numerical results for two different pairs of samples have been presented graphically.

1. INTRODUCTION

Scattering of elastic waves by a crack of finite length at the interface of two dissimilar elastic materials is important in view of its application in Geophysics and in Mechanical engineering problems. The extensive use of composite materials in modern technology has created interest in the wave propagation problems in layered media with interfacial discontinuities. The diffraction of Love waves by a crack of finite width at the interface of a layered half space was studied by Neerhoff[5]. Kuo[6] carried out numerical and analytical studies of transient response of an interfacial crack between two dissimilar orthotropic half spaces. Following the method of Mal[7], Srivastava et al.[1] also considered the low frequency aspect of the interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half space.

But high frequency solution of the diffraction of elastic waves by a crack of finite size is interesting in view of the fact that transient solution close to the wave front can be represented by an integral of the high frequency component of the solution. Green's function method together with a function-theoretic technique based upon an extended Wiener-Hopf argument has been developed by Keogh[3,4] for solving the problem of high frequency scattering of elastic waves by a Griffith crack situated in an infinite homogeneous elastic medium.

In the present paper, we have derived the high frequency solution of the diffraction of SH-wave when it interacts with a Griffith crack located at the interface of two bonded dissimilar elastic half spaces. To solve the problem, following the method of Chang[2], the problem has been formulated as an extended Wiener-Hopf equation and the asymptotic solutions for high frequencies or for wavelengths short compared to the length of the crack have been derived. Expressions for the dynamic stress intensity factor and the crack opening displacement have been obtained and the results have been illustrated graphically for two pairs of different types of material.

2. Formulation of the Problem

Let (x, y, z) be a rectangular Cartesian coordinates. Let an open crack of finite length 2l be located at the interface of two bonded dissimilar elastic semi-infinite solids lying parallel to x-axis. The x-axis is taken along the interface, y-axis vertically downwards into the medium and z-axis is perpendicular to the plane of the paper. (μ_1, ρ_1) and (μ_2, ρ_2) are coefficients of rigidity and density respectively of the upper and lower semi-infinite medium. The crack is subjected to a normally incoming antiplane shear wave originating at $y = -\infty$.

We are interested in finding the high frequency solution of the diffraction problem i.e. the solution when the length of the crack is large compared to the wave length of the incident wave.

Accordingly we shall have to solve the problem when the crack is subject to the following boundary conditions:

$$\sigma_{yz}^{(1)}(x, 0+) = \sigma_{yz}^{(2)}(x, 0-) = -P_s - P_0 e^{-\omega t}; \ |x| < l \qquad \ldots(1)$$

$$\sigma_{yz}^{(1)}(x, 0+) = \sigma_{yz}^{(2)}(x, 0-), \ |x| > l \qquad \ldots(2)$$

$$w_1(x, 0^+) = w_2(x, 0^{-1}), \ |x| > l \qquad \ldots(3)$$

where ω is the circular frequency and P_s is the static pressure.
Assume

$$w_1(x, y, t) = W_1(x, y)\, e^{-i\omega t} \qquad \ldots(4)$$

$$w_2(x, y, t) = W_2(x, y)\, e^{-i\omega t} \qquad \ldots(5)$$

where W_1 and W_2 satisfy the following two wave equations

$$\nabla^2 W_1(x, y) + k_1^2 W_1(x, y) = 0 \qquad \ldots(6)$$

$$\nabla^2 W_2(x, y) + k_2^2 W_2(x, y) = 0 \qquad \ldots(7)$$

with $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

The shear wave numbers k_1 and k_2 are related to the two shear wave velocities C_1 and C_2 of medium (1) and (2) respectively by

$$k_1 = \omega / C_1 \qquad \ldots(8)$$

$$k_2 = \omega / C_2 \qquad \ldots(9)$$

Without any loss of generality we assume that $k_2 > k_1$.

Let $$\sigma_{yz}^{(1)}(x, y, t) = \tau_{yz}^{(1)}(x, y)\, e^{-i\omega t} \quad ...(10)$$

$$\sigma_{yz}^{(2)}(x, y, t) = \tau_{yz}^{(2)}(x, y)\, e^{-i\omega t}. \quad ...(11)$$

In the boundary condition (1), P_s is the static pressure assumed to be sufficiently large so that crack faces do not come in contact during vibration. Since we are interested in the dynamic part of the stress distribution, so the boundary conditions (1), (2) and (3) may be written as

$$\tau_{yz}^{(1)}(x, 0^+) = \tau_{yz}^{(2)}(x, 0^-) = -P_0,\ |x| < l \quad ...(12)$$

$$\tau_{yz}^{(1)}(x, 0^+) = \tau_{yz}^{(2)}(x, 0^-),\quad |x| > l \quad ...(13)$$

and $$W_1(x, 0^+) = W_2(x, 0^-),\ |x| > l \quad ...(14)$$

that is

$$\mu_1 \frac{\partial W_1}{\partial y} = \mu_2 \frac{\partial W_2}{\partial y} = -P0,\ |x| < l,\ y = 0 \quad ...(15)$$

$$\mu_1 \frac{\partial W_1}{\partial y} = \mu_2 \frac{\partial W_2}{\partial y},\ |x| > l,\ y = 0 \quad ...(16)$$

and $$W_1(x, 0^+) = W_2(x, 0^-),\ |x| > l \quad ...(17)$$

In order to obtain solutions of wave equations (6) and (7) we introduce Fourier transform defined by

$$\overline{W}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(x, y)\, e^{i\alpha x}\, dx. \quad ...(18)$$

Thus we obtain the transformed wave equations as

$$\frac{d^2\overline{W}_1}{dy^2} - (\alpha^2 - k_1^2)\,\overline{W}_1 = 0 \quad ...(19)$$

$$\frac{d^2\overline{W}_2}{dy^2} - (\alpha^2 - k_2^2)\,\overline{W}_2 = 0. \quad ...(20)$$

The solutions of (19) and (20), bounded as y tends to infinity, are

$$\overline{W}_1(\alpha, y) = A_1(\alpha)\, e^{-\gamma_1 y},\ y \geq 0 \quad ...(21)$$

$$\overline{W}_2(\alpha, y) = A_2(\alpha)\, e^{\gamma_2 y},\ y \leq 0 \quad ...(22)$$

where

$$\gamma_1 = (\alpha^2 - k_1^2)^{1/2} \quad ...(23)$$

$$\gamma_2 = (\alpha^2 - k_2^2)^{1/2}. \quad ...(24)$$

Introducing for a complex α

$$G_+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{l}^{\infty} \tau_{yz}^{(1)}(x, 0)\, e^{i\alpha(x-l)}\, dx \qquad \ldots(25)$$

$$G_-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-l} \tau_{yz}^{(1)}(x, 0)\, e^{i\alpha(x+l)}\, dx \qquad \ldots(26)$$

and

$$G_1(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-l}^{l} \tau_{yz}^{(1)}(x, 0)\, e^{i\alpha x}\, dx \qquad \ldots(27)$$

the transformed stress at the interface $y = 0$ can be written as

$$\bar{\tau}_{yz}^{(1)}(\alpha, 0) = G_+(\alpha)\, e^{i\alpha l} + G_1(\alpha) + G_-(\alpha)\, e^{-i\alpha l}. \qquad \ldots(28)$$

Using the boundary condition (12) we note that

$$G_1(\alpha) = \frac{-P_0}{\sqrt{2\pi}\, i\alpha} \left[e^{i\alpha l} - e^{-i\alpha l} \right]. \qquad \ldots(29)$$

Further using the fact that

$$\bar{\tau}_{yz}^{(1)}(\alpha, 0) = -\mu_1\gamma_1 A_1(\alpha) \qquad \ldots(30)$$

we obtain from (28)

$$-\mu_1\gamma_1 A_1(\alpha) = G_+(\alpha)\, e^{i\alpha l} + G_-(\alpha)\, e^{-i\alpha l} - \frac{P0}{\sqrt{2\pi}\, i\alpha} \left[e^{i\alpha l} - e^{-i\alpha l} \right]. \qquad \ldots(31)$$

Since from (12) and (13) stress τ_{yz} is continuous at all points of the interface so we obtain

$$A_2(\alpha) = -\frac{\mu_1\gamma_1}{\mu_2\gamma_2} A_1(\alpha) \qquad \ldots(32)$$

so (21) and (22) take the forms

$$\bar{W}_1(\alpha, y) = A_1(\alpha)\, e^{-\gamma_1 y}, \; y \geq 0 \qquad \ldots(33)$$

$$\bar{W}_2(\alpha, y) = -\frac{\mu_1\gamma_1}{\mu_2\gamma_2} A_1(\alpha)\, e^{\gamma_2 y}, \; y \leq 0. \qquad \ldots(34)$$

Now $\bar{W}_1(\alpha, 0^+) - \bar{W}_2(\alpha, 0^-) = \frac{1}{\sqrt{2\pi}} \int_{-l}^{l} \left[W_1(x, 0^+) - W_2(x, 0^-) \right] e^{i\alpha x}\, dx$

$$= B(\alpha) \quad \text{(say)} \qquad \ldots(35)$$

which is the measure of the discontinuity of displacement along the surface of the crack. From (35) we get

$$A_1(\alpha) = \frac{\mu_2\gamma_2 B(\alpha)}{\mu_1\gamma_1 + \mu_2\gamma_2}. \qquad ...(36)$$

Eliminating $A_1(\alpha)$ from (31) and (36) we obtain an extended Wiener-Hopf equation, namely

$$G_+(\alpha)\, e^{i\alpha l} + G_-(\alpha)\, e^{-i\alpha l} + B(\alpha)K(\alpha)$$
$$= \frac{P_0}{\sqrt{2\pi}\, i\alpha}\left[e^{i\alpha l} - e^{-i\alpha l}\right] \qquad ...(37)$$

where

$$K(\alpha) = \frac{\mu_1\mu_2\gamma_1\gamma_2}{\mu_1\gamma_1 + \mu_2\gamma_2} = \frac{\mu_1\mu_2\,(\alpha^2 - k_1^2)^{1/2}}{(\mu_1 + \mu_2)}.\, R(\alpha) \qquad ...(38)$$

$$R(\alpha) = \frac{(\mu_1 + \mu_2)\,(\alpha^2 - k_2^2)^{1/2}}{\mu_1\,(\alpha^2 - k_1^2)^{1/2} + \mu_2\,(\alpha^2 - k_2^2)^{1/2}}. \qquad ...(39)$$

In order to solve the Wiener-Hopf equation given by (37) we assume that the branch points $\alpha = k_1$ and k_2 of $K(\alpha)$ possess a small imaginary part such that

$$k_1 = k_1 + i\,k_1' \quad \text{and} \quad k_2 = k_2 + i\,k_2'$$

where k_1' and k_2' are infinitesimally small positive quantities which would ultimately be made to tend to zero.

Now we write $K(\alpha) = K_+(\alpha)\, K_-(\alpha)$ where $K_+(\alpha)$ is analytic in the upper half plane $\text{Im}\ \alpha > -k_2'$ whereas $K_-(\alpha)$ is analytic in the lower half plane given by $\text{Im}\ \alpha < k_2'$. Since $\tau_{yz}(x, 0)$ decreases exponentially as $x \to \pm\infty$, $G_+(\alpha)$ and $G_-(\alpha)$ have the same common region of regularity as $K_+(\alpha)$ and $K_-(\alpha)$.

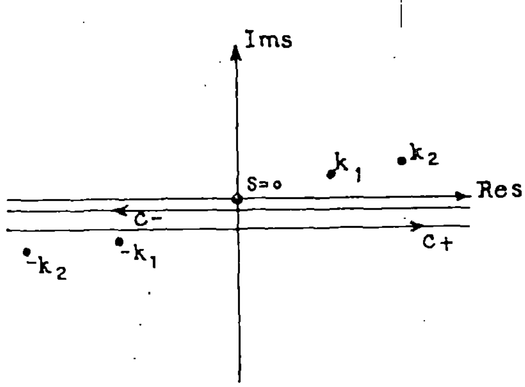

FIG. 1. Path of integration in the complex s-plane.

Now (37) can easily be expressed as two integral equations relating $G_+(\alpha)$, $G_-(\alpha)$ and $B(\alpha)$ as follows:

$$\frac{G_+(\alpha)}{K_+(\alpha)} - \frac{P_0}{\sqrt{2\pi}\, i\alpha}\left[\frac{1}{K_+(\alpha)} - \frac{1}{K_+(0)}\right]$$

$$+ \frac{1}{2\pi i}\int_{C_+} \frac{e^{-2isl}}{(s-\alpha)\,K_+(s)}\left[G_-(s) + \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

$$= -B(\alpha)\,K_-(\alpha)\,e^{-i\alpha l} + \frac{P_0}{\sqrt{2\pi}\, i\alpha\, K_+(0)} - \frac{1}{2\pi i}\cdot\int_{C_-}$$

$$\frac{e^{-2isl}}{(s-\alpha)\,K_+(s)}\left[G_-(s) + \frac{P_0}{\sqrt{2\pi}\, is}\right] ds \qquad \ldots(40)$$

and

$$\frac{G_-(\alpha)}{K_-(\alpha)} + \frac{P_0}{\sqrt{2\pi}\, i\alpha\, K_-(\alpha)} + \frac{1}{2\pi i}\int_{C_-}\frac{e^{2isl}}{(s-\alpha)\,K_-(s)}$$

$$\left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

$$= -B(\alpha)\,K_+(\alpha)\,e^{i\alpha l} - \frac{1}{2\pi i}\int_{C_+}\frac{e^{2isl}}{(s-\alpha)\,K_-(s)}\left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds \qquad \ldots(41)$$

where C_+ and C_- are the straight contours below the pole at $s = 0$ and situated within the common region of regularity of $G_+(s)$, $G_-(s)$, $K_+(s)$ and $K_-(s)$ as shown in Fig. 1.

In (40), the left-hand side is analytic in the upper half plane whereas the right-hand side is analytic in the lower-half plane and both of them are equal in the common region of analyticity of these two functions. So by analytic continuation, both sides of (40) are analytic in the whole of the s-plane. Now since

$$\tau_{yz} \sim (x \mp l)^{-1/2} \qquad \text{as } x \to \pm l$$

so $\quad G_\pm(\alpha) \sim \alpha^{-1/2} \qquad$ as $|\alpha| \to \infty$

and also $\quad K_\pm(\alpha) \sim \alpha^{1/2} \qquad$ as $|\alpha| \to \infty$

so it follows that

$$\frac{G_\pm(\alpha)}{K_\pm(\alpha)} \sim \alpha^{-1} \qquad \text{as } |\alpha| \to \infty.$$

Therefore by Liouville's theorem, both sides of (40) are equal to zero. Equation (41) can be treated similarly.

Therefore from (40) and (41) we obtain the system of integral equations given by

$$\left[G_+(\alpha) - \frac{P_0}{\sqrt{2\pi}\, i\alpha}\right] \frac{1}{K_+(\alpha)} + \frac{P_0}{\sqrt{2\pi}\, i\alpha\, K_+(0)}$$

$$+ \frac{1}{2\pi i} \int_{C_+} \frac{e^{-2isl}}{(s-\alpha)\, K_+(s)} \left[G_-(s) + \frac{P_0}{\sqrt{2\pi}\, is}\right] ds = 0 \quad \text{...(42)}$$

and

$$= \left[G_-(\alpha) + \frac{P_0}{\sqrt{2\pi}\, i\alpha}\right] \frac{1}{K_-(\alpha)} + \frac{1}{2\pi i} \int_{C_-} \frac{e^{2isl}}{(s-\alpha)\, K_-(s)}$$

$$\left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds = 0. \quad \text{...(43)}$$

Since $\tau_{yz}^{(1)}(x, 0)$ is an even function of x, so from (25) and (26) it can be shown that $G_+(-\alpha) = G_-(\alpha)$ and it has been shown in the appendix that $K_+(-\alpha) = iK_-(\alpha)$. Using these results and replacing α by $-\alpha$ and s by $-s$ in (42) it can easily be shown that equations (42) and (43) are identical. So $G_+(\alpha)$ and $G_-(\alpha)$ are to be determined from any one of the integral equation (42) or (43).

3. High Frequency Solution of the Integral Equation

To solve the integral equation (43) in the case when normalized wave number $k_1 l \gg 1$, the integration along the path C_- in (43) is replaced by the integration round the circular contour C_0 round the pole at $s = 0$ and by the integration along the contours C_{k_1} and C_{k_2} round the branch cuts through the branch points k_1 and k_2 of the function $K_-(s)$ as shown in Fig. 2.

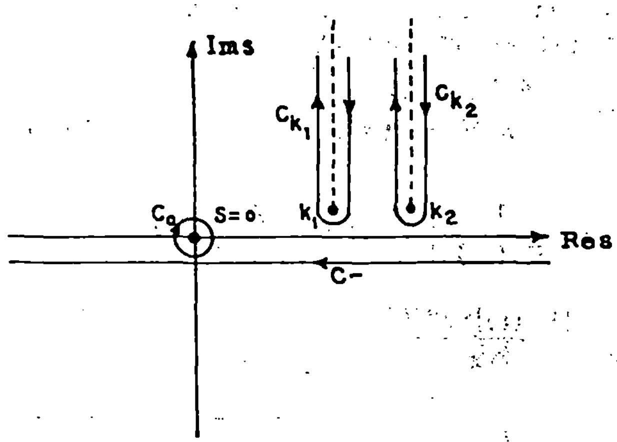

Fig. 2. Path of integration C_0, C_{k_1}, C_{k_2}.

