Chapter VI

Present study for the three equations

In this chapter we discuss the present study for the three equations (2-4),(2-8) and (2-10) for the Yang equations, Charap equations and Combined equations respectively.

First we discuss the exact solutions of the three equations obtained through Ad-hoc procedure.

Next we discuss the graphical representations of some of those exact solutions.

Then we discuss Painleve′ test of these three equations.

Finally we discuss about the chaos in the perspective of Painleve′ test and graphical representations.

6-1 : Study of exact solutions of three equations :

In this article we discuss the exact solutions of the equations.

The main procedure used here is the Ad-hoc method as discussed in article 4-1.

6-1-1 : Some exact solutions of the Yang equations (2-4)

The solutions for (2-4) presented here are those satisfy the relations,

$$\phi = \phi(\tau, \sigma) \qquad \text{---- (6-1a)}$$

$$\psi = \psi(\tau, \sigma) \qquad \text{---- (6-1b)}$$

$$\chi = \chi(\tau, \sigma) \qquad \text{---- (6-1c)}$$

$$\tau = \tau(x^1, x^2) \qquad \text{---- (6-1d)}$$

$$\sigma = \sigma(x^3, x^4) \qquad \text{---- (6-1e)}$$

It is observed that the solutions for (2-4) subjected to (6-1) are given by the solutions of the equations (for details see Appendix – A in the end of this article)

$$(\phi\phi_{\tau\tau} - \phi_\tau^2 + \psi_\tau^2 + \chi_\tau^2 + P\,\phi\,\phi_\tau)\,\psi'$$
$$+(\phi\,\phi_{\sigma\sigma} - \phi_\sigma^2 + \psi_\sigma^2 + \chi_\sigma^2 + Q\,\phi\,\phi_\sigma)\,\chi' = 0 \qquad \text{--(6-2a)}$$

$$(\phi\,\psi_{\tau\tau} - 2\,\phi_\tau\,\psi_\tau + P\,\phi\,\psi_\tau)\,\psi'$$
$$+(\phi\,\psi_{\sigma\sigma} - 2\psi_\sigma\;\phi_\sigma + Q\,\phi\,\alpha_\sigma)\,\chi' = 0 \qquad \text{---- (6-2b)}$$

$$(\phi\,\chi_{\tau\tau} - 2\,\chi_\tau\,\phi_\tau + P\,\phi\,\chi_\tau)\,\psi'$$
$$+(\phi\,\chi_{\sigma\sigma} - 2\chi_\sigma\,\phi_\sigma + Q\,\phi\,\chi_\sigma)\,\chi' = 0 \qquad \text{---- (6-2c)}$$

where $(\tau_{11} + \tau_{22}) / (\tau_1^2 + \tau_2^2) = P(\tau)$ ---- (6-2d)

$$(\tau_1^2 + \tau_2^2) = \psi'(\tau) \quad \text{---- (6-2e)}$$

and $(\sigma_{33} + \sigma_{44}) / (\sigma_3^2 + \sigma_4^2) = Q(\sigma)$ ---- (6-2f)

$$(\sigma_3^2 + \sigma_4^2) = \chi'(\sigma) \quad \text{---- (6-2g)}$$

Equation (6-2d) – (6-2g) can be rewritten as

$$(\nu_{11} + \nu_{22}) = 0 , \quad (\nu_1^2 + \nu_2^2) = R \quad \text{---(6-3a,b)}$$

$$(\delta_{33} + \delta_{44}) = 0 , \quad (\delta_3^2 + \delta_4^2) = S \quad \text{--- (6-3c,d)}$$

where, $\nu = \int [\exp(-\int P(\tau)\, d\tau)]\, d\tau$ --- (6-3e)

$$\delta = \int [\exp(-\int Q(\sigma)\, d\sigma)]\, d\sigma \quad \text{--- (6-3f)}$$

$$R = [\exp(-2\int P(\tau)\, d\tau)]\, \psi'(\tau) \quad \text{--- (6-3g)}$$

$$S = [\exp(-2\int Q(\sigma)\, d\sigma)]\, \chi'(\sigma) \quad \text{--- (6-3h)}$$

By virtue of (6-3e,f) one can consider R and S as functions of ν and δ respectively.

The solution for (6-3 a,b) are given by (Appendix – B) :

(i) $\nu = K_2 x^1 + K_3 x^2 + K_1$ --- (6-4a)

$$R = K_2^2 + K_3^2 \quad \text{--- (6-4b)}$$

Or,

(ii) $\nu = 1/(2K_4) \ln[(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2$

$$+ (\ln K_7)/(2K_4) \quad \text{--- (6-5a)}$$

$$R = 1/[(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] \quad \text{--- (6-5b)}$$

The solutions for (6-3c,d) are given by (for details see Appendix– B)

i) $\delta = K_9 x^3 + K_{10} x^4 + K_8$ --- (6-6a)

$S = K_9^2 + K_{10}^2$ --- (6-6b)

Or,

(ii) $\delta = 1/(2K_{11}) \ln[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2 + (\ln K_4)/(2K_{11})$ -(6-7a)

$S = 1/\ [\ (K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2\ ]$ --- (6-7b)

Now, without any loss of generality one can transform (τ,σ) to (ν,δ), and (6-2a) – (6-2c) lead to

$(\phi\phi_{\nu\nu} - \phi_\nu^2 + \psi_\nu^2 + \chi_\nu^2)R + (\phi\,\phi_{\delta\delta} - \phi_\delta^2 + \psi_\delta^2 + \chi_\delta^2)\,S = 0$ ---(6-8a)

$(\phi\,\psi_{\nu\nu} - 2\,\phi_\nu\,\psi_\nu)\ R + (\phi\,\psi_{\delta\delta} - 2\,\phi_\delta\psi_\delta)\,S = 0$ ---- (6-8b)

$(\phi\,\chi_{\nu\nu} - 2\,\phi_\nu\,\chi_\nu)\ R + (\phi\,\chi_{\delta\delta} - 2\,\phi_\delta\,\chi_\delta)\,S = 0$ ---- (6-8c)

where, ν,δ,R and S are given by (6-4) - (6-7).

Regarding the equations (6-2) and (6-8),the following observations may be interesting :

I. If τ and σ satisfy a set of coupled equations of the form (6-2), then any function of τ and σ also satisfies the set of coupled equations of the form (6-2).

II. For all possible equations of the form (6-2) which generate as a result of the transformation of (6-1), τ is any function of $[K_2 x^1 + K_3 x^2 + K_1]$ or $[(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2]$ and σ is any function of $[K_9 x^3 + K_{10} x^4 + K_8]$ or $[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2]$.

Moreover, all such transformed equations are equivalent to the set of equations (6-8) via (6-3).

Now consider some examples.

(i) The equations (6-2) along with,

$$\tau = \ln (K_2 x^1 + K_3 x^2 + K_1) \qquad \text{--- (6-9a)}$$

$$\sigma = \ln (K_9 x^3 + K_{10} x^4 + K_8) \qquad \text{--- (6-9b)}$$

or

$$\tau = (K_2 x^1 + K_3 x^2 + K_1)^2 \qquad \text{--- (6-10a)}$$

$$\sigma = (K_9 x^3 + K_{10} x^4 + K_8)^2 \text{ , etc.} \qquad \text{--- (6-10b)}$$

are equivalent to the equations (6-8) along with (6-4) and (6-6)

(For details see Appendix – C, Case 1 a, b)

(ii) The equations (6-2) along with,

$$\tau = [(K_4 \; x^1 + K_5)^2 + (K_4 \; x^2 + K_6)^2] \qquad \text{--- (6-11a)}$$

$$\sigma = [(K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2] \qquad \text{--- (6-11b)}$$

or

$$\tau = \exp [(K_4 \; x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] \qquad \text{--- (6-12a)}$$

$$\sigma = \exp [(K_{11}x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2] \text{ , etc.(6-12b)}$$

are equivalent to the equation(6-8) along with (6-5) and (6-7).

(For details see Appendix – C, Case 2 a, b)

(iii) The equations (6-2) along with,

$$\tau = \ln (K_2 x^1 + K_3 x^2 + K_1) \qquad \text{--- (6-13a)}$$

$$\sigma = [(K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2] \qquad \text{-- (6-13b)}$$

or

$$\tau = (K_2 x^1 + K_3 x^2 + K_1)^2 \qquad \text{--- (6-14a)}$$

$$\sigma = \exp [(K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2] \text{ ,etc.} \qquad \text{-- (6-14b)}$$

are equivalent to the equations (6-2) along with (6-4) and (6-7).

(For details see Appendix – C, Case 3 a, b)

(iv) The equations (6-2) along with,

$$\tau = [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] \quad \text{--- (6-15a)}$$

$$\sigma = \ln [K_9 x^3 + K_{10} x^4 + K_8] \quad \text{--- (6-15b)}$$

or

$$\tau = \exp [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] \quad \text{--- (6-16a)}$$

$$\sigma = [K_9 x^3 + K_{10} x^4 + K_8], \text{ etc.} \quad \text{--- (6-16b)}$$

are equivalent to the equation(6-2) along with (6-5) and (6-6).

(For details see Appendix – C, Case - 4 a, b)

The equations (6-8) reduce to an interesting form when R = constant and S = constant . After a transformation $(\nu,\delta) \rightarrow (\nu',\delta')$, where $\nu' = \nu/\sqrt{R}$ and $\delta' = \delta/\sqrt{S}$, one gets from (6-8)

$$(\phi\phi_{\nu'\nu'} - \phi_{\nu'}^2 + \psi_{\nu'}^2 + \chi_{\nu'}^2) + (\phi\phi_{\delta'\delta'} - \phi_{\delta'}^2 + \psi_{\delta'}^2 + \chi_{\delta'}^2) = 0 \quad \text{--(6-17a)}$$

$$(\phi\,\psi_{\nu'\nu'} - 2\,\phi_{\nu'}\psi_{\nu'}) + (\phi\,\psi_{\delta'\delta'} - 2\phi_{\delta'}\psi_{\delta'}) = 0 \quad \text{---(6-17b)}$$

$$(\phi\,\chi_{\nu'\nu'} - 2\,\phi_{\nu'}\chi_{\nu'}) + (\phi\,\chi_{\delta'\delta'} - 2\,\phi_{\delta'}\chi_{\delta'}) = 0 \quad \text{---- (6-17c)}$$

Finally,(6-17) can be rewritten as,

$$(\Phi_{\nu'\nu'} + \Phi_{\delta'\delta'} + [(\psi_{\nu'}^2 + \chi_{\nu'}^2) + (\psi_{\delta'}^2 + \chi_{\delta'}^2)] \exp(-2\Phi) = 0 \quad \text{--(6-18a)}$$

$$[\psi_{\nu'} \exp(-2\Phi)]_{\nu'} + [\psi_{\delta'} \exp(-2\Phi)]_{\delta'} = 0 \quad \text{--(6-18b)}$$

$$[\chi_{\nu'} \exp(-2\Phi)]_{\nu'} + [\chi_{\delta'} \exp(-2\Phi)]_{\delta'} = 0 \quad \text{--(6-18c)}$$

where, $\phi = \exp(\Phi)$ ---- (6-18d)

The set of the equations in (6-18) are conformally invariant, i.e. the form of those equations is retained under the transformation $(\nu', \delta') \to (f, q)$ where f and q are functions of ν' and δ' such that $f_{\nu'} = q_{\delta'}$ and $f_{\delta'} = -q_{\nu'}$ i.e. f and q are mutually conjugate solutions of the Laplace's equations in ν' and δ'. Hence from any solution of the equations in (6-18) one can immediately generate infinitely many other solutions of equations in (6-18) by simply replacing (ν', δ') by (f, q).

One can compare the equations (6-18) with the generalized Lund-Regge equations discussed in article 2-3, equations (2-9).

Now, we present some exact solutions of (6-18) which are considerably general in nature. Four interesting cases have been observed.

<u>Case-I</u>

For, $\psi = \psi(\phi)$, $\chi = \chi(\phi)$

This can be identified as a particular situation of the work of De and Ray [41].

Case-II

$\psi = \psi(\chi)$ when the set of three equations of (6-18) reduces to a set of two equations similar to the set of two equations of two dimensional Heisenberg Ferromagnets and can be solved using the procedure of Trimper [34] and Ray [35].

Case-III

$$\psi = K_{15}\,\chi + u(\Phi) \qquad (6\text{-}19)$$

where K_{15} is an arbitrary constant and $u(\Phi)$ is an unspecified function of Φ, $u(\Phi)\neq 0$.

Using (6-19) in (6-18b) and then using (6-17c) in the resulting expression one gets,

$$[\, u_{\nu'} \exp(-2\Phi)\,]_{\nu'} + [\, u_{\delta'} \exp(-2\Phi)\,]_{\delta'} = 0 \qquad \text{----}\ (6\text{-}20)$$

Defining, $X = \int \exp(-2\Phi)\, du$ ----(6-21)

One can reduce (6-20) to,

$$X_{\nu'\nu'} + X_{\delta'\delta'} = 0 \qquad \text{---}\ (6\text{-}22)$$

which is the Laplace's equation and standard solutions for X in terms of ν' and δ' are obtainable. With (6-19) and (6-21),(6.22) ,the equation (6-18b) now becomes equivalent to (6-18c).

Since the set of equations in (6-18) are conformally invariant , then the transformation $(\nu', \delta') \rightarrow (X, Y)$, where X and Y are mutually conjugate solutions of the Laplace's equations, keeps the form of the equations (6-18) unchanged.

But now, from (6-21) & (6-22),

$$\Phi = \Phi(X) \;, u = u(X) \qquad \text{---- (6-23)}$$

Thus, using the transformation $(\nu',\delta') \rightarrow (X,Y)$; the equations(6-19), (6-21) and (6-23), one can observe that the three equations in (6-18) reduce to two equations only and after some rearrangement can be written as,

$$\psi_x^2 + \psi_y^2 + [(2K_{15}) / (K_{15}^2+1)]\, \psi_x \exp(2\Phi)$$

$$= -[\Phi_{xx} \exp(2\Phi) + \exp(4\Phi)] / (K_{15}^2+1) \qquad \text{--- (6-24a)}$$

and $\psi_{xx} + \psi_{yy} - 2\psi_x \Phi_x = 0$, respectively ---- (6-24b)

Defining, $\chi = \Theta - [K_{15} \int \exp(2\Phi).dX] / (K_{15}^2 + 1)$ ----(6-25)

One can reduce (6-24) to,

$$\Theta_x^2 + \Theta_y^2 = M(X) \qquad \text{---- (6-26a)}$$

where, $M(X) = -[(K_{15}^2 + 1)\Phi_{xx} \exp(2\Phi) + \exp(4\Phi)]/(K_{15}^2+1)^2$--(6-26b)

and $\Theta_{xx} + \Theta_{yy} - 2\Theta_x \Phi_x = 0$, respectively ---- (6-26c)

From (6-26a) , one can examine four cases separately. However, the following three case i.e.

(i) $\Theta_x = \Theta_y = 0$

(ii) $\Theta_x \neq 0$ & $\Theta_y = 0$ and

(iii) $\Theta_x = 0$ & $\Theta_y \neq 0$

can be grouped under the head Θ_y = constant, where the constant may even take the value zero. However, $\Theta_y = 0$ represent $\chi = \chi(\phi)$ and hence

$\psi = \psi(\phi)$ from (6-19), which has been considered by De and Ray [41] previously.

To study the fourth case i.e. $\Theta_x \neq 0$, $\Theta_y \neq 0$, one can proceed as follows. Differentiating (6-26a) with respect to Y one gets,

$$\Theta_x \Theta_{xy} + \Theta_y \Theta_{yy} = 0$$

with the help of which Θ_{yy} can be eliminated from (6-26b). After some manipulation in the resulting expression and then on integration once, one gets,

$$[\, \Theta_x / \Theta_y \,] \exp(-2\Phi) = \pi(Y) \qquad \text{----(6-27)}$$

where $\pi(Y)$ is an unspecified function of Y.

This readily gives,

$$\Theta = \Theta(w)$$

where, $w = u + v$

$$u = \int \exp(2\Phi)\, dX \qquad \text{(from 6-21a)}$$

and $v = \int dY / \pi$

with the use of which in (6-26a) one gets,

$$(u_x^2 / M) + (v_y^2 / M) = 1 / \Theta_w^2 \qquad \text{---- (6-28)}$$

Differentiating (6-28) separately with respect to u and v respectively and comparing the results, one gets,

$$M\,(u_x^2 / M)_u + M\,(1/M)_u\, v_y^2 = (v_y^2)_v \qquad \text{----(6-29)}$$

Differentiating (6-29) successively with respect to u and v respectively, one finally gets,

$[M (1 / M)_u]_u \; (v_y^2)_v \; = 0$

Hence, $M\,(1/M)_u = \text{constant}$ ----(6-30a)

Or $\quad v_y \; = \text{constant}$ ----(6-30b)

That (6-30a) is not permitted in our basic assumption of (6-21a), this is shown in Appendix – D.

In the following we consider, $v_y = \text{constant}$.

Differentiating (6-28) with respect to v and using $v_y = \text{constant}$, one gets,

$[\, 1 / \Theta_w^2 \,]_w = 0$, which gives $\Theta_w = \text{constant}$.

Hence, $\Theta_y = \Theta_w\, w_y \; = \; \Theta_w\, v_y = \text{constant}$.

Thus, in this case of $\Theta_x \neq 0, \Theta_y \neq 0$, too $\Theta_y = \text{constant}$ is satisfied.

So, to find the solution for the general case, when

$\Theta_y = \text{constant} \; = K_{16}$ (say) ---(6-31)

one can proceed as follows.

Using (6-31) in (6-26b) and then integrating once one gets,

$\Theta_x = K_{17} \; \exp(2\Phi)$ ---(6-32)

where, K_{17} is an arbitrary constant of integration.

Generalizing (6-31) and (6-32) one can conclude that,

$$\Theta = K_{17} \int \exp(2\Phi)\ dX + K_{16} Y + K_{18} \qquad \text{----(6-33)}$$

with the use of (6-25) , (6-33) reduces to,

$$\chi = K_{19} \int \exp(2\Phi)\ dX + K_{16} Y + K_{18} \qquad \text{----(6-34)}$$

where, $K_{19} = K_{17} - (K_{15}) / (K_{15}^2 + 1)$

It may be noted from (6-19), (6-23) and (6-34) that both of ψ and χ become functions of Φ only, when $K_{16} = 0$. This case has been treated by De and Ray [41].

Using (6-34) in (6-24a) and rearranging one can get,

$$\Phi_{xx} = -[\ K_{19}^2 / (K_{15}^2 + 1) + 2 K_{19} K_{15} + 1\ ] \exp(2\Phi)$$
$$- K_{16}^2 (K_{15}^2 + 1) \exp(-2\Phi) \qquad \text{--- (6-35)}$$

on integration (see Appendix – E) , (6-34) and (6-35) lead to,

$$\phi = K_{25}\, \mathrm{Cn}\,(r) \qquad \text{--- (6-36a)}$$

$$\chi = K_{19} (K_{22}^2 + K_{19}^2)^{-1/2} (K_{25}^2 + K_{24}^2)^{-1/2} [(K_{25}^2 + K_{24}^2) E(r)$$
$$- K_{24}^2 r] + K_{16} Y + K_{18} \qquad \text{----(6-36b)}$$

where, $r = (K_{22}^2 + K_{19}^2)^{1/2} (K_{25}^2 + K_{24}^2)^{1/2} (X - K_{21})$ ----(6-36c)

$$K_{22} = K_{19} K_{15} + 1 \qquad \text{---- (6-36d)}$$

$$K_{23} = K_{16} K_{15} \qquad \text{----(6-36e)}$$

where, $K_{16} \neq 0$

K_{20} and K_{21} are arbitrary constants of integration.

Here,

$$K_{24}^2 = [-K_{20} - \sqrt{\{K_{20}^2 + 4(K_{23}^2 + K_{16}^2)(K_{22}^2 + K_{19}^2)\}}]/2(K_{22}^2 + K_{19}^2) \qquad \text{–(6-36f)}$$

$$K_{25}^2 = [K_{20} - \sqrt{\{K_{20}^2 + 4(K_{23}^2 + K_{16}^2)(K_{22}^2 + K_{16}^2)\}}] / 2(K_{22}^2 + K_{19}^2) \quad \text{--(6-36g)}$$

The requirement that the permitted values of ϕ lies between $+K_{25}$ to $-K_{25}$ enables one to avoid the possible singularities.

To find the value of ψ, one may put the value of χ from equation (6-36 b) and value of $u(\Phi)$ from equation (6-21a) and obtained

$$\psi = K_{19}(K_{22}^2 + K_{19}^2)^{-1/2} (K_{25}^2 + K_{24}^2)^{-1/2} (1+K_{15}) [(K_{25}^2 K_{24}^2) E(r) - K_{24}^2 r] + K_{15}K_{16} Y + K_{15} K_{18} \quad \text{----(6-36h)}$$

Case - IV

Without loss of generality one can write from equation (6-17b),

$$\psi_{v'} \exp(-2\Phi) = \zeta_{\delta'} \quad \text{----(6-37a)}$$

$$\psi_{\delta'} \exp(-2\Phi) = -\zeta_{v'} \quad \text{---- (6-37b)}$$

such that, $\psi_{v'\delta'} = \psi_{\delta'v'}$ leads to,

$$[\zeta_{v'} \exp(2\Phi)]_{v'} + [\zeta_{\delta'} \exp(2\Phi)]_{\delta'} = 0 \quad \text{----(6-38)}$$

Similarly, one can write without loss of generality from equation (6-17c),

$$[\chi_{v'} \exp(-2\Phi)] = Z_{\delta'} \quad \text{---- (6-39a)}$$

$$[\beta_{\delta'} \exp(-2\Phi)] = -Z_{v'} \quad \text{---- (6-39b)}$$

such that, $\beta_{v'\delta'} = \beta_{\delta'v'}$ leads to,

$$[Z_{v'} \exp(2\Phi)]_{v'} + [Z_{\delta'} \exp(2\Phi)]_{\delta'} = 0 \quad \text{----(6-40)}$$

Eliminating $\psi_{v'}$, $\psi_{\delta'}$, $\chi_{v'}$, $\chi_{\delta'}$ from equation (6-17a) with use of (6-37) and (6-39) one gets,

$$\Phi_{v'v'} + \Phi_{\delta'\delta'} + [(\zeta_{v'}^2 + Z_{v'}^2) + (\zeta_{\delta'}^2 + Z_{\delta'}^2)] \exp(2\Phi) = 0 \quad \text{----(6-41)}$$

In the following , we will obtain solutions of the three coupled equations (6-38),(6-40) and (6-41) using the assumption,

$$\zeta = K_{25}\, Z + m\,(\Phi) \qquad \text{----(6-42)}$$

where , K_{25} is an arbitrary real constant and $m(\Phi)$ is an unspecified function of Φ = constant.

Using (6-42) in equation (6-38) and then using (6-40) in the resulting expression one gets,

$$[\, m_{v'} \exp(\, 2\Phi\,)\,]_{v'} + [\, m_{\delta'} \exp(\, 2\Phi\,)\,]_{\delta'} = 0 \qquad \text{----(6-43)}$$

Defining, $\xi = \int \exp\,(2\Phi)\, dm$ ---(6-44a)

One can reduce (6-43) to,

$$\xi_{v'v'} + \xi_{\delta'\delta'} = 0 \qquad \text{----(6-44b)}$$

which is the Laplace's equation and standard solutions of ξ in terms of ν and δ are obtainable.

With equation (6-42) and (6-44) , the equation (6-40) now becomes equivalent to (6-38).

Since, the set of equations (6-38),(6-40) and (6-41) are conformally invariant the transformation $(\nu', \delta') \rightarrow (\xi, \eta)$, where ξ and η are mutually conjugate solutions of the Laplace's equations, keeps the form of the equations (6-38) , (6-40) and (6-41) unchanged.

But now from equation (6-44),

$\Phi = \Phi(\xi)$, $m = m(\xi)$ ----(6-45)

Thus using the transformation,$(\nu',\delta') \rightarrow (\xi,\eta)$ along with equations (6-42), (6-44a) and (6-45),one can observe that the three equations (6-38), (6-40) and (6-41) reduce to two equations only and after some rearrangement can be written as,

$$Z_\xi^2 + Z_\eta^2 + [\, 2K_{25} \, / \, (K_{25}^2 + 1)\,]\, Z_\xi \, \exp(-2\,\Phi)$$

$$= -[\, \Phi_{\xi\xi} \exp(-2\Phi) + \exp(-4\,\Phi)\,] \, / \, (K_{25}^2 + 1) \quad \text{--- (6-46a)}$$

and $Z_{\xi\xi} + Z_{\eta\eta} + 2\, Z_\xi\, \Phi_\xi = 0$, respectively ---- (6-46b)

Defining, $Z = \chi - [K_{25} / (K_{25}^2 + 1)\,] \int \exp(-2\Phi)\, d\xi$ ----(6-46c)

$\chi = \chi(\xi, \eta)$ one can rewrite (6-46) as,

$\chi_\xi^2 + \chi_\eta^2 = N(\xi)$ ----(6-47a)

where,

$N(\xi) = -[(K_{25}^2+1)\Phi_{\xi\xi} \exp(-2\Phi) + \exp(-4\Phi)]/(K_{25}^2+1)^2$--(6-47b)

and $\chi_{\xi\xi} + \chi_{\eta\eta} + 2\,\chi_\xi\, \Phi_\xi = 0$, respectively ---- (6-47c)

One may observe the similarities between the equations (6-24) and (6-46) or between the equations (6-26) and (6-47). Thus, the procedure adopted in case of (6-26) holds here also.

Proceeding from (6-47), similarly as was done in (6-26) up to (6-31), here also one can observe that,

$\chi' = \chi' (m + n)$

where, $m = \int \exp(-2\Phi)\, d\xi$

and $n = \int d\eta / \varepsilon$, ε being an unspecified function of η.

Proceeding in the similar manner as in equation (6-30), these lead to,

n_η = constant

or $N (1/ N)_m$ = constant

Similar to (6-30a), it can be shown that, $N (1/N)_m$ = constant is not permitted (as shown in Appendix – F) in following we will consider,

$$\chi' = \text{constant} = K_{26} \text{ (say)} \qquad \text{----(6-48)}$$

Proceeding similarly as was done, starting from (6-31) up to (6-34), here one can obtain the following results :

$$Z = K_{27} \int \exp (-2\Phi)\, d\xi + K_{26}\, \eta + K_{28} \qquad \text{---(6-49)}$$

Using (6-49) in (6-46a) and rearranging one can get,

$$\phi_{\xi\xi} = -\left[K_{29}^2 (K_{25}^2 + 1) + 2 K_{25} K_{29} + 1 \right] \exp(-2\Phi) - K_{26}^2 (K_{25}^2 + 1) \exp(2\Phi) \quad \text{---(6-50)}$$

Also, using (6-50), it can be shown that,

$$\xi = \pm (K_{33}^2 + K_{26}^2)^{-1/2} \int [(\phi^2 + K_{34}^2)(K_{35}^2 - \phi^2)]^{-1/2} d\phi + K_{31} \quad \text{--(6-51)}$$

Here, also one can observe that the integral in (6-51) has the form of an elliptical integral and can be expressed in terms of standard elliptic integrals.

