CHAPTER - III

DIFFRACTION PROBLEMS IN ELASTODYNAMICS

FORCED VERTICAL VIBRATION OF FOUR RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

1. INTRODUCTION

The problem of the effect of vibrating source in different forms on the surface of an elastic medium have aroused attention in view of their application in seismology and geophysics. Reissner [1937], and later Millar and Pursey [1954], treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. Analytical treatment of the dynamical response of footings and solid-structure interaction are usually available in the literature only for circular and elliptical footings, and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, and also in the study of the vibration of dams and large structures subjected to earthquakes. The problem of circular punch has been solved analytically by Awojobi and Grootenhuis [1965], Robertson [1966], Gladwell [1968] and others. Roy [1986] considered the dynamic

IN PRESS "INT. J. SOLIDS and STRUCTURES", 1996

response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. Karasudhi, Keer and Lee [1968] obtained a low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium. Wickham [1977] worked out in detail the problem of forced two-dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium. Recently, Mandal and Ghosh [1992] treated the problem of forced vertical vibration of two rigid strips on a semi-infinite elastic medium.

To improve the dynamic models of buildings and other structures, it will be fruitful to have analytic results for foundations of a more complicated nature. In what follows, the problem of vertical vibration of four rigid strips in smooth contact with a semi-infinite elastic medium has been considered. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by running wheels on a railway track. The resulting mixed boundary value problem has been reduced to the solution of quadruple integral equations, which have further been reduced to the solution of integral-differential equations. Finally, an iterative solution valid for low frequency has been obtained.

From the solution of the integral equations, the stress just below the strips and also the vertical displacement at points outside the strips on the free surface have been found. The effects of stress intensity factors at the edges of the strips and vertical displacement outside the strips have been shown by means of graphs.

2. FORMULATION OF THE PROBLEM

Consider the normal vibration of frequency ω of four rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty<X<\infty$, $Y\geq 0$, $-\infty<Z<\infty$. It is assumed that the motion is forced by prescribed displacement distribution $v_0 e^{-i\omega t}$ normal to the four infinite strips located in the region $d_1\leq|X|\leq d_2$, $d_3\leq|X|\leq d$, $Y=0$, $|Z|<\infty$, where v_0 is a constant.

Normalizing all the lengths with respect to d and putting

$$\frac{X}{d} = x, \quad \frac{Y}{d} = y, \quad \frac{Z}{d} = z, \quad \frac{d_1}{d} = a, \quad \frac{d_2}{d} = b, \quad \frac{d_3}{d} = c,$$

one finds that the rigid strips are defined by $a\leq|x|\leq b$, $c \leq|x|\leq 1$, $y=0$, $|z|<\infty$ (fig.1). With the time factor $e^{-i\omega t}$ suppressed throughout the analysis, the displacement components can be written as

$$u(x,y) = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y} \; ; \quad v(x,y) = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x} \; ; \quad w(x,y) = 0 \qquad (1)$$

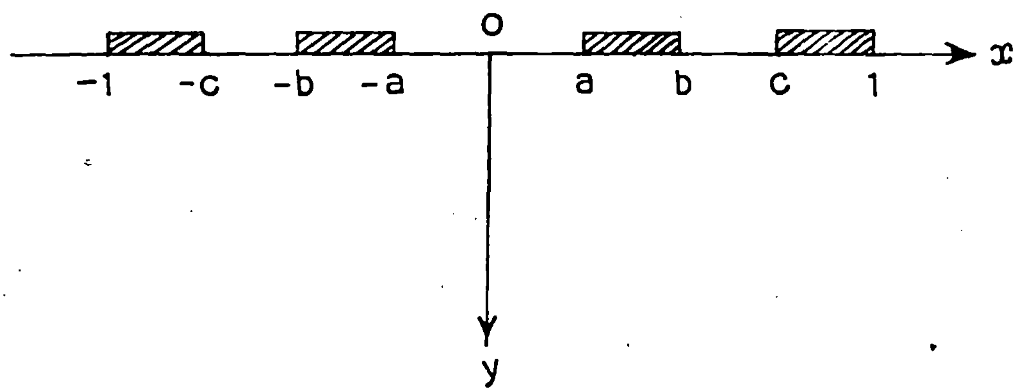

Fig. 1. Geometry of the problem.

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + m_1^2\phi = 0 .$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + m_2^2\psi = 0 \tag{2}$$

in which $m_1^2 = \dfrac{\omega^2 d^2}{c_1^2}$ and $m_2^2 = \dfrac{\omega^2 d^2}{c_2^2}$.

In terms of ϕ and ψ the stress components are

$$\tau_{xy} = \mu \left\{ 2 \frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} \right\}$$

$$\tau_{yy} = -\mu \left\{\left(m_2^2 + 2 \frac{\partial^2}{\partial x^2} \right)\phi - 2 \frac{\partial^2\psi}{\partial x\partial y} \right\} \tag{3}$$

$$\tau_{yz} = 0$$

The boundary conditions are

$$v(x,0) = v_0 \quad , \quad x \in I_2 , I_4 \tag{4}$$

$$\tau_{yy}(x,0) = 0 \quad , \quad x \in I_1 , I_3 , I_5 \tag{5}$$

$$\tau_{xy}(x,0) = 0 \quad , \quad -\infty < x < \infty \tag{6}$$

where $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,1)$, $I_5 = (1,\infty)$.

The solution of the Helmholtz equation (2) can be written as

$$\phi = 2 \int_0^\infty A(\xi)\cos\xi x \, e^{-\gamma_1 y} \, d\xi$$

$$\psi = 2 \int_0^\infty B(\xi)\sin\xi x \, e^{-\gamma_2 y} \, d\xi \qquad (7)$$

where

$$\gamma_j = \left\{ \begin{array}{ll} (\xi^2 - m_j^2)^{1/2} , & |\xi| \geq m_j \\ -i(m_j^2 - \xi^2)^{1/2} , & |\xi| \leq m_j \end{array} \right\} , \qquad j = 1,2$$

and $A(\xi)$ and $B(\xi)$ are unknown functions, to be determined from the boundary conditions.

By using the boundary condition (6) it can be shown that

$$B(\xi) = \frac{2\gamma_1\xi}{\xi^2 + \gamma_2^2} A(\xi) \qquad (8)$$

Now the displacement component v and stress τ_{yy} become

$$v(x,y) = 2 \int_0^\infty \left[\frac{2\xi^2}{2\xi^2 - m_2^2} e^{-\gamma_2 y} - e^{-\gamma_1 y} \right] A(\xi)\cos\xi x \, d\xi \qquad (9)$$

$$\tau_{yy}(x,y) = -2\mu \int_0^\infty \left[(m_2^2 - 2\xi^2) e^{-\gamma_1 y} + \frac{2\xi^2\gamma_1\gamma_2}{2\xi^2 - m_2^2} e^{-\gamma_2 y} \right] A(\xi)\cos\xi x \, d\xi \qquad (10)$$

From the boundary conditions (4) and (5) we get the following set of integral equations in $P(\xi)$:

$$\int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} P(\xi)\cos\xi x \, d\xi = \frac{1}{2} v_0 , \quad x \in I_2 , I_4 \tag{11}$$

and

$$\int_0^\infty P(\xi)\cos\xi x \, d\xi = 0 , \quad x \in I_1 , I_2 , I_5 \tag{12}$$

where

$$P(\xi) = \frac{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2}{(2\xi^2 - m_2^2)} A(\xi).$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (11) and (12) in the form

$$P(\xi) = \int_a^b t f(t^2)\cos\xi t \, dt + \int_c^1 u g(u^2)\cos\xi u \, du \tag{13}$$

where $f(t^2)$ and $g(u^2)$ are unknown functions to be determined.

By the choice of $P(\xi)$ given by (13) the relation (12) is satisfied automatically and the equation (11) becomes

$$\int_a^b tf(t^2)dt \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \cos\xi x \cos\xi t \, d\xi \; +$$

$$+ \int_c^1 ug(u^2)du \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \cos\xi x \cos\xi u \, d\xi = \frac{v_0}{2} ,$$

$$x \in I_2 \, , \, I_4 \qquad (14)$$

using the relation

$$\frac{\sin\xi x \sin\xi t}{\xi^2} = \int_0^x \int_0^t \frac{wvJ_0(\xi w) \, J_0(\xi v) \, dvdw}{(x^2-w^2)^{1/2} \, (t^2-v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{d}{dx} \int_a^b t \, f(t^2) \, dt \, \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{wv \, L_1(v,w) \, dvdw}{(x^2-w^2)^{1/2} \, (t^2-v^2)^{1/2}} \; +$$

$$+ \frac{d}{dx} \int_c^1 u \, g(u^2)du \, \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{wv \, L_1(v,w) \, dvdw}{(x^2-w^2)^{1/2} \, (u^2-v^2)^{1/2}}$$

$$= \frac{v_0}{2} \, , \; x \in I_2, \; I_4 \qquad (15)$$

where

$$L_1(v,w) = \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} J_0(\xi w)J_0(\xi v)d\xi \qquad (16)$$

By a simple contour integration technique used by Ghosh and Ghosh (1985), $L_1(v,w)$ can be written as

$$L_1(v,w) = -i\tau^2 \int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\, H_0^{(1)}(m_1\eta w)\, J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\, d\eta\ -$$

$$- 4i\tau^2 \int_0^\tau \frac{\eta^2(\eta^2-1)(\tau^2-\eta^2)^{1/2}\, H_0^{(1)}(m_1\eta w)\, J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\, d\eta\ +$$

$$+ \pi i\tau^2 \left[\frac{(\eta^2-1)^{1/2} H_0^{(1)}(m_1\eta w)\, J_0(m_1\eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0}, \quad w>v$$

$$= \frac{-i\tau^2}{16(1-\tau^2)} \left[\sum_{j=0}^{2} P_j \int_0^1 \frac{(1-\eta^2)^{1/2} H_0^{(1)}(m_1\eta w) J_0(m_1\eta v)}{\eta^2-\tau_j^2}\, d\eta\ +\right.$$

$$\left. + \sum_{j=0}^{2} S_j \int_0^\tau \frac{(\tau^2-\eta^2)^{1/2} H_0^{(1)}(m_1\eta w) J_0(m_1\eta v)}{\eta^2-\tau_j^2}\, d\eta \right] +$$

$$+ \pi i\tau^2 \left[\frac{(\eta^2-1)^{1/2} H_0^{(1)}(m_1\eta w)\, J_0(m_1\eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0}, \quad w>v \qquad (17)$$

where $\tau = \frac{m_2}{m_1} = \frac{c_1}{c_2}$, $Q_0(\eta) = (2\eta^2-\tau^2)^2 - 4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$ and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta) = 0$. τ_1, τ_2 are the roots of the equation

$$(2\eta^2-\tau^2)^2 + 4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2} = 0.$$

$Q_0'(\eta)$ denotes the derivative of $Q_0(\eta)$ with respect to η and

$$P_j = \frac{(2\tau_j^2 - \tau^2)}{\prod_i (\tau_j^2 - \tau_i^2)} ,$$

$$S_j = \frac{4\tau_j^2 (\tau_j^2 - 1)}{\prod_i (\tau_j^2 - \tau_i^2)} , \quad i,j = 0,1,2 \text{ and } i \neq j.$$

The corresponding expression for $L_1(v,w)$ for $w < v$ follows from equation (17) by interchanging w and v. For a Poisson ratio $\sigma = \frac{1}{4}$, the values of τ, τ_0, τ_1, and τ_2 are given by

$$\tau^2 = \frac{2(1-\sigma)}{(1-2\sigma)} = 3, \quad \tau_0^2 = \frac{3}{(0.9194)^2}, \quad \tau_1^2 = \frac{3}{(2+2\sqrt{3})} \text{ and } \tau_2^2 = \frac{3}{4} .$$

Hence, in this case $\tau_2 < \tau_1 < 1 < \tau < \tau_0$.

By using the series expansions of J_0 and $H_0^{(1)}$ and evaluating the integrals arising in equation (17), we obtain, after some algebraic manipulation,

$$L_1(v,w) = \frac{2}{\pi}\tau^2\left[\left(\gamma + \log\frac{m_1 w}{2} - \frac{\pi i}{2}\right)M + N - \frac{P}{4}(w^2+v^2)m_1^2\log m_1\right] + O(m_1^2)$$

$w > v.$

$$= \frac{2}{\pi}\tau^2\left[\left(\gamma + \log\frac{m_1 v}{2} - \frac{\pi i}{2}\right)M + N - \frac{P}{4}(w^2+v^2)m_1^2\log m_1\right] + O(m_1^2)$$

$w < v.$ (18)

where $\gamma = 0.5772157...$ is Euler's constant,

$$M = -\frac{\pi}{4(1-\tau^2)} \quad (19)$$

$$N = \frac{\pi}{32(1-\tau^2)}\Bigg[4\log\frac{4}{\tau} + \sum_{j=1}^{2} P_j\frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j} -$$

$$- P_0\frac{\sqrt{(\tau_0^2-1)}}{\tau_0}\log\left\{\tau_0 + \sqrt{(\tau_0^2-1)}\right\} +$$

$$+ \sum_{j=1}^{2} S_j\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j} -$$

$$- S_0\frac{\sqrt{(\tau_0^2-\tau^2)}}{\tau_0}\log\left\{\frac{\tau_0 + \sqrt{(\tau_0^2-\tau^2)}}{\tau}\right\}\Bigg], \quad (20)$$

$$P = \frac{\pi}{32(1-\tau^2)}\left[\sum_{j=0}^{2} P_j\left(\frac{1}{2} - \tau_j^2\right) + \sum_{j=0}^{2} S_j\left(\frac{\tau^2}{2} - \tau_j^2\right)\right]. \quad (21)$$

Next, differentiating both sides of the relation (14) with respect to x, we obtain

$$\int_a^b tf(t^2)dt \int_0^\infty \frac{\gamma_1 m_2^2 \xi}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \sin\xi x \sin\xi t \, d\xi \; +$$

$$+ \int_c^1 ug(u^2)du \int_0^\infty \frac{\gamma_1 m_2^2 \xi}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2} \sin\xi x \sin\xi u \, d\xi = 0,$$

$$x \in I_2 \,,\; I_4$$

Following similar procedure as done for deriving equation (15), we get

$$x \int_a^b \frac{tf(t^2)}{x^2-t^2} dt + x \int_c^1 \frac{ug(u^2)}{x^2-u^2} du$$

$$= \int_a^b t\, f(t^2)\, dt \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{wv\, L_2(v,w)\, dvdw}{(x^2-w^2)^{1/2}\, (t^2-v^2)^{1/2}} \; +$$

$$+ \int_c^1 u\, g(u^2)du \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{wv\, L_2(v,w)\, dwdv}{(x^2-w^2)^{1/2}\, (u^2-v^2)^{1/2}} \,,\; x \in I_2,\, I_4 \qquad (22)$$

where $$L_2(v,w) = \int_0^\infty \left[\xi - \frac{2\gamma_1\xi^2(m_1^2-m_2^2)}{(2\xi^2-m_2^2)^2 - 4\xi^2\gamma_1\gamma_2}\right] J_0(\xi w)\, J_0(\xi v) d\xi \qquad (23)$$

For small values of m_1 and m_2 such that $m_1 = O(m_2)$, one can use the contour integration technique mentioned above and obtain

$$L_2(v,w) = 2im_1^2(1-\tau^2)\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\eta^2\ H_0^{(1)}(m_1\eta w)\ J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4\ (\eta^2-1)(\tau^2-\eta^2)}\,d\eta$$

$$+\ 4im_1^2(1-\tau^2)\int_0^\tau -\frac{2\eta^4(\eta^2-1)(\tau^2-\eta^2)^{1/2}\ H_0^{(1)}(m_1\eta w)\ J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta\ -$$

$$-\ 2\pi im_1^2(1-\tau^2)\left[\frac{\eta^2(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)\ J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0},\quad w>v \tag{24}$$

By a process similar to the one which led to equation (18), equation (24) can be written as

$$L_2(v,w) = -\frac{4P}{\pi}\,(1-\tau^2)\ m_1^2\ \log m_1 + O(m_1^2) \tag{25}$$

where P is given by equation (21).

