Chapter - I

INTRODUCTION

1-1 Physical background :

If we consider the history of development of Physics, the present time may quite aptly be called the age of High-Energy Physics. With the commissioning of many high-energy particle accelerators in advance Laboratories of the globe, physicists are now all very eager to know the inner mysteries of the material world. So there should , of course, be proper theoretical framework to explain the new experimental observations. If not, new theories have to be developed with the urge to understand the experimental data. That is how physics progress.

One very common feature of the high energy experiments is the proliferation of number of particles and anti-particles, subject to the Conservation laws and Einstein's mass energy relationship, $E=mc^2$. For example, in case of electron-nucleon scattering with the electron energy in the range of 100 Gev (1 Gev = 10^9 eV) required, which is quite available today. Obviously, the theoretical framework that can describe such an experimental observation cannot be single-particle Quantum Mechanics, but has to be one that can deal with a many-particle system, especially creation of new particle.

Max Planck recognized the relationship between a field and many particle systems at the early beginning of the 20^{th} century by his "blackbody radiation" – concept, which is an Electromagnetic Field.

The relationship between fields and particles proposed by Planck was a hypothesis only. It took more than 30 years to translate this into a theory. This is because physicists had to wait for the development of Relativity, Quantum Mechanics and Relativistic Quantum Mechanics, which could provide the necessary theoretical background for the formulation of field theory. Today field theory is considered to be the essential language of the high-energy physics although it has been applied successfully to low energy many particle systems also.

To search for a theoretical background to explain the connection between a field and a many particle system, we have to give a prescription for the quantization of field and extract particle properties from the Quantized Field or Quantum Field.

A field before quantization may be called a Classical Field. A Classical Field may be prescribed by associating a set of observable at each space-time point. For the Classical Electromagnetic Field the set of observable or variables consist of Scalar Potential and 3-components of Vector Potential i.e. (ϕ, A_x, A_y ,A_z) at each point of space time (x,t). In the case of Quantum Field we have to elevate the variables to the status of operators. Different phases of the fields are represented by space-time co-ordinate. If the phases of the fields are altered in such an amount that is function of space and time, is called the Gauge Transformation. A Gauge Theory is such a theory in which all measurable quantities remain unchanged under a Gauge Transformation. Examples of Gauge Theories are Quantum Electrodynamics, Quantum Chromodynamics, Electro Weak Theories etc.

Most of the field theories of physical interest are highly nonlinear. This is especially true of theories like the models of Strong Interaction. To explain or describe the behavior of the particle and field one has to write some equation relating to them and try to solve them to have a relation between the particles and field. But, as we say that the physical fields are mostly nonlinear hence they cannot be solved with the help of simple algorithms. That is why one has to rely on non-perturbative methods for these. It has been found that some of these field theories possess localized, stable solutions with finite energy at classical level. It is interesting to note that in some cases the nonlinearity leads to solitionic solutions whereas in some other cases the nonlinearity leads to chaos. Our investigation is oriented along this direction in the area of nonlinear field equations.

1-2 A few words about Nonlinearity:

1-2-1 What is nonlinearity ?

Physical fields are mostly non-linear. Non-linear systems, unlike their linear counterparts, do not have closed form solutions. Hence one has to numerically simulate the behavior of the nonlinear system.

Eugene Wigner pointed out that the chief role of mathematics in physics consists not in its being an instrument (i.e. computations) but in being the language of physics(details in [1]). This role of mathematics is being served for about last few hundred years chiefly by differential equations. The general practice is to formulate the laws of physics in the form of differential equations and then to solve the differential equations in different physical situations. Though the process of getting the solutions of differential equations is just the inverse of the process of the formation of those equations, it is for several reasons much more difficult to get the solutions. Thus, to get the solutions of differential equations has become a central problem of theoretical physics. The problem has become still more difficult with the fact that differential equations arising from physical situations are mostly nonlinear.

It is not far back when the nonlinear differential equations were something of a closed chapter. The reason is that such equations are very difficult to study. Linear differential equations have the advantage that the principle of (linear) superposition holds in their cases, i.e. by adding two or more solutions, one can always get a new solution and the general solution can be expressed as linear combination of the particular solutions.