Thus eqn. (43) takes the form

$$\left[G_-(\alpha) + \frac{P_0}{\sqrt{2\pi}\, i\alpha}\right] - \frac{P_0 K_-(\alpha)}{\sqrt{2\pi}\, i\alpha K_-(0)}$$

$$+ \frac{K_-(\alpha)}{2\pi i} \int_{C_{k_1} + C_{k_1}} \frac{\exp(2isl)}{(s-\alpha) K_-(s)} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds = 0. \quad \ldots(44)$$

Now

$$\int_{C_{k_1}} \frac{\exp(2isl)}{(s-\alpha) K_-(s)} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

$$= \frac{1}{\mu_1} \int_{C_{k_1}} \frac{e^{2isl} K_+(s)}{(s-\alpha)(s^2 - k_1^2)^{1/2}} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

which can easily be evaluated when $k_1 l \gg 1$ and is found to be equal to

$$-\frac{1}{\mu_1} \sqrt{\frac{\pi}{k_1 l}} \frac{\exp(2ik_1 l) K_+(k_1) e^{i\pi/4}}{(k_1 - \alpha)} \left[G_+(k_1) - \frac{P_0}{\sqrt{2\pi}\, ik_1}\right]. \quad \ldots(45)$$

Similarly for $k_1 l \gg 1$

$$\int_{C_{k_2}} \frac{\exp(2isl)}{(s-\alpha) K_-(s)} \left[G_+(s) - \frac{P_0}{\sqrt{2\pi}\, is}\right] ds$$

$$= -\frac{1}{\mu_2} \sqrt{\frac{\pi}{k_2 l}} \cdot \frac{\exp(2ik_2 l) K_+(k_2) e^{i\pi/4}}{(k_2 - \alpha)} \left[G_+(k_2) - \frac{P_0}{\sqrt{2\pi}\, ik_2}\right] \quad \ldots(46)$$

Using the results (45) and (46) and also the relations $G_+(-\alpha) = G_-(\alpha)$ and $K_-(\alpha) = -iK_+(\alpha)$, we obtain from (44)

$$F_+(-\alpha) + \frac{A(k_1) F_+(k_1) e^{2ik_1 l}}{\mu_1 (k_1 - \alpha)\sqrt{k_1 l}} + \frac{A(k_2) F_+(k_2) e^{2ik_2 l}}{\mu_2 (k_2 - \alpha)\sqrt{k_2 l}} = C(\alpha). \quad \ldots(47)$$

where

$$F_+(\xi) = \frac{1}{K_-(-\xi)} \left[G_+(\xi) - \frac{P_0}{\sqrt{2\pi}\, i\xi}\right] \quad \ldots(48)$$

$$A(\xi) = \frac{[K_+(\xi)]^2 e^{i\pi/4}}{2\sqrt{\pi}} \quad \ldots(49)$$

and

$$C(\xi) = \frac{P_0}{\sqrt{2\pi}\, iK_-(0)\xi} \quad \ldots(50)$$

Substituting $\alpha = -k_1$ and $\alpha = -k_2$ in (47) we obtain respectively the equations

$$\left[1 + \frac{A(k_1)\, e^{2ik_1 l}}{2\mu_1\, k_1 \sqrt{k_1 l}}\right] F_+(k_1) + \frac{A(k_2)\, F_+(k_2)\, e^{2ik_2 l}}{\mu_2\, (k_1 + k_2) \sqrt{k_2 l}} = -C(k_1) \qquad \ldots(51)$$

and

$$\frac{A(k_1)\, e^{2ik_1 l}}{\mu_1\, (k_1 + k_2) \sqrt{k_1 l}} F_+(k_1) + \left[1 + \frac{A(k_2)\, e^{2ik_2 l}}{2\mu_2\, k_2 \sqrt{k_2 l}}\right] F_+(k_2) = -C(k_2). \qquad \ldots(52)$$

Now solving (51) and (52) we get

$$F_+(k_1) = C(k_1) \left[\frac{A(k_2)\, (k_1 - k_2)\, e^{2ik_2 l}}{2\mu_2\, k_2\, (k_1 + k_2) \sqrt{k_2 l}} - 1\right] L\,(k_1, k_2) \qquad \ldots(53)$$

and

$$F_+(k_2) = C(k_2) \left[\frac{A(k_1)\, (k_2 - k_1)\, e^{2ik_1 l}}{2\mu_1\, k_1\, (k_1 + k_2) \sqrt{k_1 l}} - 1\right] L\,(k_1, k_2) \qquad \ldots(54)$$

where

$$L\,(k_1, k_2) = \left[1 + \frac{A(k_1)\, e^{2ik_1 l}}{2\mu_1\, k_1 \sqrt{k_1 l}} + \frac{A(k_2)\, e^{2ik_2 l}}{2\mu_2\, k_2 \sqrt{k_2 l}} + \frac{A(k_1) A(k_2)\, (k_1 - k_2)^2\, e^{2i(k_1 + k_2) l}}{4\mu_1\, \mu_2\, k_1\, k_2\, (k_1 + k_2)^2 \sqrt{l k_1} \sqrt{l k_2}}\right]^{-1}. \qquad \ldots(55)$$

Now expanding $L(k_1, k_2)$ and neglecting higher order terms of $1/\sqrt{k_1 l}$ and $1/\sqrt{k_2 l}$ and using (47) we get

$$\begin{aligned} G_-(\alpha) = {} & -C(\alpha)\, K_-(0) + C(\alpha)\, K_-(\alpha) \\ & + \frac{K_-(\alpha) A(k_1)\, e^{2ik_1 l} \cdot C(k_1)}{\mu_1\, (k_1 - \alpha) \sqrt{k_1 l}} \left[1 - \frac{A(k_1)\, e^{2ik_1 l}}{2\mu_1\, k_1 \sqrt{k_1 l}} - \frac{A(k_2)\, k_1\, e^{2ik_2 l}}{\mu_2\, k_2 \sqrt{k_2 l}\, (k_1 + k_2)}\right] \\ & + \frac{K_-(\alpha) A(k_2)\, e^{2ik_2 l} \cdot C(k_2)}{\mu_2\, (k_2 - \alpha) \sqrt{k_2 l}} \left[1 - \frac{A(k_1)\, k_2\, e^{2ik_1 l}}{\mu_1\, k_1 \sqrt{k_1 l}\, (k_1 + k_2)} - \frac{A(k_2)\, e^{2ik_2 l}}{2\mu_2\, k_2 \sqrt{k_2 l}}\right] \end{aligned} \qquad \ldots(56)$$

Now replacing α by $-\alpha$ and using $C(-\alpha) = -C(\alpha)$. We have

$$\begin{aligned} G_+(\alpha) = {} & C(\alpha)\, K_-(0) - C(\alpha)\, K_-(-\alpha). \\ & + \frac{K_-(-\alpha) A(k_1)\, e^{2ik_1 l} \cdot C(k_1)}{\mu_1\, (k_1 + \alpha) \sqrt{k_1 l}} \left[1 - \frac{A(k_1)\, e^{2ik_1 l}}{2\mu_1\, k_1 \sqrt{k_1 l}} - \frac{A(k_2)\, k_1\, e^{2ik_1 l}}{\mu_2\, k_2 \sqrt{k_2 l}\, (k_1 + k_2)}\right] \\ & + \frac{K_-(-\alpha) A(k_2)\, e^{2ik_2 l} \cdot C(k_2)}{\mu_2\, (k_2 + \alpha) \sqrt{k_2 l}} \left[1 - \frac{A(k_1)\, k_2\, e^{2ik_1 l}}{\mu_1\, k_1 \sqrt{k_1 l}\, (k_1 + k_2)} - \frac{A(k_2)\, e^{2ik_2 l}}{2\mu_2\, k_2 \sqrt{k_2 l}}\right]. \end{aligned} \qquad \ldots(57)$$

4. Stress Intensity Factor and Crack Opening Displacement near the Crack Tips

Now as $\alpha \to \infty$

$$K_-(-\alpha) = -iK_+(\alpha) = -i(\alpha+k_1)^{1/2}\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}} \simeq -i\alpha^{1/2}\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}$$

$$\frac{K_-(-\alpha)}{\alpha+k_1} \approx -i\alpha^{-1/2}\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}$$

$$\frac{K_-(-\alpha)}{\alpha+k_2} \approx -i\alpha^{-1/2}\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}.$$

So as $\alpha \to \infty$ we get from (56) and (57)

$$\left.\begin{aligned} G_+(\alpha) &\approx S\alpha^{-1/2} + \frac{P_0}{\sqrt{2\pi}\, i\alpha} \\ \text{and} \quad & \\ G_-(\alpha) &\approx -iS\alpha^{-1/2} - \frac{P_0}{\sqrt{2\pi}\, i\alpha} \end{aligned}\right\} \qquad \ldots(58)$$

where

$$S = \frac{P_0}{\sqrt{2\pi}\,K_-(0)}\left[1 - \frac{A(k_1)\,e^{2ik_1l}}{\mu_1k_1\sqrt{k_1l}} + \frac{A(k_2)\,e^{2ik_2l}}{\mu_2k_2\sqrt{k_2l}} + \right.$$

$$\left. + \frac{1}{2}\left(\frac{A^2(k_1)\,e^{4ik_1l}}{\mu_1^2k_1^2k_1l} + \frac{A_2^2(k_2)\,e^{4ik_2l}}{\mu_2^2k_2^2k_2l}\right) + \frac{A(k_1)A(k_2)\,e^{2i(k_1+k_2)l}}{\mu_1k_1\mu_2k_2\sqrt{k_1l.k_2l}}\right]$$

$$\times\sqrt{\frac{\mu_1\mu_2}{\mu_1+\mu_2}}. \qquad \ldots(59)$$

Now from eqn. (37) using (58) and also the fact that

$$K(\alpha) \to \pm\,\alpha.\,\frac{\mu_1\mu_2}{\mu_1+\mu_2} \text{ as } \alpha \to \pm\infty \qquad \ldots(60)$$

we get

$$B(\alpha) = \frac{\pm S}{\alpha\sqrt{\alpha}}\left[ie^{-i\alpha l} - e^{i\alpha l}\right]\frac{\mu_1+\mu_2}{\mu_1\mu_2} \qquad \ldots(61)$$

$$\text{as } \alpha \to \pm\infty.$$

Taking inverse Fourier-Transform of (35) and using the results of Fresnel integrals viz.

$$\int_0^\infty \frac{\begin{matrix}\sin\\ \cos\end{matrix}(x+l)\alpha}{\sqrt{\alpha}}\,d\alpha = \sqrt{\frac{\pi}{2(x+l)}}. \qquad \ldots(62)$$

We get the displacement jump across the surface of the crack as

$$\Delta W = W_1(x, 0+) - W_2(x, 0-) = 2S_1(1-i)\sqrt{(l-x)} \quad \text{for } x \to l - 0 \qquad \ldots(63)$$

and
$$\Delta W = W_1(x, 0+) - W_2(x, 0-) = 2S_1(1-i)\sqrt{(x+l)} \quad \text{for } x \to -l + 0 \qquad \ldots(64)$$

where
$$S_1 = \frac{(\mu_1 + \mu_2)}{\mu_1 \mu_2} \cdot S. \qquad \ldots(65)$$

Next inorder to find the value of τ_{xy} near about the crack tip we use (61) in (36) and (32) and to obtain

$$A_j(\alpha) = \frac{(-1)^{j+1} \cdot S}{\mu_j \alpha \sqrt{\alpha}} \left[ie^{-i\alpha l} - e^{i\alpha l} \right], \quad (j = 1,2) \quad \text{as } \alpha \to \infty \qquad \ldots(66)$$

and
$$A_j(\alpha) = \frac{(-1)^{j+1} \cdot S}{\mu_j \alpha \sqrt{-\alpha}} \left[e^{-i\alpha l} - ie^{i\alpha l} \right], \quad (j = 1,2) \quad \text{as } \alpha \to -\infty. \qquad \ldots(67)$$

Now
$$\tau_{yz}(x, y) = \mu_j \frac{\partial W_j(x, y)}{\partial y}, \quad j = 1,2$$

$$= \mu_j \frac{\partial}{\partial y} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_j(\alpha) \exp^{(-\gamma_j |y| - i\alpha x)} d\alpha \right]. \qquad \ldots(68)$$

Substituting the values of $A_j(\alpha)$ as $|\alpha| \to \infty$, we can write the stress near about the crack tip as

$$\tau_{yz}(x, y) = \frac{S}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-\alpha|y|}}{\sqrt{\alpha}} \left[e^{i\alpha(x+l)} - ie^{i\alpha(x-l)} - ie^{-i\alpha(x+l)} + e^{-i\alpha(x-l)} \right] d\alpha$$

$$= \frac{S(1-i)}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-\alpha|y|}}{\sqrt{\alpha}} \left[\cos\alpha(x+l) - \sin\alpha(x+l) + \cos\alpha(x-l) + \sin\alpha(x-l) \right] d\alpha$$

$$= S(1-i) \left[\frac{1}{\sqrt{r_2}} \sin\frac{\phi_2}{2} + \frac{1}{\sqrt{r_1}} \cos\frac{\phi_1}{2} \right] \qquad \ldots(69)$$

near about the crack tips, where

$$r_1 = \left[(x-l)^2 + y^2 \right]^{1/2}, \ \phi_1 = \sin^{-1} \frac{|y|}{r_1} \qquad \ldots(70)$$

$$r_2 = \left[(x+l)^2 + y^2 \right]^{1/2}, \ \phi_2 = \sin^{-1} \frac{|y|}{r_2}. \qquad \ldots(71)$$

Therefore at the interface ($y = 0$) we obtain

$$\tau_{yz} \to \frac{S\,(1-i)}{\sqrt{x-l}} \quad \text{as} \quad x \to l+0 \qquad ...(72)$$

and
$$\tau_{yz} \to \frac{S\,(1-i)}{\sqrt{-(x+l)}} \quad \text{as} \quad x \to -l-0. \qquad ...(73)$$

Now the stress intensity factor is defined by

$$K = \frac{|(1-i)\,S|\sqrt{2\pi k_1}}{P_0}. \qquad ...(74)$$

The absolute value of the complex stress intensity factor defined by (74) has been plotted against $k_1\, l$ in Fig. 3 for values of $k_1\, l > 1$ for the following two sets of materials, given by

First Set:	Steel	$\rho_1 = 7.6$ gm/cm^3	$\mu_1 = 8.32 \times 10^{11}$ dyne/cm^2
	Aluminium	$\rho_2 = 2.7$ gm/cm^3	$\mu_2 = 2.63 \times 10^{11}$ dyne/cm^2
Second Set:	Wrought iron	$\rho_1 = 7.8$ gm/cm^3	$\mu_1 = 7.7 \times 10^{11}$ dyne/cm^2
	Copper	$\rho_2 = 8.96$ gm/cm^3	$\mu_2 = 4.5 \times 10^{11}$ dyne/cm^2.

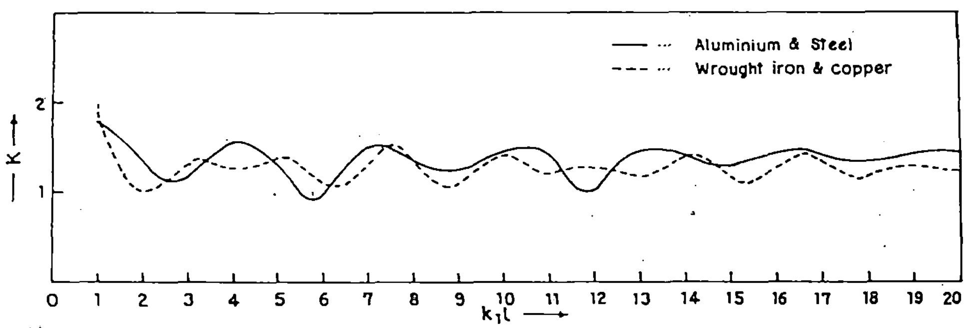

FIG. 3. Stress intensity factor K versus dimensionless frequency $k_1\, l$.

5. Crack Opening Displacement at Points Away from the Crack Tips

Next in order to obtain the displacement jump for the large values of $k_1\,(l - x)$ and $k_1\,(l+x)$ we write $G_+(\alpha)$ and $G_-(\alpha)$ from (57) and (56) respectively as

$$G_+(\alpha) = \frac{P}{\alpha} - \frac{QK_-(-\alpha)}{\alpha} + \frac{R(k_1, k_2)\,K_-(-\alpha)}{k_1+\alpha} + \frac{R(k_2, k_1)\,K_-(-\alpha)}{k_2+\alpha} \qquad ...(75)$$

and
$$G_-(\alpha) = \frac{-P}{\alpha} + \frac{QK_-(\alpha)}{\alpha} + \frac{R(k_1, k_2)\,K_-(\alpha)}{k_1-\alpha} + \frac{R(k_2, k_1)\,K_-(\alpha)}{k_2-\alpha} \qquad ...(76)$$

where $$P = \frac{P_0}{\sqrt{2\pi}.\, i} \qquad ...(77)$$

$$Q = \frac{P_0}{\sqrt{2\pi}\, iK_-(0)} = \frac{P}{K_-(0)} \qquad ...(78)$$

and $$R(k_m, k_n) = \frac{QA(k_m)\cdot e^{2ik_m l}}{\mu_m\, k_m\, \sqrt{lk_m}} \left[1 - \frac{e^{2ik_m l}\cdot A(k_m)}{\sqrt{lk_m}\, 2\mu_m k_m} - \frac{e^{2ik_n l}\cdot A(k_n)k_m}{\sqrt{lk_n}\, \mu_n k_n\, (k_m + k_n)}\right], \qquad ...(79)$$

where $m = 1$ when $n = 2$

and $m = 2$ when $n = 1$.

