Chapter 3

Relativistic Sagnac effect and Ehrenfest Paradox

3.1 Introduction

In the last chapter, where we have reviewed some aspects of Sagnac effect we have shown that two relativistic formulae for Sagnac effect exists in the literature involving the appearance of the relativistic γ factor in the formula. The present day precision in measurement of Sagnac effect is unable to decide between the two formulae and the issue may remain inconclusive for years to come. Nevertheless from the theoretical standpoint the question cannot be lightly dismissed. In this chapter we shall see that the issue is intimately connected with the Ehrenfest paradox concerning the spatial geometry of a rotating disc.

3.2 Ehrenfest Paradox

Rotation has always been a subject of fascination by its own for the relativist. It is the subject of vast and growing literature. Sometimes it has also been some source of misunderstanding leading to conceptual problems. Ehrenfest paradox is one of them. The paradox can be described as follows:

When a body moves, it is contracted along its direction of motion by a factor of γ, the Lorentz contraction factor. There will be no contraction along the directions perpendicular to the direction of motion.

Now let us consider a disc rotating with a velocity $v = \omega R$, ω and R being its angular velocity and radius respectively. The circumference of the disc (when at rest in an inertial frame) is $2\pi R$, according to Euclidean geometry. What will happen to the disc when it is rotating? Let us discuss the issue from the point of view of an observer in the inertial frame. At any point on the circumference, the linear velocity of the disc $v = \omega R$ is tangential. Thus the radius of the disc is always perpendicular to the velocity. So, according to SR, There will be no change of the radial length of the disc.

Any elementary circumferential length segment of the disc is parallel to the linear velocity, and thus will suffer Lorentz-Fitzerald contraction. The contracted length then will be $2\pi R\gamma^{-1}$. This means that circumference to diameter ratio of a disc at rest will not be maintained for a disc under rotation, violating Euclidean geometry in an *inertial frame*! Essentially this is known as Ehrenfest Paradox (EP) [1].

Paul Ehrenfest was the first to identify this problem. His intention was to understand Max Born's notion of relativistic rigidity [2]. Though generally this is called 'rotating disc problem', originally Ehrenfest proposed it for a *relativistically rigid* rotating cylinder.[1] This problem was termed by Varićak [3] as Ehrenfest Paradox(EP).

Grøn [4] is of the view that EP *motivated* Einstein *to search for a geometrical theory of gravity*. Einstein never participated in the EP debate [4] yet he proposed a resolution of EP in a letter to J. Petzold in 1919 (quoted by J. Stachel [5]).

> *A rigid circular disk at rest must break up if it is set into rotation, on account of the Lorentz contraction of the tangential fibres and the non-contraction of the radial ones. Similarly, a rigid disc in rotation (produced by casting) must explode as consequence of the inverse changes in length, if one attempts to put it at rest.*

A detail discussion of rotating disc problem and its historical development is given by Ø. Grøn in Ref. [4].

[1] Born's definition of relativistic rigidity concerns a motion where the proper length of the moving body is constant. In a latter section of this chapter we comment on this issue and its relation to EP.

3.3 Sagnac effect and Ehrenfest Paradox

In Sec. 2.5.2 we have given a relativistic analysis of Sagnac effect and mentioned that there is a mild controversy as to what is the correct relativistic formula for Sagnac effect. The formula can written as (2.5.28)

$$\delta\tau = \frac{4\pi\omega r^2}{c^2}\Gamma. \tag{3.3.1}$$

There are two claims about the value of Γ depending on whether one considers length contraction of the periphery of the disc or not. These are,

1. $\Gamma = 1$

2. $\Gamma = \gamma$

respectively, γ being the Lorentz factor. Selleri [6] and Goy [7] support the first claim pointing out the discrepancy ($\Gamma = \gamma$ is generally believed to be the *true relativistic formula* [8]). The present day precision in measurement of the Sagnac effect may be unable to decide between the two formulae, nevertheless from the theoretical and pedagogical standpoint the question cannot be ignored[2]. We shall see below that the issue is intimately connected with the EP. However for this moment we refrain from discussing the paradox any further or from giving any verdict right away regarding the discrepancy over the correct relativistic formula for Sagnac effect. In order to appreciate the question, the real physics behind the

[2]To appreciate a current perspective of this pedagogical question consider the following: The Sagnac experiment is often regarded as fundamental as the Michelson-Morley experiment so much so that in some recent papers it is claimed that one may even rederive relativity theory from some new postulates based on the Sagnac effect [9, 10]. Unfortunately the Sagnac effect unlike the Michelson-Morley result is a verification of a first order effect. To hope to derive relativity theory from the Sagnac result therefore requires the exact Sagnac formula and one cannot remain content with the approximate one.