On integration (similar as in Appendix – E) (6-49) and (6-50) lead to,

$$\phi = K_{31} \, Cn\, (r_1) \quad \text{----(6-52a)}$$

and
$$Z = \mp \left[K_{27} (K_{33}^2 + K_{26}^2)^{-1/2} \left[(K_{34}^2 + K_{35}^2) E(r_1) - K_{34}^2 r_1 \right] \right. + K_{26}\eta + K_{28} \quad \text{----(6-52b)}$$

where, $r_1 = + [(K_{33}^2+K_{26}^2)^{1/2} (K_{34}^2 + K_{35}^2)^{1/2} (\xi - K_{31})]$ ----(6-52c)

$$K_{32} = K_{29} K_{25} + 1 \quad \text{---(6-52d)}$$

$$K_{33} = K_{26} K_{25} + 1 \quad \text{---(6-52e)}$$

K_{30} and K_{31} are arbitrary constant of integration.

Here,

$$K_{34}^2 = [-K_{30} - \{K_{30}^2 + 4(K_{32}^2 + K_{29}^2)(K_{33}^2 + K_{26}^2)\}^{1/2}]/2(K_{33}^2 + K_{26}^2) \quad \text{---(6-52f)}$$

$$K_{35}^2 = [K_{30} - \{K_{30}^2 + 4(K_{32}^2 + K_{29}^2)(K_{33}^2 + K_{26}^2)\}^{1/2}]/2(K_{33}^2 + K_{26}^2) \quad \text{--(6-52g)}$$

Thus ϕ is given by (6-52a). Then Z is given by (6-52b) and m is given by (6-44a). so that ξ is given by (6-42). All these quantities are given in terms of ξ and η which are mutually conjugate solutions of the Laplace equation (6-44b).

Since the equations in (6-47) and hence those in (6-46) have been completely solved, one can now conclude that for these solutions, $\psi_{v'\delta'} = \psi_{\delta' v'}$ and $\chi_{v'\delta'} = \chi_{\delta' v'}$ are satisfied.

Hence, from (6-37) one can write,

$\psi = \int[\zeta_{\delta'} \exp(2\Phi)]dv' + \int[-\zeta_{v'}\exp(2\Phi) - (\partial/\partial\delta')\int\{\zeta_{\delta'}\exp(2\Phi)dv'] d\delta' + K_{36}$

which with the use of (6-18d) , reduce to,

$\psi = \int(\phi^2\zeta_{\delta'})dv' + \int[-\phi^2\zeta_{v'} - (\partial/\partial\delta')\int(\phi^2\zeta_{\delta'})dv'] d\delta' + K_{36}$ ----(6-53)

where K_{36} is an arbitrary constant of integration.

Similarly, from (6-39) and (6-17b) one can write,

$\chi = \int(\phi^2 Z_{\delta'})dv' + \int[-\phi^2 Z_{v'} - (\partial/\partial\delta')\int(\phi^2 Z_{\delta'})dv'] d\delta' + K_{37}$ ---(6-54)

where K_{37} is an arbitrary constant of integration.

We have seen that Z, ζ and ϕ can be expressed in terms of ζ and η which are mutually conjugate solutions of the Laplace equation (6-44b). Hence, one can conclude that (6-53) and (6-54) give ψ and χ respectively in terms of ξ and η.

The results of this article may be summarized as follows :

1) τ is any arbitrary function of $(K_2 x^1 + K_3 x^2 + K_1)$ or $[(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2]$. σ is any arbitrary function of $[K_9 x^3 + K_{10} x^4 + K_8]$ or $[(K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2]$.

2) For any such value of τ and σ the equations (2-4) can be transformed to (6-8) via (6-1) along with (6-4) to (6-7).

3) For R= constant and S= constant only (6-4) and (6-6) are permitted. However, for various complicated forms of τ and σ the equations (2-4 can be transformed to (6-18) with (6-4) and (6-6). Furthermore for R= constant and S= constant we get from (6-8) a set of equations (6-18) which are conformally invariant and is very similar in form to the generalized Lund-Regge [30,31] equations (2-9a,b) in form. Thus from any solutions of (6-18) one can generate infinitely many other solutions by virtue of transformation of the type $(\nu', \delta') \rightarrow (f, q)$ where (ν', δ') are old independent variables, (f, q) are new independent variables and are functions of (ν', δ') such that $f_{\nu'} = q_{\delta'}$ and $f_{\delta'} = -q_{\nu'}$, i.e. f and q are mutually conjugate solutions of the Laplace's equations in ν' and δ'.

4) The solutions of the equation (2-4) via (6-18) observed by us can be grouped under four cases. In spite of the fact that the cases I and II are to some extent repetitions of the previous work [34,35,41] and our

observation that infinite number of new solutions can be generated from any solution of (6-18) makes mention of these cases worthwhile in our context. A short summary of these four cases is appended below.

Case-I

Here, $\psi = \psi(\phi)$, $\chi = \chi(\phi)$ and the situations of the solutions are the particular situations of the solutions obtained by De and Ray [41] previously.

Case-II

Here $\psi = \psi\ (\chi)$ and the solutions are the particular situations of the solutions obtainable using the procedure of Trimper [34] and Ray [35].

Case-III

ϕ is obtained in equation (6-36a) where r can be found out from (6-36c); ψ and χ obtained in the equations (6-36 h) and (6-36 b) respectively. In these equations, E(r) is an odd analytic function of r and when r is increased by 2K, E(r) is reproduced, save for an additive constant given by

$$\int_0^{2K} dn^2 r \ dr$$

where, K is a complete elliptical integral of the first kind and can be given as,

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{-(1/2)} d\theta$$

$$= (½)\ \pi\ F(\ ½\ , ½\ ; 1; k^2) \quad \text{---- (6-55)}$$

when k lies in the cut plane.

The ϕ in equation (6-36 a) oscillates [56] between + K_{25} and $-K_{25}$ with a period 4K and has zero points congruent with,

$$X = (K_{19}^2 + K_{22}^2)^{-(1/2)} (K_{24}^2 + K_{25}^2)^{-(1/2)} K + K_{21} \quad \text{or}$$

$$X = 3 (K_{19}^2 + K_{22}^2)^{-(1/2)} (K_{24}^2 + K_{25}^2)^{-(1/2)} K + K_{21}$$

where, K is defined by (6-55).

Case- IV

ϕ is obtained in equation (6-52a) where, r_1 is obtained in (6-52c) , ψ and χ is obtained in equation (6-53) and (6-54). Hence Z is obtained in equation (6-52b),which also gives ξ.

Here, r_1 has the property similar to r as described above for Case- III. As we know now ψ and χ, we can obtain ρ using equation (2-4d). Once we have found ρ and ϕ, the corresponding R-gauge potentials and the R-gauge field strengths can be obtained from (2-2) and (2-3) respectively.

All such solutions represent the condition of self-duality except when ϕ is zero. Because, where ϕ is zero $F_{\mu\nu}$ becomes singular and the solutions obtained can only be treated as solutions of Yang's R-gauge equations and not self-dual solutions unless a transformation like $F'_{\mu\nu} \rightarrow U^{-1} F_{\mu\nu} U$ removes the singularities.

APPENDIX – A

From equation (6-1c) , $\phi = \phi\ (\tau, \sigma)$ which gives,

$\phi_1 = \phi_\tau\, \tau^1$ and $\phi_{11} = (\phi_{\tau\tau}\, \tau_1^{\,2} + \phi_\tau\, \tau_{11})$

Similarly, we have, ϕ_2 , ϕ_3 , ϕ_4 , ϕ_{22} , ϕ_{33} and ϕ_{44} .

Again , from (6-1a) and (6-1b) we have similar equations as ϕ's.

Using these in equation (2-4a) , keeping in mind the equation (6-1d) & (6-1e) we have,

$$(\phi\, \phi_{\tau\tau} - \phi_\tau^{\,2} + \psi_\tau^{\,2} + \chi_\tau^{\,2})(\tau_1^{\,2} + \tau_2^{\,2}) + (\phi\, \phi_{\sigma\sigma} - \phi_\sigma^{\,2} + \psi_\sigma^{\,2} + \chi_\sigma^{\,2})(\sigma_3^{\,2} + \sigma_4^{\,2})\, \phi\, \phi_\tau\, (\tau_{11} + \tau_{22}) + \phi\, \phi_\sigma\, (\sigma_{33} + \sigma_{44}) = 0 \qquad \text{---- (A-1a)}$$

Similarly, we have from equation (2-4b),

$$(\phi\, \psi_{\tau\tau} - 2\psi_\tau\, \phi_\tau)(\tau_1^{\,2} + \tau_2^{\,2}) + (\phi\, \psi_{\sigma\sigma} - 2\, \psi_\sigma\, \phi_\sigma)(\sigma_3^{\,2} + \sigma_4^{\,2}) + \phi\, \psi_\tau\, (\tau_{11} + \tau_{22}) + \phi\, \psi_\sigma\, (\sigma_{33} + \sigma_{44}) = 0 \qquad \text{---- (A-1b)}$$

Similarly, we have from equation (2-4c),

$$(\phi\, \chi_{\tau\tau} - 2\chi_\tau\, \phi_\tau)(\tau_1^{\,2} + \tau_2^{\,2}) + (\phi\, \chi_{\sigma\sigma} - 2\, \chi_\sigma\, \phi_\sigma)(\sigma_3^{\,2} + \sigma_4^{\,2}) + \phi\, \chi_\tau\, (\tau_{11} + \tau_{22}) + \phi\, \chi_\sigma\, (\sigma_{33} + \sigma_{44}) = 0 \qquad \text{---- (A-1c)}$$

Comparing the value of $(\sigma_{33} + \sigma_{44})$ from (A-1a) & (A-1c) and that from (A-1c) & (A-1b), then dividing these two equations, we have after some simplification,

$(\tau_{11} + \tau_{22}) / (\tau_1^2 + \tau_2^2)$

$= $ a function of $(\phi, \phi_\tau, \phi_\sigma, \phi_{\tau\tau}, \phi_{\sigma\sigma}, \psi_\tau, \psi_\sigma, \psi_{\tau\tau}, \psi_{\sigma\sigma}, \chi_\tau, \chi_\sigma, \chi_{\tau\tau}, \chi_{\sigma\sigma})$

The left hand side of above equation is a function of x^1 & x^2. Hence, right hand side will also be function of x^1 & x^2. But in the right hand side x^1 & x^2 does not come in the explicit form, rather they came as a function of $\tau(x^1, x^2)$ only, thus

we may write ,

$(\tau_{11} + \tau_{22}) / (\tau_1^2 + \tau_2^2)$ = an arbitrary function of τ

$$= P(\tau) \quad \text{(say)} \qquad \text{---- (A-2)}$$

In the same procedure we may arrive at,

$(\sigma_{33} + \sigma_{44}) / (\sigma_3^2 + \sigma_4^2) =$ an arbitrary function of σ

$$= Q(\sigma) \quad \text{(say)} \qquad \text{---- (A-3)}$$

Putting (A-2) and (A-3) in equation (A-1a) and using the same argument as above we arrive at,

$$(\tau_1^2 + \tau_2^2) = \psi'(\tau) \text{ and } (\sigma_3^2 + \sigma_4^2) = \chi'(\sigma)$$

Here, $\psi'(\tau)$ is an arbitrary function of τ and and $\chi'(\sigma)$ is another arbitrary function of σ. Then the equations (A-1a,b,c) reduce to (6-2 a,b,c).

APPENDIX – B

At first we consider the equations,

$$(6\text{-}3a) \quad : (\nu_{11} + \nu_{22}) = 0$$

$$(6\text{-}3b) \quad : (\nu_1^2 + \nu_2^2) = R$$

Differentiating (6-3b) first with respect to x^1, we obtain ν_{11} and then with respect to x^2 we obtain ν_{22} . Putting these values of ν_{11} and ν_{22} into equation (6-3a) we have

$$R \cdot \nu_{12} = R_\nu \nu_1 \nu_2 \qquad \text{---- (B-1)}$$

From , here we have two cases :

Case- (i) When $R_\nu = 0$

Then , R = constant.

Hence, ν should be a linear function of x^1 & x^2 i.e.

$$\nu = E(x^1) + F(x^2) + K_1'$$

Finding ν_1 & ν_2 and using (6-3b) we arrive at,

$$\nu = K_2 x^1 + K_3 x^2 + K_1 \qquad \text{--- (B-2)}$$

$$\text{with, } (K_2^2 + K_3^2) = R \qquad \text{---- (B-3)}$$

where K_1 , K_2 and K_3 are constants.

Case- (ii) when, $R_\nu \neq 0$

Writing (B-1) firstly as, $R \nu_{12} = R_1 \nu_2$ and then integrating with respect to x^1 we have

$$\nu_2 = VR \qquad \text{------- (B- 4a)}$$

Here V is an arbitrary function of x^2 only.

Again, writing (B-1) as, $R\,\nu_{12} = R_2\,\nu_1$ and then integrating with respect to x^2 we have, $\nu_1 = UR$ ------- (B- 4b)

where U is an arbitrary function of x^1 only.

Using (B- 4a,b) in equation (6-3 a,b) we have,

$$U_1 + V_2 + (U^2 + V^2)\, R_\nu = 0 \qquad \text{---- (B-5a)}$$

$$\text{and} \quad U^2 + V^2 = 1/R \qquad \text{---- (B-5b)}$$

Using (B-5b) in equation (B-5a) we arrive,

$$U_1 + V_2 = -\,(\ln R\,)_\nu \qquad \text{---- (B-6)}$$

Differentiating (B-6) separately with respect to x^1 & x^2, using (B-4a,b) and finally comparing the results we may conclude that,

$$U_1 = V_2 \qquad \text{----(B-7)}$$

which shows that left hand side is a function of x^1 only whereas right hand side is a function of x^2 only.

Hence, one can conclude from (B-7) that

$$U_1 = V_2 = K_4 \qquad \text{----(B-8)}$$

where, K_4 is an arbitrary constant.

When, $U_1 = K_4$, then, $U = K_4\, x^1 + K_5$ --- (B-9a)

When, $V_2 = K_4$, then, $V = K_4\, x^2 + K_6$ --- (B-9b)

where, K_5 & K_6 are arbitrary constants of integration.

Using (B-9a) & (B-9b) in equation (B-6) we have,

$$R = K_7 \exp(-2\ K_4 \nu) \qquad \text{----(B-10)}$$

with, $K_4 \neq 0$, $K_7 \neq 0$.

Putting this value of R from equation (B-10) in (B-4 a,b) we have,

$$[\exp(2 K_4 \nu)]_1 = 2 K_4^2\ K_7 x^1 + 2 K_4 K_5 K_7 \qquad \text{---- (B-11a)}$$

and $$[\exp(2 K_4 \nu)]_2 = 2 K_4^2\ K_7 x^2 + 2 K_4 K_5 K_7 \qquad \text{---- (B-11b)}$$

Integrating (B-11a) one gets,

$$[\exp(2 K_4 \nu)] = K_4^2\ K_7 (x^1)^2 + 2 K_4 K_5 K_7 x^1 + G(x^2) \quad \text{-- (B-12a)}$$

where, $G(x^2)$ is an any arbitrary function of x^2.

Using equation (B-12a) in (B-11b) we have,

$$[G(x^2)]_2 = 2 K_4^2\ K_7\ x^2 + 2 K_4 K_6 K_7 \qquad \text{---- (B-12b)}$$

Integrating (B-12b) with respect to x^2 then using the result in (B-12a) one finally gets,

$$\nu = [1/(2K_4)] \ln[K_4^2 K_7\{(x^1)^2+(x^2)^2\} + 2K_4K_7(K_5x^1+K_6 x^2) + K_8' \text{---(B-13)}$$

Using (B-13) in equation (B-10) we have,

$$R = K_7 / [K_4^2 K_7\{(x^1)^2 + (x^2)^2\} + 2 K_4K_7 (K_5 x^1 + K_6 x^2) + K_8'] \text{ - (B-14)}$$

where, K_8' is an arbitrary constant of integration.

Differentiating (B-13) at first with respect to x^1 & then with respect to x^2 and using them along with (B-14) in (6-3b) one gets,

$$K_8' = K_7 (K_5^2 + K_6^2) \qquad \text{---- (B-15)}$$

Using (B-15) in (B-13) and (B-14) we have,

$$\nu = [1/(2K_4)] \ln [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] + [\ln K_7/(2K_4)] \qquad \text{-(B-16)}$$

$$R = 1/ [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] \qquad \text{---(B-17)}$$

One can check then (B-16) identically satisfies (6-3a).

Similarly, using equation (6-3c) & (6-3d) we have,

(i) for, $S_\delta = 0$

$$\delta = K_9 x^3 + K_{10} x^4 + K_8 \qquad \text{---- (B-18)}$$

$$S = K_9^2 + K_{10}^2 \qquad \text{----- (B-19)}$$

(ii) for, $S_\delta \neq 0$

$$\delta = [1/(2K_{11})] \ln [(K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2] + [1/(2K_{11}) \ln (K_{14})] \qquad \text{--(B-20)}$$

$$S = 1/ [(K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2] \qquad \text{---(B-21)}$$

where K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13} and K_{14} are arbitrary constants.

Appendix –C

Case-1a

(i) $\tau = \ln (K_2 x^1 + K_3 x^2 + K_1)$ --- (C-1)

(ii) $\sigma = \ln (K_9 x^3 + K_{10} x^4 + K_8)$ ---- (C-2)

These give (6-2d) as,

$$P(\tau) = -1 \qquad \text{-----(C-3)}$$

And (6-2e) as,

$$\psi(\tau) = (K_2^2 + K_3^2) \exp(-2\tau) \qquad \text{----(C-4)}$$

In the similar process we have,

$$Q(\sigma) = -1 \qquad \text{----(C-5)}$$

and $\chi(\sigma) = (K_9^2 + K_{10}^2) \exp(-2\sigma)$ ----(C-6)

Using (C-3) in equation (6-3e) we have,

$\nu = \int \exp(\tau) d\tau = K_2 x^1 + K_3 x^2 + K_1$, which is nothing but equation (6-4a).

Again, using (C-3) and (C-4) in equation (6-3g) we have,

$R = K_2^2 + K_3^2$ which is nothing but equation (6-4b).

Similarly, on differentiation of (C-2) and with use of equation (6-2 f,g) and equation (6-3 f,h) we have equation (6-6 a,b).

Case-1b

$$\tau = (K_2 x^1 + K_3 x^2 + K_1)^2 \qquad \text{--- (C-7)}$$

$$\sigma = (K_9 x^3 + K_{10} x^4 + K_8)^2 \qquad \text{---- (C-8)}$$

Deducing similarly as in above case we have,

$P(\tau) = 1/(2\tau)$, putting this in equation (6-3 e) we have,

$$\nu = (2/K_{15})\, \tau^{1/2} + K_{16} \qquad \text{----(C-9)}$$

where , K_{16} is another constant of integration.

For, $K_{15} = 2$ and $K_{16} = 0$, equation (C-9) reduces to,

$\nu = K_2 x^1 + K_3 x^2 + K_1$ which is the equation (6-4 a)

Similarly, as in Case – 1a, we also have, here,

$\psi(\tau) = 4(K_2^2 + K_3^2)\,\tau$, which gives,

$R = K_2^2 + K_3^2$, the equation (6-4 b).

In the similar manner , starting from equation (C-8) and using (6-2 f,g) and equation (6-3 f, h), we arrive at the equation (6-6).

Case – 2a

$$\tau = (K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2 \qquad \text{--- (C-10)}$$

$$\sigma = (K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2 \qquad \text{---- (C-11)}$$

Differentiating (C-10) with respect to x^1 & x^2 , we have,

(i) from equation (6-2 d) that,

$$P(\tau) = 1/(2\tau), \qquad \text{---(C-12)}$$

which reduces (6-2 e) that,

$$\nu = [1/(2K_4)] \ln(\tau) + [1/(2K_4)] \ln K_7$$

Putting the value of τ from equation (C-10) to the above equation we arrive at the equation (6-5a).

(ii) from equation (6-2 e),

$\psi(\tau) = 4 K_4^2 \tau$ which reduces (6-3 g) with the help of equation (C-12) as,

$R = 1 / [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2]$ which is the equation (6-5b).

Similarly, starting from equation (C-11) with (6-2 f,g) and (6-3 f,h), we arrive at the equation (6-7).

Case – 2b

$$\tau = \exp [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] \qquad \text{--- (C-13)}$$

$$\sigma = \exp [(K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2] \qquad \text{---- (C-14)}$$

Differentiating (C-13) and using (6-2d), we have,

$P(\tau) = (1 + \ln \tau) / (\tau \ln \tau)$, which ultimately gives

$P(\tau) = [1/(2K_4)] \ln [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] + [1/(2K_4) \ln (K_7)]$

i.e. the equation (6-5 b) and

$R = 1 / (\ln \tau)$

$= 1 / [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2]$ which is the equation (6-5 b).

Similarly, from equation (C-14), we arrive at the equation (6-7).

Case – 3a

$$\tau = \ln [\, K_2 x^1 + K_3 x^2 + K_1 \,] \qquad \text{---- (C-15a)}$$

$$\sigma = [(K_{11} x^3 + K_{12})^2 + (K_{12} x^4 + K_{13})^2\,] \qquad \text{---- (C-15b)}$$

Starting from(C-15a) we arrive at the equation (6-4),in the same way we have (6-4) starting from (C-1) as in Case –1a .

Also, starting from (C-15b) we arrive at the equation (6-7), in the same way we have (6-4) starting from (C-1) as in Case –2a .

Case – 3b

$$\tau = (\, K_2 x^1 + K_3 x^2 + K_1 \,)^2 \qquad \text{---- (C-16a)}$$

$$\sigma = \exp\,[(K_{11} x^3 + K_{12})^2 + (K_{11} x^4 + K_{13})^2\,] \qquad \text{---- (C-16b)}$$

Equation (C-16a) gives (6-4), as equation (C-7) gives (6-4) in Case- 1b. Also equation (C-18) gives (6-7) as equation (C-14) gives (6-7) in Case – 2b.

Case – 4a

$$\tau = [(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2\,] \qquad \text{---- (C-17)}$$

$$\sigma = \ln [\, K_9 x^3 + K_{10} x^4 + K_8 \,] \qquad \text{---- (C-18)}$$

Here (C-19) leads to equation (6-5) as (C-10) to (6-5) in Case – 2a, and (C-20) leads to equation (6-6) as (C-2) to (6-6) in Case – 1a.

Case – 4b

$$\tau = \exp[(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2\,] \qquad \text{---- (C-19)}$$

$$\sigma = [\, K_9 x^3 + K_{10} x^4 + K_8 \,]^2 \qquad \text{---- (C-20)}$$

Here (C-21) gives equation (6-5) as (C-13) gives (6-5), in Case – 2b, and (C-22) gives equation (6-6) as (C-8) gives (6-6) in Case – 1b.

Appendix – D

$M\,(1/M)_u = \text{Constant} = L \text{ (say)}$ ----(D-1)

It is evident, from equation (6-9) that,

$M\,(u_x^2/M)_u + M\,(1/M)_u\, v_y^2 = (v_y^2)_v$

that, $M\,(u_x^2/M)_u = \text{constant} = C \text{ (say)}$ ---- (D-2)

Using (D-1) and (D-2) in equation (6-9) , it reduces to,

$$(v_y^2)_v = L\,(v_y^2) + C \quad \text{----(D-3)}$$

From (D-1) we get,

$$M = N_1 \exp(-Lu) \quad \text{----(D-4)}$$

where, $N_1 = \text{constant} \neq 0$

In the following it will be shown that the above equations are not satisfied simultaneously and hence, $M(1/M)_u = \text{constant}$ is not possible.

***Case – I* : when $L \neq o$**

Expanding (D-2) we have,

$$M(u_x^2)_u / M + u_x^2\; M\,(1/M)_u = C$$

Using (D-1) and then on integration we have from above equation,

$$C_1 \exp(-Lu) = C - Lu_x^2 \quad \text{---- (D-5)}$$

Where, C_1 is an arbitrary constant.

Rearranging equation (D-5) we have,

$$u_x^2 = C_2 - C_3 \exp(-L\,u) \quad \text{---- (D-6)}$$

where, $C_2 = C/L$ and $C_3 = C_1/L$.

From equation (6-21a) we have, $u_x = \exp(2\Phi)$ ---- (D-7)

which turns equation (D-6) to ,

$$\exp(-L u) = C_4 - C_5 \exp(4\Phi) \quad \text{---- (D-8)}$$

where, $C_4 = C_2 / C_3$ and $C_5 = 1 / C_3$

Putting the value of M from equation (D-4) to equation (6-26b) and using

(D-8) we have, $\phi_{xx} = C_6 \exp(2\Phi) + C_7 \exp(-2\Phi)$ ----(D-9)

where, $C_6 = [\, N_1 C_5 (K_{15}^2 + 1)^2 - 1 \,] / (K_{15}^2 + 1)$ = constant

$C_7 = -[\, N (K_{15}^2 + 1)^2 C_4 \,]$ = constant

Using (D-9) in equation (6-26b), we obtain,

$$M = C_8 \exp(4\Phi) + C_9 \quad \text{----(D-10)}$$

where, $C_8 = -[\, C_6 (K_{15}^2 + 1)^2 + 1 \,] / (K_{15}^2 + 1)^2$ = constant

$C_9 = C_7 / (K_{15}^2 + 1)^2$ = constant

Comparing equation (D-10) and (D-4) , we have,

$$N_1 \exp(-L u) = C_8 \exp(4\Phi) + C_9 \quad \text{----(D-11)}$$

Using (D-8) , (D-11) and $\exp(\Phi) = \phi$, we have on simplification,

$$\phi^4 = [C_9 - N_1 (C_2 / C_3)] / [\, C_8 + (N_1 / C_3) \,] \quad \text{---- (D-12)}$$

As, C_2 ,C_3 ,C_8 , C_9 and N_1 are constants.

(D-12) leads to the trivial solution,

$$\phi = \text{Constant} \quad \text{----(D-13)}$$

This proves that (D-1) is not possible.

Case- II **L = 0**

From equation (D-4) , $M(x)$ = constant,

which leads to x = constant. Hence (D-1) is not possible.

Appendix –E

Putting , $\Phi_{xx} = (1/2)\,(\phi_x^2)_\Phi$ in equation (6-34) and integrating we have

$$\Phi_x^2 = -[\,K_{19}^2\ (K_{15}^2+1)^2\ + K_{19}\,K_{15} + 1\,]\ \exp(2\Phi)$$

$$+ K_{16}^2\,(K_{15}^2+1)\ \exp(-2\Phi) + K_{20} \qquad \text{----(E-1)}$$

where K_{20} is an arbitrary constant of integration.

