Chapter IV

Methodology used for the present study

4-1 *Ad-hoc procedure*

While discussing about the non-linearity we mentioned that solutions of linear differential equations can be expressed as a linear combination of some particular solutions. But it is not possible in the case of non-linear equations (for details see Ames [2],[3], [4]). For this we have to use different approaches in different situations. One of the best methods for this is the method of Ad-hoc solution.

Actually it is a trial and error method, done by suitably fixed one or more parameters. Sometimes some of the parameters are expressed as functions of a particular function among the variables. For example, say, a field equation is expressed as a functions of three variables ϕ,ψ,χ . In one approach ψ and χ may be expressed as function of ϕ , i.e. $\psi = \psi(\phi)$, $\chi = \chi(\phi)$ and hence find a set of solutions (for detail, see [41]).

In another approach ϕ,ψ and χ , if be the function of x^1,x^2,x^3,x^4 ,can be expressed in different ways which may be as following:

(i) ϕ,ψ and χ are all function of a linear combination of x^1,x^2,x^3,x^4 i.e. say function of $(K_1x^1+K_2x^2+K_3x^3+K_4x^4)$, where K_i 's are constants.(see ref. [32]).

(ii) ϕ,ψ,χ are all functions of any variable, u (say), defined suitably i.e. $\phi=\phi(u),\psi=\psi(u),\chi=\chi(u)$ (see ref[32]).

(iii) ϕ,ψ,χ all may be function of say, (x^1,x^2,x^3-x^4) i.e. $\phi=\phi(x^1,x^2,x^3-x^4)$, $\psi=\psi(x^1,x^2,x^3-x^4)$, $\chi=\chi(x^1,x^2,x^3-x^4)$ (see ref [30]) and so on.

A best assumption gives a best result which tallies with some physical situation or some theoretical model (see ref[30]).

4-2 *Method of graphical representation* :

We know that the nature of a equation can be analyzed in different ways. But among them, graphical representation is the best way to interpret a result. For this we plot some of the solved exact solutions of the three equations under study viz. Yang equation (2-4) , Charap equation (2-8) and Combined equation (2-10). The technique used here is to plot ϕ,ψ and χ with respect to one of the space coordinates (others remaining constants) for a particular value of time. The whole thing is repeated for different values of time.

4-3 *Methodology for Painleve' test :*

Determining whether or not a given system of nonlinear or partial differential equations is integrable raises many fundamental issues. In particular there is the issue of what is meant by 'integrability' and the issue of how that integrability can best be determined without having resort to a complete solution of the problem. For Hamiltonian systems the notion of integrability is well defined, i.e., the existence of as many involutive first integrals as there are degrees of freedom. For non-Hamiltonian systems things are less clear-cut. Clearly the existence of integrals can lead to a reduction of the order of the system and hence to a solution in terms of an 'integration by quadratures'. However, there are equations such as the Painleve' transcendent which do not have algebraic integrals and for which an integration by quadratures is not possible. It now seems that it is possible to identify many different classes of integrable systems on the basis of their analytic structure, i.e. the types of singularities exhibited by their solutions in the complex domain. The techniques for doing this can be applied to both the ordinary and partial differential equations and furthermore can be extended to provide, in many cases, explicit solutions to the systems in questions, see for example [24,42]. It also turns out that the complex domain of non-integrable systems also contains much valuable information and overall it would seem that the study of analytic structure can provide a wide ranging and unified treatment for a large class of nonlinear problems.

Historically, the idea was originally applied by Kovalevskaya [43a] to the equations of motion of a rotating rigid body about a fixed point. By demanding that the solutions of the equations of motion have 'only movable poles' in the complex t-plane she discovered one more set of parameter values (other than those found by Euler and Lagrange earlier) for which the equations could be integrated and solved exactly in terms of the known functions. It is interesting to note that for all three choices of parameter values the equations have 'only movable poles' in the complex t-plane, and no solutions are known to this day when the equations do not have the property i.e. they have only 'movable poles' in the complex t-plane.