Again using $K_-(-\alpha) = -iK_+(\alpha)$ we get from (37)

$$B(\alpha) = -\frac{Qi\, e^{i\alpha l}}{\alpha\, K_-(\alpha)} + \frac{iR(k_1, k_2)\, e^{i\alpha l}}{(k_1+\alpha)\, K_-(\alpha)} + \frac{iR(k_2, k_1)\, e^{i\alpha l}}{(k_2+\alpha)\, K_-(\alpha)}$$

$$- \frac{Q\, e^{-i\alpha l}}{\alpha\, K_+(\alpha)} - \frac{R(k_1, k_2)\, e^{-i\alpha l}}{(k_1 - \alpha)\, K_+(\alpha)} - \frac{R(k_2, k_1)\, e^{-i\alpha l}}{(k_2 - \alpha)\, K_+(\alpha)}. \qquad ...(80)$$

From (35) we get the displacement jump across the surface of the crack as

$$W_1(x, 0^+) - W_2(x, 0^-) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(\alpha)\, e^{-i\alpha x}\, d\alpha. \qquad ...(81)$$

Now substituting the expression of $B(\alpha)$ from (80) in (81) and approximately evaluating the integrals arising in (81) term by term for large values of $k_1(l - x)$, $k_2(l - x)$, $k_1(l + x)$ and $k_2(l + x)$ and neglecting terms of order higher than $(k_1 l)^{-3/2}$ and $(k_2 l)^{-3/2}$ we obtain finally the crack opening displacement across the cracked-surface in the following form:

$$\Delta W = W_1(x, 0^+) - W_2(x, 0^-) = 2\pi\, Qi\, K_+(0) \left(\frac{1}{\mu_1 k_1} + \frac{1}{\mu_2 k_2}\right)$$

$$+ \sqrt{2}\, Qe^{-i\pi/4} \left[\left(\frac{e^{ik_1(l-x)}}{\sqrt{k_1(l-x)}} + \frac{e^{ik_1(l+x)}}{\sqrt{k_1(l+x)}}\right)\right.$$

$$\times \left(R_1 + \frac{R_1 R_{11}\, e^{2ik_1 l}}{\sqrt{2k_1 l}} + \frac{R_2 R_{21}\, e^{2ik_2 l}}{\sqrt{2k_2 l}} + \frac{R_1 (R_{11})^2\, e^{4ik_1 l}}{\sqrt{2k_1 l}\,\sqrt{2k_1 l}}\right.$$

$$+ \frac{R_2 R_{22} R_{21}\, e^{4ik_2 l}}{\sqrt{2k_2 l}\,\sqrt{2k_2 l}} + \frac{R_1 R_{12} R_{21}\, e^{2i(k_1+k_2) l}}{\sqrt{2k_1 l}\,\sqrt{2k_2 l}} + \frac{R_2 R_{21} R_{11}\, e^{2i(k_1+k_2) l}}{\sqrt{2k_1 l}\,\sqrt{2k_2 l}}$$

$$+ \left(\frac{e^{ik_2(l-x)}}{\sqrt{k_2(l-x)}} + \frac{e^{ik_2(l+x)}}{\sqrt{k_2(l+x)}}\right)$$

$$\times \left(R_2 + \frac{R_2\, R_{22}\, e^{2ik_2 l}}{\sqrt{2k_2\, l}} + \frac{R_1\, R_{12}\, e^{2ik_1 l}}{\sqrt{2k_1\, l}} + \frac{R_2\, (R_{22})^2\, e^{4ik_2 l}}{\sqrt{2k_2\, l}\ \sqrt{2k_2\, l}} + \frac{R_1\, R_{11}\, R_{12}\, e^{4ik_1 l}}{\sqrt{2k_1\, l}\ \sqrt{2k_1\, l}} + \frac{R_2\, R_{21}\, R_{12}\, e^{2i(k_1+k_2)\, l}}{\sqrt{2k_1\, l}\ \sqrt{2k_2\, l}} + \frac{R_1\, R_{12}\, R_{22}\, e^{2i(k_1+k_2)\, l}}{\sqrt{2k_1\, l}\ \sqrt{2k_2\, l}} \right) \Bigg] \qquad \ldots(82)$$

where

$$R_1 = \frac{K_+\ (k_1)}{\sqrt{2}\ \mu_1\ k_1} \qquad R_2 = \frac{K_+\ (k_2)}{\sqrt{2}\ \mu_2\ k_2}$$

$$R_{11} = \frac{D[K_+\ (k_1)]^2}{\mu_1\ (k+k_1)} \qquad R_{22} = \frac{D[K_+\ (k_2)]^2}{\mu_2\ (k_2+k_2)} \qquad \ldots(83)$$

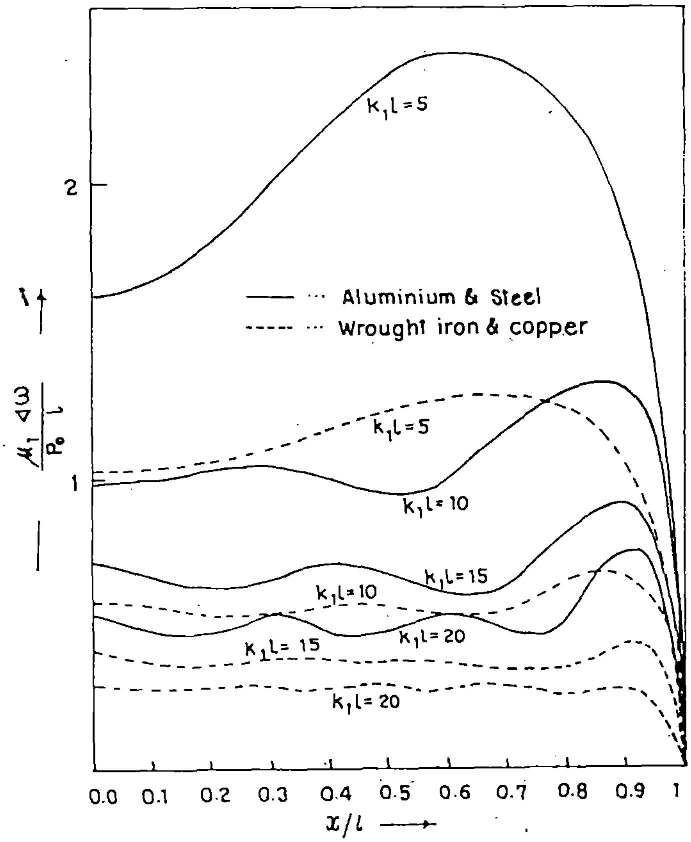

FIG. 4. Normalized crack opening displacement versus normalized distance x/l from the centre of the crack.

$$R_{21} = \frac{D.K_+(K_1)\,K_+(k_2)}{\mu_1\,(k_1+k_2)} \qquad R_{12} = \frac{DK_+(K_1)\,K_+(k_2)}{\mu_2\,(k_1+k_2)}$$

$$D = (-1)\,\frac{e^{i\pi/4}}{\sqrt{2\pi}}.$$

Expressions in (63) and (64) give the displacement jump nearabout the crack tips where as the displacement jump at points away from the crack tips are given by (82).

From these two results we can obtain the crack opening displacement at any point of the crack surface $-l < x < l, y = 0$.

Here also normalized crack opening displacement has been plotted against normalized distance x/l from the centre of the crack for two different sets of materials in Fig. 4. It is interesting to note that oscillatory nature of the crack opening displacement increases with the increase of frequencies as a result of the interference of waves inside the crack. Further we note that amplitude of the crack opening displacement decreases with the increase of frequency.

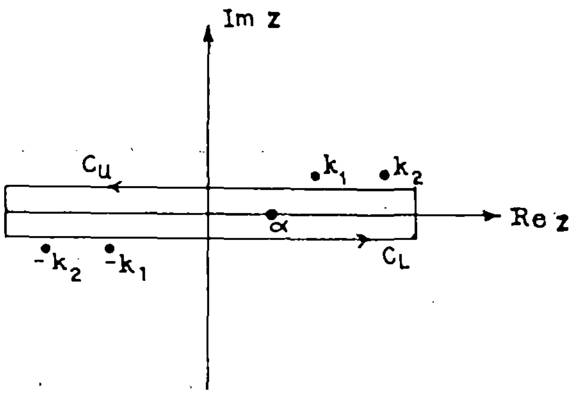

FIG. 5. Complex z-plane.

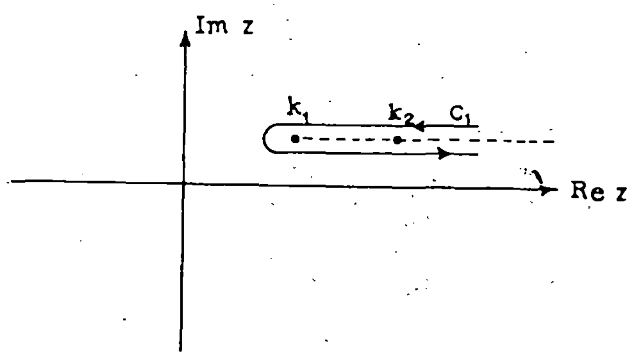

FIG. 6. Path of integration round the branch points.

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Appendix A

$$K(\alpha) = \frac{\mu_1\mu_2 (\alpha^2 - k_1^2)^{1/2}}{(\mu_1 + \mu_2)} R(\alpha)$$

where

$$R(\alpha) = \frac{(\mu_1 + \mu_2)(\alpha^2 - k_2^2)^{1/2}}{\mu_1 (\alpha^2 - k_1^2)^{1/2} + \mu_2 (\alpha^2 - k_2^2)^{1/2}}.$$

Put $$m = \frac{\mu_2}{\mu_1}.$$

Therefore

$$K(\alpha) = \frac{\mu_2 (\alpha^2 - k_1^2)^{1/2}}{1+m} R(\alpha) \qquad \text{...(A1)}$$

where

$$R(\alpha) = \frac{(1 + m)(\alpha^2 - k_2^2)^{1/2}}{(\alpha^2 - k_1^2)^{1/2} + m(\alpha^2 - k_2^2)^{1/2}} \to 1 \text{ as } |\alpha| \to \infty.$$

Now

$$R_+(\alpha)R_-(\alpha) = \frac{1}{\dfrac{m}{1+m} + \dfrac{(\alpha^2 - k_1^2)^{1/2}}{(m+1)(\alpha^2 - k_2^2)^{1/2}}}$$

Therefore

$$\log R_+(\alpha) + \log R_-(\alpha) = \log \frac{1}{\dfrac{m}{1+m} + \dfrac{(\alpha^2 - k_1^2)^{1/2}}{(m+1)(\alpha^2 - k_2^2)^{1/2}}} = \log R(\alpha)$$

$$\therefore \quad \log R_+(\alpha) = \frac{1}{2\pi i} \int_{C_L} \frac{\log R(z)}{(z-\alpha)}\, dz$$

$$= \frac{1}{2\pi i} \int_{-ic-\infty}^{-ic+\infty} \frac{\log R(z)}{(z-\alpha)}\, dz$$

where the path of integration C_L is shown in Fig. 5.
Putting $z = -z$ and using the fact that $R(z) = R(-z)$, we get

$$\log R_+(\alpha) = -\frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{\log R(z)}{(z+\alpha)}\, dz$$

$$= -\frac{1}{2\pi i} \int_{C_1} \frac{\log R(z)}{(z+\alpha)}\, dz$$

where C_1 is the contour round the branch points k_1 and k_2 as shown in Fig. 6.

So,

$$\log R_+(\alpha) = \frac{1}{2\pi i} \int_{C_1} \frac{\log \left[\dfrac{m}{m+1} + \dfrac{(z^2-k_1^2)^{1/2}}{(m+1)\,(z^2-k_2^2)^{1/2}} \right]}{(z+\alpha)}\, dz$$

$$= \frac{1}{2\pi i} \int_{k_1}^{k_2} \frac{\log \left[1 + \dfrac{i\,(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}} \right]}{(z+\alpha)}$$

$$- \frac{1}{2\pi i} \int_{k_1}^{k_2} \frac{\log \left[1 - \dfrac{i(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}} \right]}{(z+\alpha)}\, dz$$

$$= \frac{1}{\pi} \int_{k_1}^{k_2} \frac{\tan^{-1} \left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}} \right]}{(z+\alpha)}\, dz$$

$$\therefore R_+(\alpha) = \exp \left[\frac{1}{\pi} \int_{k_1}^{k_2} \frac{\tan^{-1} \left[\dfrac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}} \right]}{(z+\alpha)}\, dz \right]$$

Similarly

$$R_-(\alpha) = \exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\frac{(z^2-k_1^2)^{1/2}}{m(k^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right]$$

Therefore from (A1) we can write

$$K_+(\alpha) = \frac{\sqrt{\mu_2}(\alpha+k_1)^{1/2}}{\sqrt{(1+m)}}\cdot\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\frac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z+\alpha)}\,dz\right] \quad \text{...(A2)}$$

and

$$K_-(\alpha) = \frac{\sqrt{\mu_2}(\alpha-k_1)^{1/2}}{\sqrt{(1+m)}}\cdot\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\frac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right]. \quad \text{...(A3)}$$

Hence from (A2) and (A3) we get

$$K_+(-\alpha) = \frac{\sqrt{\mu_2}\,i(\alpha-k_1)^{1/2}}{\sqrt{(1+m)}}\cdot\exp\left[\frac{1}{\pi}\int_{k_1}^{k_2}\frac{\tan^{-1}\left[\frac{(z^2-k_1^2)^{1/2}}{m(k_2^2-z^2)^{1/2}}\right]}{(z-\alpha)}\,dz\right]$$

$$= iK_-(\alpha)$$

i.e. $K_+(-\alpha) = iK_-(\alpha)$ (A4)

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HIGH FREQUENCY SCATTERING OF PLANE HORIZONTAL SHEAR WAVES BY A GRIFFITH CRACK PROPAGATING ALONG THE BIMATERIAL INTERFACE

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Abstract—The problem of diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface is studied. In order to obtain a high frequency solution, the problem is formulated as an extended Wiener–Hopf problem. The expressions for the dynamic stress intensity factor at the crack tip and the crack opening displacement are derived for the case of wave lengths which are short compared to the length of the crack. The dynamic stress intensity factor for high frequencies is illustrated graphically for two pairs of different types of material for different crack velocities and angles of incidence.

1. INTRODUCTION

SCATTERING of elastic waves by a stationary or a moving crack of finite length at the interface of two dissimilar elastic materials is important in view of its application in fracture mechanics as well as in seismology. Recently, Takei *et al.* [1] considered the problem of diffraction of transient horizontal shear waves by a finite crack lying on a bimaterial interface. The method of solution was extended by Ueda *et al.* [2] to solve the problem of torsional impact response of a penny shaped interface crack. Srivastava *et al.* [3] also considered the low frequency aspect of the interaction of an antiplane shear wave by a Griffith crack at the interface of two bonded dissimilar elastic half spaces.

In the case of cracks of finite size, travelling at a constant velocity, loads, for mathematical simplicity, are usually assumed to be independent of time. However, in practice, structures are often required to sustain oscillating loads where the dynamic disturbances propagate through the elastic medium in the form of stress waves. The problem of diffraction of a plane harmonic polarized shear wave by a half plane crack extended under antiplane strain was first studied by Jahanshahi [4]. Later Chen and Sih [5] considered the interaction of stress waves with a semi-infinite running crack under either the plane strain or the generalized plane stress condition. Sih and Loeber [6] and Chen and Sih [7] also considered the problem of scattering of plane harmonic waves by a running crack of finite length. In both the cases the problem was reduced to a system of simultaneous Fredholm integral equations which were solved numerically.

In the present paper, we have investigated the high frequency solution of the problem of diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface. The high frequency solution of the diffraction of elastic waves by a crack of finite size is important in view of the fact that the transient solution close to the wave front can be represented by an integral of the high frequency component of the solution. In order to solve the problem, following the method of Chang [8], the problem has been formulated as an extended Wiener–Hopf equation and the asymptotic solutions for high frequencies or for wave lengths which are short compared to the length of the crack have been derived. Expressions for the dynamic stress intensity factor at the crack tip and the crack opening displacement have been derived. The dynamic stress intensity factor for high frequencies has been illustrated graphically for two pairs of different types of materials for different crack velocities and angles of incidence.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let a plane crack of width $2L$ move at a constant velocity V at the interface of two bonded dissimilar elastic semi-infinite media due to the incidence of the plane horizontal SH-wave

$$W_i = A \exp[-\{k_1(X \cos\theta_1 + Y \sin\theta_1) + \Omega T\}] \tag{1}$$

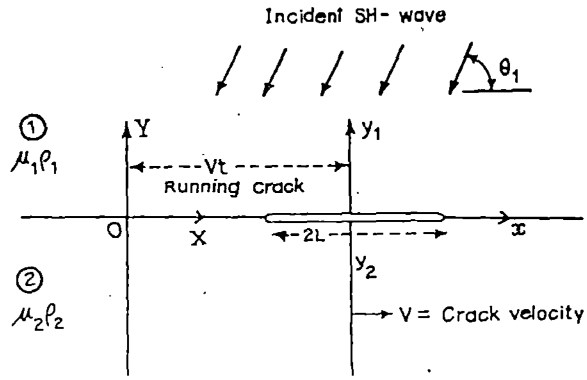

Fig. 1. Running interface crack.

in the medium. The crack lies on the bimaterial interface along $Y = 0$ with respect to the fixed rectangular co-ordinate system (X, Y, Z) as shown in Fig. 1.

We assume that the displacement and stress fields W_j, τ_{YZ_j} $(j = 1, 2)$ are

$$W_j = W_j(X, Y, T) \tag{2}$$

$$\tau_{YZ_j} = \mu_j \frac{\partial W_j(X, Y)}{\partial Y}, \tag{3}$$

in which subscripts $j = 1, 2$ refer to the upper and lower half planes, respectively, T denotes time and μ_j is the shear modulus of elasticity. The displacement W_j is governed by the classical wave equation

$$\frac{\partial^2 W_j}{\partial X^2} + \frac{\partial^2 W_j}{\partial Y^2} = \frac{1}{c_j^2}\frac{\partial^2 W}{\partial T^2}, (j = 1, 2), \tag{4}$$

where $c_j = (\mu_j/\rho_j)^{1/2}$ is the shear wave velocity and ρ_j is the density of the material. Without any loss of generality, we further assume that $c_1 > c_2$.

Due to the incident wave given by (1), reflected and transmitted waves in the absence of the crack may be written in the form

$$W_r = B \exp[-i\{k_1(X\cos\theta_1 - Y\sin\theta_1) + \Omega T\}] \tag{5}$$

and

$$W_T = C \exp[-i\{k_2(X\cos\theta_2 + Y\sin\theta_2) + \Omega T\}], \tag{6}$$

where

$$B = \frac{k_1\sin\theta_1 - mk_2\sin\theta_2}{k_1\sin\theta_1 + mk_2\sin\theta_2}A \tag{7}$$

$$C = \frac{2k_1\sin\theta_1}{k_1\sin\theta_1 + mk_2\sin\theta_2}A \tag{8}$$

$$m = \mu_2/\mu_1 \quad \text{and} \quad k_1\cos\theta_1 = k_2\cos\theta_2. \tag{9}$$

A, B, C are incident, reflected and transmitted wave amplitude, k_j is the wave number, $\Omega = k_jc_j$ is the circular frequency and θ_1, θ_2 are the angles of incidence and refraction, respectively.

A set of moving co-ordinates (x, y_j, z, t) attached to the centre of the crack moving at a constant velocity V is introduced in accordance with

$$x = X - Vt, \quad y_j = s_jY, \quad z = Z, \quad t = T, \tag{10}$$

where $s_j = (1 - M_j^2)^{1/2}$ and $M_j = V/c_j$ is the Mach number.

In terms of the translating co-ordinates x, y_j, eq. (4) becomes

$$\frac{\partial^2 W_j}{\partial x^2}+\frac{\partial^2 W_j}{\partial y_j^2}+\frac{1}{c_j^2 s_j^2}\frac{\partial}{\partial t}\left[2M_j c_j\frac{\partial W_j}{\partial x}-\frac{\partial W_j}{\partial t}\right]=0. \tag{11}$$

In the moving system (x, y, z, t) eqs (1), (5) and (6) take the form

$$e^{-i\omega t}\begin{bmatrix} W_i \\ W_r \\ W_T \end{bmatrix}=\begin{bmatrix} A\exp\left[-i\left\{k_1\left(x\cos\theta_1+\frac{y_1}{s_1}\sin\theta_1\right)+\omega t\right\}\right] \\ B\exp\left[-i\left\{k_1\left(x\cos\theta_1-\frac{y_1}{s_1}\sin\theta_1\right)+\omega t\right\}\right] \\ C\exp\left[-i\left\{k_2\left(x\cos\theta_2+\frac{y_2}{s_2}\sin\theta_2\right)+\omega t\right\}\right]\end{bmatrix}, \tag{12}$$

where $\omega=\Omega\alpha$ and $\alpha=(1+M_1\cos\theta_1)=(1+M_2\cos\theta_2)$.