The non-linear differential equations do not obey the principle of linear superposition. This is a severe loss on the part of non-linear differential equations and to obtain general exact solutions for the non-linear differential equations become more complicated. However, there exists classes of non-linear (and even linear) equations which posses non-linear superposition principles. Of course, there is no universal non-linear superposition (details in [2,3]).

As a result of these difficulties natural scientists preferred for a long time to use the assumptions like "fluid is inviscid", "given a perfect insulator" or "for constant thermal conductivity"…. etc. [4]. Such assumptions enabled them to keep themselves within the safe zone of linear differential equations. And there was not very much to reveal the mysteries of nonlinear differential equations.

The boost for the study of nonlinear partial differential equations (nPDEs) started with the work of Zabusky and Kruskal [5] in the year of 1965. Their work was stimulated by a physical problem and is also a classic example of how computational results may lead to the development of new mathematics, just as observational and experimental results have done since the time of Archimedes (for details see [6]).

1-2-2 Different approaches for understanding nonlinearity:

(a) One can seek steady-state solutions [6], i.e. waves of permanent form. A soliton is such a solution, although such a solution is in general not a soliton.

(b) One can seek the similarity solutions [6,7]. They may be found by use of dimensional analysis or of the group of transformations from one dynamically similar solution to another.

(c) One can seek this group and other groups of transformations [6,7,8]under which the nonlinear system is invariant. They are likely to underlie the character of all methods of solution of the system.

(d) One can see as many conservation laws of [9] the given system as possible. Infinity of conservation laws seems to be associated with the existence of soliton interactions.

(e) One can seek a Hamiltonian representation [10] of the given nonlinear system would have an infinite dimensional Hamiltonian representation or none at all.

(f) One can seek the Lax representation [11] of the given nonlinear partial differential equation. In this representation the given nPDE comes out as an integrability condition of the linear equations. Actually this identification is more an art than a science, because it depends upon the use of trial and error rather than an algorithm.

However, there exist at least two systematic procedures, which work in a large number of cases. The first one is due to M.J.Ablowitz, A.Ramani and H.Segur [12] where they leave some arbitrary coefficients in the linear Lax-equation and then determine to arbitrary coefficients which are compatible with the integrability condition of the linear equations. And naturally the corresponding nonlinear partial differential equations are also found out. The work of Ablowitz et al [12] is based on the pioneering work of Zakharov and Shabat [13]. Another procedure is due to Wahlquist and Estabrook [14]. In essence they try to force the nonlinear equation of interest to be an integrability condition of two linear equations containing the unknown variables and its x-derivatives as coefficients. In doing so, they obtain dimensional algebra or, to put it another way, a set of commutation relations that are not closed. One can close the algebra with the help of symmetry of the original nPDE [15,16] or with the help of some adhoc procedure analysed by Shadwick [17].

Once Lax-pair is obtained one can go for solutions through Inverse Scattering Transform (IST) [18a,b] or through Riemann-Hilbert problem [18c].

(g) One can seek a relevant Backlund transformation of the nPDE [19,20]. Backlund transformations were shown to be closely associated with the method of inverse scattering [20] and to be useful in finding multisoliton solutions.

(h) One can check using the formalism of Ablowitz, Ramani and Segur (ARS) [12] whether all ordinary differential equations (ODEs) derivable from the given nonlinear partial differential equation have Painleve' property.

(i) One can check using the formalism of Weiss, Tabor and Carnevale [21a] whether the given nonlinear partial differential equation possesses the Painleve' property in the sense of Weiss et. al [21a]. One can also try to obtain the Lax pair, the Backlund Transform, rational solutions etc. In recent times this approach is gaining more and more interest due to several reasons.

(j) In order to obtain exact solutions in a straightforward manner one can take the help of the method due to Hirota [22a]. However, recent findings indicate that Hirota's method plays a much more central role in the theory than heretofore believed [23]. It is also becoming more and more visible that the method due to Hirota and Painleve' property are closely related [22 ; 23]. One can check this in relation with the particular nPDE.