Now examining the relation (15) and (18) we assume the expressions of the functions $f(t^2)$ and $g(u^2)$ as

$$f(t^2) = f_0(t^2) + f_1(t^2)\ m_1^2\ \log m_1 + O(m_1^2)$$

$$g(u^2) = g_0(u^2) + g_1(u^2)\ m_1^2\ \log m_1 + O(m_1^2). \tag{26}$$

Putting the above expressions of $f(t^2)$ and $g(u^2)$ and the value of $L_2(v,w)$ given by (25) in equation (22) and equating the coefficients of like powers of m_1 we obtain

$$\int_a^b \frac{tf_0(t^2)}{x^2-t^2}\,dt + \int_c^1 \frac{ug_0(u^2)}{x^2-u^2}\,du = 0 \;, \quad x \in I_2 \;, \; I_4 \tag{27}$$

and $$\int_a^b \frac{tf_1(t^2)}{x^2-t^2}\,dt + \int_c^1 \frac{ug_1(u^2)}{x^2-u^2}\,du =$$

$$= -\frac{4}{\pi} P(1-\tau^2)\left[\int_a^b tf_0(t^2)dt + \int_c^1 ug_0(u^2)\,du\right], \quad x \in I_2, I_4. \tag{28}$$

Following Srivastava and Lowengrub (1970) the solutions of the above integral equations (27) can be obtained as

$$f_0(t^2) = D_1 \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{c^2-t^2}{1-t^2}\right)^{1/2} \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} -$$

$$- D_2 \left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} \frac{1}{\sqrt{(1-t^2)(c^2-t^2)}} \;, \; t \in I_2 \tag{29}$$

and

$$g_0(u^2) = D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{u^2-c^2}{1-u^2} \right)^{1/2} \frac{1}{\sqrt{(u^2-a^2)(u^2-b^2)}} +$$

$$+ D_2 \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \frac{1}{\sqrt{(u^2-c^2)(1-u^2)}} \quad , \; u \in I_4 \qquad (30)$$

where D_1 and D_2 are constants which can be calculated as follows:

We substitute the value of $L_1(v,w)$ from (18) as well as the expansions of $f(t^2)$ and $g(u^2)$ obtained from (26), (29) and (30) upto $O(m_1^2 \log m_1)$ in the equation (15). When the coefficients of like powers of m_1 from both sides of the resulting equation are equated and we get after some algebraic manipulation, the following

$$D_1 = \frac{\pi v_0}{4\tau^2} \frac{(X_2-X_1)}{(X_1X_4-X_2X_3)} \quad ; \quad D_2 = \frac{\pi v_0}{4\tau^2} \frac{(X_1-X_3)}{(X_1X_4-X_2X_3)} \qquad (31)$$

where

$$X_1 = \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left[\left\{ \left(\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2} \right) M + N \right\} (J_1+J_3) + \right.$$

$$\left. + \frac{1}{2} M J_1 \log(b^2-a^2) + M J_5 \right] \qquad (32)$$

$$X_2 = \left\{\left[\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2}\right]M + N\right\}(J_4 - J_2) - \frac{1}{2} MJ_2 \log(b^2 - a^2) + MJ_6 \quad (33)$$

$$X_3 = \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left[\left\{\left[\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2}\right]M + N\right\}(J_1 + J_3) + \right.$$

$$\left. + \frac{1}{2} MJ_3 \log(1-c^2) + MJ_7 \right] \quad (34)$$

$$X_4 = \left\{\left[\gamma + \log\frac{m_1}{2} - \frac{\pi i}{2}\right]M + N\right\}(J_4 - J_2) + \frac{1}{2} MJ_4 \log(1-c^2) - MJ_8 \quad (35)$$

$$J_1 = \int_a^b \left(\frac{c^2-t^2}{1-t^2} \right)^{1/2} \frac{tdt}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$

$$J_2 = \int_a^b \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \frac{tdt}{\sqrt{(1-t^2)(c^2-t^2)}}$$

$$J_3 = \int_c^1 \left(\frac{u^2-c^2}{1-u^2} \right)^{1/2} \frac{udu}{\sqrt{(u^2-a^2)(u^2-b^2)}}$$

$$J_4 = \int_c^1 \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \frac{udu}{\sqrt{(u^2-c^2)(1-u^2)}}$$

$$J_5 = \int_c^1 \frac{u\log\left(\sqrt{u^2-b^2}+\sqrt{u^2-a^2}\right)}{\sqrt{(u^2-a^2)(u^2-b^2)}} \left[\frac{u^2-c^2}{1-u^2}\right]^{1/2} du$$

$$J_6 = \int_c^1 \frac{u\log\left(\sqrt{u^2-b^2}+\sqrt{u^2-a^2}\right)}{\sqrt{(1-u^2)(u^2-c^2)}} \left[\frac{u^2-a^2}{u^2-b^2}\right]^{1/2} du$$

$$J_7 = \int_a^b \frac{t\log\left(\sqrt{c^2-t^2}+\sqrt{1-t^2}\right)}{\sqrt{(t^2-a^2)(b^2-t^2)}} \left[\frac{c^2-t^2}{1-t^2}\right]^{1/2} dt$$

$$J_8 = \int_a^b \frac{t\log\left(\sqrt{c^2-t^2}+\sqrt{1-t^2}\right)}{\sqrt{(1-t^2)(c^2-t^2)}} \left[\frac{t^2-a^2}{b^2-t^2}\right]^{1/2} dt.$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ on the plane y=0 can be found from the relations (10), (13), (26), (29) and (30) as

$$\tau_{yy}(x,0) = \frac{\pi\mu x}{\sqrt{(x^2-a^2)(b^2-x^2)}} \left[D_1 \left[\frac{1-a^2}{c^2-a^2}\right]^{1/2} \left[\frac{c^2-x^2}{1-x^2}\right]^{1/2} - \right.$$

$$- D_2 \frac{(x^2-a^2)}{\sqrt{(1-x^2)(c^2-x^2)}} \Big] + O(m_1^2 \log m_1), \quad x \in I_2$$

$$= \frac{\pi\mu x}{\sqrt{(x^2-c^2)(1-x^2)}} \left[D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \frac{(x^2-c^2)}{\sqrt{(x^2-a^2)(x^2-b^2)}} + \right.$$

$$\left. + D_2 \left(\frac{x^2-a^2}{x^2-b^2} \right)^{1/2} \right] + O(m_1^2 \log m_1), \quad x \in I_4 \qquad (36)$$

Defining the stress intensity factors at the edges of the strips by the relations

$$K_a = \operatorname{Lt}_{x\to a+} \left| \frac{\tau_{yy}(x,0)\sqrt{x-a}}{\pi\ \mu v_0} \right| ; \qquad K_b = \operatorname{Lt}_{x\to b-} \left| \frac{\tau_{yy}(x,0)\sqrt{b-x}}{\pi\ \mu v_0} \right|$$

$$K_c = \operatorname{Lt}_{x\to c+} \left| \frac{\tau_{yy}(x,0)\sqrt{x-c}}{\pi\ \mu v_0} \right| ; \qquad K_1 = \operatorname{Lt}_{x\to 1-} \left| \frac{\tau_{yy}(x,0)\sqrt{1-x}}{\pi\ \mu v_0} \right|$$

We get

$$K_a = \left| \frac{\sqrt{a}\ D_1/v_0}{\sqrt{2(b^2-a^2)}} \right| \qquad (37)$$

$$K_b = \left| \frac{\gamma\sqrt{b}}{\sqrt{2(b^2-a^2)}} \left\{ \frac{D_1}{v_0}\left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-b^2}{1-b^2} \right)^{1/2} - \frac{D_2}{v_0} \frac{(b^2-a^2)}{\sqrt{(1-b^2)(c^2-b^2)}} \right\} \right| \tag{38}$$

$$K_c = \left| \frac{\gamma\sqrt{c}}{\sqrt{2(1-c^2)}} \frac{D_2}{v_0} \left(\frac{c^2-a^2}{c^2-b^2} \right)^{1/2} \right| \tag{39}$$

$$K_1 = \left| \frac{1}{\sqrt{2(1-c^2)}} \left\{ \frac{(1-c^2)\, D_1}{\sqrt{(c^2-a^2)(1-b^2)}} + \left(\frac{1-a^2}{1-b^2} \right)^{1/2} D_2 \right\} \right| \tag{40}$$

The vertical displacement v(x,y) on the plane y=0 can be obtained from equations (9), (13), (26), (29), and (30) as

$$v(x,o) = \frac{4\tau^2}{\pi} \left[\left\{ \left(\gamma + \log m_1 - \frac{\pi i}{2} \right) M + N \right\} \left\{ D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} (J_1+J_3) + \right. \right.$$

$$\left. \left. + D_2(J_4-J_2) \right\} + \frac{M}{2} \left\{ (J_9+J_{11}) \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} D_1 + D_2(J_{12}-J_{10}) \right\} \right]$$

$$x \in I_1,\ I_3,\ I_5 \tag{41}$$

where

$$J_9 = \int_a^b \frac{t\log\left|t^2-x^2\right|}{\sqrt{(t^2-a^2)(b^2-t^2)}} \left(\frac{c^2-t^2}{1-t^2} \right)^{1/2} dt$$

$$J_{10} = \int_a^b \frac{t\log|t^2-x^2|}{\sqrt{(1-t^2)(c^2-t^2)}} \left[\frac{t^2-a^2}{b^2-t^2}\right]^{1/2} dt$$

$$J_{11} = \int_c^1 \frac{u\log|u^2-x^2|}{\sqrt{(u^2-a^2)(u^2-b^2)}} \left[\frac{u^2-c^2}{1-u^2}\right]^{1/2} du$$

$$J_{12} = \int_c^1 \frac{u\log|u^2-x^2|}{\sqrt{(u^2-c^2)(1-u^2)}} \left[\frac{u^2-a^2}{u^2-b^2}\right]^{1/2} du.$$

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_a, K_b, K_c and K_1 at the edges of the strips and vertical displacement $|v(x,0)/v_0|$ near about the rigid strips have been plotted against dimensionless frequency m_1 and distance x respectively for a Poisson solid ($\tau^2=3$).

It is found that whatever the lengths of the strips are, SIFs at the four edges of the strips increase with increase in the value of m_1 ($0.1 \leq m_1 \leq 0.6$).

From the graphs, it may be noted further that with a decrease in the length of the inner strip, which might be induced either by increasing 'a' or by decreasing 'b', the SIFs gradually increase (fig.2 - fig.9).

Also, a decrease in the value of the length of the outer strip, which might be induced by increasing the value of c, causes an increase in the values of the SIFs (fig.10 - fig.13), from which an interesting conclusion might be drawn: i.e, that the presence of the outer strip suppresses the SIFs at both the edges of the inner strip and the presence of the inner strip suppresses the SIFs at both the edges of the outer strip.

The vertical displacement has been plotted for different strip lengths. It is found from fig.14 - fig.16 that with the increase in the value of strip lengths, the displacement increases.

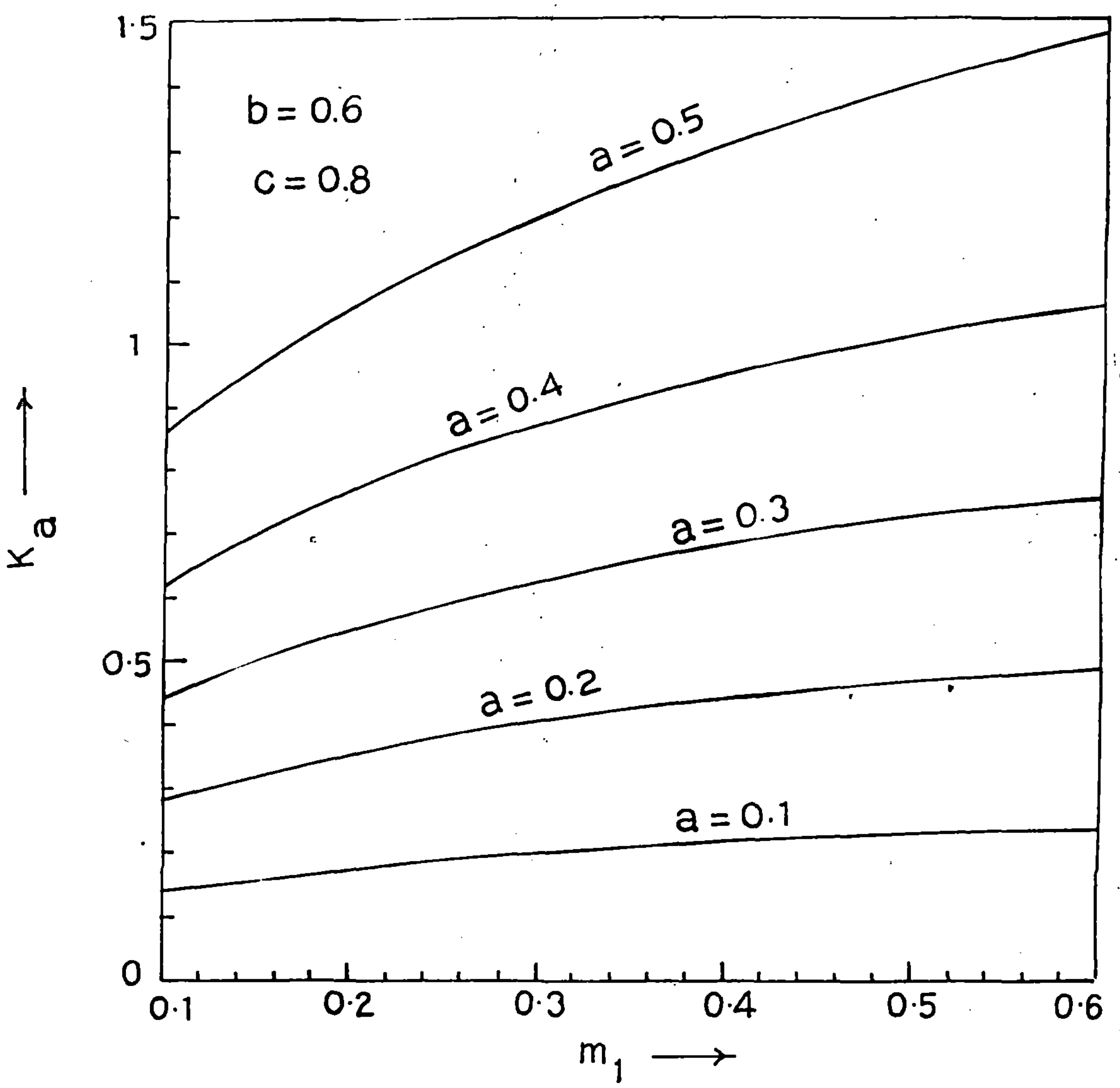

Fig. 2. Stress intensity factor K_a versus dimensionless frequency m_1 for b = 0.6, c = 0.8 and for different values of a.

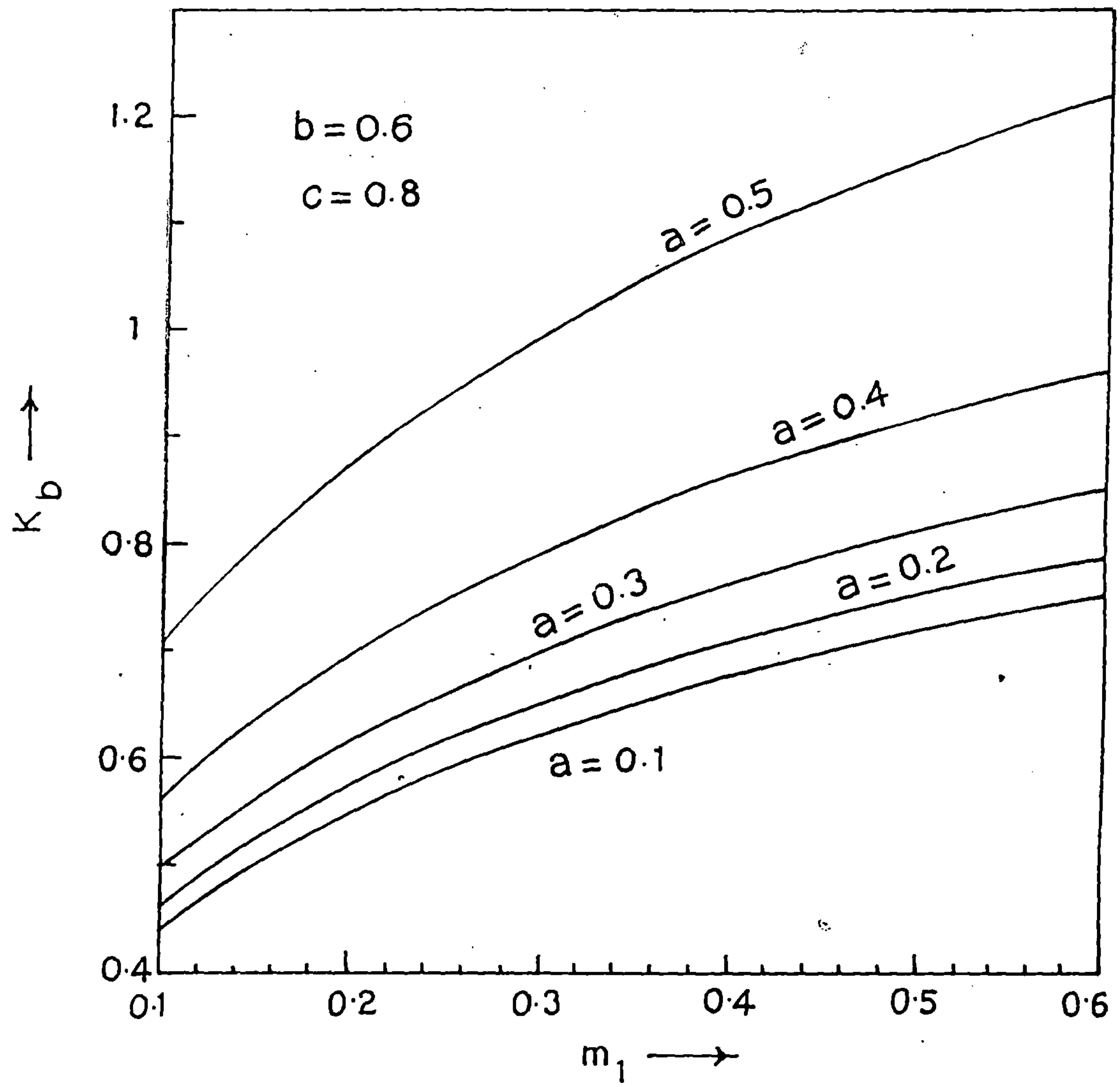

Fig. 3. Stress intensity factor K_b versus dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

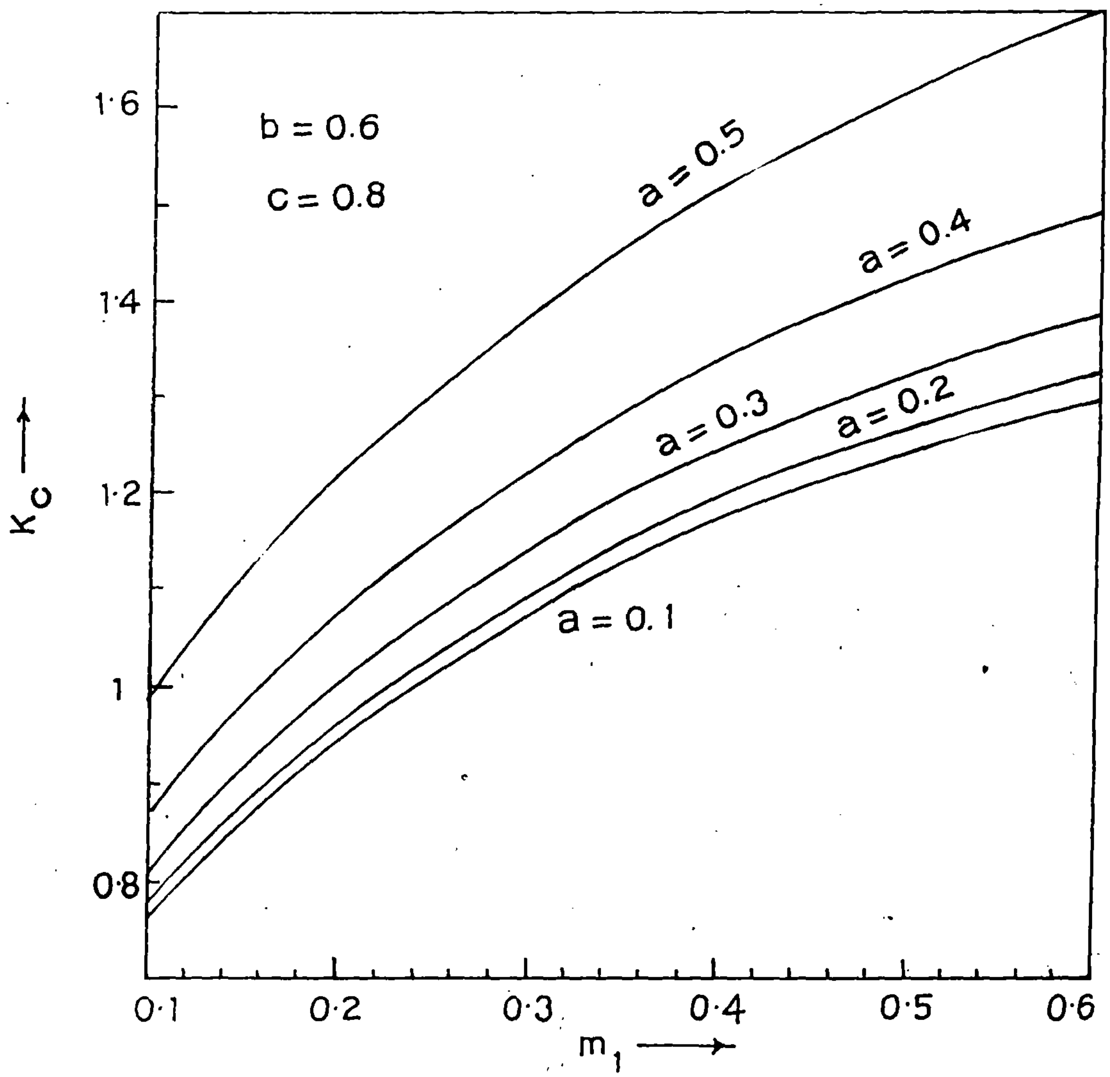

Fig. 4. Stress intensity factor K_c versus dimensionless frequency m_1 for b = 0.6, c = 0.8 and for different values of a.

