Chapter V

Review of previous work related to the present study

5-1 Review of Exact solutions

In the following we review some of the previous work regarding exact solutions of Yang equations (2-4) and Charap equations (2-8). We proposed an equation on combination of these two equations. Exact solutions for these equations have been presented in Chapter – VI (article – 6-1-3).

5-1-1 Review of the exact solutions for the Yang equations (2-4)

Before 1970 self dual solutions were known in a different form (see ref. [57]). Still then an attempt to obtain self- dual solutions using Yang's formalism has an added interest because of the simplicity and the straight forwardness of Yang's [26] formalism. The set of equations (2-4) are important from the mathematical point of view too.

Some particular solutions of equations (2-4 a,b) were given by Yang himself. Yang [26] indicated the existence of a class of solutions for equations (2-4) that satisfies

$$\rho_y = \phi_{\bar{z}} \; , \rho_z = -\phi_{\bar{y}} \; , \bar{\rho}_{\bar{y}} = \phi_z \; , \bar{\rho}_{\bar{z}} = -\phi_y \; , \phi_{y\bar{y}} + \phi_{z\bar{z}} = 0 \qquad \text{--- (5-1)}$$

Ray[35] generalized the solutions of Yang equations with the ansatz,

$$\rho_y = \psi_z \; , \rho_z = -\psi_{\bar{y}} \qquad \text{--- (5-2)}$$

where ψ is any complex function.

With (5-2) Ray [35] arrived at a new set of solutions for (2-4) given by,

$$\rho = \text{constant} \qquad \text{--- (5-3)}$$

and $$\phi = e^{\nu} \qquad \text{--- (5-4)}$$

where, ν is any real solution of $(\nu_{y\bar{y}} + \nu_{z\bar{z}}) = 0$ --- (5-5)

Again, in the same paper and (2-4) with the same ansatz as (5-2) Ray [35] gave a procedure to obtain a different solutions from the old ones. The procedure is as follows :

If (ρ_0 , ϕ_0) is any solution of Yang equations (2-4) that satisfies Yang's ansatz, i.e. $\rho_{oy} = \phi_{o\bar{z}} \; , \rho_{oz} = -\phi_{o\bar{y}}$ --- (5-6 a,b)

Where ρ_0 is complex and ϕ is real, then ρ and ϕ are given by,

$$\rho = k\,\rho_0 \qquad \text{--- (5-7)}$$

$$\phi = \pm \phi_0$$

$$= \pm a^{-1} \operatorname{Sin}(a\phi_0 + b) \qquad \text{--- (5-8)}$$

$$= \pm a^{-1} \operatorname{Sinh}(a\phi_0 + b)$$

Here, a & b are real constants and k is a complex constant with $k\bar{k} = 1$, are solutions of Yang equations (2-4).

In an another publication, De and Ray [41] found two different classes of solutions of Yang equations (2-4).

In case – I, they assumed, $\rho = \rho(\phi)$.

As $\rho = \psi + i\chi$ this means

$\psi = \psi(\phi)$, $\chi = \chi(\phi)$ --- (5-9)

where ϕ is not a constant and ψ & χ are not both constants.

Using (5-9) in Yang equations (2-4) and with a little calculation it can be shown that,

$\psi = A\gamma$ ----- (5-10)

and $\chi = B\gamma$ ---- (5-11)

where A and B are constants and γ is a function of ϕ.

With (5-10) and (5-11), one arrives after a simplification that,

$\phi = [\,C\,(A^2 + B^2)^{1/2}\,]^{-1}$ Sech V ---- (5-12a)

$\nu = [\,C\,(A^2 + B^2)\,]^{-1}\,[\,1 - C^2\,(A^2 + B^2)\,\phi^2\,]^{1/2}$ ---- (5-12b)

where C is a constant and V is a function satisfies

$V_{11} + V_{22} + V_{33} + V_{44} = 0$ --- (5-13)

Hence (5-12), (5-10), (5-11) respectively give the solutions for ϕ, ψ and χ which are the solutions of Yang equations.

<u>In case- II</u> the assumption was ,

$$\phi\,(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44}) = (\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) \quad \text{--- (5-14)}$$

Use of this equation in Yang equations , one arrives,

$$\psi_1 + \chi_2 = 0\,,\ \psi_2 - \chi_1 = 0,\ \psi_3 - \chi_4 = 0,\ \psi_4 + \chi_3 = 0 \quad \text{--- (5-15)}$$

On integration of (5-14) , one can have

$$\phi = e^V \quad \text{--- (5-16)}$$

where V satisfies (5-13).