The above list do not exhaust all possible approaches towards a given nPDE. But it is believed that they are the most prominent uptill now. The connections among the findings obtained through different approaches are not very clear even today. For a guideline in this direction one can see the work of Newell [23]. For details see ref [24] and [25].

1-2-3 An introduction to Solitons

The axon or nerve fiber in neuron (neuron: the basic structural unit of the nervous system), which may be as long as 1 meter carries the electrical signal to muscles, glands, or other neurons. But a natural question may arise, how can a small electric signal can travel along the axon over a relatively large distance without almost any loss of energy?

The possible answer is: The electrical signal travels in the form of 'solitons'.

Solitons are actually solitary waves in contrast to plane waves which consist of a train of waveforms. In case of a plane wave profile it moves in the forward direction without any change of form. But natural systems are dissipative in nature. So in reality the wave profile dies down with the corresponding loss of energy associated with it. On the other hand, a natural system can be nonlinear as well. Such a system undergoes so called self-focussing.

What happens when both of the above two effects are present? There is a chance of proper balance and getting solitary waves or even solitons.

All solitary waves are not again solitons. A solitary wave is said to be a soliton, only when it is having collision properties of solitions. The behavior of the soliton is very surprising, because it is exactly similar to linear interaction except for the phase difference.

An example of solitonic equations can be given as follows:

For linear waves the equation is : $u_t + b\, u_{xxx} = 0$

For self–focussing waves the equation is : $u_t - a\, u\, u_{xx} = 0$

Adding above two equations,

we have the equation of solitons as : $u_t - a\, u\, u_{xx} + b\, u_{xxx} = 0$,

the celebrated kdV equation.

1-2-4: <u>An introduction to Chaos</u>

As has been stated earlier, the solution to most physical problems involves setting up and solving differential equations. For example, in mechanical systems, Newton's laws provide us with the required second order differential equations whose solution gives the path taken by the system being studied. As most systems are non-linear, until recently approximations had to be used for solving these equations analytically. While solving some non-linear equations using these techniques, it is found that the dynamics is very different from that of linear equations, one notable feature being the tendency of very close initial conditions to lead to completely different motion. This is the central identifying feature of chaotic systems. Such systems therefore appear unpredictable, since small changes in initial conditions can become amplified.

Chaos is a science of everyday things: it has been implicated in areas ranging from heart failure, meteorology, economic modelling, population biology to chemical reactions, neural networks, fluid turbulence and more speculatively even manic-depressive behavior. It also seems to occur everywhere – in rising columns of cigarette smoke, in fluttering flags, in traffic jams and so on.

To specify the position of a single particle, we need its Cartesian coordinates x, y, z. Another set of three numbers is required in order to fix the corresponding components of its velocity. Thus for N particles we need 6N independent quantities. The state of the system can then be represented as a point in a 6N dimensional abstract space called phase space. The motion of the whole system then corresponds to the trajectory of this single representative phase point in this phase space. Deterministic dynamics implies that there is a unique trajectory through any given phase or state point, and it is calculable in principle. This absolute determinism was probably first recognized by the 19 th Century French mathematician Pierre Simon de Laplace. But the Laplacian determinism is now known to have serious errors. The macroscopic uncertainty, or rather the unpredictability, emerges in an entirely different and rather subtle manner out of the very deterministic nature of the classical laws. When this happens we say that we have chaos.

Now one can question how can a system be deterministic and yet become chaotic?

The answer lies in the sensitive dependence on initial conditions. The deterministic laws permit a given initial state to evolve to a unique and calculable state of the system at the future instant of time. What if the initial conditions are known only approximately? If we start identical systems from two neighboring state points, then we generally expect their trajectories to stay close by for all future times. Such a system is said to be well behaved. But if initial errors actually grow with time say exponentially, then, any two trajectories that started off at some neighboring points initially, will begin to diverge/converge rapidly accordingly as the exponent is positive/negative respectively. This is precisely what is meant by sensitive dependence on initial conditions. It makes the flow in the phase space complex almost random. For, then the approximately known initial conditions do not give the distant future states with comparable approximation. This is often referred to picturesquely as the Butterfly Effect. The flapping of a butterfly's wings in America may set off a tornado in Kolkata.