I N T R O D U C T I O N

The study of wave and vibration phenomena in elastic solids has a distinguished history of more than hundred years. Some pioneer workers in the field of wave propagation in elastic medium and vibrating bodies are Cauchy, Rayleigh, Love, Poisson, Ostrogadsky, Green, Lame, Stokes, Kelvin.

Seismology has made a tremendous progress during the last three decades, mainly because of the technological developments, which have enabled seismologist to make measurements with far greater precision and sophistication than was previously possible.

Here, some of the major progress in the field of wave propagation are given in chronological order.

1678 : Robert Hooke (England) established the stress-strain relation for elastic bodies.

1821 : Louis Nevier (France) derived the differential equations of the theory of elasticity.

1822 : Cauchy developed most of the aspects of the pure theory of elasticity including the dynamical equations of motion for a solid.

1828 : Simeo-Denis Poisson (France) predicted theoretically the existance of longitudinal and tranverse elastic waves.

1849 : George Gabriel Stokes (England) conceived the first mathematical model of an earthquake source.

1857 : First systematic attempt to apply physical principles to earthquake effects by Robert Mallet (Ireland).

1862 : Clebsch found the general theory for the free vibration of solid bodies using normal modes.

1872 : J. Hopkinson performed the first experiments on plastic waves propagation in wires.

1883 : saint Venant summarized the work on impact of earlier investigators and presented his results on transverse impact.

1883 : Rosi-Forel scale for earthquake effects published.

1885 : C. Somigliana (Italy) produced formal solutions to Navier equations for a wide class of sources and boundary conditions.

1887 : Lord Rayleigh (England) predicted the existance of elastic surface waves.

1899 : C. G. Knott (England) derived the general equations for the reflection and refraction of plane seismic waves at plane boundaries.

1903 : A. E. H. Love (England) developed the fundamental theory of point sources in an infinite elastic space.

1904 : Horance Lamb (England) made the first investigation of pulse propagation in a semi-infinite solid.

1911 : Love developed the theory of waves in a thin layer overlying a solid and showed that such waves accounted for certain anomalies in seismogram records.

1949 : Devies published an extensive theoretical and experimental study on waves in bars.

1959 : Ari Ben-Menahem (Israel) discovered that the energy release in earthquakes takes place through a propagating rupture over the causative fault.

1967 : Global seismicity patterns and earthquake generation linked to plate motions.

During the first two decades of this century the subject was not given so much importance by Mathematicians or Physicists. But later, interest in the study of waves in elastic solids attracted the attention of the researchers because of applications in the field of geophysics and engineering constructions. Since that time in seismology the wave propagation has remained an interesting area because of the need for details information on earthquake phenomena, prospecting techniques and the detection of nuclear explosions. Bullen [1963], Ewing et al [1957], Cagniard [1962], Pilant [1979] and Aki and Richards [1980] have discussed about seismic waves in their books.

During last 30-40 years the development of theory of wave propagation in elasticity has been characterized by a detailed investigation of the classical methods of mathematical analysis and the trends to obtain specific results. The solution of many of the problems in elastodynamics, which are frequently encountered in practice need advance level of mathematical technique, which may roughly be grouped into the following categories:

(a) Theory of analytic function

(b) The Fredholm integral equation

(c) The singular integral equation

(d) Integral transforms and Representations

(e) Dual integral and series equations

(f) Harmonic function. Potential theory

(g) The Drichlet and Neumann problems

(h) Green's functions

(i) The Cauchy problem

(j) Cagniard-deHoop technique

(k) Wiener - Hopf technique

(l) Riemann - Hilbert problem

(m) The method of Matched Asymptotic expansions

(n) Perturbation technique

(o) Variational method, The Ritz method

(p) The method of finite element

(q) The method of boundary element

and others.

While earlier investigation in the theory of elasticity was essentially reduced to the construction of particular solution; the invention of computer technology has led to the development of general and quite universal methods of solving the problems of this theory, namely, the boundary value problems and initial boundary value problems for systems of differential equations having partial derivatives of a definite structure.

Most of the experimental works carried out on the wave propagation are concerned with studying propagation in specimens of comparatively simple geometrical shape. The results of this experiment could be compared directly with exact or approximate theoretical predictions. The agreement, with experimental results and theoretical predictions, inspires confidence in taking up complicated problems and makes possible theoretical predictions and interpretations of observations.

The propagation of waves through homogeneous isotropic elastic materials of unbounded extension is not a subject of very complexity. The waves are either dilatiational or distortional or a combination there of. The picture changes radically as soon as there is a boundary. Interaction of two types of waves occurs, when boundary is present and this interaction presents an inherent difficulty in the solution of elastodynamic problems.

More over the effect of a free surface on the generation and propagation of waves in elastic medium has been the subject of many investigations ever since the discovery of existance of surface waves by LORD RAYLEIGH.

In general, problems which mostly attract the researchers both theoretical and experimental, in relation to the generation and propagation of waves in an elastic medium may be classified as follows;

(i) diffraction of propagating waves through the medium due to any obstacle, cavity or a crack of any shape situated some where in the medium;

(ii) reflection, refraction and diffraction of propagating waves due to mixed boundary conditions;

(iii) wave motion generated due to a punch on some bounded region of the medium;

(iv) radiation of waves i.e. the wave motions generated due to some fixed external disturbance and propagating away from the source of disturbance;

(v) wave motion generated in a medium when a source of disturbance moves along the medium.

Depending on the nature of the source of disturbance, shape of the punch or normal loading on the free surface and the presence of discontinuities in the medium, different complicated problems arise. The solution of these problems need an advance level of sophisticated mathematical techniques some of which have been mentioned earlier.

The dynamic response of an elastic half space due to an external load or punch on the free surface and also the scattering of elastic waves by a finite crack or a strip inside an elastic medium may be investigated by the use of integral transform technique.