In view of eq. (12) we take the solution of (11) as

$$W_j(x, y_j)\,e^{-i\omega t}=w_j(x, y_j)\exp[i(M_j\lambda_j x-\omega t)]. \tag{13}$$

Substitution of eq. (13) into eq. (11) yields the Helmholtz equation governing w_j:

$$\frac{\partial^2 w_j}{\partial x^2}+\frac{\partial^2 w_j}{\partial y_j^2}+\lambda_j^2 w_j=0 \quad (j=1,2), \tag{14}$$

where

$$\lambda_j=\frac{k_j\alpha}{s_j^2}.$$

Applying Fourier transform, eq. (14) can be solved and the result is

$$w_1(x, y_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}B_1(\xi)\exp[-i\xi x-(\xi^2-\lambda_1^2)^{1/2}y_1]\,d\xi, \quad y_1>0 \tag{15}$$

$$w_2(x, y_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}B_2(\xi)\exp[-i\xi x+(\xi^2-\lambda_2^2)^{1/2}y_2]\,d\xi, \quad y_2<0. \tag{16}$$

From (13), (15) and (16) we obtain the displacement components due to scattered field as

$$W_1=\frac{1}{2\pi}\int_{-\infty}^{\infty}A_1(\xi)\exp[-i\xi x-v_1 y_1]\,d\xi, \quad y_1>0 \tag{17}$$

$$W_2=\frac{1}{2\pi}\int_{-\infty}^{\infty}A_2(\xi)\exp[-i\xi x+v_2 y_2]\,d\xi, \quad y_2<0, \tag{18}$$

where

$$v_j=[(\xi+\lambda_j M_j)^2-\lambda_j^2]^{1/2}, \quad j=1,2. \tag{19}$$

$A_1(\xi)$ and $A_2(\xi)$ are the unknown quantities to be determined from the following boundary conditions:

$$\mu_1 s_1\frac{\partial W_1}{\partial y_1}=\mu_2 s_2\frac{\partial W_2}{\partial y_2}, \quad \text{for all } x,\ y=0 \tag{20}$$

$$W_1=W_2, \quad |x|>L, \quad y=0 \tag{21}$$

$$\frac{\partial W_1}{\partial y_1}+\frac{\partial W_i}{\partial y_1}+\frac{\partial W_r}{\partial y_1}=0, \quad |x|<L, \quad y=0+. \tag{22}$$

From the boundary condition (22) we obtain

$$\frac{\partial W_1}{\partial y_1} = A_1 \exp[-ik_1 x\cos\theta_1], \quad |x| < L, \qquad y = 0, \tag{23}$$

where

$$A_1 = \frac{i(A - B)k_1 \sin\theta_1}{s_1}. \tag{24}$$

Using (17), the above equation can be written as

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)\nu_1 \exp[-i\xi x]\,\mathrm{d}\xi &= -A_1 \exp[-ik_1 x\cos\theta_1], \quad -L < x < L \\ &= P(x), \quad x > L \quad \text{(say)} \\ &= Q(x), \quad x < -L \quad \text{(say)}.\end{aligned}$$

Therefore

$$A_1(\xi)\nu_1 = \exp[i\xi L]G_+(\xi) + \exp[-i\xi L]G_-(\xi) - \frac{A_1}{i(\xi - \xi_0)}[\exp\{i(\xi - \xi_0)L\} - \exp\{-i(\xi - \xi_0)L\}], \tag{25}$$

where

$$G_+(\xi) = \int_L^{\infty} P(x)\exp[i\xi(x - L)]\,\mathrm{d}x \tag{26}$$

$$G_-(\xi) = \int_{-\infty}^{-L} Q(x)\exp[i\xi(x + L)]\,\mathrm{d}x \tag{27}$$

$$\xi_0 = k_1 \cos\theta_1. \tag{28}$$

From the boundary condition (20) we obtain

$$A_2(\xi) = -\frac{M\nu_1 A_1(\xi)}{\nu_2}, \tag{29}$$

where

$$M = \frac{\mu_1 s_1}{\mu_2 s_2}. \tag{30}$$

Next using the boundary condition (21), we obtain

$$\begin{aligned}A_1(\xi) - A_2(\xi) &= \int_{-\infty}^{\infty} (W_1 - W_2)\exp[i\xi x]\,\mathrm{d}x \\ &= \int_{-L}^{L} P_1(x)\exp[i\xi x]\,\mathrm{d}x \\ &= N(\xi) \quad \text{(say)},\end{aligned} \tag{31}$$

which is the measure of the discontinuity of displacement along the surface of the crack. Now with the aid of (29) and (31), we find

$$A_1(\xi) = \frac{\nu_2 N(\xi)}{\nu_2 + M\nu_1}. \tag{32}$$

Eliminating $A_1(\xi)$ from (25) and (32) we obtain an extended Wiener–Hopf equation, namely

$$\exp[i\xi L]G_+(\xi) + \exp[-i\xi L]G_-(\xi) - N(\xi)K(\xi) = \frac{A_1}{i(\xi - \xi_0)}[\exp\{i(\xi - \xi_0)L\} - \exp\{-i(\xi - \xi_0)L\}], \tag{33}$$

where

$$K(\xi)=\frac{\nu_1\nu_2}{\nu_2+M\nu_1}=\frac{\nu_1}{1+M}R(\xi) \tag{34}$$

$$R(\xi)=\frac{(1+M)\nu_2}{\nu_2+M\nu_1}. \tag{35}$$

In order to solve the Wiener–Hopf equation given by (33) we assume that branch points $\xi=\lambda_1(1-M_1)$, $\lambda_2(1-M_2)$, $-\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ of $K(\xi)$ possess small imaginary parts, which would ultimately be made to tend to zero.

Now we write $K(\xi)=K_+(\xi)K_-(\xi)$, where $K_+(\xi)$ is analytic in the upper-half plane $\mathrm{Im}\,\xi>\mathrm{Im}[-\lambda_1(1+M_1)]$, whereas $K_-(\xi)$ is analytic in the lower-half plane given by $\mathrm{Im}\,\xi<\mathrm{Im}[\lambda_1(1-M_1)]$. The expressions of $K_+(\xi)$ and $K_-(\xi)$ are derived in the Appendix. Since $\partial W_1/\partial y_1$ decreases exponentially as $x\to\pm\infty$, $G_+(\xi)$ and $G_-(\xi)$ have the same common region of regularity as $K_+(\xi)$ and $K_-(\xi)$.

Now eq. (33) can easily be expressed as two integral equations involving $G_+(\xi)$, $G_-(\xi)$ and $N(\xi)$ as follows:

$$\frac{G_+(\xi)}{K_+(\xi)}-\frac{A_1\,\mathrm{e}^{-i\xi_0L}}{i(\xi-\xi_0)}\left[\frac{1}{K_+(\xi)}-\frac{1}{K_+(\xi_0)}\right]+\frac{1}{2\pi i}\int_{c_+}\frac{\mathrm{e}^{-2isL}}{(s-\xi)K_+(s)}\left[G_-(s)+\frac{A_1\,\mathrm{e}^{i\xi_0L}}{i(s-\xi_0)}\right]\mathrm{d}s$$

$$=N(\xi)K_-(\xi)\mathrm{e}^{-i\xi L}+\frac{A_1\,\mathrm{e}^{-i\xi_0L}}{i(\xi-\xi_0)K_+(\xi_0)}-\frac{1}{2\pi i}\int_{c_-}\frac{\mathrm{e}^{-2isL}}{(s-\xi)K_+(s)}\left[G_-(s)+\frac{A_1\,\mathrm{e}^{i\xi_0L}}{i(s-\xi_0)}\right]\mathrm{d}s, \tag{36}$$

where c_+ and c_- are the straight contours below the pole at $\xi=\xi_0$ and situated within the common region of regularity of $G_+(\xi)$, $G_-(\xi)$, $K_+(\xi)$ and $K_-(\xi)$ as shown in Fig. 2.

The left hand side of (36) is analytic in the upper-half plane whereas the right hand side is analytic in the lower-half plane and both of them are equal in the common region of analyticity of these two functions. Therefore, by analytic continuation, both sides of (36) are analytic in the whole of the s-plane. Next, by Liouville's theorem, it can be shown that both sides of (36) are equal to zero. Thus we obtain

$$\frac{1}{K_+(\xi)}\left[G_+(\xi)-\frac{A_1\,\mathrm{e}^{-i\xi_0L}}{i(\xi-\xi_0)}\right]+\frac{A_1\,\mathrm{e}^{-i\xi_0L}}{i(\xi-\xi_0)K_+(\xi_0)}$$
$$+\frac{1}{2\pi i}\int_{c_+}\frac{\mathrm{e}^{2isL}}{(s-\xi)K_+(s)}\left[G_-(s)+\frac{A_1\,\mathrm{e}^{i\xi_0L}}{i(s-\xi_0)}\right]\mathrm{d}s=0. \tag{37}$$

Similarly, we also obtain

$$\frac{1}{K_-(\xi)}\left[G_-(\xi)+\frac{A_1\,\mathrm{e}^{i\xi_0L}}{i(\xi-\xi_0)}\right]+\frac{1}{2\pi i}\int_{c_-}\frac{\mathrm{e}^{2isL}}{(s-\xi)K_-(s)}\left[G_+(s)-\frac{A_1\,\mathrm{e}^{-i\xi_0L}}{i(s-\xi_0)}\right]\mathrm{d}s=0. \tag{38}$$

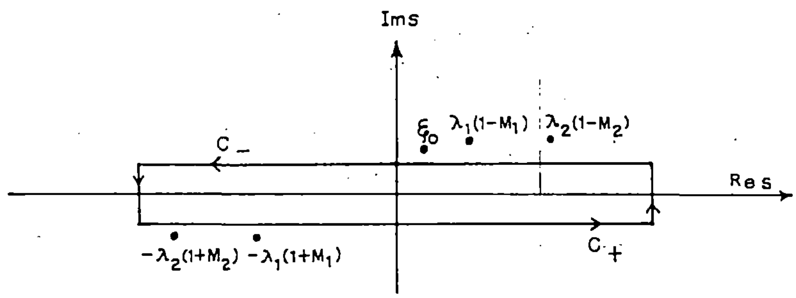

Fig. 2. Path of integration in the complex s-plane.

3. HIGH FREQUENCY SOLUTION OF THE INTEGRAL EQUATIONS

In order to obtain $G_+(\xi)$ and $G_-(\xi)$ from the integral equations (37) and (38) in the case when the normalized wave number $\lambda_1(1+M_1)L \gg 1$, the integration along the path c_+ in (37) is replaced by the integration along the loops $L_{-\lambda_1}$ and $L_{-\lambda_2}$ round the branch points $-\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ of $K_+(s)$, respectively. Also, the integration along the path c_- in (38) is replaced by the integration round the circular contour L_0, round the pole $s=\xi_0$ and by the integrations along the loops L_{λ_1} and L_{λ_2} round the branch cuts through the branch points $\lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ of the function $K_-(s)$ as shown in Fig. 3.

Finally evaluating the integrals along the straight line paths round the branch points for large values of frequency, we obtain two equations given by

$$F_\pm(\xi)+C_\pm(\xi)+\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\pm M_j)}A_\mp[\mp\lambda_j(1\pm M_j)]F_\mp[\mp\lambda_j(1\pm M_j)]}{2\{\lambda_j(1\pm M_j)-\xi\}(\lambda_jL)^{1/2}}=0, \tag{39}$$

where $\sigma_1=1$ and $\sigma_2=M$, and

$$F_\pm(\xi)=\frac{1}{K_\pm(\xi)}\left[G_\pm(\xi)\mp\frac{A_1\,e^{\mp i\xi_0L}}{i(\xi-\xi_0)}\right]$$

$$A_\pm(\xi)=\frac{i\,e^{i\pi/4}}{\pi^{1/2}}[K_\pm(\xi)]^2$$

$$C_\pm(\xi)=\frac{A_1\,e^{\mp i\xi_0L}}{i(\xi-\xi_0)K_\pm(\xi_0)}. \tag{40}$$

Now substituting $\xi=\lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ and $\xi=-\lambda_1(1+M_1)$ and $-\lambda_2(1+M_2)$ in (39) a system of linear equations of $F_+[\lambda_1(1-M_1)]$, $F_+[\lambda_2(1-M_2)]$, $F_-[-\lambda_1(1+M_1)]$ and $F_-[-\lambda_2(1+M_2)]$ are obtained. Now solving them and neglecting higher order terms of $(\lambda_1L)^{-1/2}$ and $(\lambda_2L)^{-1/2}$ we obtain, finally, after some algebraic manipulation:

$$F_\pm[\pm\lambda_k(1\mp M_k)]=-C_\pm[\pm\lambda_k(1\mp M_k)]\times\left[1-\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\mp M_k)L}A_\mp[\mp\lambda_j(1\pm M_j)]C_\mp[\mp\lambda_j(1\pm M_j)]}{2(\lambda_jL)^{1/2}\{\lambda_j(1\pm M_j)+\lambda_k(1\mp M_k)\}C_\pm[\pm\lambda_k(1\mp M_k)]}\right],\quad k=1,2. \tag{41}$$

Now using (39) we obtain from (41)

$$\begin{aligned}G_\pm(\xi)={}&\pm\frac{A_1\,e^{\mp i\xi_0L}}{i(\xi-\xi_0)}\mp\frac{A_1\,e^{\mp i\xi_0L}K_\pm(\xi)}{i(\xi-\xi_0)K_\pm(\xi_0)}\\&+\sum_{k=1}^{2}\left[\frac{\sigma_k\,e^{2i\lambda_k(1\pm M_k)L}A_\mp[\mp\lambda_k(1\pm M_k)]C_\mp[\mp\lambda_k(1\pm M_k)]K_\pm(\xi)}{2(\lambda_kL)^{1/2}\{\lambda_k(1\pm M_k)\pm\xi\}}\right.\\&\left.\times\left(1-\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\mp M_j)L}A_\pm[\pm\lambda_j(1\mp M_j)]C_\pm[\pm\lambda_j(1\mp M_j)]}{2(\lambda_jL)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\}C_\mp[\mp\lambda_k(1\pm M_k)]}\right)\right].\end{aligned} \tag{42}$$

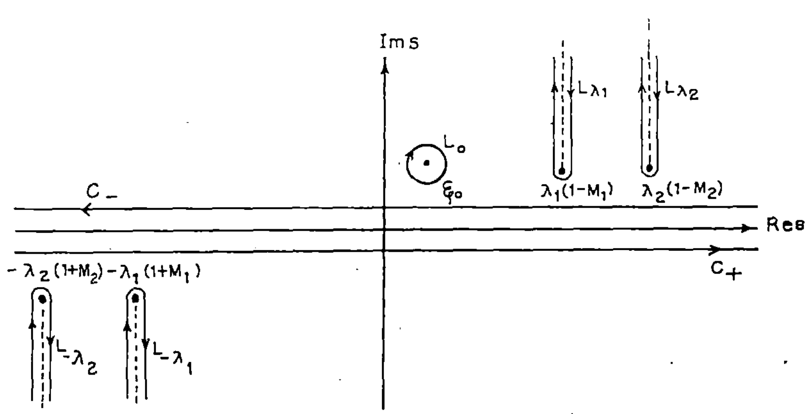

Fig. 3. Path of integration L_0, L_{λ_1}, L_{λ_2} and $L_{-\lambda_1}$, $L_{-\lambda_2}$.