Using, $\phi = \exp(\phi)$ from equation (6-18d) in (E-1) one have after inversion,

$$X_\phi = +[\ -(K_{22}^2 + K_{19})^2\ \phi^4 + K_{20}\,\phi^2 + (K_{23}^2 + K_{16}^2)] \qquad \text{---(E-2)}$$

where, $K_{22} = K_{19}\,K_{15} + 1$ and $K_{23} = K_{15}\,K_{16}$

In view of equation (6-19) , it can be shown that,

$$K_{19}^2\ (K_{15}^2+1)^2\ +\ 2\,K_{19}\,K_{15} + 1\ \neq 0$$

and $$K_{16}^2\,(K_{15}^2+1)\ \neq 0$$

On integration of (E-2) one have,

$$X = \pm(K_{19}^2 + K_{22}^2)^{-(1/2)}\int[(\phi^2 + K_{24}^2)\,(K_{25}^2 - \phi^2)]^{-(1/2)}d\phi + K_{21} \qquad \text{--(E-3)}$$

where , K_{21} is an arbitrary constant of integration and the values of K_{24} & K_{25} are given in equation (6- 52f) & (6-52g) respectively.

Now, let $\phi = \operatorname{Cos}\theta$, then equation (E-3) reduces to,

$$X = \mp(K_{19}^2+K_{22}^2)^{-(1/2)}(K_{24}^2+K_{25}^2)^{-(1/2)}\int 1/(1-k^2\,\operatorname{Sin}^2\theta)^{(1/2)}]d\theta + K_{21} \qquad \text{--(E-4)}$$

where, $K_{25}^2\,/\,(K_{24}^2 + K_{25}^2) = k^2$ (say) --(E-5)

The integral on the right hand side of equation (E-4) is Legendre's elliptical integral (details in [59]) of the first kind. Equation (E-4) can be inverted to write ϕ as Jacobi's elliptical function (details in [59]) and can be written as,

$$\phi = K_{25}\, Cn\,(r) \qquad \text{---- (E-6)}$$

$$\text{where, } r = \mp(K_{19}^2 + K_{22}^2)^{(1/2)}(K_{24}^2 + K_{25}^2)^{(1/2)}(X - K_{21}) \quad \text{--(E-7)}$$

Using (E-7) , equation (6-33) reduces to,

$$\psi = K_{19}^2 K_{25}^2 (K_{19}^2 + K_{22}^2)^{-(1/2)} (K_{19}^2 + K_{25}^2)^{-(1/2)} \int Cn^2(r)\, dr$$

$$+ K_{16}\, Y + K_{18} \qquad \text{--(E-8)}$$

$$\text{Now, } \int Cn^2(r)\, dr = (1/k^2) \int k^2 Cn^2(r)\, dr$$

$$= (1/k^2) \int [dn^2(r) - k'^2)dr$$

$$[\text{as}, dn^2(r) - k^2 Cn^2(r) = k'^2]$$

$$= (1/k^2)\, E(r) - [(1 - k^2)/k^2]\, r \qquad \text{---- (E-9)}$$

In the above we use the relations, $k^2 + k'^2 = 1$

And $\int (dn^2(r)\, dr = E(r)$ = Jacobi's function.

Using (E1-5) , we have, $(1 - k^2)/k^2 = K_{24}^2 / K_{25}^2$ -----(E-10)

Using (E-9) and (E-10) in equation (E-8) we have,

$$\chi = K_{19}(K_{19}^2 + K_{22}^2)^{-(1/2)}(K_{24}^2 + K_{25}^2)^{-(1/2)}[(K_{25}^2 + K_{24}^2)E(r) - K_{24}^2\, r]$$

$$+ K_{16}\, Y + K_{18} \qquad \text{--(E-11)}$$

Appendix – F

Using (6-46) and the value of χ , m and n , we have a equation, similar to equation (6-28) as,

$$N\,(\,m_\xi^{\,2}\,/\,N\,)_m + N\,(\,1/\,N)_m\, n_\eta^{\,2} = (n_\eta^{\,2}\,)_m \qquad \text{----(F-1)}$$

Differentiating (F-1) successively with respect to m and n respectively, we have,

$$N\,(\,1/\,N)_m = \text{ constant} = A \text{ (say)} \qquad \text{---(F-2)}$$

and $\quad n_\eta = \text{constant}$

Using (F-1) and n_η = constant, equation (F-2) reduces to,

$$N\,(m_\xi^{\,2}\,/\,N\,)_m = \text{constant} = D \text{ (say)} \qquad \text{---(F-3)}$$

Using (F-2) and (F-3) , equation (F-1) reduces to,

$$(\,n_\xi^{\,2}\,)_n = A\, n_\xi^{\,2} + D \qquad \text{---(F-4)}$$

From (F-2) we get, $N = B\,\exp(\,-\,A\,m\,)$ --- (F-5)

where, B is an integration constant $\neq 0$.

In the following it will be shown that the above equations do not satisfy simultaneously and hence, $N(1/N)_m$ = constant is not possible.

<u>*Case- I*</u> **$A \neq 0$**

Expanding (F-3) we have,

$[N (m_\xi^2)_m] / N + m_\xi^2 N (1 / N)_m = D$

Using (F-2) and then on integration, we have ,

$$m_\xi^2 = D_2 - D_3 \exp (- A m) \qquad \text{----(F-6)}$$

where, $D_2 = D / A$ and $D_3 = D' / A$, D' being integration constant.

From equation (6-43a) we have,

$$m_\xi = \exp(- 2\Phi) \qquad \text{----(F-7)}$$

which turns (F-6) to

$$\exp (- Am) = D_4 - D_5 \exp (-4\Phi) \qquad \text{----(F-8)}$$

where, $D_2 /D_3 = D_4$ and $1/D_3 = D_5$

Putting the value of N from equation (F-4) to equation (6-45b) we have,

$$\phi_{\xi\xi} = D_6 \exp(- 2\Phi) + D_7 \exp(2\Phi) \qquad \text{----(F-9)}$$

where, $D_6 = [B D_5 (K_{25}^2 + 1)^2 - 1] / (K_{25}^2 + 1)$

$D_7 = - B D_4 (K_{25}^2 + 1)$

Using (F-9) to equation (6-46b) we have,

$$N = D_8 \exp(-4 \Phi) + D_9 \qquad \text{----(F-10)}$$

where, $D_8 = - [D_6 (K_{25}^2 + 1)^2 + 1] / (K_{25}^2 + 1)^2$

$D_9 = - D_7 /(K_{25}^2 + 1)$

Comparing equation (F-4) and (F-10),

$$B \exp(-A\, m) = D_8 \exp(-4\Phi) + D_9 \qquad \text{---- (F-11)}$$

Using (F-7) to (F-6) and then using (F-11) we have,

$$\exp(4\Phi) = (B/D_3 + D_8)/(BD_2/D_3 - D_9) = \text{constant} \qquad \text{---(F-12)}$$

But, $\exp(\Phi) = \phi$, which turns (F-11) to,

$$\phi^4 = \text{constant} \qquad \text{----(F-13)}$$

with this (F-13) leads to the trivial solution, ϕ =constant .

This proves that (F-2) is not possible.

<u>Case- II</u> **A= 0**

This turns (F-5) to,

$$N(\xi) = \text{constant}$$

which leads to,

$$\xi = \text{constant.}$$

Hence, (F-2) is not possible.

6-1-2 : Some exact solutions of the Charap equations (2-8)

In article 5-1-2 we reviewed some of the solution of Charap's chiral equation (2-8) as obtained by Charap himself [29], Chanda, De and Ray [30]. After that Ray [32] obtained infinite number of solutions of the equations (2-8), where x^3 and x^4 appears in terms of $(x^3 - x^4)$ only. Here, we present another two types of exact solution and each of these two allow infinite number of solutions where the dependence on x^3 and x^4 is much more general.

One of the two types of solutions and the process, which generate that type of solution, add to the observation that there are considerable similarities between the two sets of equations, namely Charap chiral invariant field equations and Yang's R-gauge field equations. The similarities among these two equations motivated us to combine these two sets of equations, Yang (2-4) & Charap (2-8) equations and to formulate the combined Yang-Charap (Y-C) equations (2-10).

<u>First type of exact solutions</u>

In the case of first type of solutions we have applied the procedure as given in previous article 6-1-1 which was originally used for obtaining solutions for the Yang's R-gauge equations (2-4) in Euclidean space. Under similar assumptions and similar transformations the equations (2-8) lead to conformally invariant equations in a similar fashion permitting one to obtain infinitely many other solutions from any solution of these conformally invariant equations. These conformally invariant equations also closely resemble the mathematically interesting generalized Lund-Regge equations given in equations (2-9a,b).

The solutions for (2-8) presented here are those which satisfy the relations,

$$\phi = \phi\,(\,\tau\,,\sigma\,) \qquad \text{---- (6-56a)}$$

$$\psi = \psi\,(\,\tau\,,\sigma\,) \qquad \text{---- (6-56b)}$$

$$\chi = \chi\,(\,\tau\,,\sigma\,) \qquad \text{---- (6-56c)}$$

where,

$$\tau = \tau\,(x^1,x^2\,)\,,\ \sigma = \sigma(x^3,x^4\,) \qquad \text{---- (6-56d,e)}$$

Using the procedure which is exactly similar to previous article 6-1-1 and using the relations (6.56a,b,c) one can rewrite (2-8), the Charap equations as [similar to Appendix – A of article 6-1-1],

$(\phi_{\tau\tau} - \phi_\tau \beta_\tau + P\,\phi_\tau)\,R + (\phi_{\sigma\sigma} - \phi_\sigma \beta_\sigma + Q\,\phi_\sigma)\,S = 0$ ----(6-57a)

$(\psi_{\tau\tau} - \psi_\tau \beta_\tau + P\,\psi_\tau)\,R + (\psi_{\sigma\sigma} - \psi_\sigma \beta_\sigma + Q\,\psi_\sigma)\,S = 0$ --- (6-57b)

$(\chi_{\tau\tau} - \chi_\tau \beta_\tau + P\,\chi_\tau)\,R + (\chi_{\sigma\sigma} - \chi_\sigma \beta_\sigma + Q\,\chi_\sigma)\,S = 0$ ---- (6-57c)

$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)$, f_π = constant --- (6-57d)

where $(\tau_{11} + \tau_{22}) / (\tau_1^2 + \tau_2^2) = P(\tau)$ ---- (6-57e)

$(\tau_1^2 + \tau_2^2) = R(\tau)$ ---- (6-57f)

and $(\sigma_{33} - \sigma_{44}) / (\sigma_3^2 - \sigma_4^2) = Q(\sigma)$ ---- (6-57g)

$(\sigma_3^2 - \sigma_4^2) = S(\sigma)$ ---- (6-57h)

where P & R are arbitrary functions of τ and Q & S are arbitrary functions of σ.

Equations (6-57e) – (6-57h) can be rewritten as,

$(\nu_{11} + \nu_{22}) = 0$, $(\nu_1^2 + \nu_2^2) = R'$ ---(6-58a,b)

$(\delta_{33} - \delta_{44}) = 0$, $(\delta_3^2 - \delta_4^2) = S'$ --- (6-58 c, d)

where, $\nu = \int [\exp(-\int P(\tau)\,d\tau)]\,d\tau$ --- (6-58 e)

$\delta = \int [\exp(-\int Q(\sigma)\,d\sigma)]\,d\sigma$ --- (6-58f)

$R' = [\exp(-2\int P(\tau)\,d\tau)]\,R(\tau)$ --- (6-58g)

$S' = [\exp(-2\int Q(\sigma)\,d\sigma)]\,S(\sigma)$ --- (6-58h)

By virtue of (6-58e,f) one can consider R′ and S′ as functions of ν and δ respectively.

The solution for (6-58 a,b) are given by

(i) $\nu = K_2\,x^1 + K_3\,x^2 + K_1$ --- (6-59a)

$R' = K_2^2 + K_3^2$ --- (6-59b)

Or,

(ii) $\nu = 1/(2K_4) \ln[(K_4x^1 + K_5)^2 + (K_4x^2+K_6)^2] + (\ln K_7)/(2K_4)$ - (6-60a)

$R' = 1/[(K_4x^1+K_5)^2 + (K_4x^2+K_6)^2]$ --- (6-60b)

The solutions for (6-58 c, d) are given by :

i) $\delta = K_9 x^3 + K_{10} x^4 + K_8$ --- (6-61a)

$S' = K_9^2 - K_{10}^2$ --- (6-61b)

Or,

(ii) $\delta = 1/(2K_{11}) \ln[(K_{11}x^3 + K_{12})^2 - (K_{11}x^4 - K_{13})^2] + (\ln K_{14})/(2K_{11})$ - (6-62a)

$S' = 1/\ [(K_{11} x^3 + K_{12})^2 - (K_{11} x^4 - K_{13})^2]$ --- (6-62b)

where, K_i , $i = 1$ to 14 , are arbitrary constants.

Now, without any loss of generality one can transform (τ,σ) to (ν,δ) when the equations (6-57 a,b,c) lead to ,

$(\phi_{\nu\nu} - \phi_\nu\ \beta_\nu)\, R' + (\phi_{\delta\delta} - \phi_\delta\ \beta_\delta)\, S' = 0$ ---(6-63a)

$(\psi_{\nu\nu} - \psi_\nu\ \beta_\nu)\, R' + (\psi_{\delta\delta} - \psi_\delta\ \beta_\delta)\, S' = 0$ ---- (6-63b)

$(\chi_{\nu\nu} - \chi_\nu\ \beta_\nu)\, R' + (\chi_{\delta\delta} - \chi_\delta\ \beta_\delta)\, S' = 0$ ---- (6-63c)

$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)$, $f_\pi =$ constant --- (6-63d)

where, ν,δ,R' and S' are given by (6-59) - (6-62).

Regarding the equations (6-57) and (6-63), the following observations may be interesting :

I. If τ and σ satisfies a set of coupled equations of the form (6-57), then any function of τ and σ also satisfies the set of coupled equations of the form (6-57).

II. For all possible equations of the form (6-57) which generate as a result of the transformation of (6-56), τ is any function of $[K_2 x^1 + K_3 x^2 + K_1]$ or $[(K_4 x^1+K_5)^2 + (K_4 x^2 + K_6)^2]$ and σ is any function of $[K_9 x^3+K_{10} x^4+K_8]$ or $[(K_{11} x^3+ K_{12})^2 - (K_{11} x^4 - K_{13})^2]$.

Moreover, all such transformed equivalent to the set of equations (6-63) via (6-58). Some helpful examples in this regard are given in previous article from equation (6-9) to (6-16).

The equations (6-63) reduce to an interesting form when R' = constant and S' = constant . After a transformation $(\nu , \delta) \rightarrow (\nu' , \delta')$ where $\nu' = \nu/ \sqrt{R'}$ and $\delta' = \delta/ \sqrt{S'}$, one gets from (6-63) ,

$$\phi_{\nu'\nu'} + \phi_{\delta'\delta'} = \phi_{\nu'} \beta_{\nu'} + \phi_{\delta'} \beta_{\delta'} \qquad \text{--(6-64a)}$$

$$\psi_{\nu'\nu'} + \psi_{\delta'\delta'} = \psi_{\nu'} \beta_{\nu'} + \psi_{\delta'} \beta_{\delta'} \qquad \text{--(6-64b)}$$

$$\chi_{\nu'\nu'} + \chi_{\delta'\delta'} = \chi_{\nu'} \beta_{\nu'} + \chi_{\delta'} \beta_{\delta'} \qquad \text{--(6-64c)}$$

$$\beta = \ln [\phi^2 + \psi^2 + \chi^2 + f_\pi^2] \qquad \text{--(6-64d)}$$

and f_π is a constant.

where $\nu' = (\nu/\sqrt{R'}) = (K_2 x^1 + K_3 x^2 + K_1) / \sqrt{(K_2^2 + K_3^2)}$ -- (6-64e)

and $\delta' = (\delta/\sqrt{S'}) = (K_9 x^3 + K_{10} x^4 + K_8) / \sqrt{(K_9^2 - K_{10}^2)}$ -- (6-64f)

The relations (6-64e,f) follow from the fact that for R' = constant, S'= constant, ν and δ are given by (6-59a,b) and (6-61 a,b) respectively.

That the equations (6-64) are consistent with the constraint R'=constant and S'=constant can be checked by putting the following in equation (2-8):

$$\phi = \phi\,(\nu',\delta')$$

$$\psi = \psi\,(\nu',\delta')$$

$$\chi = \chi\,(\nu',\delta')$$

where ν' and δ' are given by (6-64 e, f). This reduces Charap equation (2-8) exactly to (5-29) as described in article 5-1-2.

The set of the equations in (6-64) are conformally invariant, i.e. the form of those equations is retained under the transformation $(\nu',\delta') \rightarrow (f, q)$, where f and q are functions of ν' and δ' such that $f_{\nu}' = q_{\delta}'$ and $f_{\delta}' = -q_{\nu}'$ i.e. f and q are mutually conjugate solutions of the Laplace's equations in ν' and δ'.

Hence from any solution of the equations in (6-64) one can immediately generate infinitely many other solutions of equations in (6-64) by simply replacing (ν', δ') by (f, q).

It is interesting to note that equation (6-64) are of the same form as one of the two generalized Lund-Regge equations (2-9 a,b) as discussed previously. With $g = 0$, the equations (2-9) reduce to a conformally invariant set of equations, a particular example of which is the physically interesting equations of two dimensional Heisenberg Ferromagnets[34,35]. Again the equations (6-64) and (5-28) are exactly same form as (5-29a) as presented by Chanda, De and Ray [30] and Ray [32] . We can use their solutions for obtaining the solutions for (6-64).However the following points should be kept in mind :

i) As given in Review 5-1-2, the work of Ray [32] , the independent variables for (5-28 a,b,c) are x^1 & x^2 , whereas the independent variables for (6-64) are ν' and δ' where $\nu' = \nu / \sqrt{R'}$ and $\delta' = \delta / \sqrt{S'}$. ν , R' and δ, S' are given in (6-59) - (6-62).

Hence for obtaining solutions for the equations (6-64) one has to replace x^1 and x^2 in the solutions for (5-28) by ν' and δ'.

ii) The arbitrary constants of integration in the solutions for (5-28) (of article 5-1-2 for the work of Ray [32]) are functions of ($x^3 - x^4$). But for the equations (6-64) they are pure constants.

iii) For solutions of (2-8) via the solutions of (6-64) one has R'=constant and S'= constant. Hence such solutions for (2-8) are found not for all x^1,x^2,x^3,x^4 but only on the two dimensional surfaces R'=constant and S'=constant.

Second type of exact solutions

Here we have sought a class of solutions by changing variables to functions of the space-time coordinates which are restricted in the following way.

$$(x^2, x^2, x^3, x^4) \rightarrow (X, Y, Z, W) \quad \text{----(6-65a)}$$

such that, $X_1 = Y_2$, $\quad X_2 = -Y_1$ ----(6-65b)

$$Z_3 = W_4, \qquad Z_4 = W_3 \quad \text{---(6-65c)}$$

So that $\quad X_{11} + X_{22} = 0$, $Y_{11} + Y_{22} = 0$ ----(6-65d)

$$Z_{33} - Z_{44} = 0 \text{ , } W_{33} - W_{44} = 0 \quad \text{----(6-65e)}$$

Some examples of X and Y satisfying (6-65b) are as follows :

$$X = x^1 \text{ , } Y = x^2 \quad \text{---- (6-66a)}$$

$$X = (x^1)^2 - (x^2)^2 \text{ , } Y = 2\,(x^1)\,(x^2), \quad \text{---- (6-66b)}$$

$$X = (x^1)^3 - 3\,(x^1)\,(x^2)^2 \text{ ,} Y = 3\,(x^1)^2\,(x^2) - (x^2)^3. \quad \text{---- (6-66c)}$$

And, some examples of Z and W satisfying (6-65c) are as follows :

$$Z = x^3, W = x^4 \quad \text{---- (6-67a)}$$

$$Z = (\text{Sin } x^3)\,(\text{Sin } x^4) \text{ , } W = -\,(\text{Cos } x^3)\,(\text{Cos } x^4) \quad \text{---- (6-67b)}$$

$$Z = (\text{Sinh } x^3)\,(\text{Sinh } x^4) \text{,} W = (\text{Cosh } x^3)\,(\text{Cosh } x^4) \quad \text{---- (6-67c)}$$

As a consequence of the transformation (6-65) the equations (2-8) reduce to :

$$(\phi_{xx} + \phi_{yy} - \phi_x\,\beta_x - \phi_y\,\beta_y)\,(X_1^2 + X_2^2) +$$

$$(\phi_{zz} - \phi_{ww} - \phi_z\,\beta_z + \phi_w\,\beta_w)\,(Z_3^2 - Z_4^2) = 0 \quad \text{----(6-68a)}$$

$$(\psi_{xx} + \psi_{yy} - \psi_x\,\beta_x - \psi_y\,\beta_y)\,(X_1^2 + X_2^2) +$$

$$(\psi_{zz} - \psi_{ww} - \psi_z\,\beta_z + \psi_w\,\beta_w)\,(Z_3^2 - Z_4^2) = 0 \quad \text{---(6-68b)}$$

$$(\chi_{xx} + \chi_{yy} - \chi_x \beta_x - \chi_y \beta_y) (X_1^2 + X_2^2) +$$

$$(\chi_{zz} - \chi_{ww} - \chi_z \beta_z + \chi_w \beta_w) (Z_3^2 - Z_4^2) = 0 \quad \text{----(6-68c)}$$

$$\beta = \ln [\phi^2 + \psi^2 + \chi^2 + f_\pi^2] \quad \text{----(6-68d)}$$

From (6-68) one gets two distinctly separate cases.

The first case is given by ($Z_3^2 - Z_4^2$) = 0 , which has been considered by Chanda, De & Ray [30] and Ray [32] and we are not discussed here.

The second case is given by ($Z_3^2 - Z_4^2$) ≠ 0.

Here we have considered a particular situation of this case which is given by the simultaneous satisfaction of the following sets of equations:

$$\phi_{xx} + \phi_{yy} = \phi_x \beta_x + \phi_y \beta_y \quad \text{--(6-69a)}$$

$$\psi_{xx} + \psi_{yy} = \psi_x \beta_x + \psi_y \beta_y \quad \text{--(6-69b)}$$

$$\phi_{xx} + \phi_{yy} = \phi_x \beta_x + \phi_y \beta_y \quad \text{--(6-69c)}$$

$$\beta = \ln [\phi^2 + \psi^2 + \chi^2 + f_\pi^2] \quad \text{--(6-69d)}$$

and

$$\phi_{zz} - \phi_{ww} = \phi_z \beta_z - \phi_w \beta_w \quad \text{--(6-70a)}$$

$$\psi_{zz} - \psi_{ww} = \psi_z \beta_z - \psi_w \beta_w \quad \text{--(6-70b)}$$

$$\chi_{zz} - \chi_{ww} = \chi_z \beta_z - \chi_w \beta_w \quad \text{--(6-70c)}$$

$$\beta = \ln [\phi^2 + \psi^2 + \chi^2 + f_\pi^2] \quad \text{--(6-70d)}$$

Now the equations (6-69) and (6-70) remain invariant in form under the transformation

$$(X,Y,Z,W) \rightarrow (X',Y',Z',W')$$

where $\quad X_x' = Y_y' \,, \qquad X_y' = -Y_x' \qquad$ ----(6-71a)

$$Z_z' = W_w', \qquad Z_w' = W_z' \qquad \text{----(6-71b)}$$

$$X' = X'(X,Y) \,,\; Y' = Y'(X,Y) \qquad \text{----(6-71c)}$$

$$Z' = Z'(Z,W) \,,\; W' = W'(Z,W) \qquad \text{----(6-71d)}$$

Examples of such X',Y',Z',W' are same in form to those given for X,Y,Z,W as given for (6-65).

Hence from any solution for (2-8) which simultaneously satisfy (6-70) and (6-71) one can generate infinitely many other solutions of (2-8) simply by replacing (X,Y,Z,W) by (X',Y',Z',W').

The equations (6-69) are of the same form as that of (5-3) as in 5-1-2 . The equations (6-70) reduces to the equations (6-69) under the transformation $W \rightarrow i\,W$ and the replacement of (Z,W) by (X,Y) . Thus if the equations (6-70) is a set coupled equations in two-dimensional space-time continuum where W is time like and Z is space like, then equation (6-69) is its Euclidean counterpart, where both X and Y are space like.

One simple way for obtaining solutions for (2-8) which satisfy simultaneously equations (6-69) and (6-70) is to follow the algorithm given below:

1)Obtain any solution for (6-69) using the formalism of Chanda, De and Ray [30] and Ray [32].

2) Express the arbitrary constants in that solution as functions of Z and W such that the role of X and Z or Y and W remain symmetrical and the final solution satisfies (6-69) and (6-70) simultaneously.

It may be checked easily that the existence of terms like XZ,YW etc. are not permitted . In the following we use a particular situation of the solutions reported in Case-I of the work of Chanda, Ray and De [30] for demonstrating the algorithm stated above.

Here the solution for (6-69) can be written as

(For detail see Appendix – G at the end of this article)

$$\phi = f_\pi \tan \ (2 f_\pi X - G) \quad \text{---- (6-72a)}$$

$$\psi = f_\pi \operatorname{Sec} \ (2 f_\pi X - G) \ \operatorname{Cos} (AX + BY + C) \quad \text{---- (6-72b)}$$

$$\chi = f_\pi \operatorname{Sec} \ (2 f_\pi X - G) \ \operatorname{Sin} (AX + BY + C) \quad \text{---- (6-72c)}$$

$$A^2 + B^2 = 4 f_\pi^2 \quad \text{----(6-72d)}$$

Where A,B,C,G are arbitrary constants. X, Y satisfy (6-65b), (6-65d), (6-71b) and (6-71d).

Now following the algorithm stated above we can write:

$$\phi = f_\pi \tan\ [\ 2\, f_\pi\, (X+Z)\,] \qquad \text{---- (6-73a)}$$

$$\psi = f_\pi\ \text{Sec}\,[\, 2\, f_\pi\, (X+Z)\,]\ \text{Cos}\,(\, CX + DY + EZ + FW) \text{ -- (6-73b)}$$

$$\chi = f_\pi\ \text{Sec}\,[2\, f_\pi\, (X+Z)\,]\ \ \text{Sin}\,(\, CX + DY + EZ + FW) \text{ -- (6-73c)}$$

$$C^2 + D^2 = 4\, f_\pi^{\ 2} \qquad \text{----(6-73d)}$$

$$E^2 - F^2 = 4\, f_\pi^{\ 2} \qquad \text{----(6-73e)}$$

where X,Y,Z,W satisfy (6-65) and (6-71). C,D,E,F are all arbitrary constants.

Equations (6-73) identically satisfy the equations (2-8). That is (6-73) represent an exact solution of (2-8) whereform we obtain infinitely many other solutions with the use of (6-65) and (6-71).

