CHAPTER - I

RING SOURCE PROBLEMS

SPECTRAL REPRESENTATION OF A CERTAIN CLASS OF SELF-ADJOINT DIFFERENTIAL OPERATORS AND ITS APPLICATION TO AXISYMMETRIC BOUNDARY VALUE PROBLEMS IN ELASTODYNAMICS

1. INTRODUCTION

In this work an inategral representation of the Dirac delta function required for solving the axisymmetric boundary value problem has been derived first. This representation is particularly suitable for problems where mixed boundary conditions are encountered. Following Friedmann [1966], by contour integration of a suitable Green's function, integral representation of $\delta(R - R_0)$ $(R,R_0 > 1)$ has been derived. This representation has been used to solve a particular type of axisymmetric problem in elastodynamics.

The problem treated is that of a semi-infinite elastic body containing a circular cylindrical cavity, whose axis is perpendicular to the plane surface. The semi-infinite medium is subjected to an axisymmetric concentric torque applied dynamically as a step function in time at the plane surface.

At first Lamb [1904] investigated the classical normal loading problem of an elastic half-space. Similar type of problem was

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investigated by Eason [1964], Mitra [1964], Chakraborty and De [1971] and many others. They are all point source problems in a homogeneous semi-infinite medium.

The propagation of elastic waves, due to applied boundary tractions, in semi-infinite media containing internal boundaries has as yet not been studied to any large extent.

An earlier and comprehensive survey of the field is given by Scott and Miklowitz [1964]. Recently this type of work has been done by Johnson and Parnes [1977].

We have solved the problem of the SH-type of elastic wave propagation in the semi-infinite medium due to a ring source producing SH-waves in the presence of a circular cylindrical cavity (case 1). The problem of SH-wave propagation in the presence of rigid circular cylindrical inclusion in the semi-infinite medium due to the ring source has also been treated in the case 2.

2. INTEGRAL REPRESENTATION OF A DIRAC DELTA FUNCTION

Consider the operator L with λ as a complex parameter, where

$$L \equiv \frac{d}{dr}\left[r\frac{d}{dr} \right] + \lambda r - \frac{1}{r} \tag{1}$$

whose domain, D, is the set of all twice-differentiable functions $u(r)$, $a < r < \infty$ such that

(i) $r \frac{du}{dr} - u = 0$ at $r = a > 0$

(ii) the behaviour of u as $r \longrightarrow \infty$ is that of an outgoing wave.

The solutions of $LG_1 = 0$ which satisfy (i) are

$$G_1 = A_1\left[J_1\left(\sqrt{\lambda}r\right)Y_2\left(\sqrt{\lambda}a\right) - Y_1\left(\sqrt{\lambda}r\right)J_2\left(\sqrt{\lambda}a\right)\right], \quad a<r<r_0, \qquad (2)$$

Where A_1 is an arbitrary constant and J_n and Y_n are the Bessel functions of the first and second kind, respectively.

Again the function G_2 which will satisfy $LG_2 = 0$ and the condition (ii) can be written as

$$G_2 = A_2 H_1^{(1)}\left(\sqrt{\lambda}r\right) \qquad (a < r_0 < r < \infty), \qquad (3)$$

where A_2 is an arbitrary constant and $H_n^{(1)}$ is the Hankel function of the first kind of order n.

From Eqs. (2) and (3) the Green's function G satisfying the equation $LG = -\delta(r - r_0)$ and the conditions (i) and (ii) mentioned above is given by (e.f. Friedmann [1966])

$$G(r,r_0;\lambda) =$$

$$= -\frac{\pi H_1^{(1)}(\sqrt{\lambda}r_0)}{2H_2^{(1)}(\sqrt{\lambda}a)}\left[J_1(\sqrt{\lambda}r)Y_2(\sqrt{\lambda}a) - Y_1(\sqrt{\lambda}r)J_2(\sqrt{\lambda}a)\right]H(r_0-r) -$$

$$-\frac{\pi H_1^{(1)}(\sqrt{\lambda}r)}{2H_2^{(1)}(\sqrt{\lambda}a)}\left[J_1(\sqrt{\lambda}r_0)Y_2(\sqrt{\lambda}a) - Y_1(\sqrt{\lambda}r_0)J_2(\sqrt{\lambda}a)\right]H(r-r_0),$$

$$0 < \arg\lambda < 2\pi \quad . \tag{4}$$

Now consider

$$\frac{1}{2\pi i}\oint G(r,r_0;\lambda)r\,d\lambda, \tag{5}$$

where the contour of integration in the λ-plane is shown in Fig. 1. Since G has a branch point at $\lambda = 0$, we introduce a branch cut in the complex λ-plane along the positive real axis and then take the contour as a large circle of radius R_1^2, having the centre at $\lambda = 0$, not crossing the branch cut. In terms of Hankel functions Eq. (4) can be written as

$$\frac{\pi}{4i}\left[H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\ \frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)} - H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(2)}(\sqrt{\lambda}r)\right]H(r_0-r) \quad +$$

$$+\frac{\pi}{4i}\left[H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\ \frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)} - H_1^{(1)}(\sqrt{\lambda}r)H_1^{(2)}(\sqrt{\lambda}r_0)\right]H(r-r_0).$$

(6)

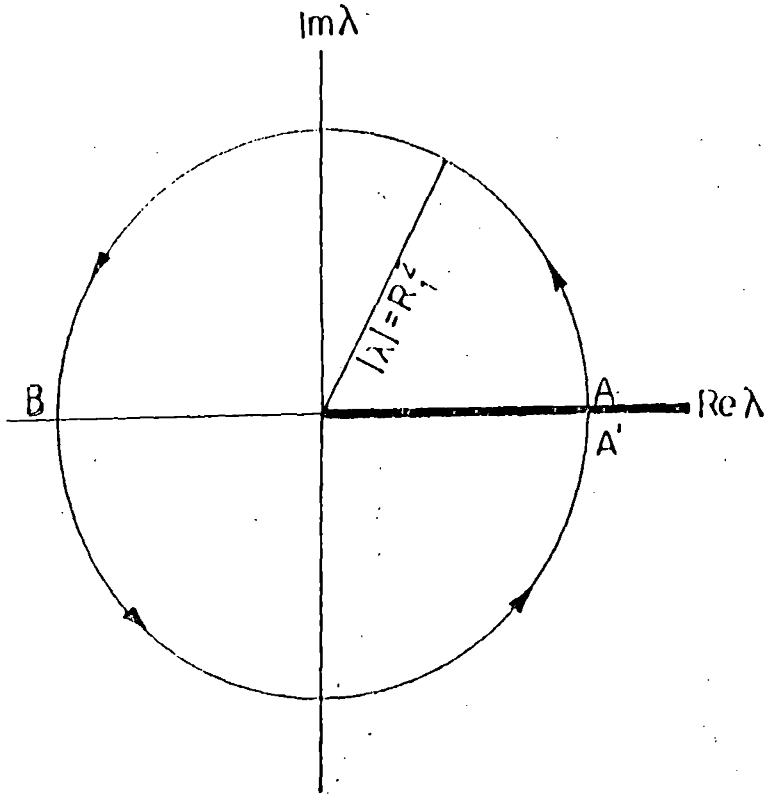

FIG. 1. Circular contour of integration ABA' in the λ-plane.

For large $|z|$, the asymptotic behaviour of $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ are (Lebedev [1965])

$$H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\ \exp\left[\ i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right],$$

$$H_n^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\ \exp\left[\ -i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]. \tag{7}$$

Thus, for large values of $|\lambda|$, from the relations (7) we obtain

$$H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(1)}(\sqrt{\lambda}r)\frac{H_2^{(2)}(\sqrt{\lambda}a)}{H_2^{(1)}(\sqrt{\lambda}a)} \sim \frac{2}{\pi\sqrt{\lambda r r_0}}\ \exp\left[i\sqrt{\lambda}(r + r_0 - 2a) + i\pi\right],$$

$$H_1^{(1)}(\sqrt{\lambda}r_0)H_1^{(2)}(\sqrt{\lambda}r) \sim \frac{2}{\pi\sqrt{\lambda r r_0}}\ \exp\left[\ i\sqrt{\lambda}\ (r_0 - r)\right], \tag{8}$$

$$H_1^{(1)}(\sqrt{\lambda}r)H_1^{(2)}(\sqrt{\lambda}r_0) \sim \frac{2}{\pi\sqrt{\lambda r r_0}}\ \exp\left[\ i\sqrt{\lambda}\ (r - r_0)\right].$$

If we put $\lambda = k^2$, then the circle in the λ-plane becomes a semi-circular arc C of radius R_1 in the upper half of the k-plane (shown in Fig.2.) Consequently, for large values of R_1 the integral

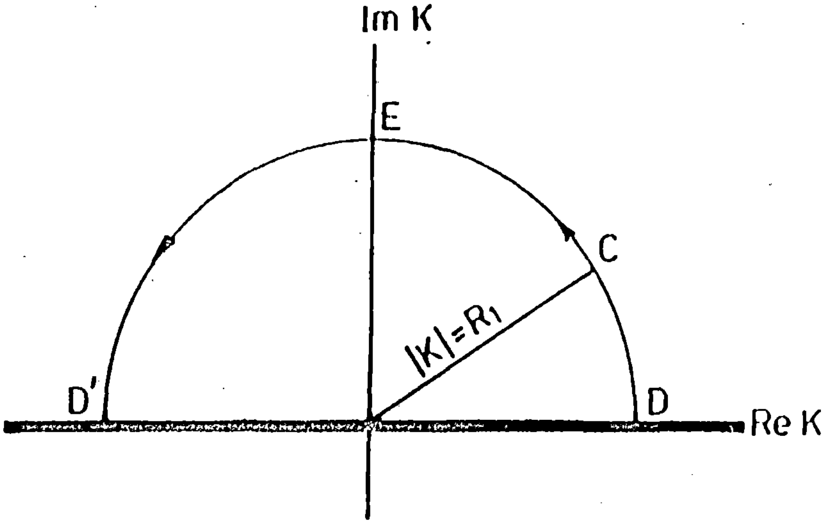

Fig. 2. DED′ - the semi-circular path of integration C in the K-plane.

(5) can be written as

$$\frac{1}{2\pi}\sqrt{\frac{r}{r_o}}\int_C\Big[\exp\{ik(r_o-r)\}H(r_o-r)+\exp\{ik(r-r_o)\}H(r-r_o)\Big]dk-$$

$$-\frac{1}{2\pi}\int_C\sqrt{\frac{r}{r_o}}\ \exp\{ik(r+r_o-2a)\}dk$$

$$=-\frac{1}{2\pi}\sqrt{\frac{r}{r_o}}\int_{-R_1}^{R_1}\exp(ik|r-r_o|)dk+$$

$$+\frac{1}{2\pi}\sqrt{\frac{r}{r_o}}\int_{-R_1}^{R_1}\exp\{ik(r+r_o-2a)\}dk$$

$$=-\frac{1}{\pi}\sqrt{\frac{r}{r_o}}\ \frac{\sin R_1(r-r_o)}{r-r_o}+\frac{1}{\pi}\sqrt{\frac{r}{r_o}}\ \frac{\sin R_1(r+r_o-2a)}{r+r_o-2a}\ . \quad (9)$$

Our object is to show that the integral (5) represents $-\delta(r-r_o)$ when $R_1\to\infty$. To justify the statement, consider a testing function $\phi(r)$, in D which is continuous, has a continuous derivative of order two and vanishes outside a finite interval. Then, from the relations (5) and (9)

$$\lim_{R_1 \to \infty} \int_a^\infty \phi(r) \frac{1}{2\pi i} \oint G(r, r_0; \lambda) r \, d\lambda \, dr$$

$$= - \lim_{R_1 \to \infty} \frac{1}{\pi} \int_a^\infty \phi(r) \sqrt{\frac{r}{r_0}} \frac{\sin R_1 (r - r_0) dr}{(r - r_0)} +$$

$$+ \lim_{R_1 \to \infty} \frac{1}{\pi} \int_a^\infty \phi(r) \sqrt{\frac{r}{r_0}} \frac{\sin R_1 (r + r_0 - 2a) dr}{(r + r_0 - 2a)}$$

$$= - \phi(r_0),$$

where we have used the result of Dirichlet integral and Riemann-Lebesgue Lemma (Whittaker and Watson [1963]).

Therefore

$$\lim_{R_1 \to \infty} \frac{1}{2\pi i} \oint G(r, r_0; \lambda) r \, d\lambda = - \delta(r - r_0).$$

To obtain an alternative integral representation, which will be useful for our subsequent application in physical problems, we consider the contour Γ (Fig.3) consisting of the real axis from $k = \rho$ to $k = R_1$, where $0 < \rho < R_1$; a semi-circle C of radius R_1 above the real axis; the real axis again from $- R_1$ to $- \rho$; and finally a semi-circle γ of radius ρ above the real axis with the centre at the origin. We take ρ small and R_1 large.

The integrand $2G(r, r_0, k^2)\, kr$ has no singularity inside the contour Γ , and so the value of the integral

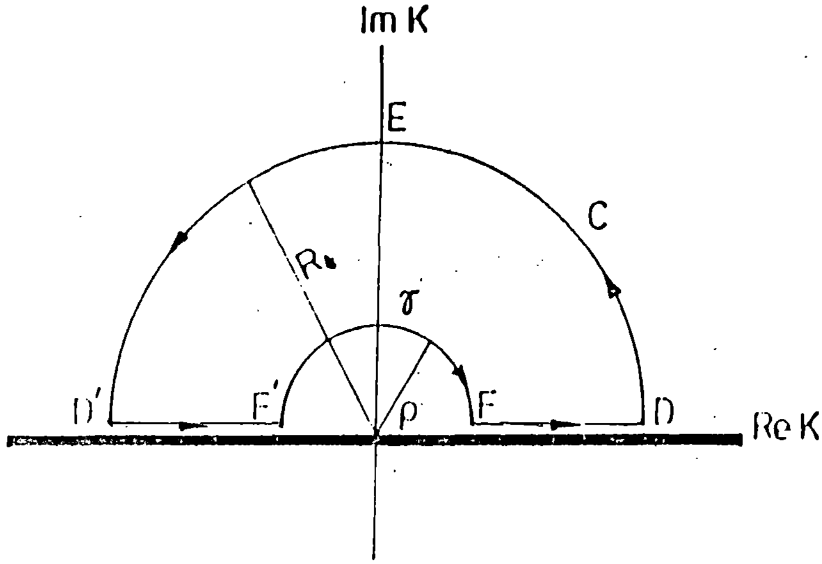

Fig. 3. FDED'F'F- thr path of integration Γ in the K-plane.

$$\frac{1}{2\pi i}\int_{\Gamma} G(r,r_o;k^2)2krdk = 0,$$

i.e.

$$\frac{1}{2\pi i}\int_{C} G(r,r_o;k^2)2krdk = -\frac{1}{2\pi i}\int_{\rho}^{R_1} G(r,r_o;u^2)2urdu +$$

$$+\frac{1}{2\pi i}\int_{\rho}^{R_1} G(r,r_o;e^{2\pi i}u^2)2rudu -$$

$$-\frac{1}{2\pi}\int_{0}^{\pi} G(r,r_o;\rho^2 e^{2i\theta})2r\rho^2 e^{2i\theta}d\theta. \qquad (10)$$

The behaviour of $Y_n(z)$ for small values of $|z|$ is described by the formula (Lebedev [1965])

$$Y_n(z) \sim -\frac{2^n\Gamma(n)}{\pi z^n}$$

and $J_n(z)$ is bounded for small values of $|z|$ when n is a positive integer. Using these results we conclude

$$|\ G(r,r_o;\rho^2 e^{2i\theta})\rho\ |$$

is bounded for small values of ρ. Hence

$$\lim_{\rho\to 0}\ \frac{1}{\pi}\int_{0}^{\pi} G(r,r_o;\rho^2 e^{2i\theta})e^{2i\theta}\rho^2 rd\theta = 0.$$

Letting $\rho \to 0$ and $R_1 \to \infty$ in (10), we get

$$\delta(r - r_0) = -\lim_{R_1 \to \infty} \frac{1}{2\pi i} \int_c G(r, r_0; k^2) 2krdk$$

$$= \frac{1}{2\pi i} \int_0^\infty \left[G(r, r_0; k^2) - G(r, r_0; k^2 e^{2i\pi}) \right] 2krdk. \quad (11)$$

From Eq. (4)

$$G(r, r_0; k^2) - G(r, r_0; k^2 e^{2i\pi}) =$$

$$= -\frac{\pi}{2}\left[\frac{J_1(kr_0)+iY_1(kr_0)}{J_2(ka)+iY_2(ka)} - \frac{J_1(kr_0)-iY_1(kr_0)}{J_2(ka) - iY_2(ka)} \right] \times$$

$$\times \left[J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)\right] H(r_0-r) -$$

$$- \frac{\pi}{2} \left[\frac{J_1(kr)+iY_1(kr)}{J_2(ka)+iY_2(ka)} - \frac{J_1(kr)-iY_1(kr)}{J_2(ka)-iY_2(ka)} \right] \times$$

$$\times \left[J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)\right]H(r - r_0)$$

$$= i\pi \frac{\left[J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)\right]\left[J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)\right]}{J_2^2(ka)+Y_2^2(ka)}$$

Substituting this expression in Eq. (11), we get

$$\delta(r-r_0) =$$

$$= \int_0^\infty \frac{\left[J_1(kr_0)Y_2(ka)-Y_1(kr_0)J_2(ka)\right]\left[J_1(kr)Y_2(ka)-Y_1(kr)J_2(ka)\right]}{J_2^2(ka) + Y_2^2(ka)} rkdk \tag{12}$$

Substituting $r/a = R$, $r_0/a = R_0$ and $ka = \gamma$, Eq.(12) can be written as

$$\delta(R-R_0) =$$

$$= \int_0^\infty \frac{\left[J_1(\gamma R_0)Y_2(\gamma)-Y_1(\gamma R_0)J_2(\gamma)\right]\left[J_1(\gamma R)Y_2(\gamma)-Y_1(\gamma R)J_2(\gamma)\right]}{J_2^2(\gamma) + Y_2^2(\gamma)} R\gamma d\gamma \tag{13}$$

Since $\delta(R-R_0)$ is symmetric with respect to R and R_0, then, on the right hand side of Eq. (13), R and R_0 can be interchanged. So we write

$$\delta(R-R_0) =$$

$$= R_0 \int_0^\infty \frac{\gamma\left[J_1(\gamma R_0)Y_2(\gamma)-Y_1(\gamma R_0)J_2(\gamma)\right]\left[J_1(\gamma R)Y_2(\gamma)-Y_1(\gamma R)J_2(\gamma)\right]}{J_2^2(\gamma) + Y_2^2(\gamma)} d\gamma . \tag{14}$$

3. FORMULATION AND GENERAL SOLUTION (CASE - 1)

Case 1. We shall now use the integral representation of the delta function given by Eq. (13) to derive the time dependent response of an isotropic linearly elastic half-space containing a cylindrical cavity of radius a due to a ring source. The axis of the cylider considered as the z-axis, which is perpendicular to the plane surface, is directed downwards (Fig.4). A torque is applied on the free surface of the half-space over the rim of a concentric circle of radius $r = r_o$ ($r_o > a$) for $t \geq 0$. Therefore on the cavity surface r = a

$$\tau_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) = 0 \qquad (15)$$

and on the plane surface z = 0

$$\tau_{\theta z} = \mu \frac{\partial u_\theta}{\partial z} = \delta(r - r_o)H(t) \qquad (a<r<\infty,\ r_o>a), \qquad (16)$$

where μ is Lame's constant, δ is the Dirac delta function and H is the unit step function.