The equation (5-15) may be expressed as ,

$$\psi = U(x^1, x^2) + F(x^3, x^4) \quad \text{--- (5-17)}$$

$$\chi = V(x^1, x^2) + G(x^3, x^4) \quad \text{--- (5-18)}$$

where,

$$\partial U(x^1,x^2)/\partial x^1 + \partial V(x^1, x^2)/\partial x^2 = 0$$

$$\partial U(x^1,x^2)/\partial x^2 - \partial V(x^1, x^2)/\partial x^1 = 0$$

$$\partial F(x^3,x^4)/\partial x^3 - \partial G(x^3, x^4)/\partial x^4 = 0$$

$$\partial F(x^3,x^4)/\partial x^4 + \partial G(x^3, x^4)/\partial x^3 = 0$$

Thus in Case – II , equations (5-16), (5-17) and (5-18) respectively give ϕ, ψ and χ , which are the solutions of Yang equations.

In another publication Chanda and Ray [58] further generalized the solutions obtained by Yang. These solutions included some particular cases of their generalizations reported by De & Ray [41] and Ray [32] as discussed earlier.

Chanda and Ray [58] assumed that,

$$\phi = \phi (\tau) \qquad \text{---- (5-19a)}$$

$$\psi = a \xi - b \eta + f \qquad \text{--- (5-19b)}$$

$$\chi = a \eta + b \xi + g \qquad \text{--- (5-19c)}$$

where a and b are real arbitrary constants; ξ , η and τ are functions of (x^1,x^2,x^3,x^4); f,g and ϕ are functions of τ. Here ξ , η and τ satisfy,

$$\xi_1 + \eta_2 = \tau_3 \qquad \text{--- (5.20a)}$$

$$\xi_2 - \eta_1 = \tau_4 \qquad \text{--- (5.20b)}$$

$$\xi_3 - \eta_4 = - \tau_1 \qquad \text{--- (5.20c)}$$

$$\xi_4 + \eta_3 = - \tau_2 \qquad \text{--- (5.20d)}$$

Equations (5-20) represent the ansatz used by Yang [26].

Using (5-19) in the Yang equation (2-4) and on integration, one have,

$$f = c \int \phi^2 \, d\tau + p \qquad \text{--- (5-21a)}$$

$$g = d \int \phi^2 \, d\tau + q \qquad \text{--- (5-21b)}$$

$$\tau = \pm \int [- (c^2 + d^2) \phi^4 + g \phi^2 + (a^2 + b^2)]^{-½} d\phi + h \qquad \text{---- (5-22)}$$

where c ,d ,p ,q , g and h are real arbitrary integration constants.

Using (5-21) and (5-22) Chanda and Ray [58] arrived at the results given below.

In contrast with Yang's solutions for Yang equations (2-4) given by,

$$\rho = \sigma \qquad \text{--- (5-23a)}$$

$$\phi = \tau \qquad \text{--- (5-23b)}$$

such that,

$\sigma_y = \tau_{\bar{z}}$, $\sigma_z = -\tau_{\bar{y}}$

Here they [58] had the following results :

$$\rho = U\sigma + V(c^2 + d^2)^{-½} (p^2+q^2)^{-½} [(p^2+q^2) E(u) - p^2 u] + W \quad \text{--- (5-24)}$$

where, $E = \int dn^2 u \, du$ = Jacobi's function (details in ref. [59])

$$u = (c^2 + d^2)^{½} (p^2+q^2)^{½} (\tau - H)$$

$$U = a + ib, \; V = c + id, \; W = p + iq$$

where a, b,c, d, p, q are all real arbitrary constants, and p & q are given as,

$$p^2 = \frac{-g + [g^2 + 4(a^2 + b^2)(c^2 + d^2)]^{½}}{2(c^2 + d^2)}$$

$$q^2 = \frac{g + [g^2 + 4(a^2 + b^2)(c^2 + d^2)]^{½}}{2(c^2 + d^2)}$$

and ϕ corresponding to (5-24) is given by with $U \neq 0$, $V=0$ as,

$$\phi = \pm (a^2 + b^2)^{½} (\tau - h) \quad \text{--- (5-25 a)}$$

$$\phi = \pm r^{-1}(a^2 + b^2)^{½} \, \text{Sin} (r\tau - rh) \text{ where } r < 0 \quad \text{--- (5-25 b)}$$

$$\phi = \pm r^{-1}(a^2 + b^2)^{½} \, \text{Sinh} (r\tau - rh) \text{ where } r > 0 \quad \text{--- (5-25 c)}$$

and when $U \neq 0$, $V \neq 0$,

$$\phi = q \, \text{Cn}(u) \quad \text{--- (5-26)}$$

The situation U = 0 may be observed as a special case of the work of De and Ray [41] as described earlier. In 6-1-1, we present some new solutions of Yang equations.