That the existence of terms like XZ,YW etc. is not permitted as stated in the algorithm with the help of this example :

$$\phi = f_\pi \tan\ [\ 2\, f_\pi\, (X+Z)\,] \qquad \text{---- (6-74a)}$$

$$\psi = f_\pi \text{Sec}[2f_\pi(X+Z)]\ \ \text{Cos}\,(AZX+BWY+CX+DY+EZ+FW) \text{ - (6-74b)}$$

$$\chi = f_\pi\ \text{Sec}[2f_\pi\,(X+Z)]\ \ \text{Sin}(AZX+BWY+CX+DY+EZ+FW) \text{ --(6-74c)}$$

This when put in (2-8) leads to,

$$(\,A\,Z + C\,)^2 + (\,B\,W + D\,)^2 = 4\, f_\pi^{\ 2}$$

$$(\,A\,Z + E\,)^2 - (\,B\,Y + F\,)^2 = 4\, f_\pi^{\ 2} \quad \text{, which are not permitted.}$$

Finally, one can note some similarities and differences between the solutions presented in the work of Chanda , De & Ray [30] and Ray [32] and the solutions presented in this section. They are as follows :

(i) In both the cases we get infinite number of solutions.

(ii) In both the cases the dependence on x^1 and x^2 appears in the same fashion.

(iii) In the solutions obtained by Chanda, De & Ray [30] and Ray [32] the dependence on x^3 and x^4 appears in terms of arbitrary functions of (x^3-x^4) only. In the solutions presented in this section the dependence on x^3 and x^4 appears in terms of Z and W which satisfy (6-65c) sand (6-65e).

For example, (6-72) is a solution of (2-8) obtained by Chanda, De and Ray [30] . On the otherhand, (6-73) is also a solution of (2-8) obtained with the help of the algorithm given here. In (6-72) the dependence on x^1 and x^2 comes through X and Y (given in 6-65) and the dependence on x^3 & x^4 comes through A,B,C,G which are all arbitrary functions of $(x^3 - x^4)$. In (6-73) the dependence of x^1 and x^2 remains the same and comes through X and Y given in (6-65), whereas the dependence of x^3 & x^4 comes through Z and W given in (6-65) where C,D,E, F are all arbitrary constants. Thus we see that (6-73) is a more general version of (6-72).

Transformations of the dependent variables

The equations (2-8) admit an invariance under transformation of the dependent variables. For example, they remain invariant under the transformation :

$$\psi = \bar{c}\,[(\bar{c})^2+1]^{-(1/2)}Z + [(\bar{c})^2+1]^{-(1/2)}\xi \quad \text{------ (6-75a)}$$

$$\chi = [(\bar{c})^2+1]^{-(1/2)}Z - \bar{c}\,[(\bar{c})^2+1]^{-(1/2)}\xi \quad \text{------ (6-75b)}$$

where $\bar{c}$ is an arbitrary constant, Z and ξ are arbitrary functions of (x^1,x^2,x^3,x^4).

So, if one can get any set of solutions for equations (2-8) , it is easy to generate new solutions with the help of the simple relations in (6-75) where Z and ξ may be treated as the old solution of ψ and χ respectively.

In this way we can generate infinite number of solutions for the equations (2-8). Furthermore it may be noted that the equations (2-8) are symmetric in ϕ,ψ and χ. Hence at the time of generating new solutions, any two out of ϕ,ψ and χ can be chosen.

The results of the <u>article 6-1-2</u> may be summarized to give as follows :

<u>For first type of solution as given in (6-56)</u>

(i) τ is arbitrary function of $(K_2 x^1 + K_3 x^2 + K_1)$ or $[(K_4 x^1+K_5)^2+(K_4x^2+ K_6)^2]$ and σ is any arbitrary function of $(K_9x^1+K_{10} x^4 + K_8)$ or $[(K_{11} x^3 +K_{12})^2 -(K_{11}x^4-K_{13})^2]$.

(ii) For any such value of τ and σ the equations (2-8) can be transformed to (6-63) via (6-58) along with (6-59) – (6-52).

(iii) For R' = constant and S' = constant only (6-59) and (6-61) are permitted. However, for various complicated forms of τ and σ the equations (2-8) can be transformed to (6-63) with R' = constant and S'=constant via (6-58) along with (6-59) and (6-61). Furthermore for R'= constant and S' = constant, we get from (6-63) a set of equations (6-64) which are conformally invariant and is very similar to the generalized Lund-Regge equations (2-9a,b) (Corenes [31]; Ray [33]). Thus from any solutions of (6-64) one can generate infinitely many other solutions by virtue of transformations of the type $(\nu',\delta')\rightarrow(f,q)$ where (ν',δ') are old independent variables, (f,q) are new independent variables, and (f,q) are functions of (ν',δ') such that $f_{\nu}' = q_{\delta}'$ and $f_{\delta}' = - q_{\nu}'$, i.e. f and q are mutually conjugate solutions of the Laplace equations in ν' and δ'.

(iv) The solutions of the equations (2-8) via (6-64) are readily available from the work of Chanda, De and Ray [30] and Ray [32] by replacing x^1 and x^2 by ν' and δ' respectively and making the arbitrary constants of integration as pure constants in the solutions reported by them.

(v) Similarity of the above the observations with those for Yang's Euclidean R-gauge equations (2-4), justifies the motivation expressed in the introduction and encourages one to pursue the possible generalization of the Yang's Euclidean R-gauge equations (2-4), the nonlinear chiral invariant field equations (5-2) and the generalized Lund-Regge equations (2-9) in future.

For second type of solutions as given in (6-65)

It is represented by those which satisfy (6-69) and (6-70) simultaneously. Such solutions can be obtained with the help of a simple algorithm which has been exemplified in the solutions (6-71) and (6-73). The equations (6-69) and (6-70) remain invariant under the transformation (6-71). Hence from any solution of (2-8) which satisfy (6-69) and (6-70) simultaneously one can generate infinitely many other solutions of (2-8) with the help of (6-71). It has been observed that in the solutions obtained by Chanda, De and Ray [30] and Ray [32] the dependence on x^3 and x^4 appears in terms of arbitrary functions of $(x^3 - x^4)$ only. In the solutions presented here as "Type-2", the dependence on x^3 and x^4 appears in terms of Y and Z which satisfy(6-65c) and (6-65e). However, in both the cases the dependence on $x^1$and x^2 appears in the same fashion.

We have also shown that the equations (2-8) admit invariance for a transformation of the dependent variables.

Appendix – G

Here we use the ansatz of Chanda, De and Ray [30] which is $\beta = \beta(\phi)$.

This lead to

$$\phi = \int (f_\pi^2 + \phi^2 + \alpha^2)\, dx, \quad \phi = \phi(x), \alpha = \alpha(x) \qquad \text{----(G-1a)}$$

$$\alpha_{\phi\phi} = (A^2/\alpha^3) + [\,(B^2\alpha)/(f_\pi^2 + \phi^2 + \alpha^2)^2\,] \qquad \text{----(G-1b)}$$

$$\psi = \alpha \operatorname{Cos}\theta \qquad \text{----(G-1c)}$$

$$\chi = \alpha \operatorname{Sin}\theta \qquad \text{----(G-1d)}$$

where

$$\theta = A \int [\,(f_\pi^2 + \phi^2 + \alpha^2)/\alpha^2\,]\; dx \;+ BY + C \qquad \text{--- (G-1e)}$$

where A,B and C are arbitrary constants, X and Y satisfy (6-65 b) and (6-65 d). From (G-1a,b) one get in principle ϕ and α in terms of X. Then θ is expressed in terms of X and Y in (G-1e). Finally one gets ψ and χ from (G-1c,d).

In one particular case of (G-1) one gets ϕ, ψ and χ in compact form which is given below :

$$\phi = f_\pi \tan(\,2 f_\pi X - G\,) \qquad \text{----(G-2a)}$$

$$\psi = f_\pi \operatorname{Sec}(\,2 f_\pi X - G\,)\;\operatorname{Cos}(\,AX + BY + C\,) \qquad \text{----(G-2b)}$$

$$\chi = f_\pi \operatorname{Sec}(\,2 f_\pi X - G\,)\;\operatorname{Sin}(\,AX + BY + C\,) \qquad \text{----(G-2c)}$$

$$A^2 + B^2 = 4 f_\pi^2 \qquad \text{----(G-2d)}$$

where A,B,C,G are arbitrary constants.

6-1-3 : Some exact solutions of the Combined equation (2-10)

Just as in case of the other two equations viz Yang equation in (2-4) and Charap equation in (2-8) we here also adopt the Ad-hoc procedure as described in chapter (4-1).

The form of the combined equation is given in equation (2-10). To find the exact solution the ansatz used here is given by :

$$\phi = \phi(u)\,,\ \psi = \psi\,(u)\,,\ \chi = \chi\,(u) \qquad \text{---- (6-76)}$$

where u is an unspecified function of x^1, x^2, x^3, x^4.

One of the motivations of analyzing the ansatz (6-76) is that Ray[32] & De and Ray [41] obtained physical solutions of Yang equations (2-4) and Charap's equation (2-8) respectively using the ansatz (6-76) in both cases. [The ansatz used by De and Ray [41] was $\psi = \psi\,(\phi)$, $\chi = \chi(\phi)$. However, $\phi = \phi(u)$ can be rewritten as $u = u(\phi)$ when (6-76) reduces to $\psi = \psi(\phi)$, $\chi=\chi(\phi)$].

The solutions have presented in two cases and each case again has two parts. In the first part of Case- I , solutions of the Charap equations (2-8) obtained by Ray [32] for the ansatz (6-76) along with the procedure have been described. And in the second part of Case- I the solutions of Extended Charap equations [equations (2-10) with $\varepsilon = -1$] with $K' = 1$, $K'' = 1$] for the ansatz (6-76) have been obtained.

Similarly, in the first part of Case- II the solutions of Yang equations (2-4) obtained by De and Ray [41] for the ansatz (6-76) have been rediscovered. And in the second part of Case –II the solutions of extended Yang equations [equation (2-10) with $\varepsilon = 1$) with $K' = 1, K'' = 1$ for the ansatz (6-76) have been obtained.

The procedure adopted by Ray [32] for obtaining the solutions of (2-8) for the ansatz (6-76) has been used for obtaining all the solutions mentioned in this chapter. This has been done with the purpose of visualizing the effect of combining the two sets of equations.

<u>Case – I</u> : Here we discuss, solutions of Charap equations obtained from Combined equations in the 1st part and, the Extended Charap equations in 2nd part as obtained from Combined equations.

<u>Part – 1 (Solutions of the Charap equations (2-8) :</u>

The solutions presented here are due to Ray [32]

After the use of (6-76) the equations (2-8) reduce to

$$(u_{11}+u_{22}+u_{33}+u_{44}) + A\,(u_1^2+u_2^2+u_3^2-u_4^2) = 0 \quad \text{----(6-77a)}$$

$$(u_{11}+u_{22}+u_{33}+u_{44}) + D\,(u_1^2+u_2^2+u_3^2-u_4^2) = 0 \quad \text{----(6-77b)}$$

$$(u_{11}+u_{22}+u_{33}+u_{44}) + E\,(u_1^2+u_2^2+u_3^2-u_4^2) = 0 \quad \text{----(6-77c)}$$

$$A(u) = (\phi_{uu}/\phi_u) - 2(\phi\phi_u+\psi\psi_u+\chi\chi_u)\,/\,(f_\pi^2+\phi^2+\psi^2+\chi^2) \quad \text{---(6-77d)}$$

$$D(u) = (\psi_{uu}/\psi_u) - 2(\phi\phi_u+\psi\psi_u+\chi\chi_u)\,/(f_\pi^2+\phi^2+\psi^2+\chi^2) \quad \text{--(6-77e)}$$

$$E(u) = (\chi_{uu}/\chi_u) - 2(\phi\phi_u+\psi\psi_u+\chi\chi_u)\,/\,(f_\pi^2+\phi^2+\psi^2+\chi^2) \quad \text{--(6-77f)}$$

So that either

$$A = D = E \qquad \text{----(6-78)}$$

Or $$u_1^2 + u_2^2 + u_3^2 - u_4^2 = 0 \quad \text{and} \qquad \text{----(6-79a)}$$

$$u_{11} + u_{22} + u_{33} + u_{44} = 0 \qquad \text{----(6-79b)}$$

Equations (6-79) have simple solutions and are given in the work of Ray [32] and Ghosh, Ray & Chanda [60].

The general solutions of (6-78) are given by :

$$\psi = K_1 \phi + K_2 \qquad \text{--- (6-80a)}$$

$$\chi = K_3 \phi + K_4 \qquad \text{----(6-80b)}$$

where K_i are arbitrary constants of integration.

Let us now define consistent with (6-76) and without any loss of generality

$$\phi = K_5 v + K_6 \qquad \text{---- (6-81)}$$

where 'v' is some unspecified function of u. K_5 and K_6 are arbitrary constants.

Since u has been defined in (6-76) to be an unspecified function of x^1, x^2, x^3, x^4 one can conclude till now that v is an unspecified function of x^1, x^2, x^3, x^4.

Putting (6-81) in (6-80 a,b) one gets

$$\phi = K_5 v + K_6 \qquad \text{---- (6-82a)}$$

$$\psi = K_7 \phi + K_8 \qquad \text{--- (6-82b)}$$

$$\chi = K_9 \phi + K_{10} \qquad \text{----(6-82c)}$$

where $K_1 K_5 = K_7$, $K_3 K_5 = K_9$, $K_1 K_6 + K_2 = K_8$, $K_3 K_6 + K_4 = K_{10}$.

The use of (6-82) reduces the equations (2-8) to a single equation given by

$$(v_{11} + v_{22} + v_{33} - v_{44}) + A' (v_1^2 + v_2^2 + v_3^2 - v_4^2) = 0 \quad \text{----(6-83a)}$$

$$A' = (d/dv)\ln[1/\{(K_5^2+K_7^2+K_9^2)v^2+2(K_5K_6+K_7K_8+K_9K_{10})v$$
$$+(f_\pi^2+K_6^2+K_8^2+K_{10}^2)\}] \qquad \text{---(6-83b)}$$

The equation (6-83a) can be rewritten as

$$\xi_{11} + \xi_{22} + \xi_{33} - \xi_{44} = 0 \qquad \text{---- (6-84a)}$$

where $\xi = \int [\, \exp \{ A' (v)\, dv \} \,]\, dv + K_0$ ---- (6-84b)

where K_0 is an arbitrary constant of integration. Using (6-83b) in (6-84b) one can write

$$\xi = [\int dv/\{(K_5^2+K_7^2+K_9^2)v^2+2(K_5K_6+K_7K_8+K_9K_{10})v$$
$$+(f_\pi^2+K_6^2+K_8^2+K_{10}^2)\}] + K_{10} \qquad \text{--- (6-85)}$$

In order to obtain a compact form of the solutions without the loss of generality we choose $K_6 = 0$, $K_8 = 0$, $K_{10} = 0$ when (6-85) reduces to

$$v = \{f_\pi / \sqrt{(K_5^2+K_7^2+K_9^2)}\}\tan[\{f_\pi /\sqrt{(K_5^2+K_7^2+K_9^2)}\}] (\xi - K_0) \quad \text{--(6-86)}$$

Finally ϕ,ψ and χ are given by [from (6-82a,b,c) with $K_6=0, K_8=0, K_{10}=0$ and $K_0 = 0$]

$$\phi = \{(K_3 f_\pi) / \sqrt{(K_5^2 + K_7^2 + K_9^2)}\} \tan [\{f_\pi / \sqrt{(K_5^2 + K_7^2 + K_9^2)}] \xi \quad \text{---(6-87a)}$$

$$\psi = \{(K_4 f_\pi) / \sqrt{(K_5^2 + K_7^2 + K_9^2)}\} \tan [\{f_\pi / \sqrt{(K_5^2 + K_7^2 + K_9^2)}] \xi \quad \text{--(6-87b)}$$

$$\chi = \{(K_5 f_\pi) / \sqrt{(K_5^2 + K_7^2 + K_9^2)}\} \tan [\{f_\pi / \sqrt{(K_5^2 + K_7^2 + K_9^2)}] \xi \quad \text{---(6-87c)}$$

where ξ is given by (6-84a).

A particular solution of (6-84a) is given by [32]

$$\xi = \{ (\text{Sin}\,\gamma / \gamma) \} \text{ Cos } t, \; t = x^4 \quad \text{---(6-88a)}$$

$$\gamma^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \quad \text{---(6-88b)}$$

$$\max (\text{Sin}\,\gamma / \gamma) < [\pi / \{ 2 f_\pi \sqrt{(K_5^2 + K_7^2 + K_9^2} \}] \quad \text{---(6-88c)}$$

Part –II [***Solutions of the extended Charap equations***

(Equation (2-10) with $\varepsilon = -1$) for $K'=1, K''=1$]

After the use of (6-76) the equation (2-10) (with $\varepsilon = -1$, $K' = 1$, $K'' = 1$) reduce to

$$(u_{11} + u_{22} + u_{33} - u_{44}) + A (u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0 \quad \text{----(6-89a)}$$

$$(u_{11} + u_{22} + u_{33} - u_{44}) + D (u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0 \quad \text{---(6-89b)}$$

$$(u_{11} + u_{22} + u_{33} - u_{44}) + E (u_1^2 + u_2^2 + u_3^2 - u_4^2) = 0 \quad \text{--(6-89c)}$$

where

$$A(u) = (\phi_{uu} / \phi_u) - [\{(\phi_u^2 - \psi_u^2 - \chi_u^2) / (\phi\phi_u)\} + \{ 2 (\phi\phi_u + \psi\psi_u + \chi\chi_u) / (f_\pi^2 + \phi^2 + \psi^2 + \chi^2) \}] \quad \text{--(6-89d)}$$

$$D(u) = (\psi_{uu}/\psi_u) - [(2\phi_u/\phi)$$

$$+ \{2(\phi\phi_u + \psi\psi_u + \chi\chi_u)/(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)\}] \quad \text{----(6-89e)}$$

$$E(u) = (\chi_{uu}/\chi_u) - [(2\phi_u/\phi)$$

$$+ \{2(\phi\phi_u + \psi\psi_u + \chi\chi_u)/(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)\}] \quad \text{----(6-89f)}$$

So that either

$$A = D = E \quad \text{----(6-90)}$$

Or $$u_1^2 + u_2^2 + u_3^2 - u_4^2 = 0 \quad \text{and} \quad \text{---(6-91a)}$$

$$u_{11} + u_{22} + u_{33} + u_{44} = 0 \quad \text{---(6-91b)}$$

The equations (6-91) are same as those numbered (6-79). In the following the situation given by (6-90) has been discussed. Considering D = E in (6-90) one arrives at

$$\chi = K_{11}\psi + K_{12} \quad \text{---- (6-92)}$$

where K_{11} and K_{12} are arbitrary constants.

Let us define

$$\psi = K_{13} v + K_{14} \quad \text{----- (6-93)}$$

where

(i) v is some unspecified function of u.

(ii) K_{13}, K_{14} are arbitrary constants.

Since u has been defined in (6-76) and it is an unspecified function of x^1, x^2, x^3, x^4 one can conclude till now that v is an unspecified function of x^1, x^2, x^3, x^4.

Putting (6-93) in (6-92) one gets

$$\chi = K_{15}\, v + K_{16} \qquad \text{---- (6-94)}$$

where $K_{11}\, K_{13} = K_{15}$, $K_{11}\, K_{14} + K_{12} = K_{16}$.

Now, from (6-76) we have $\phi = \phi(u)$, where ϕ is an unspecified function of u. Here we set $v = v(u)$, where v is an unspecified function of u. Hence one can write without any loss of generality

$$\phi = \phi(v) \qquad \text{---- (6-95)}$$

The use of (6-93), (6-94) and (6-95) reduce the equations (2-10) [*with* $\varepsilon = -1$, $K' = 1$, $K'' = 1$] to two equations given by

$$(v_{11} + v_{22} + v_{33} - v_{44}) + A'\,(v_1^{\,2} + v_2^{\,2} + v_3^{\,2} - v_4^{\,2}) = 0 \qquad \text{--(6-96a)}$$

$$(v_{11} + v_{22} + v_{33} - v_{44}) + D'\,(v_1^{\,2} + v_2^{\,2} + v_3^{\,2} - v_4^{\,2}) = 0 \qquad \text{-(6-96b)}$$

where

$$A' = (2\phi_{vv} / \phi_v) - (\phi_v^{\,2} - K_{13}^{\,2} - K_{15}^{\,2}) / (\phi\phi_v) - F' \qquad \text{--- (6-97a)}$$

$$D' = -2\,(\phi_v / \phi) - F' \qquad \text{--- (6-97b)}$$

$$F' = \frac{2\{\phi\phi_v + (K_{13}^{\,2} + K_{15}^{\,2})\, v + K_{13} K_{14} + K_{15} K_{16}\}}{f_\pi^{\,2} + \phi^2 + (K_{13}^{\,2} + K_{15}^{\,2})\, v^2 + 2(K_{13} K_{14} + K_{15} K_{16})\, v + (K_{14}^{\,2} + K_{16}^{\,2})} \qquad \text{--(6-97c)}$$

Just as in the above the possibility other than

$$(v_{11} + v_{22} + v_{33} - v_{44}) = 0 \qquad \text{---(6-98a)}$$

$$(v_1^{\,2} + v_2^{\,2} + v_3^{\,2} - v_4^{\,2}) = 0 \qquad \text{---(6-98b)}$$

requires that

$$A' = D' \qquad \text{--- (6-99)}$$

From (6-99) one gets

$$\phi\phi_{vv} + \phi_v^2 + (K_{13}^2 + K_{15}^2) = 0 \qquad \text{---(6-100)}$$

which on integration leads to

$$\phi = \sqrt{[K_{18} + 2 K_{17} v - (K_{13}^2 + K_{15}^2) v^2]} \qquad \text{--- (6-101)}$$

where K_{17} and K_{18} are arbitrary constants of integration.

Thus when ϕ,ψ and χ are given by (6-101) ,(6-93) and (6-94) respectively the equations (2-10) [*with* $\varepsilon = -1, K_1 = 1, K_2 = 1$] reduce to a single equation given by :

$$(v_{11}+v_{22}+ v_{33} - v_{44}) + A'' (v_1^2 +v_2^2+v_3^2 - v_4^2) = 0 \qquad \text{---(6-102a)}$$

$$A'' = \frac{d}{dv} \ln \frac{1}{\phi^2 \{f_\pi^2+\phi^2+ (K_{13}^2+ K_{15}^2) v^2 + 2(K_{13} K_{14}+K_{15}K_{16}) v + (K_{14}^2 + K_{16}^2)\}}$$

$$\text{---(6-102b)}$$

Equations (6-102) can again be rewritten as

$$\xi_{11} + \xi_{22} + \xi_{33} - \xi_{44} = 0 \qquad \text{---- (6-103a)}$$

where $\xi = \int [\exp \{ A'' (v) dv \}] dv + K_{20}$ --- (6-103b)

and here K_{20} is an arbitrary constant of integration and $A''(v)$ is given by (6-102b).

Using (6-102b) and (6-101) in (6-103b) one gets

$$\xi = [\int dv / (G.H)] + K_{20} \qquad \text{---- (6-104)}$$

with $G = K_{18} + 2 K_{17} v - (K_{13}^2 + K_{15}^2) v^2$

$H = f_\pi^2+K_{18}+2K_{17} v +2(K_{13}K_{14}+K_{15}K_{16}) v +(K_{14}^2 + K_{16}^2)$.

Here K_{20} is an arbitrary constant of integration.

Then having a compact form of the solutions without the loss of much generality we choose $K_{17} = 0$,$K_{14} = 0$, $K_{16} = 0$ when (6-104) reduces to

$$\xi = [\, 1/\{(f_\pi^2 + K_{18})\,(K_{14}^2 + K_{16}^2)\}] \,.\, \int [\, dv/\,(K_{19}^2 - v^2)] + K_{20} \qquad \text{--(6-105a)}$$

where $K_{19}^2 = K_{18} \,/\, (K_{13}^2 + K_{15}^2)$ --- (6-105b)

For $K_{17} = 0$ the equation (6-101) along with (6-105b) reduces to

$$\phi = \sqrt{(K_{18}/K_{19})\,(K_{19}^2 - v^2)} \qquad \text{---(6-106)}$$

Since ϕ is real one must have $K_{19} > v$.

Then integrating the right hand side of (6-105) one can write v in terms of ξ as follows :

$$v = K_{19} \tanh [\, K_{19}\,(f_\pi^2 + K_{18})\,(K_{13}^2 + K_{15}^2)\,(\xi - K_{20})] \qquad \text{--- (6-107)}$$

Finally ϕ,ψ and χ are given by *[from(6-93),(6-94), (6-106)and (6-107) with $K_{14} = 0$, $K_{16} = 0$, $K_{20} = 0$]*

$$\phi = \sqrt{(K_{18})}\ \text{Sech}\, [\, K_{19}(\, f_\pi^2 + K_{18})\,(K_{13}^2 + K_{15}^2)\,\xi\,] \qquad \text{---(6-108a)}$$

$$\psi = (K_{13}K_{19}) \tanh [\, K_{19}(\, f_\pi^2 + K_{18})\,(K_{13}^2 + K_{15}^2)\,\xi\,] \qquad \text{---(6-108b)}$$

$$\chi = (K_{15}\, K_{19})\ \tanh [\, K_{19}(\, f_\pi^2 + K_{18})\,(K_{13}^2 + K_{15}^2)\,\xi\,] \qquad \text{---(6-108c)}$$

where ξ satisfies (6-103a) and a particular example of such ξ is given by (6-88).

Comparing (6-87) and (6-108) one can easily observe that the solutions of the Charap equations (2-8) and the extended Charap equations [*equations (2-10) with $\varepsilon = -1$, $K' = 1$, $K'' = 1$*] differ considerably. One can also notice that both solutions are physical in nature.

Case – II

Here we discuss the solutions of Yang's equations as obtained from Combined equations and the Extended Yang equations as obtained from Combined equations. Here the solutions have been obtained using the techniques identical to those for Case- I . Arbitrary constants of this case which are similar to those of the Case – I have been chosen in the same fashion to be equal to zero.

Part – 1 (Solutions of the Yang equations (2-4) :

The solutions presented here are due to De and Ray [41] and are given by :

$$\phi = \sqrt{K_{27}} \quad \text{Sech}\,[\, K_{29}\,(K_{22}^2 + K_{24}^2)\,\xi\,] \qquad \text{---- (6-109a)}$$

$$\psi = K_{22}\,K_{29}\ \tanh\,[\, K_{29}\,(K_{22}^2 + K_{24}^2)\,\xi\,] \qquad \text{---- (6-109b)}$$

$$\chi = K_{24}\,K_{29}\ \tanh\,[\, K_{29}\,(K_{22}^2 + K_{24}^2)\,\xi\,] \qquad \text{---- (6-109c)}$$

where ξ satisfies

$$\xi_{11} + \xi_{22} + \xi_{33} - \xi_{44} = 0 \qquad \text{---- (6-110)}$$

and a particular example of such ξ is

$$\xi = \{\, (\text{Sin}\,\gamma \,/\, \gamma\,)\,\}\ \text{Cos}\ t,\ t = x^4 \qquad \text{---(6-111a)}$$

$$\gamma^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \qquad \text{---(6-111b)}$$

$$K_{29}^2 = K_{27} \,/\, (K_{22}^2 + K_{24}^2) \qquad \text{---(6-111c)}$$

Part – 2 (Solutions of the extended Yang equations)

[equations (2-10) with $\varepsilon = 1$ for $K' = 1, K'' = 1$]

Here the solutions are given by :

$$\phi = \sqrt{(K_{35})}\ \mathrm{Sech}\ [\ K_{36}(\ f_\pi^2 + K_{35}\ )\ (K_{30}^2 + K_{32}^2\ )\ \xi\ ] \quad \text{---- (6-112a)}$$

$$\psi = (K_{30}K_{36})\ \tanh\ [\ K_{36}(\ f_\pi^2 + K_{35}\ )\ (K_{30}^2 + K_{32}^2\ )\ \xi\ ] \quad \text{---- (6-112b)}$$

$$\chi = (K_{32}\ K_{36})\ \tanh\ [\ K_{36}(\ f_\pi^2 + K_{35}\ )\ (K_{30}^2 + K_{32}^2\ )\ \xi\ ] \quad \text{--- (6-112c)}$$

where ξ satisfies (6-110) and a particular example of such ξ is given by (6-111).