Now the only non-zero equation of motion is

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} = \frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial t^2}, \qquad (17)$$

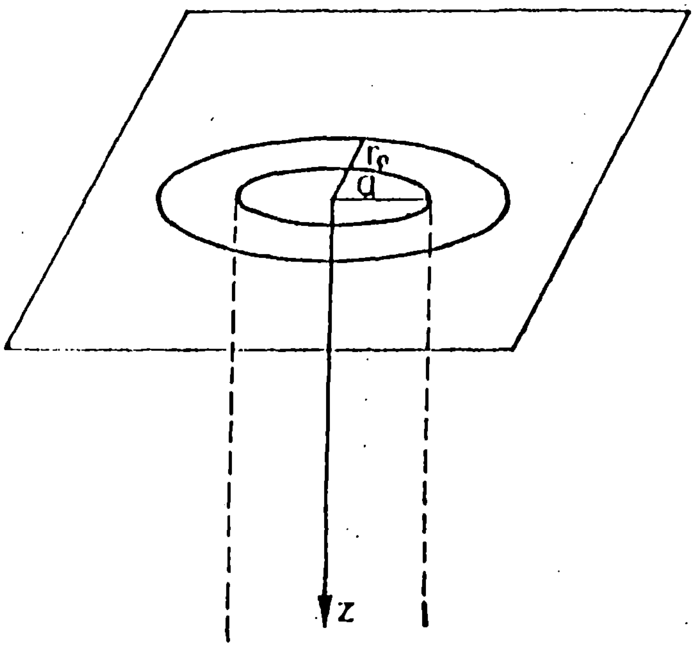

Fig. 4. Geometry of the problem.

where $\beta = \sqrt{\mu/\rho}$ is the shear wave velocity.

Changing the independent variables (r,z,t) to the no-dimensional variables (R,Z,τ) defined by

$$R = \frac{r}{a}\ ,\quad Z = \frac{z}{a}\ ,\quad \tau = \frac{\beta t}{a}\ ,\quad R_o = \frac{r_o}{a} \tag{18}$$

the above equation reduces to

$$\frac{\partial^2 u_\theta}{\partial R^2} + \frac{1}{R}\frac{\partial u_\theta}{\partial R} + \frac{\partial^2 u_\theta}{\partial Z^2} - \frac{u_\theta}{R^2} = \frac{\partial^2 u_\theta}{\partial \tau^2} \tag{19}$$

and boundary conditions become

$$\tau_{r\theta} = \frac{\mu}{a}\left[\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right] = 0 \quad \text{on } R = 1 \tag{20}$$

and

$$\tau_{\theta z} = \frac{\mu}{a}\frac{\partial u_\theta}{\partial z} = \frac{1}{a}\,\delta(R - R_o)H(t) \quad \text{on} \quad Z = 0. \tag{21}$$

Now, taking the Laplace transform with respect to nondimensional time (τ) and assuming the homogeneous initial conditions

$$u_\theta(R,Z,0) = \frac{\partial u_\theta(R,Z,0)}{\partial t} = 0 \quad \text{at} \quad t = 0$$

Eq. (19) takes the form

$$\frac{\partial^2 \tilde{u}_\theta}{\partial R^2} + \frac{1}{R}\frac{\partial \tilde{u}_\theta}{\partial R} + \frac{\partial^2 \tilde{u}_\theta}{\partial Z^2} - \frac{\tilde{u}_\theta}{R^2} = s^2 \tilde{u}_\theta , \tag{22}$$

where
$$\tilde{u}_\theta = \int_0^\infty u_\theta e^{-s\tau} d\tau . \tag{23}$$

Take solution of Eq. (22) in the form

$$\tilde{u}_\theta(R,Z,s) = \int_0^\infty \left[A_1(\gamma)J_1(\gamma R)+B_1(\gamma)Y_1(\gamma R) \right] e^{-\sqrt{s^2+\gamma^2}\, Z} d\gamma , \tag{24}$$

where γ is real, J_1 and Y_1 are Bessel functions of the first and second kind respectively.

Using the boundary condition (20), we obtain

$$B_1(\gamma) = - A_1(\gamma) \frac{J_2(\gamma)}{Y_2(\gamma)} . \tag{25}$$

Substituting the value of $B_1(\gamma)$ in Eq. (24), we have

$$\tilde{u}_\theta(R,Z,s) = \int_0^\infty A(\gamma)\left[J_1(\gamma R)Y_2(\gamma)-J_2(\gamma)Y_1(\gamma R)\right]e^{-\sqrt{s^2+\gamma^2}\, Z} d\gamma , \tag{26}$$

where
$$A(\gamma) = \frac{A_1(\gamma)}{Y_2(\gamma)} . \tag{27}$$

Therefore the transformed stress component reduces to

$$\tilde{\tau}_{\theta Z} = \frac{\mu}{a} \int_0^{\infty} A(\gamma)\ (\gamma^2+s^2)^{1/2}\ C_2(\gamma R)\ e^{-\sqrt{\gamma^2+s^2}\ Z}\ d\gamma , \qquad (28)$$

where $\quad C_2(\gamma R) = J_2(\gamma)\ Y_1(\gamma R) - Y_2(\gamma)\ J_1(\gamma R). \qquad (29)$

Now, using the representation (29), Eq. (14) becomes

$$\delta(R-R_0) = R_0 \int_0^{\infty} \frac{\gamma C_2(\gamma R)\ C_2(\gamma R_0)}{J_2^2(\gamma) + Y_2^2(\gamma)}\ d\gamma . \qquad (30)$$

Using Eqs. (21), (28) and (30), the value of $A(\gamma)$ is obtained as

$$A(\gamma) = \frac{R_0}{\mu s} \frac{\gamma C_2(\gamma R_0)}{(s^2+\gamma^2)^{1/2}\ \{J_2^2(\gamma) + Y_2^2(\gamma)\}} . \qquad (31)$$

Therefore $\tilde{u}_\theta$ becomes

$$\tilde{u}_\theta(R,Z,s) = - \frac{R_0}{\mu s} \int_0^{\infty} \frac{\gamma C_2(\gamma R)\ C_2(\gamma R_0)}{(\gamma^2+s^2)^{1/2}\ \{J_2^2(\gamma) + Y_2^2(\gamma)\}}\ e^{-\sqrt{\gamma^2+s^2}\ Z} d\gamma . \qquad (32)$$

On the plane boundary Z = 0

$$\tilde{u}_\theta(R,0,s) = - \frac{R_0}{\mu s} \int_0^{\infty} \frac{\gamma C_2(\gamma R)\ C_2(\gamma R_0)}{(\gamma^2+s^2)^{1/2}\ \{J_2^2(\gamma) + Y_2^2(\gamma)\}}\ d\gamma . \qquad (33)$$

Now, introducing the change of the variable $\gamma = s\zeta$ into the above expression (33), we obtain

$$\tilde{u}_{\theta}(R,0,s) = -\frac{R_o}{\mu}\int_o^{\infty}\frac{\zeta C_2(s\zeta R)\ C_2(s\zeta R_o)}{(\zeta^2+1)^{1/2}\ \{J_2^2(s\zeta) + Y_2^2(s\zeta)\}}\,d\zeta . \qquad (34)$$

Next, using

$$J_n(s\zeta R) = \frac{H_n^{(1)}(s\zeta R) + H_n^{(2)}(s\zeta R)}{2} \qquad (35)$$

and

$$Y_n(s\zeta R) = \frac{H_n^{(1)}(s\zeta R) - H_n^{(2)}(s\zeta R)}{2i}, \qquad (35')$$

we obtain

$$C_2(s\zeta R) = J_2(s\zeta)Y_1(s\zeta R) - Y_2(s\zeta)J_1(s\zeta R)$$

$$= \frac{1}{2i}\left[H_1^{(1)}(s\zeta R)\ H_2^{(2)}(s\zeta) - H_1^{(2)}(s\zeta R)\ H_2^{(1)}(s\zeta)\right] \qquad (36)$$

and

$$C_2(s\zeta R_o) = \frac{1}{2i}\left[H_1^{(1)}(s\zeta R_o)\ H_2^{(2)}(s\zeta) - H_1^{(2)}(s\zeta R_o)\ H_2^{(1)}(s\zeta)\right]. \qquad (36')$$

Also

$$J_2^2(s\zeta) + Y_2^2(s\zeta) = H_2^{(1)}(s\zeta)\ H_2^{(2)}(s\zeta). \qquad (36'')$$

Therefore, Eq.(34) becomes

$$\tilde{u}_{\theta}(R,0,s) = -\frac{R_o}{4\mu}\int_o^{\infty}\frac{\zeta}{\sqrt{(\zeta^2+1)}}\,F(R,R_o,s\zeta)\,d\zeta , \qquad (37)$$

where

$$F(R,R_0,s\zeta) = F_1(R,R_0,s\zeta) + F_2(R,R_0,s\zeta)$$

$$= F_1(R_0,R,s\zeta) + F_2(R_0,R.s\zeta)$$

$$= F(R_0,R,s\zeta) \qquad (38)$$

and

$$F_1(\alpha,\beta,s\zeta) = H_1^{(2)}(s\zeta\beta)\left\{H_1^{(1)}(s\zeta\alpha) - H_1^{(2)}(s\zeta\alpha)\,\frac{H_2^{(1)}(s\zeta)}{H_2^{(2)}(s\zeta)}\right\}, \qquad (38')$$

$$F_2(\alpha,\beta,s\zeta) = H_1^{(1)}(s\zeta\beta)\left\{H_1^{(2)}(s\zeta\alpha) - H_1^{(1)}(s\zeta\alpha)\,\frac{H_2^{(2)}(s\zeta)}{H_2^{(1)}(s\zeta)}\right\}. \qquad (38'')$$

Using the asymptotic values of the Hankel functions for a large argument, it can be shown that

$$\frac{\zeta\; F_1(R,R_0,s\zeta)}{\sqrt{(\zeta^2+1)}} \longrightarrow \frac{2}{\pi s\zeta\sqrt{RR_0}}\left[e^{-is\zeta(R_0-R)} + e^{-is\zeta(R+R_0-2)}\right] \qquad (39)$$

as $|s\zeta| \longrightarrow \infty$, showing that $\dfrac{\zeta\; F_1(R,R_0,s\zeta)}{\sqrt{(\zeta^2+1)}}$ vanishes over a large circular arc in the forth quadrant of the complex ζ-plane for $R < R_0$.

Also

$$\frac{\zeta F_2(R,R_0,s\zeta)}{\sqrt{(\zeta^2+1)}} \longrightarrow \frac{2}{\pi s\zeta \sqrt{RR_0}} \left[e^{is\zeta(R_0-R)} + e^{is\zeta(R+R_0-2)} \right] \quad (39')$$

showing that $\frac{\zeta F_2(R,R_0,s\zeta)}{\sqrt{(\zeta^2+1)}}$ vanishes over a large circular arc in the first quadrant of the complex ζ-plane for $R<R_0$. Therefore, for $R > R_0$,

$$\frac{\zeta F_2(R_0,R,s\zeta)}{\sqrt{(\zeta^2+1)}} \quad \text{and} \quad \frac{\zeta F_1(R_0,R,s\zeta)}{\sqrt{(\zeta^2+1)}}$$

vanish over large circular arcs in the first and fourth quadrants, respectively, of the complex ζ-plane.

Denoting the responses for field points inside ($R < R_0$) and outside ($R > R_0$) the source by the subscripts I and O respectively, we have for points inside the source ($R < R_0$)

$$\tilde{u}_{\theta I}(R,0,s) = -\frac{R_0}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} \left[F_2(R,R_0,s\zeta) + F_1(R,R_0,s\zeta) \right] d\zeta \quad (40)$$

and for points outside the source ($R > R_0$)

$$\tilde{u}_{\theta O}(R,0,s) = -\frac{R_0}{4\mu}\int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} \left[F_2(R_0,R,s\zeta) + F_1(R_0,R,s\zeta) \right] d\zeta . \quad (40')$$

In order to evaluate

$$- \frac{R_0}{4\mu} \int_0^\infty \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_2(R,R_0,s\zeta)\, d\zeta , \tag{41}$$

which is the first part of $\tilde{u}_{\theta I}(R,0,s)$ we note first that the integrand has branch points at $\zeta = \pm i$ and also has a branch point at the origin of coordinates due to the presence of Hankel functions in the integrand. The integrand has also poles which correspond to the zeros of $H_2^{(1)}(s\zeta)$. From Eq. (32) we note that in order that $\tilde{u}_\theta(R,Z,s)$ may be finite for large positive values of Z, $(\zeta^2+1)^{1/2}$ should have a positive real part on the path of integration. Accordingly, we draw cuts parallel to the real axis from $+i$ to $-\infty+i$ and from $-i$ to $\infty-i$ to satisfy our requirement. A cut along the negative real axis from the origin is also drawn to make Hankel functions single valued

$$- \frac{R_0}{4\mu} \frac{\zeta}{\sqrt{(\zeta^2+1)}} F_2(R,R_0,s\zeta)$$

is now integrated along the quadrant of a large circle lying in the first quadrant of the complex ζ-plane as shown in Fig. 5a. Since poles of the integrand are out side the path of integration, the integral (41) becomes

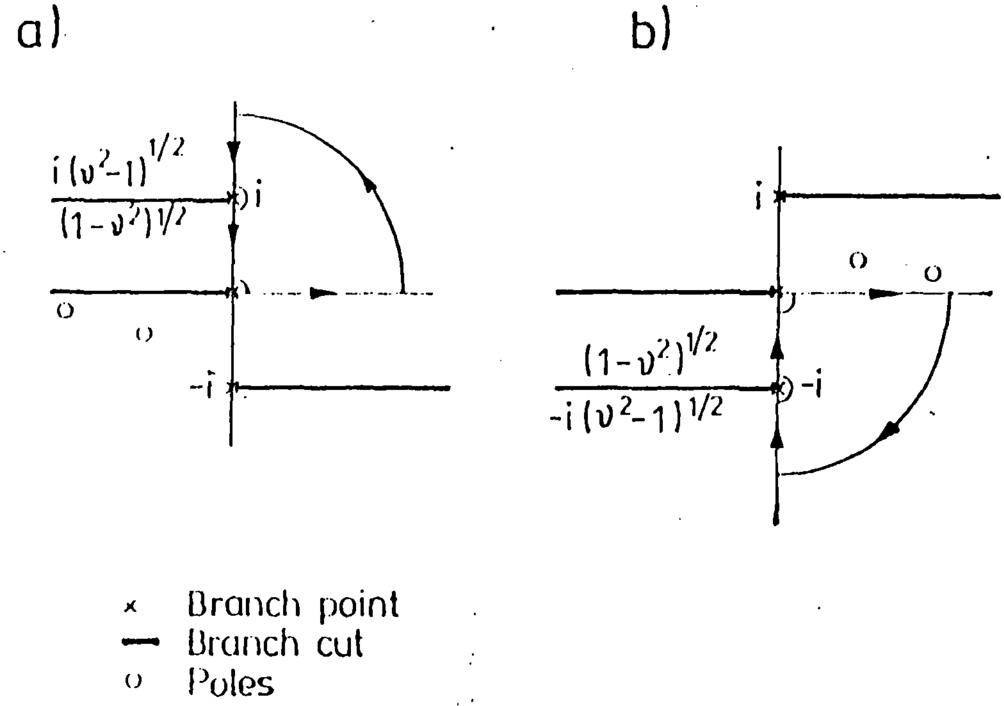

Fig. 5. Integration paths in the complex ζ-plane.

$$\frac{R_0}{4\mu}\left[\int_0^1 \frac{v}{\sqrt{(1-v^2)}} F_2(R,R_0,isv)dv + \right.$$

$$\left. + \int_1^\infty \frac{v}{i\sqrt{(v^2-1)}} F_2(R,R_0,isv)dv \right] . \qquad (42)$$

Using the relations

$$\begin{aligned}
H_1^{(1)}(iv) &= -\frac{2}{\pi} K_1(v), \\
H_1^{(2)}(iv) &= \frac{2}{\pi} K_1(v) + 2iI_1(v), \\
H_2^{(1)}(iv) &= \frac{2i}{\pi} K_2(v), \\
H_2^{(2)}(iv) &= -2I_2(v) - \frac{2i}{\pi} K_2(v),
\end{aligned} \qquad (43)$$

we have

$$F_2(R,R_0,isv) = -\frac{4i}{\pi} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}. \qquad (44)$$

Therfore, the expression (42) becomes

$$-\frac{iR_0}{\mu\pi}\int_0^1 \frac{v}{\sqrt{(1-v^2)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv \; -$$

$$-\frac{R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv. \qquad (45)$$

The second part of $\tilde{u}_{\theta I}(R,0,s)$ is equal to

$$-\frac{R_0}{4\mu}\int_0^{\infty}\frac{\zeta}{\sqrt{(\zeta^2+1)}}F_1(R,R_0,s\zeta)d\zeta \qquad (46)$$

we draw cuts from +i to ∞+i and from -i to $-\infty$-i as shown in Fig. (5b). A cut from the origin along the negative real axis is also drawn to make Hankel functions single valued.