The analysis mentioned above is commonly known as 'Painleve' analysis'. It has been seen that almost all the integrable equations pass the 'Painleve' test' which has been designed based on the principles of 'Painleve' analysis'. On the other hand almost all the nonintegrable equations fail the 'Painleve' test' and at the same time show chaotic property. Thus a Painleve' test may offer a very reliable indirect indication of the existence of chaos. In the subsequent sections we have discussed in greater detail, Painleve' tests for Ordinary and partial differential equations, the relation between the Painleve' test and integrability, and the relation between Painleve' test and chaos.

a)*Painleve' test for the Ordinary differential equations(ODE s)*

The singularities of an ODE can be classified as (i) fixed and (ii) movable. While the location of the former is fixed by the nature of the coefficients of the ODE, the later is a function of the integration constant or initial condition.

Consider the linear first order ODE

$(dW / dz) + (z^{-2}) w = 0$

It has the solution,

$w = C \exp(1/z)$, C is arbitrary constant.

So, $z = 0$ is the fixed (essential) singular point. Further, the location of this singularity is fixed by the nature of the coefficient of the given ODE.

For a more general case we consider a linear second order differential equation :

$(d^2w / dz^2) + p(z) (dw/dz) + q(z) w = 0$

Its general solution is $w(z) = Aw_1(z) + Bw_2(z)$

Where A and B are two arbitrary constants depending on the initial data, and $w_1(z)$ & $w_2(z)$ are two independent solutions.

In general, for an n-th order linear ODE

$(d^nw / dz^n) + P_1(z) (d^{n-1} w / dz^{n-1}) + \ldots + P_n(z) w = 0$,

where $P_i(z)$, $i = 1,2,3, \ldots n$, are all analytic at $z = z'$ and admit 'n' linearly independent solutions in the neighbourhood of z', so that the general solution may be written as

$$w(z) = \sum_{i=1}^{n} C_i w_i(z)$$

where C_i's are integration constants. Here the singularities of the solution must be located at the singularities of the coefficients $P_i(z)$ which are all fixed and that they do not depend on the constants of integration C_i, where $i = 1,2,3, \ldots n$.

However, if the equation is 'nonlinear', a different kind of singularities can appear. For example, we consider

$(dw/dz) + w^2 = 0$, which has the solution

$w = (z - z_0)^{-1}$, where z_0 is a constant of integration.

Thus , at $z = z_0$, w has a singularity, a pole of order one, it is movable because its location depends on z_0.

Now, we may give a precise definition for the " Painleve' property" :

1) A 'critical point' is a branch point or an essential singularity in the solution of the ODE.
2) A 'critical point' is 'movable' if its location in the complex plane depends on the constants of integration of the ODE.
3) A family of solutions of the ODE without movable critical points has P-property: here P stands for Painleve'.
4) The ODE is of P-type if all its solutions have this property.

Ablowitz, Ramani and Segur [12] have developed a test to determine whether an ODE (or a system of ODEs) satisfies the necessary conditions of having Painleve' property. In many cases this algorithm seems to be simpler than the α-method of Painleve' and his coworkers (details in ref.

[44]), which also determines whether an ODE satisfies the necessary conditions of having Painleve' property. It is rather similar to the method of Kovalevskaya [43a].

Algorithm for a single nth order ODE :

For the n-th order ODE

$$(d^n w / dz^n) = F(z ; w, dw/dz, \ldots, d^{n-1} w / dz^{n-1}) \qquad \text{--- (4-1)}$$

where F is analytic in z and rational in its other arguments. The algorithm is like the following. There are basically three steps to the algorithm.

Step 1 : Find the dominant behaviour.

Look for a solution of (4-1) in the form

$$w \sim a_0 (z - z_0)^p \qquad \text{--- (4-2)}$$

where $Re(p) < 0$ and z_0 arbitrary.

Substituting (4-2) in (4-1) shows that for certain values of p, two or more terms may balance (depending on a_0), and the rest can be ignored as $z \to z_0$. For each such choice of p, the terms which can balance are called the 'leading terms' (or sometimes, 'dominant terms'). Requiring that the leading terms do balance (usually) determines a_0. If any of the possible p's is not an integer, the equation is not of the P-type. Otherwise, one has to go on with the second step.

Step 2 : Find the resonance

If all possible p's are integers, then for each p, (4-2) may represent the first term in the Laurent series, valid in a deleted neighbourhood of a movable pole. In this case, a solution of (4-1) is

$$w(z) = (z - z_0)^p \sum_{i=1}^{n} a_j (z - z_0)^j \quad , \quad 0 < |z - z_0| < R \qquad \text{--- (4-3)}$$

Here z_o is an arbitrary constants. If (n −1) of the co-efficients $\{a_j\}$ are also arbitrary, these are the n constants of integration of the ODE, and (4-3) is the general solution in the deleted neighbourhood. The powers of $(z - z_0)$ at which these arbitrary constants enter are called resonances.