Comparing (6-109) and (6-112) one observes that solutions of Yang equation (2-4) and extended Yang equations [equations (2-10) with ε=1, K'=1, K''=1) for the ansatz (6-76) are same in form. Both the solutions are physical in nature.

<u>The results of the article 6-1-3 may be summarized as:</u>

For the exact solutions of the combined equation (2-10) we obtain two basic solutions:

i) Exact solutions in compact form for the Yang equations (2-4) and the extended Yang equation [Equations (2-10) with $\varepsilon=1, K'=1, K''=1$] are given by (6-109) and (6-112) respectively.

They are same in form.

ii) Exact solutions in compact form for the Charap equations (2-8) and the extended Charap equations [equations (2-10) with $\varepsilon=-1, K'=1, K''=1$] are given by (6-87) and (6-108) respectively.

They differ considerably.

6-2 Graphical representation of some exact solutions for the three equations under study

In this article we represent some of the exact solutions of the Yang equation (2-4) , Charap equation (2-8) and their combined equation (2-10) graphically.

In chapter 6-1-3 , we integrated the equations (2-10 a,b,c) with a particular ansatz [$\phi=\phi(u)$, $\Psi=\Psi(u)$, $\chi=\chi(u)$ *, where u is an unspecified function of* x^1,x^2,x^3,x^4*] for* $k'=1$ *and* $k''=1$ *and expressed the solutions in terms of known functions for some particular values of the arbitrary constants of integration. Here those solutions have been represented graphically.*

In order to do that we have plotted the dependence of ϕ,Ψ and χ *on* x^1 *co-ordinate (keeping* $x^2=0, x^3=0$ *) at different values of* x^4 *(i.e. time). This does not lead to much loss of generality as the solutions have exact symmetrical of dependence on* x^1,x^2,x^3*. In article 4-2 we mentioned the method, used here.*

In the following we separately represent graphically some of the exact solutions of the three equations (2-4),(2-8) and (2-10) referred to above.

6-2-1 Graphical representation of some exact solutions for Yang equations (2-4) :

In article 6-1-3 we discuss for the exact solutions of Yang equations (2-4) from the combined equations (2-10). These are given in equations (6-109a,b,c) and may be rewritten as

$$\phi = \sqrt{K_{27}} \;\; \mathrm{Sech}\,[\, K_{29}\,(K_{22}^2 + K_{24}^2)\,\xi\,]$$

$$\Psi = K_{22}\,K_{29} \;\; \tanh\,[\, K_{29}\,(K_{22}^2 + K_{24}^2)\,\xi\,]$$

$$\chi = K_{24}\,K_{29} \;\; \tanh\,[\, K_{29}\,(K_{22}^2 + K_{24}^2)\,\xi\,]$$

where $\xi = [\,(\mathrm{Sin}\,\tau)\,/\,\tau\,]\;\; \mathrm{Cosh}\,t\,,\, t = x^4$,

$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, K_i are constants.

From the equations (6-109 a,b,c) it is clear that ϕ represents a spreading wave with solitary profile which tends to zero as $t \to \infty$, and, Ψ & χ are spreading wave packets.

For all graphical representations discuss here, we made

(a) $x^2 = 0$ & $x^3 = 0$ and

(b) values of x^1 taken as X-axis are in multiple of $\pi/2$ &

values of x^4 (i.e. time) taken as Y-axis are in multiple of $\pi/4$

The graphical representations for (6-109 a) are given in Fig. 1(a) to Fig. 1(h) and that for (6-109 b & c) are given in Fig. 2(a) to 2(h). Fig. 1(a) is the graph corresponding to time,$x^4 = 0$, fig 1(b) is corresponding to $x^4 = 1$ and so on. Fig 1(h) is the expanded graph corresponding to $x^4 = 7$.

Similarly, fig. 2(a) is the graph corresponding to time, $x^4 = 0$, fig 2(b) is corresponding to $x^4 = 1$ and so on. Fig 2(i) is the expanded graph corresponding to $x^4 = 7$.

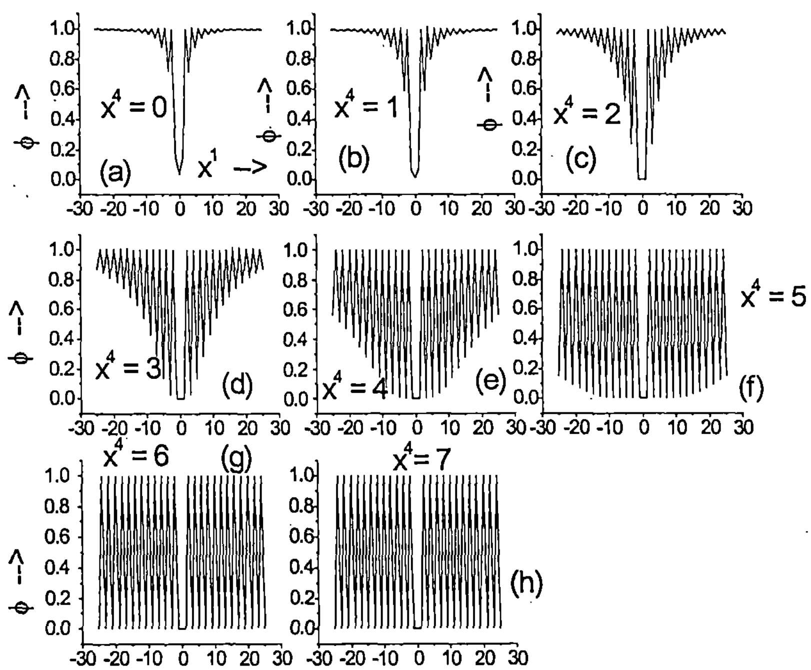

Figure 1 (a-h). Solutions (6-109a) for ϕ in Yang equations (2-4).

Solutions (6-112 a) for extended Yang equations (2-10)

(with $\varepsilon = 1, k' = 1, k'' = 1$) have the same form.

Following is the graph of figure - 1(h) with larger range of x^1 in the X-axis.

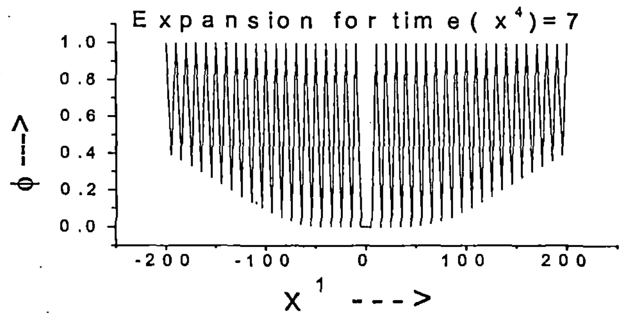

Figure 1 (i). Solutions (6-109a) for ϕ in Yang equations (2-4).

[Expanded representation of Figure 1 (h)]

It is interesting to see that the Fig-1(h) does not represent a plane wave. It has still a solitary profile. Thus, one can say that here ϕ represents a spreading wave with solitary profile. The profile tends to disappear as time tends to infinity.

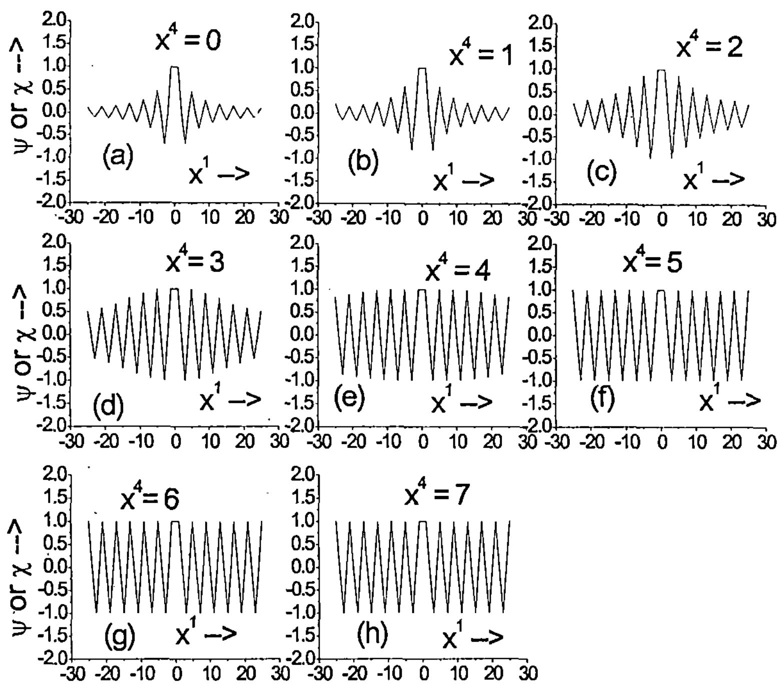

Figure 2(a-h). Solutions (6-109b,c) for ψ & χ in Yang equations (2-4).

Solutions (6-112 b,c) for extended Yang equations (2-10).

(with $\varepsilon = 1, k' = 1, k'' = 1$) have the same form.

Following is the graph of figure - 2(h) with a larger range of x^1, where we take more x^1 in the X-axis. Here again we see that ψ and χ actually represent spreading wave packets.

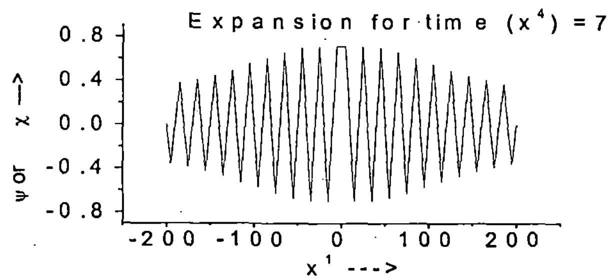

Figure 2 (i). Solutions (6-109 b,c) for ψ & χ in Yang equations (2-4).

In general, the observations are :

For the figure- 1(a) to 1(h) with different values of x^4

The solutions are basically localized. Initially only the central peak is prominent. Gradually other peaks grow. However, the profile is always of solitary waves. After some large value of time, $t = x^4$, the solution seems to be of plane wave type. But if one observes the solution profile over large x^1 then the same solitary profile is observable.

For the fig.2 (a) to 2 (h) i.e. ψ or χ vs x^1 with different values of x^7.

The solutions are basically localized. In both the cases the profile is of 'photon type'. Gradually the profile starts extending over space. However, at any stage the profile remains to be 'photon type'.

6-2-2 Graphical representation of some exact solutions for Charap equations (2-8) :

In article 6-1-3 we discuss for the exact solutions of Charap equations (2-8) from the combined equations (2-10). These are given in equations (6-87a,b,c) and may be rewritten as

$$\phi = \{(K_3 f_\pi) / \sqrt{(K_5^2 + K_7^2 + K_9^2)}\} \tan [\{ f_\pi / \sqrt{(K_5^2 + K_7^2 + K_9^2)}] \ \xi$$

$$\psi = \{ (K_4 f_\pi) / \sqrt{(K_5^2 + K_7^2 + K_9^2)}\} \ \tan [\{ f_\pi / \sqrt{(K_5^2 + K_7^2 + K_9^2)}] \ \xi$$

$$\chi = \{ (K_5 f_\pi) / \sqrt{(K_5^2 + K_7^2 + K_9^2)} \} \ \tan [\{ f_\pi / \sqrt{(K_5^2 + K_7^2 + K_9^2)}] \ \xi$$

where $\xi = [(\mathrm{Sin}\ \tau) / \tau] \ \mathrm{Cos}\ t , t = x^4$,

$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, K_i are constants, f_π = constant.

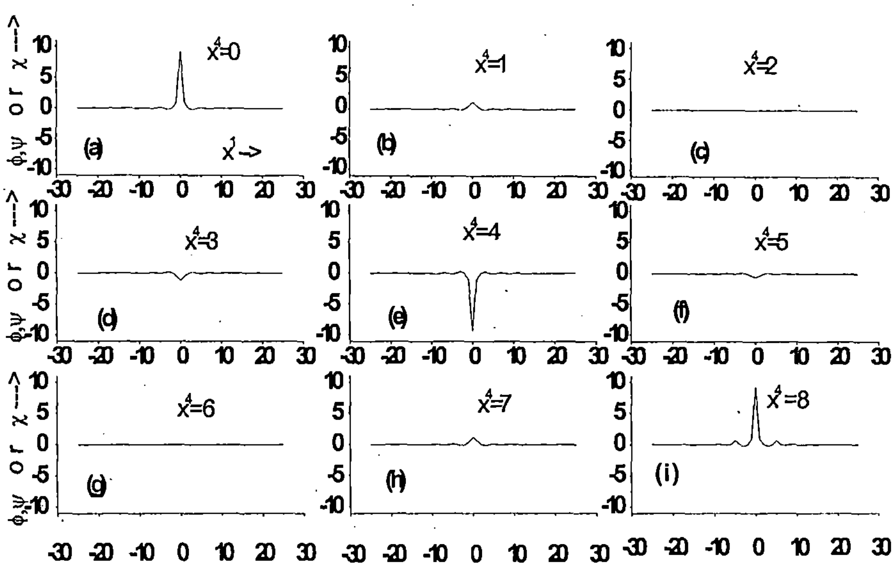

Figure 3(a–i). Solutions(6–87a,b,c) for ϕ,ψ & χ in Charap equations (2–8).

Here all of ϕ,Ψ and χ are solitary waves with oscillatory profile. These can be observed clearly from the graphical representations in Fig (3a) to (3 i).

The solutions are basically localized. Here all of ϕ,ψ and χ have same type of solutions. The profiles are always 'solitary waves'. Only the central peak is prominent and the total profile oscillates.

6-2-3 Graphical representation of some exact solutions for Combined equations (2-10) :

6-2-3 (A) : Graphical study of the equation for some exact solutions for "Extended Yang Equations"

[i.e. combined equation (2-10), with $\varepsilon = 1, k' = 1, k'' = 1$]

In article 6-1-3 we discuss for the exact solutions of the combined equations (2-10). These are given in equations (6-112) which can be rewritten as

$$\phi = \sqrt{(K_{35})} \quad \text{Sech}\,[\,K_{36}(\,f_\pi^2 + K_{35}\,)\,(K_{30}^2 + K_{32}^2\,)\,\xi\,]$$

$$\psi = (K_{30}K_{36}) \quad \tanh\,[\,K_{36}(\,f_\pi^2 + K_{35}\,)\,(K_{30}^2 + K_{32}^2\,)\,\xi\,]$$

$$\chi = (K_{32}\,K_{36}) \quad \tanh\,[\,K_{36}(\,f_\pi^2 + K_{35}\,)\,(K_{30}^2 + K_{32}^2\,)\,\xi\,]$$

where $\xi = [\,(\text{Sin}\,\tau)\,/\,\tau\,]\ \text{Cosh}\,t\,,\ t = x^4\,,$

$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, K_i are constants, f_π = constant.

The equation (6-112) have basically the same structure as (6-109) [*the solutions of Yang equation*] . The graphical representations of equations (6-112 a) are of the same type as those given in Fig 1(a) to Fig –1(h).

Also, the graphical representations of equations (6-112 b,c) are of the same type as those given in Fig 2(a) - 2 (h).

Therefore we have not plotted (6-112 a,b,c) separately.

6-2-3 (B) : ***Graphical study of the equation for some exact solutions for "Extended Charap Equations"***

[i.e. combined equation (2-10), with $\varepsilon = -1, k' = 1, k'' = 1$]

Solutions for **"Extended Charap Equations"** [i.e. combined equation (2-10), with $\varepsilon = -1, k' = 1, k'' = 1$] given in equation (6-108) which can be rewritten as .

$$\phi = \sqrt{(K_{18})} \quad \text{Sech} [K_{19}(f_\pi^2 + K_{18}) (K_{13}^2 + K_{15}^2) \xi]$$

$$\psi = (K_{13}K_{19}) \quad \tanh [K_{19}(f_\pi^2 + K_{18}) (K_{13}^2 + K_{15}^2) \xi]$$

$$\chi = (K_{15}\, K_{19}) \quad \tanh [K_{19}(f_\pi^2 + K_{18}) (K_{13}^2 + K_{15}^2) \xi]$$

where $\xi = [(\text{Sin}\, \tau) / \tau]$ $\text{Cos}\, t$, $t = x^4$,

$\tau^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, K_i are constants.

The graphical representation for (6-108 a) are given in Fig 4(a) to 4(i) and that for (6-108 b,c) are given in Fig 5(a) to Fig 5(i).

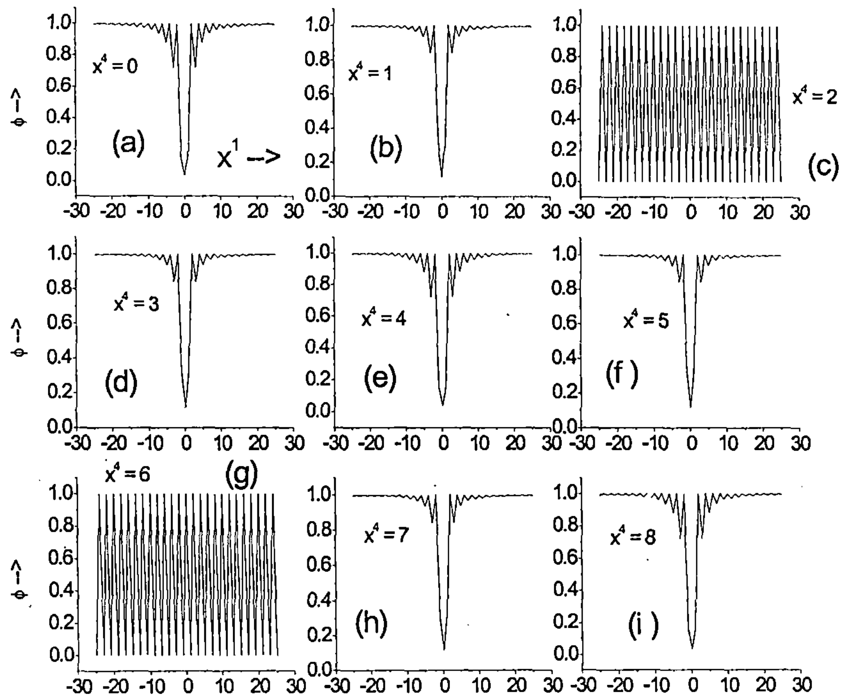

Figure 4 (a-i). Solutions (6-108a) of ϕ in Extended Charap equations (2-10) (with $\varepsilon = -1$, $k' = 1$, $k'' = 1$)

From the above Fig-4 , it is seen that the solutions for ϕ are no longer oscillatory. It is now a localized wave with solitary profile which becomes plain wave periodically and abruptly.

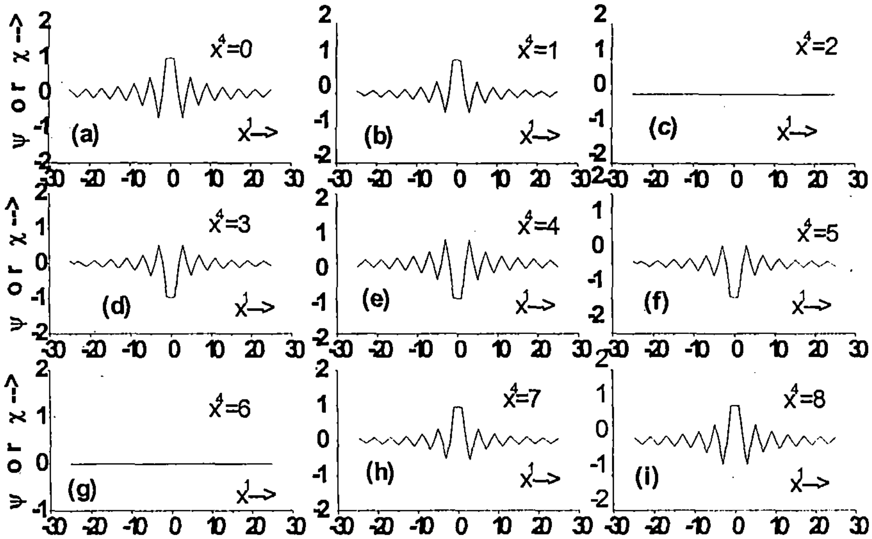

Figure 5 (a–i). Solutions (6–108 b,c) of ψ & χ in Extended Charap equations (2–10) (with $\varepsilon = -1$, $k' = 1$, $k'' = 1$)

Again, from the above Fig-5 it is seen that the solutions have photon-like structure. That is , ψ and χ are wave packets which are oscillatory in nature. Moreover, they become zero periodically and abruptly.

Thus from Fig-4 and Fig- 5, we can say that all of ϕ,ψ and χ are basically localized.

***We can summarise the physical characteristics observable from the article (6-2) as*:**

They are as follows:

(i) Spreading wave with solitary profile which tends to vanish as time tends to infinity [in case of Yang equations(2-4) and combined (extended Yang) equations (2-10)].
(From the Fig- 1 & Fig. – 2).

(ii) Solitary wave with oscillatory profile[Charap equations (2-8)].
(From the Fig-3).

(iii) Localized wave with solitary profile which becomes plane wave periodically & abruptly and wave packet/s which are oscillatory in nature and become zero periodically & abruptly [Combined (extended Charap) equation (2-10)].
(From the Fig- 4 & Fig. – 5).

(iv) All the solutions are basically localised in character.

6-3 Painleve' test of the three equations under study and discussion of Chaos

In this section we discuss on Painleve' test for the three equations (2-4),(2-8) and (2-10) for the Yang, Charap and Combined equations respectively. In section 4-3, we discuss the method of Painleve' analysis. Along with the analysis we here also discuss truncation of the Series, auto-Backlund transformation and exact solutions of the equations.

Here we also discuss the correlation between the existence of Painleve' property and the existence or absence of Chaotic behaviour from the Painleve' test and Graphical representations of exact solutions for the Yang equations (2-4), Charap equations (2-8) and their combined form (2-10).

6-3-1 On a revisit to the Painleve' test for integrability and exact solutions for Yang equations

This article revisits some observations of Jimbo, Kruskal and Miwa [49] regarding Yang's self-dual equations (2-4) for SU(2) gauge fields . Where it was shown that the equations pass the Painleve' test for integrability in the sense of Weiss et al [21a]. Ward [46] used a completely different approach, complicated indeed, and arrived at the same conclusion. Both the investigations [21a, 49] used the complex form of the equations (2-4) and neither of them reported any solution obtainable from the analysis.

In review (Article 5-2-1) we discussed the results of the Painleve analysis for the Yang equations as done by Jimbo, Kruskal and Miwa [49]. Here the analysis has been done starting from the real form of the same equations and keeping the singularity manifold completely general in nature.

Here it has been found that the real form of Yang's self-dual equations passes the Painleve' test for integrability in the sense of Weiss et al [21a] and admit truncation of series leading to non-trivial exact solutions obtained previously and Auto-Backlund transformation between two pairs of these solutions (see for example the work of Larsen [55] and

Roy Chowdhury [56]). The important aspect of our analysis is that we have analyzed the equation keeping the singularity manifold completely general, whereas Jimbo, Kruskal and Miwa [49] analyzed the same equation with a restricted nature of singularity manifold.

The discussion here are subdivided into three main parts:

Part – I : Painleve' test for integrability of the equations (2-4) in the sense of Weiss et al.

Part –II :Truncation, Backlùnd Transformation and exact solutions.

Part – III : Revisit to solutions obtained previously as in chapter 6-1-1 for the exact solutions of the Yang equations.

In the following we discuss these three parts in details.

Part – I

Painleve' test for integrability of the equations (2-4) in the sense of Weiss et al [21a]

For the equations (2-4) we define the singularity manifold given by:

$$u = u\,(x^1, x^2, x^3, x^4) \;\; = 0 \qquad \text{--- (6-113)}$$

and set

$$\phi = u^{\alpha} \sum_{j=0}^{\infty} \phi_j\, u^j\;,\quad \psi = u^{\beta} \sum_{j=0}^{\infty} \psi_j\, u^j\,,\;\; \chi = u^{\gamma} \sum_{j=0}^{\infty} \chi_j\, u^j \qquad \text{--- (6-114 a,b,c)}$$

where $\phi(x^1, x^2, x^3, x^4)$,ψ (x^1, x^2, x^3, x^4) , χ (x^1, x^2, x^3, x^4) are a set of solutions of (2-4); ϕ_j ,ψ_j , χ_j are all analytic functions of (x^1, x^2, x^3, x^4) in the neighborhood of the manifold (6-113); $\phi_0 \neq 0, \psi_0 \neq 0$, $\chi_0 \neq 0$.

The test may be divided into three main steps after the substitution of (6-114) in the differential equations concerned, i.e. (2-4) :

I. Make the leading order analysis [where one gets all possible α,β,γ , ϕ_0 , ψ_0 and χ_0 in (6-114)]

II. Define the recursion relations for u_j for leading orders obtained in step I and determine the resonance positions (those values of j for which some or all of the relations are not defined).

III. Check whether the expansions allow requisite number of arbitrary functions at the resonance positions.

Step I. ***<u>Leading order analysis :</u>***

We assume,

$$\phi \sim \phi_0 u^{\alpha}, \quad \psi \sim \psi_0 u^{\beta}, \chi \sim \chi_0 u^{\gamma} \qquad \text{----- (6-115 a,b,c)}$$

We substitute (6-115 a,b,c) in (2-4) and equate the coefficients of the negative powers of u (considering that all of α,β and γ are negative).

This leads to $\alpha = -1, \beta = -1, \gamma = -1$ --- (6-116 a,b,c)

And $\phi_0^2 + \psi_0^2 + \chi_0^2 = 0$ --- (6-116 d)

so that

$$\phi = u^{-1} \sum_{j=0}^{\infty} \phi_j u^j, \quad \psi = u^{-1} \sum_{j=0}^{\infty} \psi_j u^j, \quad \chi = u^{-1} \sum_{j=0}^{\infty} \chi_j u^j \qquad \text{--- (6-117 a,b,c)}$$

where $\phi_0^2 + \psi_0^2 + \chi_0^2 = 0$ --- (6-117 d)

One can notice the similarity between the equation (6-117d) obtained here and the equation (3) of the publication of Jimbo, Kruskal and Miwa [49]. Again, all the terms in their case equivalent to α,β and γ were equal to (– 1) as has been obtained here.

II. ***<u>Resonance positions</u>*** :

We directly substitute (6-117) in (2-4). We have not written explicitly the recursion relations because of their involved structure. In order to have an idea of the style of writing one can consult the recursion relations of the work of Chanda and Roychowdhury [61].