The propagation of waves due to the application of loads at the boundary of a semi-infinite medium was first considered by Lamb [1904], who studied the axisymmetric propagation of a pulse created by transient normal point load on the surface of the half-space. By means of Fourier Synthesis of steady state solutions, Lamb showed the predominant character of the Rayleigh wave response. Later, Sauter [1950] derived a closed form solution by means of an integral superposition of plane harmonic waves. Many authors have subsequently viewed and reviewed the problems which deal with the disturbance produced by a point or line source acting on the surface or buried in an elastic half-space by means of Laplace transform. Pekeris [1955] derived the exact expression for the vertical and horizontal components of the displacement on the surface of a uniform elastic half-space due to a point load with step function time variation, situated on the surface and also at a finite depth below the surface. Thiruvenkatachar [1955] derived the exact expression for the Laplace transform of the displacement over a circular region which is more realistic physically. Knopoff and Gilbert [1959] and Lang [1961] derived the wave front approximation by the application of saddle point method to the Laplace transformed solution and limit theorems of Tauberian type. While

Cagniard [1962] developed powerful technique of finding the Laplace inversion for this class of problems. Mitra [1964] investigated this type of problem in detail, verified Pekaris's result and pointed out that Cagniard's method can be applied more widely than either Pekaris's or Chao's method. This type of problem was then investigated by Eason [1964, 1966], Mitra [1964], Chakraborty and De [1971], Gakenheimer [1971], Ghosh [1971] and many others. All these are axisymmetric problems.

Very few wave propagation problems of non-axisymmetric type have been solved. Chao [1960] derived the exact solution for the half-space problem in which the disturbance is due to a tangential surface point load. Pekeris and Longman [1958] investigated the motion of the surface of a uniform elastic half-space produced by the application of torque pulse at a point below the surface. Using a modification of Cagniard's method, Gakenheimer and Miklowitz [1969] analysed transient excitation of the elastic half-space by a point load travelling on the surface. All these non-axisymmetric problems deal with the point load.

For the problems dealing with the ring load we refer Maiti [1978], Ghosh [1980-81] and others. Maiti [1978] treated the problem of asymmetric finite source, examined the effect of a half-space of impulsive shearing traction over a circular portion of the surface. The formal solution is obtained by expressing the displacement components in terms of scalar and vector potentials and using Laplace and double Fourier transforms. The inverse transforms are evaluated by modified Cagniard's techinque which yields the solution within and on the half-space in a closed integral form. Ghosh [1980-81] treated the problem of disturbance in an elastic semi-infinite medium due to the torsonal motion of a circular ring source on the free surface of homogeneous and inhomogeneous medium. Using Laplace transform and the Hankel transform and the Laplace inversion by Cagniard's method the integrals for displacement are evaluated numerically.

On the other hand Pal and Ghosh [1987] considered the elliptic ring load propagating over the free surface of a semi-infinite medium. The expression for displacement at points on the free surface has been derived in integral form by the application of Cagniard-de-Hoop technique for different values of the rate of increase of the major and minor axes of the elliptic ring source The displacement jumps across the different wave fronts have also been derived. A comprehensive survey of the field due to extended source problems has been given by Scott and Miklowitz [1964].

The problems relating to the propagation of elastic waves, due to applied boundary tractions, in semi-infinite media containing internal boundaries are of immense importance in seismology and geophysics rather than of point source problems in homogeneous semi-infinite medium. This type of problem was first considered by Johnson and Parnes [1977]. The problem, they treated, is that of a semi-infinite elastic body containing a rigid lubricted inclusion whose axis is perpendicular to the plane surface subjected to an axisymmetric concentric line load applied dynamically as a step function in time at the plane surface. The dynamic problem was formulated interms of two potential functions which satisfy uncoupled two dimensional wave equations with coupled boundary conditions. Using Laplace transform, the integral solution for the transformed stress and displacement fields throughout the medium are obtained. The behaviour near the wave fronts was analyzed and singularities at the load were determined.

This type of work has been treated by Pal, Ghosh and Chowdhuri [1985]. They solved the problem of SH-type of elastic wave propagating in the semi-infinite medium due to a ring source producing SH-waves in presence of circular cylindrical cavity as well as circular cylindrical inclusion in the semi-infinite medium.

The diffraction of elastic waves by cracks is the most interesting branch of elastodynamics. Normally cracks are present in all structural materials, either as natural defects or as a result of fabrication processes. In many cases, the cracks are sufficiently small so that their presence does not significantly reduce the strength of the material. In other cases, however, the cracks are large enough, or they may become large enough through fatigue, stress corrosion cracking, etc., so that they must be taken into account in determining the strength. The body of knowledge which has been developed for the analysis of stresses in cracked solids is known generally as fracture mechanics. In recent years problems of diffraction of elastic waves by cracks are of considerable importance in view of their application in seismology and geophysics. Indeed in geophysical stratifications, faults occur at the interfaces while in manufactured laminates imperfections occur at the interface of the adjoining layers. This stress singularity near the edge of finite crack at the bimaterial interface is important in view of its practical application. Also the results of research on dynamic crack propagation are relevent in numerous applications. For example, a primary objective in engineering structures is to avoid a running fracture, or at least to arrest a running crack once it is initiated. Indeed there are several kinds of large engineering structures in which rapid crack growth is a definite possibility. In earth science, study of the elastic field near about the propagating fault is also important from the stand point of earthquake engineering.

Whithin the framework of a continuum model, such as the homogeneous, isotropic linearly elastic continuum, the classic analytical problem of fracture mechanics consists of the computation of the fields of stress and deformation in the vicinity of the tip of a crack, together with the application of a fracture criterion. In a conventional analysis inertia (or dynamic) effects are neglected and the analytical work is quasi-static in nature.

Because of loading conditions and material characteristics, however, there are many fracture mechanics problems which can not be viewed as being quasi-static and for which the inertia of the material must be taken into account. Also inertia effects become of importance if the propagation of the crack is so fast, as for example in essentially brittle fracture, that rapid motions are generated in the medium. The label "dynamic loading" is attached to the effects of inertia on fracture due to rapidly applied loads.

There are two broad classes of fracture mechanics problems that have to be treated as dynamic problems. These are concerned with

1. cracked bodies subjected to rapidly varying loads,
2. bodies containing rapidly propagating cracks.

In both the cases the crack tip is an environment disturbed by wave motion.

Impact and vibration problems fall into the first class of dynamic problems. In the analysis of such problems it is often found that at inhomogeneities in a body the dynamic stresses are higher than the stresses computed from the corresponding problem of static equilibrium. This effect occurs when a propagating mechanical disturbance strikes a cavity. The dynamic stress "overshoot" is especially pronounced if the cavity contains a sharp edge. For a crack the intensity of the stress field in the vicinity of the crack tip can be significantly affected by dynamic effects.

In view of the dynamic amplification, it is conceivable that there are cases for which fracture at a crack tip does not occur under a gradually applied system of loads, but where a crack does indeed propagate when the same system of loads is rapidly applied, and gives rise to wave which strike the crack tip.