For each (p,a) from step 1, construct a simplified equation that retains only the leading terms of the original equation. Substitute

$$w = a_0 (z - z_0)^p + \beta (z - z_0)^{p+r} \qquad \text{--- (4-4)}$$

into the simplified equation.

To leading order in β, this equation reduces to

$$Q(r)\, \beta (z - z_0)^q = 0 \, , \, q \geq p + r - n \qquad \text{---- (4-5)}$$

If the highest derivatives of the original equation is a leading term, $q = p + r - n$, and Q(r) is a polynomial of order n. If not, $q > p + r - n$, and the order of polynomial Q(r) equals the order of the highest derivative among the leading terms (< n).

The roots of $Q(r) = 0$ determine the resonances.

(a) One root is always (-1).It corresponding to the arbitrary constant z_0.

(b) If a_0 is arbitrary in step 1, another root is (0).

(c) Ignore any root with $Re(r) < 0$, because they violate the hypothesis that $(z - z_0)^p$ is the dominant term in the expansion near z_0.

(d) Any root with $Re(r) > 0$, but r not a real integer, indicates a (movable) branch point at $z = z_0$. There is no need to continue the algorithm, but it remains to prove that the equation actually has a branch point.

(e) If for every possible (p,a) from step 1, all the roots of $Q(r)$ (except -1 and possibly 0) are positive real integers, then there are no algebraic branch points. Proceed to step 3 to check for logarithmic branch points.

(f) To represent the general solution of the nth order ODE in the neighbourhood of a movable pole, $Q(r)$ must have $(n - 1)$ non-negative distant roots, all real integers. If for 'every' (p,a) from step 1, $Q(r)$ has fewer than $(n - 1)$ such roots, then none of the local solutions is general. This suggests (4-2) misses an essential part of the solution.

Step 3: Find the constants of integration :

For a given (p,a) from step 1, let $r_1 \le r_2 \le \ldots \le r_s$ denote the positive integer roots of Q(r) ; ($s \le n - 1$). Substitute

$$w = a_0 (z - z_0)^p + \sum_{j=1}^{r} a_j (z - z_0)^{p+j} \qquad \text{--- (4-6)}$$

into the full equation (4-1).

The coefficient of $(z - z_0)^{p+j-n}$ which must vanish identically, is

$$Q(j)\, a_j - R_j (z_0, a_0, a_1 \ldots a_{j-1}) = 0 \qquad \text{--- (4-7)}$$

i) For $j < r_1$, (4-7) determines a_j.

ii) For $j = r_1$, (4-7) becomes

$$Q(r_1) \cdot a_{r1} - R_{r1} (z_0, a_0, a_1 \ldots a_{r-1}) = 0, \text{ if}$$
$$R_{r1} (z_0, a_0, a_1 \ldots a_{r-1}) \ne 0, \qquad \text{--- (4-8)}$$

Then (4-7) cannot be satisfied .

iii) If it happens that (4-8) is false (i.e. , $R_{r1} = 0$), then a_{r1} is an arbitrary constant of integration. Proceed to the next coefficient.

iv) Any resonance that is a multiple root of Q (r) represents a (movable) logarithmic branch point with an arbitrary coefficient. If the assumed representation is asymptotic as $z \to z_0$, the equation is not of P-type.

v) At each nonresonant power, (4-7) determines a_j. At each resonance, either $R_{r1} \neq 0$, logarithmic terms must be introduced in (4-6), and the equation is not of P-type,

or $R_{r1} = 0$ and a_{r1} is an arbitrary constant of integration.

vi) If no logarithms are introduced at any of the resonances, one could in principle compute all the terms in the series. However, because the recursion relations are nonlinear, it is usually not feasible to determine the region of convergence of the series, as one does in a linear problem. An alternative is to prove directly from the ODE that each arbitrary constants (a_j), is the coefficient of an analytic function.

vii) If no logarithms are introduced at any of the resonances for all possible (p,a) from step 1, then the equation has met the necessary conditions to be of P-type (under the assumption that $p< 0$).

b) Painleve' test for the Partial differential equations (PDEs)

There are two approaches based on two conjectures for this purpose:

The first approach is developed by Ablowitz, Ramani and Segur [12] who introduced the idea of the Painleve' property in connection with the complete integrability of nonlinear PDE. They conjectured that every nonlinear ODE obtained by an exact reduction (i.e. through a similarity transformation) of a completely integrable non-linear PDE (i.e. solvable by the inverse scattering method, see ref. [18]) has the Painleve' property perhaps after a transformation of variables, that its general solution can have no movable singular points other than poles. Furthermore, they proposed that a consequence of the conjecture was an explicit test of whether or not a given nonlinear PDE may be solvable by the Inverse Scattering method. Partial proofs of this conjecture have been given by Ablowitz, Ramani and Segur [12a] and McLeod and Olver [45].

The drawbacks of the approach of Ablowitz, Ramani and Segur [12] are :
(i) the problem of identifying all possible reductions and
(ii) the inability to provide further information about actual solutions to the equations under study.

Weiss, Tabor and Carnevale [21a] proposed to overcome the drawbacks. They developed a conjecture for Painleve' test that can be applied directly to PDEs without any need for reductions.
According to Weiss, Tabor and Carnevale [21a] a nonlinear PDE is said to possess the Painleve' property if the solutions of the nonlinear PDE are single-valued in the neighbourhood of a movable singularity manifold.

In their approach Weiss et. al [21a] looked for solutions u = u(x,t) for a given nonlinear PDE in the form

$$u(x,t) = [\phi(x,t)]^{-\alpha} \sum_{j=0}^{\infty} u_j(x,t) [\phi(x,t)]^j \qquad \text{---- (4-9)}$$

in the neighbourhood of a noncharacteristic singularity manifold defined by $\phi(x,t) = 0$, where $\phi(x,t)$ and the expansion 'co-efficient' $u_j(x,t)$ are analytic functions of (x,t) in the neighbourhood of the manifold defined by $\phi(x,t) = 0$, and α is an integer.

According to Cauchy-Kovalevskaya theorem [43a] such an expansion of the (general) solution must have as many arbitrary functions as the order of the system.

However, several authors have questioned the rigorous extent of both of these two approaches. Ward [46] suggested an approach, which seems to be more rigorous than the above two approaches and at the same time too complicated to be applied in actual situations.

According to the "Painleve' conjecture" due to Weiss et. al [21a] those nonlinear PDE which have the Painleve' property in the sense of Weiss et. al [21a] are completely integrable (i.e. solvable by the inverse scattering method). At this point it may be mentioned that there are completely integrable equations such as the Harry-Dym equation $u_t + u^3 u_{xxx} = 0$ which are known to possess the weak-Painleve' property. A system is said to show the weak-Painleve' property when the system has solution $u = u(x,t)$ in the form

$$u(x,t) = \phi(x,t)^{-\alpha'/d} \sum_{j=0}^{\infty} u_j(x,t) [\phi(x,t)]^{j/d} \qquad \text{--- (4-10)}$$

instead of (4-10) , α' and d are positive integers and $d \neq 1$.

The requirement as imposed by the Cauchy-Kovalevskaya theorem [43a] remains the same, that is, the expansion (4-10) of the (general) solution must have as many arbitrary functions as the order of the system.

With the example of the Harry-Dym equation in hand one could reasonably assert that if a PDE or a system of PDE has a weak-Painleve' expansion then it is still a candidate for being completely integrable. But that may not be always true.

It has been observed that the higher KdV equation $u_t + u^3 u_x + u_{xxx} = 0$ also possesses the weak-Painleve' property through there is other evidence against its complete integrability. It is interesting to observe that the Harry-Dym equation can be transformed into integrable equations having the Painleve' property in order to distinguish between completely integrable and non-integrable PDE in general.

A suitable algorithm for the Painleve' test of partial differential equations in the sense of Weiss, Tabor and Carnevale [21a] involved three essential steps. They are as follows.

<u>Step 1</u> : Leading order analysis

For a specific PDE it is necessary to identify all possible values for α and then find what the form of resulting Laurent-like series (see ref. [47]). For this we is substitute $u \sim u_0 \, \phi^{\alpha}$ in the equations concerned shows that for certain values of α, two or more terms balance (depending on u_0). For each such choice of α, the terms which can balance are called the 'leading terms' (or sometimes, 'dominant terms'). Equating the co-efficients of the dominant terms to zero and solving the resulting algebraic equation for u_0 can determine the value of u_0.