Sagnac effect may be brought out first by delinking any other effect that may be present due to the rotation of the disc from some *pure* Sagnac effect.

Let us first recognize that the essential content of the Sagnac experiment lies in its ability to detect acceleration of the experimental platform by comparing the round-trip times for lights traveling parallel or antiparallel to the motion of the platform. It is therefore expected that the acceleration need not have to be due to rotation alone; a suitably modified Sagnac type experiment should as well be able to detect the change of the direction of motion of a platform which is allowed to move or shuttle along a straight line. In the next section we shall propose such a thought experiment which will mimic the optical Sagnac experiment in almost all respect but with the difference that the motion of the experimental platform will not involve rotation. The outcome of the experiment will be called the *pure* Sagnac effect. This will give us a perspective which will enable us to appreciate the connection between the Sagnac effect and the Ehrenfest paradox. We shall see below that the formula obtained for the *pure* Sagnac effect may or may not be modified when rotation is introduced. This modification or the lack of it will depend on the way the Ehrenfest paradox is resolved [11, 12].

It is interesting to note that no author has ever explicitly mentioned any role of the EP in the derivation of the SD. In Sec. 3.8 we shall argue that the so-called kinematic resolution of the EP is based on some implicit assumptions regarding the behaviour of the solid discs when set into rotation. In order to prepare the background of this argument, in Secs. 3.6 and 3.7 we shall consider another version of the linear Sagnac experiment and analysis involving a non-rigid frame of reference of a special kind. We shall see that the Sagnac type experiment performed on such platforms gives the usually quoted formula for the SD. Significance of these observations among other things will be discussed in

Sec. 3.8 and finally will be summarized in Sec. 3.9.

3.4 Sagnac effect in a Linear Platform

The title of this section may appear to be misleading at the first reading, because Sagnac experiment detects, and is used to detect rotation. The derivation also involves rotation. But a close scrutiny of the derivations will reveal that what the Sagnac effect essentially detects is the acceleration of the platform, of which rotation is a special case. Sagnac experiment detects acceleration of the experimental platform by comparing times for lights traveling parallel or antiparallel to the motion of the platform. Thus the acceleration need not have to be due to rotation alone; a suitably modified form of Sagnac experiment should be able to detect the change of direction of platform with linear motion as well. Here we describe such a gedanken experiment [13, 14] which will mimic the optical Sagnac experiment in almost all respect but with the difference that the motion of the experimental platform will not involve rotation. Though Sagnac effect without rotation may seem to be a contradiction in terms one must recognise that the basic ingredients of Sagnac experiment that will also be included in our modified (gedanken) Sagnac experiment essentially are

1. Light signal complete round trips with the help of mirrors.
2. The experimental platform moves with a uniform *speed* (in case of rotational platform, this speed is tangential to the circumference)
3. Light travels parallel (or antiparallel) to the direction of motion of the platform throughout during their round trips.
4. The difference of the round trip times (the SD) for the parallel and the antiparallel light signals will be measured on board the platform.

The outcome of this experiment is called *pure* Sagnac effect, as no other effect than the acceleration that may have their present in case of its rotational analogue, such as Coriolis force or centrifugal force (both are essentially non-inertial forces) will not be present in this modification. The reason behind introducing this linear analogue of Sagnac experiment is that delinking this rotation-specific forces will enable us to appreciate the connection between the Sagnac effect and the EP. It will also be shown later that the formula obtained for linear version of the Sagnac effect (we call it *linear Sagnac effect* (LSE)) may or may not be modified when rotation is introduced depending on the way EP is resolved. In next two sections we shall discuss two versions of LSE [13, 14]. In the following sections we shall discuss a resolution of EP and show that the resolution EP will determine the value of Γ (3.3.1) to find the correct SD formula.

It is interesting to note that no author has ever explicitly mentioned any role of EP in the derivation SD. Indeed we shall argue in a later section that the so-called kinematic resolution of the EP is based on some implicit assumptions regarding the behaviour of the solid discs when set into rotation.