The second class of problems is equally important. Indeed, there are several kinds of large engineering structure in which rapid crack growth is a definite possibility. Examples are gas transmission pipelines, ship hulls, aircraft fuselages and nueclear reactor components. Dynamic effects affect the stress fields near rapidly propagating cracks, and hence the conditions for further unstable crack propagation or for crack arrest. Another area to which the analysis of rapidly propagating cracks is relevant is the study of earthquake mechanisims.

There have been a number of comprehensive review articles in the general area of elastodynamic fracture mechanics. some of them are Achenbach [1972], Freund [1975], Achenbach [1976], Freund [1976] and Kanninen [1978].

At present, dynamic fracture mechanics solutions are largely confined to conditions where Linear Elastic Fracture Mechanics (LEFM) is valid. These are appropriate when the plastic deformation attending the crack tip is small enough to be dominated by the elastic field surrounding it. Problems of crack growth initiation under impact loads and of rapid unstable crack propagation and arrest can be treated with LEFM by using dynamically computed fields of stress and deformation. Engineering structures requring protection against the possibility of large-scale catastropic crack propagation are, however, generally constructed of ductile, tough materials. For the initiation of crack growth, LEFM procedures can give only approximately correct predictions for such materials. The elastic-plastic treatments required to give precise results have not yet been developed in a completely acceptable manner, even under static conditions.

The shapes of the cracks which have been studied uptil now are as follows :

(i) Semi-infinite plane cracks;

(ii) Finite Griffith cracks;

(iii) Penny shaped and annular cracks;

(iv) Non-planar cracks.

A transient problem involving the sudden appearance of a semi-infinite crack in a stretched elastic plate was considered by Maue [1954]. Baker [1962] studied the problem of a semi-infinite crack suddenly appearing and growing at constant velocity in a stretched elastic body. The mixed boundary value problem is solved by transform methods including the Weiner-Hopf and Cagniard techniques. Atkinson and List [1978] considered the steady state semi-infinite crack propagation into media with spatially varying elastic properties. The quasi-static problem of an infinite elastic medium weakened by an infinite number of semi-infinite, rectilinear, parallel and equally spaced cracks which are subjected to identical loads satisfying the conditions of amtiplane state of strain was solved by Matczynski [1973]. Sarkar, Ghosh and Mandal [1991] studied the problem of scattering of horizontally polarized shear wave by a semi-infinite crack running with uniform velocity along the interface of two dissimilar semi-infinite elastic media.

The powerful technique to solve the diffraction problem of semi-infinite crack is the Wiener-Hopf [Noble 1958] technique.

The in-plane problem of finite Griffith crack propagating at a constant velocity under a uniform load was first solved by Yoffe [1951]. Sih [1968] has also provided a Riemann-Hilbert formulation of the same problem where both in-plane extensional and antiplane shear loads were considered.

Other references treating elastodynamic problem involving a single finite Griffith crack are Loeber and Sih [1967]. Ang and Knopoff [1964]. Mal [1970, 1972], Chang [1971], Kassir and Bandyopadhyay [1983], Kassir and Tse [1983]. Loeber and Sih [1967] solved the problem of diffraction of antiplane shear waves by a finite crack by using integral transform method. Kassir and Bondyopadhyay [1983] considered the problem of impact response of a cracked orthotropic medium. Laplace and Fourier transforms were employed to reduce the transient problem to the solution of standard integral equation in the Laplace transform plane and was solved by Laplace inversion technique [Krylov et al, 1957]; Miller and Guy [1966].

The problems of finite Griffith crack lying at the interface of two dissimilar elastic media have been studied by Srivastava, Palaiya and Karaulia [1980], Nishida, Shindo and Atsumi [1984] and Bostrom [1987]. Bostrom [1987] used the method of Krenk and Schmidt [1982] to solve the two-dimensional scalar problem of scattering of elastic waves under antiplane strain from an interface crack between two elastic half-spaces. Sih and Chen [1980] analyzed the dynamic response of a layered composite containing a Griffith crack under normal and shear impact.

The problems of diffraction of elastic waves become more complicated when boundaries are present in the medium. Chen [1978] considered the problem of dynamic response of a central crack in a finite elastic strip. The crack was assumed to appear suddenly when the strip is being stretched at its two ends. The problem was solved by Laplace and Fourier transform technique. Some other references are Srivastava, Gupta and Palaiya [1981], Srivastava, Palaiya and Karaulia [1983], Shindo, Nozaki and Higaki [1986], De and Patra [1990].

High frequency solution of the diffraction of elastic waves by a crack of finite size is interesting in view of the fact that transient solution close to the wave front can be represented by an integral of the high frequency componant of the solution. Green's function method together with a function-theoretic technique based upon an extended Wiener-Hopf argument has been developed by Keogh [1985 a, 1985 b] for solving the problem of high frequency scattering of elastic waves by a Griffith crack situated in an infinite homogeneous elastic medium. Pal and Ghosh [1990] considered the problem of diffraction of normally incident antiplane shear waves by a crack of finite length situated at the interface of two bonded dissimilar elastic half-spaces. The problem is reduced to the solution of a Wiener-Hopf problem. The expressions for the stress intensity factor and the crack opening displacement have been derived for the case of wave-lengths short compared to the length of the crack. Recently Pal and Ghosh [1993] have investigated the high frequency solution of the problem of diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface. Following the method of Chang [1971], the problem has been formulated as an extended Wiener-Hopf equation and the asymptotic solutions for high frequencies or for wave lengths which are short compared to the length of the crack have been derived. Expressions for the dynamic stress intensity factor at the crack tip and the crack opening displacement have been derived.

Vibratory motion of a body on an elastic half-plane was treated by Karasudhi, Keer and Lee [1968]. They considered the vertical, horizontal and rocking vibrations of a body on the surface of an otherwise unloaded half-plane. The problem was formulated so that shearing stress vanishes over the entire surface, and an oscillating displacement is prescribed in the loaded region. The problems were mixed with respect to the prescribed displacement and the remaining stress. Each case led to a mixed boundary value problem represented by dual integral equations which were reduced to a single Fredholm integral equation.

Wickham [1977] studied the problem of the forced two dimensional oscillations of a rigid strip in smooth contact with a semi-infinite elastic solid. He reduced the mixed boundary value problem with the help of Green's function to Fredholm integral equation of the first kind involving displacement boundary conditions. Using Noble's [1962] method, this equation was reduced to Fredholm integral equation of the second kind with a kernel which was small in the low frequency limit. Then applying the method of iteration, a simple explicit long-wave asymptotic formula for the normal stress in terms of the prescribed displacement and dimensionless wave number K was rigorously derived.