The recursion relation for ϕ_j , ψ_j , χ_j for the equations are

$$[T]\begin{bmatrix}\phi_m\\ \psi_m\\ \chi_m\end{bmatrix} = [\text{ other terms with } \phi_j, \psi_j, \chi_j \text{ and their derivatives, where } j < m\ ]$$

It is written as,

$$[T] = \begin{bmatrix}(m^2 - m + 2)\,\phi_0 & -2(m-1)\psi_0 & -2(m-1)\chi_0\\ 2m\psi_0 & M(m-1)\phi_0 & 0\\ 2m\chi_0 & 0 & m(m-1)\phi_0\end{bmatrix}$$

Resonance are those values of m = R, for which det [T] = 0,which yields,

$$\phi_0^3\ [R^2(R-1)^2(R+1)\,(R-2)\,] = 0 \qquad \text{--- (6-118)}$$

i.e. Poles are, R = (−1) , (0, 0) , (1, 1) , (2) which are the exactly same as given by Jimbo,Kruskal and Miwa as discussed in review article.

(i) R = −1 indicates that the singularity manifold defined in (6-133) is required to be arbitrary.

(ii) R=0,0 indicate that all of the coefficients ϕ_0 ; ψ_0 and χ_0 are required to be arbitrary.

(iii) R= 1,1 indicate that any two of the coefficients ϕ_1, ψ_1 and χ_1 are required to be arbitrary.

(iv) R = 2 indicates that any one of ϕ_2 , ψ_2 and χ_2 are required to be arbitrary.

III. ***To check whether the expansions allow requisite number of arbitrary functions at the resonance positions*** :

(i) The singularity manifold u = 0 , by defination, is arbitrary.

(ii) There is only one equation involving ϕ_0 , ψ_0 , χ_0 which is given by (6-117d). Hence any two of ϕ_0 , ψ_0 , χ_0 are arbitrary.

(iii) We get only one equation involving ϕ_1 , ψ_1 , χ_1 , ϕ_0 , ψ_0 , χ_0. Hence two out of ϕ_1 , ψ_1 and χ_1 can be kept arbitrary when the third is determined in terms of those two arbitrary functions and ϕ_0 ,ψ_0 ,χ_0 .

(iv) We get only two equations involving ϕ_2 , ψ_2 , χ_2 , ϕ_1 , ψ_1 , χ_1 , ϕ_0,ψ_0 ,χ_0. Hence only one out of ϕ_2 , ψ_2 , χ_2 can be kept arbitrary. The second and the third out of ϕ_2,ψ_2,χ_2 is determined in terms of that arbitrary function and ϕ_1,ψ_1,χ_1 , ϕ_0 , ψ_0 , χ_0.

With these observations one can conclude that the equations (2-4) pass the Painleve' test for integrability in the sense of Weiss et al.

Part - 2

Truncation of the Series (6-114 a,b,c), Auto-Backlund Transformation and exact solutions

Here we forcefully make the coefficients ϕ_j, ψ_j, χ_j of u^{j-1} in the expansion (6-117 a,b,c) zero for $j>1$.

The coefficients ϕ_1 , ψ_1 and χ_1 in (6-117 a,b,c) are rewritten as p,q,r respectively in order to differentiate them from $\partial\phi/\partial x^1$, $\partial\psi/\partial x^1$, $\partial\chi/\partial x^1$ etc.

Then from (6-117 a,b,c) one gets

$$\phi = \phi_0 u^{-1} + p,\ \psi = \psi_0 u^{-1} + q,\ \chi = \chi_0 u^{-1} + r \qquad \text{---- (6-119 a,b,c)}$$

Putting these in (2- 4) one gets:

$$A u^{-4} + C\ u^{-3} + E\ u^{-2} + F\ u^{-1} + G \quad = 0 \quad \text{---- (6-120 a)}$$

$$A' u^{-4} + C' u^{-3} + E' u^{-2} + F' u^{-1} + G' \quad = 0 \quad \text{---- (6-121 a)}$$

$$A'' u^{-4} + C'' u^{-3} + E'' u^{-2} + F'' u^{-1} + G'' = 0 \quad \text{---- (6-122 a)}$$

where,

$$A = (u_1^2 + u_2^2 + u_3^2 + u_4^2)(\phi_0^2 + \psi_0^2 + \chi_0^2) \quad \text{--- (6-120b)}$$

$$\begin{aligned} C = [\, & 2\phi_0\, p\, (u_1^2 + u_2^2 + u_3^2 + u_4^2) - \phi_0^2 (u_{11} + u_{22} + u_{33} + u_{44}) \\ & - 2\psi_0(\psi_{01}u_1 + \psi_{02}u_2 + \psi_{03}u_3 + \psi_{04}u_4) - 2\chi_0(\chi_{01}u_1 + \chi_{02}u_2 + \chi_{03}u_3 + \chi_{04}u_4) \\ & + 2\psi_0(\chi_{01}u_2 - \chi_{02}u_1 - \chi_{03}u_4 + \chi_{04}u_3) \\ & - 2\chi_0(\psi_{01}u_2 - \psi_{02}u_1 - \psi_{03}u_4 + \psi_{04}u_3) \,] \end{aligned} \qquad \text{--- (6-120 c)}$$

$$E = [\phi_0(\phi_{011}+\phi_{022}+\phi_{033}+\phi_{044}) - 2p(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4)$$

$$-(\phi_{01}^2+\phi_{02}^2+\phi_{03}^2+\phi_{04}^2)+(\psi_{01}^2+\psi_{02}^2+\psi_{03}^2+\psi_{04}^2)$$

$$+(\chi_{01}^2+\chi_{02}^2+\chi_{03}^2+\chi_{04}^2) - \phi_0\, p(u_{11}+u_{22}+u_{33}+u_{44})$$

$$+2\phi_0(u_1p_1+u_2p_2+u_3p_3+u_4p_4) - 2\psi_0(u_1q_1+u_2q_2+u_3q_3+u_4q_4)$$

$$-2\chi_0(u_1r_1+u_2r_2+u_3r_3+u_4r_4) - 2\psi_0(u_1r_2-u_2r_1-u_3r_4+u_4r_3)$$

$$+2\chi_0(u_1q_2-u_2q_1-u_3q_4+u_4q_3) + 2(\psi_{01}\chi_{02}-\psi_{02}\chi_{01}-\psi_{03}\chi_{04}+\psi_{04}\chi_{03})$$

$$-2p(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4)] \qquad \text{--- (6-120 d)}$$

$$F = [\phi_0(p_{11}+p_{22}+p_{33}+p_{44}) + p(\phi_{011}+\phi_{022}+\phi_{033}+\phi_{044})$$

$$-2(\phi_{01}p_1+\phi_{02}p_2+\phi_{03}p_3+\phi_{04}p_4) + 2(\phi_{01}q_1+\psi_{02}q_2+\psi_{03}q_3+\psi_{04}q_4)$$

$$+2(\chi_{01}r_1+\chi_{02}r_2+\chi_{03}r_3+\chi_{04}r_4) + 2(\psi_{01}r_2-\psi_{02}r_1-\psi_{03}r_4+\psi_{04}r_3)$$

$$-2(\chi_{01}q_2-\chi_{02}q_1-\chi_{03}q_4+\chi_{04}q_3)] \qquad \text{--- (6-120e)}$$

$$G = p(p_{11}+p_{22}+p_{33}+p_{44}) - (p_1^2+p_2^2+p_3^2+p_4^2)$$

$$+(q_1^2+q_2^2+q_3^2+q_4^2)+(r_1^2+r_2^2+r_3^2+r_4^2)$$

$$+2(q_1r_2-q_2r_1-q_3r_4+q_4r_3)] \qquad \text{--- (6-120 f)}$$

Also, $A' = 0$ --- (6-121b)

$$C' = 2\psi_0\, p(u_1^2+u_2^2+u_3^2+u_4^2) - \phi_0\psi_0(u_{11}+u_{22}+u_{33}+u_{44})$$

$$+2\psi_0(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4) - 2\phi_0(\chi_{01}u_2-\chi_{02}u_1-\chi_{03}u_4+\chi_{04}u_3)$$

$$+2\chi_0(\phi_{01}u_2-\phi_{02}u_1-\phi_{03}u_4+\phi_{04}u_3) \qquad \text{--- (6-121c)}$$

$$E' = \phi_0 (\psi_{011}+\psi_{022}+\psi_{033}+\psi_{044}) - 2p(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3+\psi_{04}u_4)$$

$$- \psi_0 p(u_{11}+u_{22}+u_{33}+u_{44}) + 2\phi_0 (u_1q_1+u_2q_2+u_3q_3+u_4q_4)$$

$$-2(\phi_{01}\psi_{01}+\phi_{02}\psi_{02}+\phi_{03}\psi_{03}+\phi_{04}\psi_{04})+2\psi_0 (u_1p_1+u_2p_2+u_3p_3+u_4p_4)$$

$$+ 2\phi_0 (u_1r_2-u_2r_1-u_3r_4+u_4r_3) - 2\chi_0 (u_1p_2-u_2p_1-u_3p_4+u_4p_3))$$

$$-2(\phi_{01}\chi_{02}-\phi_{02}\chi_{01}-\phi_{03}\chi_{04}+\phi_{04}\chi_{03}) \qquad \text{--- (6-121d)}$$

$$F' = \phi_0 (q_{11}+q_{22}+q_{33}+q_{44}) + p(\psi_{011}+\psi_{022}+\psi_{033}+\psi_{044})$$

$$- 2(\phi_{01}q_1+\phi_{02}q_2+\phi_{03}q_3+\phi_{04}q_4) - 2(\psi_{01}p_1+\psi_{02}p_2+\psi_{03}p_3+\psi_{04}p_4)$$

$$- 2(\phi_{01}r_2-\phi_{02}r_1-\phi_{03}r_4+\phi_{04}r_3)$$

$$+ 2(\chi_{01}p_2-\chi_{02}p_1-\chi_{03}p_4+\chi_{04}p_3) \qquad \text{--- (6-121e)}$$

$$G' = p(q_{11}+q_{22}+q_{33}+q_{44}) - 2(p_1r_2-p_2r_1-p_3r_4+p_4r_3)$$

$$- 2(p_1q_1+p_2q_2+p_3q_3+p_4q_4) \qquad \text{--- (6-121 f)}$$

And, $A'' = 0$ ---- (6-122 b)

$$C'' = 2\chi_0 p(u_1^2+u_2^2+u_3^2+u_4^2) - \phi_0\chi_0 (u_{11}+u_{22}+u_{33}+u_{44})$$

$$+2\chi_0(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4) + 2\phi_0(\psi_{01}u_2-\psi_{02}u_1-\psi_{03}u_4+\psi_{04}u_3)$$

$$- 2\psi_0 (\phi_{01}u_2-\phi_{02}u_1-\phi_{03}u_4+\phi_{04}u_3) \qquad \text{--- (6-122 c)}$$

$$E'' = \phi_0(\chi_{011}+\chi_{022}+\chi_{033}+\chi_{044}) - 2p(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3+\chi_{04}u_4)$$

$$- \chi_0 p(u_{11}+u_{22}+u_{33}+u_{44}) + 2\phi_0 (u_1r_1+u_2r_2+u_3r_3+u_4r_4)$$

$$- 2(\phi_{01}\chi_{01}+\phi_{02}\chi_{02}+\phi_{03}\chi_{03}+\phi_{04}\chi_{04}) + 2\chi_0 (u_1p_1+u_2p_2+u_3p_3+u_4p_4)$$

$$- 2\phi_0 (u_1q_2-u_2q_1-u_3q_4+u_4q_3) + 2\psi_0 (u_1p_2-u_2p_1-u_3p_4+u_4p_3)$$

$$+ 2(\phi_{01}\psi_{02}-\phi_{02}\psi_{01}-\phi_{03}\psi_{04}+\phi_{04}\psi_{03}) \qquad \text{--- (6-122 d)}$$

$$F'' = \phi_0 (r_{11}+ r_{22}+ r_{33}+ r_{44}) + p(\chi_{011}+\chi_{022}+\chi_{033}+\chi_{044})$$

$$- 2(\phi_{01}r_1+ \phi_{02}r_2 +\phi_{03}r_3+\phi_{04}r_4)$$

$$- 2 (\chi_{01}p_1+\chi_{02}p_2+\chi_{03} p_3+\chi_{04}p_4) + 2(\phi_{01}q_2 - \phi_{02} q_1 - \phi_{03}q_4 +\phi_{04}q_3)$$

$$- 2 (\psi_{01}p_2 - \psi_{02}p_1 - \psi_{03}p_4 + \psi_{04}p_3) \qquad \text{--- (6-122 e)}$$

$$G'' = p (r_{11} + r_{22} + r_{33} + r_{44}) + 2 (p_1q_2 - p_2q_1 - p_3q_4+ p_4q_3)$$

$$- 2 (p_1r_1 + p_2r_2 + p_3r_3 + p_4r_4) \qquad \text{---- (6-122 f)}$$

As has been stated in relation to checking of resonance position that the equations

(i) $C=0, C'=0, C''=0$ are equivalent to only one equation and

(ii) $E=0, E'=0, E''=0$ are equivalent to two equations.

subject to the condition given by $\phi_0^2 + \psi_0^2 + \chi_0^2 = 0$

Normally these conditions are written as over determined system by equating the coefficients of u^{-j} separately to zero.

But, here if we do so we get

$$\phi_0^2 + \psi_0^2 + \chi_0^2 = 0 \qquad \text{---- (6-123)}$$

which is valid only when ϕ_0, ψ_0 and χ_0 are complex quantities. At the time of checking the existence of Painleve' property we obtained the same result [i.e. (6-123)] at the leading order stage.

The same kind of expression was obtained by Jimbo, Kruskal and Miwa (in their paper having equation no. 3) [57] . That creates no problem in the infinite Laurent- like expansion (6-114). However, when we truncate the series we must assure that ultimately ϕ,ψ and χ are real.

It has been shown in the Appendix – H (at the end of this article) that if we start with the co-efficients of u equal to zero and all of p, q, r equal to zero then we are left with solutions of ϕ,ψ and χ which involve the complex quantity $\sqrt{-1}$. This happens even when all of p,q,r are not equal to zero.

Part – 3 :

Revisit to solutions obtained previously as in chapter 6-1-1 for the exact solutions of the Yang equations.

In the following we show that two types of solutions obtained previously in article (6-1-1) when discussed about the exact solutions of the Yang equations and Chanda & Ray [58] can be rediscovered from (6-119) to (6-122).

Solutions as reported in article (6-1-1)

Here the solutions can be obtained from (6-119) to (6-122) with

$$p = 0,\ q = 0,\ r = 0 \qquad \text{---- (6-124)}$$

and assumption that

$$(\phi_0 / u) = a(v)\ ,\ (\psi_0 / u) = h(v)\ ,\ (\chi_0 / u) = c(v) \quad \text{--- (6-125a,b,c)}$$

where a,h,c are functions of v and $v = v\ (x^1,x^2,x^3,x^4)$.

If one uses the expressions (6-124) and (6-125) in (6-119) one arrives exactly at $\phi = a(v),\ \psi = h(v)\ ,\ \chi = \chi(v)$

which is the same as the equation (2-4).

It can be checked that starting from (6-120) – (6-122) with p=0, q=0 , r = 0 and (6-124) one can also arrive as the equations (2-4).

Solutions reported by Chanda and Ray [58]

Here the solutions can be obtained from (6-119) to (6-122) with the use of the assumption that

$p \neq 0 , q \neq 0 , r \neq 0$

$(\phi_0 / u) = a'(p) , (\psi_0 / u) = h'(p) , (\chi_0 / u) = c'(p)$ ---- (6-126a,b,c)

where a',h',c' are functions of p.

Hence we get from (6-119)

$\phi = a'(p) + p = a(p)$, where a is another function of p --- (6-127a)

$\psi = h'(p) + q$ --- (6-127b)

$\chi = c'(p) + r$ --- (6-127c)

where (p,q,r) and (ϕ,ψ,χ) are two separate sets of solutions of the equation (2-4).

Yang [26] noted that if p,q,r are three functions of x^1,x^2,x^3 and x^4 such that

$q_1 + r_2 = p_3$ ---- (6-128a)

$q_2 - r_1 = p_4$ ---- (6-128b)

$q_3 - r_4 = -p_1$ ---- (6-128c)

$q_4 + r_3 = -p_2$ ---- (6-128d)

then a set of solutions of Yang equations (2-4) is obtained by setting

$\phi = p \ , \ \psi = q \ , \chi = r$ --- (6-129 a,b,c)

Yang [29] indicated some particular solutions of (6-128) as well.

Chanda and Ray [58] obtained some new solutions using

$$\phi = a''(p) \qquad \text{---- (6-130a)}$$

$$\psi = h''(p) + M\,q - N\,r \qquad \text{---- (6-130b)}$$

$$\chi = c''(p) + M\,r + N\,q \qquad \text{---- (6-130c)}$$

where (i) M and N are real arbitrary constants.

(ii) p, q and r are functions of (x^1, x^2, x^3, x^4) and satisfy (6-128)

(iii) a'', h'', c'' are functions of p.

It is easy to identify that (6-127) is a particular situation of (6-130) where M=1,N=0, $a''=a'$, $b''=b'$,$c''=c'$.It may be checked that with(6-127),(6-128) and (6-129) one can arrive from the equation (6-120)- (6-122) to the equation (2.4) of Chanda and Ray [58] through straightforward manipulation. Thus, the equations (6-127) with (6-128) can be treated as an Auto-Backlund transformation between two sets of solutions where one set of solutions (p,q,r) are given by (6-128) and another set of solutions (ϕ,ψ,χ) are given by the equations (3) of Chanda and Ray [58] with A=1,B=0 in the equation (2.4) of the same paper.

Thus the article (6-3-1) may be summarized as follows :

Yang's self-dual equations for SU (2) gauge fields (expressed in real form ; equation 2-4) pass the Painleve' test for integrability in the sense of Weiss et al.[21a], Jimbo, Kruskal and Miwa [49] analyzed the complex form of (2-4) of the same equations and found that the equation pass the Painleve' test for integrability. However, they did not show whether the equations allowed truncation of series. Here we have shown that the equations (2-4) , admit truncation of series. The truncation procedure leads to previously obtained two pairs non-trivial exact solutions and the Auto-Backlund transformation between those pairs solutions. Another important aspect of our analysis is that Jimbo, Kruskal and Miwa [49] analysed the equation with a restricted nature of the singularity manifold. Here we have analysed the equation keeping the singularity manifold completely general.

Appendix – H

Let us start from the trivial solution $p = 0$, $q = 0$, $r = 0$.

One gets from (6-119 a,b,c)

$$\phi = \phi_0\, u^{-1}\ ,\quad \psi = \psi_0\, u^{-1}\ ,\quad \chi = \chi_0\, u^{-1} \qquad \text{---- (H -1 a, b, c)}$$

Equating the different co-efficient of u^{-j} in (6-120) – (6-122) separately to zero one finally gets

$$\phi_0^{\,2} + \psi_0^{\,2} + \chi_0^{\,2} = 0 \qquad \text{--- (H - 2 a)}$$

$$\phi_0\,(u_{11} + u_{22} + u_{33} + u_{44})$$
$$= 2\,(\phi_{01}\, u_1 + \phi_{02}\, u_2 + \phi_{03}\, u_3 + \phi_{04}\, u_4) + (2u_2/\psi_0)\,(\chi_0\, \phi_{01} - \phi_0\, \chi_{01})$$
$$+ (2u_1/\psi_0)\,(\phi_0\chi_{02} - \chi_0\, \phi_{02}) + (2u_4/\psi_0)\,(\phi_0\chi_{03} - \chi_0\phi_{03}) \qquad \text{-- (H -2 b)}$$

$$\phi_0\,(\psi_{011} + \psi_{022} + \psi_{033} + \psi_{044})$$
$$= 2(\phi_{01}\, \psi_{01} + \phi_{02}\, \psi_{02} + \phi_{03}\, \psi_{03} + \phi_{04}\, \psi_{04})$$
$$+2(\phi_{01}\, \chi_{02} - \phi_{02}\, \chi_{01} - \phi_{03}\, \chi_{04} + \phi_{04}\, \chi_{03}) \qquad \text{---(H -2c)}$$

$$\phi_0\,(\chi_{011} + \chi_{022} + \chi_{033} + \chi_{044})$$
$$= 2(\phi_{01}\chi_{01} + \phi_{02}\chi_{02} + \phi_{03}\chi_{03} + \phi_{04}\chi_{04})$$
$$+ 2\,(\psi_{01}\phi_{02} - \psi_{02}\phi_{01} - \psi_{03}\phi_{04} - \psi_{04}\phi_{03}) \qquad \text{---(H - 2d)}$$

In (H-2) there are four equations and four unknowns,

namely ϕ_0, ψ_0, χ_0 and u.

Let us assume, again,

$$\psi_0 = \chi_0 \qquad \text{----- (H – 3)}$$

when we get

[as a result of the requirement of consistency between (H-2c) and (H -2d)]

$$\phi_0^2 = -2\,\psi_0^2 \qquad \text{---- (H - 4a)}$$

$$\begin{aligned}\phi_0\,(u_{11} + u_{22} + u_{33} + u_{44})\\ &= 2(\phi_{01}\,u_1 + \phi_{02}\,u_2 + \phi_{03}\,u_3 + \phi_{04}\,u_4) - (2u_2/\psi_0)(\psi_0\,\phi_{01} - \phi_0\psi_{01})\\ &\quad - (2u_1/\psi_0)(\phi_0\,\psi_{02} - \psi_0\,\phi_{02}) - (2u_3/\psi_0)(\psi_0\phi_{04} - \phi_0\psi_{04})\\ &\quad - (2u_4/\psi_0)(\phi_0\,\psi_{03} - \psi_0\,\phi_{03}) \qquad \text{---- (H - 4b)}\end{aligned}$$

$$\begin{aligned}\phi_0\,(\psi_{011} + \psi_{022} + \psi_{033} + \psi_{044})\\ &= 2\,(\phi_{01}\,\psi_{01} + \phi_{02}\,\psi_{02} + \phi_{03}\,\psi_{03} + \phi_{04}\,\psi_{04}) \qquad \text{---(H- 4c)}\end{aligned}$$

$$\phi_{01}\,\psi_{02} - \phi_{02}\,\psi_{01} + \phi_{04}\,\psi_{03} - \phi_{03}\,\psi_{04} = 0 \qquad \text{-- (H - 4d)}$$

With a little manipulation (H-4) lead to

$$\phi_0^2 = -2\psi_0^2 \qquad \text{--- (H-5a)}$$

$$\phi_0\,(u_{11} + u_{22} + u_{33} + u_{44}) = 2(\phi_{01}u_1 + \phi_{02}u_2 + \phi_{03}u_3 + \phi_{04}\,u_4) \qquad \text{---(H-5b)}$$

$$\begin{aligned}\phi_0\,(\psi_{011} + \psi_{022} + \psi_{033} + \psi_{044})\\ &= 2\,(\phi_{01}\,\psi_{01} + \phi_{02}\,\psi_{02} + \phi_{03}\,\psi_{03} + \phi_{04}\,\psi_{04}) \qquad \text{--- (H-5c)}\end{aligned}$$

Again we have three equations and three unknown.

From (H-5a) and (H-5c) one can write,

$$\phi_0 = (\sqrt{2}\, i) / \zeta \qquad \text{----- (H-6a)}$$

$$\psi_0 = 1 / \zeta \qquad \text{---- (H-6b)}$$

where ζ satisfies $\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0$ ---- (H-6c)

Using (H-6) in (H-5) we get

$$u = \int (dU / \zeta^2) \qquad \text{---- (H-7)}$$

where U satisfies $U_{11} + U_{22} + U_{33} + U_{44} = 0$.

Hence finally one can conclude

$$\phi = (2i/\zeta) \;/\; (\int dU / \zeta^2) \qquad \text{---(H-8a)}$$

$$\psi = (1/\zeta) \;/\; (\int dU / \zeta^2) \qquad \text{----(H-8b)}$$

$$\chi = (1/\zeta) \;/\; (\int dU / \zeta^2) \qquad \text{----(H-8c)}$$

where ζ and U satisfy

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0 \qquad \text{----(H-9a)}$$

and $U_{11} + U_{22} + U_{33} + U_{44} = 0$ respectively ---(H-9b)

The equations (H-8) demand that

$$U = W(\zeta) \qquad \text{----(H -10)}$$

where W is a function of ζ.

With the use of (H-10) the equations (H-9) reduce to

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0 \qquad \text{---- (H-11a)}$$

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2 = 0 \qquad \text{----(H-11b)}$$

which has the solution

$$\zeta = P\,x^1 + Q\,x^2 + R\,x^3 + S\,x^4 + T \qquad \text{----(H-12)}$$

Here P,Q, R,S,T are all arbitrary constants. Finally , ϕ , ψ and χ are given by (H-8), (H-10) and (H-12).

Obviously another solution of (H-9) can be written as

$$U = H\,\zeta \qquad \text{---(H–13)}$$

where H is an arbitrary constant.

With (H-13) one can arrive from (H-8) at

$$\phi = (2\,H\,i \,/\, \zeta\,)\,(\,\ln\zeta\,) \qquad \text{--- (H-14a)}$$

$$\psi = (\,H \,/\, \zeta\,)\,(\,\ln\zeta\,) \qquad \text{--- (H-14b)}$$

$$\chi = (\,H \,/\, \zeta\,)\,(\,\ln\zeta\,) \qquad \text{--- (H-14c)}$$

6-3-2 Painleve' test for integrability and exact solutions for Charap equations

Unlike in the case of Yang equations (2-4) no previous work on the Painleve' test for Charap equation (2-8) is known. In this article we have observed that the field equations for Charap Chiral invariant model of the pion dynamics pass the test for integrability in the sense of Painleve' analysis due to Weiss et. al. [21a].

The results showing that the equations (2-8) pass the Painleve' test for integrability (in the sense of Weiss et al) and admit truncation of series leading to Auto-Backlund transformation between two pairs of exact solutions wherefrom non-trivial exact solutions can be rediscovered add to the importance of Charap's equations (2-8).

The discussion in this article also has three parts :

Part – 1 : Here we discuss Painleve' test for integrability of the equations (2-8) in the sense of Weiss et al.

Part – 2 :Truncation , Auto-Buckland Transformation and exact solutions.

Part – 3 : Revisit to solutions obtained previously as in article (6-1-2) for the Charap equations (2-8).

Part – 1

Painleve' test for integrability of the equations (2-8) in the sense of Weiss et al. [21a]

For the equations (2-8) we define the singularity manifold given by

$$u = u\,(x^1, x^2, x^3, x^4) = 0 \qquad \text{---- (6-133)}$$

and set

$$\phi = u^{\alpha} \sum_{j=0}^{\infty} \phi_j\, u^{j},\ \psi = u^{\beta} \sum_{j=0}^{\infty} \psi_j\, u^{j},\ \chi = u^{\gamma} \sum_{j=0}^{\infty} \chi_j\, u^{j} \qquad \text{---- (6-134 a,b,c)}$$

where $\phi(x^1,x^2, x^3, x^4)$, $\psi(x^1,x^2, x^3,x^4)$, $\chi(x^1, x^2,x^3, x^4)$ are a set of solutions of (1.4); ϕ_j, ψ_j, χ_j are all analytic functions of (x^1, x^2, x^3, x^4) in the neighborhood of the manifold (2.1);

$\phi_0 \neq 0$, $\psi_0 \neq 0$, $\chi_0 \neq 0$.