3.5 Linear Sagnac Effect – I

In optical Sagnac experiment, one essentially compares the round trip times of two light signals one of which propagates parallel and the other travel antiparallel to the direction of motion of the edge of the rotating disc on which all the measurements are carried out. The gedanken linear version of the Sagnac effect is proposed so that it essentially contains this characteristic of Sagnac effect. Two separate experiments will be considered for the parallel and the antiparallel journeys.

Let us consider a *rigid* linear platform A is made such that it can move in

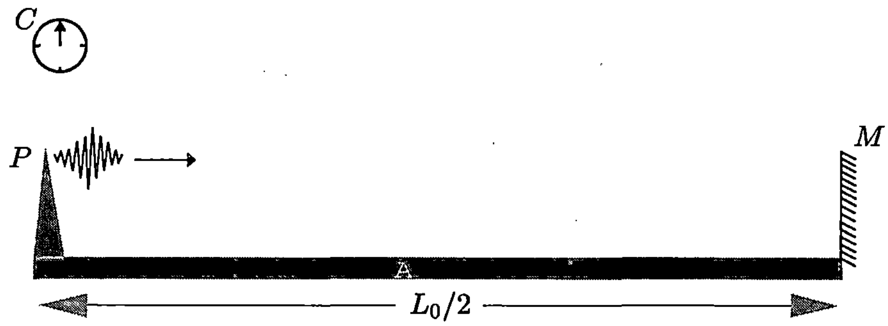

Figure 3.1: Experimental Arrangement

either direction with a constant linear velocity $\mathbf{v}$. Also, it is possible to change the motion to the opposite direction as and when needed. The length of the rigid rod, in its rest frame is assumed to be $L_0/2$. At one end of the rigid rod (right end in Fig. 3.1) there is a mirror M. A sensor is attached to the mirror. Whenever light falls on the mirror, the sensor passes a massage to a mechanism set to the platform and the platform at once reverses its direction of motion and starts to move in the opposite direction. This reversal of direction is assumed to occur in no time, *i.e.* there will be no time lag. This mechanism will assure that the light motion will be parallel or antiparallel throughout its journey to the direction of motion.

At the other end, there is a light source P. A certain mechanism is set so that whenever light pulse is emitted from P, the platform starts to move in a predefined direction.

There is a clock (C) attached to the source which records the round trip time of the light pulse. Each experiment is divided in two phases. At the first phase of the first experiment (Fig. 3.2) when light is emitted from the source and moves toward the mirror, the rod starts to move toward right (along the positive $x-$direction) with a constant linear velocity $\mathbf{v}$. After a certain time, the light pulse reaches the mirror and is reflected by it. At this phase, the velocity of light is along the

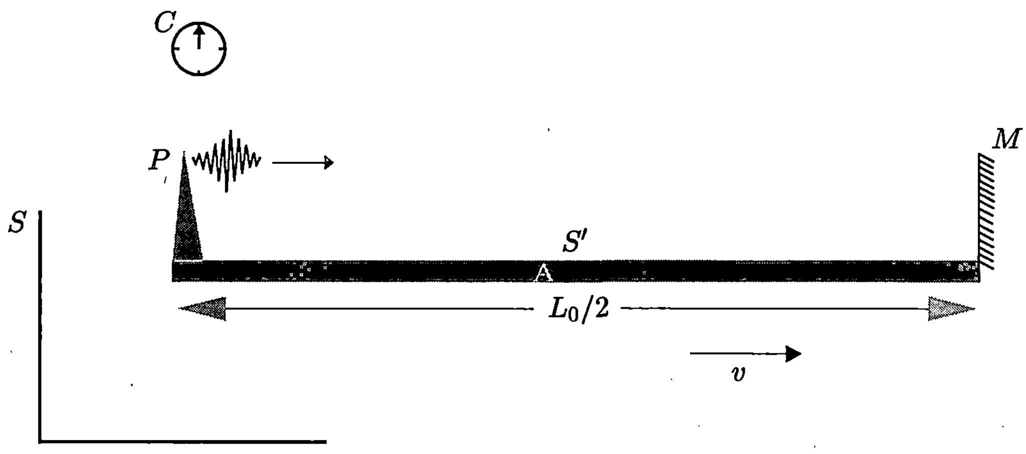

Figure 3.2: Linear Sagnac Effect–I phase I

negative x-direction. By some mechanism attached to the mirror, just the moment the light beam reaches the mirror, the whole platform reverses its direction of motion and starts to move along the negative $x-$axis (Fig. 3.3) with a constant velocity $-\mathbf{v}$. This is the second phase of the first experiment.