5-1-2 Review of the exact solutions for the Charap equations (2-8)

Charap [29] himself suggested some exact solutions of the classical non-linear field equations for the chiral invariant model of pion dynamics(2-8). He [29] obtained solutions of (2-8) under the assumption that ϕ,ψ and χ are all functions of $(K_1 x^1 + K_2 x^2 + K_3 x^3 + K_4 x^4)$ where K_μ is any arbitrary constant-four-vector as stated above. But such solutions give non-vanishing derivatives of ϕ,ψ and χ at infinity and hence cannot represent soliton solutions.

Ray [32] integrated the equations (2-8) under a weaker assumption that there exists some function u, such that ϕ,ψ and χ are functions of u i.e.

$$\phi = \phi(u) \qquad \text{--- (5-27a)}$$

$$\psi = \psi(u) \qquad \text{--- (5-27b)}$$

$$\chi = \chi(u) \qquad \text{--- (5-27c)}$$

$$u = u(x^1, x^2, x^3, x^4) \qquad \text{--- (5-27d)}$$

Ray [32] demonstrated that these solutions include the soliton solution as a special case. We rediscovered and afterwards offered graphical representation of these solutions. Details have been presented in Chapter – VI (article : 6-1-3 and 6-2).

In this publication Ray [32] presented another type of solutions of the equations, by assuming,

$$\phi = \phi\,(x^1, x^2, x^3 - x^4) \qquad \text{--- (5-28a)}$$

$$\psi = \psi\,(x^1, x^2, x^3 - x^4) \qquad \text{--- (5-28b)}$$

$$\chi = \chi\,(x^1, x^2, x^3 - x^4) \qquad \text{--- (5-28c)}$$

When one uses (5-27) in the Charap equation in the form (2-5) , the later reduces to,

$$\phi_{11} + \phi_{22} = \beta_1\ \phi_1 + \beta_2\ \phi_2 \qquad \text{---- (5-29a)}$$

$$\psi_{11} + \psi_{22} = \beta_1\ \psi_1 + \beta_2\ \psi_2 \qquad \text{---- (5-29b)}$$

$$\chi_{11} + \chi_{22} = \beta_1\ \chi_1 + \beta_2\ \chi_2 \qquad \text{---- (5-29c)}$$

where, $\beta = \ln\,(\, f_\pi^{\,2} + \phi^{\,2} + \psi^2 + \chi^2\,)$ ---- (5-29d)

f_π = Constant ---- (5-29e)

Though Ray [32] failed to solve (5-29) completely he presented some particular solutions of it.

For the first type of solution he assumed,

$$\phi = \phi\,(\,\alpha,\ x^3 - x^4\,) \qquad \text{--- (5-30a)}$$

$$\psi = \psi\,(\,\alpha\,, x^3 - x^4) \qquad \text{--- (5-30b)}$$

$$\chi = \chi\,(\,\alpha\,,\ x^3 - x^4) \qquad \text{--- (5-30c)}$$

where, $\alpha = \alpha\,(x^1, x^2, x^3 - x^4)$.

Proceeding similarly as (5-27) , he arrived at,

$$\phi = g\alpha + h\,,\ \psi = i\alpha + j\,,\ \chi = k\alpha + m \qquad \text{--- (5-31a,b,c)}$$

where g, h, i, j, k and m are functions of $(x^3 - x^4)$ and α is given as,

$$\xi = \int d\alpha \, / \, [\, (g^2 + i^2 + k^2)\, \alpha^2 + 2\alpha(gh + ij + km) + (h^2 + j^2 + m^2 + f_\pi^2)] \quad \text{--(5-31d)}$$

where ξ satisfies, $\xi_{11} + \xi_{22} = 0$.