4. CRACK OPENING DISPLACEMENT AT POINTS AWAY FROM THE CRACK TIPS

In order to obtain the displacement jump for the large values of $\lambda_1(L-x)$, $\lambda_2(L-x)$, $\lambda_1(L+x)$ and $\lambda_2(L+x)$, we can write $G_+(\xi)$ and $G_-(\xi)$ from (42) as

$$G_\pm(\xi) = \pm\frac{P_\pm}{\xi-\xi_0} \mp \frac{Q_\pm K_\pm(\xi)}{\xi-\xi_0} + \sum_{k=1}^{2}\frac{K_\pm(\xi)R_\pm^{(k)}}{\{\lambda_k(1\pm M_k)\pm\xi\}}, \tag{43}$$

where

$$P_\pm = \frac{A_1\,\mathrm{e}^{\mp i\xi_0 L}}{i} \tag{44}$$

$$Q_\pm = \frac{A_1\,\mathrm{e}^{\mp i\xi_0 L}}{iK_\pm(\xi_0)} = \frac{P_\pm}{K_\pm(\xi_0)} \tag{45}$$

$$R_\pm^{(k)} = \frac{\sigma_k\,\mathrm{e}^{2i\lambda_k(1\pm M_k)L}A_\mp[\mp\lambda_k(1\pm M_k)]C_\mp[\mp\lambda_k(1\pm M_k)]}{2(\lambda_k L)^{1/2}} \times\left(1-\sum_{j=1}^{2}\frac{\sigma_j\,\mathrm{e}^{2i\lambda_j(1\mp M_j)L}A_\pm[\pm\lambda_j(1\mp M_j)]C_\pm[\pm\lambda_j(1\mp M_j)]}{2(\lambda_j L)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\}C_\mp[\mp\lambda_k(1\pm M_k)]}\right). \tag{46}$$

Now we obtain from (33)

$$N(\xi) = -\frac{Q_+\,\mathrm{e}^{i\xi L}}{(\xi-\xi_0)K_-(\xi)} + \frac{R_+^{(1)}\,\mathrm{e}^{i\xi L}}{\{\xi+\lambda_1(1+M_1)\}K_-(\xi)} + \frac{R_+^{(2)}\,\mathrm{e}^{i\xi L}}{\{\xi+\lambda_2(1+M_2)\}K_-(\xi)} + \frac{Q_-\,\mathrm{e}^{-i\xi L}}{(\xi-\xi_0)K_+(\xi)} - \frac{R_-^{(1)}\,\mathrm{e}^{-i\xi L}}{\{\xi-\lambda_1(1-M_1)\}K_+(\xi)} - \frac{R_-^{(2)}\,\mathrm{e}^{-i\xi L}}{\{\xi-\lambda_2(1-M_2)\}K_+(\xi)}. \tag{47}$$

From (31) we obtain the displacement jump across the surface of the crack as

$$W_1(x,0+) - W_2(x,0-) = \frac{1}{2\pi}\int_{-\infty}^{\infty} N(\xi)\,\mathrm{e}^{-i\xi x}\,\mathrm{d}\xi. \tag{48}$$

Substituting the expression of $N(\xi)$ from (47) in (48) and approximately evaluating the integrals arising in (48) term by term for large values of $\lambda_1(L-x)$, $\lambda_2(L-x)$, $\lambda_1(L+x)$, and $\lambda_2(L+x)$, and neglecting terms of order higher than $(\lambda_1 L)^{-3/2}$ and $(\lambda_2 L)^{-3/2}$, we finally obtain the crack opening displacement across the cracked surface at points away from the crack tips in the following form:

$$\Delta W = W_1(x,0+) - W_2(x,0-) = -iQ_+K_+(\xi_0)\,\mathrm{e}^{i\xi_0(L-x)} \times\left[\frac{1}{\{(\xi_0+\lambda_1M_1)^2-\lambda_1^2\}^{1/2}} + \frac{M}{\{(\xi_0+\lambda_2M_2)^2-\lambda_2^2\}^{1/2}}\right] - \frac{\mathrm{e}^{-i\pi/4}}{\pi^{1/2}}[T_+ - T_-], \tag{49}$$

where

$$T_\pm = \sum_{k=1}^{2}\frac{\sigma_k\,\mathrm{e}^{i\lambda_k(1\mp M_k)(L\mp x)}}{\{\lambda_k(L\mp x)\}^{1/2}}\left[\frac{Q_\pm K_\pm[\pm\lambda_k(1\mp M_k)]}{2^{1/2}[\lambda_k(1\mp M_k)\mp\xi_0]} - \sum_{j=1}^{2}\frac{\sigma_j A_\mp[\mp\lambda_j(1\pm M_j)]K_\pm[\pm\lambda_k(1\mp M_k)]}{2(2\lambda_j L)^{1/2}\{\lambda_k(1\mp M_k)+\lambda_j(1\pm M_j)\}}\left(\frac{Q_\mp\,\mathrm{e}^{2i\lambda_j(1\pm M_j)L}}{\{\lambda_j(1\pm M_j)\pm\xi_0\}} - \sum_{r=1}^{2}\frac{\sigma_r A_\pm[\pm\lambda_r(1\mp M_r)]Q_\pm\,\mathrm{e}^{2i[\lambda_r(1\mp M_r)+\lambda_j(1\pm M_j)]L}}{2(\lambda_r L)^{1/2}\{\lambda_r(1\mp M_r)+\lambda_j(1\pm M_j)\}\{\lambda_r(1\mp M_r)\mp\xi_0\}}\right)\right]. \tag{50}$$

5. STRESS INTENSITY FACTOR AND CRACK OPENING DISPLACEMENT NEAR THE CRACK TIPS

Now considering the behaviour of ξ at infinity we obtain from (42)

$$G_\pm(\xi) \simeq \pm\frac{A_1\,\mathrm{e}^{\mp i\xi_0 L}}{i(\xi-\xi_0)} + S_\pm\xi^{-1/2} \quad \text{as } \xi\to\infty, \tag{51}$$

where

$$S_{\pm} = \frac{1}{(1+M)^{1/2}}\left[\mp\frac{A_1\,e^{\mp i\xi_0 L}}{iK_{\pm}(\xi_0)} \pm \sum_{k=1}^{2} \frac{\sigma_k\,e^{2i\lambda_k(1\pm M_k)L}A_{\mp}[\mp\lambda_k(1\pm M_k)]C_{\mp}[\mp\lambda_k(1\pm M_k)]}{2(\lambda_k L)^{1/2}}\right.$$

$$\left.\times\left(1-\sum_{j=1}^{2}\frac{\sigma_j\,e^{2i\lambda_j(1\mp M_j)L}A_{\pm}[\pm\lambda_j(1\mp M_j)]C_{\pm}[\pm\lambda_j(1\mp M_j)]}{2(\lambda_j L)^{1/2}\{\lambda_j(1\mp M_j)+\lambda_k(1\pm M_k)\}C_{\mp}[\mp\lambda_k(1\pm M_k)]}\right)\right]. \tag{52}$$

Now, from eq. (33), using (51) and also the fact that

$$K(\xi)\to\pm\frac{\xi}{1+M} \quad \text{as } \xi\to\pm\infty, \tag{53}$$

we obtain

$$N(\xi) = \frac{1+M}{\pm\xi(\xi)^{1/2}}[S_+\,e^{i\xi L} + S_-\,e^{-i\xi L}] \quad \text{as } \xi\to\pm\infty. \tag{54}$$

Taking the inverse Fourier transform of (31) and using the results of Fresnel integrals, *viz.*

$$\int_0^\infty \frac{\begin{matrix}\sin\\ \cos\end{matrix}(x+L)\alpha}{(\alpha)^{1/2}}\,d\alpha = \left[\frac{\pi}{2(x+L)}\right]^{1/2}, \tag{55}$$

we obtain the displacement jump across the surface of the crack as

$$\Delta W = W_1(x,0+) - W_2(x,0-) = -(1+M)(1+i)S_-\left[\frac{2(x+L)}{\pi}\right]^{1/2} \quad \text{for } x\to -L+0 \tag{56}$$

$$= -(1+M)(1-i)S_+\left[\frac{2(L-x)}{\pi}\right]^{1/2} \quad \text{for } x\to L-0. \tag{57}$$

Expressions (56) and (57) give the displacement jump near to the crack tips, whereas the displacement jump away from the crack tips is given by (49).

Next, in order to find the value of τ_{yz} near to the crack tip we use (54) in (32) and (29) and obtain

$$A_j(\xi) = \frac{(-1)^{j+1}\sigma_j}{\xi(\xi)^{1/2}}[S_+\,e^{i\xi L} + S_-\,e^{-i\xi L}], \quad j=1,2 \quad \text{as } \xi\to\infty \tag{58}$$

$$A_j(\xi) = \frac{i(-1)^{j+1}\sigma_j}{\xi(-\xi)^{1/2}}[S_+\,e^{i\xi L} - S_-\,e^{-i\xi L}], \quad j=1,2 \quad \text{as } \xi\to-\infty. \tag{59}$$

Now

$$\tau_{yz}(x,y_j) = \mu_j\frac{\partial W_j(x,y_j)}{\partial y} = \mu_j s_j\frac{\partial W_j(x,y_j)}{\partial y_j} = \frac{\mu_j s_j}{2\pi}\frac{\partial}{\partial y_j}\left[\int_{-\infty}^{\infty} A_j(\xi)\,e^{-i\xi x - \nu_j|y_j|}\,d\xi\right]. \tag{60}$$

Now substituting the values of $A_j(\xi)$ as $|\xi|\to\infty$ in (60) and integrating, we obtain the stress near to the crack tip as

$$\tau_{yz}(x,y_1) = -\frac{\mu_1 s_1}{(2\pi)^{1/2}}\left[(1-i)S_+\frac{\cos(\psi_1/2)}{r_1^{1/2}} + (1+i)S_-\frac{\sin(\psi_2/2)}{r_2^{1/2}}\right] \tag{61}$$

and

$$\tau_{yz}(x,y_2) = -\frac{\mu_1 s_1}{(2\pi)^{1/2}}\left[(1-i)S_+\frac{\cos(\phi_1/2)}{d_1^{1/2}} + (1+i)S_-\frac{\cos(\phi_2/2)}{d_2^{1/2}}\right], \tag{62}$$

where

$$r_1 = \{(x-L)^2 + y_1^2\}^{1/2}, \quad \psi_1 = \sin^{-1}\frac{|y_1|}{r_1}$$

$$r_2 = \{(x+L)^2 + y_1^2\}^{1/2}, \quad \psi_2 = \sin^{-1}\frac{|y_1|}{r_2}$$

$$d_1 = \{(x-L)^2 + y_2^2\}^{1/2}, \quad \phi_1 = \sin^{-1}\frac{|y_2|}{d_1}$$

$$d_2 = \{(x+L)^2 + y_2^2\}^{1/2}, \quad \phi_2 = \sin^{-1}\frac{|y_2|}{d_2}. \tag{63}$$

Therefore at the interface ($y = 0$) near to the right-hand crack vertex, we obtain

$$\tau_{yz} \to -\frac{\mu_1 s_1 (1-i) S_+}{\{2\pi(x-L)\}^{1/2}} \quad \text{as } x \to L + 0. \tag{64}$$

Now the normalized dynamic stress intensity factor K at the crack tip $x = L$ is defined by

$$K = \left|\frac{[2\pi k_1 (x-L)]^{1/2}\tau_{yz}}{\mu_1 A_1}\right| = s_1 \left|\frac{(1-i)S_+(k_1)^{1/2}}{A_1}\right| \quad \text{for } x \to L + 0, \tag{65}$$

where A_1 is given by (24).

The absolute values of the complex stress intensity factor defined by (65) have been plotted against $k_1 L$ in Fig. 4 for values $k_1 L > 1$ for different values of the Mach number M_2 and the angle of incidence for the following sets of materials:

first set:	steel	$\rho_1 = 7.6$ gm/cm^3,	$\mu_1 = 8.32 \times 10^{11}$ dyne/cm^2
	aluminium	$\rho_2 = 2.7$ gm/cm^3,	$\mu_2 = 2.63 \times 10^{11}$ dyne/cm^2
second set:	wrought iron	$\rho_1 = 7.8$ gm/cm^3,	$\mu_1 = 7.7 \times 10^{11}$ dyne/cm^2
	copper	$\rho_2 = 8.96$ gm/cm^3,	$\mu_2 = 4.5 \times 10^{11}$ dyne/cm^2.

As the Mach number $M_2 \to 0$ the stress intensity factor K tends to the value of the stress intensity factor corresponding to the stationary crack. The problem for $\theta_1 = \pi/2$ and $M_2 = 0.0$ was solved earlier by Pal and Ghosh [9]. The graph of stress intensity factor vs $k_1 L$ corresponding to $\theta_1 = \pi/2$ and $M_2 = 0.0$ as given in Fig. 4a is found to coincide exactly with that given by Pal and

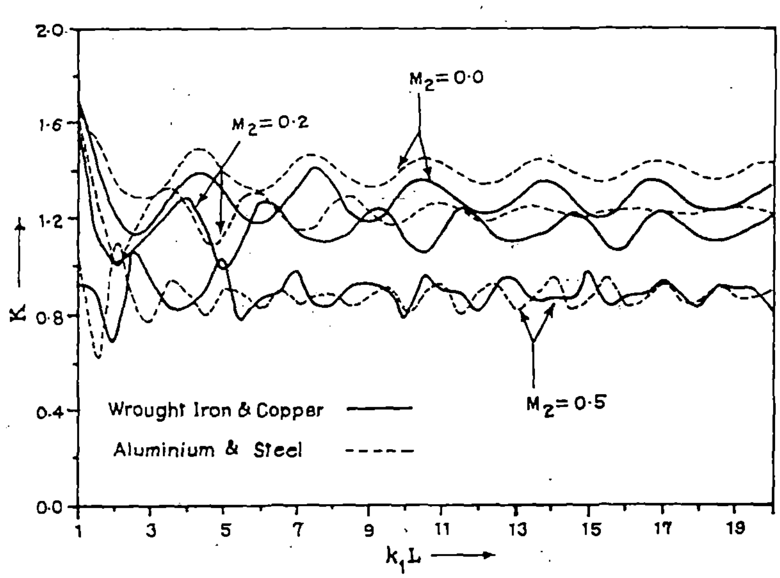

Fig. 4(a) (*caption overleaf*)

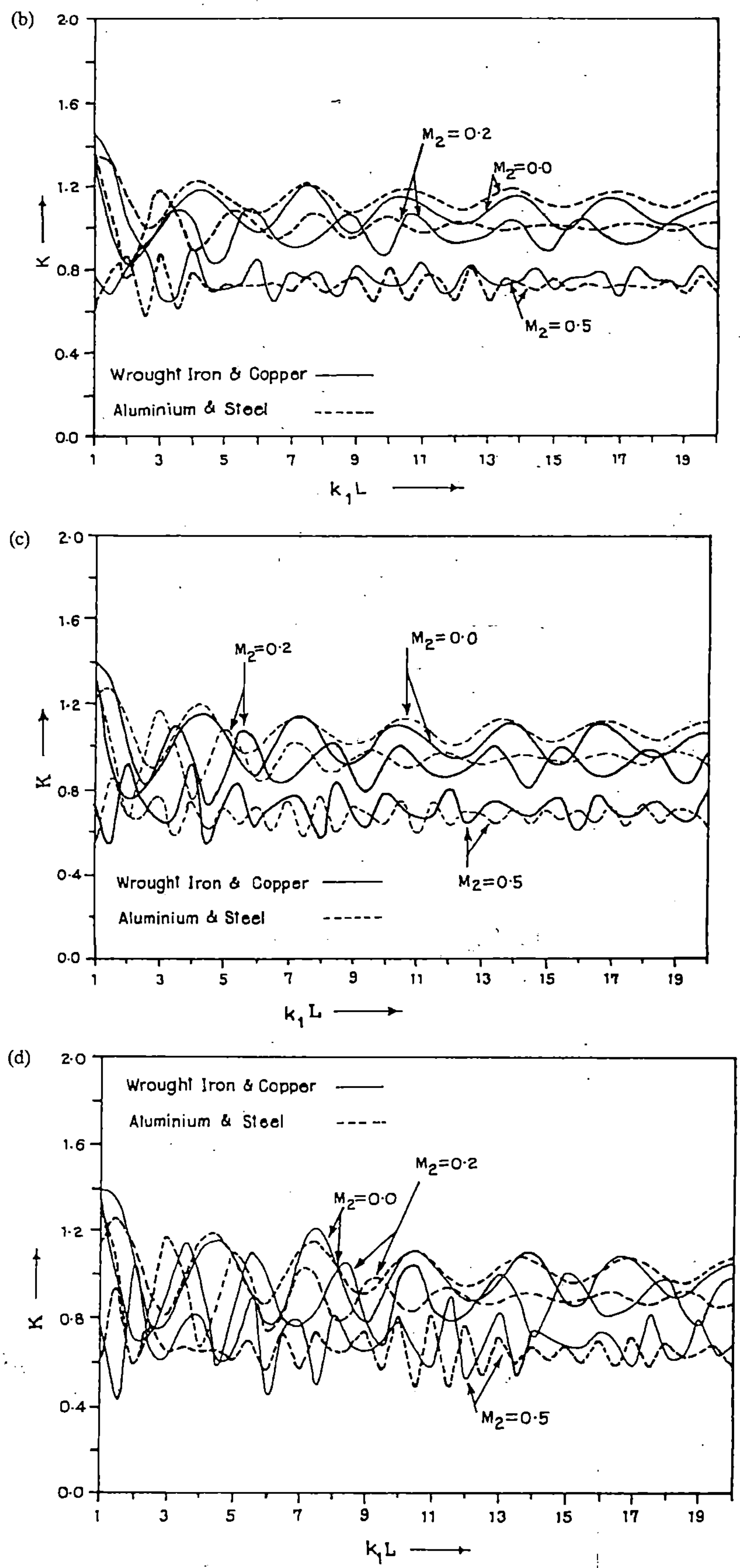

Fig. 4. Stress intensity factor K versus dimensionless $k_1 L$. (a) $\theta_1 = \pi/2$. (b) $\theta_1 = \pi/3$. (c) $\theta_1 = \pi/4$. (d) $\theta_1 = \pi/6$.

Ghosh [9]. It is interesting to note that for both pairs of materials, as M_2 increases, the peaks of the curves of stress intensity factors decrease in magnitude and occur at lower values of $k_1 L$. Further, it may be noted that for any fixed value of M_2 the stress intensity factor decreases with the decrease in the value of the angle of incidence.

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APPENDIX

$$K(\xi) = \frac{\{(\xi + \lambda_1 M_1)^2 - \lambda_1^2\}^{1/2}}{1 + M} R(\xi), \tag{A1}$$

where

$$M = \frac{\mu_1 s_1}{\mu_2 s_2}$$

$$R(\xi) = \frac{(1 + M)\{(\xi + \lambda_2 M_2)^2 - \lambda_2^2\}^{1/2}}{M\{(\xi + \lambda_1 M_1)^2 - \lambda_1^2\}^{1/2} + \{(\xi + \lambda_2 M_2)^2 - \lambda_2^2\}^{1/2}} \to 1 \quad \text{as } |\xi| \to \infty.$$

Now

$$R_+(\xi) R_-(\xi) = \frac{1}{\dfrac{1}{1+M} + \dfrac{M\{(\xi + \lambda_1 M_1)^2 - \lambda_1^2\}^{1/2}}{(1+M)\{(\xi + \lambda_2 M_2)^2 - \lambda_2^2\}^{1/2}}}.$$

Taking logs on both sides

$$\log R(\xi) = \log R_+(\xi) + \log R_-(\xi) = \frac{1}{2\pi i} \int_{c_L + c_U} \frac{\log R(\eta)}{\eta - \xi} \, d\eta,$$

where the paths of integration c_L and c_U are as shown in Fig. A1. Therefore

$$\log R_+(\xi) = \frac{1}{2\pi i} \int_{c_L} \frac{\log R(\eta)}{\eta - \xi} \, d\eta$$

$$\log R_-(\xi) = \frac{1}{2\pi i} \int_{c_U} \frac{\log R(\eta)}{\eta - \xi} \, d\eta$$

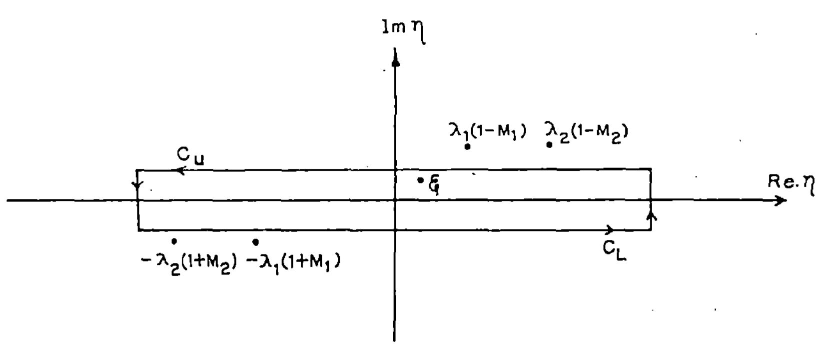

Fig. A1. Complex η-plane.