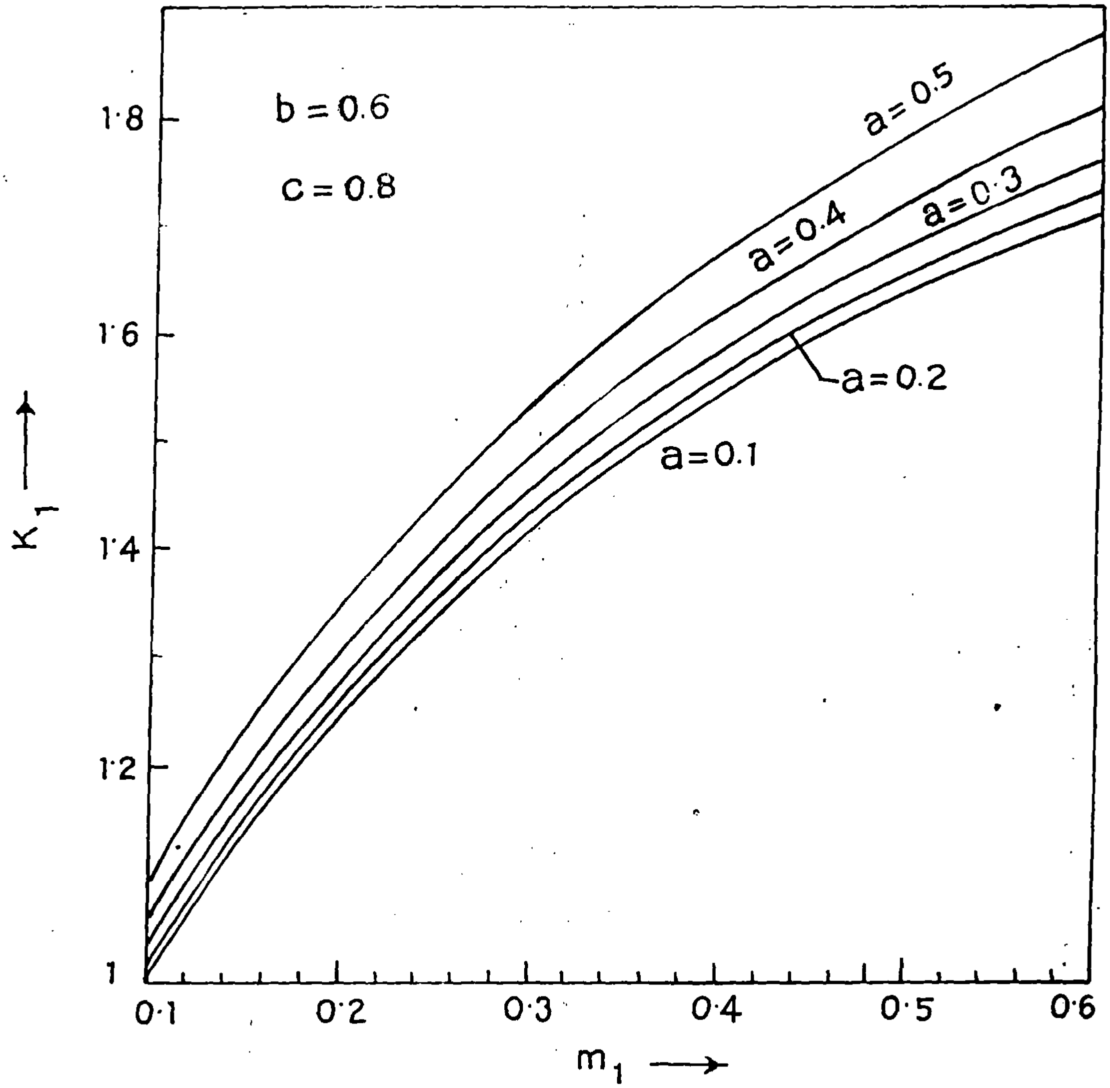

Fig. 5. Stress intensity factor K_1 versus dimensionless frequency m_1 for b = 0.6, c = 0.8 and for different values of a.

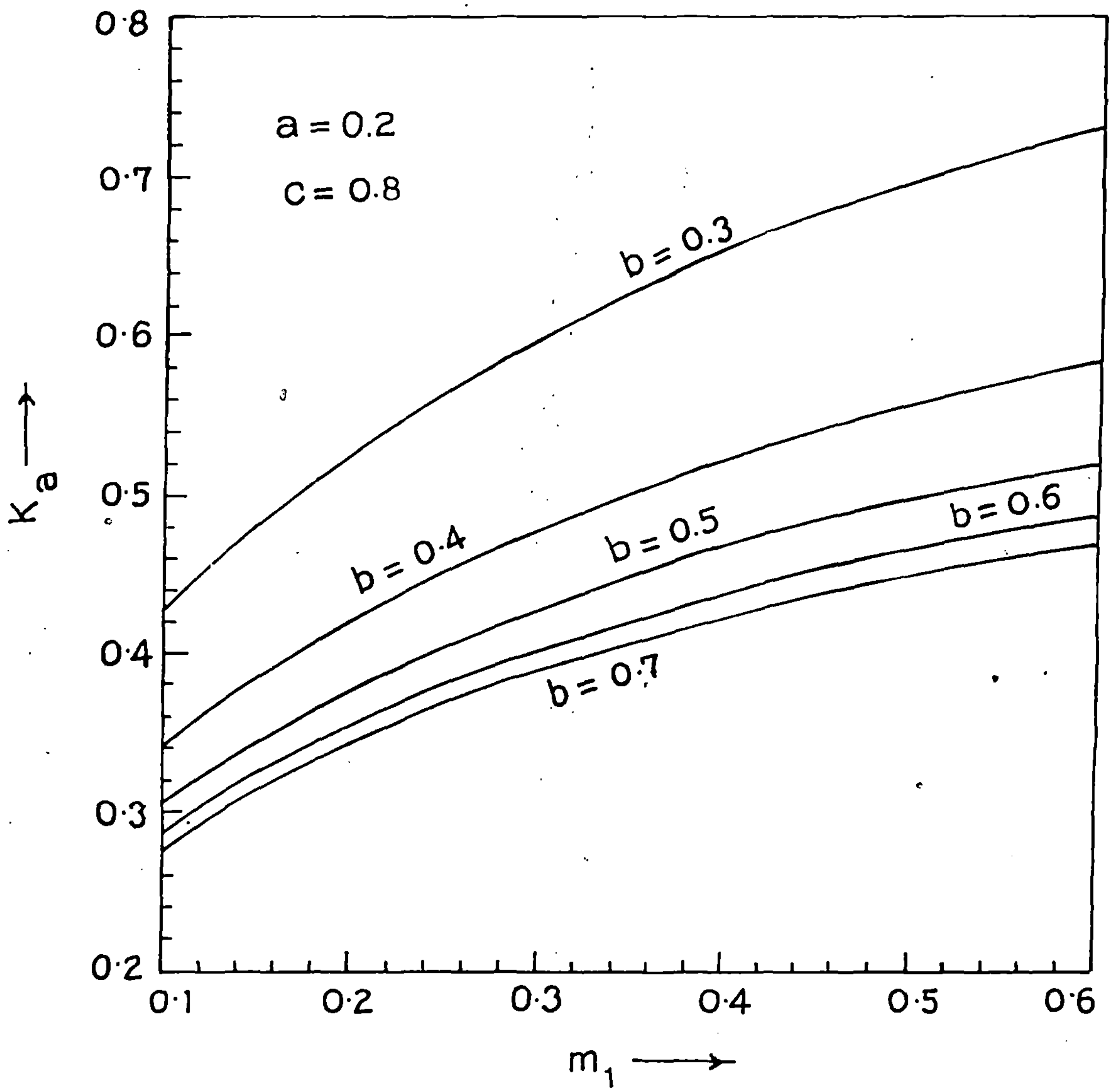

Fig. 6. Stress intensity factor K_a versus dimensionless frequency m_1 for a = 0.2, c = 0.8 and for different values of b.

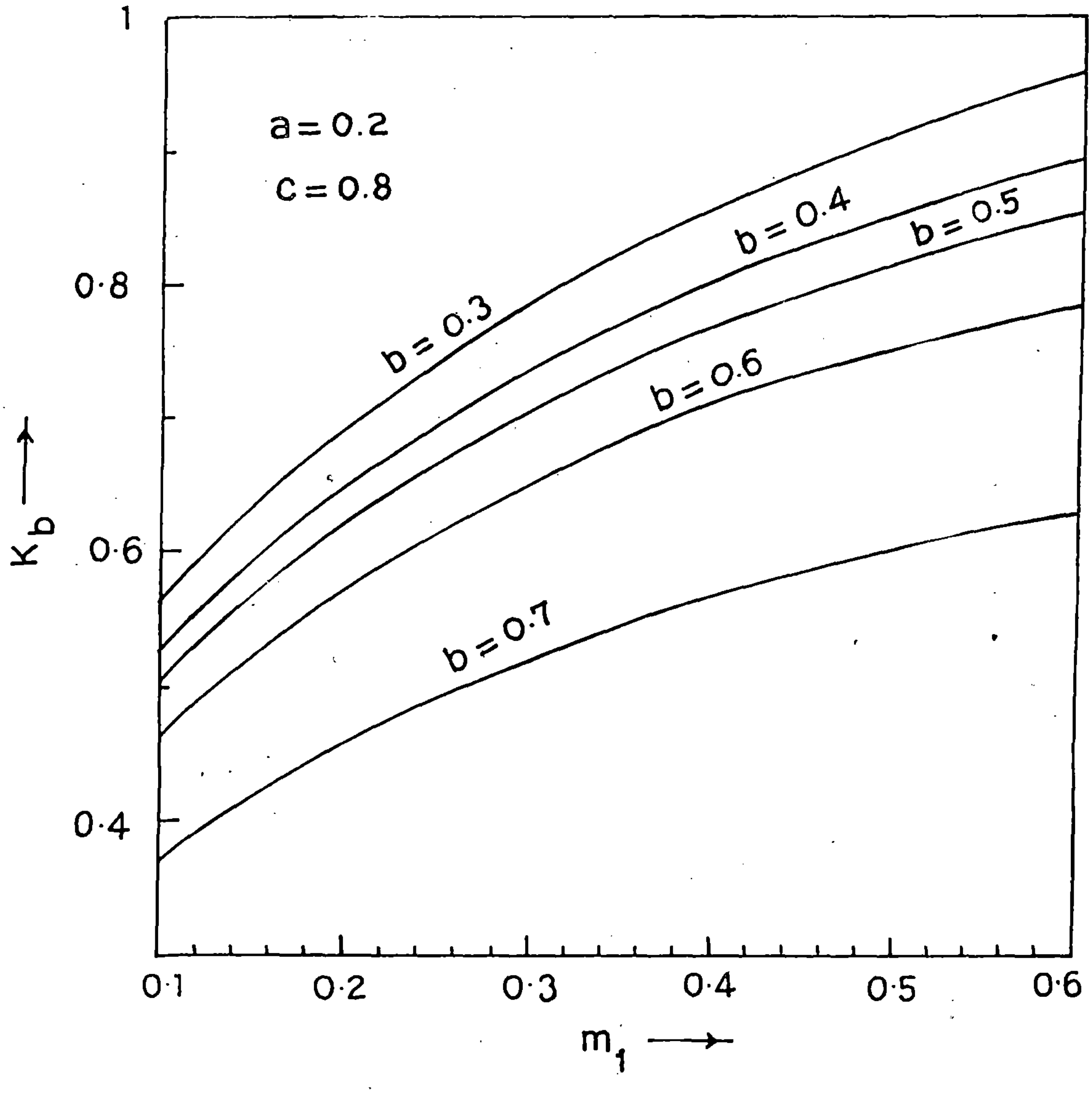

Fig. 7. Stress intensity factor K_b versus dimensionless frequency m_1 for a = 0.2, c = 0.8 and for different values of b.

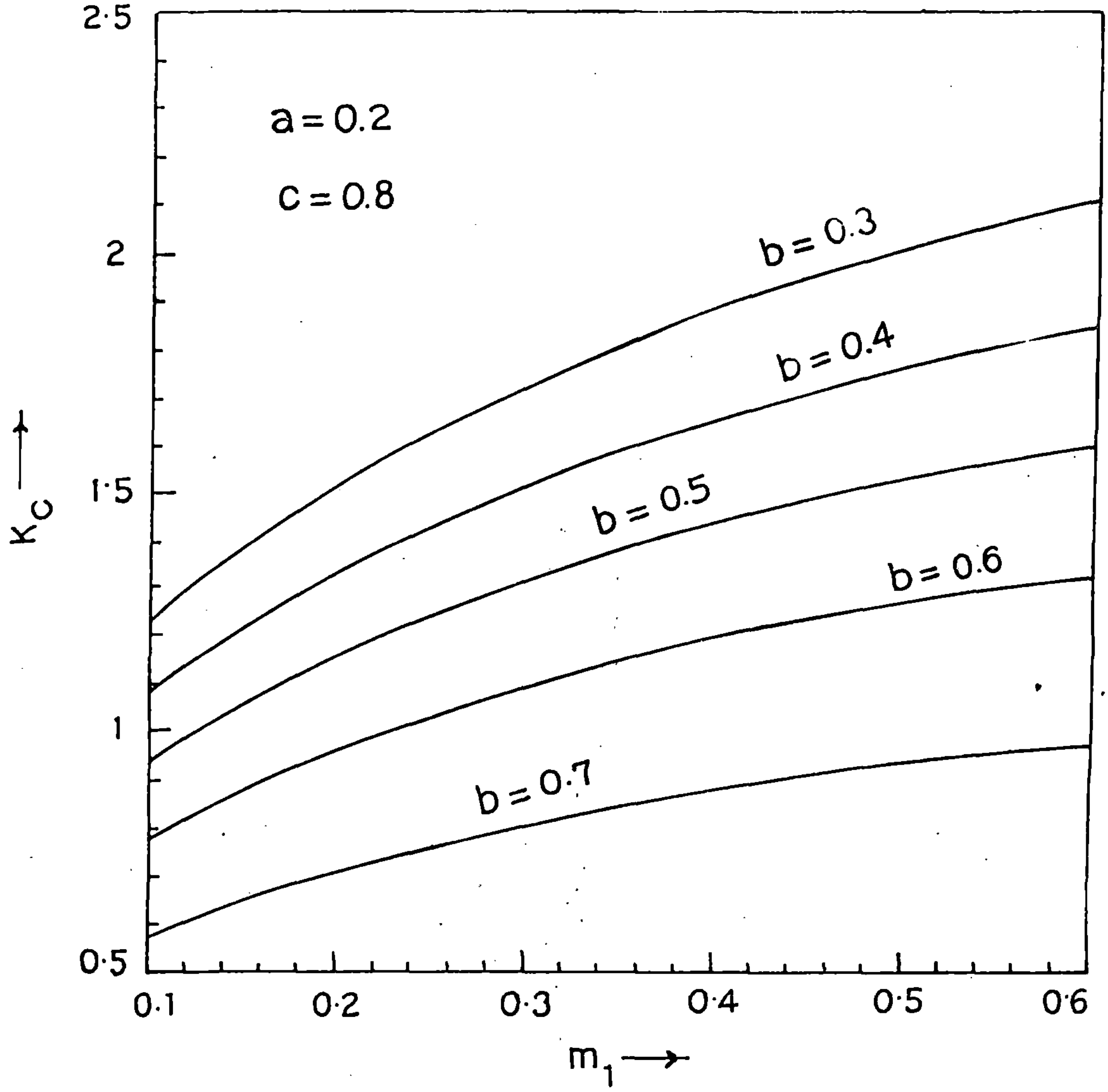

Fig. 8. Stress intensity factor K_c versus dimensionless frequency m_1 for a = 0.2, c = 0.8 and for different values of b.

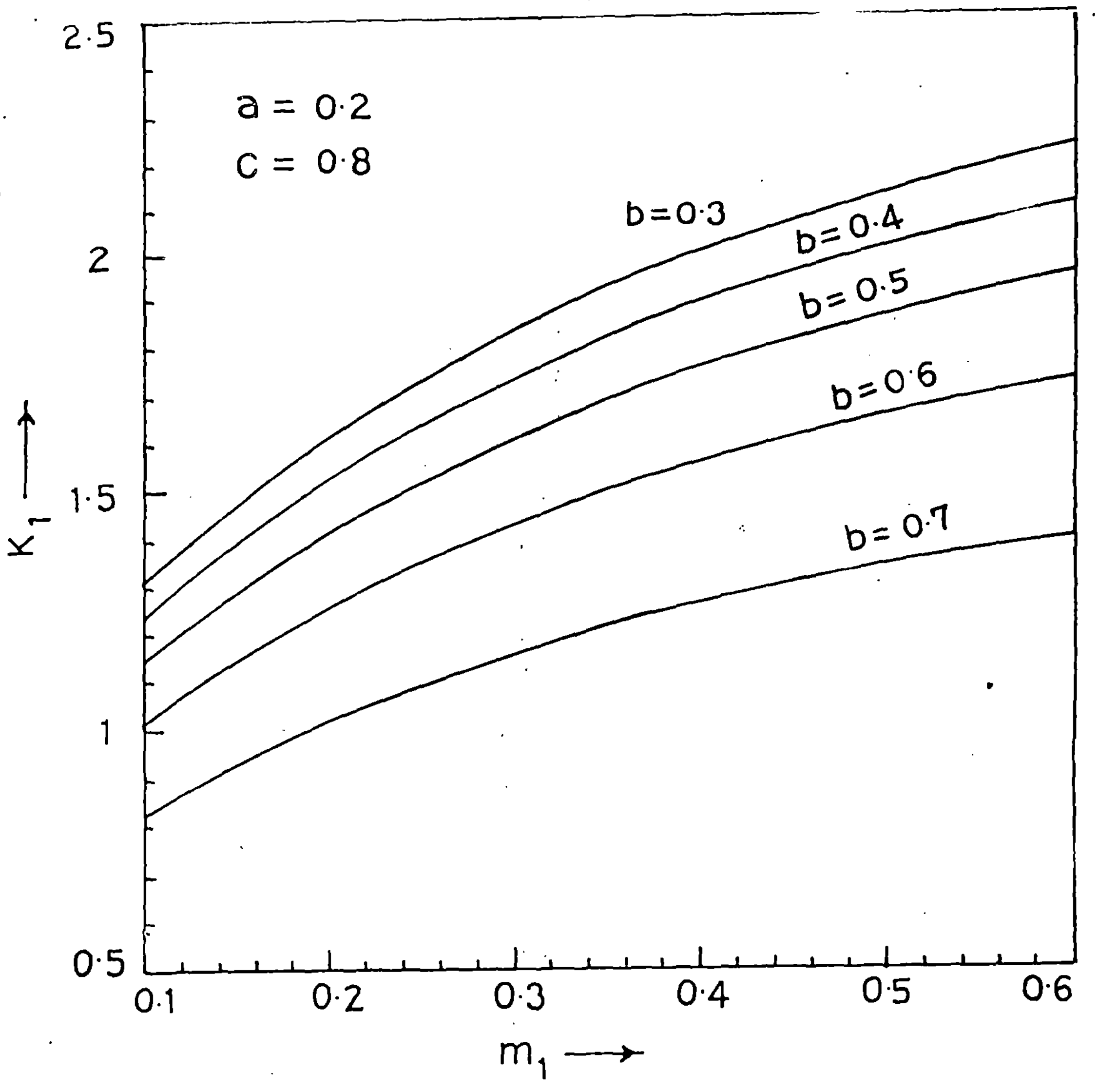

Fig. 9. Stress intensity factor K_1 versus dimensionless frequency m_1 for a = 0.2, c = 0.8 and for different values of b.