From Figs. 3.2 and 3.3 it is clear that the motion of light is always parallel to the direction of motion. Let us call the whole experiment (Figs. 3.2 and 3.3) parallel linear Sagnac experiment (PLSE). This is analogous to the co-rotating beam of light in the rotational optical Sagnac effect.

The frame S in the figures is the laboratory frame, S' and S'' are the frames comoving with the platform throughout the first phase and the second phase of the experiment respectively. A frame K (not shown in the figures) may be considered to be attached to the platform A. The frame S is obviously inertial from definition. S' and S'' are also inertial frames because throughout each phase, the platform moves with a constant velocity $\pm\mathbf{v}$. But the frame K attached to the platform throughout both the phases is certainly a non-inertial frame because at one point

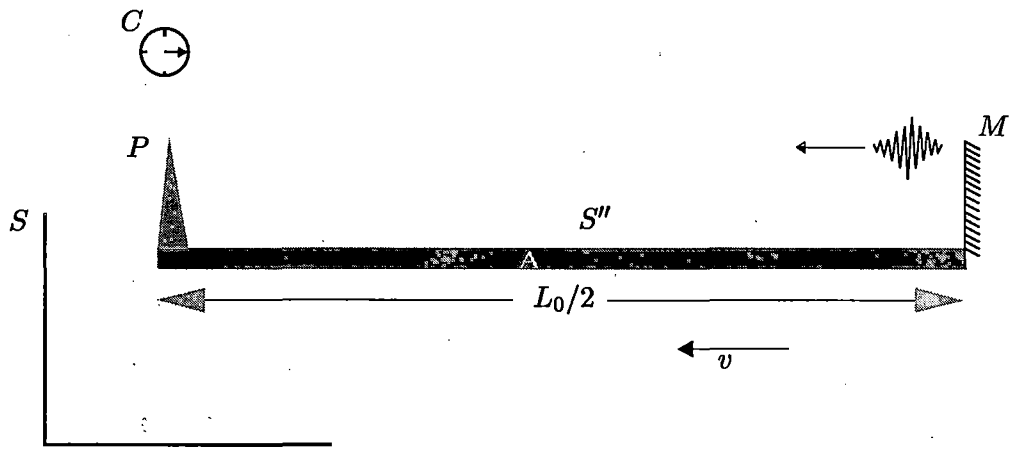

Figure 3.3: Linear Sagnac Effect–I phase II

of the experiment it suffers an acceleration, velocity changes from $+\mathbf{v}$ to $-\mathbf{v}$, though for an infinitesimal time.

A similar arrangement is made for the antiparallel linear Sagnac effect (ALSE). (see Fig. 3.4 and Fig. 3.5) Here in the first phase of the second experiment when light is emitted by the source and traverses towards the mirror M along the positive $x-$axis, the platform A moves along negative $x$direction with constant velocity $-\mathbf{v}$ (Fig. 3.4)

In the second phase, light after the reflection moves towards negative $x-$axis while the platform moves along the positive $x-$axis (Fig. 3.5)

From Figs. 3.4 and 3.5 it is clear that light motion is antiparallel to the motion of the platform throughout this experiment. This is analogous to the counterrotating beam of the rotational Sagnac effect. As in the previous case, S, S' and S'' are inertial, K is non-inertial. Though we used S' and S'' in both the cases they are certainly not the same.