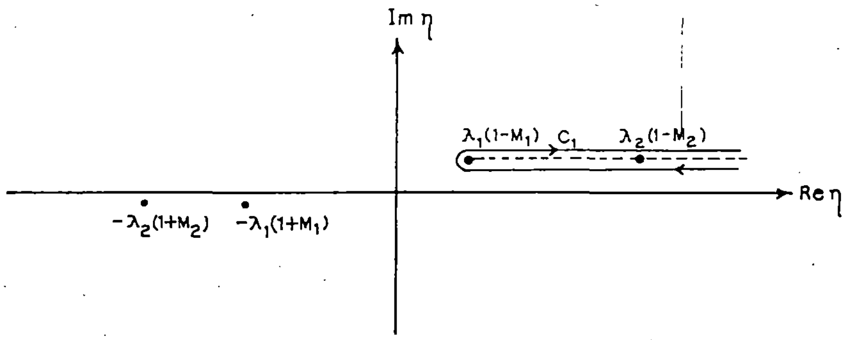

Fig. A2. Path of integration round the branch points.

or

$$\log R_+(\xi) = \frac{1}{2\pi i}\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty} \frac{\log R(\eta)}{\eta-\xi}\,d\eta.$$

Putting $\eta = -\eta$

$$\log R_+(\xi) = \frac{1}{2\pi i}\int_{i\varepsilon+\infty}^{i\varepsilon-\infty} \frac{\log R(-\eta)}{\eta+\xi}\,d\eta$$

$$\log R_-(\xi) = \frac{1}{2\pi i}\int_{i\varepsilon+\infty}^{i\varepsilon-\infty} \frac{\log R(\eta)}{\eta-\xi}\,d\eta,$$

therefore

$$\log R_-(\xi) = \frac{1}{2\pi i}\int_{c_1} \frac{1}{(\eta-\xi)}\log\left[\frac{1}{\dfrac{1}{1+M}+\dfrac{M\{(\eta+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{(\eta+\lambda_2 M_2)^2-\lambda_2^2\}^{1/2}}}\right]d\eta,$$

where c_1 is the contour round the branch points $\lambda_1(1-M_1)$ and $\lambda_2(1-M_2)$ as shown in Fig. A2. Therefore

$$\log R_-(\xi) = \frac{1}{2\pi i}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)} \frac{1}{(\eta-\xi)}\left[\log\left(\frac{1}{1+M}+\frac{iM\{(\eta+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{\lambda_2^2-(\eta+\lambda_2 M_2)^2\}^{1/2}}\right)\right.$$
$$\left.-\log\left(\frac{1}{1+M}-\frac{iM\{(\eta+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{(1+M)\{\lambda_2^2-(\eta+\lambda_2 M_2)^2\}^{1/2}}\right)\right]d\eta$$
$$= \frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)} \frac{1}{(\eta-\xi)}\tan^{-1}\left[\frac{M\{(\eta+\lambda_1 M_1)^2-\lambda_1^2\}^{1/2}}{\{\lambda_2^2-(\eta+\lambda_2 M_2)^2\}^{1/2}}\right]d\eta,$$

and therefore

$$R_-(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)} \frac{1}{(\eta-\xi)}\tan^{-1}\left(\frac{M[(\eta+\lambda_1 M_1)^2-\lambda_1^2]^{1/2}}{[\lambda_2^2-(\eta+\lambda_2 M_2)^2]^{1/2}}\right)d\eta\right].$$

Similarly

$$R_+(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1(1+M_1)}^{\lambda_2(1+M_2)} \frac{1}{(\eta+\xi)}\tan^{-1}\left(\frac{M[(\eta-\lambda_1 M_1)^2-\lambda_1^2]^{1/2}}{[\lambda_2^2-(\eta-\lambda_2 M_2)^2]^{1/2}}\right)d\eta\right].$$

Therefore from (A1) we can write

$$K_+(\xi) = \left[\frac{\xi+\lambda_1(1+M_1)}{(1+M)}\right]^{1/2}\exp\left[\frac{1}{\pi}\int_{\lambda_1(1+M_1)}^{\lambda_2(1+M_2)} \frac{1}{(\eta+\xi)}\tan^{-1}\left(\frac{M[(\eta-\lambda_1 M_1)^2-\lambda_1^2]^{1/2}}{[\lambda_2^2-(\eta-\lambda_2 M_2)^2]^{1/2}}\right)d\eta\right] \quad \text{(A2)}$$

and

$$K_-(\xi) = \left[\frac{\xi-\lambda_1(1-M_1)}{(1+M)}\right]^{1/2}\exp\left[\frac{1}{\pi}\int_{\lambda_1(1-M_1)}^{\lambda_2(1-M_2)} \frac{1}{(\eta-\xi)}\tan^{-1}\left(\frac{M[(\eta+\lambda_1 M_1)^2-\lambda_1^2]^{1/2}}{[\lambda_2^2-(\eta+\lambda_2 M_2)^2]^{1/2}}\right)d\eta\right]. \quad \text{(A3)}$$

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FORCED VERTICAL VIBRATION OF FOUR RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

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Abstract—In this paper, the problem of two-dimensional oscillations of four rigid strips, situated on a homogeneous isotropic semi-infinite elastic solid and forced by a specified normal component of the displacement has been considered. The mixed boundary value problem of determining the unknown stress distribution just below the strips and vertical displacement outside the strips has been converted to the determination of the solution of quadruple integral equations by the use of Fourier transform. An iterative solution of these integral equations valid for low frequency has been found by the application of the finite Hilbert transform. The normal stress just below the strips and the vertical displacement away from the strips have been obtained. Finally, graphs are presented which illustrate the salient features of the displacement and stress intensity factors at the edges of the strips. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The problem of the effect of vibrating source in different forms on the surface of an elastic medium have aroused attention in view of their application in seismology and geophysics. Reissner (1937), and later Millar and Pursey (1954), treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. Analytical treatment of the dynamical response of footings and solid-structure interaction are usually available in the literature only for circular and elliptical footings, and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, and also in the study of the vibration of dams and large structures subjected to earthquakes. The problem of circular punch has been solved analytically by Awojobi and Grootenhuis (1965), Robertson (1966), Gladwell (1968) and others. Roy (1986) considered the dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. Karasudhi *et al.* (1968) obtained a low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium. Wickham (1977) worked out in detail the problem of forced two-dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium. Recently, Mandal and Ghosh (1992) treated the problem of forced vertical vibration of two rigid strips on a semi-infinite elastic medium.

To improve the dynamic models of buildings and other structures, it will be fruitful to have analytic results for foundations of a more complicated nature. In what follows, the problem of vertical vibration of four rigid strips in smooth contact with a semi-infinite elastic medium has been considered. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by running wheels on a railway track. The resulting mixed boundary value problem has been reduced to the solution of

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quadruple integral equations, which have further been reduced to the solution of integral-differential equations. Finally, an iterative solution valid for low frequency has been obtained.

From the solution of the integral equations, the stress just below the strips and also the vertical displacement at points outside the strips on the free surface have been found. The effects of stress intensity factors at the edges of the strips and vertical displacement outside the strips have been shown by means of graphs.

2. FORMULATION OF THE PROBLEM

Consider the normal vibration of frequency ω of four rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty < X < \infty$, $Y \geqslant 0$, $-\infty < Z < \infty$. It is assumed that the motion is forced by prescribed displacement distribution $v_0 e^{-i\omega t}$ normal to the four infinite strips located in the region $d_1 \leqslant |X| \leqslant d_2$, $d_3 \leqslant |X| \leqslant d$, $Y = 0$, $|Z| < \infty$, where v_0 is a constant.

Normalizing all the lengths with respect to d and putting $X/d = x$, $Y/d = y$, $Z/d = z$, $d_1/d = a$, $d_2/d = b$, $d_3/d = c$, one finds that the rigid strips are defined by $a \leqslant |x| \leqslant b$, $c \leqslant |x| \leqslant 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

With the time factor $e^{-i\omega t}$ suppressed throughout the analysis, the displacement components can be written as

$$u(x,y) = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}; \quad v(x,y) = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x}; \quad w(x,y) = 0 \tag{1}$$

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + m_1^2\phi &= 0 \\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + m_2^2\psi &= 0 \end{aligned} \tag{2}$$

in which

$$m_1^2 = \frac{\omega^2 d^2}{c_1^2} \quad \text{and} \quad m_2^2 = \frac{\omega^2 d^2}{c_2^2}.$$

In terms of ϕ and ψ the stress components are

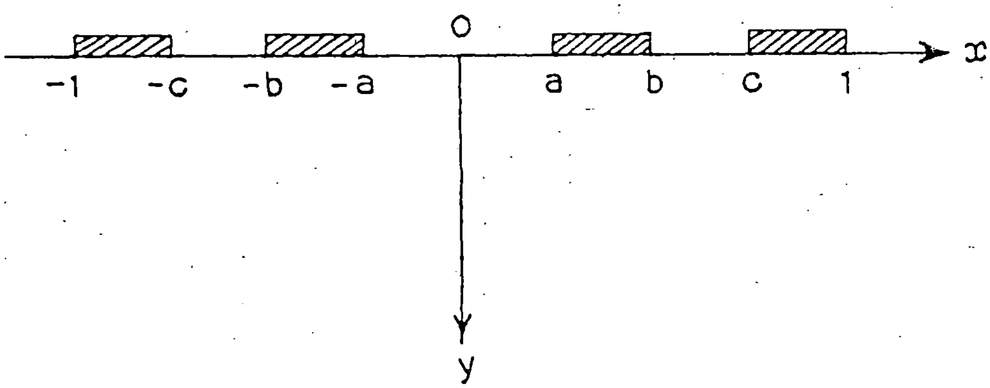

Fig. 1. Geometry of the problem.

$$\tau_{xy} = \mu\left\{2\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right\}$$

$$\tau_{yy} = -\mu\left\{\left(m_2^2+2\frac{\partial^2}{\partial x^2}\right)\phi-2\frac{\partial^2\psi}{\partial x\partial y}\right\}$$

$$\tau_{yz} = 0. \tag{3}$$

The boundary conditions are

$$v(x,0) = v_0, \quad x\in I_2, I_4 \tag{4}$$

$$\tau_{yy}(x,0) = 0, \quad x\in I_1, I_3, I_5 \tag{5}$$

$$\tau_{xy}(x,0) = 0, \quad -\infty < x < \infty \tag{6}$$

where $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,1)$, $I_5 = (1,\infty)$. The solution of the Helmholtz equation (2) can be written as

$$\phi = 2\int_0^\infty A(\xi)\cos\xi x e^{-\gamma_1 y}\,d\xi$$

$$\psi = 2\int_0^\infty B(\xi)\sin\xi x e^{-\gamma_2 y}\,d\xi \tag{7}$$

where

$$\gamma_j = \begin{cases}(\xi^2-m_j^2)^{1/2}, & |\xi|\geqslant m_j\\ -i(m_j^2-\xi^2)^{1/2}, & |\xi|\leqslant m_j\end{cases}, \quad j = 1,2$$

and $A(\xi)$ and $B(\xi)$ are unknown functions to be determined from the boundary conditions.

By using the boundary condition (6), it can be shown that

$$B(\xi) = \frac{2\gamma_1\xi}{\xi^2+\gamma_2^2}A(\xi). \tag{8}$$

Now the displacement component v and stress τ_{yy} become

$$v(x,y) = 2\int_0^\infty\left[\frac{2\xi^2}{2\xi^2-m_2^2}e^{-\gamma_2 y}-e^{-\gamma_1 y}\right]A(\xi)\cos\xi x\,d\xi \tag{9}$$

$$\tau_{yy}(x,y) = -2\mu\int_0^\infty\left[(m_2^2-2\xi^2)e^{-\gamma_1 y}+\frac{2\xi^2\gamma_1\gamma_2}{2\xi^2-m_2^2}e^{-\gamma_2 y}\right]A(\xi)\cos\xi x\,d\xi. \tag{10}$$

From the boundary conditions (4) and (5) we get the following set of integral equations in $P(\xi)$:

$$\int_0^\infty\frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}P(\xi)\cos\xi x\,d\xi = \frac{1}{2}v_0, \quad x\in I_2, I_4 \tag{11}$$

and

$$\int_0^\infty P(\xi)\cos \xi x \, d\xi = 0, \quad x \in I_1, I_2, I_5 \tag{12}$$

where

$$P(\xi) = \frac{(2\xi^2 - m_2^2)^2 + 4\xi^2\gamma_1\gamma_2}{(2\xi^2 - m_2^2)} A(\xi).$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (11) and (12) in the form

$$P(\xi) = \int_a^b t f(t^2)\cos \xi t \, dt + \int_c^1 u g(u^2) \cos \xi u \, du \tag{13}$$

where $f(t^2)$ and $g(u^2)$ are unknown functions to be determined.

By the choice of $P(\xi)$ given by eqn (13) the relation (12) is satisfied automatically and eqn (11) becomes

$$\int_a^b t f(t^2)\, dt \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2 - m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \cos \xi x \cos \xi t \, d\xi$$

$$+ \int_c^1 u g(u^2)\, du \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2 - m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \cos \xi x \cos \xi u \, d\xi = \frac{v_0}{2}, \quad x \in I_2, I_4 \tag{14}$$

using the relation

$$\frac{\sin \xi x \sin \xi t}{\xi^2} = \int_0^x \int_0^t \frac{wv J_0(\xi w) J_0(\xi v) \, dv \, dw}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{d}{dx}\int_a^b t f(t^2)\, dt \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{wv L_1(v, w)\, dv \, dw}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}} +$$

$$+ \frac{d}{dx}\int_c^1 u g(u^2)\, du \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{wv L_1(v, w)\, dv \, dw}{(x^2 - w^2)^{1/2}(u^2 - v^2)^{1/2}} = \frac{v_0}{2}, \quad x \in I_2, I_4 \tag{15}$$

where

$$L_1(v, w) = \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2 - m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} J_0(\xi w) J_0(\xi v)\, d\xi. \tag{16}$$

By a simple contour integration technique used by Ghosh and Ghosh (1985), $L_1(v, w)$ can be written as

$$L_1(v, w) = -i\tau^2 \int_0^1 \frac{(1 - \eta^2)^{1/2}(2\eta^2 - \tau^2)^2 H_0^{(1)}(m_1 \eta w) J_0(m_1 \eta v)}{(2\eta^2 - \tau^2)^4 + 16\eta^4(\eta^2 - 1)(\tau^2 - \eta^2)} d\eta$$

$$- 4i\tau^2 \int_0^1 \frac{\eta^2(\eta^2 - 1)(\tau^2 - \eta^2)^{1/2} H_0^{(1)}(m_1 \eta w) J_0(m_1 \eta v)}{(2\eta^2 - \tau^2)^4 + 16\eta^4(\eta^2 - 1)(\tau^2 - \eta^2)} d\eta$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v$$

$$=\frac{-i\tau^2}{16(1-\tau^2)}\left[\sum_{j=0}^{2}P_j\int_0^1\frac{(1-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{\eta^2-\tau_j^2}\,d\eta\right.$$

$$\left.+\sum_{j=0}^{2}S_j\int_0^{\tau}\frac{(\tau^2-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{\eta^2-\tau_j^2}\,d\eta\right]$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v \tag{17}$$

where

$$\tau=\frac{m_2}{m_1}=\frac{c_1}{c_2}, \quad Q_0(\eta)=(2\eta^2-\tau^2)^2-4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$$

and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta)=0$. τ_1, τ_2 are the roots of the equation

$$(2\eta^2-\tau^2)^2+4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}=0.$$

$Q_0'(\eta)$ denotes the derivative of $Q_0(\eta)$ with respect to η and

$$P_j=\frac{(2\tau_j^2-\tau^2)}{\prod_i(\tau_j^2-\tau_i^2)}, \quad S_j=\frac{4\tau_j^2(\tau_j^2-1)}{\prod_i(\tau_j^2-\tau_i^2)}, \quad i,j=0,1,2 \quad \text{and} \quad i\neq j.$$

τ_i^2

The corresponding expression for $L_1(v,w)$ for $w<v$ follows from eqn (17) by interchanging w and v. For a Poisson ratio $\sigma=\frac{1}{4}$, the values of τ, τ_0, τ_1 and τ_2 are given by

$$\tau^2=\frac{2(1-\sigma)}{(1-2\sigma)}=3, \quad \tau_0^2=\frac{3}{(0.9194)^2}, \quad \tau_1^2=\frac{3}{(2+2\sqrt{3})} \quad \text{and} \quad \tau_2^2=\frac{3}{4}.$$

Hence, in this case $\tau_2<\tau_1<1<\tau<\tau_0$.

By using the series expansions of J_0 and $H_0^{(1)}$, and evaluating the integrals arising in eqn (17), we obtain, after some algebraic manipulation,

$$L_1(v,w)=\frac{2}{\pi}\tau^2\left[\left(\gamma+\log\frac{m_1w}{2}-\frac{\pi i}{2}\right)M+N-\frac{P}{4}(w^2+v^2)m_1^2\log m_1\right]+O(m_1^2) \quad w>v.$$

$$=\frac{2}{\pi}\tau^2\left[\left(\gamma+\log\frac{m_1v}{2}-\frac{\pi i}{2}\right)M+N-\frac{P}{4}(w^2+v^2)m_1^2\log m_1\right]+O(m_1^2) \quad w<v, \tag{18}$$

where $\gamma=0.5772157\ldots$ is Euler's constant,

$$M\equiv-\frac{\pi}{4(1-\tau^2)} \tag{19}$$

$$N = \frac{\pi}{32(1-\tau^2)}\left[4\log\frac{4}{\tau} + \sum_{j=1}^{2} P_j \frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j} - P_0\frac{\sqrt{(\tau_0^2-1)}}{\tau_0}\right.$$
$$\log\{\tau_0 - \sqrt{(\tau_0^2-1)}\} + \sum_{j=1}^{2} S_j \frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j}$$
$$\left. - S_0\frac{\sqrt{(\tau_0^2-\tau^2)}}{\tau_0}\log\left\{\frac{\tau_0+\sqrt{(\tau_0^2-\tau^2)}}{\tau}\right\}\right], \quad (20)$$

$$P = \frac{\pi}{32(1-\tau^2)}\left[\sum_{j=0}^{2} P_j\left(\frac{1}{2}-\tau_j^2\right) + \sum_{j=0}^{2} S_j\left(\frac{\tau^2}{2}-\tau_j^2\right)\right]. \quad (21)$$

Next, differentiating both sides of relation (14) with respect to x, we obtain

$$\int_a^b tf(t^2)\,dt\int_0^\infty \frac{\gamma_1 m_2^2\xi}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\sin\xi x\sin\xi t\,d\xi$$
$$+\int_c^1 ug(u^2)\,du\int_0^\infty \frac{\gamma_1 m_2^2\xi}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\sin\xi x\sin\xi u\,d\xi = 0, \quad x\in I_2, I_4.$$

Following a similar procedure as for deriving eqn (15), we get

$$x\int_a^b \frac{tf(t^2)}{x^2-t^2}dt + x\int_c^1 \frac{ug(u^2)}{x^2-u^2}du = \int_a^b tf(t^2)\,dt\frac{\partial}{\partial t}\int_0^x\int_0^t \frac{wvL_2(v,w)\,dv\,dw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$
$$+\int_c^1 ug(u^2)\,du\frac{\partial}{\partial u}\int_0^x\int_0^u \frac{wvL_2(v,w)\,dw\,dv}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}}, \quad x\in I_2, I_4 \quad (22)$$

where

$$L_2(v,w) = \int_0^\infty\left[\xi - \frac{2\gamma_1\xi^2(m_1^2-m_2^2)}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\right]J_0(\xi w)J_0(\xi v)\,d\xi. \quad (23)$$

For small values of m_1 and m_2 such that $m_1 = O(m_2)$, one can use the contour integration technique mentioned above and obtain

$$L_2(v,w) = 2im_1^2(1-\tau^2)\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\eta^2 H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}d\eta$$
$$+4im_1^2(1-\tau^2)\int_0^\tau -\frac{2\eta^4(\eta^2-1)(\tau^2-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}d\eta$$
$$-2\pi im_1^2(1-\tau^2)\left[\frac{\eta^2(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v \quad (24)$$

By a process similar to the one which led to eqn (18), eqn (24) can be written as

$$L_2(v,w) = -\frac{4P}{\pi}(1-\tau^2)m_1^2\log m_1 + O(m_1^2) \quad (25)$$

where P is given by eqn (21).