Taking a quadrant of a large circular contour in the fourth quadrant (Fig. (5b)) and noting that the poles of $F_1(R,R_0,s\zeta)$ lie outside the contour, the integral (46) takes the form

$$\frac{R_0}{4\mu}\left[\int_0^{1}\frac{v}{\sqrt{(1-v^2)}}F_1(R,R_0,-isv)dv - \right.$$

$$\left. -\int_1^{\infty}\frac{v}{i\sqrt{(v^2-1)}}F_1(R,R_0,-isv)dv\right]. \qquad (47)$$

Using the relations

$$H_1^{(1)}(-iv) = \frac{2}{\pi}K_1(v) - 2iI_1(v),$$

$$H_1^{(2)}(-iv) = -\frac{2}{\pi}K_1(v),$$

$$H_2^{(1)}(-iv) = -2I_2(v) + \frac{2i}{\pi} K_2(v),$$

$$H_2^{(2)}(-iv) = -\frac{2i}{\pi} K_2(v), \qquad (48)$$

the expression (47) becomes

$$\frac{iR_0}{\mu\pi}\int_0^1 \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{ I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv \; -$$

$$-\frac{R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0)\left\{ I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv. \qquad (49)$$

Adding the relations (45) and (49), we obtain

$$\tilde{u}_{\theta I}(R,0,s) = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR_0) \times$$

$$\times \left\{I_1(svR)+K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}dv \qquad (50)$$

Similarly, it can be shown that

$$\tilde{u}_{\theta O}(R,0,s) = -\frac{2R_0}{\mu\pi}\int_1^\infty \frac{v}{\sqrt{(v^2-1)}} K_1(svR) \times$$

$$\times \left\{ I_1(svR_0) + K_1(svR_0)\frac{I_2(sv)}{K_2(sv)}\right\}dv. \qquad (50')$$

Laplace inversion of the relations (50) is now taken to obtain the displacement of points inside the source. Therefore

$$u_{\theta I}(R,0,\tau) = -\frac{1}{2\pi i}\frac{2R_0}{\mu\pi}\int_{Br} e^{\tau s}ds \int_1^{\infty} \frac{v}{\sqrt{(v^2-1)}}\tilde{E}(sv)dv, \qquad (51)$$

where

$$\tilde{E}(sv) = K_1(svR_0)\left\{I_1(svR) + K_1(svR)\frac{I_2(sv)}{K_2(sv)}\right\}. \qquad (52)$$

Introducing the change of variable $p = sv$, and changing the order of integration

$$u_{\theta I}(R,0,\tau) = -\frac{2R_0}{\mu\pi}\int_1^{\infty}\frac{1}{\sqrt{(v^2-1)}}dv\left[\frac{1}{2\pi i}\int_{Br} e^{(\tau/v)p}\tilde{E}(p)\,dp\right]$$

$$= -\frac{2R_0}{\mu\pi}\int_1^{\infty}\frac{1}{\sqrt{(v^2-1)}}E(\tau/v)\,dv, \qquad (53)$$

where $\qquad E(\tau/v) = \mathscr{L}^{-1}\{\tilde{E}(p)\}.$

We note that $\tilde{E}(p)$ possesses no poles and is analytic for $p > 0$. It has a branch point at the origin and therefore a cut is drawn from the origin along the negative real axis of the complex p-plane in order to make $\tilde{E}(p)$ single valued.

Drawing a large semi-circular contour to the right of the

Bromwich path AB in the complex p-plane, we conclude that $E(\tau/v) = 0$ if the integral

$$\frac{1}{2\pi i} \int_{BC'A} \tilde{E}(p)\, e^{(\tau/v)p}\, dp = 0$$

over the semi-circular arc BC'A (Fig. 6).

Now

$$E(p) = -\frac{1}{2\pi i} \int_{BC'A} \tilde{E}(p) e^{(\tau/v)p}\, dp$$

$$= -\frac{1}{2\pi i} \int_{BC'A} K_1(pR_0) I_1(pR) e^{(\tau/v)p} dp -$$

$$- \frac{1}{2\pi i} \int_{BC'A} K_1(pR_0) K_1(pR) \frac{I_2(p)}{K_2(p)} e^{(\tau/v)p}\, dp. \qquad (54)$$

Since

$$e^{(\tau/v)p} K_1(pR_0) I_1(pR) \sim \frac{1}{2p\sqrt{RR_0}} e^{\left[\frac{\tau}{v} - (R_0 - R)\right] p}$$

and

$$e^{(\tau/v)p} K_1(pR_0) I_1(pR) \frac{I_2(p)}{K_2(p)} \sim \frac{1}{2p\sqrt{RR_0}} e^{\left[\frac{\tau}{v} - (R + R_0 - 2)\right] p} \quad \text{as } |p| \rightarrow \infty$$

then the first integral on the right hand side of Eq.(54) vanishes for $0 < \tau/v < (R_0 - R)$, whereas the second integral vanishes for $0 < \tau/v < (R + R_0 - 2)$.

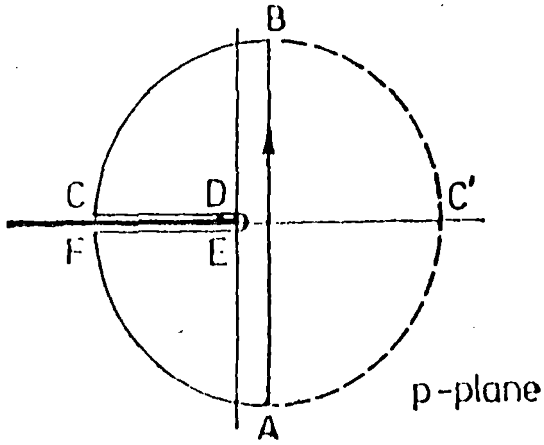

Fig. 6. Laplace inversion contour.

Therefore

$$E(\tau/v) = \begin{cases} 0, & \text{for } 0 < \tau/v < (R_0 - R), \\ E^D(\tau/v), & \text{for } (R_0 - R) < \tau/v < (R + R_0 - 2), \\ E^R(\tau/v), & \text{for } (R + R_0 - 2) < \tau/v. \end{cases} \quad (55)$$

where

$$E^D(\tau/v) = \mathscr{L}^{-1} [K_1(pR_0) I_1(pR)],$$

$$(56)$$

$$E^R(\tau/v) = \mathscr{L}^{-1} \left[K_1(pR_0) I_1(pR) + K_1(pR_0) K_1(pR) \frac{I_2(p)}{K_2(p)} \right].$$

For value of τ/v lying in the range $(R_0 - R) < \tau/v < (R + R_0 - 2)$

$$E(\tau/v) = E^D(\tau/v) = \frac{1}{2\pi i} \int_{Br} K_1(pR_0) I_1(pR) e^{(\tau/v)p} dp. \quad (57)$$

Therefore, putting $\tau/v = (R_0 - R + y)$, where $y > 0$

$$E^D(R_0 - R + y) = \frac{1}{2\pi i} \int_{Br} \left[K_1(pR_0) e^{pR_0} \right] \left[I_1(pR) e^{-pR} \right] e^{yp} dp.$$

From the Laplace inversion table Erdelyi [1954], we find that

$$\mathscr{L}^{-1} \left[K_1(pR_0) e^{pR_0} \right] = \frac{H(y) (y+R_0)}{R_0 \{y(y+2R_0\}^{1/2}},$$

and

$$\mathscr{L}^{-1}\left[I_1(pR)\, e^{-pR} \right] = \frac{[H(y) - H(y-2R)]\,(R-y)}{\pi R\,\{y(2R - y)\}^{1/2}}.$$

So by the convolution theorem

$$E^D(R_0-R+y) = \int_0^y \frac{[H(\eta)-H(\eta-2R)]\; H(y-\eta)(R-\eta)(y-\eta+R_0)}{\pi RR_0\;\,[\eta(2R-\eta)(y-\eta)(y-\eta+2R_0)]^{1/2}}\, d\eta. \quad (58)$$

For τ/v lying in the range $(R_0-R) < \tau/v < (R+R_0-2)$, τ/v must be less than $(R+R_0)$, i.e. $y < 2R$.

Therefore we can write

$$E^D(R_0-R+y) = \int_0^y \frac{(R-\eta)(y-\eta+R_0)}{\pi RR_0\;\,[\eta(2R-\eta)(y-\eta)(y-\eta+2R_0)]^{1/2}}\, d\eta.$$

So

$$E(\tau/v) = E^D(\tau/v) =$$

$$= \int_0^{\tau/v-(R_0-R)} \frac{(R-\eta)(\tau/v+R-\eta)\; d\eta}{\pi RR_0\,[\eta(2R-\eta)(\tau/v-R_0+R-\eta)(\tau/v+R_0+R-\eta)]^{1/2}} \quad (59)$$

for $(R_0-R) < (\tau/v) < (R+R_0-2)$.

For values of τ/v satisfying the condition $\tau/v > R+R_0-2$,

$$E(\tau/v) = E^{R}(\tau/v) =$$

$$= \frac{1}{2\pi i} \int_{Br} \left[K_1(pR_o)\ I_1(pR) + K_1(pR_o)\ K_1(pR)\ \frac{I_2(p)}{K_2(p)} \right] e^{(\tau/v)p}\ dp. \quad (60)$$

This integral is equal to the integral along the large semi-circular arc on the left side of the Bromwich path AB plus the integral on the two sides of the branch cut (Fig.6). Since the integral on the large semi-circular arc vanishes, then Eq. (60) becomes

$$E(\tau/v) = \frac{1}{2\pi i} \left[-\int_0^{\infty} \tilde{E}(\eta e^{i\pi}) e^{-(\tau/v)\eta} d\eta \; + \right.$$

$$\left. + \int_0^{\infty} \tilde{E}(\eta e^{-i\pi}) e^{-(\tau/v)\eta} d\eta \right]. \quad (61)$$

Using the relations

$$I_\nu(\eta e^{\pm i\pi}) = e^{\pm i\nu\pi}\ I_\nu(\eta),$$

and

$$K_\nu(\eta e^{\pm i\pi}) = e^{\mp i\nu\pi}\ K_\nu(\eta) \pm i\pi\ I_\nu(\eta),$$

we obtain (for $\tau/v > R+R_o-2$)

$$E(\tau/v) = E^R(\tau/v) = -\int_0^\infty \frac{U_2(R,\eta)\, U_2(R_0,\eta)\, e^{-(\tau/v)\eta}}{K_2^2(\eta) + \pi^2 I_2^2(\eta)}\, d\eta, \qquad (62)$$

where $\quad U_2(x,\eta) = K_2(\eta)\, I_1(x,\eta) + I_2(\eta)\, K_1(x,\eta).$

Substituting these values of $E(\tau/v)$ in Eq. (53), we obtain

$$u_{\theta I}(R,0,\tau) =$$

$$= -\frac{2R_0}{\mu\pi}\left[\left\{H\left(t - \frac{r_0-r}{\beta}\right) - H\left(t - \frac{r+r_0-2a}{\beta}\right)\right\}\int_1^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}}\, E^D(\tau/v)\, dv + \right.$$

$$\left. + H\left(t - \frac{r+r_0-2a}{\beta}\right)\left\{\int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R_0-R}} \frac{1}{\sqrt{v^2-1}}\, E^D(\tau/v) dv + \int_1^{\frac{\tau}{R+R_0-2}} \frac{1}{\sqrt{v^2-1}}\, E^R(\tau/v) dv\right\}\right],$$

(63)

where the values of $E^D(\tau/v)$ and $E^R(\tau/v)$ are given in Eqs. (59) and (62), respectively.

Similarly, taking the inverse Laplace transform of Eq. (40′), the displacement $u_{\theta O}(R,0,\tau)$ on the free surface outside the ring source can be derived and it is found that

$u_{\theta O}(R,0,\tau) =$

$$= -\frac{2R_o}{\mu\pi}\left[\left\{H\left(t-\frac{r-r_o}{\beta}\right)-H\left(t-\frac{r+r_o-2a}{\beta}\right)\right\}\int_1^{\frac{\tau}{R-R_o}}\frac{1}{\sqrt{v^2-1}}F^D(\tau/v)\,dv\right. +$$

$$\left.+\,H\left(t-\frac{r+r_o-2a}{\beta}\right)\left\{\int_{\frac{\tau}{R+R_o-2}}^{\frac{\tau}{R-R_o}}\frac{1}{\sqrt{v^2-1}}F^D(\tau/v)dv+\int_1^{\frac{\tau}{R+R_o-2}}\frac{1}{\sqrt{v^2-1}}F^R(\tau/v)dv\right\}\right],$$

(63')

where $F^R(\tau/v) = E^R(\tau/v)$, and

$$F^D(\tau/v) = \int_0^{\tau/v-(R-R_o)}\frac{(R_o-\eta)(\tau/v+R_o-\eta)d\eta}{\pi RR_o[\eta(2R_o-\eta)(\tau/v-R+R_o-\eta)(\tau/v+R+R_o-\eta)]^{1/2}}\,.$$

(64)

First, the integrals of Eq. (63) are the displacements due to a direct wave from the ring source before the arrival of the waves reflected from the wall of the cylindrical cavity. The last two integrals together give the displacement after the arrival of the reflected wave.

In order to obtain the response in the vicinity of the SH-wave front, we consider the displacement profile immediately behind the direct outgoing SH-wave. Accordingly, we shall have to study the first integral of Eq. (63′) because it gives the response of the direct SH-wave before the arrival of the reflected wave front.

Let $R_s = R_0 + \tau$ and $R_s^- = R_s - \varepsilon R_0$ where R_s and R_s^- denote points at and immediately behind the SH-wave front, respectively, ε is a small positive quantity.

Then

$$\frac{\tau}{R_s - R_0} = 1 \tag{65}$$

and

$$\frac{\tau}{R_s^- - R_0} = \left[1 + \frac{\varepsilon R_0}{\tau} \right] = q(\tau). \text{ (say)} \tag{65′}$$

Substituting these values in the first integral of Eq. (63′), we obtain

$$u_{\theta 0}(R_s, 0, \tau) = 0,$$

and

$$u_{\theta 0}(R_s^-, 0, \tau) = -\frac{2R_0}{\mu\pi} \int_1^{q(\tau)} \frac{1}{\sqrt{(v-1)}} \left\{ \frac{1}{\sqrt{v+1}} F^D(R^-, R_0, \tau/v) \right\} dv.$$

Therefore, we can write

$$u_{\theta 0}(R_s^-, 0, \tau) = -\frac{2R_0}{\mu\,\pi} \int_1^{q(\tau)} \frac{1}{\sqrt{v-1}} V(v)dv, \tag{66}$$

where V(v) is analytic portion of the integrand. For small value of

ε expanding V(v) by the Taylor's series about the point v = 1 and integrating term by term, we obtain

$$u_{\theta 0}(R_s^-, 0,\tau) \simeq - \frac{4R}{\mu\pi} V(1) \left[\frac{R_0}{\tau} \right]^{1/2} \varepsilon^{1/2} = A\, \varepsilon^{1/2} \text{ (say)}, \qquad (67)$$

where A is a constant.

It therefore follows that the displacement component is continuous i.e. there is no jump in displacement across the direct SH-wave front.

Next, in order to consider the behaviour of response just under the ring source, it should be remembered that the integral representations of transformed displacements given by Eqs. (50) were derived from Eqs. (40) assuming that $R \neq R_0$. For $R = R_0$ the integrals along large quarter circles in the first and fourth quadrants should be reexamined. In this case it is found that though the contributions from the integrals along large circular arcs in the first and fourth quadrants are not separately zero, but the combined sum of the integrals along the large arcs in the first and fourth quadrants of the ζ-plane (Fig. 5a and 5b) vanishes. So the transformed displacements for $R = R_0$ are also given by Eqs. (50). Making $R \rightarrow R_0\pm$, it can easily be shown by help of Eqs. (50) that the displacement has no jump discontinuity across the ring source.

Therefore, in order to derive the nature of the displacement as $R \rightarrow R_o$, any one of the relations (63) may be studied. Consider, for example, the displacement at field points outside the source given by (63′). As $R \rightarrow R_o$, the upper limit of integration $\tau/(R-R_o)$ $\rightarrow \infty$.

Further, as

$$v \rightarrow \frac{\tau}{R-R_o} \rightarrow \infty,$$

$$\frac{1}{\sqrt{v^2-1}} \rightarrow \frac{1}{v} \tag{68}$$

and

$$F^D(\tau/v) \rightarrow \frac{1}{2R_o}. \tag{68'}$$

Thus, from Eq. (63′)

$$\lim_{R \rightarrow R_o} u_{\theta 0}(R,0,\tau) = -\frac{2R_o}{\mu\,\pi} \int_N^{\frac{\tau}{R-R_o}} \frac{1}{v}\,\frac{1}{2R_o}\,dv + \tag{69}$$

+ a finite quantity, where N is large.

The integral is found to contribute a logarithmic singularity to the displacement just on the ring source.