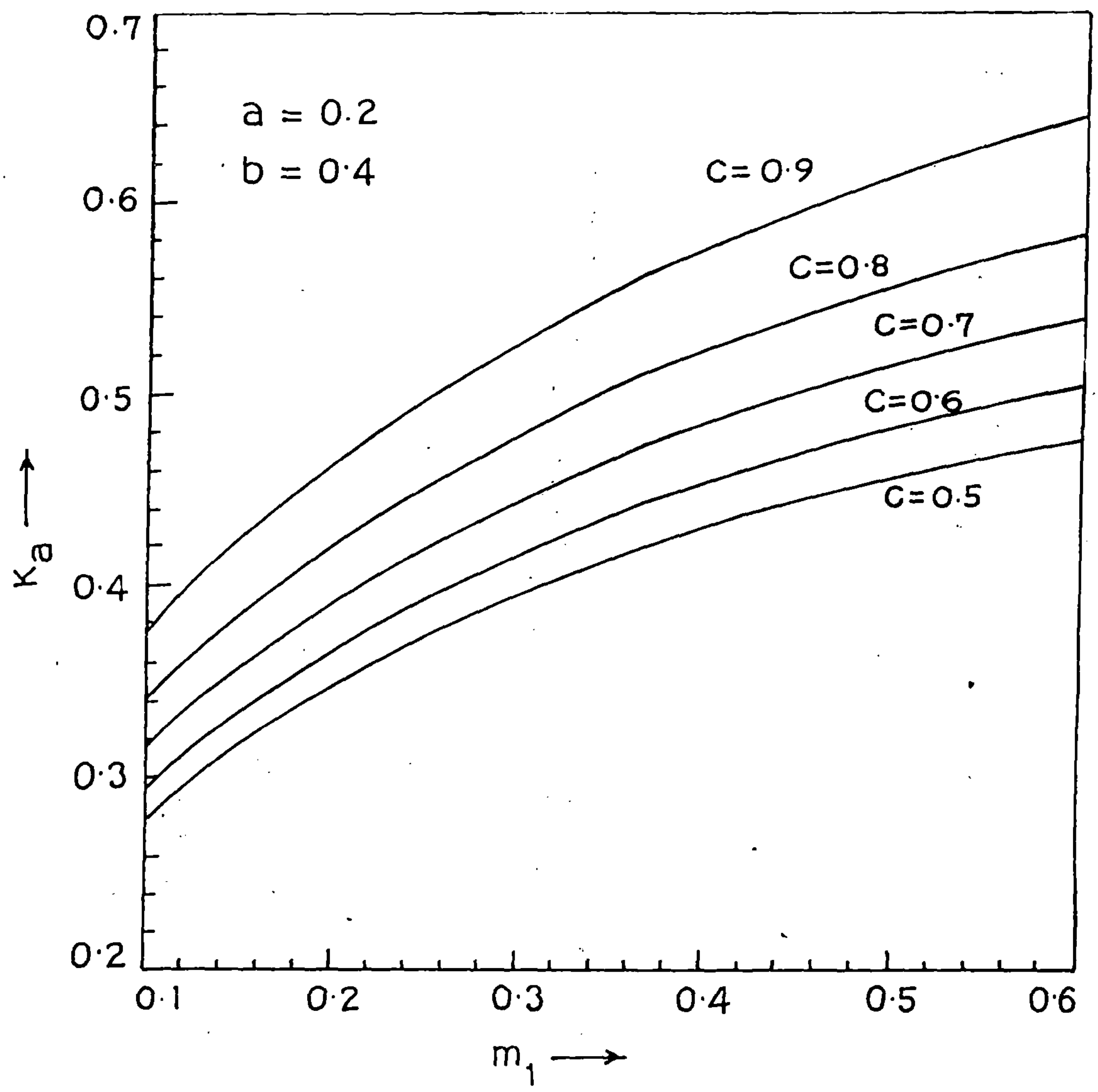

Fig. 10. Stress intensity factor K_a versus dimensionless frequency m_1 for a = 0.2, b = 0.4 and for different values of c.

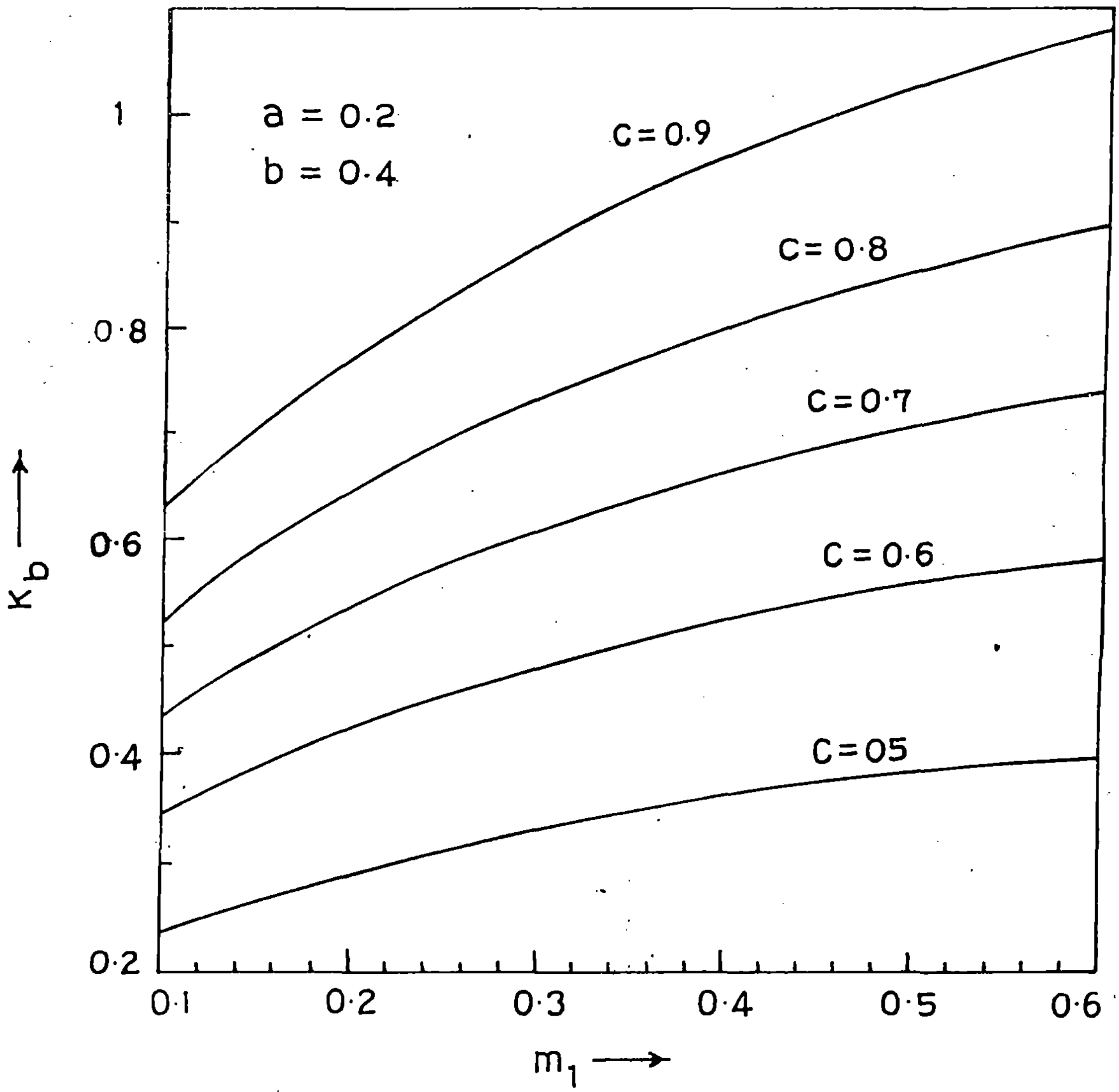

Fig. 11. Stress intensity factor K_b versus dimensionless frequency m_1 for a = 0.2, b = 0.4 and for different values of c.

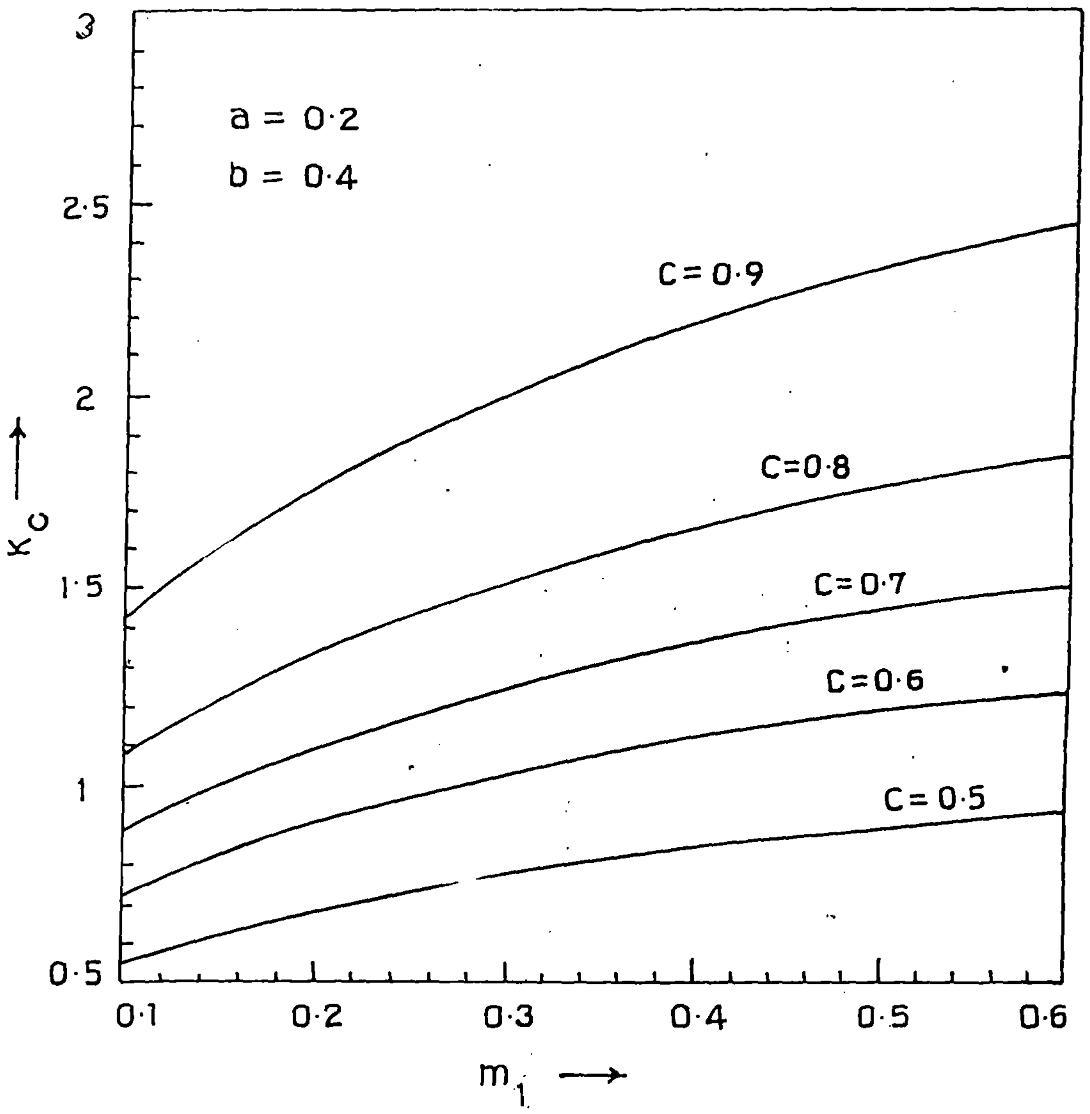

Fig. 12. Stress intensity factor K_c versus dimensionless frequency m_1 for a = 0.2, b = 0.4 and for different values of c.

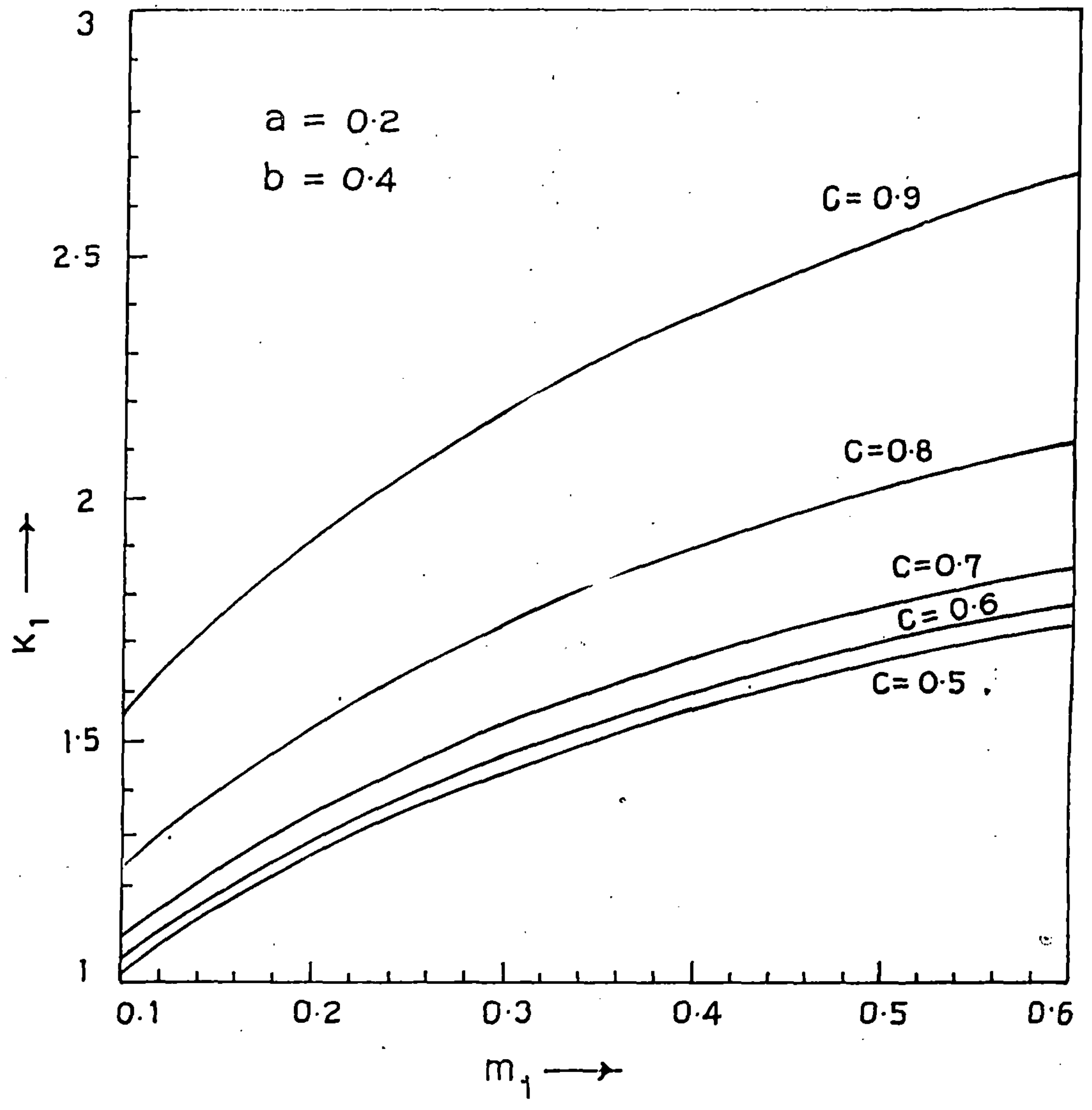

Fig. 13. Stress intensity factor K_1 versus dimensionless frequency m_1 for a = 0.2, b = 0.4 and for different values of c.