Rocking motion of a rigid strip on a semi-infinite elastic medium has been studied by Ghosh and Ghosh [1985] by using a different technique. The forced rocking of the strip about the horizontal axis has been reduced to a solution of a dual integral equation. Following Tranter's [1968] method the dual integral equation was solved for low frequency oscillations by reducing the equation to a system of linear algebraic equations.

Studies of single Griffith crack as well as two parallel and coplanar Griffith cracks have been made by Mal [1970], Jain and Kanwal [1972] and Itou [1978, 1980 a, 1980 b]. The coresponding problems of diffraction by a single and two parallel rigid strips have been solved by Wickham [1977], Jain and Kanwal [1972] and Mandal and Ghosh [1992] respectively. And three dimensional problem of moving crack was considered by Itou [1979]. In most of the cases the problems were solved by integral equation technique.

The problem involving single Griffith crack in orthotropic medium was investigated by Kassir and Bandopadhyay [1983], Shindo et al [1986] and De and Patra [1990]. Sindo et al [1991] have investigated the impact response of symmetric edge cracks in an orthotropic strip. Mandal and Ghosh [1994] considered the problem of interaction of elastic waves with a periodic array of coplanar Griffith cracks in an orthotropic elastic medium.

Recently Mandal, Pal and Ghosh [1996 a] considered the two-dimensional problems of diffraction of elastic waves by four coplanar parallel rigid strips embedded in an infinite orthotropic medium. The five part mixed boundary value problem is reduced to the solution of a set of integral equations. The normal stress under the strips and displacement out side the strips were derived in close analytical form. In another paper, Mandal, Pal and Ghosh [1996 b] considered the vertical vibration of four rigid strips in smooth contact with a semi-infinite elastic medium. The resulting mixed boundary value problem has been reduced to the solution of quadruple integral equations, which have further been reduced to the solution of a integro-differential equations. An iterative solution valid for low frequency has been obtained. From the solution, the stress just below the strips and also the vertical displacement at points outside the trips on the free surface have been found.

In case of low frequency oscillations Noble's [1963] method of solving dual integral equations, Tranter's [1968] technique for solving dual integral equations, Matched Asymptotic Expansion, and variational principle are found to be very useful techniques.

Different techniques have been applied by many authors to tackle these type of crack problems. From these stand point, these problems may be divided into two categories : one for low frequency oscillation of the source or long wave scattering or transmission and the other for high frequency oscillation or short wave scattering or transmission in the medium. The term long and short are used in comparison to the region of the source of distrubance or the size of the crack or strip etc. inside the medium to the wave length of disturbance. The useful techniques for low frequency scattering are due to Noble [1963] and Tranter [1968]. In case of high frequency oscillations Wiener-Hopf [Noble, 1958] technique and Keller's [1958] geometrical theory are found to be most suitable.

Here we briefly discuss some of the useful methods.

GREEN'S FUNCTIONS :

The general theory of linear equations suggests two methods which can be used to solve the equation of the type

$$Lu = f \tag{1}$$

where L is an ordinary linear differential operator, f a known function, and u the unknown function.

One method is to find the operator inverse to L, that is, to find an operator L^{-1} such that the product $L^{-1}L$ is the identity operator. We shall find that the inverse of a differential operator is an integral operator. The kernel of that integral operator will be called the Green's function of the differential operator. The techniques which we shall provide for finding the Green's function use a tool which has proved valuable in many branches of applied mathematics, namely, the Dirac δ-function.

Inverse of a differential operator can be obtained, following Friedman [1966], Roach [1982], as follows:

Suppose that ψ and ϕ are testing functions and consider the equation

$$L\psi = \phi \tag{2}$$

Here we assume that the inverse operator L^{-1} is an integral operator with some kernel $G(x,t)$ such that

$$L^{-1}\phi = \int G(x,t)\,\phi(t)\,dt. \tag{3}$$

Now we permit $G(x,t)$ to be symbolic function. Applying the differential operator L to both sides of this equation, we get

$$LL^{-1}\phi = \phi = \int L\,G\,\phi\,dt. \tag{4}$$

This equation will be satisfied if we find g such that

$$LG = \delta(x-t), \tag{5}$$

where the differentiation is to be understood as symbolic differentiation.

To illustrate the method of inverting an operator, we consider the special case when

$$L = \frac{d^2}{dx^2} \quad ;$$

then (5) becomes

$$\frac{d^2}{dx^2} G(x,t) = \delta(x-t) \tag{6}$$

This equation can be solved by straightforward integration and using the fact that the δ-function is the derivative of the Heaviside unit function and we get

$$\frac{d}{dx} G(x,t) = H(x-t) + \alpha(t) \tag{7}$$

where $\alpha(t)$ is an arbitrary function.

Integrating again, we get

$$G(x,t) = \int H(x-t)dt + x\alpha(t) + \beta(t)$$

$$= (x-t)H(x-t) + x\alpha(t) + \beta(t), \tag{8}$$

where $\beta(t)$ is another arbitrary function. It can be proved that any symbolic function which is a solution of (6) may be written in the form (8). Note that $G(x,t)$ is a continuous, piecewise, differentiable function, and note also that if $f(x)$ is an integrable function which vanishes outside a finite interval, then it is easy to show that the function

$$u(x) = \int G(x,t)\, f(t)\, dt \tag{9}$$

satisfies the differential equation

$$\frac{d^2u}{dx^2} = f(x) \tag{10}$$

By the suitable choice of the function $\alpha(t)$ and $\beta(t)$ we can in general find a solution of (10) which satisfies two conditions. Thus, to find a solution of (10) which satisfies the conditions $u(0) = u(1) = 0$, we proceed as follows :

From (9) we have

$$u(x) = \int_{-\infty}^{x} (x-t)\, f(t)\, dt + x \int_{-\infty}^{\infty} \alpha(t)\, f(t)\, dt + \int_{-\infty}^{\infty} \beta(t)\, f(t)\, dt. \tag{11}$$

Substituting $x = 0$ and $x = 1$ in (11) we get

$$0 = -\int_{-\infty}^{0} t\, f(t)\, dt + 0 + \int_{-\infty}^{\infty} \beta(t)\, f(t)\, dt \tag{12}$$

$$0 = \int_{-\infty}^{1} (1-t)\, f(t)\, dt + \int_{-\infty}^{\infty} \alpha(t)\, f(t)\, dt + \int_{-\infty}^{\infty} \beta(t)\, f(t)\, dt. \tag{13}$$

