Chapter II

The nonlinear field equations understudy and motivation for the study

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The equations under study are ———

(i) *Yang Euclidean SU (2) R-gauge equations [26].*

(ii) *Charap equations [27] for chiral invariant model of the pion dynamics and*

(iii) *A combined equations [28] of these two equations.*

In the following we discuss about the first two equations. After that we discuss the motivation that leads to combined equations of both of them.

2-1 Yang Euclidean SU(2) R-gauge field equations :

At the classical level, the mathematical description of Spin-1 particle started with Maxwell's famous equations for electromagnetism, and their generalization in 1954 due to Yang and Mills. The symmetry group associated with electromagnetism is Abelian. In an Abelian group one can apply transformations in any order without changing the result. On the other hand, the symmetry group associated with the corresponding generalization due to Yang and Mills is non-Abelian. In a non-Abelian group the same two transformations done in two different orders give different results. Together, all these theories are sometimes called 'Gauge Theories'. According to Yang and Mills "A change in gauge means a change of phase factor $\psi \rightarrow \psi'$, $\psi' = (\exp i\alpha)\,\psi$, a change that is devoid of any physical consequences, Since ψ may depend on x, y, z and t, the relative phase factor of ψ at two different space-time points is therefore completely arbitrary. In other words, the arbitrariness in choosing the phase factor is local in character".

The corresponding quantum theory was formulated for electromagnetism by Feynman, Schwinger and Tomonaga in 1940's and for the Yang-Mills generalization by 't Hoot and Veltman in the 1970's. Feynman et al received the Nobel Prize in October 1965, while 't Hooft and Veltman were awarded the Nobel Prize in October 1999. Quantum electromagnetism describes the photon and its interactions with charged particles, while quantum Yang-Mills theory describes W and Z bosons and gluons (the carriers of weak and strong nuclear forces) and their interaction. The combination of all these theories makes up a single large theory called the 'Standard Model' of particle interactions, which is a quantum gauge theory.

In this perspective we describe the equations due to Yang [26].

Many years ago Weyl suggested that electromagnetic field can be formulated in terms of an Abelian gauge transformation. Then the idea was also extended to non-abelian transformation. One might call such formulations as differential formulations. C.N.Yang is the first to formulate the gauge field in an integral formalism, which is superior to the differential formalism as it allows for natural developments of additional concepts. It further allows a mathematical and physical discussion of the gravitational field as a gauge field, resulting in equations related, but not identical, to Einstein's.

There has been great interest in recent years in sourceless gauge fields. A self-dual gauge field is sourceless. The dynamical behavior of gauge theories is great importance in particle physics specially to understand the chaotic behavior of field theories. Early investigation along these lines is to understand the structure of the field theoretic vacuum and asymptotic states of the theory with particular reference to strong interaction physics. This was motivated by the belief that there was a connection between color confinement and chaos in quantum dynamics. Again it was seen that gauge theories support solitons.

If the system are considered to be conservative and described by Hamiltonians, non-integrability of the evolution equation implies chaos. Under this circumstances Yang [26] take the following equations. They are Laplace-like equations for three real variables or two variables one real and one complex and are obtained when the condition of self-duality for a SU(2) R-gauge field on Euclidean four-dimensional flat space is integrated once.

The equations are given by,

$$\phi(\phi_{y\bar{y}}+\phi_{z\bar{z}}) - \phi_y\phi_{\bar{y}} - \phi_z\phi_{\bar{z}} + \rho_y\bar{\rho}_{\bar{y}} + \rho_z\bar{\rho}_{\bar{z}} = 0 \quad \text{--- (2-1a)}$$

$$\phi(\rho_{y\bar{y}} + \rho_{z\bar{z}}) - 2\rho_y\phi_{\bar{y}} - 2\rho_z\phi_{\bar{z}} = 0 \quad \text{--- (2-1b)}$$

where an over bar denotes the complex conjugate, ϕ and ρ are functions of y, $\bar{y}$, z, $\bar{z}$, ϕ is real, ρ is complex and $\sqrt{2}\,y = x^1 + i\,x^2$, $\sqrt{2}z = x^3 - i\,x^4$, x^1, x^2, x^3, x^4 are real.

Once one has found ρ and ϕ; the corresponding R-gauge potentials are given by [26]

$$\phi\vec{b}_y = (\,i\rho_y,\ \rho_y, -i\phi_y),\quad \phi\vec{b}_y = (-i\,\bar{\rho}_{\bar{y}},\ \bar{\rho}_y\,,\ i\phi_y) \quad \text{--- (2-2a,b)}$$

$\phi \vec{b}_z = (i\rho_z, \ \rho_z, -i\phi_z), \qquad \phi \vec{b}_{\bar{z}} = (-i\,\bar{\rho}_{\bar{z}}, \ \bar{\rho}_{\bar{z}}, \ i\phi_z)$ ---(2-2c,d)

and R-gauge field strengths $F_{\mu\nu}$ are given by [26]

$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu B_\nu + B_\nu B_\mu$ ----- (2-3a)

$B_\mu = b^i_\mu \ X_i$ --- (2-3b)

and $X_i = -(\tfrac{1}{2})\, i\sigma_i$ --- (2-3c)

where σ_i are 2×2 Pauli matrices.