Now examining relations (15) and (18), we assume the expressions of the functions $f(t^2)$ and $g(u^2)$ as

$$f(t^2) = f_0(t^2) + f_1(t^2) m_1^2 \log m_1 + O(m_1^2)$$
$$g(u^2) = g_0(u^2) + g_1(u^2) m_1^2 \log m_1 + O(m_1^2). \tag{26}$$

Putting the above expressions of $f(t^2)$ and $g(u^2)$, and the value of $L_2(v, w)$ given by eqn (25) in eqn, (22) and equating the coefficients of like powers of m_1 we obtain

$$\int_a^b \frac{t f_0(t^2)}{x^2 - t^2} \mathrm{d}t + \int_c^1 \frac{u g_0(u^2)}{x^2 - u^2} \mathrm{d}u = 0, \quad x \in I_2, I_4 \tag{27}$$

and

$$\int_a^b \frac{t f_1(t^2)}{x^2 - t^2} \mathrm{d}t + \int_c^1 \frac{u g_1(u^2)}{x^2 - u^2} \mathrm{d}u = -\frac{4}{\pi} P(1-\tau^2) \left[\int_a^b t f_0(t^2)\, \mathrm{d}t + \int_c^1 u g_0(u^2)\, \mathrm{d}u \right], \quad x \in I_2, I_4. \tag{28}$$

Following Srivastava and Lowengrub (1970), the solutions of the above integral equations (27) can be obtained as

$$f_0(t^2) = D_1 \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{c^2-t^2}{1-t^2}\right)^{1/2} \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$
$$- D_2 \left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} \frac{1}{\sqrt{(1-t^2)(c^2-t^2)}}, \quad t \in I_2 \tag{29}$$

and

$$g_0(u^2) = D_1 \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{u^2-c^2}{1-u^2}\right)^{1/2} \frac{1}{\sqrt{(u^2-a^2)(u^2-b^2)}}$$
$$+ D_2 \left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2} \frac{1}{\sqrt{(u^2-c^2)(1-u^2)}}, \quad u \in I_4 \tag{30}$$

where D_1 and D_2 are constants which can be calculated as follows:

We substitute the value of $L_1(v, w)$ from eqn (18), as well as the expansions of $f(t^2)$ and $g(u^2)$ obtained from eqns (26), (29) and (30) up to $O(m_1^2 \log m_1)$ in eqn (15). When the coefficients of like powers of m_1 from both sides of the resulting equation are equated, after some algebraic manipulation we get the following

$$D_1 = \frac{\pi v_0}{4\tau^2} \frac{(X_2 - X_1)}{(X_1 X_4 - X_2 X_3)}; \quad D_2 = \frac{\pi v_0}{4\tau^2} \frac{(X_1 - X_3)}{(X_1 X_4 - X_2 X_3)} \tag{31}$$

where

$$X_1 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left[\left\{ \left(\gamma + \log\frac{m_1}{2} - \frac{\pi \mathrm{i}}{2}\right) M + N \right\} (J_1 + J_3) + \frac{1}{2} M J_1 \log(b^2 - a^2) + M J_5 \right] \tag{32}$$

$$X_2 = \left\{ \left(\gamma + \log\frac{m_1}{2} - \frac{\pi \mathrm{i}}{2}\right) M + N \right\} (J_4 - J_2) - \frac{1}{2} M J_2 \log(b^2 - a^2) + M J_6 \tag{33}$$

$$X_3 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left[\left\{\left(\gamma+\log\frac{m_1}{2}-\frac{\pi i}{2}\right)M+N\right\}(J_1+J_3)+\frac{1}{2}MJ_3\log(1-c^2)+MJ_7\right] \tag{34}$$

$$X_4 = \left\{\left(\gamma+\log\frac{m_1}{2}-\frac{\pi i}{2}\right)M+N\right\}(J_4-J_2)+\frac{1}{2}MJ_4\log(1-c^2)-MJ_8$$

$$J_1 = \int_a^b\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}\frac{t\,dt}{\sqrt{(t^2-a^2)(b^2-t^2)}};\quad J_2 = \int_a^b\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\frac{t\,dt}{\sqrt{(1-t^2)(c^2-t^2)}}$$

$$J_3 = \int_c^1\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}\frac{u\,du}{\sqrt{(u^2-a^2)(u^2-b^2)}};\quad J_4 = \int_c^1\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\frac{u\,du}{\sqrt{(u^2-c^2)(1-u^2)}}$$

$$J_5 = \int_c^1\frac{u\log(\sqrt{u^2-b^2}+\sqrt{u^2-a^2})}{\sqrt{(u^2-a^2)(u^2-b^2)}}\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}du$$

$$J_6 = \int_c^1\frac{u\log(\sqrt{u^2-b^2}+\sqrt{u^2-a^2})}{\sqrt{(1-u^2)(u^2-c^2)}}\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}du$$

$$J_7 = \int_a^b\frac{t\log(\sqrt{c^2-t^2}+\sqrt{1-t^2})}{\sqrt{(t^2-a^2)(b^2-t^2)}}\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}dt$$

$$J_8 = \int_a^b\frac{t\log(\sqrt{c^2-t^2}+\sqrt{1-t^2})}{\sqrt{(1-t^2)(c^2-t^2)}}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}dt. \tag{35}$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x, y)$ on the plane $y = 0$ can be found from the relations (10), (13), (26), (29) and (30) as

$$\tau_{yy}(x,0) = \frac{\pi\mu x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\left[D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-x^2}{1-x^2}\right)^{1/2} - D_2\frac{(x^2-a^2)}{\sqrt{(1-x^2)(c^2-x^2)}}\right]+O(m_1^2\log m_1),\quad x\in I_2$$

$$= \frac{\pi\mu x}{\sqrt{(x^2-c^2)(1-x^2)}}\left[D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\frac{(x^2-c^2)}{\sqrt{(x^2-a^2)(x^2-b^2)}} + D_2\left(\frac{x^2-a^2}{x^2-b^2}\right)^{1/2}\right]+O(m_1^2\log m_1),\quad x\in I_4. \tag{36}$$

Defining the stress intensity factors at the edges of the strips by the relations

$$K_a = \underset{x\to a^+}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{x-a}}{\pi\mu v_0}\right|;\quad K_b = \underset{x\to b}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{b-x}}{\pi\mu v_0}\right|$$

$$K_c = \underset{x\to c^+}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{x-c}}{\pi\mu v_0}\right|;\quad K_1 = \underset{x\to 1}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{1-x}}{\pi\mu v_0}\right|.$$

We get

$$K_a = \left|\frac{\sqrt{a}D_1/v_0}{\sqrt{2(b^2-a^2)}}\right| \tag{37}$$

$$K_b = \left|\frac{\sqrt{b}}{\sqrt{2(b^2-a^2)}}\left\{\frac{D_1}{v_0}\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-b^2}{1-b^2}\right)^{1/2} - \frac{D_2}{v_0}\frac{(b^2-a^2)}{\sqrt{(1-b^2)(c^2-b^2)}}\right\}\right| \tag{38}$$

$$K_c = \left|\frac{\sqrt{c}}{\sqrt{2(1-c^2)}}\frac{D_2}{v_0}\left(\frac{c^2-a^2}{c^2-b^2}\right)^{1/2}\right| \tag{39}$$

$$K_1 = \left|\frac{1}{\sqrt{2(1-c^2)}}\left\{\frac{(1-c^2)D_1}{\sqrt{(c^2-a^2)(1-b^2)}} + \left(\frac{1-a^2}{1-b^2}\right)^{1/2} D_2\right\}\right|. \tag{40}$$

The vertical displacement $v(x, y)$ on the plane $y = 0$ can be obtained from eqns (9), (13), (26), (29) and (30) as

$$v(x,0) = \frac{4\tau^2}{\pi}\left[\left\{\left(\gamma+\log m_1-\frac{\pi i}{2}\right)M+N\right\}\left\{D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}(J_1+J_3)\right.\right.$$
$$\left.\left.+D_2(J_4-J_2)\right\}+\frac{M}{2}\left\{(J_9+J_{11})\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}D_1+D_2(J_{12}-J_{10})\right\}\right] \quad x\in I_1, I_3, I_5 \tag{41}$$

where

$$J_9 = \int_a^b \frac{t\log|t^2-x^2|}{\sqrt{(t^2-a^2)(b^2-t^2)}}\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2} dt$$

$$J_{10} = \int_a^b \frac{t\log|t^2-x^2|}{\sqrt{(1-t^2)(c^2-t^2)}}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} dt$$

$$J_{11} = \int_c^1 \frac{u\log|u^2-x^2|}{\sqrt{(u^2-a^2)(u^2-b^2)}}\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2} du$$

$$J_{12} = \int_c^1 \frac{u\log|u^2-x^2|}{\sqrt{(u^2-c^2)(1-u^2)}}\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2} du.$$

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_a, K_b, K_c and K_1 at the edges of the strips and vertical displacement $|v(x, 0)/v_0|$ near the rigid strips have been plotted against dimensionless frequency m_1 and distance x, respectively, for a Poisson solid ($\tau^2 = 3$).

It is found that whatever the lengths of the strips are, SIFs at the four edges of the strips increase with an increase in the value of m_1 ($0.1 \leqslant m_1 \leqslant 0.6$).

From the graphs, it may be further noted that with a decrease in the length of the inner strip, which might be induced either by increasing "a" or by decreasing "b" the SIFs gradually increase (Figs 2–9).

Also, a decrease in the value of the length of the outer strip, which might be induced by increasing the value of c, causes an increase in the values of the SIFs (Figs 10–13), from which an interesting conclusion might be drawn: i.e. that the presence of the outer strip suppresses the SIFs at both the edges of the inner strip and the presence of the inner strip suppresses the SIFs at both the edges of the outer strip.

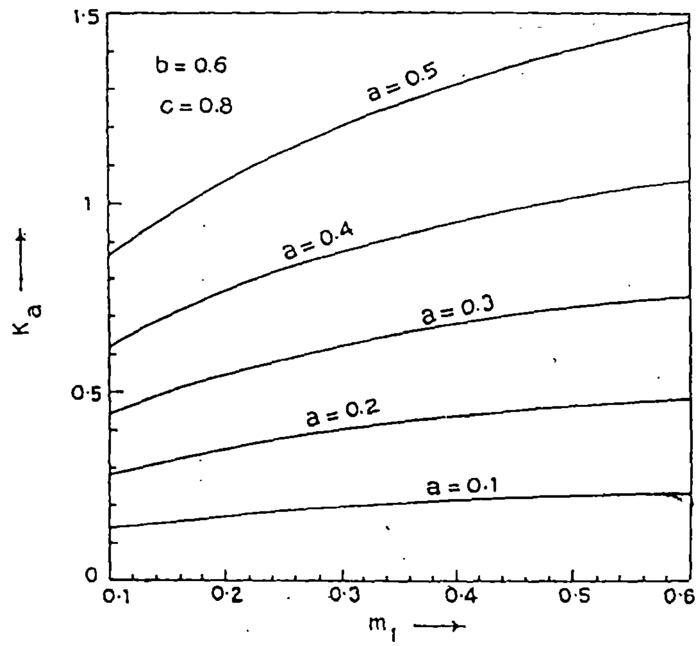

Fig. 2. Stress intensity factor K_a vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

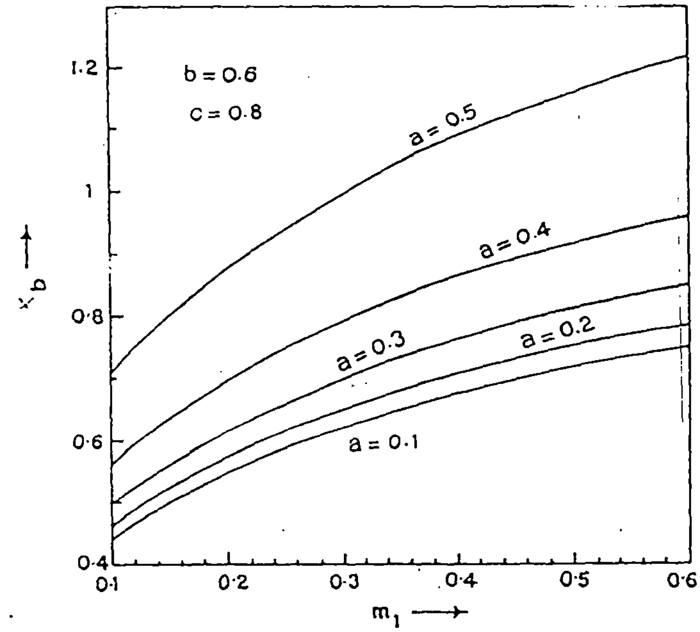

Fig. 3. Stress intensity factor K_b vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

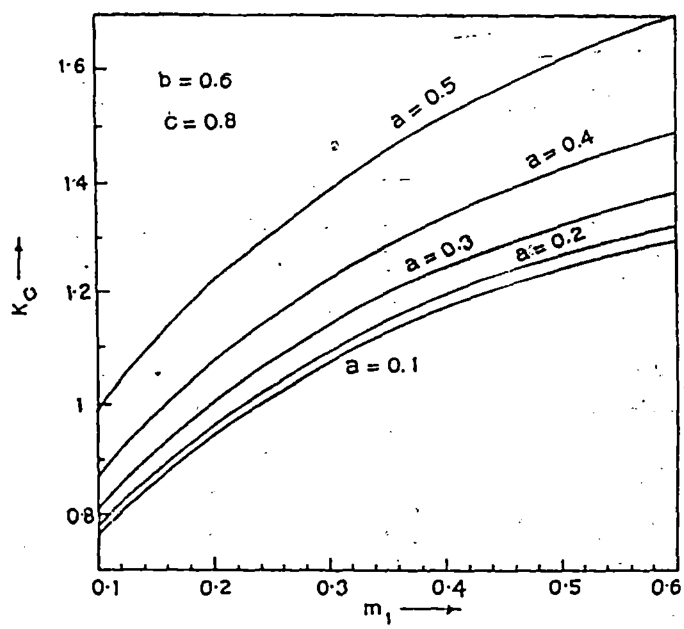

Fig. 4. Stress intensity factor K_c vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

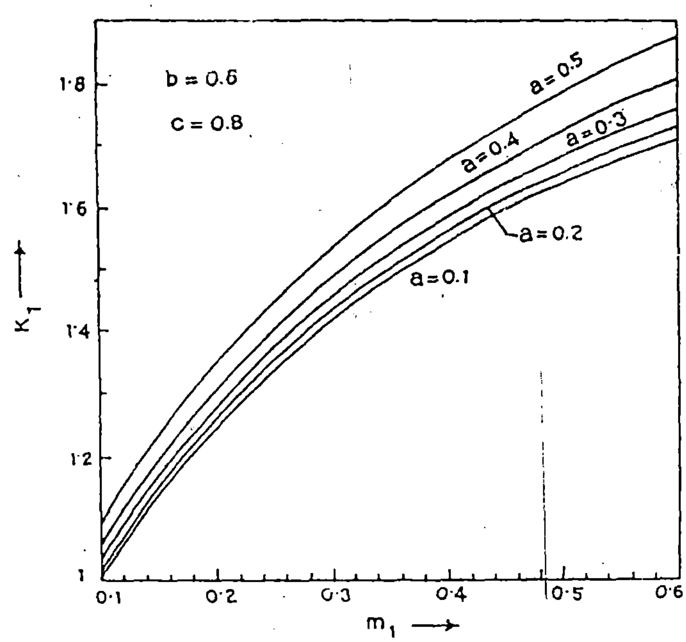

Fig. 5. Stress intensity factor K_1 vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

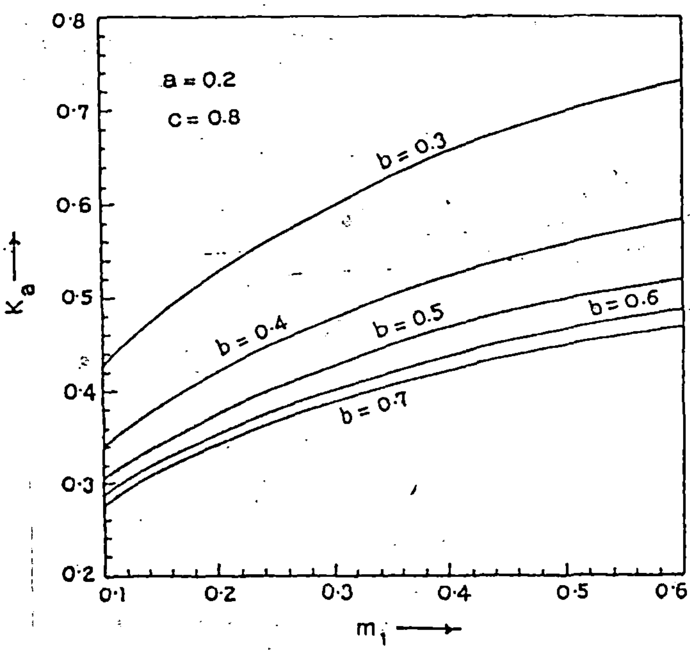

Fig. 6. Stress intensity factor K_a vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