From equation (12) we get

$$\beta(t) = t\ H(-t), \tag{14}$$

and then from (13) we obtain

$$\alpha(t) = -\ 1 + t\ H(t), \qquad -\infty \le t \le 1 \tag{15}$$

$$= \ 0\ , \qquad \text{for all other values of } t.$$

Substituting (14) and (15) in (9) we get

$$u(x) = \int_0^x (x-t)\ f(t)\ dt - x \int_0^1 (1-t)\ f(t)\ dt. \tag{16}$$

So, in this case the kernel (Green's function)

$$G(x,t) = (x-t)\ H(x-t) - x\ (1-t), \qquad 0 \le x, \quad t \le 1 \tag{17}$$

also satisfies the boundary conditions

$$G(0,t) = G(1,t) = 0 \tag{18}$$

The Other Method is to find the spectral representation of L by studing the solution of the equation

$$Lu = \lambda u, \tag{19}$$

where λ is an arbitrary constant.

Let L be an ordinary self-adjoint differential operator and suppose that u_1, u_2, ... are its eigenfunctions and λ_1, λ_2, ... the corresponding eigenvalues. We shall also suppose that the eigenfunctions span the domain of the given operator, and that, in consequence, any square integrable function u(x) may be expanded as

$$u(x) = \sum \alpha_k u_k(x), \tag{20}$$

where $\alpha_k = (u_k, u).$ (21)

Now, it follows that

$$Lu(x) = \sum \alpha_k \lambda_k u_k(x) \tag{22}$$

and if f(x) denotes a function which is analytic in a region containing the eigenvalues, we define

$$f(L)u(x) = \sum f(\lambda_k)\alpha_k u_k(x). \tag{23}$$

For the particular case when

$f(t) = (\lambda - t)^{-1}$ we obtain

$$\left(\frac{1}{\lambda - L} \right) u(x) = \sum \frac{\alpha_k u_k(x)}{\lambda - \lambda_k} . \tag{24}$$

The left hand side of (24) can be expressed in terms of the Green's function for the differential operator $L-\lambda$. Therefore, we put

$$w(x) = (\lambda - L)^{-1} u(x);$$

and we have $(L - \lambda)w = - u.$

If $G(x,\xi,\lambda)$ is the Green's function for the operator $L-\lambda$, we have

$$w(x) = - \int G(x,\xi,\lambda)\ u(\xi)d\xi\ , \tag{25}$$

and consequently,

$$\left(\frac{1}{\lambda - L}\right)u(x) = - \int G(x,\xi,\lambda)\ u(\xi)d\xi \tag{26}$$

Now, integrating (24) over a large circle of radius R in the complex λ-plane, we get

$$\frac{1}{2\pi i}\int \frac{u(x)}{\lambda - L}\, d\lambda = \sum \frac{1}{2\pi i}\int \frac{\alpha_k u_k(x)}{\lambda - \lambda_k}\, d\lambda\ . \tag{27}$$

Now, as the radius of the circle approaches infinity, the right-hand side of (27) includes more and more residues, and we obtain, bearing in mind that necessarily u is also a function of λ,

$$\underset{R \to \infty}{\mathrm{Lt}}\ \frac{1}{2\pi i}\int \frac{u(x)}{L - \lambda}\, d\lambda = - \sum \alpha_k u_k(x) = - u(x)\ . \tag{28}$$

This result, which connects the Green's function with the eigenfunctions, was obtained, by making a great many assumptions, such as that the eigenfunctions were known and that they were complete. In practice, we try to work it backwards. We start with a knowledge of the Green's function $G(x,\xi;\lambda)$ for the operator $L-\lambda$;

then we consider the following integral in the complex λ-plane;

$$\frac{1}{2\pi i}\int \frac{u(x)}{L-\lambda}\, d\lambda = \frac{1}{2\pi i}\int d\lambda \int G(x,\xi;\lambda)u(\xi)d\xi , \qquad (29)$$

and then, by evaluating it in terms of residues, we hope to get (28), that is, an expansion of u(x) in terms of the eigenfunctions of L.

CAGNIARD-DEHOOP TRANSFORMATION :

Following Pilant [1979] Cagniard-deHoop technique can better be explined taking an example. We find a solution of the inhomogeneous scalar wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2\phi}{\partial t^2} = -2\pi\delta(x)\delta(z)\delta(t)$$

$$= -\frac{\delta(r)\delta(t)}{r} \qquad (30)$$

Taking a Laplace transform with respect to time, we get

$$\frac{\partial^2\bar{\phi}}{\partial x^2} + \frac{\partial^2\bar{\phi}}{\partial z^2} - \frac{s^z}{V^2}\bar{\phi} = -2\pi\delta(x)\delta(z), \qquad (31)$$

where $$\bar{\phi} = \int_0^\infty \phi(x,z,t)\, e^{-st}\, dt. \qquad (32)$$

In order to simplify what is to come, we shall take a slightly modified Fourier transform with respect to x, i.e.,

$$\bar{\bar{\phi}}(q,z,s) = \int_{-\infty}^{\infty} \bar{\phi}(x,z,s)\, e^{-isqx/v}\, dx \,, \tag{33}$$

with the inverse

$$\bar{\phi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\bar{\phi}}(q,z,s)\, e^{isqx/v}\, d(sq/v). \tag{34}$$

This gives

$$-(sq/v)^2\, \bar{\bar{\phi}} + \partial^2 \bar{\bar{\phi}} / \partial z^2 - (s/v)^2\, \bar{\bar{\phi}} = -2\pi\delta(z) \tag{35}$$

Finally, taking a two-sided Laplace transform with respect to z, we have

$$\left\{ p^2 - (s/v)^2 (q^2 + 1) \right\} \bar{\bar{\bar{\phi}}} = -2\pi, \tag{36}$$

where $$\bar{\bar{\bar{\phi}}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\bar{\phi}}(q,z,s)\, e^{-pz}\, dz$$

Inverting with respect to p, we have

$$\bar{\bar{\phi}} = (\pi v / s)\, e^{-(s/v)(q^2+1)^{1/2}|z|}\, (q^2+1)^{-1/2} \tag{37}$$

Inverting with respect to q, we obtain

$$\bar{\phi}(x,z,s) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-(s/v)(q^2+1)^{1/2}|z|}\,(q^2+1)^{-1/2}\, e^{isqx/v}\, dq$$

$$= K_0(sr/v) \tag{38}$$

The expression (38) is just the integral representation of the Macdonald function $K_0(sr/v)$.