All such solutions represent the condition of self-duality except when ϕ is zero. Because when ϕ is zero $F_{\mu\nu}$ becomes singular and the solutions can only be treated as solutions of Yang's R-gauge equations and not self-dual solutions unless a transformation like $F'_{\mu\nu} \to U^{-1} F_{\mu\nu} U$ removes the singularities.

When written in terms of real variables the equations in (2-1) read

$\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44} =$

$(1/\phi)(\phi_1^{\,2} + \phi_2^{\,2} + \phi_3^{\,2} + \phi_4^{\,2}) - (1/\phi)(\psi_1^{\,2} + \psi_2^{\,2} + \psi_3^{\,2} + \psi_4^{\,2})$

$-(1/\phi)(\chi_1^{\,2} + \chi_2^{\,2} + \chi_3^{\,2} + \chi_4^{\,2}) - (2/\phi)(\psi_1\chi_2 - \psi_2\chi_1 + \psi_4\chi_3 - \psi_3\chi_4)$ --- (2-4a)

$\psi_{11} + \psi_{22} + \psi_{33} + \psi_{44} =$

$(2/\phi)(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \phi_4\psi_4) + (2/\phi)(\phi\chi_2 - \phi_2\chi_1 + \phi_4\chi_3 - \phi_3\chi_4)$ – (2-4b)

$\chi_{11} + \chi_{22} + \chi_{33} + \chi_{44} =$

$(2/\phi)(\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \phi_4\chi_4) + (2/\phi)(\phi_2\psi_1 - \phi_1\psi_2 + \phi_3\psi_4 - \phi_4\psi_3)$ – (2-4c)

where $\rho = \psi + i\chi$ --- (2-4d)

$\phi_1 \equiv \partial\phi/\partial x^1, \ \phi_{11} \equiv \partial^2\phi/\partial(x^1)^2$ etc.

2-2 Charap Chiral field equations of pion dynamics :

John M Charap considered a simple model analogous of the Chiral invariant dynamics of Zero mass mesons. A system of massless particles could obey laws such as 'the number of left hand and right hand particles are each conserved' or 'the iso-spins carried by the left-hand particles and by the right hand particles are each conserved'. The category ' left hand particles' includes their charge conjugates, namely, right-hand antiparticles. Such conservation laws follow from symmetries of the form of a product of an internal symmetry group for the left-hand particles (that is, their field) and another for the right hand particles. Such a symmetry group is called Chiral (meaning 'handed') . In the model due to Charap instead of considering a multiplet of fields, with the appropriate gradient self-couplings of a Chiral invariant theory, work has been done with a system having only a finite number of degrees of freedom [27]. For a particular choice of chiral transformation, called tangential parameterization (Charap [29]) the field equations for the chiral invariant model of the pion dynamics take the form (Charap, [27])

$$\Box \phi = \eta^{\mu\nu} (\partial\phi / \partial x^{\mu}).(\partial\beta / \partial x^{\nu}) \qquad \text{----- (2-5a)}$$

$$\Box \psi = \eta^{\mu\nu} (\partial\psi / \partial x^{\mu}).(\partial\beta / \partial x^{\nu}) \qquad \text{----- (2-5b)}$$

$$\Box \chi = \eta^{\mu\nu} (\partial\chi / \partial x^{\mu}).(\partial\beta / \partial x^{\nu}) \qquad \text{----- (2-5c)}$$

Where, $\eta^{\mu\nu} = 0$ for $\mu \neq \nu$

$= 1$ for $\mu = \nu \neq 4$ ----- (2-5d)

$= -1$ for $\mu = \nu = 4$

$$\beta = \ln (f_\pi^2 + \phi^2 + \psi^2 + \chi^2) \qquad \text{----- (2-5e)}$$

$$f_\pi = \text{Constant} \qquad \text{----- (2-5f)}$$

The Lagrangian is given by

$$L = (1/2) (g_{11}\, \partial_\mu \phi\, \partial^\mu \phi + g_{22}\, \partial_\mu \psi\, \partial^\mu \psi + g_{33}\, \partial_\mu \chi\, \partial^\mu \chi$$

$$+ 2 g_{12}\, \partial_\mu \phi\, \partial^\mu \phi + 2 g_{13}\, \partial_\mu \phi\, \partial^\mu \chi + 2 g_{23}\, \partial_\mu \psi\, \partial^\mu \chi) \qquad \text{---- (2-6)}$$

where the g_{ij} are such that $\Gamma_{ij}{}^{1}$, the Christopher symbols, take the form

that $\Gamma_{ij}{}^{1} = - (f_\pi^2 + \phi^2 + \psi^2 + \chi^2)^{-1} (\partial_i^1 \phi_i + \partial_i^1 \phi_j)$ ----- (2-7)

In (2-7) ϕ_1, ϕ_2 and ϕ_3 represent ϕ, ψ and χ respectively.