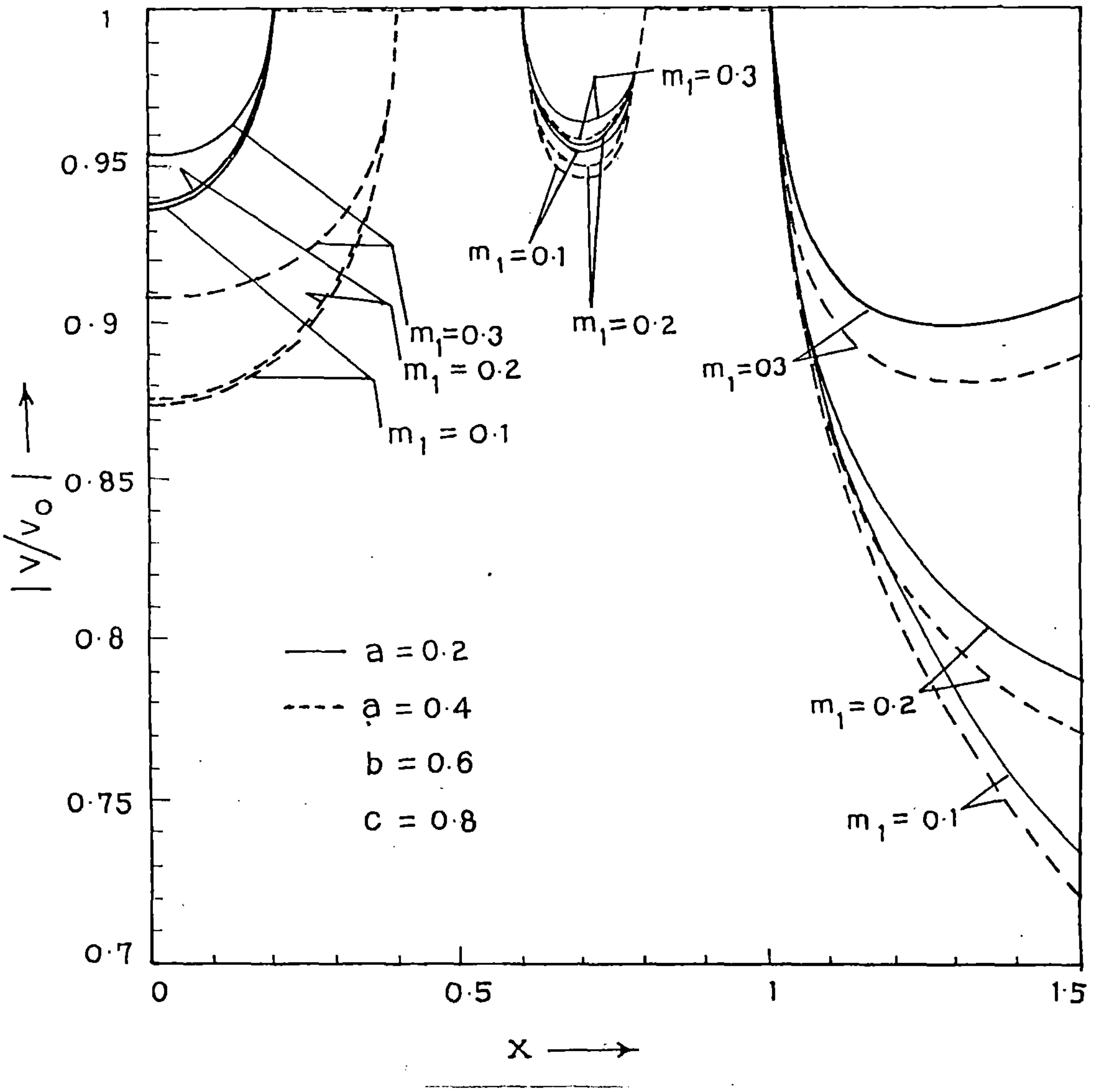

Fig. 14. Vertical displacement $|v(x,0)/v_0|$ versus dimensionless distance x for b = 0.6, c = 0.8, a = 0.2, 0.4 and for m_1 = 0.1, 0.2, 0.3.

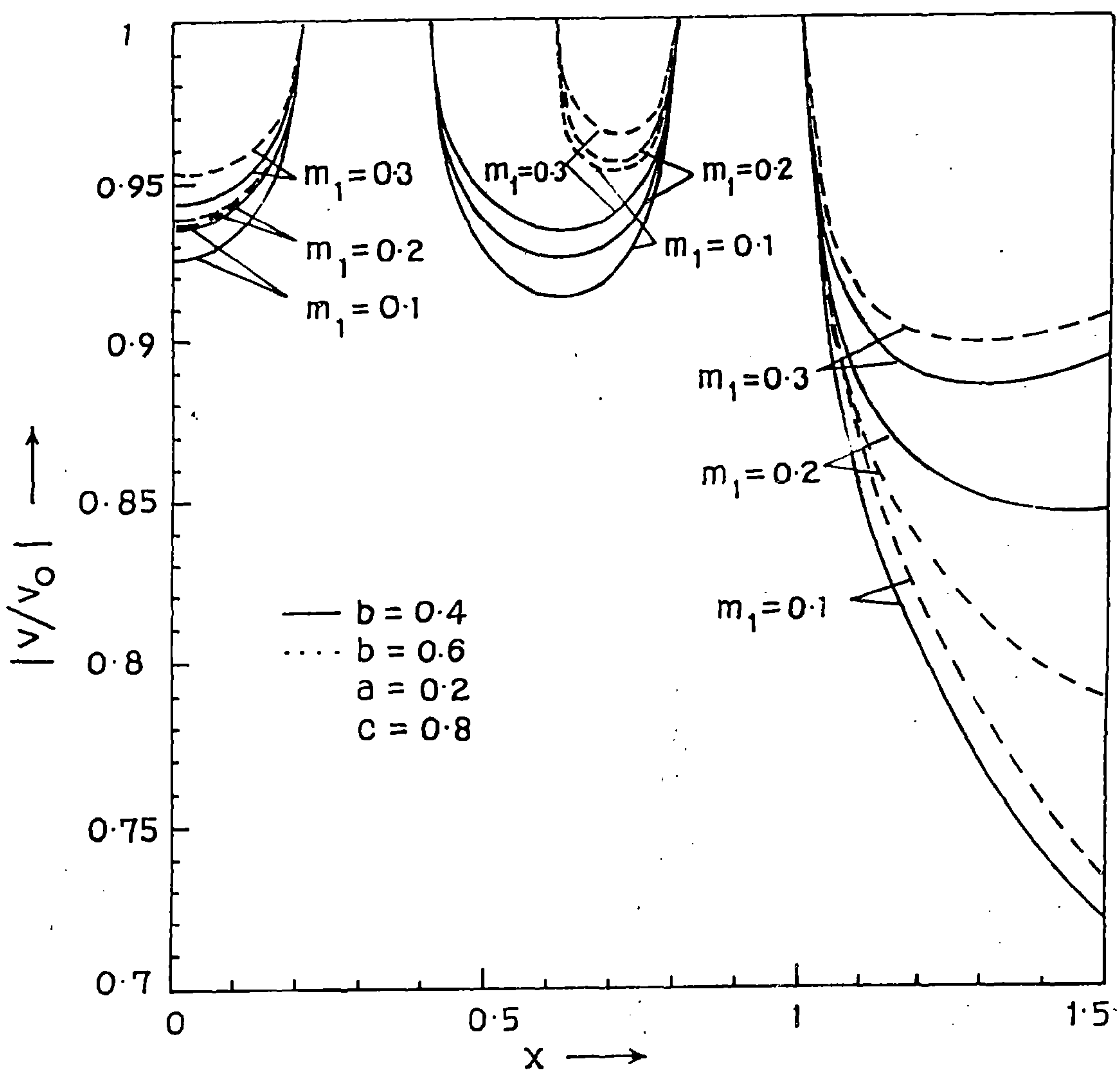

Fig. 15. Vertical displacement $| v(x,0)/v_0 |$ versus dimensionless distance x for a = 0.2, c = 0.8, b = 0.4, 0.6 and for m_1 = 0.1, 0.2, 0.3.

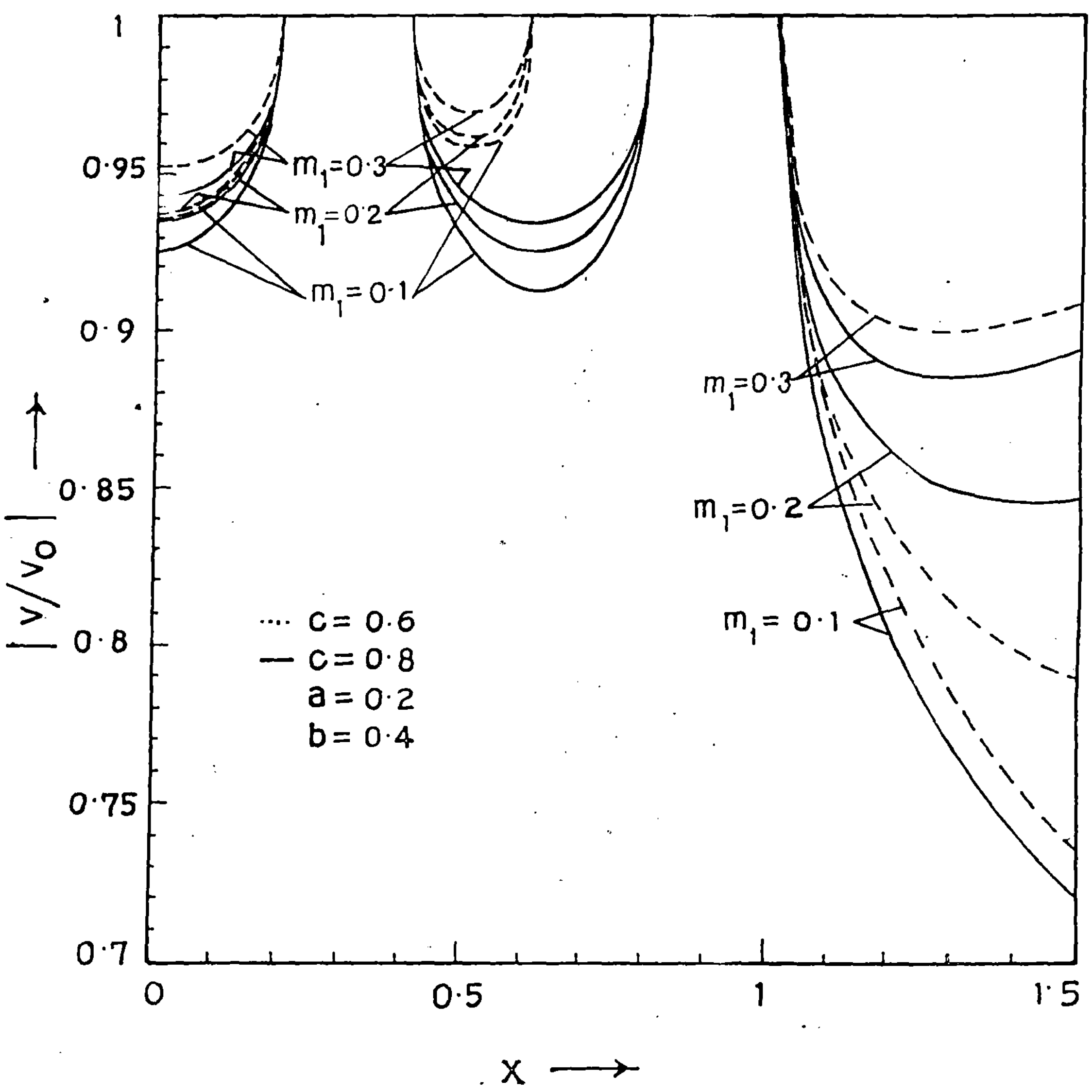

Fig. 16. Vertical displacement $| v(x,0)/v_0 |$ versus dimensionless distance x for a = 0.2, b = 0.4, c = 0.6, 0.8 and for m_1 = 0.1, 0.2, 0.3.

DIFFRACTION OF ELASTIC WAVES BY FOUR RIGID STRIPS EMBEDDED IN AN INFINITE ORTHOTROPIC MEDIUM

1. INTRODUCTION

In recent years, the study of the problems involving cracks or inclusions in composite and anisotropic materials has gained much importance. The problems of diffraction of elastic waves by cracks or inclusions have aroused attention in the field of fracture mechanics in view of their application in Seismology and Geophysics. Studies of a single Griffith crack as well as two parallel and coplanar Griffith cracks have been made by Mal [1970], Jain and Kanwal [1972] and Itou [1980]. The corresponding problems of diffraction by a single and two parallel rigid strips have been solved by Wickham [1977], Jain and Kanwal [1972] and Mandal and Ghosh [1992] respectively. In most of the cases the problems were solved by the integral equation technique, but the solutions of interesting problems involving the scattering of elastic waves by more than two coplanar Griffith cracks or strips are still lacking. The problem involving single Griffith crack in orthotropic medium was investigated by Kassir and Bandyopadhya [1983], Shindo et al

IN PRESS "JOURNAL OF ENGINEERING MATHEMATICS", 1996.

[1986] and De and Patra [1990]. Shindo et al [1991] have investigated the impact response of symmetric edge cracks in an orthotropic strip. Mandal and Ghosh [1994] considered the problem of interaction of elastic waves with a periodic array of coplanar Griffith cracks in an orthotropic elastic medium. The problem of scattering of elastic waves by a circular crack in transversely isotropic medium was investigated by Kundu and Bostrom [1991].

In our case, we have considered the two-dimensional problems of diffraction of elastic waves by four coplanar parallel rigid strips embedded in an infinite orthotropic medium. The five part mixed boundary value problem was reduced to the solution of a set of integral equations. Following the technique developed by Srivastava and Lowengrub [1970], the integral equations were solved. The normal stress under the strips and displacement outside the strips were derived in closed analytical form. To display the influence of the material orthotropy numerical values of stress intensity factors at the edges of the strips and vertical displacement have been plotted against dimensionless frequency and distance respectively for several orthotropic materials. This type of problem is important in view of their application in detecting the presence of inhomogeneities embedded in material structure and in seismology while studing the scattering of elastic waves by inhomogeneities like rigid hard rocks inside the earth.

2. FORMULATION OF THE PROBLEM

Consider the diffraction of normally incident longitudinal wave by four coplanar and parallel rigid strips embedded in an infinite orthotropic elastic medium and the strips occupy the region $d_1 \leq |x_1| \leq d_2$, $d_3 \leq |x_1| \leq d$, $x_2=0$, $|x_3|<\infty$. Let E_i, μ_{ij} and ν_{ij} $(i,j=1,2,3)$ denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the x_1, x_2, x_3 directions which coincide with the axes of material orthotropy. Normalizing all lengths with respect to 'd' and putting $x_1/d=x$, $x_2/d=y$, $x_3/d=z$, $d_1/d=a$, $d_2/d=b$, $d_3/d=c$, the rigid strips are defined by $a \leq |x| \leq b$, $c \leq |x| \leq 1$, $y=0$, $|z|<\infty$ (Fig.1).

Let a time harmonic wave given by $u_i=0$ and $v_i=v_0 \exp[\, i(ky-\omega t)]$ where $k=\omega d/c_s \sqrt{c_{22}}$, $c_s=(\mu_{12}/\rho)^{1/2}$ and v_0 is a constant, travelling in the direction of positive y-axis be incident normally on the strips. The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy} / \mu_{12} = c_{12} \frac{\partial u}{\partial x} + c_{22} \frac{\partial v}{\partial y}$$

$$\tau_{xy} / \mu_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (1)$$

where c_{ij} $(i,j = 1,2)$ are nondimensional parameters related to the elastic constants by the relations

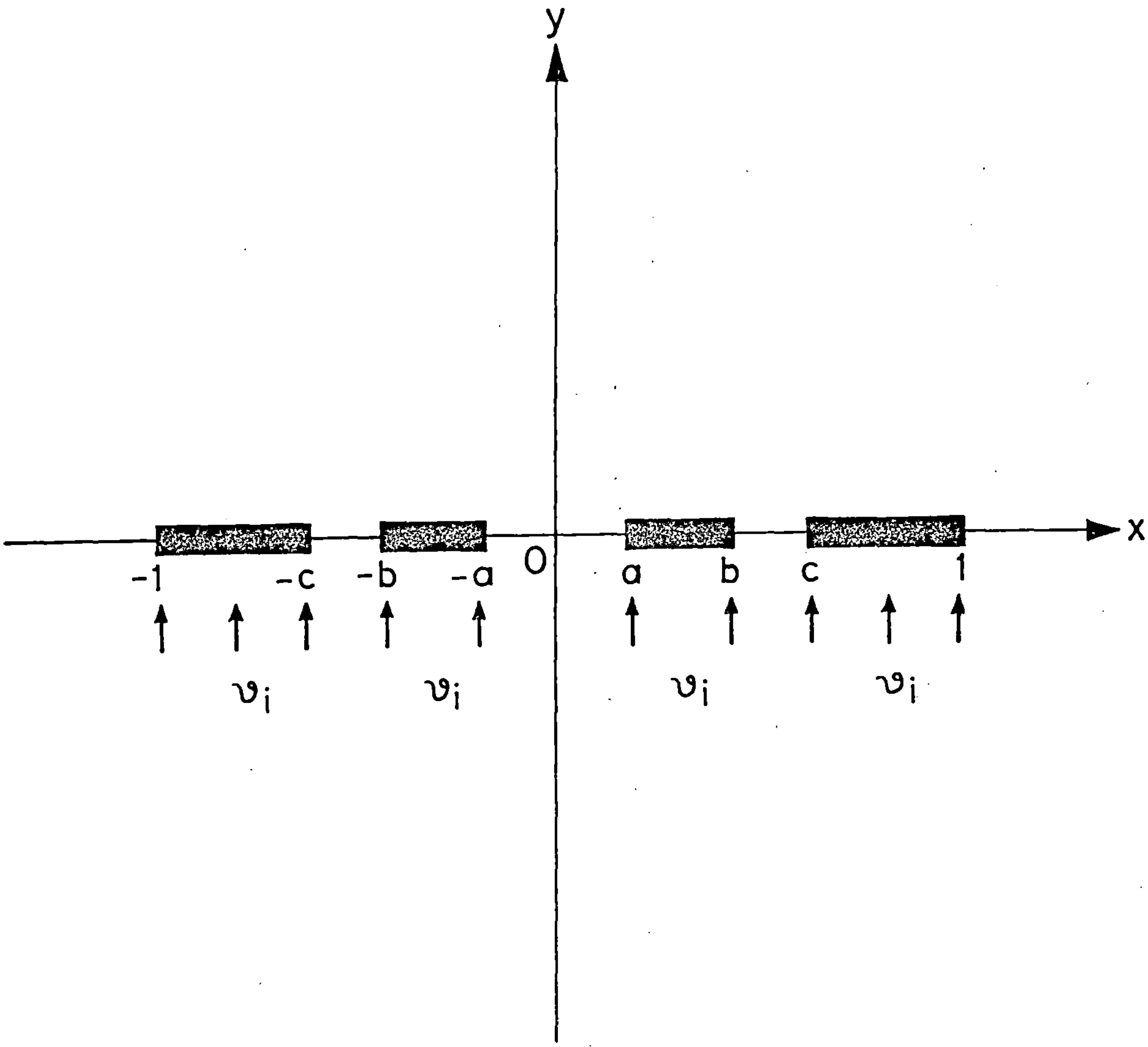

Fig. 1. Geometry of the strips and incident field.

$$c_{11} = E_1 / \mu_{12} (1 - \nu_{12}^2 E_2/E_1)$$

$$c_{22} = E_2 / \mu_{12} (1 - \nu_{12}^2 E_2/E_1) = c_{11}E_2/E_1 \qquad (2)$$

$$c_{12} = \nu_{12}E_2 / \mu_{12} (1 - \nu_{12}^2 E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11}$$

The constants E_i and ν_{ij} satisfy the Maxwell's relation

$$\nu_{ij} / E_i = \nu_{ji} / E_j \ .$$

The equations of motion for orthotropic material, interms of displacements are

$$c_{11} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1 + c_{12})\frac{\partial^2 v}{\partial x \partial y} = \frac{d^2}{c_s^2} \frac{\partial^2 u}{\partial t^2} \qquad (3)$$

$$c_{22} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1 + c_{12})\frac{\partial^2 u}{\partial x \partial y} = \frac{d^2}{c_s^2} \frac{\partial^2 v}{\partial t^2}$$

where u, v are the displacement components of the scattered field (Fig.2).