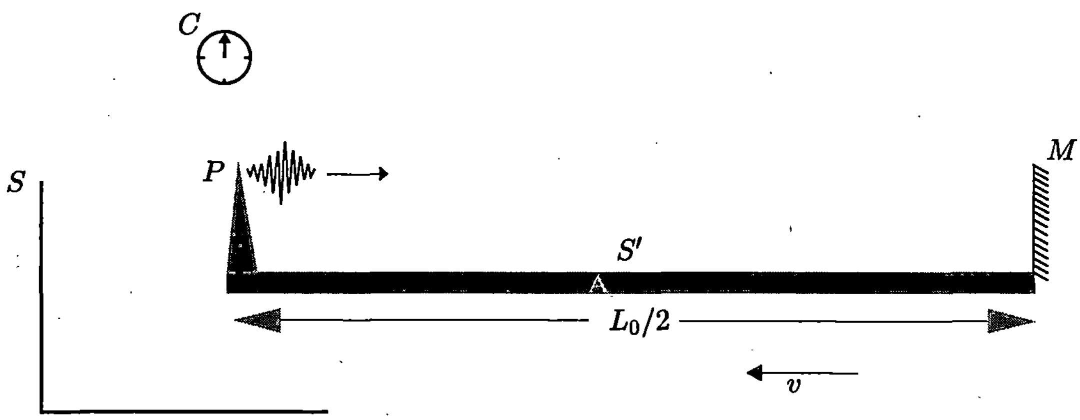

Figure 3.4: Linear Sagnac Effect–II phase I

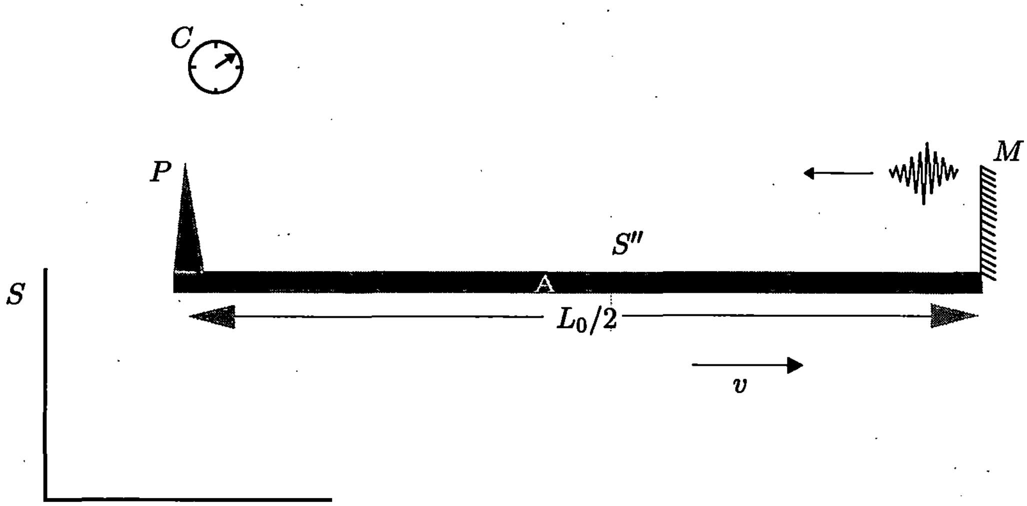

Figure 3.5: Linear Sagnac Effect–II phase II

Now, we can write for the motion of the light pulse, using Lorentz transformation

$$\mathrm{d}t_{\text{lab}} = \gamma \left(\mathrm{d}t \pm \frac{v\mathrm{d}x}{c^2} \right), \tag{3.5.1}$$

where t_{lab} is the co-ordinate time of the global inertial frame of the laboratory and t and x refer to the local time and spatial co-ordinate of the instantaneous inertial frames S' and S'' and the positive and negative sign apply for the forward and reverse journeys of the rod respectively.

For the first experiment where light is always moving parallel to the motion of the rod, we obtain the round trip time $\Delta t_{\text{lab}}(1)$ by integrating Eq. (3.5.1)

$$\begin{aligned}\gamma^{-1}\,\Delta t_{\text{lab}}(1) &= \int \mathrm{d}t + \int_0^{\frac{L_0}{2}} \frac{|v|\mathrm{d}x}{c^2} - \int_{\frac{L_0}{2}}^{0} \frac{|v|\mathrm{d}x}{c^2} \\ &= \int \mathrm{d}t + \frac{L_0|v|}{c^2}.\end{aligned} \tag{3.5.2}$$

Similarly we can find the round trip time $\Delta t_{\text{lab}}(2)$ for the experiment where light moves antiparallel to the motion of the rod throughout the experiment

$$\begin{aligned}\gamma^{-1}\,\Delta t_{\text{lab}}(2) &= \int \mathrm{d}t - \int_0^{\frac{L_0}{2}} \frac{|v|\mathrm{d}x}{c^2} + \int_{\frac{L_0}{2}}^{0} \frac{|v|\mathrm{d}x}{c^2} \\ &= \int \mathrm{d}t - \frac{L_0|v|}{c^2}.\end{aligned} \tag{3.5.3}$$