Now, the test may be divided into three main steps after the substitution of (6-134) in the differential equations concerned :

I. Make the leading order analysis [where one gets all-possible $\alpha,\beta,\gamma,\phi_0,\psi_0,\chi_0$ in (6-134)]

II. Define the recursion relations for u_j for the leading orders obtained in step I and determine the resonance positions (those values of j for which the relations are not defined).

III. Check whether the expansions allow requisite number of arbitrary functions at the resonance positions.

I. <u>Leading order analysis</u> :

We assume

$$\phi \sim \phi_0 u^{\alpha} \quad , \psi \sim \psi_0 u^{\beta} \ , \ \chi \sim \chi_0 u^{\gamma} \qquad \text{---- (6-135 a,b,c)}$$

We substitute (6-135 a,b,c) in (2-8) and equate the coefficients of the negative powers of u (considering that all of α,β and γ are negative). This leads to $\alpha = -1$, $\beta = -1$, $\gamma = -1$ so that

$$\phi = u^{\alpha} \sum_{j=0}^{\infty} \phi_j u^j \ , \ \psi = u^{\beta} \sum_{j=0}^{\infty} \psi_j u^j \ , \chi = u^{\gamma} \sum_{j=0}^{\infty} \chi_j u^j \qquad \text{---- (6-136 a,b,c)}$$

ϕ_0 = arbitrary, ψ_0 = arbitrary, χ_0 = arbitrary.

II. <u>Resonance positions</u>

We directly substitute (6-136) in (2-8). We have not written explicitly the recursion relation because of their involved structure. In order to have an idea one can consult the recursion relations of the work of Chanda and Roychowdhury [5]. The recursion relation for ϕ_j , ψ_j , χ_j are

$$[\,T\,] \begin{bmatrix} \phi_m \\ \\ \psi_m \\ \\ \chi_m \end{bmatrix} = \text{[other terms with } \phi_j , \psi_j , \chi_j \text{ and their derivatives, where } j < m \text{]}$$

where , [T] is the system matrix. It is written as,

$$[T] = \begin{bmatrix} m(m+1)\phi_0^2+m(m-1)\psi_0^2+m(m-1)\chi_0^2 & 2m\phi_0\psi_0 & 2m\phi_0\chi_0 \\ 2m\phi_0\psi_0 & m(m-1)\phi_0^2+m(m+1)\psi_0^2+m(m-1)\chi_0^2 & 2m\psi_0\chi_0 \\ 2m\phi_0\chi_0 & 2m\psi_0\chi_0 & m(m-1)\phi_0^2+m(m+1)\psi_0^2+m(m-1)\chi_0^2 \end{bmatrix}$$

Resonances are those values of $m = R$, for which $\det [T] = 0$, which yields,

$$R^3(R+1)(R-1)^2 [\, \phi_0^6+\psi_0^6+\chi_0^6+3\,\phi_0^2\,\psi_0^4+3\,\phi_0^2\,\chi_0^4+3\,\phi_0^4\,\psi_0^2 +3\phi_0^4\chi_0^2+3\phi_0^2\psi_0^2\chi_0^2+3\,\psi_0^2\,\chi_0^4+3\psi_0^4\,\chi_0^2\,] = 0$$

i.e. Poles are, $R = (-1), (0, 0, 0), (1, 1)$.

(i) $R = -1$ indicates that the singularity manifold defined in (6-133) is required to be arbitrary.

(ii) $R=0,0,0$ indicate that all of the coefficients ϕ_0, ψ_0 and χ_0 are required to be arbitrary.

(ii) $R= 1,1$ indicate that any two of the coefficients ϕ_1, ψ_1 and χ_1 are required to be arbitrary.

III. <u>To check whether the expansions allow requisite number of arbitrary functions at the resonance positions :</u>

(i) The singularity manifold, by definition, is arbitrary.

(ii) The terms involving ϕ_0, ψ_0 and χ_0 cancel each other. Hence all of ϕ_0, ψ_0 and χ_0 are arbitrary.

(iii) We get only one equation involving ϕ_1, ψ_1, χ_1, ϕ_0, ψ_0, χ_0.

Hence two of ϕ_1 ,ψ_1 ,χ_1 can be kept arbitrary when the third is determined in terms of those arbitrary functions and ϕ_0 , ψ_0 , χ_0 .

With above observations one can conclude that the equations (2-8) pass the Painleve' test for integrability in the sense of Weiss et al.

Part - 2

Truncation of the series (6-134 a,b,c), Auto-Buckland Transformation and exact solutions

Here we forcefully make the coefficients ϕ_j ,ψ_j ,χ_j of u^{j-1} in the expansions (6-136 a,b,c) zero for $j > 1$. The coefficients ϕ_1 , ψ_1 and χ_1 in (6-136 a,b,c) are rewritten as p,q and r respectively in order to differentiate them from $(\partial\phi/\partial x^1)$,$(\partial\psi/\partial x^1)$, $(\partial\chi/\partial x^1)$.

Then from (6-136 a,b,c) one gets ,

$$\phi = \phi_0 u^{-1} + p \,,\ \psi = \psi_0 u^{-1} + q \,,\ \chi = \chi_0 u^{-1} + r \qquad \text{-- (6-137 a,b,c)}$$

Putting these in (2-8) one gets :

$$P\ u^{-5} + Q\ u^{-4} + C\ u^{-3} + E\ u^{-2} + F u^{-1} + G = 0 \quad \text{-- (6-138)}$$

$$P'\ u^{-5} + Q'\ u^{-4} + C'\ u^{-3} + E'\ u^{-2} + F' u^{-1} + G' = 0 \quad \text{-- (6-139)}$$

$$P''u^{-5} + Q''u^{-4} + C''u^{-3} + E''\ u^{-2} + F''\ u^{-1} + G'' = 0 \quad \text{-- (6-140)}$$

where P, Q,C, E,F,G etc are given below.

Here,

$$P = 0 \,,\ P' = 0 \,,\ P'' = 0 \qquad \text{---- (6-141 a,b,c)}$$

$$Q = 2(\phi_0 p+\psi_0 q+\chi_0 r)(u_1^2+u_2^2+u_3^2-u_4^2) - (\phi_0^2+\psi_0^2+\chi_0^2)(u_1^2+u_2^2+u_3^2-u_4^2)$$

$$+2\phi_0(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3-\phi_{04}u_4)+2\psi_0(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3-\psi_{04}u_4)$$

$$+2\chi_0(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3-\chi_{04}u_4) = 0 \qquad \text{--- (6-142 a)}$$

$$Q' = Q\,,\ Q'' = Q \qquad \text{---- (6-142 b,c)}$$

$$C = 2\phi_0 e^{\beta}(u_1^2+u_2^2+u_3^2-u_4^2)+\phi_0\,[2p(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3-\phi_{04}u_4)$$

$$+2q(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3-\psi_{04}u_4)+2r(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3-\chi_{04}u_4)]$$

$$-2A(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3-\phi_{04}u_4)-2\phi_0(\phi_{01}^2+\phi_{02}^2+\phi_{03}^2-\phi_{04}^2)$$

$$-2\psi_0(\phi_{01}\psi_{01}+\phi_{02}\psi_{02}+\phi_{03}\psi_{03}-\phi_{04}\psi_{04})-2\chi_0(\phi_{01}\chi_{01}+\phi_{02}\chi_{02}+\phi_{03}\chi_{03}-\phi_{04}\chi_{04})$$

$$+2\phi_0\,[\,\phi_0(u_1p_1+u_2p_2+u_3p_3-u_4p_4)+\psi_0(u_1q_1+u_2q_2+u_3q_3-u_4q_4)$$

$$+\chi_0(u_1r_1+u_2r_2+u_3r_3-u_4r_4)]$$

$$+2(\phi_0^2+\psi_0^2+\chi_0^2)(u_1p_1+u_2p_2+u_3p_3-u_4p_4)$$

$$-2\phi_0(\phi_0 p+\psi_0 q+\chi_0 r)(u_{11}+u_{22}+u_{33}-u_{44})$$

$$+(\phi_0^2+\psi_0^2+\chi_0^2)(\phi_{011}+\phi_{022}+\phi_{033}-\phi_{044}) \qquad \text{---- (6-143a)}$$

$$C' = 2\psi_0 e^{\beta'}(u_1^2+u_2^2+u_3^2-u_4^2)+\psi_0\,[2p(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3-\phi_{04}u_4)$$

$$+2q(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3-\psi_{04}u_4)+2r(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3-\chi_{04}u_4)\,]$$

$$-2(\phi_0 p+\psi_0 q+\chi_0 r)(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3-\psi_{04}u_4)-2\psi_0(\psi_{01}^2+\psi_{02}^2+\psi_{03}^2-\psi_{04}^2)$$

$$-2\phi_0(\phi_{01}\psi_{01}+\phi_{02}\psi_{02}+\phi_{03}\psi_{03}-\phi_{04}\psi_{04})-2\chi_0(\psi_{01}\chi_{01}+\psi_{02}\chi_{02}+\psi_{03}\chi_{03}-\psi_{04}\chi_{04})$$

$$+2\psi_0\,[\phi_0(u_1p_1+u_2p_2+u_3p_3-u_4p_4)+\psi(u_1q_1+u_2q_2+u_3q_3-u_4q_4)$$

$$+\chi_0(u_1r_1+u_2r_2+u_3r_3-u_4r_4)\,]$$

$$+2(\phi_0^2+\psi_0^2+\chi_0^2)(u_1q_1+u_2q_2+u_3q_3-u_4q_4)-2\psi_0(\phi_0 p+\psi_0 q+\chi_0 r)(u_{11}+u_{22}+u_{33}-u_{44})$$

$$+(\phi_0^2+\psi_0^2+\chi_0^2)(\psi_{011}+\psi_{022}+\psi_{033}-\psi_{044}) \qquad \text{---- (6-143 b)}$$

$$C'' = 2\chi_0 e^{\beta'} (u_1^2+u_2^2+u_3^2-u_4^2) +\chi_0 [2p(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3-\phi_{04} u_4)$$

$$+2q(\psi_{01}u_1+ \psi_{02} u_2+\psi_{03}u_3 -\psi_{04}u_4)+2r(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3-\chi_{04}u_4)]$$

$$- 2 (\phi_0 p + \psi_0 q + \chi_0 r) (\chi_{01} u_1+ \chi_{02} u_2 +\chi_{03} u_3 - \chi_{04} u_4)$$

$$- 2\chi_0 (\chi_{01}^2+\chi_{02}^2+\chi_{03}^2 -\chi_{04}^2) - 2 \phi_0 (\phi_{01}\chi_{01}+\phi_{02}\chi_{02}+\phi_{03}\chi_{03} - \phi_{04}\chi_{04})$$

$$- 2\psi_0 (\psi_{01} \chi_{01}+ \psi_{02} \chi_{02} + \psi_{03} \chi_{03} - \psi_{04} \chi_{04})$$

$$+ 2\chi_0 [\phi_0(u_1p_1+u_2p_2+u_3p_3 -u_4p_4) + \psi_0(u_1q_1+u_2q_2 + u_3 q_3 - u_4 q_4)$$

$$+\chi_0(u_1r_1+u_2 r_2+u_3 r_3 -u_4r_4)] + 2(\phi_0^2+\psi_0^2+\chi_0^2) (u_1r_1+u_2 r_2+u_3 r_3-u_4r_4)$$

$$-2\chi_0 (\phi_0 p + \psi_0 q + \chi_0 r) (u_{11} + u_{22} + u_{33} - u_{44})$$

$$+ (\phi_0^2 + \psi_0^2 + \chi_0^2) (\chi_{011} + \chi_{022} + \chi_{033} - \chi_{044}) \qquad \text{---- (6-143 c)}$$

$$E = -2 e^{\beta'}(\phi_{01}u+\phi_{02}u_2+\phi_{03}u_3-\phi_{04}u_4) -\phi_0^2e^{\beta'}(u_{11}+u_{22}+ u_{33} - u_{44})$$

$$+ 2 (\phi_0 p + \psi_0 q + \chi_0 r) (\phi_{011} + \phi_{022} + \phi_{033} - \phi_{044})$$

$$+(\phi_0^2+\psi_0^2+\chi_0^2)(p_{11}+p_{22}+p_{33}-p_{44})$$

$$+2(\phi_0p+\psi_0q+\chi_0r)(u_1p_1+u_2p_2+u_3p_3-u_4p_4)$$

$$-2\phi_0(\phi_{01}p_1+\phi_{02}p_2+\phi_{03}p_3-\phi_{04} p_4) -2\psi_0(\phi_{01}q_1+\phi_{02}q_2+\phi_{03}q_3 -\phi_{04} q_4)$$

$$- 2\chi_0 (\phi_{01}r_1+\phi_{02} r_2+\phi_{03} r_3 -\phi_{04} r_4)+2p\phi_0(u_1p_1+u_2p_2+u_3p_3 - u_4 p_4)$$

$$+ 2q \phi_0 (u_1q_1+u_2q_2 +u_3q_3 -u_4q_4)+ 2r\phi_0(u_1r_1+u_2r_2+u_3r_3 -u_4r_4)$$

$$-2p(\phi_{01}^2+\phi_{02}^2+\phi_{03}^2-\phi_{04}^2)-2q(\phi_{01}\psi_{01}+\phi_{02}\psi_{02} +\phi_{03} \psi_{03} -\phi_{04} \psi_{04})$$

$$-2r(\phi_{01}\chi_0+\phi_{02}\chi_{02}+\phi_{03}\chi_{03}-\phi_{04}\chi_{04})-2\phi_0(\phi_{01}p_1+\phi_{02}p_2+\phi_{03}p_3-\phi_{04}p_4)$$

$$-2\psi_0(\psi_{01}p_1+\psi_{02}p_2+\psi_{03}p_3-\psi_{04}p_4)$$

$$-2\chi_0(\chi_{01}p_1+\chi_{02} p_2+\chi_{03}p_3-\chi_{04}p_4) \qquad \text{---- (6-144a)}$$

$$E' = -2e^{\beta'}(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3-\psi_{04}u_4) - \psi_0{}^2e^{\beta'}(u_{11}+u_{22}+u_{33}-u_{44})$$

$$+2(\phi_0 p+\psi_0 q+\chi_0 r)(\psi_{011}+\psi_{022}+\psi_{033}-\psi_{044})$$

$$+(\phi_0{}^2+\psi_0{}^2+\chi_0{}^2)(q_{11}+q_{22}+q_{33}-q_{44})$$

$$+2(\phi_0 p+\psi_0 q+\chi_0 r)(u_1q_1+u_2q_2+u_3q_3-u_4q_4)$$

$$-2\phi_0(\psi_{01}p_1+\psi_{02}p_2+\psi_{03}p_3-\psi_{04}p_4)-2\psi_0(\psi_{01}q_1+\psi_{02}q_2+\psi_{03}q_3-\psi_{04}q_4)$$

$$-2\chi_0(\psi_{01}r_1+\psi_{02}r_2+\psi_{03}r_3-\psi_{04}r_4)+2p\psi_0(u_1p_1+u_2p_2+u_3p_3-u_4p_4)$$

$$+2q\phi_0(u_1q_1+u_2q_2+u_3q_3-u_4q_4)+2r\psi_0(u_1r_1+u_2r_2+u_3r_3-u_4r_4)$$

$$-2p(\phi_{01}\psi_{01}+\phi_{02}\psi_{02}+\phi_{03}\psi_{03}-\phi_{04}\psi_{04})-2q(\psi_{01}{}^2+\psi_{02}{}^2+\psi_{03}{}^2-\psi_{04}{}^2)$$

$$-2r(\psi_{01}\chi_{01}+\psi_{02}\chi_{02}+\psi_{03}\chi_{03}-\psi_{04}\chi_{04})-2\phi_0(\phi_{01}q_1+\phi_{02}q_2+\phi_{03}q_3-\phi_{04}q_4)$$

$$-2\psi_0(\psi_{01}q_1+\psi_{02}q_2+\psi_{03}q_3-\psi_{04}q_4)$$

$$-2\chi_0(\chi_{01}q_1+\chi_{02}q_2+\chi_{03}q_3-\chi_{04}q_4) \qquad \text{---- (6-144b)}$$

$$E'' = -2e^{\beta'}(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3-\chi_{04}u_4) - \chi_0{}^2e^{\beta'}(u_{11}+u_{22}+u_{33}-u_{44})$$

$$+2(\phi_0 p+\psi_0 q+\chi_0 r)(\chi_{011}+\chi_{022}+\chi_{033}-\chi_{044})+(\phi_0{}^2+\psi_0{}^2+\chi_0{}^2)(r_{11}+r_{22}\ r_{33}-r_{44})$$

$$+2(\phi_0 p+\psi_0 q+\chi_0 r)(u_1r_1+u_2r_2+u_3r_3-u_4r_4)$$

$$-2\phi_0(\chi_{01}p_1+\chi_{02}p_2+\chi_{03}p_3-\chi_{04}p_4)$$

$$-2\psi_0(\chi_{01}q_1+\chi_{02}q_2+\chi_{03}q_3-\chi_{04}q_4)-2\chi_0(\chi_{01}r_1+\chi_{02}r_2+\chi_{03}r_3-\chi_{04}r_4)$$

$$+2p\chi_0(u_1p_1+u_2p_2+u_3p_3-u_4p_4)+2q\chi_0(u_1q_1+u_2q_2+u_3q_3-u_4q_4)$$

$$+2r\chi_0(u_1r_1+u_2r_2+u_3r_3-u_4r_4)-2p(\phi_{01}\chi_{01}+\phi_{02}\chi_{02}+\phi_{03}\chi_{03}-\phi_{04}\chi_{04})$$

$$-2q(\psi_{01}\chi_{01}+\psi_{02}\chi_{02}+\psi_{03}\chi_{03}-\psi_{04}\chi_{04})-2r(\chi_{01}{}^2+\chi_{02}{}^2+\chi_{03}{}^2-\chi_{04}{}^2)$$

$$-2\phi_0(\phi_{01}r_1+\phi_{02}r_2+\phi_{03}r_3-\phi_{04}r_4)-2\psi_0(\psi_{01}r_1+\psi_{02}r_2+\psi_{03}r_3-\psi_{04}r_4)$$

$$-2\chi_0(\chi_{01}r_1+\chi_{02}r_2+\chi_{03}r_3-\chi_{04}r_4) \qquad \text{---- (6-144c)}$$

$$F = e^{\beta'}(\phi_{011} + \phi_{022} + \phi_{033} - \phi_{044})$$

$$+ 2(\phi_0 p + \psi_0 q + \chi_0 r)(p_{11} + p_{22} + p_{33} - p_{44})$$

$$- 2p(\phi_{01}p_1 + \phi_{02}p_2 + \phi_{03}p_3 - \phi_{04}p_4) - 2q(\phi_{01}q_1 + \phi_{02}q_2 + \phi_{03}q_3 - \phi_{04}q_4)$$

$$- 2r(\phi_{01}r_1 + \phi_{02}r_2 + \phi_{03}r_3 - \phi_{04}r_4) - 2p(\phi_{01}p_1 + \phi_{02}p_2 + \phi_{03}p_3 - \phi_{04}p_4)$$

$$- 2q(\psi_{01}p_1 + \psi_{02}p_2 + \psi_{03}p_3 - \psi_{04}p_4) - 2r(\chi_{01}p_1 + \chi_{02}p_2 + \chi_{03}p_3 - \chi_{04}p_4)$$

$$- 2\phi_0(p_1^2 + p_2^2 + p_3^2 - p_4^2) - 2\psi_0(p_1q_1 + p_2q_2 + p_3q_3 - p_4q_4)$$

$$- 2\chi_0(p_1r_1 + p_2r_2 + p_3r_3 - p_4r_4) \qquad \text{---- (6-145a)}$$

$$F' = e^{\beta'}(\psi_{011} + \psi_{022} + \psi_{033} - \psi_{044}) + 2(\phi_0 p + \psi_0 q + \chi_0 r)(q_{11} + q_{22} + q_{33} - q_{44})$$

$$-2p(\psi_{01}p_1 + \psi_{02}p_2 + \psi_{03}p_3 - \psi_{04}p_4) - 2q(\psi_{01}q_1 + \psi_{02}q_2 + \psi_{03}q_3 - \psi_{04}q_4)$$

$$- 2r(\psi_{01}r_1 + \psi_{02}r_2 + \psi_{03}r_3 - \psi_{04}r_4) - 2p(\phi_{01}q_1 + \phi_{02}q_2 + \phi_{03}q_3 - \phi_{04}q_4)$$

$$- 2q(\psi_{01}q_1 + \psi_{02}q_2 + \psi_{03}q_3 - \psi_{04}q_4) - 2r(\chi_{01}q_1 + \chi_{02}q_2 + \chi_{03}q_3 - \chi_{04}q_4)$$

$$- 2\phi_0(p_1q_1 + p_2q_2 + p_3q_3 - p_4q_4) - 2\psi_0(q_1^2 + q_2^2 + q_3^2 - q_4^2)$$

$$- 2\chi_0(q_1r_1 + q_2r_2 + q_3r_3 - q_4r_4) \qquad \text{---- (6-145b)}$$

$$F'' = e^{\beta'}(\chi_{011} + \chi_{022} + \chi_{033} - \chi_{044}) + 2(\phi_0 p + \psi_0 q + \chi_0 r)(r_{11} + r_{22} + r_{33} - r_{44})$$

$$-2p(\chi_{01}p_1 + \chi_{02}p_2 + \chi_{03}p_3 - \chi_{04}p_4) - 2q(\chi_{01}q_1 + \chi_{02}q_2 + \chi_{03}q_3 - \chi_{04}q_4)$$

$$-2r(\chi_{01}r_1 + \chi_{02}r_2 + \chi_{03}r_3 - \chi_{04}r_4) - 2p(\phi_{01}r_1 + \phi_{02}r_2 + \phi_{03}r_3 - \phi_{04}r_4)$$

$$-2q(\psi_{01}r_1 + \psi_{02}r_2 + \psi_{03}r_3 - \psi_{04}r_4) - 2r(\chi_{01}r_1 + \chi_{02}r_2 + \chi_{03}r_3 - \chi_{04}r_4)$$

$$- 2\phi_0(p_1r_1 + p_2r_2 + p_3r_3 - p_4r_4) - 2\psi_0(q_1r_1 + q_2r_2 + p_3q_3 - p_4q_4)$$

$$- 2\chi_0(r_1^2 + r_2^2 + r_3^2 - r_4^2) \qquad \text{--- (6-145c)}$$

$$G = (p_{11}+p_{22}+p_{33}-p_{44}) - e^{-\beta'}[2p(p_1^2+p_2^2+p_3^2-p_4^2)$$

$$+2q(p_1q_1+p_2q_2+p_3q_3-p_4q_4)+2(p_1r_1+p_2r_2+p_3r_3-p_4r_4)] \quad \text{----} \ (6\text{-}146a)$$

$$G' = (q_{11}+q_{22}+q_{33}-q_{44}) - e^{-\beta'}[2q(q_1^2+q_2^2+q_3^2-q_4^2)$$

$$+2p(p_1q_1+p_2q_2+p_3q_3-p_4q_4)+2r(q_1r_1+q_2r_2+q_3r_3-q_4r_4)] \quad \text{--}(6\text{-}146b)$$

$$G'' = (r_{11}+r_{22}+r_{33}-r_{44}) - e^{-\beta'}[2r(r_1^2+r_2^2+r_3^2-r_4^2)$$

$$+2p(p_1r_1+p_2r_2+p_3r_3-p_4r_4)+2q(q_1r_1+q_2r_2+q_3r_3-q_4r_4)] \quad \text{--}(6\text{-}146c)$$

where $\beta' = f_\pi^2 + p^2 + q^2 + r^2$

Now, if one has $G = 0$, $G' = 0$, $G'' = 0$ then one can say that the equations in (6-141) represent Auto-Backlund transformation [55], [56] between two pairs of solutions of (2-8) given by (ϕ,ψ,χ) and (p,q,r) subject to the condition :

$$P\,u^{-5} + Q\,u^{-4} + C\,u^{-3} + E\,u^{-2} + F\,u^{-1} + G = 0 \quad \text{-----} \ (6\text{-}147a)$$

$$P'\,u^{-5} + Q'\,u^{-4} + C'\,u^{-3} + E'\,u^{-2} + F'\,u^{-1} + G' = 0 \quad \text{---} \ (6\text{-}147b)$$

$$P''\,u^{-5} + Q''\,u^{-4} + C''\,u^{-3} + E''\,u^{-2} + F''\,u^{-1} + G'' = 0 \quad \text{-} \ (6\text{-}147c)$$

where P,Q,C,E,E,F etc are given by (6-141) to (6-146).

It would have been nice if Auto-Backlund transformation between two sets of non-trivial solution could be shown. At this stage the complicacy of the system did not allow us to achieve that goal. However, in the following section we have shown the Auto-Backlund transformation between a set of trivial solution (p=0,q=0,r=0) and the non-trivial solutions (ϕ,ψ,χ) reported in articles (5-1-2) and (6-1-2).

Part - 3

Rediscovery of solutions obtained previously by

(Ray [32] and in article 6-1-2) :

Normally an over determined system is obtained by equating the co-efficient of u^{-j} in (6-138), (6-139),(6-140) separately to zero. However, at the time of rediscovering previous solutions with p=0,q=0,r=0 it is found that the act of equating the coefficients of u^{-j} separately zero imposes a very strong condition which cannot be satisfied. Therefore we have kept (6-147) as such and made p=0,q=0,r=0 so that $G=0, G' = 0$, $G'' = 0$ are automatically satisfied.

Solutions reported in the 1st part of Ray [32]

Here the solutions can be obtained from (6-141),(6-148),

($p = 0$, $q = 0$, $r = 0$) and the assumption

$(\phi_0/u) = a(w), (\psi_0/u) = m(w), (\chi_0/u) = n(w)$ ---- (6-148a,b,c)

where a, m , n are functions of w , $w = w(x^1,x^2,x^3,x^4)$.

we get, $\phi = a(w)$, $\psi = m(w)$, $\chi = n(w)$ ---- (6-149 a,b,c)

which is the similar to (5-27) as discussed in article (5-1-2), proposed by Ray [32].

Solutions reported in the 2^{nd} part of Ray [32]

Here the solutions can be obtained from (6-137) , (6-146) , (p=0,q=0,r=0) and the assumption

$(\phi_0/u) = a'(w'),\ (\psi_0/u) = m'(w'),\ (\chi_0/u) = n'(w')$ --- (6-150a,b,c)

where a', m', n' are functions of w', $w' = w'(x^1, x^2, x^3 - x^4)$.