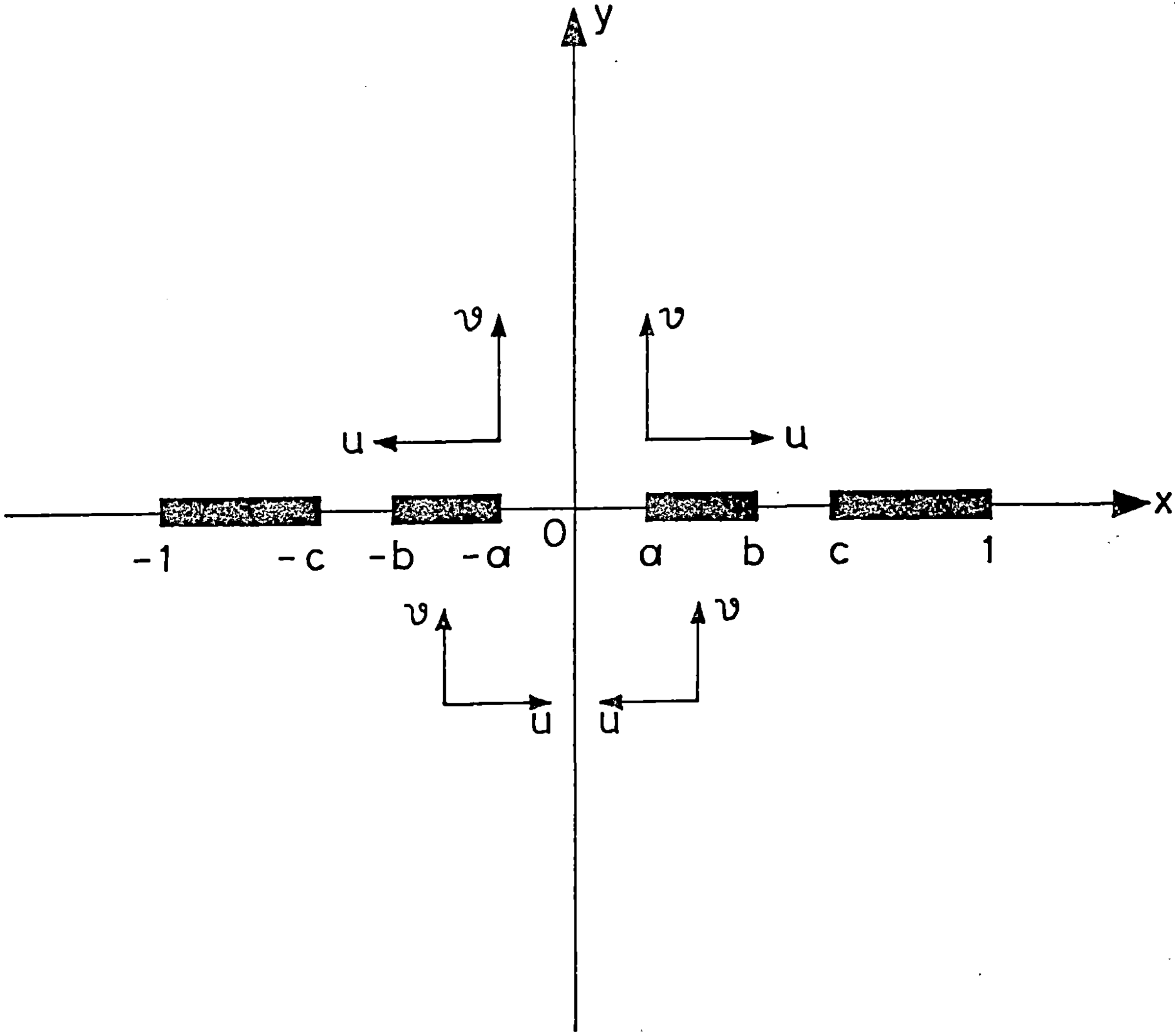

Fig. 2. Displacement components of the scattered field.

The boundary conditions are

(i) $u(x,y,t) = 0$, $v(x,y,t) + v_i(x,y,t) = 0$ across $y=0$ on the surface of the strips.

(ii) u and v are continuous across $y=0$ for $|x| < \infty$.

(iii) τ_{yy}, τ_{xy} are continuous across $y=0$ outside the strips.

Further, the scattered field should satisfy the radiation condition at infinity. Substituting $u(x,y,t) = u(x,y)\exp(-i\omega t)$ and $v(x,y,t) = v(x,y)\exp(-i\omega t)$ our problem reduces to the solution of the equations

$$c_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1 + c_{12})\frac{\partial^2 v}{\partial x \partial y} + \frac{d^2\omega^2}{c_s^2}u = 0$$

and

$$c_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1 + c_{12})\frac{\partial^2 u}{\partial x \partial y} + \frac{d^2\omega^2}{c_s^2}v = 0 \qquad (4)$$

Boundary conditions on u and v suggest that u and v are odd and even functions of y respectively. Accordingly, equations (4) are to be solved subject to the boundary conditions

$$v(x.0) = -v_0 , \qquad x \in I_2 , I_4 \tag{5}$$

$$\tau_{yy}(x,0) = 0 , \qquad x \in I_1 , I_3 , I_5 \tag{6}$$

$$u(x,0) = 0 , \qquad |x| < \infty \tag{7}$$

with $I_1 = (0,a)$, $I_2 = (a,b)$, $I_3 = (b,c)$, $I_4 = (c,1)$, $I_5 = (1,\infty)$.

Henceforth the time factor $\exp(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of equations (4) are taken as

$$u(x,y) = \pm \frac{2}{\pi} \int_0^\infty \left[A_1(\xi)\exp(-\gamma_1|y|) + A_2(\xi)\exp(-\gamma_2|y|)\right] \sin\xi x \, d\xi , \quad y \gtrless 0 \tag{8}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1 A_1(\xi)\exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\exp(-\gamma_2|y|)\right] \cos\xi x \, d\xi , \tag{9}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1 + c_{12})\gamma_i} , \qquad i = 1,2 , \qquad k_s^2 = \frac{d^2\omega^2}{c_s^2} \tag{10}$$

and $A_i(\xi)$ $(i = 1,2)$ are the unknowns to be solved, γ_1^2 and γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0 \tag{11}$$

From the boundary condition (7) it is found that

$$A_2(\xi) = - A_1(\xi).$$

Therefore displacements u, v and stresses τ_{yy}, τ_{xy} finally can be written as

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty} \left[\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)\right]A_1(\xi)\sin\xi x \, d\xi , \quad y>0 \qquad (12)$$

$$v(x,y) = \frac{2}{\pi}\int_0^{\infty} \frac{1}{\xi}\left[\alpha_1\exp(-\gamma_1|y|) - \alpha_2\exp(-\gamma_2|y|)\right]A_1(\xi)\cos\xi x \, d\xi \qquad (13)$$

$$\tau_{yy}/\mu_{12} = \frac{2}{\pi}\int_0^{\infty} \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\exp(-\gamma_1|y|) - \right.$$

$$\left. - \left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\exp(-\gamma_2|y|)\right]A_1(\xi)\cos\xi x \, d\xi , \quad y>0 \qquad (14)$$

$$\tau_{xy}/\mu_{12} = - \frac{2}{\pi}\int_0^{\infty} \left[(\gamma_1 + \alpha_1)\exp(-\gamma_1|y|) - \right.$$

$$\left. - (\gamma_2 + \alpha_2)\exp(-\gamma_2|y|)\right]A_1(\xi)\sin\xi x \, d\xi \qquad (15)$$

Next putting

$$A(\xi) = \frac{\alpha_1\gamma_1 - \alpha_2\gamma_2}{\xi} A_1(\xi)$$

the boundary conditions (5) and (6) lead to the following integral equations in $A(\xi)$:

$$\int_0^\infty \left[\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right] A(\xi) \cos\xi x \, d\xi = -\frac{\pi}{2} v_0 , \quad x \in I_2 , I_4 \tag{16}$$

and

$$\int_0^\infty A(\xi) \cos\xi x \, d\xi = 0 , \quad x \in I_1 , I_3 , I_5 \tag{17}$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (16) and (17) in the form

$$A(\xi) = \int_a^b tf(t^2)\cos\xi t \, dt + \int_c^1 ug(u^2)\cos\xi u \, du \tag{18}$$

where $f(t^2)$ and $g(u^2)$ are unknown functions to be determined.

By the choice of $A(\xi)$ given by (18) the relation (17) is satisfied automatically and the equation (16) becomes

$$\int_a^b tf(t^2)dt \int_0^\infty \left[\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right] \cos\xi x \cos\xi t \, d\xi \quad +$$

$$+ \int_c^1 ug(u^2)du \int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) \cos\xi x \cos\xi u \, d\xi$$

$$= -\frac{\pi}{2} v_0 , \quad x \in I_2, I_4 \qquad (19)$$

Using the relation

$$\frac{\sin\xi x \sin\xi t}{\xi^2} = \int_0^x \int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)dvdw}{(x^2-w^2)^{1/2} (t^2-v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{d}{dx} \int_a^b tf(t^2)dt \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{vwL_1(v,w) \, dw \, dv}{(x^2-w^2)^{1/2} (t^2-v^2)^{1/2}} +$$

$$+ \frac{d}{dx} \int_c^1 ug(u^2)du \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{vwL_1(v,w) \, dw \, dv}{(x^2-w^2)^{1/2} (u^2-v^2)^{1/2}}$$

$$= -\frac{\pi}{2} v_0 , \quad x \in I_2, I_4 \qquad (20)$$

where $$L_1(v,w) = \int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) J_0(\xi w) \, J_0(\xi v) \, d\xi . \qquad (21)$$

By a contour integration technique (Mandal and Ghosh [1994]) the infinite integral in $L_1(v,w)$ can be converted to the following finite integrals

$$L_1(v,w) = -i \left[\int_0^{1/\sqrt{c}_{11}} \frac{c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\, d\eta \right. -$$

$$\left. - \int_{1/\sqrt{c}_{11}}^{1} \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2}{\bar{\gamma}_2'^2(\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\, d\eta \right], \quad w>v \quad (22)$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2} \left\{ R_1 - (R_1^2 - 4R_2)^{1/2} \right\} \right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2} \left\{ R_1 + (R_1^2 - 4R_2)^{1/2} \right\} \right]^{1/2}$$

$$\bar{\gamma}_1' = \left[\frac{1}{2} \left\{ -R_1 + (R_1^2 + 4R_3)^{1/2} \right\} \right]^{1/2}$$

$$\bar{\gamma}_2' = \left[\frac{1}{2} \left\{ R_1 + (R_1^2 + 4R_3)^{1/2} \right\} \right]^{1/2}$$

$$R_1 = \frac{1}{c_{22}} \left\{ (c_{12}^2 + 2c_{12} - c_{12}c_{22})\, \eta^2 + (1 + c_{22}) \right\}$$

$$R_2 = \frac{c_{11}}{c_{22}} (1 - \eta^2) \left[\frac{1}{c_{11}} - \eta^2 \right]$$

$$R_3 = \frac{c_{11}}{c_{22}} (1 - \eta^2) \left[\eta^2 - \frac{1}{c_{11}} \right] \quad (23)$$

The corresponding expression of $L_1(v,w)$ for $w<v$ follows from (22) by interchanging w and v.

Substituting the series expansion of $J_0()$ and $H_0^{(1)}()$ for small k_s, in (22) we find after some algebraic manipulation

$$L_1(v,w) = \frac{2}{\pi}\left[\left(\gamma+\log(k_s w/2) - \frac{\pi i}{2}\right)M + N - \frac{(w^2+v^2)}{4} Rk_s^2 \log k_s\right] + O(k_s^2) \quad , \ w>v$$

$$= \frac{2}{\pi}\left[\left(\gamma+\log(k_s v/2) - \frac{\pi i}{2}\right)M + N - \frac{(w^2+v^2)}{4} Rk_s^2 \log k_s\right] + O(k_s^2) \quad , \ v>w \qquad (24)$$

where γ = 0.5772157....... is Euler's constant,

$$M = \int_0^{1/\sqrt{c}_{11}} \frac{c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\,d\eta - \int_{1/\sqrt{c}_{11}}^{1} \frac{c_{11}\eta^2-1+\bar{\gamma}_2'^2}{\bar{\gamma}_2'^2(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\,d\eta \qquad (25)$$

$$N = \int_0^{1/\sqrt{c}_{11}} \frac{c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\log\eta\,d\eta - \int_{1/\sqrt{c}_{11}}^{1} \frac{c_{11}\eta^2-1+\bar{\gamma}_2'^2}{\bar{\gamma}_2'^2(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\log\eta\,d\eta \qquad (26)$$

and

$$R = \int_0^{1/\sqrt{c}_{11}} \frac{\eta^2(c_{11}\eta^2-1-\bar{\gamma}_1\bar{\gamma}_2)}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1+\bar{\gamma}_2)}\,d\eta - \int_{1/\sqrt{c}_{11}}^{1} \frac{\eta^2(c_{11}\eta^2-1+\bar{\gamma}_2'^2)}{\bar{\gamma}_2'^2(\bar{\gamma}_1'^2+\bar{\gamma}_2'^2)}\,d\eta \qquad (27)$$

Now differentiating both sides of the relation (19) with respect to x we obtain

$$\int_a^b tf(t^2)dt \int_0^{\infty} \xi \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) \sin\xi x \cos\xi t \, d\xi \; +$$

$$+ \int_c^1 ug(u^2)du \int_0^{\infty} \xi \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) \sin\xi x \cos\xi u \, d\xi = 0 \;, \quad x \in I_2, I_4$$

Following similar procedure as done for deriving equation (20), we obtain

$$x\int_a^b \frac{tf(t^2)}{(x^2-t^2)} dt + x\int_c^1 \frac{ug(u^2)}{(x^2-u^2)} du$$

$$= \int_a^b tf(t^2)dt \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{vwL_2(v,w) \, dw \, dv}{(x^2-w^2)^{1/2} \; (t^2-v^2)^{1/2}} +$$

$$+ \int_c^1 ug(u^2)du \frac{\partial}{\partial u} \int_0^x \int_0^u \frac{vwL_2(v,w) \, dw \, dv}{(x^2-w^2)^{1/2} \; (u^2-v^2)^{1/2}}$$

$$= 0, \qquad x \in I_2, I_4 \tag{28}$$

where

$$L_2(v,w) = \int_0^{\infty} \left[\xi - \frac{\xi^2}{\theta} \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2} \right) \right] J_0(\xi w) \; J_0(\xi v) d\xi \tag{29}$$

$$\Theta = \frac{c_{11} + N_1 N_2}{N_1 + N_2} \tag{30}$$

$$N_1^2 = \frac{1}{2c_{22}} \left[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}} \right] \tag{31}$$

and

$$N_2^2 = \frac{1}{2c_{22}} \left[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}} \right].$$

We use the contour integration technique mentioned earlier and get from (29)

$$L_2(v,w) = \frac{ik_s^2}{\Theta} \left[\int_0^{1/\sqrt{c}_{11}} \frac{\eta^2 (c_{11}\eta^2 - 1 - \bar{\gamma}_1 \bar{\gamma}_2)}{\bar{\gamma}_1 \bar{\gamma}_2 (\bar{\gamma}_1 + \bar{\gamma}_2)} J_0(k_s \eta v) H_0^{(1)}(k_s \eta w)\, d\eta \right. -$$

$$\left. - \int_{1/\sqrt{c}_{11}}^{1} \frac{\eta^2 (c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2)}{\bar{\gamma}_2' (\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)} J_0(k_s \eta v) H_0^{(1)}(k_s \eta w)\, d\eta \right], \quad w>v \tag{32}$$

By the process similar to the one which led to the equation (24), (32) for small values of k_s can be written as

$$L_2(v,w) = -\frac{2}{\pi} P\, k_s^2 \log k_s + O(k_s^2) \tag{33}$$

where $P = \frac{1}{\Theta} R$ and R is given by (27).

Now, let us consider

$$f(t^2) = f_0(t^2) + k_s^2 \log(k_s)\, f_1(t^2) + O(k_s^2)$$

and $$g(u^2) = g_0(u^2) + k_s^2 \log(k_s)\, g_1(u^2) + O(k_s^2) \qquad (34)$$

Putting the above expressions of $f(t^2)$, $g(u^2)$ and the value of $L_2(v,w)$ given by (33) in the equation (28) and equating the coefficients of like powers of k_s we obtain,

$$\int_a^b \frac{tf_0(t^2)}{(x^2 - t^2)}\,dt + \int_c^1 \frac{ug_0(u^2)}{(x^2 - u^2)}\,du = 0 , \quad x \in I_2 , I_4 \qquad (35)$$

and $$\int_a^b \frac{tf_1(t^2)}{(x^2 - t^2)}\,dt + \int_c^1 \frac{ug_1(u^2)}{(x^2 - u^2)}\,du$$

$$= - \frac{2P}{\pi} \left[\int_a^b tf_0(t^2)dt + \int_c^1 ug_0(u^2)du \right], \quad x \in I_2, I_4 \qquad (36)$$

Following Srivastava and Lowengrub [1970] the solutions of the above integral equation (35) can be obtained as

$$f_0(t^2) = D_1 \left[\frac{1 - a^2}{c^2 - a^2} \right]^{1/2} \left[\frac{c^2 - t^2}{1 - t^2} \right]^{1/2} \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} -$$

$$- D_2 \left[\frac{t^2 - a^2}{b^2 - t^2} \right]^{1/2} \frac{1}{\sqrt{(1 - t^2)(c^2-t^2)}} , \quad x \in I_2 \qquad (37)$$

and $$g_0(u^2) = D_1 \left[\frac{1 - a^2}{c^2 - a^2} \right]^{1/2} \left[\frac{u^2 - c^2}{1 - u^2} \right]^{1/2} \frac{1}{\sqrt{(u^2-a^2)(u^2-b^2)}} +$$

$$+ D_2 \left[\frac{u^2 - a^2}{u^2 - b^2} \right]^{1/2} \frac{1}{\sqrt{(1 - u^2)(u^2-c^2)}} , \quad x \in I_4 \qquad (38)$$

where D_1 and D_2 are constants which can be calculaed as follows.