In second case, he assumed that,

$$\phi = \gamma\,(x^2, x^3 - x^4)\, p\,(x^1, x^3 - x^4) \quad \text{--- (5-32a)}$$

$$\psi = \gamma\,(x^2, x^3 - x^4)\, q\,(x^1, x^3 - x^4) \quad \text{--- (5-32b)}$$

$$\chi = \gamma\,(x^2, x^3 - x^4)\, r\,(x^1, x^3 - x^4) \quad \text{--- (5-32c)}$$

such that $(p^2 + q^2 + r^2) = 1$ and arrived at

$$p = A \; \text{Cos}\, nx^1 \; + \; B \; \text{Sin}\, nx^1 \quad \text{--- (5-33a)}$$

$$q = C \; \text{Cos}\, nx^1 \; + \; D \; \text{Sin}\, nx^1 \quad \text{--- (5-33b)}$$

$$r = E \; \text{Cos}\, nx^1 \; + \; F \; \text{Sin}\, nx^1 \quad \text{--- (5-33c)}$$

where n, A,B,C,D,E and F are functions of $(x^3 - x^4)$ satisfying

$$A^2 + C^2 + E^2 = 1$$

$$B^2 + D^2 + F^2 = 1$$

$$AB + CD + EF = 0 \text{ and}$$

$$\int d\gamma \, / [\exp\{2/(f_\pi^2 + \gamma^2)\} \, \{H + 2n^2 \int \gamma \exp(-4/(f_\pi^2 + \gamma^2))\, d\gamma\}^{1/2}\,]$$

$$= \pm\, x^2 + 1$$

Here, H and I are also arbitrary functions of $(x^3 - x^4)$.

In another publication Chanda, De and Ray [30] further generalized the solutions under the assumptions (5-28) and (5-29) . They classified the solutions under two broad cases.

Case – I :

Here assumption was

$$\beta = \beta(\phi) \qquad \text{----- (5-34a)}$$

where β is given in equation (5-29d).

The solutions are given by,

$$\psi = \alpha \,\mathrm{Cos}\,\theta \qquad \text{--- (5-34b)}$$

$$\chi = \alpha \,\mathrm{Sin}\,\theta \qquad \text{--- (5-34c)}$$

where $\theta = \theta(X,Y)$, X and Y are the solution of Laplace's equations

$$X_{11} + X_{22} = 0,\ Y_{11} + Y_{22} = 0 \qquad \text{--- (5-35a,b)}$$

Such that $X_1 = Y_2$ and $X_2 = -Y_1$ --- (5-36a,b)

i.e., Y is the conjugate solution to X,

$$\theta = \int (Ae^{\beta} / \alpha^2)\, dX + BY + C \qquad \text{--- (5-37)}$$

where B and C are constants,

$$X = \int e^{-\beta}\, d\phi \qquad \text{--- (5-38)}$$

$$\phi_X = f_\pi^2 + \phi^2 + \alpha^2 \qquad \text{--- (5-39)}$$

and $\alpha_{\phi\phi} = (A^2 / \alpha^3) + [B^2\alpha / (f_\pi^2 + \phi^2 + \alpha^2)^2]$ --- (5-40)

Equations (5-39) and (5-40) give ϕ and α, in principle, in terms of X. From (5-38) β is known, in principle, in terms of X.

Then θ is known, in principle, in terms of X and Y via (5-37). And ψ and χ are given, in principle, in terms of X via (5-34a,b).

<u>Case – I I</u>

Here the assumption was,

$$\psi^2 + \chi^2 = (f_\pi^2 + \phi^2)\,\xi^2(\sigma) \qquad \text{--- (5-41)}$$

where $\xi(\sigma)$ = some unspecified function of σ.

Then, it turns out that,

$$\psi = (f_\pi^2 + \phi^2)^{1/2}\,\xi(\sigma)\,\mathrm{Cos}\,\Theta \qquad \text{---- (5-42a)}$$

$$\chi = (f_\pi^2 + \phi^2)^{1/2}\,\xi(\sigma)\,\mathrm{Sin}\,\Theta \qquad \text{---- (5-42b)}$$

$$\Theta = \Theta(\sigma, \phi) \qquad \text{--- (5-42c)}$$

$$\beta = \ln(f_\pi^2 + \phi^2) + \ln(1 + \xi^2) \qquad \text{--- (5-42d)}$$

With a rigorous calculation they arrived at,

$$\phi = f_\pi \tan(q\, f_\pi + k\, f_\pi) \qquad \text{--- (5-43a)}$$

where, k is a constant and q is a solution of Laplace's equation,

$$q_{11} + q_{22} = 0 \qquad \text{--- (5-43b)}$$

For ψ and χ, σ may be expressed as,

$$\sigma = (1/2)\int dz / [HZ(1+Z)^4 - E^2(1+Z)^4 - (F^2 - f_\pi^2)/(1+Z^3)\,Z]^{1/2} + \text{constant} \quad \text{---(5-44a)}$$

$$\Theta = E\int [\,(1+\xi^2)^2 / \xi^2\,]\,d\sigma + (F/f_\pi)\tan^{-1}(\phi/f_\pi) + G \qquad \text{--- (5-44b)}$$

where $Z = \xi^2$ and E,F,G,H are constants.

