CHAPTER-VII

7.1: Non-linear analysis of thermo-elastic plates with varying flexural rigidity.

The governing differential equations for non-linear analysis of thermoelastic plates with varying flexural rigidity are derived in chapter III and which are given by equations (207) and (208). The required boundary conditions are also given in chapter III. To solve the differential equations one may take the deflection and stress function in the following separable forms:

$$w = W(X,Y)\, F(t)$$

$$\phi = \phi(X,Y)\, F^{2}(t)$$

where the spatial parts $W(X,Y)$ and $\phi(X,Y)$ take the prescribed values on the boundary and $F(t)$ is an unknown function of time.

(i) Deflection under mechanical loading:

To solve equations (207) and (208) under uniform load 'q' one has to start with the neglection of the inertial term in equation (207) and to assume the deflection functions as follows:

Case - I; Simply-supported plate, $$w = \sum_{mm'=1,3,..} W_{mm'} \cos(m\pi x/a)\cos(m'\pi y/b) \quad (314)$$

Case-II; Clamped edge, $$w = \tfrac{1}{4} F(t) \sum_{m,m'=1,3,5..} W_{mm'} \left[1 - \cos(2m\pi x/a)\right] \left[1 - \cos(2m'\pi y/b)\right] \quad (315)$$

Expressions (314) and (315) obviously satisfy the required boundary conditions. Here, $\sum W_{mm'}$ defines the central deflection of the plate. Again, a suitable form of Airy's stress function compatible with the boundary conditions may be taken as,

$$\phi = \tfrac{1}{2} P_x Y^2 + \tfrac{1}{2} P_Y X^2 + \sum_{m,m'=1}^{\infty} \phi_{mm'} \left\{(a/b)\cos(2m\pi x/a) + (b/a)\cos(2m'\pi y/b)\right\} \quad (316)$$

For the sake of simplicity one may start for the first modal solution with $m = m' = 1$.

Substituting equation (316) in equations ~~(a), (b), (k), (l)~~ 155 – 158 (given in chapter-III), one gets,

for case-I with immovable edges,

$$P_x = \tfrac{1}{2}\cdot\tfrac{1}{4}\left[\frac{(1+\nu R^2)\,\Pi^2\,E\,W_{11}^2}{(1-\nu^2)\quad a^2}\right] \tag{317}$$

$$P_y = \tfrac{1}{2}\cdot\tfrac{1}{4}\left[\frac{(R^2+\nu)\Pi^2 E\,W_{11}^2}{(1-\nu^2)\quad b^2}\right] \tag{318}$$

for clamped (II) and immovable edges,

$$P_x = \tfrac{3}{4}\cdot\tfrac{1}{4}\cdot\tfrac{1}{2}\left[\frac{(1+\nu R^2)\quad \Pi^2\,E\,W_{11}^2}{(1-\nu^2)\quad a^2}\right] \tag{319}$$

$$P_y = \tfrac{3}{4}\cdot\tfrac{1}{4}\cdot\tfrac{1}{2}\left[\frac{(R^2+\nu)\quad \Pi^2 E\,W_{11}^2}{(1-\nu^2)\quad b^2}\right] \tag{320}$$

for movable edges, $P_x = P_y = 0$, here, $R = a/b$.

Now, substituting equation (316) in equation (208) and equating the similar terms on both sides of the expression thus obtained one gets,

$$\phi_{11} = -(EhW_{11}^2)/16 \tag{321}$$

Substituting equations (314) and (316) in equation (207) and applying Galerkin's technique and carrying out necessary inte-

grations one finally obtains the following equation determining (W_{11}/h).

$$C_{11}(\bar{w}) + C_2(\bar{w})^3 = L\,Q^* \qquad (322)$$

where, $C_{11} = (1-\alpha)(1+R^2)^2 - \alpha(1+R^2)^2/2\pi^2$

$$+ 2\alpha(1+R^2)/\pi^2 - \alpha(1+R^2)/2\pi^2$$

$$C_2 = (9/8)(1-\nu^2)(1+R^4) + (3/2)\left\{(1+\nu R^2) + (R^2+\nu)R^4\right\}$$

$$Q^* = qa^4/Eh^4 \quad ; \quad L = 48(1-\nu^2)/\pi^6 \quad ; \quad \bar{w} = W_{11}/h \,.$$

Again, on substitution of equations (315) and (316) in equation (207) and proceeding in the same maner as described in the case of simple support and utilizing equations (319-320) one gets,

$$C_1^* \bar{w} + C_2^* \bar{w}^3 = L^* Q \qquad (323)$$

where, $C_1^* = (1-\alpha)(3+2R^2+3R^4) - \alpha(3+2R^2+3R^4)$

$$+ 4\alpha\,(0.4016 + 0.0349R^2)$$

$$C_2^* = (9/4)(1-\nu^2) + (27/32)\left\{(1+\nu R^2) + (R^2+\nu)R^4\right\}$$

$$L^* = 3(1-\nu^2)/\pi^4 \quad ; \quad Q = Q^*$$

The equations (322) and (323) completely determine the deflections simply supported and clamped square and rectangular elastic plates under a temperature distribution.