We get, $\phi = a'(x^1, x^2, x^3 - x^4)$ ---- (6-151a)

$\psi = m'(x^1, x^2, x^3 - x^4)$ ---- (6-151b)

$\chi = n'(x^1, x^2, x^3 - x^4)$ ---- (6-151c)

which is the same as discussed in (5-27) , proposed in the 2^{nd} part of Ray [32].

Solutions reported in article 6-1-2 in equation (6-56)

Here the solutions can be obtained from (6-137) , (6-146) , (p=0,q=0,r=0) and the assumption

$(\phi_0/u) = a''(\tau,\sigma),(\psi_0/u) = m''(\tau,\sigma)$,

$(\chi_0/u) = n''(\tau,\sigma)$ --- (6-152 a,b,c)

where a'', m'', n'' are functions of (τ,σ).

$\tau = \tau(x^1, x^2),\ \sigma = \sigma(x^3, x^4)$,

we get, $\phi = a''(\tau,\sigma)$ ---- (6-153a)

$\psi = m''(\tau,\sigma)$ ---- (6-153b)

$\chi = n''(\tau,\sigma)$ ---- (6-153c)

which is the same as (6-56) of article (6-1-2), discussed previously.

Summary of this article 6-3-2 can be given as :

The field equations for Charap's chiral invariant model of the pion dynamics pass the Painleve' test for complete integrability in the sense of Weiss et al. The truncation procedure of the same analysis leads to Auto-Bucklund transformation between two pairs of solutions. With the help of this transformation non-trivial solutions have been rediscovered. However only the transformation between a set of trivial solutions and another set of non-trivial solutions could be demonstrated. The transformation between two sets of non-trivial solutions remains to be demonstrated.

6-3-3 Discussion on Painleve' test for integrability for combined equations

This article gives a discussion of Painleve' analysis of the combined equation (2-10). In article 6-1-3 , we discuss about some of the exact solutions of this combined equation.

In article 6-2-1 we revisited the Painleve' test for the integrability of the Yang's equations for SU(2) gauge fields in the sense of Jimbo, Kruskal and Miwa [49]. Jimbo, Kruskal and Miwa analysed the complex form of the equations with a rather restricted form of singularity manifold. They did not discuss about exact solutions in that context. But, in those article (6-2-1) we analyzed the same equations starting from the real form of the same equations and keeping the singularity manifold completely general in nature. It has been found that the equations, in real form, pass the Painleve' test for integrability. The truncation procedure of the same analysis leads to non-trivial exact solutions obtained previously and Auto-Backlund transformation between two pairs of those solutions.

In article 6-2-2 we demonstrate that the field equations for Charap's chiral invariant model of the pion dynamics [i.e. the equation (2-8)] pass the Painleve' test for complete integrability in the sense of Weiss et al [21a]. The truncation procedure of the same analysis leads to Auto-Backlund transformation between two pairs of solutions. With the help of this transformation non-trivial exact solutions have been rediscovered as obtained previously in article 6-1-2.

In this article we apply the same procedure as done for equation (2-4) and (2-8). While doing the Painleve' test for integrability for the combined equation it appears that some new mathematical behaviors, not known in the literature, are observable in this case. For example, in the leading order analysis two possibilities merge into only one possibility. With the use of this result the system matrix again leads to a determinant having zero value, irrespective of the exponent of the Laurent like expansion used for Painleve' analysis and the value of j in

$$\Phi = u^{\alpha} \sum_{j=0}^{\infty} \Phi_j u^j \qquad \text{--- (6-154)}$$

where $\Phi_0 \neq 0$, $\Phi_j = \Phi_j (z_1 , \ldots, z_n)$ and $u = u(z_1 , z_2 , z_3 , \ldots\ldots , z_n)$ are analytic functions of (z_j) in the neighborhood of the manifold and $u(z_1 , z_2 , z_3 , \ldots\ldots , z_n) = 0$ is the singularity manifold.

To discuss about the Painleve' test for the integrability of the equations (2-10) in the sense of Weiss et al. [21a] we assume that $k' = 1$, $k'' = 1$ in the equation , for the sake of simplicity . Again the results given here remain independent of ε. Or in other words, the observations of this analysis are the same for extended Yang-equations and extended Charap equations.

For the combined equations (2-10) with $k' = 1, k'' = 1$, we define the singularity manifold given by

$$u = u(x^1, x^2, x^3, x^4) = 0 \qquad \text{--- (6-155)}$$

and set

$$\phi = u^{\alpha} \sum_{j=0}^{\infty} \phi_j u^j, \quad \psi = u^{\beta} \sum_{j=0}^{\infty} \psi_j u^j, \quad \chi = u^{\gamma} \sum_{j=0}^{\infty} \chi_j u^j \qquad \text{----- (6-156 a,b,c)}$$

where $\phi(x^1, x^2, x^3, x^4)$, $\psi(x^1, x^2, x^3, x^4)$, $\chi(x^1, x^2, x^3, x^4)$ are a set of solutions of (2-10); ϕ_j, ψ_j, χ_j are all analytic functions of (x^1, x^2, x^3, x^4) in the neighborhood of the manifold (6-155); $\phi_0 \neq 0, \psi_0 \neq 0, \chi_0 \neq 0$.

The test may be divided into three main steps after the substitution of (6-156) in the differential equations concerned, i.e. (2-10) with $k' = 1$ & $k''=1$.

I. Make the leading order analysis [where one gets all possible α, β, γ, ϕ_0, ψ_0 and χ_0 in (2-10)]

II Define the recursion relations for u_j for leading orders obtained in step I and determine the resonance positions (those values of j for which some or all of the relations are not defined).

III. Check whether the expansions allow requisite number of arbitrary functions at the resonance positions.

I. Leading order analysis :

We assume,

$$\phi \sim \phi_0\, u^{\alpha}, \quad \psi \sim \psi_0\, u^{\beta}, \quad \chi \sim \chi_0\, u^{\gamma} \qquad \text{----- (6-157 a,b,c)}$$

We substitute (6-157a,b,c) in (2-10 a,b,c) respectively and equate the coefficients of the negative powers of u (considering that all of α,β and γ are negative).

This leads to $\alpha = \beta = \gamma$

From the leading order terms of equations (2-10 a,b,c) one gets,

$$(\phi_0^2 + \psi_0^2 + \chi_0^2)\,[\,-\alpha\,(2\alpha+1)\,\phi_0^2 + \alpha^2\,\psi_0^2 + \alpha^2\,\chi_0^2\,] = 0 \quad \text{-- (6-158a)}$$

$$\phi_0\,\psi_0\,(\phi_0^2 + \psi_0^2 + \chi_0^2)\,[\,\alpha\,(\alpha-1) - 4\,\alpha^2\,] = 0 \quad \text{--- (6-158b)}$$

$$\phi_0\,\chi_0\,(\phi_0^2 + \psi_0^2 + \chi_0^2)\,[\,\alpha\,(\alpha-1) - 4\,\alpha^2\,] = 0 \quad \text{--- (6-158c)}$$

From (6-158) there appear two possibilities. The first possibility appears to be ,

$$(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0 \qquad \text{--- (6-159)}$$

and the second possibility appears to be the combination of three equations given by,

$$[\,-\alpha\,(2\alpha+1)\,\phi_0^2 + \alpha^2\,\psi_0^2 + \alpha^2\,\chi_0^2\,] = 0 \quad \text{--- (6-160a)}$$

$$[\,\alpha\,(\alpha-1) - 4\,\alpha^2\,] = 0 \quad \text{--- (6-160b)}$$

$$[\,\alpha\,(\alpha-1) - 4\,\alpha^2\,] = 0 \quad \text{--- (6-160c)}$$

It is interesting to note that the equations (6-16 b, c) lead to $\alpha = -1/3$ and if one puts this value (i.e. $\alpha = -1/3$) in the equation (6-160a), one again arrives at

$(\phi_0^2+\psi_0^2+\chi_0^2) = 0$, which is same as (6-159).

So finally we get,

$$\phi = u^{\alpha} \sum_{j=0}^{\infty} \phi_j u^j \;,\quad \psi = u^{\alpha} \sum_{j=0}^{\infty} \psi_j u^j ,\; \chi = u^{\alpha} \sum_{j=0}^{\infty} \chi_j u^j \qquad \text{--- (6-161 a,b,c)}$$

$$\text{with} \quad (\phi_0^2 + \psi_0^2 + \chi_0^2) = 0 \,,\; \alpha = \text{arbitrary} \qquad \text{---- (6-162 d, e)}$$

II. Resonance positions :

If we directly substitute (6-161 a,b,c) in equation (2-10) and then equate the co-efficients of powers of u in the various terms and thereby we observe the behaviour of the expansion co-efficients, then the recursion relation for ϕ_j, ψ_j and χ_j are

$$[T] \begin{bmatrix} \phi_m \\ \psi_m \\ \chi_m \end{bmatrix} = \text{[others terms with } \phi_j, \psi_j, \chi_j \text{ and their derivatives where } j<m] \qquad \text{--- (6-162)}$$

Where [T] is the system matrix.

It is written as,

$$[T] = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \qquad \text{---- (6-163)}$$

where,

$$A_1 = \{ (m+\alpha)(m+\alpha-1) - 6\alpha(m+\alpha) + 3\alpha(\alpha-1) - 6\alpha^2 \} \phi_0^3$$

$$+ \{ (m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha) + \alpha(\alpha-1) \} \phi_0 \psi_0^2$$

$$+ \{ (m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha) + \alpha(\alpha-1) \} \phi_0 \chi_0^2$$

$$B_1 = \{ 2\alpha(m+\alpha) + 2\alpha^2 \} \psi_0^3 + \{ 2\alpha(\alpha-1) - 4\alpha^2 \} \phi_0^2 \psi_0$$

$$+ \{ 2\alpha(m+\alpha) + 2\alpha^2 \} \phi_0 \chi_0^2$$

$$C_1 = \{ 2\alpha(m+\alpha) + 2\alpha^2 \} \chi_0^3 + \{ 2\alpha(\alpha-1) - 4\alpha^2 \} \phi_0^2 \chi_0$$

$$+ \{ 2\alpha(m+\alpha) + 2\alpha^2 \} \psi_0^2 \chi_0$$

$$A_2 = \{\alpha(\alpha-1) - 2\alpha(m+\alpha) - 2\alpha^2\}\psi_0^3 + \{3\alpha(\alpha-1) - 4\alpha(m+\alpha) - 8\alpha^2\}\phi_0^2 \psi_0$$

$$+ \{ \alpha(\alpha-1) - 2\alpha(m+\alpha) - 2\alpha^2 \} \psi_0 \chi_0^2$$

$$B_2 = \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha)\}\phi_0^3$$

$$+ \{(m+\alpha)(m+\alpha-1) - 6\alpha(m+\alpha) + 2\alpha(\alpha-1) - 6\alpha^2\} \phi_0 \psi_0^2$$

$$+ \{ (m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha) \} \phi_0 \chi_0^2$$

$$C_2 = \{ 2\alpha(\alpha-1) - 2\alpha(m+\alpha) - 6\alpha^2 \} \phi_0 \psi_0 \chi_0$$

$$A_3 = \{\alpha(\alpha-1) - 2\alpha(m+\alpha) - 2\alpha^2\}\chi_0^3 + \{3\alpha(\alpha-1) - 4\alpha(m+\alpha) - 8\alpha^2\}\phi_0^2\chi_0$$

$$+ \{\alpha(\alpha-1) - 2\alpha(m+\alpha) - 2\alpha^2\}\psi_0^2\chi_0$$

$$B_3 = \{2\alpha(\alpha-1) - 2\alpha(m+\alpha) - 6\alpha^2\}\phi_0\psi_0\chi_0$$

$$C_3 = \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha)\}\phi_0^3 + \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha)\}\phi_0\psi_0^2$$

$$+\{(m+\alpha)(m+\alpha-1) - 6\alpha(m+\alpha) + 2\alpha(\alpha-1) - 6\alpha^2\}\phi_0\chi_0^2$$

Applying (6-161d).i.e. $(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0$, the system matrix [T] reduces to a simpler form given by :

$$[T] = \begin{bmatrix} G\phi_0^3 & G\phi_0^2\psi_0 & G\phi_0^2\chi_0 \\ G\phi_0^2\psi_0 & G\phi_0\psi_0^2 & G\phi_0\psi_0\chi_0 \\ G\phi_0^2\chi_0 & G\phi_0\psi_0\chi_0 & G\phi_0\chi_0^2 \end{bmatrix}$$

where $G = -[6\alpha^2 + 2\alpha(m+1)]^3$

Obviously, the determinants for the matrix become zero irrespective of value of α and the value of m. Because the determinant takes the form

$$|T| = -[6\alpha^2 + 2\alpha(m+1)]^3\phi_0\psi_0\chi_0 \begin{vmatrix} \phi_0^2 & \phi_0\psi_0 & \phi_0\chi_0 \\ \phi_0^2 & \phi_0\psi_0 & \phi_0\chi_0 \\ \phi_0^2 & \phi_0\psi_0 & \phi_0\chi_0 \end{vmatrix} = 0 \text{ --- (6-164)}$$

Hence, we get no information about the resonance position. Only, we can say that, $m = -1$ is a resonance position corresponding to the arbitrariness of the singularity manifold given by $u=0$ defined in equation (6-155).

Again it appears by virtue of (6-161d) that any two out of ϕ_0 , ψ_0 , χ_0 can be kept arbitrary. That means, there is a possibility of having $m = 0 , 0$. However, this has to be confirmed in the next section.

III. Further investigation for the resonance position:

For m=1 ,Along with $(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0$ in (6-163) we arrive at a situation for the three equation of (2-10a,b,c) with $k' = 1$ and $k'' = 1$ as,

$$A\,\phi_1 + B\,\psi_1 + C\,\chi_1 = f' / \phi_0 \qquad \text{---- (6-165a)}$$

$$A\,\phi_1 + B\,\psi_1 + C\,\chi_1 = f'' / \psi_0 \qquad \text{---- (6-165b)}$$

$$A\,\phi_1 + B\,\psi_1 + C\,\chi_1 = f''' / \chi_0 \qquad \text{--- (6-165c)}$$

where f', f'' & f''' are functions of ϕ_0 , ψ_0 & χ_0 and their derivatives.

And, $$A = [- (6\,\alpha^2 + 4\,\alpha)^3 \;\phi_0]\;\phi_0$$

$$B = [- (6\,\alpha^2 + 4\,\alpha)^3 \;\phi_0]\;\psi_0$$

$$C = [- (6\,\alpha^2 + 4\,\alpha)^3 \;\phi_0]\;\chi_0$$

On simple manipulation we get from (2.11)

$$\frac{f'}{\phi_0} = \frac{f''}{\psi_0} = \frac{f'''}{\chi_0} = D \text{ (say)} \qquad \text{---- (6-166)}$$

where, $D = [1/(u_1^2+u_2^2+u_3^2+u_4^2)]\,[(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4)(2\phi_0^2)$

$+(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3+\psi_{04}u_4)\,(2\alpha\,\phi_0\,\psi_0)$

$+\;(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3+\chi_{04}u_4)\,(2\,\alpha\,\phi_0\,\chi_0)\,]$

Thus, (6-165 a,b,c) become identical and we have the possibility of getting resonance point at $m = (1,1)$. Definitely, at this stage we can say that the resonance point $m = (0,0)$ is confirmed.

<u>For m=2</u> : Along with $(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0$ in (6-163) we arrive a situation for the three equation of (2-10 a,b,c) with $k' = 1$ and $k'' = 1$ given by ,

$$A'\,\phi_2 + B'\,\psi_2 + C'\,\chi_2 = g'/\phi_0 \qquad \text{---- (6-167a)}$$

$$A'\,\phi_2 + B'\,\psi_2 + C'\,\chi_2 = g''/\psi_0 \qquad \text{---- (6-167b)}$$

$$A'\,\phi_2 + B'\,\psi_2 + C'\,\chi_2 = g'''/\chi_0 \qquad \text{---- (6-167c)}$$

where g',g'' & g''' are function of ϕ_1 , ψ_1 , χ_1 , ϕ_0 , ψ_0 & χ_0 and their derivatives.

And, $A' = [-(6\,\alpha^2 + 6\,\alpha)\,\phi_0]\;\phi_0$

$B' = [-(6\,\alpha^2 + 6\,\alpha)\,\phi_0]\;\psi_0$

$C' = [-(6\,\alpha^2 + 6\,\alpha)\,\phi_0]\;\chi_0$

where

$$g'/\phi_0 = [1/(u_1^2+u_2^2+u_3^2+u_4^2)]\ [(\phi_0\, p+\psi_0 q+\chi_0 r)(-2\alpha\phi_0^2)(u_{11}+u_{22}+u_{33}+u_{44})$$

$$+2\alpha\phi_0^3(u_1p_1+u_2\,p_2+u_3\,p_3+u_4\,p_4)+2\alpha\phi_0^2\,\psi_0\,(u_1q_1+u_2\,q_2+u_3\,q_3+u_4\,q_4)$$

$$+2\alpha\phi_0^2\,\chi_0\,(u_1r_1+u_2\,r_2+u_3\,r_3+u_4\,r_4)$$

$$+\{(2\alpha-1)\phi_0^2\,p+(4\alpha+2)\phi_0\,(\phi_0\,p+\psi_0 q+\chi_0\,r)\}(\phi_{01}u_1+\phi_{02}\,u_2+\phi_{03}u_3+\phi_{04}\,u_4)$$

$$+\{2\phi_0\,\psi_0\,p-4\alpha\psi_0\,(\phi_0\,p+\psi_0 q+\chi_0\,r)\}(\psi_{01}u_1+\psi_{02}\,u_2+\psi_{03}\,u_3+\psi_{04}\,u_4)$$

$$+\{2\alpha\phi_0\,(2\chi_0\,p+\phi_0\,r)+2\phi_0\chi_0\,p-4\alpha\chi_0\,(\phi_0\,p+\psi_0 q+\chi_0 r)\}(\chi_{01}u_1\chi_{02}\,u_2+\chi_{03}\,u_3+\chi_{04}\,u_4)$$

$$+2\,\phi_0^2(\phi_{01}^{\ 2}+\phi_{02}^{\ 2}+\phi_{03}^{\ 2}+\phi_{04}^{\ 2})+2\phi_0\psi_0\,(\phi_{01}\psi_{01}+\phi_{02}\,\psi_{02}+\phi_{03}\,\psi_{03}+\phi_{04}\,\psi_{04})$$

$$+2\phi_0\,\chi_0\,(\phi_{01}\chi_{01}+\phi_{02}\,\chi_{02}+\phi_{03}\,\chi_{03}+\phi_{04}\,\chi_{04})\ ]$$

$$g''/\psi_0=[1/(u_1^2+u_2^2+u_3^2+u_4^2)]\ [(\phi_0 p+\psi_0 q+\chi_0\,r)(-2\alpha\phi_0\psi_0)\,(u_{11}+u_{22}+u_{33}+u_{44})$$

$$+2\alpha\phi_0^2\psi_0(u_1p_1+u_2\,p_2+u_3\,p_3+u_4\,p_4)+2\alpha\phi_0\psi_0^2\,(u_1q_1+u_2q_2+u_3q_3+u_4\,q_4)$$

$$+2\alpha\phi_0\,\psi_0\,\chi_0\,(u_1r_1+u_2\,r_2+u_3\,r_3+u_4\,r_4)$$

$$+\{4\alpha\psi_0(\phi_0\,p+\psi_0 q+\chi_0\,r)+4\alpha\phi_0\,\psi_0\,p+2(\alpha+1)\phi_0^2 q\}(\phi_{01}u_1+\phi_{02}\,u_2+\phi_{03}\,u_3+\phi_{04}\,u_4)$$

$$+\{(2\alpha+2)\phi_0^2 p-2\alpha\chi_0^2 p+4\alpha\phi_0\,\psi_0 q+(4\alpha+2)\phi_0\chi_0\,r)\}(\psi_{01}u_1+\psi_{02}\,u_2+\psi_{03}u_3+\psi_{04}\,u_4)$$

$$+\{2\alpha\,\psi_0\,\chi_0\,p+2\,\alpha\,\phi_0\,\chi_0 q\ +2(\alpha+1)\,\phi_0\psi_0\,r\ \}(\chi_{01}u_1+\chi_{02}\,u_2+\chi_{03}\,u_3+\chi_{04}\,u_4)$$

$$+2\phi_0^2\,(\phi_{01}\psi_{01}+\phi_{02}\,\psi_{02}+\phi_{03}\,\psi_{03}+\phi_{04}\,\psi_{04})+2\phi_0\psi_0\,(\psi_{01}^{\ 2}+\psi_{02}^{\ 2}+\psi_{03}^{\ 2}+\psi_{04}^{\ 2})$$

$$+2\phi_0\,\chi_0\,(\phi_{01}\chi_{01}+\phi_{02}\,\chi_{02}+\phi_{03}\,\chi_{03}+\phi_{04}\,\chi_{04})\ ]$$

$$g'''/\chi_0 = [1/(u_1^2+u_2^2+u_3^2+u_4^2)]\,[(\phi_0 p+\psi_0 q+\chi_0\, r)(-2\alpha\phi_0\chi_0)(u_{11}+u_{22}+u_{33}+u_{44})$$

$$+2\alpha\phi_0^2\chi_0(u_1p_1+u_2\,p_2+u_3\,p_3+u_4p_4)+2\alpha\phi_0\psi_0\chi_0\,(u_1q_1+u_2q_2+u_3q_3+u_4\,q_4)$$

$$+\,2\alpha\phi_0\chi_0^2\,(u_1r_1+u_2\,r_2+u_3\,r_3+u_4\,r_4)$$

$$+\{4\alpha\chi_0\,(\phi_0\,p+\psi_0 q+\chi_0\,r)+4\alpha\phi_0\chi_0\,p+2(\alpha+1)^2\phi_0^2\,r\}(\phi_{01}u_1+\phi_{02}\,u_2+\phi_{03}\,u_3+\phi_{04}\,u_4)$$

$$+\{\,2\alpha\,\psi_0\chi_0\,p+2\alpha\,\phi_0\chi_0\,q+2(\alpha+1)\phi_0\,\psi_0\,r\,\}(\psi_{01}u_1+\psi_{02}\,u_2+\psi_{03}\,u_3+\psi_{04}\,u_4)$$

$$+\{2\alpha p(2\phi_0^2+\chi_0^2)+2\phi_0^2p+4\alpha\phi_0\,\chi_0 r+(2\alpha+4)\phi_0\psi_0\,q\}(\chi_{01}u_1+\chi_{02}\,u_2+\chi_{03}\,u_3+\chi_{04}\,u_4)$$

$$+2\phi_0^2(\phi_{01}\chi_{01}+\phi_{02}\chi_{02}+\phi_{03}\chi_{03}+\phi_{04}\chi_{04})+2\phi_0\psi_0(\psi_{01}\chi_{01}+\psi_{02}\,\chi_{02}+\psi_{03}\chi_{03}+\psi_{04}\,\chi_{04})$$

$$+\,2\,\phi_0\,\chi_0\,(\chi_{01}^{\;2}+\chi_{02}^{\;2}+\chi_{03}^{\;2}+\chi_{04}^{\;2})\,]$$

So, here we see that,

$$\frac{g'}{\phi_0} \neq \frac{g''}{\psi_0} \neq \frac{g'''}{\chi_0}$$

However, the left hand sides of (6-167) are identical hence one should forcefully have

$$\frac{g'}{\phi_0} = \frac{g''}{\psi_0} = \frac{g'''}{\chi_0} \qquad \text{--- (6-168)}$$

which generates two equations in ϕ_1 , ψ_1 , χ_1, ϕ_0 , ψ_0 , χ_0 and their derivatives. So finally one has three distinct equations in ϕ_1 , ψ_1 and χ_1 – one is given by (6-167) and the other two are given by (6-168). Thus, none of ϕ_1 , ψ_1 , χ_1 can be kept arbitrary.

Thus , there cannot be any resonance at m = 1.

Thus even at this stage we do not have any conclusive inference regarding leading order, resonance position and existence of requisite number of arbitrary functions. In this way one has to proceed indefinitely. For the same reason it has not been possible to truncate the series and to obtain Auto-Buckland transformation which we discussed in previous article 6-3-1 and 6-3-2 for the Yang equations (2-4) and Charap equations (2-8).

The article 6-3-3 may be summarized as follows :

The combined equations (2-10) come from the combination of Yang SU(2) gauge field equation (2-4) and Charap equation of pion dynamics (2-8). These two parent equation pass the Painleve' test in the sense that number of arbitrary functions in the Laurent like expansion (6-154) is equal to the number of resonance points.

But for the Combined equations given by (2-10 a,b,c), of course, with $k' = 1$, $k'' = 1$,we arrive at peculiar situations regarding the Painleve' test (in the sense Weiss et al). They do not allow none of the stages (leading order analysis, resonance calculation and checking of the existence of requisite number of arbitrary functions) to be conclusive.

6-3-4 Discussion of Chaos in the perspective of Painleve′ test and Graphical representation of some exact solutions

The observations of the Painleve′ test and Graphical representations of exact solutions for the Yang equations (2-4), Charap equations (2-8) and the combined equations (2-10) as in the previous articles are given below:

Characteristics / Equations	Leading order Analysis	Resonance Calculation	Existence of number of arbitrary functions	Graphical representation of exact solution
	I	II	III	IV
Yang [equation 2-4]	one arrives at $\phi_0^2+\psi_0^2+\chi_0^2=0$ (*)	Clear result	Clear existence	Initial solitary profile which tends to vanish as time tends to infinity and spreading wave packet
Extended Yang [equation 2-10 with $\varepsilon=1, k'=1, k''=1$]	one arrives at $\phi_0^2+\psi_0^2+\chi_0^2=0$ (*)	Clear result	Clear existence	Initial solitary profile which tends to vanish as time tends to infinity and spreading wave packet
Charap [equation 2-8]	Clear result	Clear result	Clear existence	Existence of solitonic solution with oscillatory profile
Extended Charap [equation 2-10 with $\varepsilon=-1, k'=1, k''=1$]	one arrives at $\phi_0^2+\psi_0^2+\chi_0^2=0$ (*)	Not Conclusive	Not Conclusive	Solitary profile (for ϕ) and wave packet (for ψ and χ) that pass through stages of plane wave and zero value periodically & abruptly

(*) This is valid only when ϕ_0, ψ_0, χ_0 are complex quantities.
One cannot execute the truncation procedure with this result.

From the above we can add to the observation of several other authors (see for example ref [62]) that there is

i) a correlation between the existence of Painleve' property and the absence of Chaotic behaviour, and

ii) a correlation between the absence of Painleve' property and presence of Chaotic behaviour.

Normally the existence of Painleve′ property is defined by the presence of requisite number of arbitrary function in the Laurent-like expansion (1.1) at the resonance points. Our observation is that this does not assume one about the existence of physical behaviour free from chaos or specifically the existence of solitonic behaviour. In this context we like to introduce a conjecture which is given as follows :

> The basic requirement for the existence of the regular behaviour as stated above is the existence of requisite number of arbitrary functions at the resonance points in the Laurent like expansion used for the Painleve′ test.
>
> In addition to that the Laurent like expansion must be well behaved at all the stages – namely, leading order analysis, resonance calculation and checking of the existence of the requisite number of arbitrary functions.