4. FORMULATION AND GENERAL SOLUTION (CASE - 2)

Case. 2. In this case the problem considered is the same in all respects with the first, except that the cavity of the radius a has been replaced by a rigid cylindrical inclusion of the same radius. The cylindrical inclusion being in welded contact with the elastic half-space, there is no relative displacement at the interface. In this case, the condition on the cylindrical boundary is $u_\theta=0$ on r = a. In order to solve this problem, we take the solution in this form:

$$\tilde{u}_\theta(R,Z,s) =$$

$$= \int_0^\infty \left[A_2(\gamma)J_1(\gamma R) + B_2(\gamma)Y_1(\gamma R) \right] e^{-\sqrt{\gamma^2+s^2}\, Z}\, d\gamma, \qquad (70)$$

where $\tilde{u}_\theta(R,Z,s)$ is the Laplace transform of $u_\theta(R,Z,t)$ with respect to t. Now, using the boundary condition

$$\tilde{u}_\theta = 0 \quad \text{on} \quad R = 1,$$

we have

$$B_2(\gamma) = - A_2(\gamma) \frac{J_1(\gamma)}{Y_1(\gamma)} \qquad (71)$$

so $\tilde{u}_\theta$ becomes

$$\tilde{u}_\theta(R,Z,s) =$$

$$= \int_0^\infty A^1(\gamma)\left[J_1(\gamma R)Y_1(\gamma) - J_1(\gamma)Y_1(\gamma R)\right]e^{-\sqrt{\gamma^2+s^2}\,Z}\,d\gamma, \qquad (72)$$

where $$A^1(\gamma) = \frac{A_2(\gamma)}{Y_1(\gamma)}.$$

Therefore, the transformed stress component on the free surface Z = 0 is

$$\tilde{\tau}_{\theta Z}(R,0,s) = -\frac{\mu}{a}\int_0^\infty A^1(\gamma)\sqrt{\gamma^2+s^2}\;C_1(\gamma R)\,d\gamma, \qquad (73)$$

where

$$C_1(\gamma R) = J_1(\gamma R)Y_1(\gamma) - J_1(\gamma)Y_1(\gamma R), \qquad (74)$$

$\tilde{\tau}_{\theta Z}(R,0,s)$ should be equal to $\frac{1}{as}\delta(R - R_0)$. In this case, the required integral representation of the delta function can be obtained from the following expansion formula given by Titchmarsh [1962]:

$$f(r) = \int_0^\infty \frac{\zeta\,[J_1(\zeta r)Y_1(\zeta a) - J_1(\zeta a)Y_1(\zeta r)]}{J_1^2(\zeta a) + Y_1^2(\zeta a)}\,d\zeta \quad \times$$

$$\times \int_a^\infty \xi f(\xi)\,[J_1(\zeta\xi)Y_1(\zeta a) - J_1(\zeta a)Y_1(\zeta\xi)]\,d\xi, \qquad (75)$$

where f(r) is a suitably restricted arbitrary function.

Putting $f(r) = \delta(r-r_0)$,

$f(\xi) = \delta(\xi-r_0)$, where $r_0 > a > 0$,

we get

$$\delta(r-r_0) =$$

$$= r_0\int_0^\infty \frac{\zeta\,[J_1(\zeta r)Y_1(\zeta a)-J_1(\zeta a)Y_1(\zeta r)][J_1(\zeta r_0)Y_1(\zeta a)-J_1(\zeta a)Y_1(\zeta r_0)]}{J_1^2(\zeta a) + Y_1^2(\zeta a)}\,d\zeta . \tag{76}$$

Now putting, $\dfrac{r}{a} = R$, $\dfrac{r_0}{a} = R_0$, $\zeta a = \gamma$, we have

$$\delta(R-R_0) =$$

$$= R_0\int_0^\infty \frac{\gamma\,[J_1(\gamma R)Y_1(\gamma)-J_1(\gamma)Y_1(\gamma R)][J_1(\gamma R_0)Y_1(\gamma)-J_1(\gamma)Y_1(\gamma R_0)]}{J_1^2(\gamma) + Y_1^2(\gamma)}\,d\gamma ,$$

so by the relation (74)

$$\delta(R-R_0) = R_0\int_0^\infty \frac{\gamma\; C_1(\gamma R)\; C_1(\gamma R_0)}{J_1^2(\gamma) + Y_1^2(\gamma)}\,d\gamma . \tag{77}$$

This result can also be obtained by the following technique already developed in Section-2 of this paper.

Now, we find the value of $A^1(\gamma)$ as

$$A^1(\gamma) = \frac{R_o}{\mu s} \frac{\gamma\; C_1(\gamma R_o)}{\sqrt{\gamma^2+s^2}} \; \frac{1}{J_1^2(\gamma)+Y_1^2(\gamma)} \; . \tag{78}$$

Therefore $\tilde{u}_\theta$ becomes

$$\tilde{u}_\theta(R,0,s) = \frac{R_o}{\mu s} \int_0^\infty \frac{\gamma\; C_1(\gamma R)\; C_1(\gamma R_o)}{\sqrt{\gamma^2+s^2}\; \{ J_1^2(\gamma) + Y_1^2(\gamma)\}} \, d\gamma \, . \tag{79}$$

Carrying on a similar procedure as followed to obtain the displacement in the case 1, we find that in this case

$$u_{\theta I}(R,0,\tau) =$$

$$= \frac{2R_o}{\mu\pi} \left[\left\{ H\left(t - \frac{r_o - r}{\beta}\right) - H\left(t - \frac{r+r_o-2a}{\beta}\right) \right\} \int_1^{\frac{\tau}{R_o - R}} \frac{1}{\sqrt{v^2-1}} E^D(\tau/v)\, dv \; + \right.$$

$$\left. + H\left(t - \frac{r+r_o-2a}{\beta}\right) \left\{ \int_{\frac{\tau}{R+R_o-2}}^{\frac{\tau}{R_o - R}} \frac{1}{\sqrt{v^2-1}} E^D(\tau/v) dv + \int_1^{\frac{\tau}{R+R_o-2}} \frac{1}{\sqrt{v^2-1}} E_1^R(\tau/v) dv \right\} \right]$$

(80)

and

$$u_{\theta 0}(R,0,\tau) =$$

$$= \frac{2R_0}{\mu\pi}\left[\left\{H\left(t - \frac{r-r_0}{\beta}\right) - H\left(t - \frac{r+r_0-2a}{\beta}\right)\right\} \int_1^{\frac{\tau}{R-R_0}} \frac{1}{\sqrt{v^2-1}} F^D(\tau/v)dv + \right.$$

$$\left. + H\left(t - \frac{r+r_0-2a}{\beta}\right)\left\{ \int_{\frac{\tau}{R+R_0-2}}^{\frac{\tau}{R-R_0}} \frac{1}{\sqrt{v^2-1}} F^D(\tau/v)dv + \int_1^{\frac{\tau}{R-R_0+2}} \frac{1}{\sqrt{v^2-1}} F_1^R(\tau/v)dv\right\}\right], \tag{81}$$

where $E^D(\tau/v)$ and $F^D(\tau/v)$ are respectively given by Eq. (59) and (64) and

$$E_1^R(\tau/v) = F_1^R(\tau/v) = -\int_0^\infty \frac{U_1(R,\eta)\ U_1(R_0,\eta)\ e^{-(\tau/v)\eta}}{K_1^2(\eta) + \pi^2 I_1^2(\eta)} d\eta \tag{82}$$

where

$$U_1(x,\eta) = K_1(\eta)\ I_1(x\eta) - I_1(\eta)\ K_1(x\eta). \tag{83}$$

WAVES IN A SEMI-INFINITE ELASTIC MEDIUM DUE TO AN EXPANDING ELLIPTIC RING SOURCE ON THE FREE SURFACE

1. INTRODUCTION

Since Lamb's original study of the elasitc wave produced by a time-dependent point force acting normally to the surface of an elastic half-space, many authors have elaborated on his work. Aggarwal and Abolw [1967] discussed the exact solution of a class of half-space pulse propagation problems generated by impulsive sources. Gakenheimer and Miklowitz [1969] used a modification of Cagniard's method [1962] to discuss the disturbance created by a moving point load. In case of finite sources, the most widely discussed model is that of a circular ring or disc load. Mitra [1964], Tupholme [1970] and Roy [1975] have studied the various aspects of the same problem. Elastic waves due to uniformly expanding disc or ring loads on the free surface of a semi-infinite medium have been studied extensively by Gakenheimer [1971]. The axisymmetric problem of the determination of the displacement due to a stress discontinuity over a uniformly expanding circular region at a certain depth below the free surface has been studied by Ghosh [1971].

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However exact evaluation of the displacement field for finite source other than the circular model does not seem to have been attempted much in the literature. Burridge and Willis [1969] obtained a solution for radiation from a growing elliptical crack in an anisotropic medium. The problem of an elliptical shear crack growing in prestressed medium has been solved by Richards [1973] by the Cagniard-de Hoop Method. Roy [1981] also attempted the same technique to slove the problem of elastic wave propagation due to prescribed normal stress over an elliptic area on the free surface of an elastic half-space.

In our problem, we have considered the propagation of elastic waves due to an expanding elliptical ring load over the free surface of a semi-infinite medium. The expression for displacement at points on the free surface has been derived in integral form by the application of Cagniard-de Hoop technique for different values of the rate of increase of the major and minor axes of the elliptic ring source. The displacement jumps across the different wave fronts have also been derived.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let an elliptic ring load P acting normal to the surface of an elastic half-space emanating from the origin of co-ordinates expand in such a way that the rates of increase of the major and minor axes of the ellipse are a and b respectively, a and b being constants. Major and minor axes of the ellipse are taken to coincide with the x and y-axes of co-ordinates where as z-axis is taken vertically downwards into the medium (Fig. 1.). Thus we have on z = 0

$$\tau_{zz} = -\frac{P\,\delta\left[t - (x^2a^{-2} + y^2b^{-2})^{1/2}\right]}{(x^2a^{-2} + y^2b^{-2})^{1/2}} \tag{1}$$

$$\tau_{xz} = \tau_{yz} = 0$$

where P is constant and δ is the Dirac delta function.

The displacement field inside the elastic medium ($z \geq 0$) is given interms of potentials ϕ and ψ as

$$\vec{u} = \nabla\phi + \nabla \times \nabla \times (e_z\psi)$$

where

$$\nabla^2\phi = \frac{1}{c_d^2}\frac{\partial^2\phi}{\partial t^2}, \qquad \nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2} \tag{2}$$

e_x, e_y, e_z are unit vectors along co-ordinate axes and c_d and c_s are the p- and s-wave velocities of the medium.

In order to obtain solutions of wave equations (2), we introduce Laplace transform with respect to t and denote it by bar and also introduce bilateral Fourier transform with respect to x

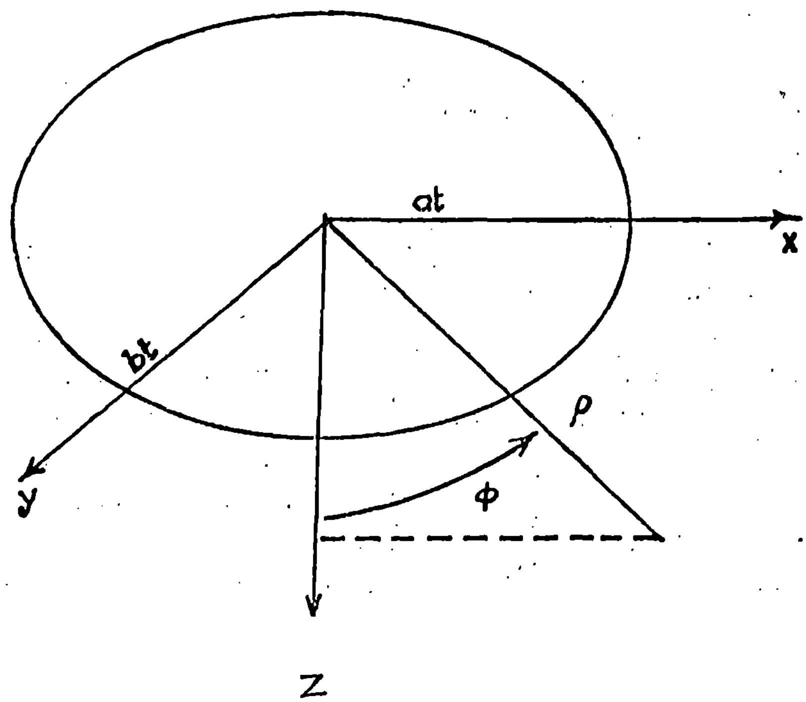


Fig. 1. Geometry of the problem.

and y to supress the time parameter t and the x, y space co-ordinates. Taking Laplace transform with respect to t(-) and also bilateral Fourier transform with respect to x and y ($\cong$), the transformed boundary conditions are

$$\tilde{\tilde{\tau}}_{zz} = - \frac{Pab}{(a^2\xi^2 + b^2\eta^2 + s^2)^{1/2}} , \quad \tilde{\tilde{\tau}}_{xz} = \tilde{\tilde{\tau}}_{yz} = 0 \qquad (3)$$

Then satisfying the transformed boundary conditions (3) and performing the inverse Fourier transform, the Laplace transformed displacement field can be written as

$$\bar{u}_j(x,y,z,s) = \bar{u}_{jd}(x,y,z,s) + \bar{u}_{js}(x,y,z,s) \qquad (4)$$

$$\text{for } j = x,y,z$$

where

$$\bar{u}_{j\alpha_1}(x,y,z,s) =$$

$$= 1/2\pi\mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{j\alpha_1}(\xi,\eta,s)\exp[-\zeta_{\alpha_1} z + i(\xi x+\eta y)]d\xi\, d\eta \qquad (5)$$

$$\text{for } \alpha_1 = d,s$$

and

$$F_{xd}(\xi,\eta,s) = - i\xi\zeta_{\upsilon}G, \quad F_{xs}(\xi,\eta,s) = 2i\xi\zeta_d\zeta_s G,$$

$$F_{yd}(\xi,\eta,s) = - i\eta\zeta_{\upsilon}G, \quad F_{ys}(\xi,\eta,s) = 2i\eta\, \zeta_d\zeta_s G,$$

$$F_{Zd}(\xi,\eta,s) = \zeta_d \zeta_o G, \qquad F_{Zs}(\xi,\eta,s) = -2(\xi^2+\eta^2)\,\zeta_d G,$$

$$G = \frac{Pab}{(s^2 + r^2)^{1/2}\, T}, \qquad T = \zeta_o^2 - 4\,\zeta_d \zeta_s (\xi^2 + \eta^2)$$

$$r^2 = a^2\xi^2 + b^2\eta^2, \tag{6}$$

$$\zeta_d = (\xi^2+\eta^2+k_d^2)^{1/2}, \qquad \zeta_s = (\xi^2+\eta^2+k_s^2)^{1/2},$$

$$\zeta_o = k_s^2 + 2(\xi^2 + \eta^2), \qquad k_d = \frac{s}{c_d}, \qquad k_s = \frac{s}{c_s}.$$

Now the De-Hoop transformation,

$$\xi = s/c_d(q \cos\theta - w \sin\theta), \quad \eta = s/c_d(q \sin\theta + w \cos\theta) \tag{7}$$

where $\theta = \tan^{-1} y/x$,

is applied into (5). The Laplace transformed displacement field (5) can be written as

$$\bar{u}_{j\alpha_1}(R,Z,s) = 1/2\pi\mu \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{j\alpha_1}(q,w,s)\exp[-s/c_d(m_\alpha Z - iqR)]\,\frac{s^2}{c_d^2}\,dq\,dw$$

(8)

where

$$F_{xd}(q,w,s) = -\frac{i\,Pab\,(q\cos\theta - w\sin\theta)m_o}{s.s/c_d(E_1 + O)^{1/2}.N},$$

$$F_{xs}(q,w,s) = \frac{2i\ Pab\ (q \cos\theta - w \sin\theta) m_d m_s}{s.s/c_d(E_1 + O)^{1/2}.N},$$

$$F_{yd}(q,w,s) = -\frac{i\ Pab\ (q \sin\theta + w \cos\theta) m_o}{s.s/c_d(E_1 + O)^{1/2}.N.}$$

$$F_{ys}(q,w,s) = \frac{2i\ Pab\ (q \sin\theta + w \cos\theta)\ m_d m_s}{s.s/c_d(E_1 + O)^{1/2}.N.},$$

$$F_{zd}(q,w,s) = \frac{Pab\ m_d m_o}{s.s/c_d(E_1 + O)^{1/2}.N.},$$

$$F_{zs}(q,w,s) = -\frac{2\ Pab\ (q^2 + w^2) m_d}{s.s/c_d(E_1 + O)^{1/2}.N.},$$

$$m_d = (q^2 + w^2 + 1)^{1/2}, \qquad m_s = (q^2 + w^2 + l^2)^{1/2},$$

$$m_o = l^2 + 2(q^2 + w^2), \qquad N = m_o^2 - 4 m_d m_s (q^2 + w^2),$$

$$E_1 = (1 + q^2 D + w^2 F), \qquad D = \frac{a^2}{c_d^2}\cos^2\theta + \frac{b^2}{c_d^2}\sin^2\theta,$$

$$F = \frac{a^2}{c_d^2}\sin^2\theta + \frac{b^2}{c_d^2}\cos^2\theta, \qquad O = -2qw \sin\theta \cos\theta (a^2 - b^2)/c_d^2,$$

$$l = c_d/c_s, \text{ and } R^2 = x^2 + y^2. \tag{9}$$

For mathematical simplicity we confine our attention to the derivation of the displacement field at any point on the xz-plane. Obviously the displacement at any point on any plane through the z-axis can then easily be visualized. Accordingly in order to obtain the displacement at any point on the xz-plane, we put $\theta = 0$ in (8) which then takes the form

$$\bar{u}_{j\alpha_1}(x,z,s) = \frac{Pab}{2\pi\mu c_d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{Re}\left[K_{j\alpha_1}(q,w)\, e^{-\frac{s}{c_d}(m_\alpha z - iqx)} \right] dq\, dw \tag{10}$$

where

$$K_{xd}(q,w) = -\frac{iqm_o}{E^{1/2}.N}, \qquad K_{xs}(q,w) = \frac{2iqm_d m_s}{E^{1/2}.N.},$$

$$K_{yd}(q,w) = -\frac{iwm_o}{E^{1/2}.N}, \qquad K_{ys}(q,w) = \frac{2iwm_d m_s}{E^{1/2}.N.}, \tag{11}$$

$$K_{zd}(q,w) = \frac{m_d m_o}{E^{1/2}.N}, \qquad K_{zs}(q,w) = -\frac{2m_d(q^2 + w^2)}{E^{1/2}.N},$$

and

$$E = (c_d^2 + a^2q^2 + b^2w^2)/c_d^2.$$

3. DILATATIONAL CONTRIBUTION

From (10) $\bar{u}_{zd}$ is converted to the Laplace transform of a known function by mapping $(m_d z - iqx)/c_d$ into t through a contour integration in a complex q-plane.