Chanda and Ray [58] demonstrated that

$$\int (1 + \xi^2)\, d\sigma = \gamma \qquad \text{--- (5-44c)}$$

where q and γ are mutually conjugate solution of Laplace's equation,

i.e. $q_1 = \gamma_2$, $q_2 = -\gamma_1$,

Specifically, $\gamma_{11} + \gamma_{22} = 0$ --- (5-44d)

Using (5-44) in (5-42 a,b) we have ψ and χ.

Thus three unknown quantities ϕ, ψ and χ for Charap equation (2-8) can be expressed with the help of (5-43), (5-42a) and (5-42b) respectively. Where q is given by (5-43b). σ and Θ are given by (5-44).

5-2 Review of Painleve' test

Before going into the review it may be stated that ***no previous work on Painleve' test for Charap equations (2-8) is known.*** *The combined Yang-Charap (Y-C) equations (2-10) have been proposed by us.* ***The Painleve' test for the Combined equations has been presented in Chapter VI (article – 6-1-3).***

5-2-1 Review of the Painleve' test for the Yang equations (2-4)

Jimbo, Kruskal and Miwa [49] adopted the algorithm of Weiss, Tabor and Carnevale [21a] and showed that the Yang equations (2-4) pass the Painleve' test for integrability . Ward [49] used a completely different approach, complicated indeed, and arrived at the same conclusion. Both the investigations [46,49] used the complex form of the equations and neither of them reported any solutions obtainable from the analysis. In the following we discuss the basic algorithm used by Weiss et. al. [21a].

In equation (2-4) they put,

$$\phi = \sum_{j=0}^{\infty} \phi_j u^{j-m}, \rho = \sum_{j=0}^{\infty} \rho_j u^{j-m}, \bar{\rho} = \sum_{j=0}^{\infty} \bar{\rho}_j u^{j-m} \qquad \text{--- (5-45)}$$

where, $u = u(y, \bar{y}, z, \bar{z})$, $m \geq 1$

Upon substitution, they obtain the leading terms as,

$$(\phi_0{}^2 + m \rho_0 \bar{\rho}_0) \eta = 0, (m-1) \phi_0 \rho_0 \eta = 0, (m-1) \phi_0 \bar{\rho} \eta = 0$$

with $\eta = u_y u_{\bar{y}} + u_z u_{\bar{z}}$

Hereafter they assume that $\eta \neq 0$ so that the variety $\{u = 0\}$ is not characteristic.

Then they have $m = 1$, and

$(\phi_0^2 + \rho_0 . \bar{\rho}_0) = 0$

The other possibility, $\phi_0 = 0$, $\rho_0 \bar{\rho}_0 = 0$

is ruled out by checking then leading terms of (5-45).

At this stage u , ρ_0 and $\bar{\rho}_0$ are arbitrary.

Equating the coefficients of u^{j-4} ($j \geq 0$) they obtain the recursion relation for ϕ_n ,ρ, $\bar{\rho}$ given by

$$\begin{vmatrix} (j^2-j+2)\phi_0\eta & -(n-\bar{1})\rho_0\eta & -(n-1)\rho_0\eta \\ 2j\rho_0\eta & n(n-1)\phi_0\eta & 0 \\ 2j\bar{\rho}_0\eta & 0 & n(n-1)\phi_0\eta \end{vmatrix} \begin{vmatrix} \phi_j \\ \rho_j \\ \bar{\rho}_j \end{vmatrix} = \begin{vmatrix} A_j \\ B_j \\ C_j \end{vmatrix}$$

where A_j, B_j and B_j are given in terms of ϕ_n , ρ_n and ρ_n

$(0 \leq n < j-1)$.

They [49] shown that the self duality equations (5-45) allow solutions of the form (5-45) with six arbitrary functions u, ρ_0, $\bar{\rho}_0$, ρ_1 , $\bar{\rho}_1$ and ϕ_2 , corresponding to the roots, (-1) , (0, 0), (1 , 1) and (2) respectively.

In 6-3-1 we revisit the Painleve' test of this Yang equations in a different approach.