When α comes out to be integers only then the equations under study are candidates for full-Painleve' property. Here the complete expansion will be of the form given by (4.9).

Sometimes α turns out to be zero. As observed by Clarkson [48], this is an indication of a logarithmic term $\log(\phi)$ in the leading order of the expansion. However, this branch point structure does not spoil Painleve' structure when the concerned field variables occur in the equations in the derivative form only thereby generating the pole structure in the equations.

As mentioned earlier α comes out to be (α' / d) then the equations under study are candidates of Weak-Painleve' property. Here the complete expansion will be of the form given by (4.10).

Step 2 : Resonance analysis

If all possible α's are integers, then for each (α, u_0) one can substitute (4-4) in the PDE concerned and get the recursion relations for u_j, $j = 0,1,2\ldots$ Those values of j for which the recursion relations are not defined are called resonance positions and at those positions in the series expansion one should be able to have those coefficients arbitrary. Just as in the case of ODEs for equations having full-Painleve' property '$j = -1$' and for equations having weak Painleve' property '$j = -d$' seems always to be a resonance and corresponding to the arbitrary singularity manifold defined by $\phi = 0$.

The computations can be simplified by a modification due to Jimbo,Kruskal and Miwa [49] which asserts that without loss of generality ϕ may have the special form $\phi(x,t) = x - f(t)$, and also that without any loss of generality one can write $u_j = u_j(t)$. It may be noted that such a modification seems to be valid only for those nonlinear PDE for which a necessary and sufficient condition for $\phi = 0$ to define a noncharacteristics manifold is that $\partial\phi/\partial x \neq 0$. In addition to that references already cited one can consult the review work of Tabor and Gibbon [50].

Steeb and Louw [51] have studied significance of negative resonance for ordinary differential equations. There they studied an expansion around infinity and made interesting observations. In the case of PDEs one could assume a solution of the partial differential equation with assumption that

$$u = \sum_{j=0}^{\infty} u_{-j} \, \phi^{-j+\alpha} \qquad \text{etc.}$$

and use them in the equation concerned in order to get the significance of negative resonances. An important point regarding rational resonances is that the algebraic constants of motion correspond to rational resonances in case of ODEs [52a,b]. This can be extended to PDEs [52c].

Step 3 : Arbitrary functions

At this stage one checks whether one actually gets the introduction of arbitrary function at the resonance positions. This is ascertained only when the compatibility conditions that arise at the resonance positions, $j = r$ are identically satisfied for all the results at $j < r$.

For a coupled system and for equations (or systems) where complex conjugate of an unknown function appears explicitly the formalism is generalized in the same way as in the case of ODEs.

(c) Painleve′ property and integrability of PDEs **:**

For partial differential equations, which are infinite dimensional, integrability is shown by the existence of infinite number of integrals in involution (for detail see ref [22c]) and hence infinite number of conservation laws. Zakharov and Faddeev [20a] demonstrated a procedure to find integrals via inverse scattering transform for the KdV equation. One may consult the work of Flaschka and Newell [20b] in this context. The consequences of this work are such that once an isospectral problem has been found for a PDE then a direct connection can be established between the existences of multiple soliton solutions. The multiple soliton solutions can be found by the Inverse Scattering Transform (IST) method [18] or by other methods [14,15,22]. In the light of these developments one can say that a PDE is integrable when there exists a nontrivial Lax Pair [11,14,15,18] for that PDE (or that system of PDEs) [21m].

We have mentioned in the previous section that, Ablowitz, Ramani and Segur [12a] introduced the idea of the Painleve' property in connection with the integrability of PDEs. They conjectured that every nonlinear ordinary differential equation obtained by an exact reduction (e.g. through a similarity transformation) of an integrable non-linear partial differential equation has the Painleve' property perhaps after a transformation of variables. Furthermore, according to them, one consequence of this conjecture (ARS conjecture) is an explicit test of whether or not a given PDE may be of IST class; namely, reduce it to an ODE, and determine whether the ODE is of P-type. However this approach cannot generate no further information.

The great advantage of the ARS conjecture is that it is simple to apply, for example see [12a,45]. Its drawback is that in order to test the integrability of a PDE, one has to ask what one means by testing 'all' the ordinary differential equations associated with its symmetries (details in ref. [23]). It would be better if one could directly attack and test the partial differential equation itself. Weiss, Tabor and Carnevale [21a] contributed in that direction. They defined a 'Painleve' property' for PDEs which is a natural extension of the 'Painleve' property' for ODEs and conjectured that a PDE having Painleve' property is integrable [21a,b].