The singularities of the integrand of $\bar{u}_{zd}$ are branch points at

$$q = S_d^{\pm} = \pm\ i(w^2 + 1)^{1/2}, \qquad q = S_s^{\pm} = \pm\ i(w^2 + l^2)^{1/2},$$

$$q = S_c^{\pm} = \pm\ i\ \frac{(w^2 b^2 + c_d^2)^{1/2}}{a}, \tag{12}$$

and the poles at (12)

$$q = S_R^{\pm} = \pm\ i(w^2 + \gamma_R^2)^{1/2}.$$

The poles at $q = S_R^{\pm}$ correspond to the zeros of the Rayleigh function N, where $\gamma_R = c_d/c_R$ and c_R is the Rayleigh surface wave speed. The contours of integration in the q-plane are shown in Fig. 2(a,b,c) which also show the positions of singularities lying in the upper half of the q-plane.

Since the positions of the singularities and the transformed contour of integration depend on different values of a and b, three different cases arise for the evaluation of u_{zd}.

(a) Case $a > b > C_d$.

The q-plane for $a > b > C_d$ is shown in Fig. 2(a). The contour $q = q_d^{\pm}$ in the q-plane, is found by solving

$$t = (m_d Z - iqx)/c_d \tag{13}$$

for q, where t is real , we get

$$q = q_d^{\pm} = i\tau \sin\phi \pm(\tau^2 - \tau_{wd}^2)^{1/2} \cos\phi \tag{14}$$

for $\tau > \tau_{wd}$, where

$$\tau_{wd} = (w^2 + 1)^{1/2}, \quad \tau = c_d t/\rho \tag{15}$$

and (ρ,ϕ) are the polar coordinates in the xz-plane as shown in Fig.1. Equations (14) define one branch of a hyperbola with vertex at $q = i(w^2 + 1)^{1/2} x/\rho$, which is parametrically described by the dimensionless time parameter τ as τ varies from τ_{wd} towards infinity.

As shown in Fig. 2(a), the contour of integration has two possible configurations in the q-plane, depending upon ϕ and w.

For the case(1) given by:

Case(1) : $\phi < \phi_{da}$ and $0 < \omega < \infty$

or

$$\phi_{da} < \phi < \phi_{ba} \text{ and } W_{da} < W < \infty \tag{16}$$

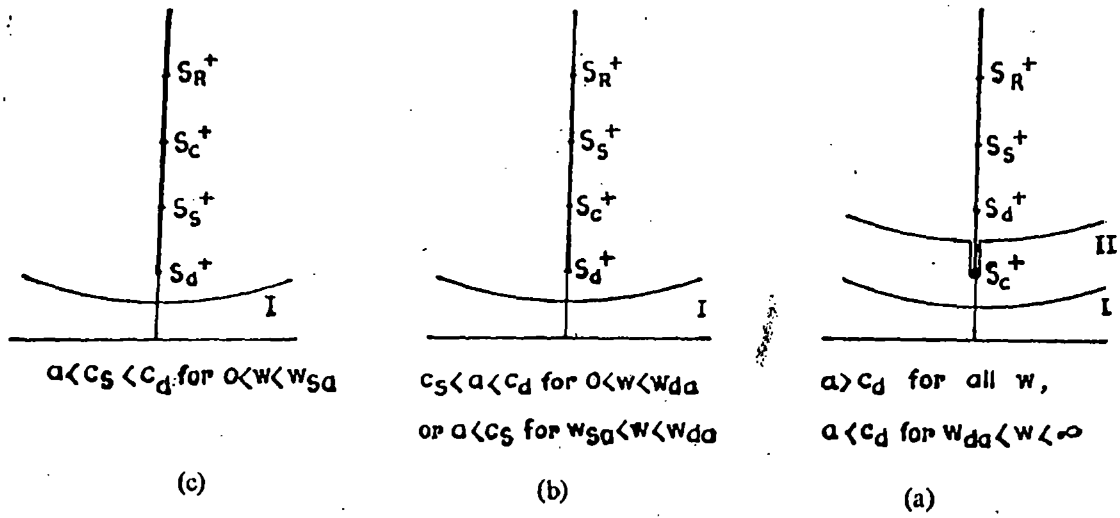


Fig. 2. Cagniard paths of integration in the q-plane.

where $\phi_{da} = \sin^{-1} C_d/a$, $\phi_{ba} = \sin^{-1} b/a$

and

$$W_{da} = \left[\frac{C_d^2 - a^2\sin^2\phi}{a^2\sin^2\phi - b^2} \right]^{1/2}, \tag{17}$$

the vertex of the path $q = q_d^{\pm}$ does not lie on the branch cuts and hence the path of integration contour is simply $q = q_d^{\pm}$ and is denoted by I. But for the case (2) given by :

$$\text{Case (2):} \quad \phi_{da} < \phi < \phi_{ba} \text{ and } 0 < w < w_{da}$$
$$\text{or } \phi > \phi_{ba} \text{ and } 0 < w < \infty \tag{18}$$

the vertex of the path $q = q_d^{\pm}$ lies on the branch cut between the branch points $q = S_c^+$ and $q = S_d^+$. Hence the integration contour is given by $q = q_d^{\pm}$ for $\tau > \tau_{wd}$ which is denoted by II, plus

$$q = q_{da} = i\tau\sin\phi - i(\tau_{wd}^2 - \tau^2)^{1/2}\cos\phi \tag{19}$$

for $\tau_{wda} < \tau < \tau_{wd}$, where

$$\tau_{wda} = \frac{1}{a}\left[\left\{w^2(a^2 - b^2)+(a^2 - C_d^2)\right\}^{1/2}\cos\phi + (w^2b^2 + C_d^2)^{1/2}\sin\phi\right]. \tag{20}$$

Transferring the path of integration from the real q-axis to the Cagniard's path we obtain

$$\bar{u}_{zd}(\rho,\phi,s) = \frac{2\ Pab}{\pi\mu C_d}\left[\int_0^{\infty}\int_{t_{vd}}^{\infty} \mathrm{Re}\left[k_{zd}(q_d^+,w)\frac{dq_d^+}{dt}\right]e^{-st}\,dt\,dw\ +\right.$$

$$+\ H(\phi_{ba} - \phi)H(\phi - \phi_{da})\int_0^{w_{da}}\int_{t_{vda}}^{t_{vd}} \mathrm{Re}\left[k_{zd}(q_{da}\ ,w)\frac{dq_{da}}{dt}\right]e^{-st}\,dtdw\ +$$

$$\left. +\ H(\phi - \phi_{ba})\int_0^{\infty}\int_{t_{vda}}^{t_{vd}} \mathrm{Re}\left[k_{zd}(q_{da}\ ,w)\frac{dq_{da}}{dt}\right]e^{-st}\,dtdw\right] \qquad (21)$$

where $t_{vd} = (\rho/C_d)\tau_{vd}$ and $t_{vda} = (\rho/C_d)\tau_{vda}$. The first term of (21) is the contribution from $q_d^{\pm}$ and the second and third terms are the contributions from q_{da}.

Now interchanging the order of integration in (21) and inverting the Laplace transform, we find that

$$u_{zd}(\rho,\phi,\tau) = \frac{2\ Pab}{\pi\mu C_d}\left[\ H(\tau - 1)\int_0^{T_d} \mathrm{Re}\left[k_{zd}(q_d^+,w)\frac{dq_d^+}{dt}\right]dw\ +\right.$$

$$+\ H(\phi - \phi_{da})H(\phi_{ba} - \phi)H(\tau - \tau_{da})H(\tau'_{da} - \tau)\ \times$$

$$\times\int_{A'_{da}}^{T_{da}} \mathrm{Re}\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right]dw\ +$$

$$+ H(\phi - \phi_{ba})H(\tau - \tau_{da}) \times$$

$$\times \int_{A^{o}_{da}}^{T_{da}} \mathrm{Re}\left[k_{zd}(q_{da}, w) \frac{dq_{da}}{dt} \right] dw \Bigg] \qquad (22)$$

where

$$\left.\begin{aligned} A'_{da} &= \begin{cases} 0 \text{ for } \tau_{da} < \tau < 1 \\ T_d \text{ for } 1 < \tau < \tau'_{da} \end{cases} \\ A^{o}_{da} &= \begin{cases} 0 \text{ for } \tau_{da} < \tau < 1 \\ T_d \text{ for } \tau > 1 \end{cases} \end{aligned}\right\} \qquad (23)$$

$$T_d = (\tau^2 - 1)^{1/2} \qquad (24)$$

$$T_{da} = \left[\frac{X_d - \{Y_d - (a^2\cos^2\phi - b^2)Z_d\}^{1/2}}{(a^2\cos^2\phi - b^2)^2} \right]^{1/2} \qquad (25)$$

$$X_d = \tau^{o}_d\, b^2\sin^2\phi + (a^2 - b^2)\tau_d\cos^2\phi$$

$$Y_d = \tau^{o^2}_d b^4 \sin^4\phi + (a^2 - b^2)^2\, \tau^2_d\cos^4\phi +$$

$$+ 2(a^2 - b^2)b^2\tau_d\tau^{o}_d \sin^2\phi \cos^2\phi$$

$$Z_d = (\tau_d - 2C^2_d \sin^2\phi)^2 - 4C^2_d(a^2 - C^2_d) \sin^2\phi \cos^2\phi$$

$$\tau_d = a^2\tau^2 + (C^2_d - a^2\cos^2\phi)$$

$$\tau_d^0 = a^2\tau^2 - (C_d^2 - a^2\cos^2\phi) \qquad (26)$$

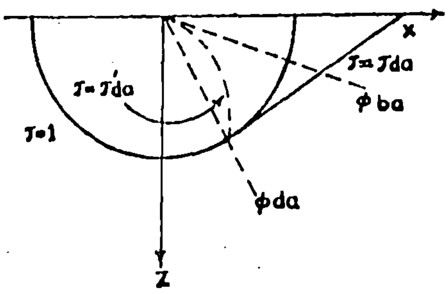
$$\tau_{d\alpha} = \frac{1}{a}\left[(a^2 - C_d^2)^{1/2} \cos\phi + C_d \sin\phi \right], \qquad (27)$$

$$\tau'_{d\alpha} = \left[\frac{C_d^2 - b^2}{a^2 \sin^2\phi - b^2} \right]^{1/2} . \qquad (28)$$

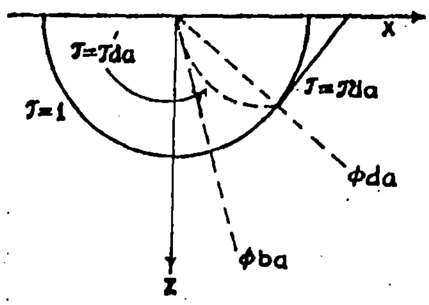
The first term in u_{zd} is due to the dilatational motion behind hemispherical wave front at $\tau = 1$ and the second and third terms are due to the dilatational motion behind the conical wave front at $\tau=\tau_{d\alpha}$ for $\phi > \phi_{d\alpha}$. These wave fronts are shown in Fig. 3(a), $\tau = \tau'_{d\alpha}$ shown in Fig 3(a) by a dashed curve, is not a wave front because it is not a characteristic surface for governing wave equation for the dilatational motion. Similar non characteristic surfaces were found by Gakenheimer and Miklowitz [1969] for a point load travelling on an elastic half-space and also by Aggarwal and Ablow [1967] for the motion of an acoustic half-space due to an expanding surface load. They proved explicitly that their solution was analytic over the surfaces. The same thing can be proved in our case also.

(b) Case $a > c_d > b$

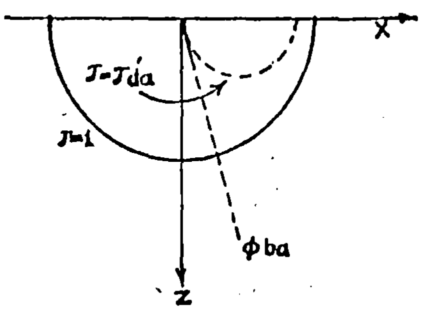
In this case, the path of integration with respect to q transforms to the simple path given by contour I (Fig.2(a)) for all



3 (a) for $\alpha > b > c_d$



3 (b) for $\alpha > c_d > b$



3 (c) for $\alpha < c_d$

FIG. 3. Wave patten for dilatational motion.

w when $\phi < \phi_{ba}$ and also for $0<w<w_{da}$ when $\phi_{ba}<\phi < \phi_{da}$, whereas the path of integration with respect to q transform to the contour II (Fig.2(a)) for $w_{da} < w < \infty$ when $\phi_{ba}<\phi<\phi_{da}$ and also for all w when $\phi > \phi_{da}$. The remaining details of inverting $\bar{u}_{zd}$ for $a > c_d > b$ are exactly the same as for $a > b > c_d$, and one can easily find that

$$u_{zd}(\rho,\phi,\tau) = \frac{2\ Pab}{\pi\mu c_d}\left[H(\tau - 1)\int_0^{\tau_d} Re\left[k_{zd}(q_d^+,w)\frac{dq_d^+}{dt}\right]dw + \right.$$

$$+ H(\phi - \phi_{ba})H(\phi_{da} - \phi)H(\tau - \tau_{da}) \times$$

$$\times \int_{\tau_d}^{\tau_{da}} Re\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right]dw +$$

$$+ H(\phi - \phi_{da})H(\tau - \tau_{da}) \times$$

$$\left. \times \int_{A_{da}^{o}}^{\tau_{da}} Re\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right]dw\right] \qquad (29)$$

where A_{da}^{o} is given by (23). The wave geometry associated with this expression is shown in Fig.3(b).

(c) Case $a < c_d$

For this case the path of integration with respect to q transform to the simple path given by contour I [Figs. 2(b),2(c)] for all w when $\phi<\phi_{ba}$ and also for $0<w<w_{da}$ when $\phi>\phi_{ba}$, whereas the path of integration with respect to q transforms to the contour II [Fig.2 (a)] for $w_{da}<w<\infty$ when $\phi>\phi_{ba}$. Note that in this case the angle ϕ_{da} does not arise. Now proceding as the case $a>b>c_d$ for inverting $\bar{u}_{zd}$ we get

$$u_{zd}(\rho,\phi,\tau) = \frac{2\,Pab}{\pi\mu c_d}\left[H(\tau - 1)\int_0^{T_d} \mathrm{Re}\left[k_{zd}(q_d^+,w)\,\frac{dq_d^+}{dt}\right]dw \;+\right.$$

$$\left. + H(\phi - \phi_{ba})H(\tau - \tau'_{da}) \int_{T_d}^{T_{da}} \mathrm{Re}\left[k_{zd}(q_{da},w)\,\frac{dq_{da}}{dt}\right]dw \right]. \qquad (30)$$

The wave geometry associated with this expression is shown in Fig.3(c). As expected physically, contribution due to the conical wave front does not exist for this case.