Explicitly the Charap equation can be written as :

$$\phi_{11} + \phi_{22} + \phi_{33} - \phi_{44} = 2 \phi [\exp(-\beta)] (\phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2)$$

$$+ 2 \psi [\exp(-\beta)] (\phi_1 \psi_1 + \phi_2 \psi_2 + \phi_3 \psi_3 - \phi_4 \psi_4)$$

$$+ 2 \chi [\exp(-\beta)] (\phi_1 \chi_1 + \phi_2 \chi_2 + \phi_3 \chi_3 - \phi_4 \chi_4) \qquad \text{---- (2-8a)}$$

$$\psi_{11} + \psi_{22} + \psi_{33} - \psi_{44} = 2 \psi [\exp(-\beta)] (\psi_1^2 + \psi_2^2 + \psi_3^2 - \psi_4^2)$$

$$+ 2 \phi [\exp(-\beta)] (\phi_1 \psi_1 + \phi_2 \psi_2 + \phi_3 \psi_3 - \phi_4 \psi_4)$$

$$+ 2 \chi [\exp(-\beta)] (\psi_1 \chi_1 + \psi_2 \chi_2 + \psi_3 \chi_3 - \psi_4 \chi_4) \qquad \text{--- (2-8b)}$$

$$\chi_{11} + \chi_{22} + \chi_{33} - \chi_{44} = 2 \chi [\exp(-\beta)] (\chi_1^2 + \chi_2^2 + \chi_3^2 - \chi_4^2)$$

$$+ 2 \phi [\exp(-\beta)] (\phi_1 \chi_1 + \phi_2 \chi_2 + \phi_3 \chi_3 - \phi_4 \chi_4)$$

$$+ 2 \psi [\exp(-\beta)] (\psi_1 \chi_1 + \psi_2 \chi_2 + \psi_3 \chi_3 - \psi_4 \chi_4) \qquad \text{----(2-8c)}$$

where $\beta = \ln (f_\pi^2 + \phi^2 + \psi^2 + \chi^2)$

2-3: Motivation for the combination of the Yang equations (2-4) and Charap equations (2-8)

In addition to the physical significance described above the equations (2-4) due to Yang and equations (2-8) due to Charap have some mathematically interesting characteristics as well. It has been observed that there is considerable similarity between these two sets of equations. First, these two sets of equations are similar in form. Second, both of them allow (i) reduction to equations in two independent variables which are conformally invariant equations permitting one to obtain infinitely many other solutions from any solution of these conformally invariant equations and (ii) those reduced equations closely resemble the generalized Lund-Regge equations (discussed in [30],[31] ,[32],[33]).

The generalized Lund-Regge equations are,

$$\theta_{11} + \theta_{22} - 4\, g(\theta) + h(\theta)\, (\lambda_1^2 + \lambda_2^2) = 0 \qquad \text{----- (2-9a)}$$

$$[\, \lambda_1 \exp(-\textstyle\int p(\theta) d\theta\,]_1 + [\, \lambda_2 \exp(-\textstyle\int p(\theta)\, d\theta\,]_2 = 0 \qquad \text{----- (2-9b)}$$

where $\theta = \theta\,(x^1, x^2)$, $\lambda = \lambda\,(x^1, x^2)$, $\theta_1 = (\partial\theta/\partial x^1)$ and so on.

With $g = 0$, the equations (2-9) reduce to a conformably invariant set of equations, a particular example of which is the physically interesting equations of two dimensional Heisenberg ferromagnets (discussed in [34],[35]). The reduced form of equations (2-9) mentioned above which originate from Yang equations (2-4) closely resembles this situation. There is, however, at least a difference that there are two equations for the Heisenberg ferromagnets, whereas the equations (2-4) obtained from those due to Yang [26] have three equations. On the other hand, all of the reduced equations mentioned above [30] which originate from Charap equations (2-8) are of the same form as one of the two equations in (2-9). However, the similarity between the solutions of the nonlinear sigma model and self-dual gauge fields is well known and, for example, forms the basis for the Atiyah-Manton [36] approach to the construction of approximate solutions to the Skyrme model [36]. This has been quite widely applied to the study of nucleon-nucleon interactions in that model. For a recent example, see Leese, Manton & Schroers [37].