Cagniard-deHoop transformation involves the following change of variable :

$$\cos\theta\,(q^2+1)^{1/2} - iq\sin\theta = \tau = vt/r, \tag{39}$$

where $r\cos\theta = z$, $r\sin\theta = x$, and τ is the reduced time variable as shown in Fig. 1. Note that r-θ system is not standard cylindrical co-ordinates. The inverse of this transformation is

$$q(\tau) = i\tau\sin\theta + \cos\theta\,(\tau^2-1)^{1/2}\;; \tag{40}$$

Therefore

$$\frac{dq}{d\tau} = i\sin\theta + \frac{\tau\cos\theta}{(\tau^2-1)^{1/2}} = \frac{(q^2+1)^{1/2}}{(\tau^2-1)^{1/2}}, \tag{41}$$

The last expression comes from solving for $(q^2+1)^{1/2}$ from (39) while substituting (40) for q. Taking account of the symmetry

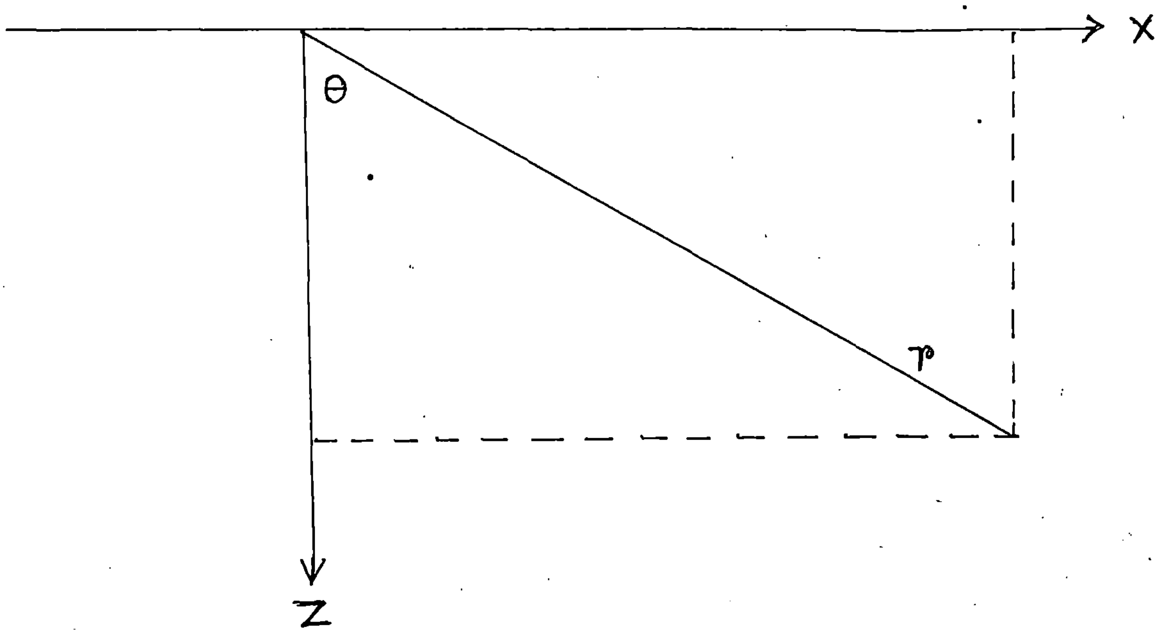

Fig. 1. Two dimensional co-ordinate systems.

of the real and imaginary parts of $\exp(isqx/v)$, we can write (38) as

$$\bar{\phi} = \mathrm{Re}\left[\int_0^\infty \frac{e^{-(s/v)(q^2+1)^{1/2}|z| + isqx/v}}{(q^2+1)^{1/2}}\, dq \right] \tag{42}$$

we can now write this using (41) in terms of the new variable "τ" and obtain

$$\bar{\phi}(x,z,s) = \mathrm{Re}\left[\int_?^? \frac{e^{-st}}{(q^2+1)^{1/2}} \frac{dq}{dt} \frac{v}{r}\, dt \right] \tag{43}$$

$$= \mathrm{Re}\left[\int_?^? \frac{e^{-st}}{(\tau^2-1)^{1/2}} \frac{v}{r}\, dt \right]$$

Equation (43) can now be recognized as the Laplace transform of the function

$$= \mathrm{Re}\left[\frac{1}{(\tau^2-1)^{1/2}} \frac{v}{r} \right]$$

looked at as a function of the time variable "τ". However, we have to look at a few details before we can say that this identification is valid and place proper limits on the integral. First of all, we want to look at the path q takes as we let the variable τ run from

0 to ∞ . For $\tau = 0$, we have that $q = - i \cos\theta$ where the sign has been choosen in (40) to satisfy (39). The variable q then moves up the imaginary axis to $q = i \sin\theta$, and then branches out into the first quadrant along a hyperbola as defined by (40) and along an asymptote at an angle θ as in Fig. 2(a). Inasmuch as the singularities of (42) are branch points at $q = \pm i$, we see that the original path can be deformed into the dashed line path as in Fig. 2(b). However, on the vertical segment from 0 to $i\sin\theta$ we see that the integrand of (42) has no real part. Consequently the limits on (43) may be written

$$\bar{\phi}(x,z,s) = \mathrm{Re}\left[\int_{r/v}^{\infty} \frac{e^{-st}}{(\tau^2-1)^{1/2}} \frac{v}{r}\, dt \right] \tag{44}$$

By inspection we have

$$\phi = \frac{1}{(t^2 - r^2/v^2)^{1/2}} H(t - r/v), \tag{45}$$

where H is the Heaviside Unit Step Function defined by

$$\begin{aligned} H(x) &= 1, & x > 0 \\ &= 1/2, & x = 0 \\ &= 0, & x < 0 \end{aligned} \tag{46}$$

There is a sharp wavefront associated with the response to a delta-function source, but in two dimensions we also have a tail