Summary

Combining (22), (29) and (30) one finds that u_{zd} can be written as one expression for all value of a and b.

$$u_{zd}(\rho,\phi,\tau) = \frac{2\ Pab}{\pi\mu c_d}\left[H(\tau - 1)\int_0^{T_d} Re\left[k_{zd}(q_d^+,w)\frac{dq_d^+}{dt}\right] dw + \right.$$

$$+ \left[H(\tau - \tau_{da})H(\phi - \phi_{da})\left\{H(b - c_d) + \right.\right.$$

$$\left. + H(a-c_d)H(c_d-b)\right\} + H(\tau - \tau'_{da})H(\phi - \phi_{ba})\left\{H(a - c_d) \times \right.$$

$$\left.\left. \times H(c_d - b)H(\phi_{da} - \phi) + H(c_d - a)\right\}\right] \times$$

$$\left. \times \int_{A_{da}}^{T_{da}} Re\left[k_{zd}(q_{da},w)\frac{dq_{da}}{dt}\right] dw \right] \tag{31}$$

where

$$A_{da} \begin{cases} = 0 \text{ for } \tau_{da} < \tau < 1 \\ = T_d \text{ for } 1 < \tau < \tau'_{da} \\ = T_{da} \text{ for } \tau > \tau'_{da} \end{cases} \quad \text{for } \phi_{da} < \phi < \phi_{ba},\ a > b > c_d$$

$$\begin{cases} = 0 \text{ for } \tau_{da} < \tau < 1 \\ = T_d \text{ for } 1 < \tau \end{cases} \quad \begin{array}{l} \text{for } \phi > \phi_{ba}, a > b > c_d \\ \text{for } \phi > \phi_{da},\ a > c_d > b \end{array}$$

$$= T_d \text{ for } \tau > \tau'_{da} \quad \left\{ \begin{array}{l} \text{for } \phi_{ba} < \phi < \phi_{da}, a > c_d > b \\ \text{for } \phi > \phi_{ba}, a < c_d . \end{array}\right. \tag{32}$$

4. EQUIVOLUMINAL CONTRIBUTIONS

Inversion of $\bar{u}_{zs}$ is complicated than the inversion of $\bar{u}_{zd}$ because of the appearence of head waves (Von-Schmidt waves) otherwise it is same as $\bar{u}_{zd}$. Here the integration contour has more configurations in the q-plane though the singularities are the same. Here the hyperbola $q = q_s^{\pm}$ arises in a similar way to $q = q_d^{\pm}$, but its vertex can lie on the branch cut between the branch points at $q = S_d^+$ and $q = S_s^+$ and at $q = S_c^+$ and $q = S_s^+$ as well as between $q = S_c^+$ and $q = S_d^+$, depending on the values of w, ϕ, a and b. In this case, the straight line contour lying along the imaginary q-axis is denoted by $q_{s\alpha}$ which is similar to $q_{d\alpha}$ appearing in the dilatational contributions. Now omiting details of inverting $\bar{u}_{zs}$, one can easily find

$$u_{zs}(\rho,\phi,\tau) = \frac{4\ Pab}{\pi\mu c_d}\left[H(\tau - 1)\int_0^{T_s} \mathrm{Re}\left[k_{zs}(q_s^+,w)\ \frac{dq_s^+}{dt} \right] dw \ + \right.$$

$$+ \left[H(\tau - \tau_{s\alpha})H(\phi - \phi_{s\alpha})\{H(b - c_s) + H(c_s - b)H(a - c_s)\} + \right.$$

$$+ H(\tau - \tau'_{s\alpha})H(\phi - \phi_{b\alpha})\ \{H(c_s - b)H(\phi_{s\alpha} - \phi)\ \times$$

$$\times\ H(a - c_s) + H(c_s - a)\}]\ \times$$

$$\times \int_{A_{sa}}^{T_{sa}} \mathrm{Re}\left[k_{zs}(q_{sa},w) \frac{dq_{sa}}{dt} \right] dw +$$

$$+ H(\tau - \tau_{sd})H(\tau'_{sd} - \tau)H(\phi - \phi_{sd}) \times$$

$$\times \int_{A_{sd}}^{T_{sd}} \mathrm{Re}\left[k_{zs}(q_{sa},w) \frac{dq_{sa}}{dt} \right] dw \Bigg] \qquad (33)$$

for $0 \le \rho < \infty$, $0 \le \phi < \pi/2$,

$0 \le \tau < \infty$, $0 \le a < \infty$ and $0 \le b < \infty$, $a > b$

where

$= 0$ for $\tau_{sa} < \tau < 1$; $= T_s$ for $1 < \tau < \tau'_{sa}$ — for $\phi_{sa} < \phi < \phi_{ba}, a > c_d, a > b > c_s, ac_s > bc_d$; $\phi_{sa} < \phi < \phi_{sd}, a > c_d, a > b > c_s, ac_s < bc_d$; $\phi_{sa} < \phi < \phi_{abs}, c_d > a > b > c_s$

$= 0$ for $\tau_{sa} < \tau < 1$; $= T_s$ for $\tau > 1$ — for $\phi_{ba} < \phi < \phi_{sd}, a > b > c_d, ac_s > bc_d$; $\phi_{sa} < \phi < \phi_{sd}, a > c_d > c_s > b$

$= 0$ for $\tau_{sa} < \tau < \tau_{sd}$; $= T_{sd}$ for $\tau_{sd} < \tau < \tau'_{sd}$; $= T_s$ for $\tau > \tau'_{sd}$ — for $\phi > \phi_{sd}, a > b > c_d$, $ac_s > bc_d$; $\phi > \phi_{sd}, a > c_d > c_s > b$

$= 0$ for $\tau_{sa} < \tau < \tau_{sd}$; $= T_{sd}$ for $\tau_{sd} < \tau < \tau'_{sd}$ — for $\phi > \phi_{sd}, a > b > c_d, ac_s < bc_d$

$$
A_{sa}\left\{\begin{array}{ll}
= T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa} & \\[2ex]
= T_s \text{ for } \tau > \tau'_{sa} & \left\{\begin{array}{l} \phi_{ba} < \phi < \phi_{sa},\ a > c_d > c_s > b \\ \phi_{ba} < \phi < \phi_{abs},\ c_d > a > c_s > b \\ \phi_{ba} < \phi < \phi_{abs},\ a < c_s \end{array}\right. \\[2ex]
\left.\begin{array}{l} = T_s \text{ for } \tau'_{sa} < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau > \tau'_{sd} \end{array}\right\} & \begin{array}{l} \phi_{abs} < \phi < \phi_{sa},\ c_d > a > c_s > b \\ \phi > \phi_{abs},\ a < c_s \end{array} \\[2ex]
\left.\begin{array}{l} = 0 \text{ for } \tau_{sa} < \tau < 1 \\ = T_s \text{ for } 1 < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau > \tau'_{sd} \end{array}\right\} & \begin{array}{l} \phi > \phi_{sa},\ c_d > a > c_s > b,\ \alpha > \beta \\ \phi_{sa} < \phi < \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\ \phi > \phi_{ba},\ c_d > a > b > c_s,\ \alpha > \beta \\ \phi_{ba} < \phi < \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \end{array} \\[2ex]
\left.\begin{array}{l} = 0 \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau > \tau'_{sd} \end{array}\right\} & \begin{array}{l} \phi > \phi_x,\ c_d > a > c_s > b,\ \beta > \alpha > \gamma' \\ \phi > \phi_x,\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\ \phi > \phi_{ba},\ c_d > a > b > c_s,\ \alpha < \gamma \end{array} \\[2ex]
\left.\begin{array}{l} = 0 \text{ for } \tau_{sa} < \tau < 1 \\ = T_s \text{ for } 1 < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa}. \end{array}\right\} & \begin{array}{l} \phi_{abs} < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \alpha > \beta \\ \phi_{abs} < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \beta > \alpha > \gamma \\ \phi_{abs} < \phi < \phi_x,\ c_d > a > b > c_s,\ \alpha < \gamma \end{array} \\[2ex]
\left.\begin{array}{l} = 0 \text{ for } \tau_{sa} < \tau < \tau'_{sda} \\ = T_{sd} \text{ for } \tau'_{sda} < \tau < \tau'_{sd} \\ = T_s \text{ for } \tau'_{sd} < \tau < \tau'_{sa} \end{array}\right\} & \phi_x < \phi < \phi_{ba},\ c_d > a > b > c_s,\ \alpha < \gamma .
\end{array}\right. \tag{34}
$$

$$
A_{sd}\left\{
\begin{array}{lll}
= 0 & \text{for } \tau_{sd}<\tau<1 & \left[\begin{array}{l} \phi>\phi_{sd},\ a>b>c_d \\ \phi>\phi_{sd},\ a>c_d>c_s>b \\ \phi_{sd}<\phi<\phi_{abs},\ c_d>a>c_s>b \\ \phi_{sd}<\phi<\phi_{sa},\ c_d>a>b>c_s \\ \phi_{sd}<\phi<\phi_{abs},\ a<c_s \end{array}\right. \\
= T_s & \text{for } 1<\tau<\tau'_{sd} & \\
& & \\
= 0 & \text{for } \tau_{sd}<\tau<1 & \left.\begin{array}{l} \phi_{abs}<\phi<\phi_{sa},\ c_d>a>c_s>b \\ \phi>\phi_{abs},\ a<c_s \end{array}\right. \\
= T_s & \text{for } 1<\tau<\tau'_{sa} & \\
= T_{sa} & \text{for } \tau'_{sa}<\tau<\tau'_{sda} & \\
= T_s & \text{for } \tau'_{sda}<\tau<\tau'_{sd} & \\
& & \\
= 0 & \text{for } \tau_{sd}<\tau<\tau_{sa} & \left.\begin{array}{l} \phi>\phi_{sa},\ c_d>a>c_s>b,\ \alpha>\beta \\ \phi_{sa}<\phi<\phi_{x},\ c_d>a>c_s>b,\ \beta>\alpha>\gamma' \\ \phi>\phi_{abs},\ c_d>a>b>c_s,\ \alpha>\beta \\ \phi_{abs}<\phi<\phi_{x},\ c_d>a>b>c_s,\ \beta>\alpha>\gamma \\ \phi_{abs}<\phi<\phi_{x},\ c_d>a>b>c_s,\ \alpha<\gamma \end{array}\right. \\
= T_{sa} & \text{for } \tau_{sa}<\tau<\tau'_{sda} & \\
= T_s & \text{for } \tau'_{sda}<\tau<\tau'_{sd} & \\
& & \\
= 0 & \text{for } \tau_{sd}<\tau<\tau_{sa} & \left.\begin{array}{l} \phi>\phi_{x},\ c_d>a>c_s>b,\ \beta>\alpha>\gamma' \\ \phi>\phi_{x},\ c_d>a>b>c_s,\ \beta>\alpha>\gamma \\ \phi>\phi_{x},\ c_d>a>b>c_s,\ \alpha<\gamma \end{array}\right. \\
= T_{sa} & \text{for } \tau_{sa}<\tau<\tau'_{sda} & \\
= 0 & \text{for } \tau'_{sda}<\tau<1 & \\
= T_s & \text{for } 1<\tau<\tau'_{sd} & \\
= 0 & \text{for } \tau_{sd}<\tau<\tau_{sa} & \left.\begin{array}{l} \phi_{sa}<\phi<\phi_{abs},\ c_d>a>b>c_s \end{array}\right. \\
= T_{sa} & \text{for } \tau_{sa}<\tau<\tau'_{sa} & \\
= T_s & \text{for } \tau'_{sa}<\tau<\tau'_{sd} &
\end{array}\right.
\qquad (35)
$$

and also where

$$T_s = (\tau^2 - l^2)^{1/2} \quad (36)$$

$$T_{sa} = \left[\frac{X_s - \{Y_s - (a^2\cos^2\phi - b^2)^2 Z_s\}^{1/2}}{(a^2\cos^2\phi - b^2)^2} \right]^{1/2} \quad (37)$$

$$X_s = \tau_s^0 b^2 \sin^2\phi + (a^2 - b^2)\,\tau_s \cos^2\phi$$

$$Y_s = \tau_s^{0\,2} b^4 \sin^4\phi + (a^2 - b^2)^2 \tau_s^2 \cos^4\phi +$$

$$+ 2(a^2 - b^2) b^2 \tau_s \tau_s^0 \sin^2\phi \cos^2\phi$$

$$Z_s = (\tau_s - 2c_d^2 \sin^2\phi)^2 - 4\, l^2\, c_d^2 (a^2 - c_s^2) \sin^2\phi \cos^2\phi$$

$$\tau_s = a^2\tau^2 + l^2(c_s^2 - a^2\cos^2\phi)$$

$$\tau_s^0 = a^2\tau^2 - l^2(c_s^2 - a^2\cos^2\phi) \quad (38)$$

$$T_{sd} = \left[\left\{(\tau - \tau_{sd})\operatorname{cosec}\phi + 1\right\}^2 - 1\right]^{1/2} \quad (39)$$

$$\tau_{sa} = 1/a\left[l(a^2 - c_s^2)^{1/2}\cos\phi + c_d \sin\phi\right] \quad (40)$$

$$\tau_{sd} = \left[(l^2 - 1)^{1/2} \cos\phi + \sin\phi\right] \quad (41)$$

$$\tau'_{sa} = \left[\frac{l^2 (b^2 - c_s^2)}{b^2 - a^2 \sin^2 \phi} \right]^{1/2} \tag{42}$$

$$\tau'_{sd} = (l^2 - 1)^{1/2} \sec\phi \tag{43}$$

$$\tau'_{sda} = \left[(l^2 - 1)^{1/2} \cos\phi + \left(\frac{c_d^2 - b^2}{a^2 - b^2} \right)^{1/2} \sin \phi \right] \tag{44}$$

$$\phi_{sa} = \sin^{-1} c_s/a, \ \phi_{sd} = \sin^{-1} c_s/c_d, \ \phi_{ba} = \sin^{-1} b/a \tag{45}$$

$$\phi_{abs} = \sin^{-1} \left(\frac{c_d^2 - b^2}{l^2 (a^2 - b^2) + c_d^2 - a^2} \right)^{1/2} \tag{46}$$

$$\phi_x = \sin^{-1} \left[\frac{(a^2 - b^2)^{1/2} \left[l (c_d^2 - b^2)^{1/2} + (l^2 - 1)^{1/2} (c_d^2 - a^2)^{1/2} \right]}{l^2 (a^2 - b^2) + c_d^2 - a^2} \right] \tag{47}$$

$$\alpha = \left(\frac{c_d^2 - a^2}{a^2 - b^2} \right)^{1/2}, \qquad \beta = (l^2 - 1)^{1/2},$$

$$\gamma = \frac{b}{a} (l^2 - 1)^{1/2} - \frac{1}{a} (c_d^2 - b^2)^{1/2}, \tag{48}$$

$$\gamma' = \frac{c_s}{a} (l^2 - 1)^{1/2} - \frac{l}{a} \left[\frac{a^2 - c_s^2}{a^2 - b^2} (c_d^2 - b^2) \right]^{1/2}$$

$$q_s^{\pm} = i \ \tau \ \sin\phi \pm (\tau^2 - \tau_{ws}^2)^{1/2} \cos\phi \tag{49}$$

$$\tau_{vs} = (w^2 + 1^2)^{1/2} \qquad (50)$$

$$q_{s\alpha} = i\tau \sin\phi - i(\tau_{vs}^2 - \tau^2)^{1/2} \cos\phi \qquad (51)$$

The first term in the expression (33) is the equivoluminal motion behind the hemispherical wave front at $\tau = 1$ and the second is due to the equivoluminal motion behind the conical wave front at $\tau = \tau_{s\alpha}$. The third term in u_{zs} represents the equivoluminal motion due to the head wave fronts at $\tau = \tau_{sd}$. The wave fronts $\tau = \tau_{sd}$ for $\phi > \phi_{sd}$ and $\tau = \tau_{s\alpha}$ are shown in Figs. 4(a-l).

The equations $\tau = \tau'_{s\alpha}$, $\tau = \tau'_{sd}$ and $\tau = \tau'_{sd\alpha}$ are shown in Fig. 4 by dashed curves which are similar to $\tau = \tau'_{d\alpha}$ appearing in the u_{zd}. These dashed curved surfaces are not considered as wave fronts because it can be shown that displacements and their derivatives are continuous across these surfaces.

5. WAVE FRONT EXPANSIONS

The wave forms of the solution given in (31) and (33) are evaluted by approximate estimation of the integrals in the neighbourhood of the first arrival of the different waves. To facilitate this evaluation we put

$$w = [A^2 + (B^2 - A^2)\sin^2\alpha]^{1/2} \qquad (52)$$

in the integrals arising in u_{zd} and u_{zs} where A and B are respectively the lower and upper limits of the particular integral in question, and the range of integration with respect to α is from 0 to $\pi/2$.

Now for the first integral of (31), we put $w = T_d \sin\alpha$ and hence for $\tau \to 1+$, we find that for any value of a,

$$w \to 0, \quad q_d^+ \longrightarrow i\sin\phi, \qquad \frac{dq_d^+}{dt} \longrightarrow \frac{c_d}{\rho}\,\frac{\cos\phi}{T_d\cos\alpha},$$

$$m_d \to \cos\phi, \quad m_s \to (1^2 - \sin^2\phi)^{1/2}, \quad m_o \to (1^2 - 2\sin^2\phi),$$

$$E^{1/2} \to \frac{1}{c_d}(c_d^2 - a^2\sin^2\phi)^{1/2}, \text{ for } \phi < \phi_{da} \tag{53}$$

$$\to \frac{i}{c_d}(a^2\sin^2\phi - c_d^2)^{1/2}, \text{ for } \phi > \phi_{da},$$

$$N \to N_1$$

where $N_1 = (1^2 - 2\sin^2\phi)^2 + 4\sin^2\phi\cos\phi(1^2 - \sin^2\phi)^{1/2}$. (54)

Substituting these approximate values in the first integral of (31) one can find, for $\phi < \phi_{da}$

$$[u_z] \to N_{z1} \text{ as } \tau \to 1+ \tag{55}$$

where

$$N_{z1} = \frac{Pabc_d\cos^2\phi(1^2 - 2\sin^2\phi)}{\mu\rho\ (c_d^2 - a^2\sin^2\phi)^{1/2}.N_1}. \tag{56}$$

Again in the second integral of (31) we put $w = T_{d\alpha} \sin\alpha$ and as $\tau \to 1-$ for $\phi > \phi_{d\alpha}$ we find that

$$q_{d\alpha} \to i \sin\phi - i \cos\phi \, T_{d\alpha} \sin\alpha$$

$$\frac{dq_{d\alpha}}{dt} \to \frac{ic_d}{\rho} \cdot \frac{T_{d\alpha} \sin\alpha \sin\phi + \cos\phi}{(T_{d\alpha}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} \cdot \tag{57}$$

Puting these values in the second integral of (31), we get

$$\int_0^{\pi/2} \mathrm{Re}\left[k_{zd}(i \sin\phi - i \cos\phi \, T_{d\alpha} \sin\alpha, T_{d\alpha} \sin\alpha) \frac{ic_d}{\rho} \times \right.$$

$$\left. \times \frac{T_{d\alpha} \sin\alpha \sin\phi + \cos\phi}{(T_{d\alpha}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{d\alpha} \cos\alpha \, d\alpha \tag{58}$$

$$= \int_0^{\epsilon} \mathrm{Re}\left[k_{zd}(i \sin\phi - i \cos\phi \, T_{d\alpha} \sin\alpha, T_{d\alpha} \sin\alpha) \frac{ic_d}{\rho} \times \right.$$

$$\left. \times \frac{T_{d\alpha} \sin\alpha \sin\phi + \cos\phi}{(T_{d\alpha}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{d\alpha} \cos\alpha \, d\alpha +$$

$$+ \int_{\epsilon}^{\pi/2} \mathrm{Re}\left[k_{zd}(i \sin\phi - i \cos\phi \, T_{d\alpha} \sin\alpha, T_{d\alpha} \sin\alpha) \frac{ic_d}{\rho} \times \right.$$

$$\left. \times \frac{T_{d\alpha} \sin\alpha \sin\phi + \cos\phi}{(T_{d\alpha}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{d\alpha} \cos\alpha \, d\alpha \tag{59}$$

where ϵ is very small.