(ii) Free vibration:

To solve equation (207) under dynamic loading one has to start with the neglection of the term involving 'q' and then assume the deflection functions as,

$$w = F(t) \sum_{m,m'} \text{Cos}(m\pi x/a)\ \text{Cos}(m'\pi y/b), \text{ for case-I,} \quad (324)$$

$$w = \tfrac{1}{4} F(t) \sum_{m,m'} \left[1 - \cos(2m\pi x/a)\right]\left[1 - \cos(2m'\pi y/b)\right], \text{ for case-II,} \quad (325)$$

Now, proceeding in the same maner as described earlier for the static case with assumed deflection and stress functions given in equations (324,325,316) and applying Galerkin's technique one finally obtains,

$$\frac{d^2F}{dt^2} + A^2F(t) + B^2F^3(t) = 0 \quad (326)$$

where, $C_1^*/Z^* = A^2$, $C_2^*/Z^* = B^2$, $Z^* = (9/16)\rho h a^4/D_o$,

for clamped edge;

$C_1/Z = A^2$, $C_2/Z = B^2$, $Z = \rho h a^4/4D_o$,

for simply-supported edge.

The exact solution of the modal equation (326) is well known and is given by,

$$F(t) = \bar{w}\ C_n\ (\omega t, K) \tag{327}$$

where $\bar{w}$ is the non-dimensional amplitude, C_n is the elliptic cosine and the non-linear frequency,

$$\omega^* = \left[A^2 + B^2(\bar{w})^2\right]^{\frac{1}{2}} \tag{328}$$

The solution for F(t) in terms of the elliptic cosine has a period

$$T^*/T = \frac{2\,A\,K(g)}{\Pi\left[A^2 + B^2(\bar{w})^2\right]^{\frac{1}{2}}} \tag{329}$$

where, K(g) is the complete elliptic integral of the first kind and T is the linear period given by $T = 2\Pi/A$.

Also, $g^2 = B^2(\bar{w})^2 / 2\left[A^2 + B(w)^2\right]$

Numerical results are computed for both cases and are presented in tables (29-32).

TABLE - XXIX

Temperature effect on dynamic response of simply-supported rectangular thermoelastic plates. Immovable edge.

Aspect ratio 'R'	Temp. para. α ↓	ω^*/ω $\bar{M}$=0.0 →	0.25	0.5	0.75	1.0
1.0	0.00	1.0000	1.0343	1.1308	1.2756	1.4544
	0.25	1.0000	1.0601	1.2229	1.4543	1.7270
	0.50	1.0000	1.0868	1.3134	1.6221	1.9748
	0.75	1.0000	1.1599	1.5433	2.0271	2.5549
	1.00	1.0000	2.1609	3.9595	5.8331	7.7273
1.5	0.00	1.0000	1.0544	1.2031	1.4165	1.6702
	0.25	1.0000	1.0716	1.2624	1.5283	1.8370
	0.50	1.0000	1.1047	1.3718	1.7275	2.1278
	0.75	1.0000	1.2683	1.6418	2.1944	2.7897
	1.00	1.0000	2.9214	5.5802	8.2953	11.0252
2.0	0.00	1.0000	1.0997	1.3554	1.6981	2.0853
	0.25	1.0000	1.1301	1.4521	1.8693	2.3312
	0.50	1.0000	1.1875	1.6250	2.1660	2.7501
	0.75	1.0000	1.3375	2.0385	2.8460	3.6907
	1.00	1.0000	3.3669	6.5073	9.6965	12.8986
0.5	0.00	1.0000	1.0432	1.1633	1.3397	1.5534
	0.25	1.0000	1.0551	1.2052	1.4207	1.6769
	0.50	1.0000	1.0758	1.2766	1.5547	1.8760
	0.75	1.0000	1.1219	1.4265	1.8245	2.2671
	1.00	1.0000	1.3136	1.9753	2.7440	3.5508

T A B L E - XXX

Temperature effect on dynamic response of clamped rectangular and square thermoelastic plates.

Aspect ratio R	Temp. para. α ↓	Edge cond.	ω^*/ω $\bar{w}$ → 0.0	0.5	0.75	1.0
	0.00	Immovable	1.0000	1.1180	1.1644	1.2778
0.5	0.25	"	1.0000	1.1211	1.2562	1.4240
	0.50	"	1.0000	1.2977	1.5935	1.9329
	0.00	Movable	1.0000	1.0538	1.1175	1.2010
0.5	0.25	"	1.0000	1.1086	1.1850	1.3109
	0.50	"	1.0000	1.2158	1.4409	1.7068
	0.00	Immovable	1.0000	1.0712	1.1539	1.2608
1.0	0.25	"	1.0000	1.1187	1.2514	1.4164
	0.50	"	1.0000	1.3634	1.7123	2.1059
	0.00	Movable	1.0000	1.0469	1.1027	1.1764
1.0	0.25	"	1.0000	1.0788	1.1699	1.2866
	0.50	"	1.0000	1.2487	1.5027	1.7991
	0.00	Immovable	1.0000	1.1406	1.2951	1.4846
2.0	0.25	"	1.0000	1.2335	1.4743	1.7567
	0.50	"	1.0000	1.7180	2.3218	2.9675
	0.00	Movable	1.0000	1.0538	1.1175	1.2010
2.0	0.25	"	1.0000	1.0916	1.1963	1.3291
	0.50	"	1.0000	1.3104	1.6167	1.9669

T A B L E - XXXI

Static deflections of simply-supported thermoelastic plates.