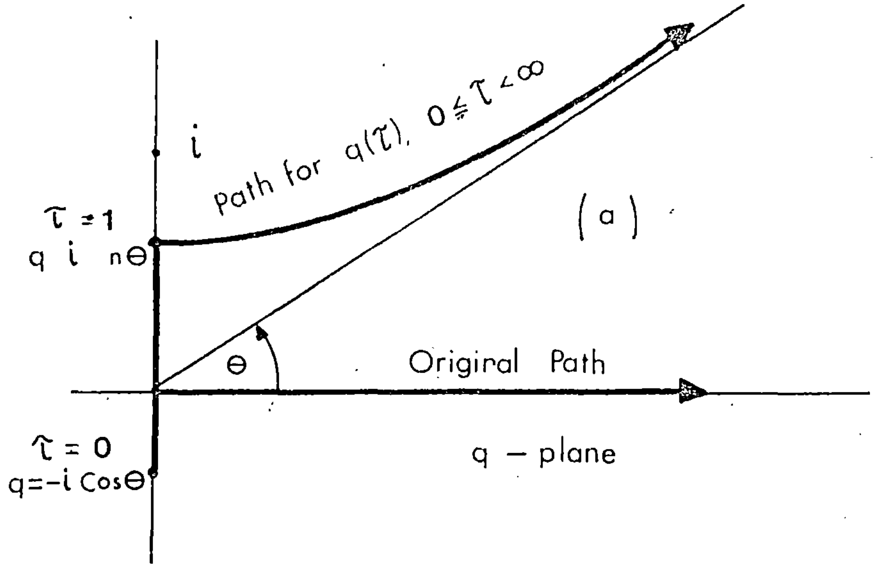

Fig. 2(a). The relationship between the original path of integration in (42) and the path which q takes as τ varies between zero and infinity.

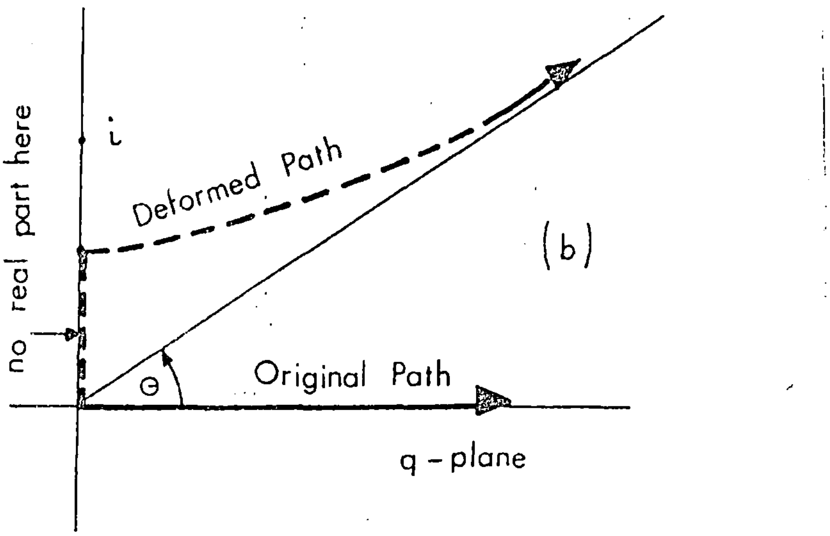

Fig. 2(b) The relationship between the original path and the deformed path (Cagniard Path) in the complex q-plane.

associated with the waveform in contrast to the delta-function which has zero width.

INTEGRAL TRANSFORM TECHNIQUE :

As the equations of motion in the theory of elasticity are partial differential equations which may be discussed with reference either to Helmholtz equation or to Laplace's equation, the method of integral transform is one of the most effective methods for solving such equations as application of this method to such equations results in the lowering of the dimension of an equation by one. There are several forms of integral transform and the choice of an integral transform depends on the structure of the equation and the geometry of the domain.

The integral transform $\bar{f}(\alpha)$ of a function $f(x)$ defined on an interval (a,∞) is an expression of the form

$$\bar{f}(\alpha) = \int_a^\infty f(x)\ K(x,\alpha)\ dx \tag{47}$$

where a is a real number and α is a complex parameter varying over some region D of the complex plane. $K(x,\alpha)$ is called the kernel of the transformation. The transformation (47) becomes particularly useful if it possesses inverse mapping. In that case one can express $f(x)$ in terms of its integral transform by

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma} \overline{f}(\alpha)\ M(x,\alpha)\ d\alpha \tag{48}$$

where $M(x,\alpha)$ is a suitable function defined in $a < x < \infty$ and $\alpha \in D$ and is called the kernel of the inverse transform, which is defined for all x in the interval (a,∞). The complex Γ is a suitable path of integration in D. After reducing the governing partial differential equation, the reduced problem can be solved for $\overline{f}(\alpha)$. The solution of the original equation can be expressed in terms of the inverse integral, which may then be evaluated. The inversion from the transformed space to the space of actual variables usually involved very complicated integrations. In many cases even the numerical integration can not be performed successfully because of the highly oscillatory character of the integrands (cf. Eringen and Suhubi [1975], chap.7; Achenbach [1975], chap.7). In particular, mixed boundary value problems like the dynamic response of a punch on an elastic half-space and the problem involving the presence of a crack or a strip inside an elastic medium may be reduced to Fredholm integral equation of first kind or to dual integral equations.

HILBERT TRANSFORM TECHNIQUE :

If $P(y) \in L_2(a,b)$, then the equation

$$\int_a^b \frac{h(x)}{x-y}\, dx = \pi\, P(y), \qquad y \in (a,b) \tag{49}$$

has the solution

$$h(x) = \frac{1}{\pi}\left[\frac{x-a}{b-x}\right]^{1/2} \int_a^b \left[\frac{b-y}{y-a}\right]^{1/2} \frac{P(y)}{x-y}\, dy + \frac{C}{\sqrt{(x-a)(b-x)}} \tag{50}$$

where C is an arbitrary constant, and the first term belongs to the class $L_2(a,b)$.

Using the above theorem, we find that the solution to the integral equation

$$\int_a^b \frac{2xh(x^2)}{x^2-y^2}\, dx = \pi P(y), \qquad y \in (a,b) \tag{51}$$

(provided that P satisfies the conditions of the above theorem) is given by

$$h(x^2) = \frac{1}{\pi}\left[\frac{x^2-a^2}{b^2-x^2}\right]^{1/2} \int_a^b \left[\frac{b^2-y^2}{y^2-a^2}\right]^{1/2} \frac{2yP(y)}{x^2-y^2}\, dy + \frac{C}{\sqrt{(x^2-a^2)(b^2-x^2)}}$$

where C is an arbitrary constant.