Since the main contribution to the integral (58) as $\tau \rightarrow 1$ arises from the first integral of (59) as $\tau \rightarrow 1$, so for the evaluation of (58) as $\tau \rightarrow 1$, we consider the approximate value of the integral given by

$$\int_0^{\epsilon} \mathrm{Re}\left[k_{zd}(i \sin\phi - i \cos\phi\, T_{d\alpha} \sin\alpha,\ T_{d\alpha} \sin\alpha)\, \frac{ic_d}{\rho} \times \right.$$

$$\left. \times \frac{T_{d\alpha} \sin\alpha \sin\phi + \cos\phi}{(T_{d\alpha}^2 \sin^2\alpha + 1 - \tau^2)^{1/2}} \right] T_{d\alpha} \cos\alpha\, d\alpha \qquad (60)$$

as $\tau \rightarrow 1$.

Since ϵ is very small so α is also small. So for the evaluation of the integral (60) as $\tau \rightarrow 1$ we also use the fact that $\alpha \rightarrow 0$, from which we get,

$$w \rightarrow 0,\ q_{d\alpha} \rightarrow i \sin\phi,\ m_d \rightarrow \cos\phi,\ m_s \rightarrow (1^2 - \sin^2\phi)^{1/2},$$

$$m_0 \rightarrow (1^2 - 2 \sin^2\phi), \qquad (61)$$

$$N \rightarrow N_1,\ E^{1/2} \rightarrow i/c_d(a^2 \sin^2\phi - c_d^2)^{1/2} \text{ for } \phi > \phi_{d\alpha}.$$

Now substituting these approximate values in (60) and integrating we obtain the approximate value of the integral as

$$- \frac{c_d^2 \cos^2\phi(1^2 - 2\sin^2\phi)}{\rho(a^2\sin^2\phi - c_d^2)^{1/2}.N_1} \log|\tau - 1| \quad \text{when } \tau \longrightarrow 1. \qquad (62)$$

So for $\phi > \phi_{d\alpha}$

$$[u_z] \longrightarrow N'_{z4} \log|\tau - 1| \text{ as } \tau \longrightarrow 1 \qquad (63)$$

where

$$N'_{z4} = - \frac{2Pabc_d \cos^2\phi(1^2 - 2\sin^2\phi)}{\pi\mu\rho\,(a^2 \sin^2\phi - c_d^2)^{1/2}.N_1} . \qquad (64)$$

In order to obtain the value of u_{zd} as $\tau \longrightarrow \tau_{d\alpha}$ we put

$$w^2 = A_{d\alpha}^2 + (T_{d\alpha}^2 - A_{d\alpha}^2)\sin^2\alpha.$$

in the second integral of (31).

When $\tau \longrightarrow \tau_{d\alpha}+$, we find that

$$w \longrightarrow 0$$

$$q_{d\alpha} \longrightarrow i\,\frac{c_d}{a}$$

$$dq_{d\alpha}/dt \longrightarrow iA'$$

where $$A' = \frac{c_d}{\rho a}\left(\frac{a^2 - c_d^2}{1 - \tau_{d\alpha}^2}\right)^{1/2} \text{ for } a > c_d,$$

$$m_d \to 1/a(a^2 - c_d^2)^{1/2} \text{ for } a > c_d, \tag{65}$$

$$m_s \to \frac{l}{a}(a^2 - c_s^2)^{1/2}, \quad m_\alpha \to \frac{l^2}{a^2}(a^2 - 2c_s^2),$$

$$N \to N_2$$

where $$N_2 = 1/a^4\left[l^4(a^2 - 2c_s^2)^2 + 4lc_d^2(a^2 - c_d^2)^{1/2}(a^2-c_s^2)^{1/2}\right]$$

$$E^{1/2} \to iK^{1/2}(\tau - \tau_{d\alpha})^{1/2}$$

where

$$K = \frac{2a}{c_d}\,\frac{\cos^2\alpha(a^2 - c_d^2)^{1/2}}{\left[(a^2 - c_d^2)^{1/2}\sin\phi - c_d\cos\phi\right]} \qquad \text{for } a > c_d.$$

Using these approximate values in the second integral of (31) we find that for $a > c_d$

$$[u_z] \to N_{z4} \text{ as } \tau \to \tau_{d\alpha}+ \tag{66}$$

where

$$N_{z4} = \frac{2Pab}{\pi\mu c_d a^3}\,\frac{l^2(a^2 - c_d^2)^{1/2}(a^2 - 2c_s^2)A'C^{1/2}}{(2KA)^{1/2}N_2} \tag{67}$$

where $$C = 8a^2 c_d \tau_{d\alpha} (a^2 - c_d^2)^{1/2} \sin\phi \cos\phi$$

$$A = a^2(a^2-b^2)\cos^2\phi\,\tau_{d\alpha}(\tau_{d\alpha} + \tau^0_{d\alpha}) + a^2b^2\sin^2\phi\,\tau_{d\alpha}(\tau_{d\alpha} - \tau^0_{d\alpha})$$

$$\tau^0_{d\alpha} = \frac{1}{a}\left[c_d \sin\phi - (a^2-c_d^2)^{1/2} \cos\phi \right] \tag{68}$$

It may be noted that conical wave front $\tau = \tau_{d\alpha}$ does not arise for $a < c_d$.

Next when $\phi < \phi_{s\alpha}$, for the evaluation of u_{zs} as $\tau \to 1$, we put $w = T_s \sin\alpha$ in the first integral of (33). When $\tau \to 1$, we find that in the above integral

$$w \to 0$$

$$q_s^+ \to i1 \sin\phi$$

$$\frac{dq_s^+}{dt} \to \frac{c_d}{\rho} \frac{1 \cos\phi}{T_s \cos\alpha}$$

$$(q^2+w^2) \to -1^2 \sin^2\phi$$

$$m_d \to (1-1^2\sin^2\phi)^{1/2}$$

$$m_s \to 1 \cos\phi$$

$$m_0 \rightarrow l^2(\cos^2\phi - \sin^2\phi)$$

$$E^{1/2} \rightarrow \frac{1}{c_s}(c_s^2 - a^2\sin^2\phi)^{1/2} \quad \text{for } \phi < \phi_{sa}$$

$$\rightarrow \frac{i}{c_s}(a^2\sin^2\phi - c_s^2)^{1/2} \quad \text{for } \phi > \phi_{sa}$$

$$N \rightarrow l^3 N_s$$

where $N_s = \left[l(\cos^2\phi - \sin^2\phi)^2 + 4\sin^2\phi\cos\phi(1 - l^2\sin^2\phi)^{1/2} \right]$.

Using these approximate values in the first integral of (33) one can find for all values of a and b,

$$[u_z] \rightarrow N_{z2} \quad \text{for } \phi < \phi_{sa} \text{ as } \tau \rightarrow 1 \tag{70}$$

where

$$N_{z2} = -\frac{2pabc_s}{\mu\rho}\,\frac{\sin^2\phi\cos\phi\,(1-l^2\sin^2\phi)^{1/2}}{(c_s^2-a^2\sin^2\phi)^{1/2}.N_s} \tag{71}$$

For $\phi > \phi_{sa}$, considering approximate evaluation of last two integrals of (33) as $\tau \rightarrow 1$ it can be shown that for the case $a > b > c_d$

$$u_z \rightarrow N'_{z5} \log|\tau - 1| \quad \text{for } \phi_{sa} < \phi < \phi_{sd} \text{ as } \tau \rightarrow 1 \tag{72}$$

$$u_z \rightarrow N'_{z3} \log|\tau - 1| \quad \text{for } \phi > \phi_{sd} \text{ as } \tau \rightarrow 1 \tag{73}$$

and for the case $c_d > a > b > c_s$,

$$u_z \longrightarrow N'_{z6} \log|\tau - 1| \text{ for } \phi_{sd} < \phi < \phi_{sa} \quad \text{as } \tau \longrightarrow 1 \qquad (74)$$

$$u_z \longrightarrow N'_{z3} \log|\tau - 1| \text{ for } \phi > \phi_{sa} \quad \text{as } \tau \longrightarrow 1 \qquad (75)$$

and also for the case $c_s > a > b$,

$$u_z \longrightarrow N'_{z6} \log|\tau - 1| \text{ for } \phi > \phi_{sd} \quad \text{as } \tau \longrightarrow 1 \qquad (76)$$

where

$$N'_{z5} = \frac{2pabc_s}{\pi\mu\rho} \frac{\sin^2\phi \cos\phi(1-l^2\sin^2\phi)^{1/2}}{(a^2\sin^2\phi - c_s^2)^{1/2} N_3} \qquad (77)$$

$$N'_{z3} = \frac{8pabc_s}{\pi\mu\rho} \frac{\sin^4\phi \cos^2\phi \, (l^2\sin^2\phi - 1)}{(a^2\sin^2\phi - c_s^2)^{1/2} N_4} \qquad (78)$$

$$N'_{z6} = - \frac{2pabc_d}{\pi\mu\rho} \frac{\sin^2\phi \cos\phi \, (l^2\sin^2\phi - 1)^{1/2}(\cos^2\phi - \sin^2\phi)^2}{(c_s^2 - a^2\sin^2\phi)^{1/2} N_4} \qquad (79)$$

$$N_4 = \left[l^2(\cos^2\phi - \sin^2\phi)^4 + 16\sin^4\phi \cos^2\phi(l^2\sin^2\phi - 1) \right] \qquad (80)$$

For the approximate evaluation of the displacements at the wave fronts $\tau = \tau_{sa}$ and $\tau = \tau_{sd}$ we follow similar procedure as followed for the evaluation of u_{zd} as $\tau \longrightarrow \tau_{da}$ and we find that

$$[u_z] \longrightarrow N_{z5} \quad \text{as } \tau \longrightarrow \tau_{sa} \text{ for } a > c_d \tag{81}$$

$$[u_z] \longrightarrow N_{z6} \quad \text{as } \tau \longrightarrow \tau_{sa} \text{ for } c_d > a > c_s \tag{82}$$

$$[u_z] \longrightarrow N_{z3}(\tau - \tau_{sd})^{3/2} \quad \text{as } \tau \longrightarrow \tau_{sd} \text{ for } a > c_d \tag{83}$$

$$[u_z] \longrightarrow N_{z7}(\tau - \tau_{sd}) \quad \text{as } \tau \longrightarrow \tau_{sd} \text{ for } a < c_d \tag{84}$$

where

$$N_{z5} = - \frac{4Pbc_d \; A_s' \; [(a^2-c_d^2)D_s]^{1/2}}{\pi\mu a^2 (2K_s B_s A_s)^{1/2}} \tag{85}$$

$$N_{z6} = - \frac{16Pa^2 bc_d^3(c_d^2-a^2) \; A_s' \; [(a^2-c_s^2)D_s]^{1/2}}{\pi\mu(2K_s l^2 A_s)^{1/2} \; [l^6(a^2-2c_s^2)^4 - 16c_d^4(c_d^2-a^2)(a^2-c_s^2)]} \tag{86}$$

$$N_{z3} = - \frac{4Pab}{\pi\mu} A_{sd} \; B_{sd}^2 \; B_{sd}' \; A_{sd}' \left[\frac{2 \operatorname{cosec}\phi}{a^2 - c_d^2} \right]^{1/2} \tag{87}$$

$$N_{z7} = \frac{4Pab}{\pi\mu} A_{sd} \; B_{sd}^2 \; A_{sd}' \left[\frac{2 \operatorname{cosec}\phi}{c_d^2 - a^2} \right]^{1/2} \tag{88}$$

$$A_s' = \frac{lc_d \; (a^2 - c_s^2)^{1/2}}{\rho \; [\; l(a^2-c_s^2)^{1/2} \sin\phi - c_d \cos\phi \;]} \tag{89}$$

$$D_s = 8a^2 l c_d \tau_{sa} \sin\phi \cos\phi \, (a^2 - c_s^2)^{1/2} \qquad (90)$$

$$B_s = \frac{1}{a^4} \left[l^3 (a^2 - 2c_s^2)^2 + 4c_d^2 \left\{ (a^2 - c_d^2)(a^2 - c_s^2) \right\}^{1/2} \right] \qquad (91)$$

$$A_s = \left[\tau_{sa} a^2 b^2 (\tau_{sa} - \tau_{sa}^{o}) \sin^2\phi + (a^2 - b^2) a^2 \cos^2\phi (\tau_{sa} + \tau_{sa}^{o}) \right] \qquad (92)$$

$$A_{sd} = \frac{\pi}{4} \left[\frac{2(l^2-1)^{1/2}}{(l^2-1)^{1/2} \sin\phi - \cos\phi} \right]^{1/2} \qquad (93)$$

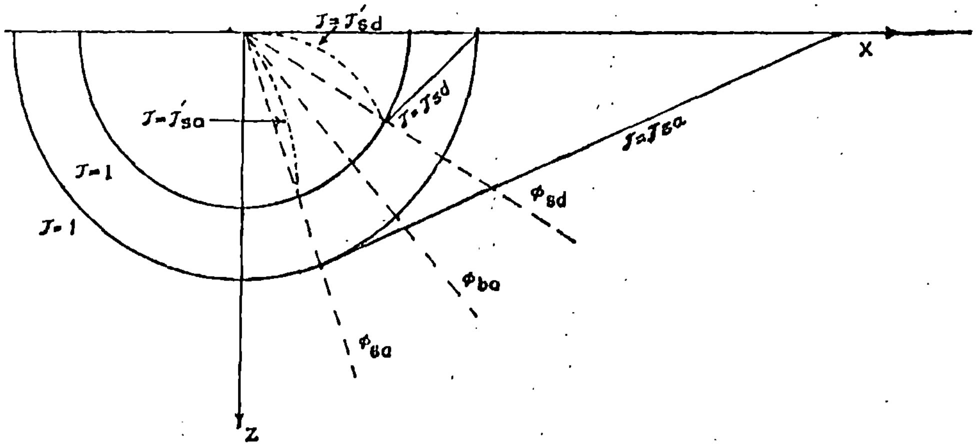
$$B_{sd} = (l^2 - 2)^{-1} \qquad (94)$$

$$B'_{sd} = 4 \, A_{sd} \, (l^2 - 1)^{1/2} \, B_{sd}^2 \qquad (95)$$

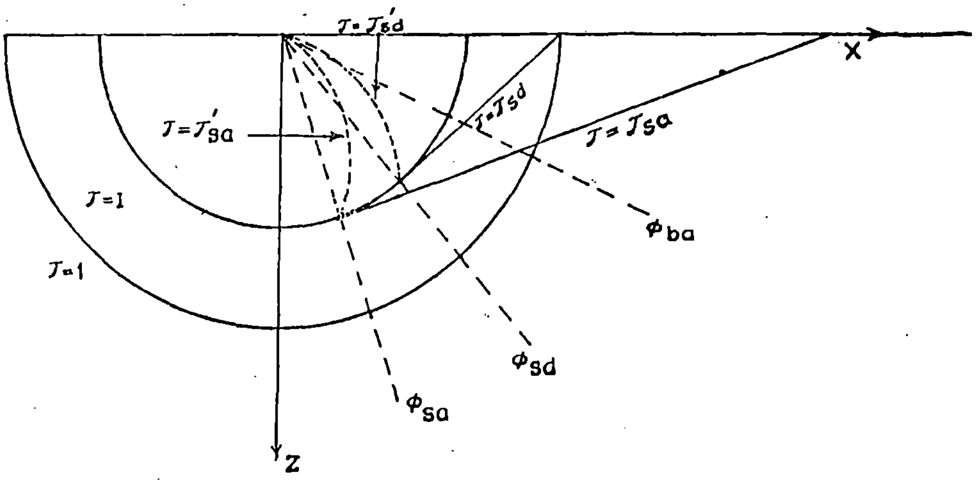
$$A'_{sd} = \frac{c_d}{\rho} (l^2-1)^{1/2} \left[(l^2-1)^{1/2} \sin\phi - \cos\phi \right]^{-1} \qquad (96)$$

In these expressions the notations $[u_z]$ stands for the change in u_z across a wave front and N_{z1} etc. are wave front coefficients.

It may also be noted that if we put a = b in this problem, it reduces to the problem of uniformly expanding circular ring source and in that case our derived results coincide with the results given in the paper of Gakenheimer [1971].

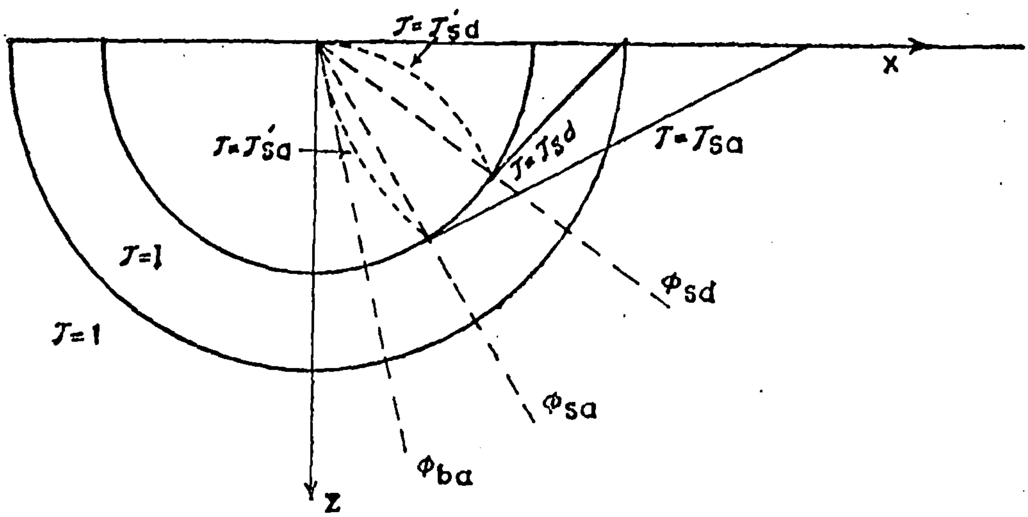


4 (a) for $a > c_d$, $a > b > c_s$, $a c_s > b c_d$.