We have also elaborated in the earlier section how Weiss, Tabor and Carnevale [21a] developed a conjecture for Painleve' test that can be applied directly to PDEs without any need for reduction. Their conjecture provides us with a test (Painleve' test) with the help of which we can identify an integrable PDE , It is true that proper mathematical founding is still lacking regarding the connection between the Painleve' test and integrability. But, experience [21] has shown that the test works with sufficient certainty. In a recent review Weiss [21m] proposes that the 'Painleve' test' is a sufficient condition for integrability. He also stresses that the Painleve' property is a statement about how the solutions behave as functional manifold and not a statement about the data itself.

However, the extent of rigor of both the approaches due to Ablowitz et. al. and Weiss et. al. has been questioned by several authors. See, for example, [46,48].

Ward suggested [46] an approach, which seems to be more rigorous than the above two approaches and at the same time too complicated to be applied in actual situations. Actually in ref. [46] it could not be observed how one might determine whether the KdV equation possess the Painleve' property in the sense due to Ward [46]. Further, the Painleve' property in the sense of Weiss et. al. [21a] requires all movable singularity manifolds to be single valued, whether characteristic or not. On the other hand, according to the observation of Ward [46] the direct consideration of expansions about characteristic manifolds cannot be allowed in the definition of the Painleve' property since, for linear systems, 'bad' singularities propagate along characteristics. For general systems, expansions about characteristics, when they exist, introduce certain arbitrary data (details in ref. [53]). If the data is 'bad', the expansion is still required to be a single valued 'functional' of that data.. The Painleve' property is a statement of how the solutions behave as functional of the data in a neighbourhood of a singularity and not a statement about the data itself.

In, general, to verify that an equation has the Painleve' property it is necessary to show that 'all' the allowed singularities are single valued (as functional of the data).

For an equation having the Painleve' property Weiss [21] proposes to calculate the Backlund transformations, Lax pair, Modified equations and Miura transformations through the expansions of the solutions about the singularity manifold. The 'Singular Manifold Method' consists in truncating the Laurent Psi series [47] after the constant level term. By construction, this forms a possible Backlund transformation. Depending on the distribution of 'resonances', generally over determined the recursion relations for the coefficients of the Laurent expansion define systems of equations. Reduction of this system to consistent form defines the Backlund- Darboux transformation, the Schwarzian form of the modified equation and the related Miura transformation to the original system. The Lax pair can be found by linearizing the Miura transformation and modified equation, using the invariance of the Schwarzian derivative under the Moebius group to motivate the substitution for 'linear' variables. The invariance under the Moebius group and the 'discrete' symmetries of the modified equations are found to constitute a nontrivial Backlund transformation for these systems. Success in these directions is noteworthy [21].

However, Lax pair is not obtainable for coupled systems even with the Weiss, Tabor and Carnevale [21a] formalism.

The present understanding of integrability has a focus on another interesting physical feature called the existence of Chaos. Rather, there has been a strong evidence that a system of equations which pass the Painleve' test is free from the existence of Chaos. See for example, the work of Bindu, Bambah, Lakshmibala,Mukku and Sriram [54d]. They have used both the numerical methods and Painleve' test. They observed that Chern-Simons-Higgs (CSH) models pass the Painleve' test and at the same time show absence of Chaotic behaviour in phase plot. They further observed that the CSH models with the introduction of kinetic term (Maxwell term), known as the Maxwell-Chern-Simons-Higgs (MCSH) models, do not pass the Painleve' test and at the same time show the clear indications of Chaotic behaviour in phase plot. Our research work reported in the present thesis also has observed (i) a correlation between the existence of Painleve property and absence of Chaotic behaviour and (ii) a correlation between the absence of Painleve' property and presence of Chaotic behaviour.

(d)Painleve′ analysis, truncation and Auto-Buckland Transformation:

Weiss, Tabor and Carnevale [21a] introduced the algorithm for testing the existence of Painleve′ property for the nonlinear partial differential equations. In addition to that developed a procedure for truncating the Laurent like expansion at the most singular term. This most often lead to auto-Buckland transformation between two pairs of solutions [21a],[55],[56].