THE WIENER-HOPF TECHNIQUE :

Let a function $\phi(z)$ analytic in the interval $y_- < \text{Im } z < y_+$ be defined in the plane of a complex variable z. It is required to express $\phi(z)$ in the form

$$\phi(z) = \phi_+(z)\, \phi_-(z) \tag{52}$$

where $\phi_+(z)$ and $\phi_-(z)$ are functions analytic in the half-plane Im z $> y_-$ and the half-plane Im z $< y_+$ respectively. The problem is called factorization problem. In a more general case, it is required to define two functions $\phi_+(z)$ and $\phi_-(z)$ which are analytic in the same half-planes respectively and which satisfy the following relation in the interval

$$A(z)\phi_+(z) + B(z)\phi_-(z) + C(z) = 0 \tag{53}$$

where A(z), B(z) and C(z) are given analytic functions in the interval. It is obvious that if C(z) = 0, we obtain the representation (52) after the corresponding changes in the notation.

Let us assume that the function $\phi(z)$ which is to be factorised does not have any zeros in the interval $y_- < \text{Im } z < y_+$ and tends to infinity as $x \longrightarrow \infty$. In this case, neither of the functions $\phi_+(z)$ and $\phi_-(z)$ will have any zero, and we can take the logarithm of both sides of the relation (52)

$$\log \phi(z) = \log \phi_+(z) + \log \phi_-(z) \tag{54}$$

The function $F(z) = \log \phi(z)$ satisfies the condition

$$\left| F(x+iy) \right| < C \left|x\right|^{-P}, \quad (P > 0 \quad \text{for} \quad x \longrightarrow \infty) \tag{55}$$

and hence the relation (54) can always be solved with the help of the transformation

$$F(z) = F_+(z) + F_-(z) \tag{56}$$

Finally, we get

$$\phi(z) = e^{F_+(z)} \, . \, e^{F_+(z)} = \phi_+(z)\phi_-(z) \tag{57}$$

If the function $\phi(z)$ has zeros in the intervals we must consider a new function

$$\phi_1(z) = \frac{(z^2+b^2)^{N/2} \quad \phi(z)}{\prod_{i=1}^{N_1} (z - z_i)^{\alpha_i}} \tag{58}$$

where z_i and α_i are the zeros, their multiplicity in the interval $N_1 \leq N$, where N is the total number of zeros, $b > (y_+, \; y_-)$. The factor in the numerator of (58) ensures that the properties of auxiliary functions are conserved at infinity.

Let us now consider the relation (53) and carry out its factorisation into L_+ and $1/L_-$ for the same interval of the ratio A/B. The relation (53) can be represented in the form

$$L_+(z)\phi_+(z) + L_-(z)\phi_-(z) + L_-(z)C(z)/B(z) = 0 \tag{59}$$

The expression $L_-(z)C(z)/B(z)$ can be represented in the following form in accordance with (56)

$$E_+(z) + E_-(z)$$

where $\phi_+(z)$ and $\phi_-(z)$ are functions analytic in the half-plane $y > y_-$ and the half-plane $y < y_+$ respectively. Taking this into account, we get

$$L_+(z)\phi_+(z) + E_+(z) = - L_-(z)\phi_-(z) - E_-(z) \tag{60}$$

It follows from the generalized Liouville's theorem that the left as well as right hand side of (60) represents the same polynomial $P_n(z)$ of nth degree.

Wiener-Hopf technique and different other techniques for solving partial differential equation arising in Solid Mechanics have been elaborately discussed by Duffy [1994] in his book.

The thesis presented here consists of some boundary value problems in elastodynamics involving wave propagation due to some finite source or cracks. The work has been presented in three chapters. The first chapter deals with problems on moving source on the free surface.

The problems on scattering of waves by moving interface crack have been presented in the second chapter.

The third chapter deals with the diffraction problems in elastic medium.

The summary of the thesis is presented here chapter wise.

The first problem of chapter-1 has been formulated as follows:

We have considered the problem of the SH-type of elastic wave propagation in the semi-infinite medium due to a ring source producing SH-waves in the presence of a circular cylindrical cavity and the problem of SH-wave propagation in the presence of rigid circular cylindrical inclusion in the semi-infinite medium due a ring source.

An integral representation of the Dirac delta function required for solving the above axisymmetric boundary value problem has been derived first.

In the second problem of chapter-1, an elliptic ring load emanating from the origin of co-ordinates at $t = 0$ is assumed to expand on the free-surface of an elastic half-space. The displacement at points on the free-surface has been derived in integral form by Cagniard-De Hoop technique. Displacement jumps across different wave fronts have also been derived.

In chapter-2, the problem of diffraction of normally incident antiplane shear wave by a crack of finite length situated at the interface of two bonded dissimilar elastic half-spaces has been considered in the first problem. The problem is reduced to the solution of a Wiener-Hopf equation. The expressions for the stress intensity factor (SIF) and the crack opening displacement have been derived for the case of wave length short compared to the length of the crack. The numerical results for two different pairs of samples have been presented graphically.

In the second problem of this chapter, the diffraction of horizontally polarized shear waves by a finite crack moving on a bimaterial interface is studied. In order to obtain a high frequency solution, the problem is formulated as an extended Wiener-Hopf problem. The expressions for the dynamic stress intansity factor at the crack tip and the crack opening displacement are derived for the case of wave lengths which are short compared to the length of the crack. The dynamic stress intensity factor for high frequencies is illustrated graphically for two pairs of different types of material for different crack velocities and angles of incidence.

In chapter-3, first paper deals with the problem of two dimansional oscillations of four rigid strips, situated on a homogeneous isotropic semi-infinite elasitic solid and forced by a specified normal component of the displacement. The mixed boundary value problem of determining the unknown stress distribution just below the strips and vertical displacement outside the strips has been converted to the determination of the solution of quadruple integral equations by the use of Fourier transform. An iterative solution of these integral equations valid for low frequency has been found by the application of the finite Hilbert transform. The normal stress just below the strips and the vertical displacement away from the strips have been obtained. Finally graphs are presented which illustrate the salient features of the displacement and stress intensity factors at the edges of the strips.

The last problem of this chapter deals with the elastodynamic response of four coplanar rigid strips embedded in an infinite orthotropic medium due to elastic waves incident normally on the strips. The resulting mixed boundary value problem has been solved by Integral Equation method. The normal stress and the vertical displacement have been derived in closed analytic form. Numerical values of stress intensity factors at the edges of the strips and the vertical displacement at points in the plane of the strips for several orthotropic materials have been calculated and plotted graphically.

With this much of introduction, we now present the thesis chapterwise. References given in the thesis do not include all the previous workers in this line. But attempt has been made to include most of them.