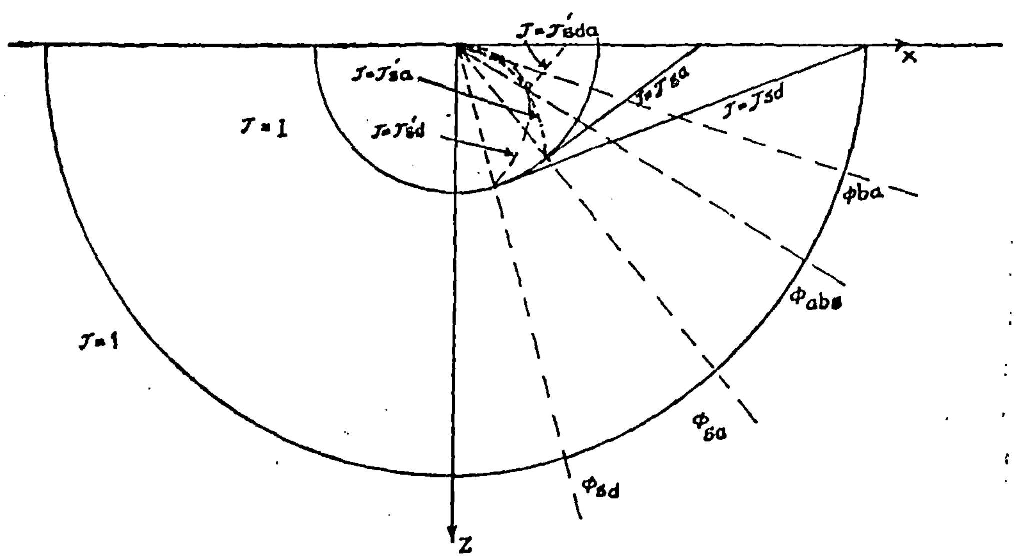


4 (b) for $a > c_d$, $a > b > c_s$, $a c_s < b c_d$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

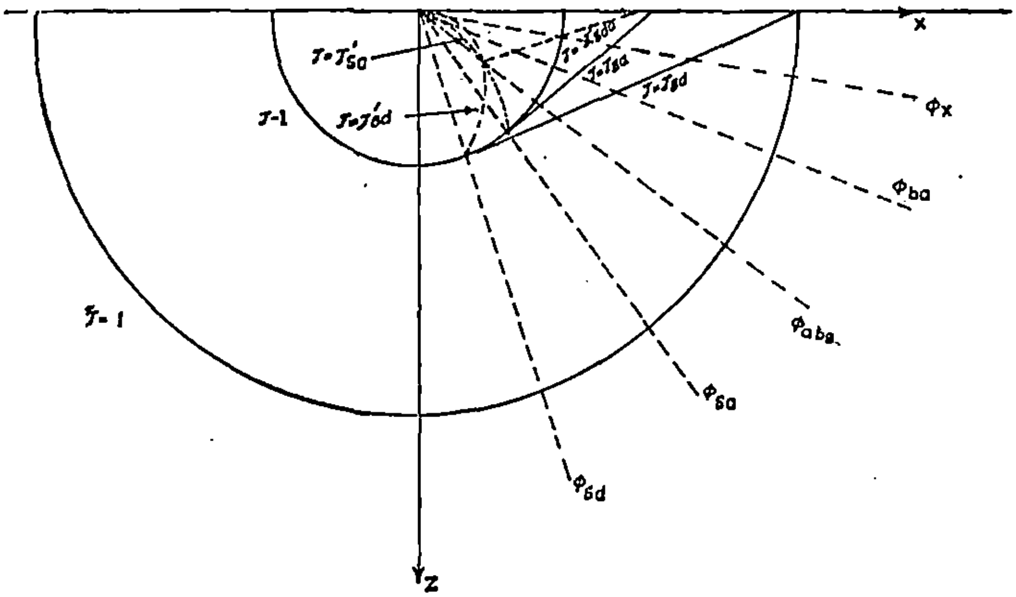


4 (c) for $a > c_d > c_s > b$.

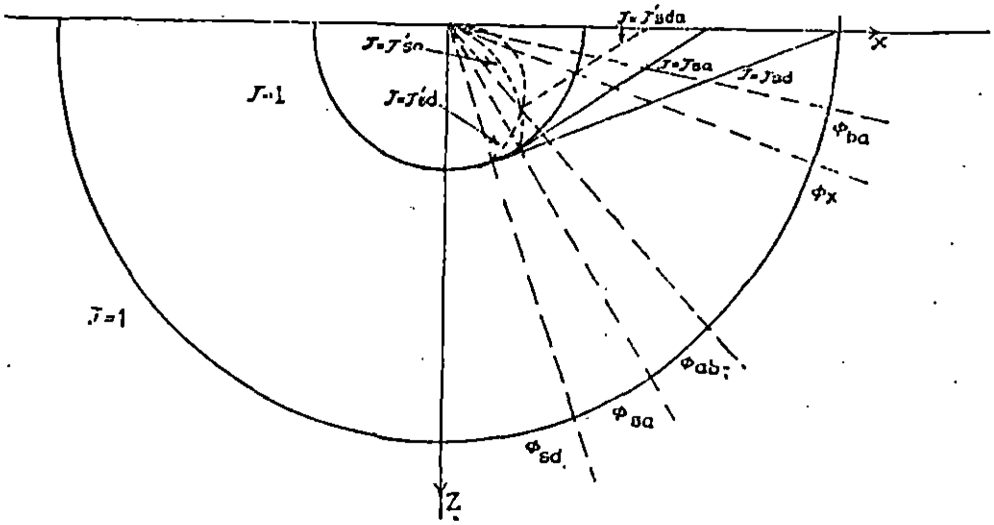


4 (d) for $c_d > a > b > c_s$, $\alpha > \beta$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

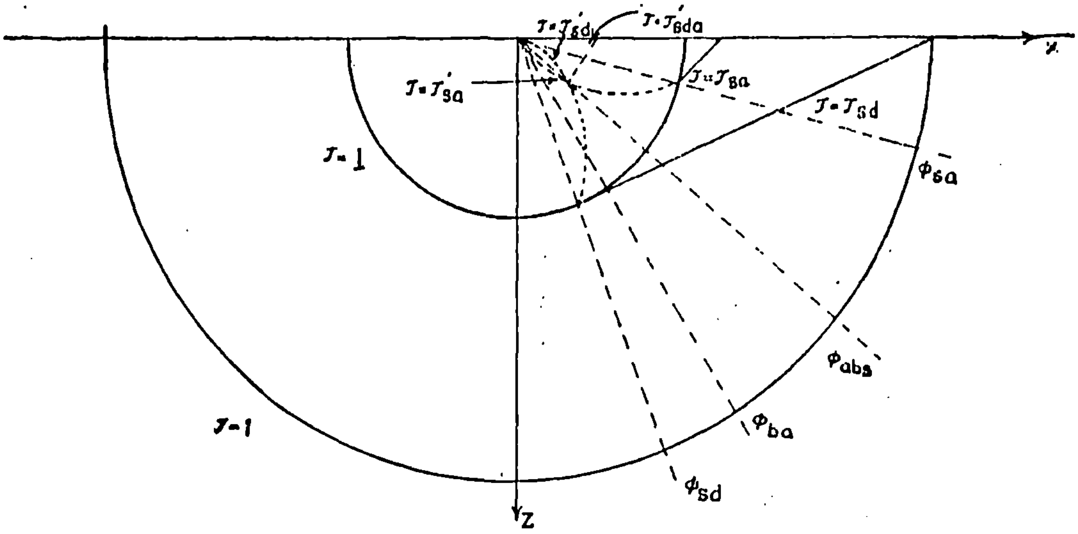


4 (e) for $c_d > a > b > c_s$, $\beta > \alpha > \gamma$.

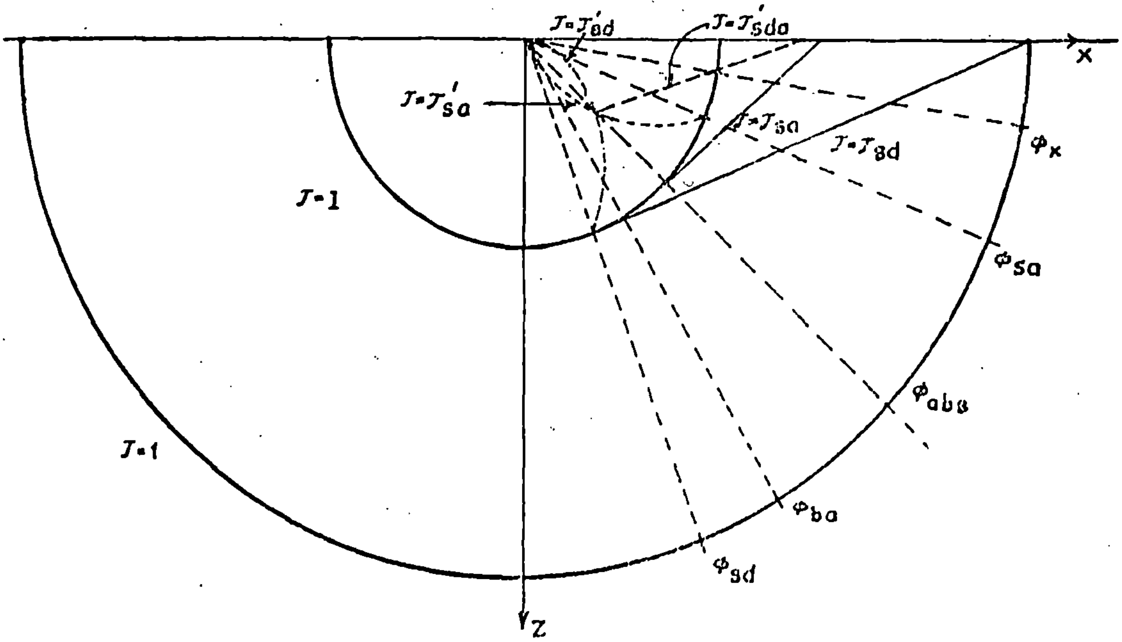


4 (f) for $c_d > a > b > c_s$, $\alpha < \gamma$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

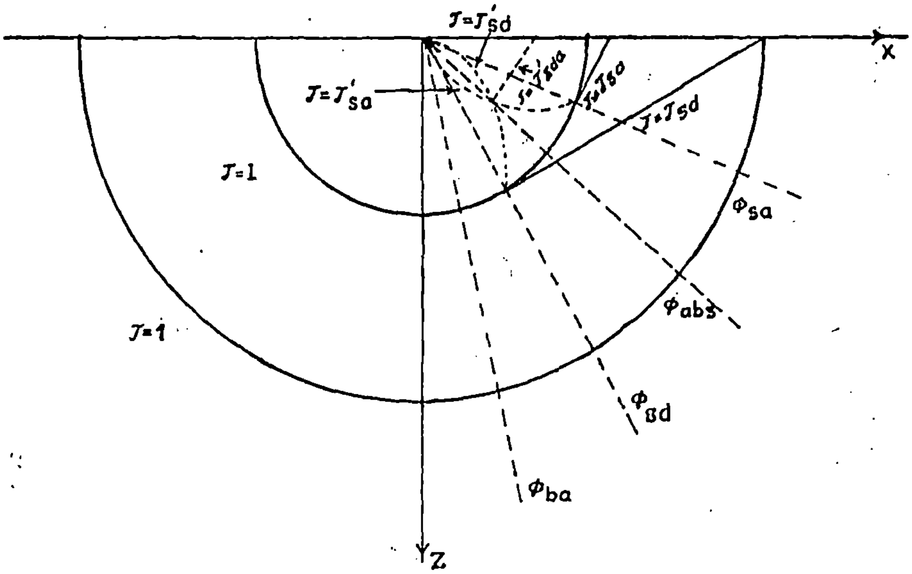


4 (g) for $c_d > a > c_s > b$, $\alpha > \beta$, $ac_s < bc_d$.

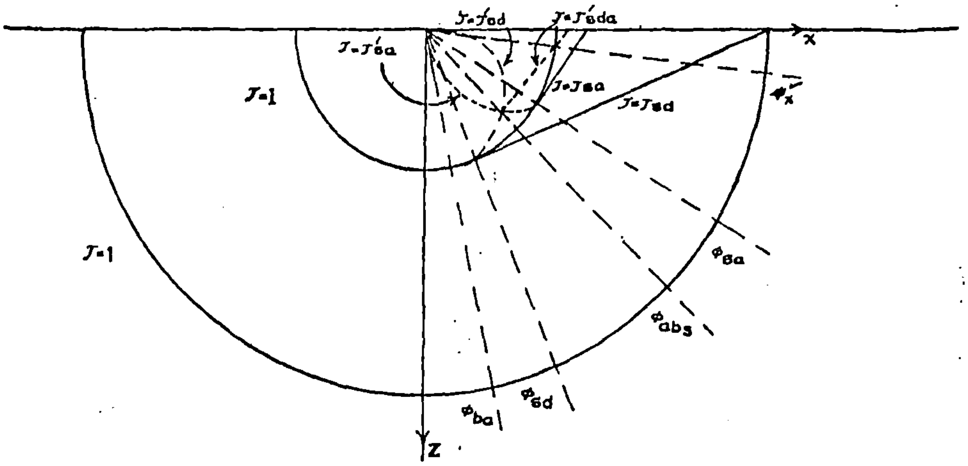


4 (h) for $c_d > a > c_s > b$, $\beta > \alpha > \gamma'$, $ac_s < bc_d$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.

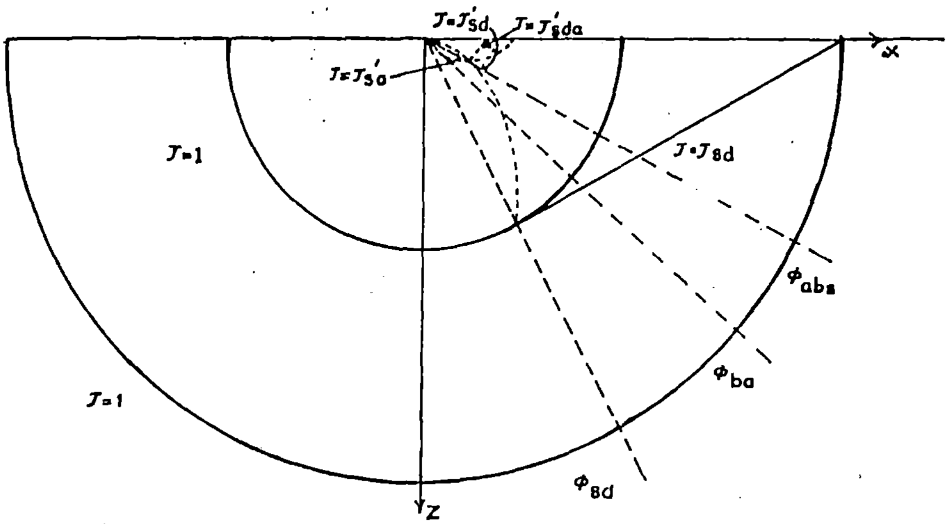


4 (i) for $c_d > a > c_s > b$, $\alpha > \beta$, $a\,c_s > b\,c_d$.

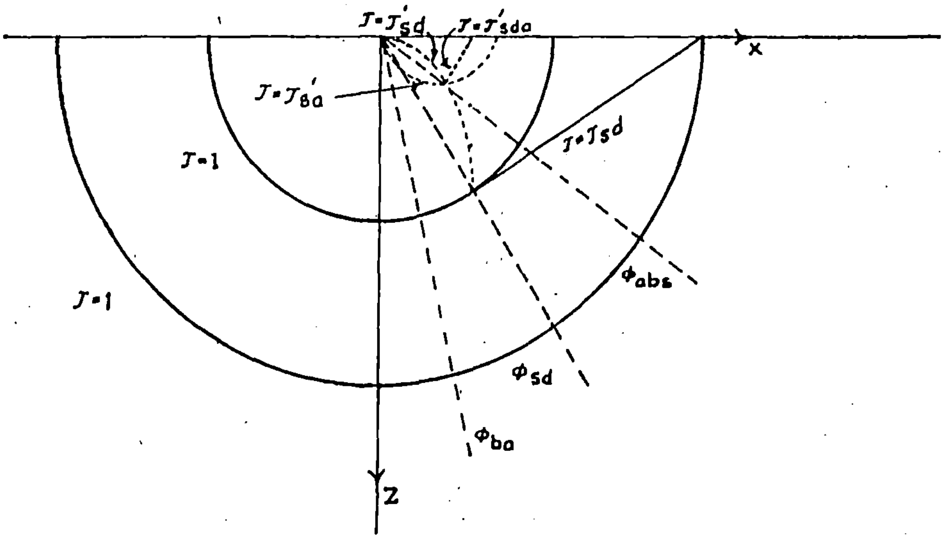


4 (j) for $c_d > a > c_s > b$, $\beta > \alpha > \gamma'$, $ac_t > bc_d$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.



4 (k) for $a < c_s$, $ac_s < bc_d$.



4 (l) for $a < c_s$, $ac_s > b\,c_d$.

Fig. 4. Wave pattern for equivoluminal and head wave motion.