These observations have inspired us to study the general set of equations having Yang equations (2-4) which were obtained at the time of discussing the condition of self-duality for SU(2) gauge fields on Euclidean space and Charap equations (2-8) for chiral invariant pion dynamics under tangential parameterizations as the particular cases.

Since the solutions to some of the particular forms of the generalized equations have been shown to be physical in nature they may be useful in a wide area of physical research such as field theories and particle physics particularly with chiral models and Skyrme models in relation to solition solutions and spatio-temporal chaos (see references [38,39]).

2-4: Formulation of the Combined equations

Based on the discussion in the previous article, we combine the equations (2-4) due to Yang & (2-8) due to Charap and write the equations in which all the terms of (2-4) and (2-8) are present as:

$$\phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\,\phi_{44} =$$

$$k'[\,(1/\phi)\,(\phi_1^{\,2} + \phi_2^{\,2} + \phi_3^{\,2} + \varepsilon\phi_4^{\,2}) - (1/\phi)\,(\psi_1^{\,2} + \psi_2^{\,2} + \psi_3^{\,2} + \varepsilon\,\psi_4^{\,2})$$

$$- (1/\phi)\,(\chi_1^{\,2} + \chi_2^{\,2} + \chi_3^{\,2} + \varepsilon\chi_4^{\,2}) - (\,2/\phi)\,(\,\psi_1\chi_2 - \psi_2\,\chi_1 + \psi_4\,\chi_3 - \psi_3\,\chi_4)\,]$$

$$+k''\{2\phi[\exp(-\beta)](\phi_1^2+\phi_2^2+\phi_3^2+\varepsilon\phi_4^2)+2\psi[\exp(-\beta)](\phi_1\psi_1+\phi_2\psi_2+\phi_3\psi_3+\varepsilon\phi_4\psi_4)$$

$$+\,2\chi[\exp(-\beta)]\,(\,\phi_1\,\chi_1 + \phi_2\,\chi_2 + \phi_3\,\chi_3 + \varepsilon\,\phi_4\,\chi_4)\,\} \qquad \text{--- (2-10a)}$$

$$\psi_{11} + \psi_{22} + \psi_{33} + \varepsilon\psi_{44} =$$

$$k'[(2/\phi)(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\,\psi_3 + \varepsilon\phi_4\,\psi_4) + (2/\phi)\,(\phi_1\chi_2 - \phi_2\chi_1 + \phi_4\chi_3 - \phi_3\chi_4)]$$

$$+k''\{2\psi[\exp(-\beta)](\psi_1^2+\psi_2^2+\psi_3^2+\varepsilon\psi_4^2)2\phi[\exp(-\beta)](\phi_1\psi_1+\phi_2\psi_2+\phi_3\psi_3+\varepsilon\phi_4\psi_4)$$

$$+2\chi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\,\chi_3 + \varepsilon\,\psi_4\,\chi_4)\} \qquad \text{-- (2-10b)}$$

$$\chi_{11} + \chi_{22} + \chi_{33} + \varepsilon\chi_{44} =$$

$$k'[(2/\phi)\,(\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \varepsilon\,\phi_4\chi_4) + (2/\phi)\,(\phi_2\psi_1 - \phi_1\,\psi_2 + \phi_3\,\psi_4 - \phi_4\psi_3)]$$

$$+k''\{2\chi[\exp(-\beta)](\chi_1^2+\chi_2^2+\chi_3^2+\varepsilon\chi_4^2)+2\phi[\exp(-\beta)](\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3+\varepsilon\phi_4\chi_4)$$

$$+\;2\psi[\exp(-\beta)]\,(\,\psi_1\,\chi_1 + \psi_2\,\chi_2 + \psi_3\chi_3 + \varepsilon\,\psi_4\,\chi_4)\} \qquad \text{-- (2-10c)}$$

where $\beta = \ln(\,f_\pi^2 + \phi^2 + \psi^2 + \chi^2)$ and k', k'' are arbitrary constants.

The equations (2-10) reduce to (2-4), the Yang equations, when $\varepsilon=1, k'=1, k''=0$ and to (2-8), the Charap equations, when $\varepsilon=-1, k'=0, k''=1$.

For convenience we call the equations (2-4) as the 'Yang-equations', the equations (2-8) as the 'Charap equations' and the equations (2-10) as the 'Combined Yang-Charap (Y-C) equations'.

For $\varepsilon=1$, $k'\neq0$, $k''\neq0$ the equations (2-10) are 'Extended Yang equations'. And for $\varepsilon=-1, k'\neq0, k''\neq 0$ the equations (2-10) are 'Extended Charap equations'.