We substitute the value of $L_1(v,w)$ from (24) as well as the expansion of $f(t^2)$ and $g(u^2)$ obtained from (34), (37) and (38) up to $O(k_s^2 \log k_s)$ in the equation (20). When the coefficients of like powers of k_s from both sides of the resulting equation are equated we get after some manipulation, the following results:

$$D_1 = - v_0 \frac{\pi^2}{4} \frac{(X_4 - X_2)}{(X_1 X_4 - X_2 X_3)} \quad ; \qquad D_2 = - v_0 \frac{\pi^2}{4} \frac{(X_3 - X_1)}{(X_2 X_3 - X_1 X_4)} \qquad (39)$$

Where

$$X_1 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left[\left\{\left[\gamma+\log(k_s/2)-\frac{\pi i}{2}\right]M + N\right\}(J_1+J_3) + \right.$$

$$\left. + \frac{1}{2}MJ_1\log(b^2-a^2) + MJ_5\right] \qquad (40)$$

$$X_2 = \left\{\left[\gamma + \log(k_s/2) - \frac{\pi i}{2}\right]M + N\right\}(J_4-J_2) -$$

$$- \frac{1}{2}MJ_2\log(b^2-a^2) + MJ_6 \qquad (41)$$

$$X_3 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left[\left\{\left[\gamma+\log(k_s/2)-\frac{\pi i}{2}\right]M + N\right\}(J_1+J_3) + \right.$$

$$\left. + \frac{1}{2}MJ_3\log(1-c^2) + MJ_7\right] \qquad (42)$$

$$X_4 = \left\{\left[\gamma + \log(k_s/2) - \frac{\pi i}{2}\right]M + N\right\}(J_4-J_2) +$$

$$+ \frac{1}{2}MJ_4\log(1-c^2) - MJ_8 \qquad (43)$$

$$J_1 = \int_a^b \left(\frac{c^2-t^2}{1-t^2}\right)^{1/2} \frac{tdt}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$

$$J_2 = \int_a^b \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \frac{tdt}{\sqrt{(1-t^2)(c^2-t^2)}}$$

$$J_3 = \int_c^1 \left(\frac{u^2-c^2}{1-u^2} \right)^{1/2} \frac{udu}{\sqrt{(u^2-a^2)(u^2-b^2)}}$$

$$J_4 = \int_c^1 \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \frac{udu}{\sqrt{(1-u^2)(u^2-c^2)}}$$

$$J_5 = \int_c^1 \left(\frac{u^2-c^2}{1-u^2} \right)^{1/2} \frac{u\log\left(\sqrt{u^2-b^2} + \sqrt{u^2-a^2}\right)}{\sqrt{(u^2-a^2)(u^2-b^2)}} du$$

$$J_6 = \int_c^1 \left(\frac{u^2-a^2}{u^2-b^2} \right)^{1/2} \frac{u\log\left(\sqrt{u^2-b^2} + \sqrt{u^2-a^2}\right)}{\sqrt{(1-u^2)(u^2-c^2)}} du$$

$$J_7 = \int_a^b \left(\frac{c^2-t^2}{1-t^2} \right)^{1/2} \frac{t\log\left(\sqrt{c^2-t^2} + \sqrt{1-t^2}\right)}{\sqrt{(t^2-a^2)(b^2-t^2)}} dt$$

$$J_8 = \int_a^b \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \frac{t\log\left(\sqrt{c^2-t^2} + \sqrt{1-t^2}\right)}{\sqrt{(1-t^2)(c^2-t^2)}} dt$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ on the plane $y = 0$ can be found from the relations (14), (18), (34),(37) and (38) as

$$\tau_{yy}(x,0) = - \frac{\mu_{12} c_{22} x}{\sqrt{(x^2-a^2)(b^2-x^2)}} \left\{ D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-x^2}{1-x^2} \right)^{1/2} - \frac{D_2 (x^2-a^2)}{\sqrt{(1-x^2)(c^2-x^2)}} \right\} + O(k_s^2 \log k_s) , \quad x \in I_2$$

$$= - \frac{\mu_{12} c_{22} x}{\sqrt{(x^2-c^2)(1-x^2)}} \left\{ D_1 \left[\frac{1-a^2}{c^2-a^2} \right]^{1/2} \frac{(x^2-c^2)}{\sqrt{(x^2-a^2)(x^2-b^2)}} + D_2 \left(\frac{x^2-a^2}{x^2-b^2} \right)^{1/2} \right\} + O(k_s^2 \log k_s) , \quad x \in I_4 \qquad (44)$$

Defining the stress intensity factors at the edges of the strips by the relations

$$K_a = \operatorname*{Lt}_{x \to a+} \left| \frac{\tau_{yy}(x,0) \sqrt{(x-a)}}{v_0 \mu_{12}} \right|$$

$$K_b = \underset{x \to b-}{Lt} \left| \frac{\tau_{yy}(x,0)\sqrt{(b-x)}}{v_0\mu_{12}} \right|$$

$$K_c = \underset{x \to c+}{Lt} \left| \frac{\tau_{yy}(x,0)\sqrt{(x-c)}}{v_0\mu_{12}} \right|$$

$$K_1 = \underset{x \to 1-}{Lt} \left| \frac{\tau_{yy}(x,0)\sqrt{(1-x)}}{v_0\mu_{12}} \right|$$

we get

$$K_a = \left| \frac{c_{22}\sqrt{a}\ D_1}{\sqrt{2(b^2-a^2)}} \right| \qquad (45)$$

$$K_b = \left| \frac{c_{22}\sqrt{b}}{\sqrt{2(b^2-a^2)}} \left\{ D_1 \left(\frac{1-a^2}{c^2-a^2} \right)^{1/2} \left(\frac{c^2-b^2}{1-b^2} \right)^{1/2} - \frac{D_2(b^2-a^2)}{\sqrt{(1-b^2)(c^2-b^2)}} \right\} \right| \qquad (46)$$

$$K_c = \left| \frac{c_{22}\sqrt{c}}{\sqrt{2(1-c^2)}}\ D_2 \left(\frac{c^2-a^2}{c^2-b^2} \right)^{1/2} \right| \qquad (47)$$

$$K_1 = \left| \frac{c_{22}}{\sqrt{2(1-c^2)}} \left\{ \frac{D_1(1-c^2)}{\sqrt{(1-b^2)(c^2-a^2)}} + D_2 \left(\frac{1-a^2}{1-b^2} \right)^{1/2} \right\} \right| \qquad (48)$$

The vertical displacement $v(x,y)$ on the plane $y = 0$ can be obtained from equations (13), (18), (34), (37) and (38) as

$$v(x,0) = \frac{4}{\pi^2}\Bigg[\left\{ \left[\gamma + \log(k_s) - \frac{\pi i}{2} \right] M + N \right\} \times$$

$$\times \left\{ D_1 \left[\frac{1 - a^2}{c^2 - a^2} \right]^{1/2} (J_1 + J_2) + D_2 (J_4 - J_2) \right\} +$$

$$+ \frac{M}{2} \left\{ D_1 \left[\frac{1 - a^2}{c^2 - a^2} \right]^{1/2} (J_9 + J_{11}) + D_2 (J_{12} - J_{10}) \right\} \Bigg] ,$$

$$x \in I_1 , I_3 , I_5 \qquad (49)$$

where

$$J_9 = \int_a^b \left[\frac{c^2 - t^2}{1 - t^2} \right]^{1/2} \frac{t \log|t^2 - x^2|}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} dt$$

$$J_{10} = \int_a^b \left[\frac{t^2 - a^2}{b^2 - t^2} \right]^{1/2} \frac{t \log|t^2 - x^2|}{\sqrt{(1 - t^2)(c^2 - t^2)}} dt$$

$$J_{11} = \int_c^1 \left[\frac{u^2 - c^2}{1 - u^2} \right]^{1/2} \frac{u \log|u^2 - x^2|}{\sqrt{(u^2 - a^2)(u^2 - b^2)}} du$$

$$J_{12} = \int_c^1 \left[\frac{u^2 - a^2}{u^2 - b^2} \right]^{1/2} \frac{u \log|u^2 - x^2|}{\sqrt{(u^2 - c^2)(1 - u^2)}} du$$

In order to obtain the solution of the problem corresponding to two rigid strips taking b $\longrightarrow$ c we find from (37) and (38) that in this particular case

$$f_0(t^2) = g_0(t^2) = D_1 \left(\frac{1 - a^2}{c^2 - a^2} \right)^{1/2} \frac{1}{\sqrt{(t^2 - a^2)(1 - t^2)}} -$$

$$- D_2 \left(\frac{t^2 - a^2}{1 - t^2} \right)^{1/2} \frac{1}{b^2 - t^2}, \quad a \leq t \leq 1.$$

It can further be shown that $X_1 = X_3$ so that

$$D_2 = 0 \text{ and } D_1 = - \frac{V_0 \pi^2}{4 X_1},$$

where

$$X_1 = \frac{\pi}{2} \left(\frac{1 - a^2}{c^2 - a^2} \right)^{1/2} \left[\left\{ \gamma + \log(k_s/2) - \frac{\pi i}{2} + \log(1 - a^2)^{1/2} \right\} M + N \right]$$

It can easily be shown that in the isotropic case this result is identical with result given by Jain and Kanwal [1972].

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_a, K_b, K_c and K_1 given by (45) - (48) at the edges of the strips and vertical displacement $|v(x,0)/v_0|$ near about the rigid strips have been plotted against dimensionless frequency k_s and distance x respectively for three different types of orthotropic materials whose engineering constants have been listed in table 1.

TABLE - 1. ENGINEERING ELASTIC CONSTANTS

	E_1(Pa)	E_2(Pa)	μ_{12}(Pa)	γ_{12}
Type I	Modulite II Graphite-Epoxi Composite :			
	15.3×10^9	$1{\cdot}58.0\times10^9$	5.52×10^9	0.033
Type II	E-Type Glass-Epoxi Composite :			
	9.79×10^9	42.3×10^9	3.66×10^9	0.063
Type III	Stainless Steel-Aluminium Composite :			
	79.76×10^9	85.91×10^9	30.02×10^9	0.31

It is found that whatever the lengths of the strips are, SIFs at the four edges of the strips increase with increase in the value of

k_s $(0.1 \le k_s \le 0.6)$. From the graphs, it may be noted further that with a decrease in the length of the inner strip, which might be induced either by increasing 'a' or by decreasing 'b', the SIF K_a at the innermost edge gradually decreases, wheareas the SIFs at the other edges show just the opposite behavior (Fig.3 - Fig.4).

Also, a decrease in the value of the length of the outer strip, which might be induced by increasing the value of 'c', causes an increase in the values of the SIFs (Fig.5) from which an interesting conclusion might be drawn : i.e., the presence of the inner strip suppresses the SIFs at both edges of the outer strip and the presence of the outer strip suppresses the SIFs at the edges of the inner strip.

The SIF K_a has been plotted (Fig. 6) for different orthotropic materials to show the effect of material orthotropy. Similar effect are being seen for other SIFs.

The vertical displacement has been plotted for different strip lengths. It is found from Fig.7 - Fig.9 that with the increase in the value of strip length, the displacement increases.

For a fixed material the variation of displacement with frequency is found to be insignificant.

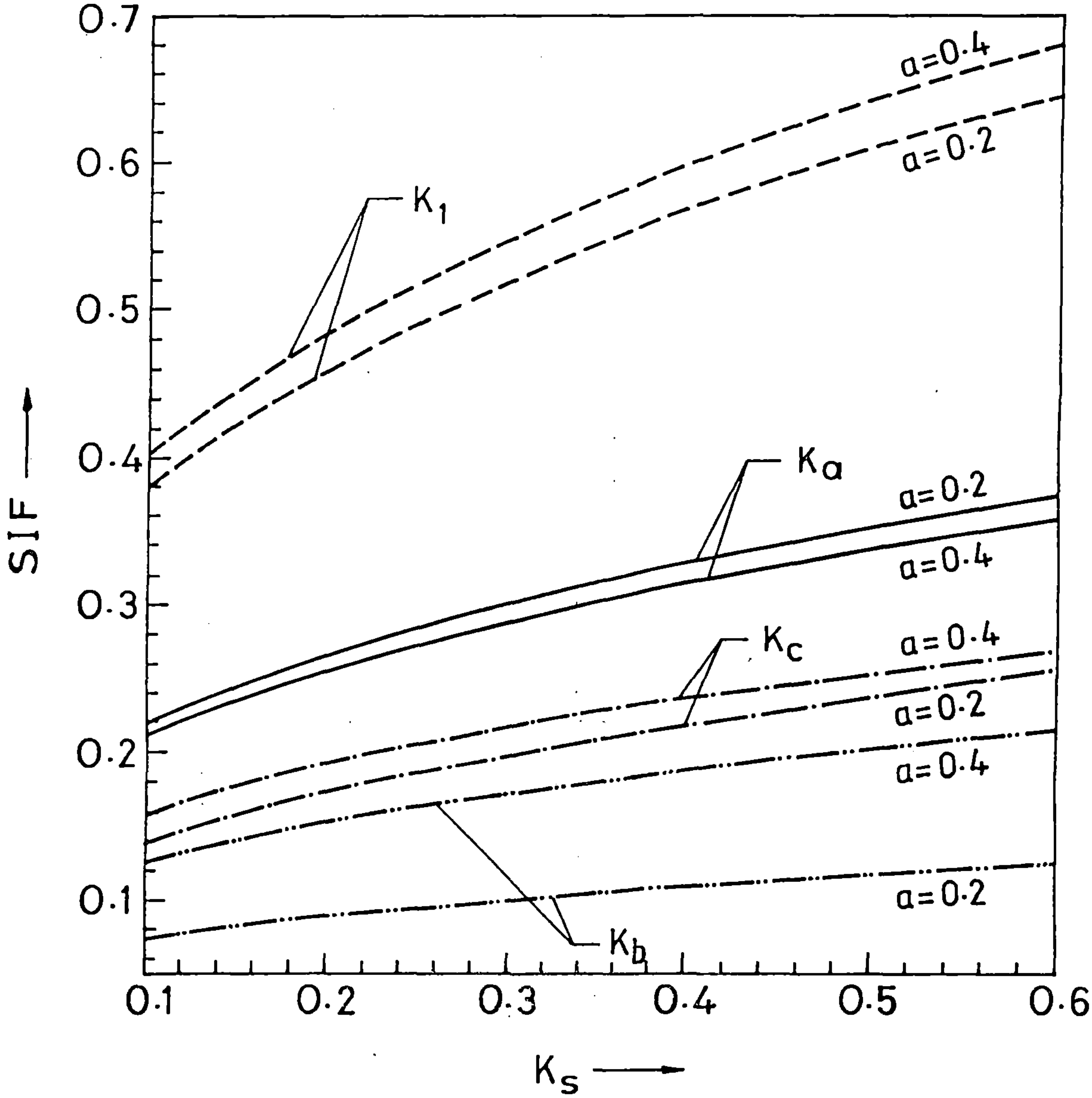

Fig. 3. Stress intensity factors vs. frequency k_s for generalized plane stress.

(for material of type III).

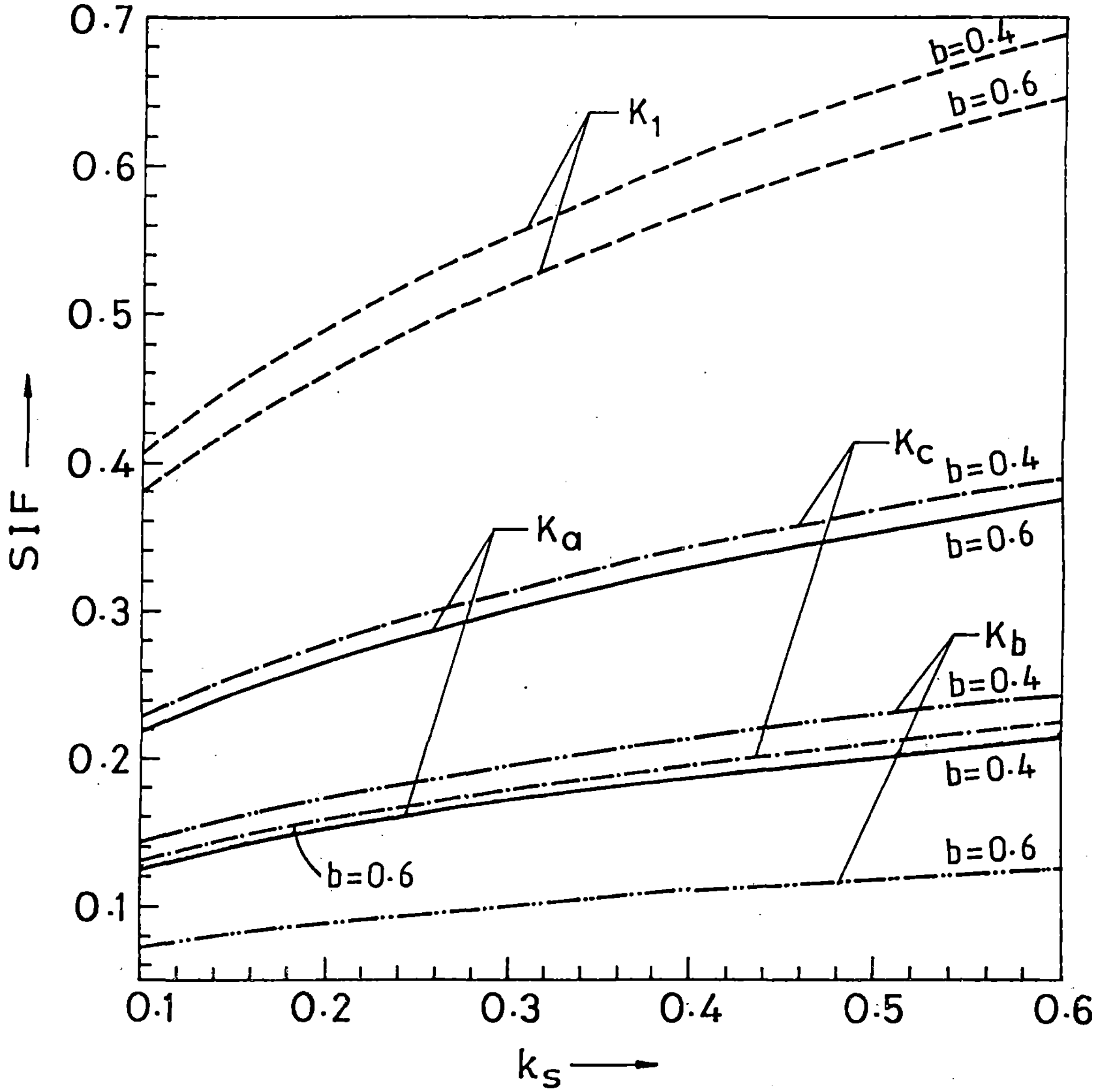

Fig. 4. Stress intensity factors vs. frequency k_s for generalized plane stress.

(for material of type III).