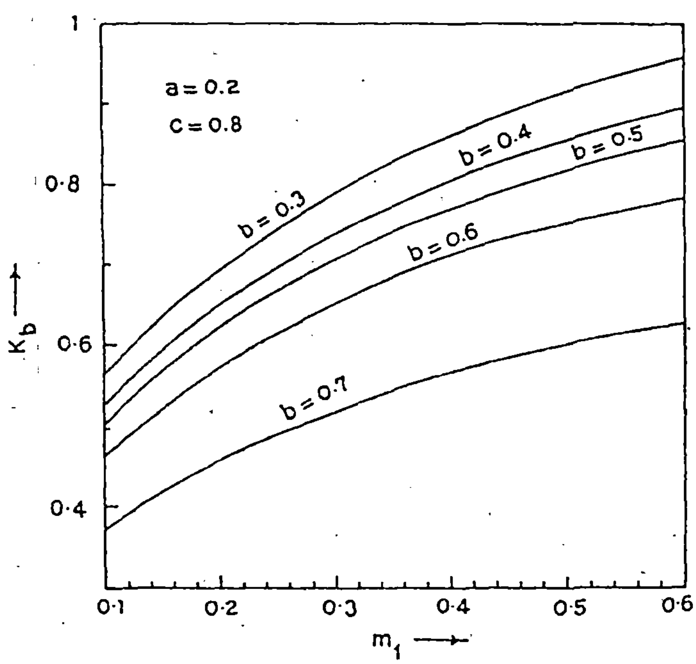

Fig. 7. Stress intensity factor K_b vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

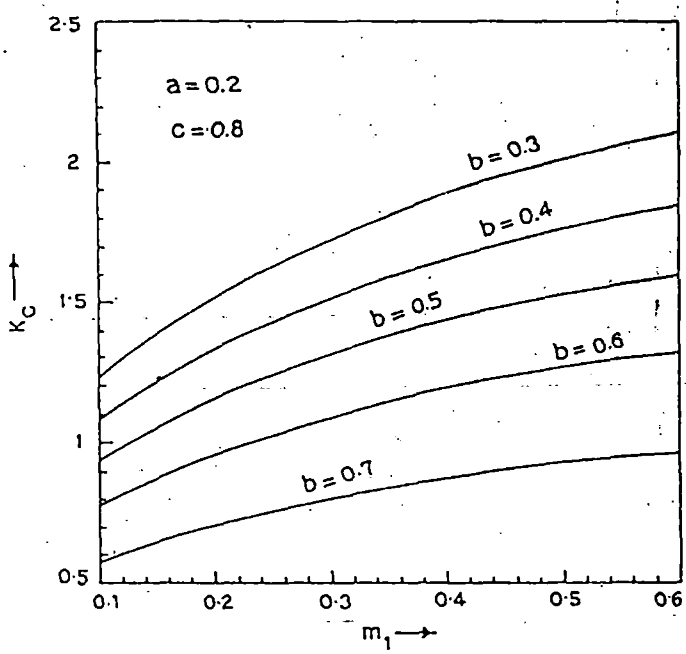

Fig. 8. Stress intensity factor K_c vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

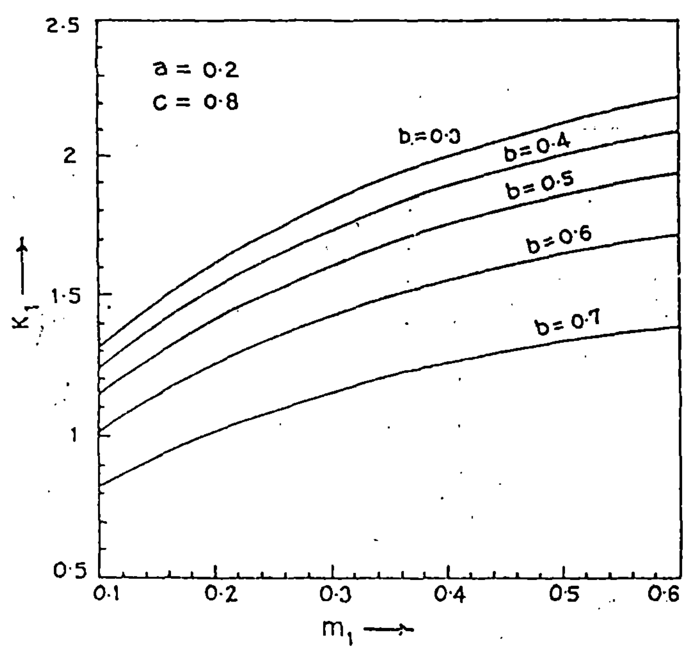

Fig. 9. Stress intensity factor K_1 vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

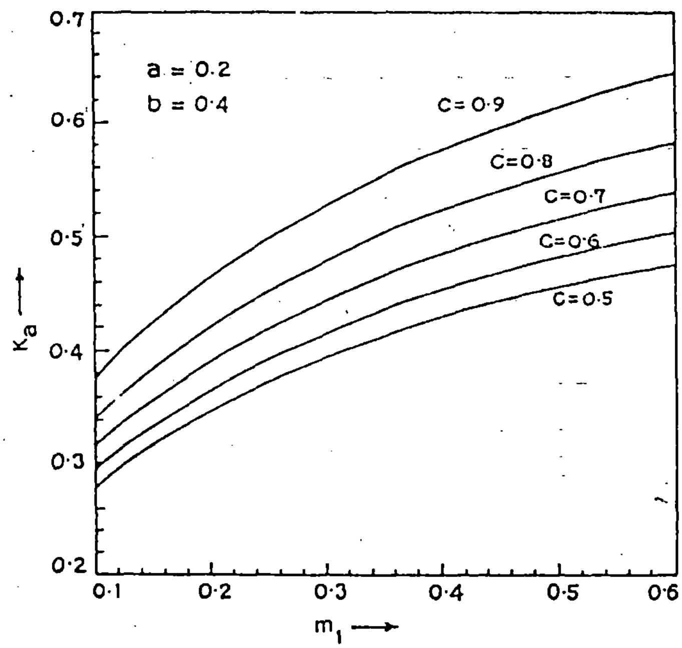

Fig. 10. Stress intensity factor K_a vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

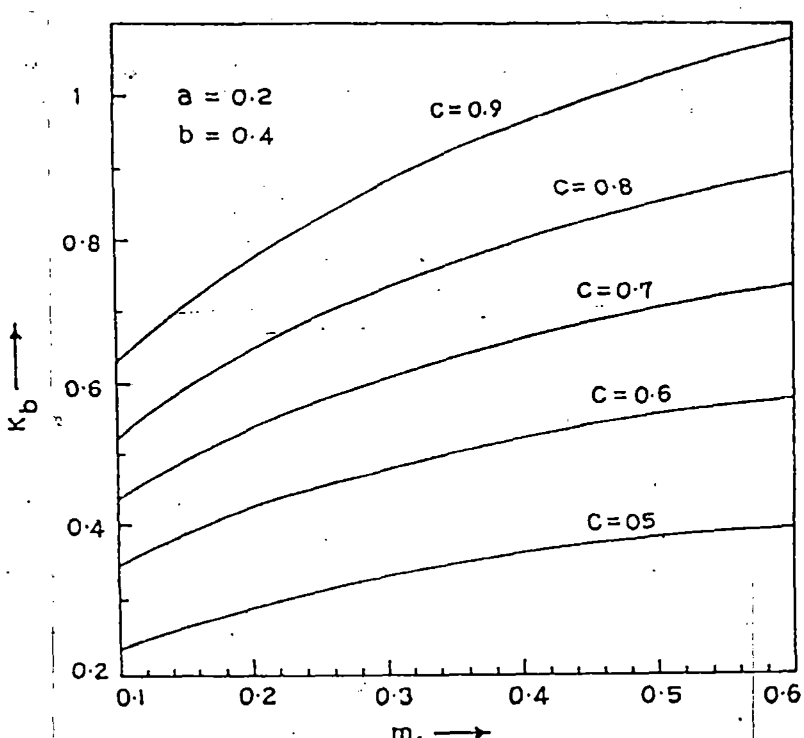

Fig. 11. Stress intensity factor K_b vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

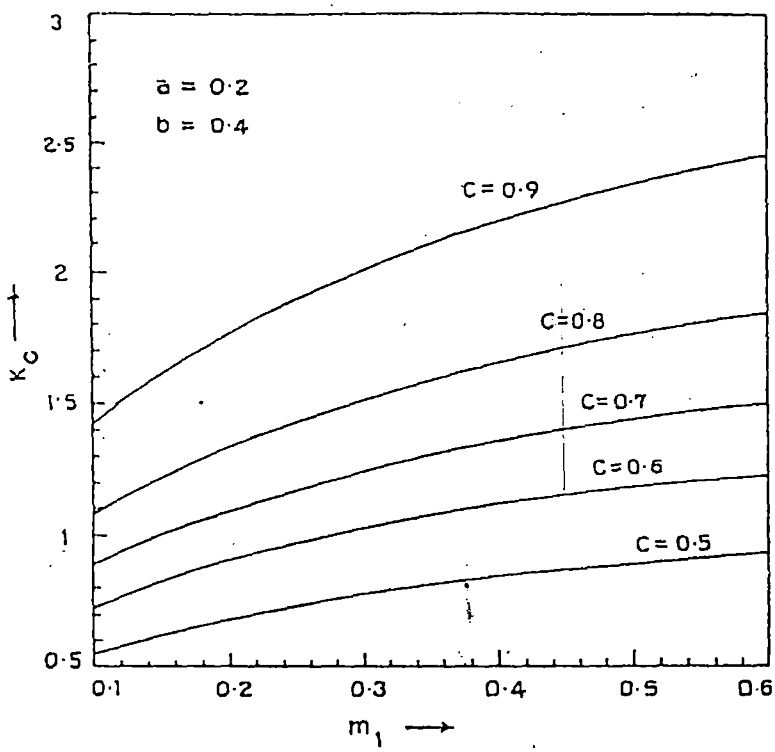

Fig. 12. Stress intensity factor K_c vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

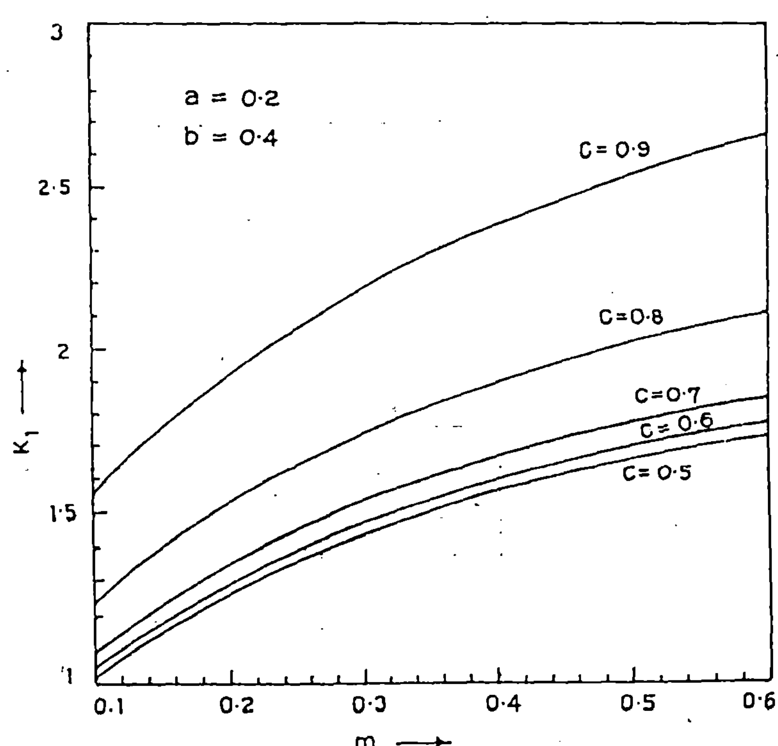

Fig. 13. Stress intensity factor K_1 vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

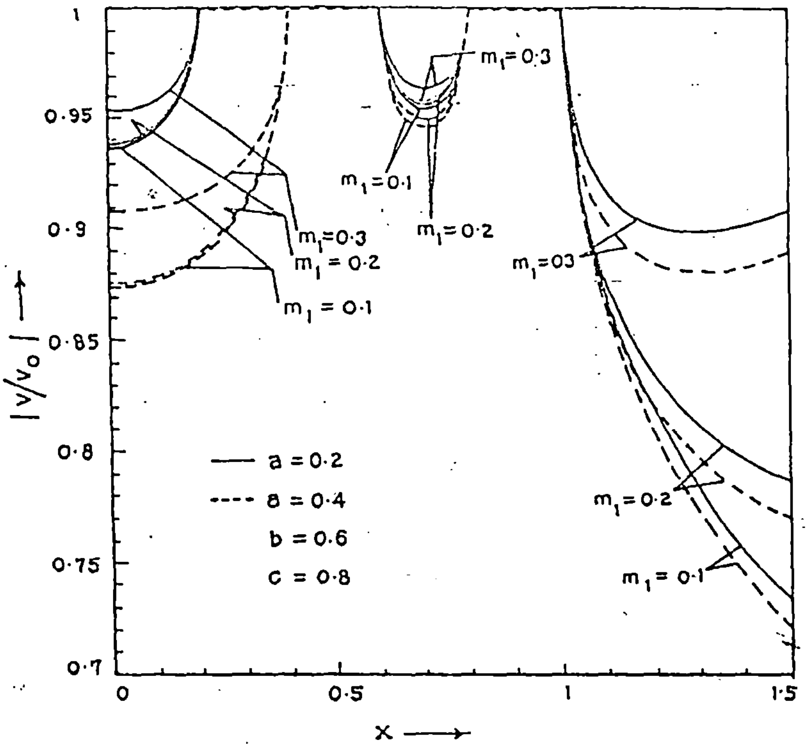

Fig. 14. Vertical displacement $|v(x,0)/v_0|$ vs dimensionless distance x for $b = 0.6$, $c = 0.8$, $a = 0.2$, 0.4 and for $m_1 = 0.1, 0.2, 0.3$.

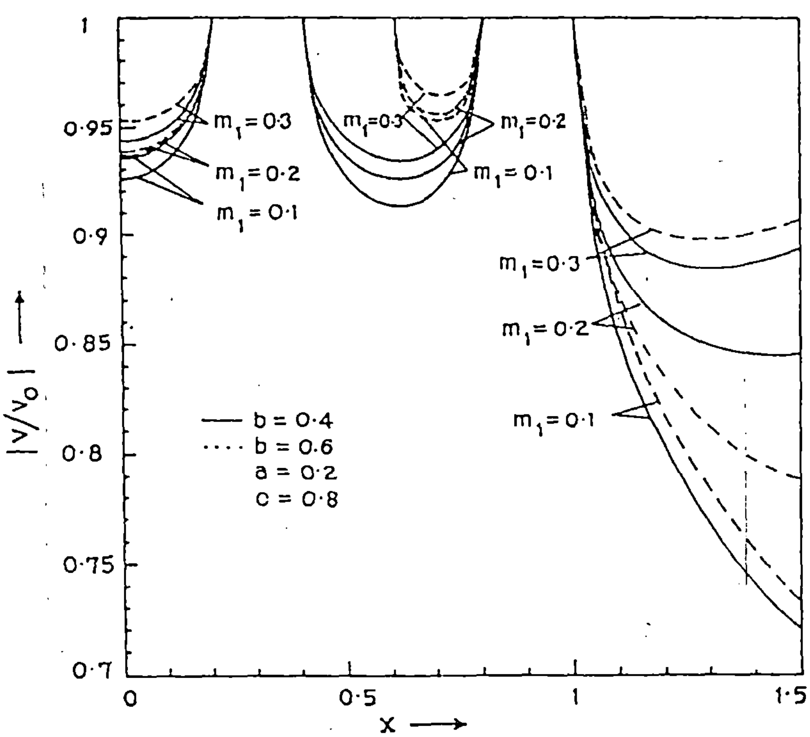

Fig. 15. Vertical displacement $|v(x,0)/v_0|$ vs dimensionless distance x for $a = 0.2$, $c = 0.8$, $b = 0.4$, 0.6 and for $m_1 = 0.1, 0.2, 0.3$.

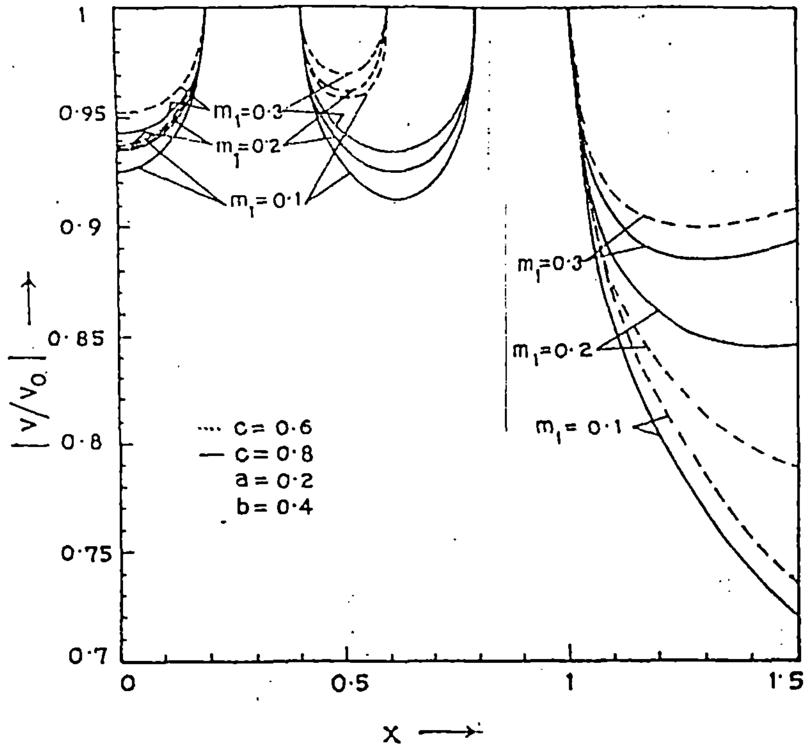

Fig. 16. Vertical displacement $|v(x,0)v_0|$ vs dimensionless distance x for $a = 0.2$, $b = 0.4$, $c = 0.6$, 0.8 and for $m_1 = 0.1, 0.2, 0.3$.

The vertical displacement has been plotted for different strip lengths. It is found from Figs 14–16 that with an increase in value of strip lengths, the displacement increases.

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