Aspect Ratio 'R'	Edge cond.	α	w	Q / Q* = qa^4/Eh^4 0.0	0.5	1.0	1.5	2.0
	Immovable	0.0	0.0	14.8667	52.9342	137.4029	291.4736	
0.5	"	0.25	0.0	16.0366	62.2930	168.9888	366.3444	
	"	0.50	0.0	18.2212	79.7698	227.9731	506.1584	
	Movable	0.0	0.0	12.9525	28.7687	85.7175	168.9600	
0.5	"	0.25	0.0	13.5432	42.3466	101.6664	206.7648	
	"	0.5	0.0	14.6462	51.1698	131.4480	227.3584	
	Immovable	0.0	0.0	15.0887	54.7096	143.3949	305.6788	
1.0	"	0.25	0.0	16.4062	65.2498	178.9681	389.9984	
	"	0.50	0.0	18.9750	85.8000	215.3250	554.4000	
	Movable	0.0	0.0	12.4074	33.2596	71.0011	134.0768	
1.0	"	0.25	0.0	12.8612	36.8896	83.2624	163.1168	
	"	0.50	0.0	13.7466	43.9648	107.1312	219.7184	
	Immovable	0.0	0.0	24.6290	131.0320	400.9830	916.2560	
	"	0.25	0.0	29.2957	168.3660	526.9882	1214.9280	
2.0	"	0.50	0.0	38.8222	244.5784	784.2020	1824.6270	
	Movable	0.0	0.0	12.9143	37.3142	84.6854	166.5136	
	"	0.25	0.0	13.5698	42.5590	102.3866	208.4720	
2.0	"	0.50	0.0	14.9080	53.2642	138.5167	294.1136	

T A B L E - XXXII

Static deflections of clamped thermoelastic plates.

Aspect ratio 'R'	Edge cond.	Temp. para. α	$\bar{w}$	$Q = qa^4/Eh^4$ 0.0	0.5	1.0	1.5	2.0
	Immovable	0.0	0.0	21.5544	65.6357	154.7705	311.4857	
0.5	"	0.25	0.0	23.8987	84.3898	218.0656	461.5184	
	"	0.50	0.0	34.0363	165.4902	491.7793	1110.3212	
	Movable	0.0	0.0	20.4250	56.6000	124.2765	239.2035	
0.5	"	0.25	0.0	22.0640	69.7119	168.5277	344.0949	
	"	0.50	0.0	29.1519	126.4156	359.9026	797.7248	
	Immovable	0.0	0.0	21.2982	63.5851	147.8499	295.0813	
1.0	"	0.25	0.0	23.7710	83.3682	214.6173	453.3446	
	"	0.50	0.0	38.1819	198.6552	603.7110	1375.6409	
	Movable	0.0	0.0	20.0775	53.8200	114.8928	216.9600	
1.0	"	0.25	0.0	21.6884	66.7073	158.3871	320.0582	
	"	0.50	0.0	31.0721	141.7777	411.7473	920.6160	
	Immovable	0.0	0.0	24.9436	92.7487	246.2767	528.3894	
	"	0.25	0.0	30.1781	134.6250	387.6092	863.3997	
2.0	"	0.50	0.0	64.1169	406.1354	1250.5572	3050.4800	
	Movable	0.0	0.0	20.4250	56.6000	124.2765	239.2350	
2.0	"	0.25	0.0	22.3487	71.9903	176.2173	362.3225	
	"	0.50	0.0	34.8212	171.7700	512.9737	1160.5600	

O B S E R V A T I O N S A N D D I S C U S S I O N S

The numerical results, both for clamped and simply-supported square and rectangular thermo-elastic plates with movable and immovable edges, have been computed for a certain range of temperature and those are presented in tables (29 - 32). The results are in close agreement with those obtained by Yamaki for non-thermal case. Yamaki's results are given in chapter IV, where non-homogeneous square and rectangular thin elastic plates are considered to investigate their static and dynamic behaviour. It is important to note that in the present study the ratio of non-linear to linear time periods decreases with the increase of ' α ', the temperature parameter. The thermal contribution to non-linearity becomes more significant for high values of the amplitude causing hardening type non-linearity. It is observed that the static deflections for both movable and immovable edge conditions increase monotonically with temperature parameter as what is expected.

For better numerical results some more terms of the series in equations(314-316) should be considered.