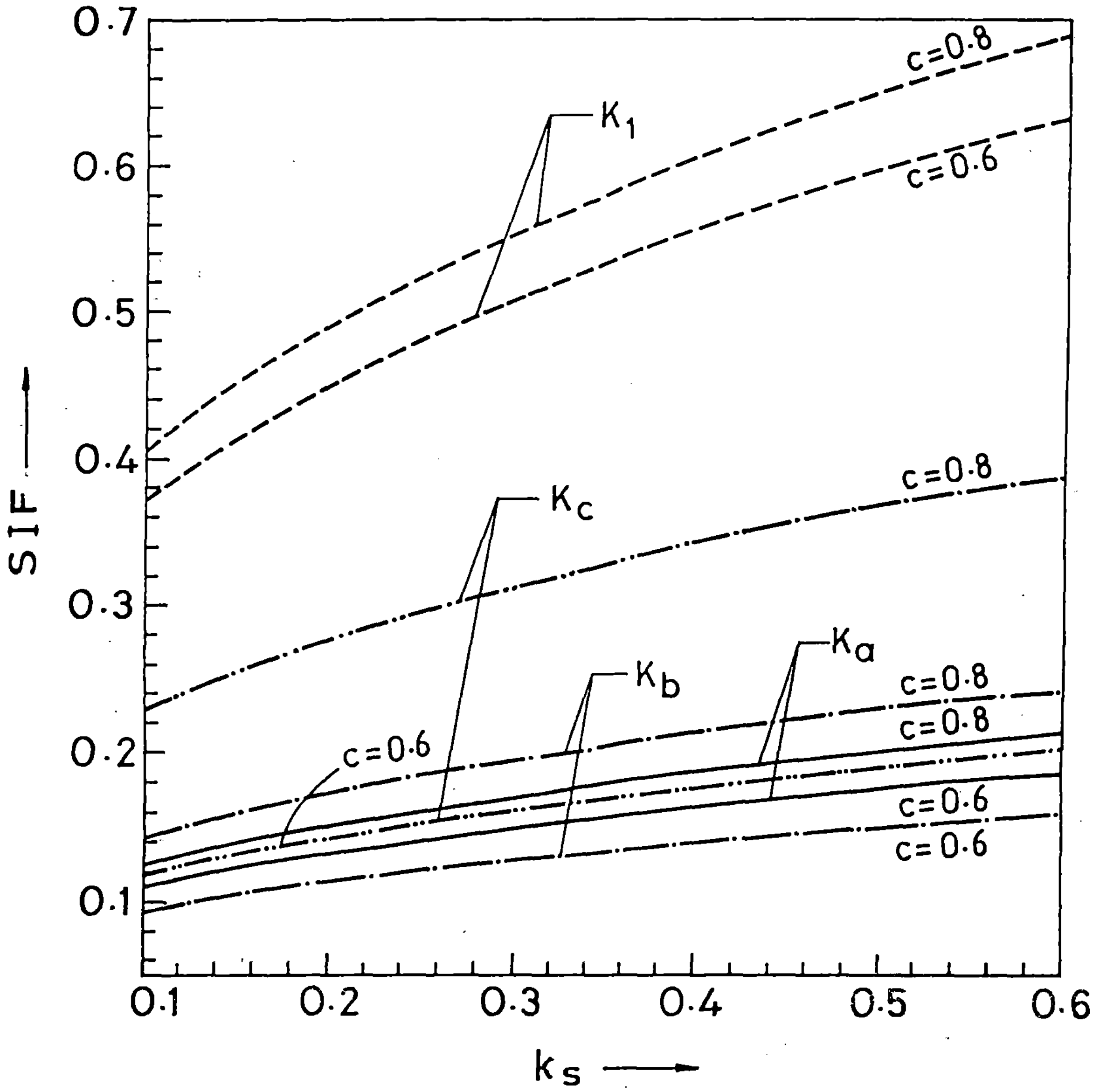

Fig. 5. Stress intensity factors vs. frequency k_s for generalized plane stress. (for material of type III).

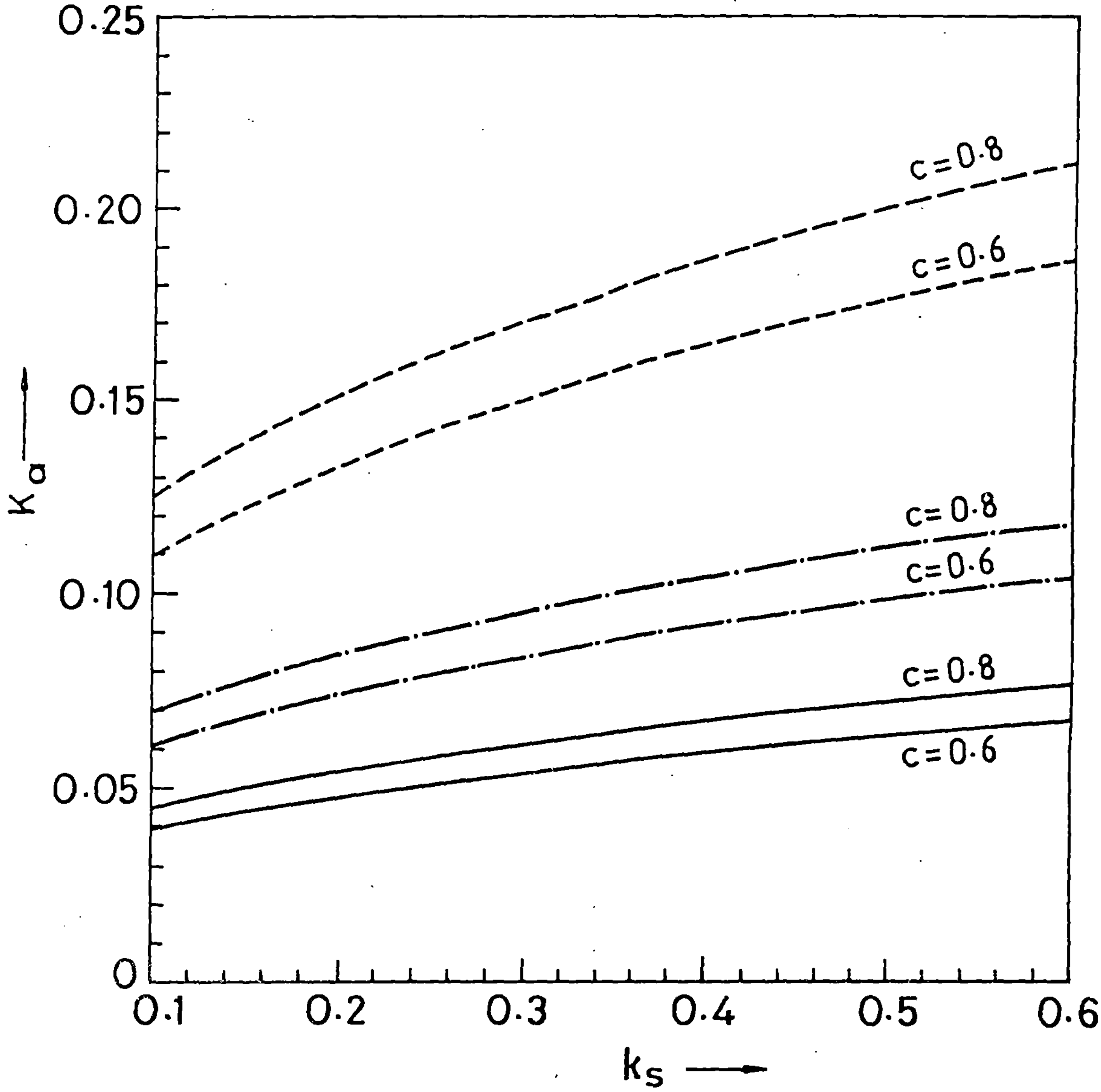

Fig. 6. Stress intensity factor K_a vs. frequency k_s for generalized plane stress.

(——— Type I, -.-.-. Type II, ----- Type III).

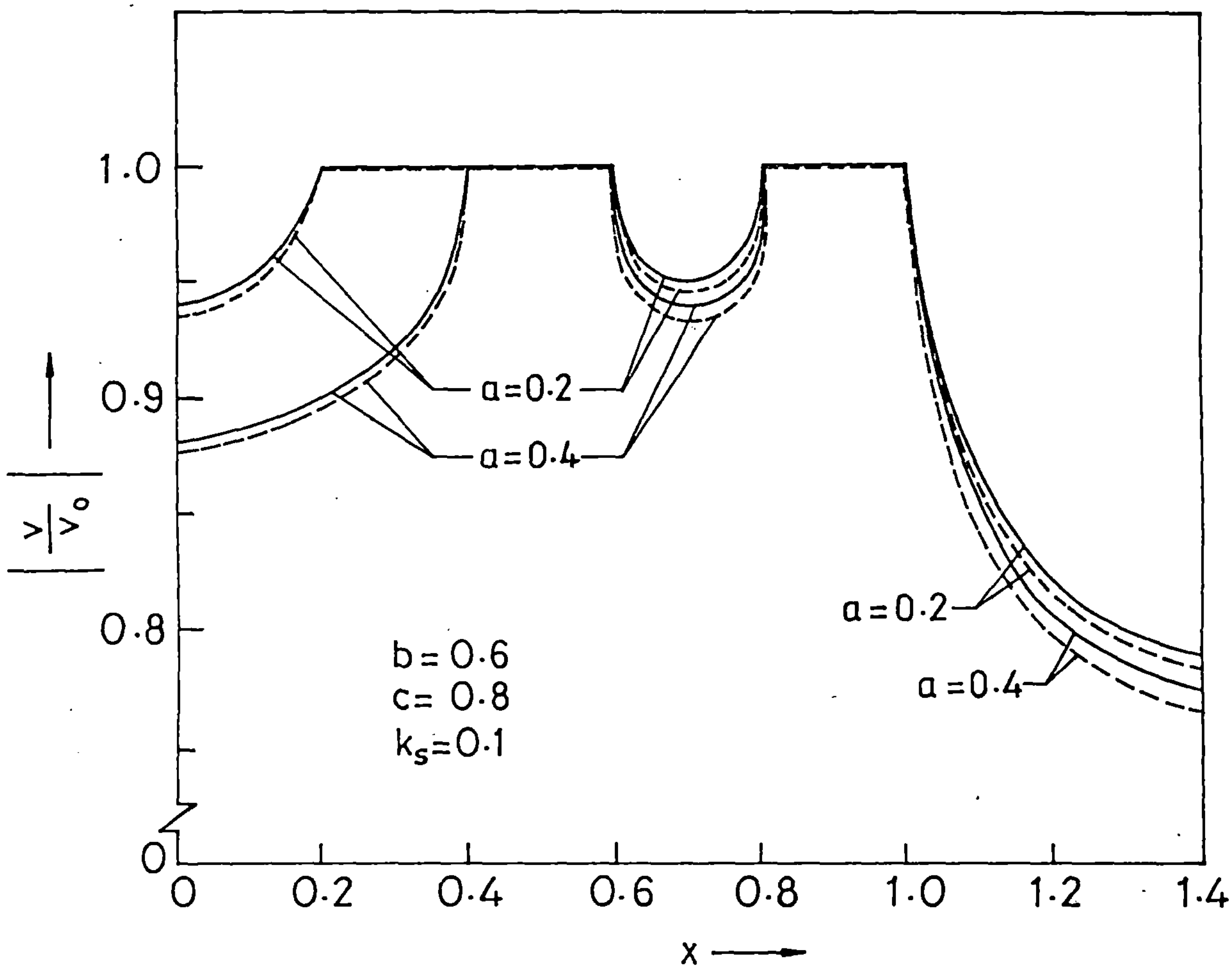

Fig. 7. Vertical displacement $|v/v_0|$ vs. distance x

for generalized plane stress.

(——— Type I, ----- Type II).

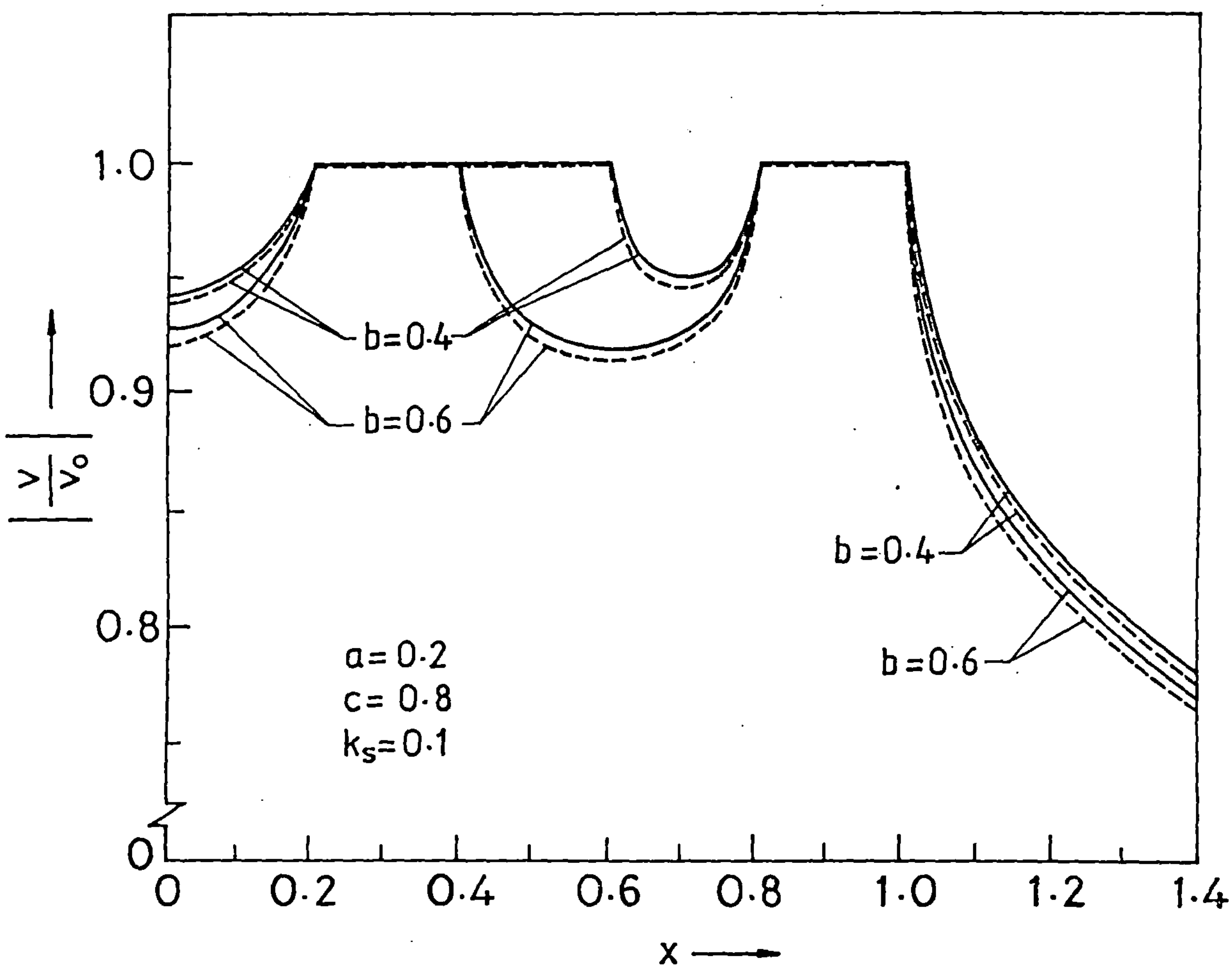

Fig. 8. Vertical displacement $|v/v_0|$ vs. distance x for generalized plane stress.

(——— Type I, ----- Type II).

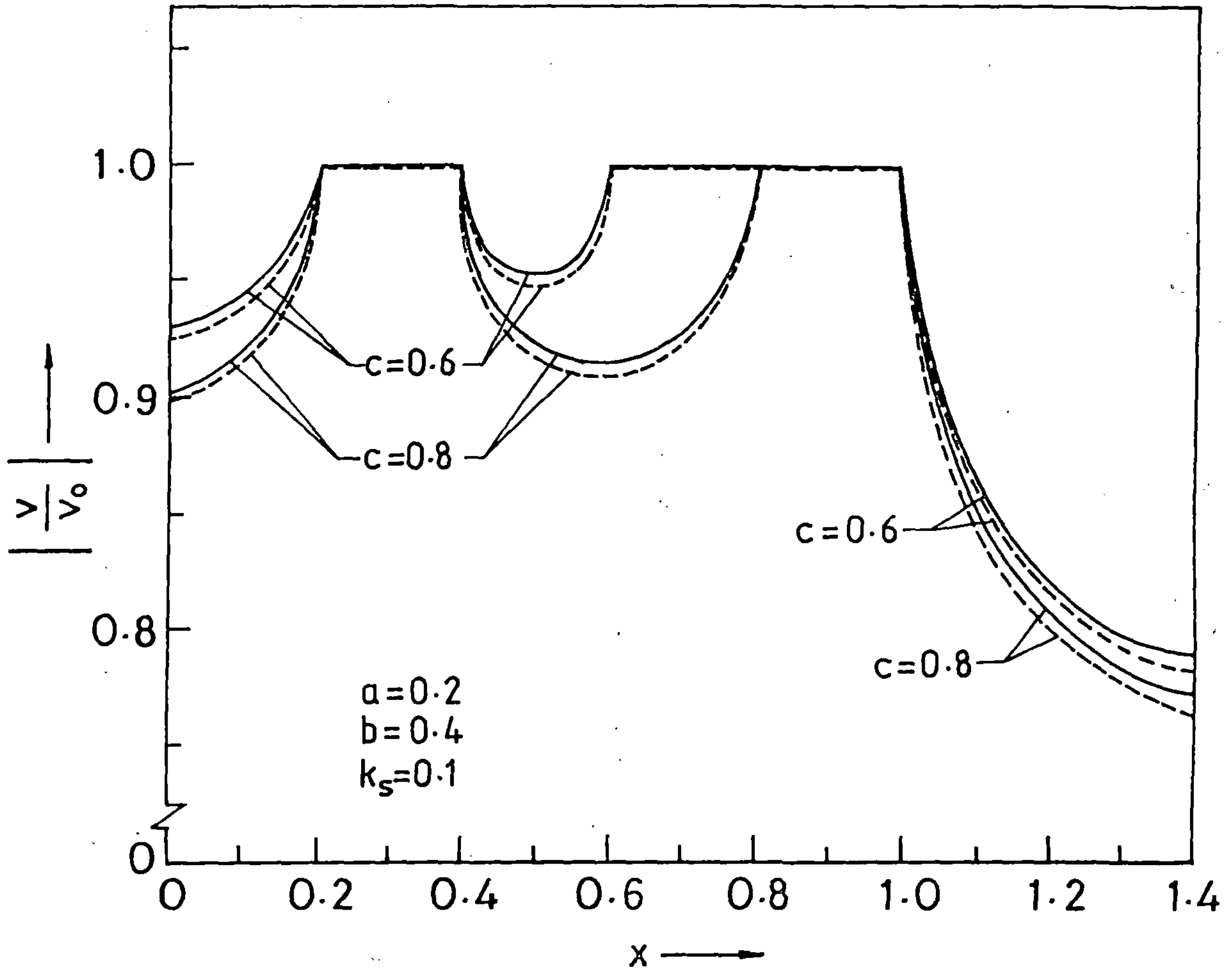

Fig. 9. Vertical displacement $|v/v_0|$ vs. distance x for generalized plane stress.

(——— Type I, ----- Type II).