A look at Eq. (3.5.1) will clarify the nature of $\int \mathrm{d}t$ in Eqs. (3.5.2) and (3.5.3). S' being an inertial frame, the observe in S' will measure the time of arrival of light from the source to the mirror to be $L_0/2c$ because the velocity of light is c in S' , $L_0/2$ being the length of the rod. The same is true for an observer in S'' who measures the time of flight of light from the mirror to the source. Thus the total time of flight of light measured by two independent observers in S' and S'' in two different phases of the experiment is

$$2 \times \frac{L_0}{2c} = \frac{L_0}{c}.$$

This is $\int dt$ and is equal for both the experiments.

Now, if $\Delta\tau = \int d\tau$ is the time of flight of light for the round trip as measured by an observer in K (remember, K is comoving with rod and suffers direction reversal) then, we may write, by virtue of time dilation effect

$$\Delta\tau = \gamma^{-1}\Delta t_{\text{lab}}$$

Eqs. (3.5.2) and (3.5.3) may now be written as

$$\Delta\tau(1) = \int dt + \frac{L_0|v|}{c^2}, \quad (3.5.4)$$

$$\Delta\tau(2) = \int dt - \frac{L_0|v|}{c^2}. \quad (3.5.5)$$

Difference of these times gives for Sagnac delay for this linear gedanken version of Sagnac experiment

$$\delta\tau = \frac{2L_0|v|}{c^2}. \quad (3.5.6)$$

Note that $L_0/2$ is the proper length of the rod thus L_0 is a constant. Thus the SD in this case is linear in v.

Also note that all the mathematical steps that are involved in this derivation smoothly go over to that leading to rotational Sagnac effect. Except there instead of two inertial frames S' and S'', one has to consider an infinite number of momentary inertial frames for the calculations [15, 16]. Since no other effect (that might be present because of rotational motion of the disc in the usual Sagnac experiment) is involved in this linear arrangement the time delay expressed by the above relation may be said to be the result of *pure* Sagnac effect.

3.6 Linear Sagnac Effect – II

For linear Sagnac experiment type II, we shall slightly modify the experiment type I. Here instead of considering a light source and a mirror fixed on a rigid rod of proper length $L_0/2$, we consider two aircrafts (unbonded, *i.e.* not tied together) separated by a distance $L/2$ initially at rest with the laboratory.

Suppose that these aircrafts are programmed in such a way that they can move in any direction always preserving a constant separation $L/2$ with respect to the laboratory. To a casual observer in the laboratory there will appear to be a bond between the objects because of the programmed constant separation of the two, but in reality they are unbonded. One may term this apparent bond between the aircrafts as *software bond* as distinguished from the bond that exists between any molecules in a solid body [17].

The two experiments in two phases described in the previous section can now be repeated by placing the light source to one aircraft and the reflector to the other. For the first phase of the experiment assume that the aircrafts are accelerated from rest and finally the system moves with constant velocity v along the positive $x-$direction.

Suppose now a light pulse from a source attached to the first aircraft travel towards the second one on the right and falls on the right and falls on the mirror attached to it. As soon as the light pulse falls on the mirror, the *software bonded* system starts moving in the negative $x-$direction. The reflected pulse of light also travels in this direction and the time of transit for the light pulse for parallel propagation is recorded upon return to the source.

In the second experiment almost the whole programme repeated except now, at the time of emission of the light pulse, the motion of the aircraft system is along the negative $x-$direction although the light pulse travels toward the right that subsequently falls on the mirror attached to the second aircraft. The direction of motion of the unbonded system is reversed as light falls on the mirror and travels to the left to be recorded at the position of the source again. In this arrangement, propagation of light remains antiparallel throughout its entire journey.

One may take a look at all the figures in the previous section. But the rigid rod will not be their and the separation of the source and the mirror (mounted on two aircrafts) is $L/2$ in the S frame. As the aircrafts are programmed frame S, the calculation for round trip time of flight for the light pulse, unlike in the previous rigid platform experiment, should be done in the laboratory frame.

Though the experiment is performed for linear motion, the kinematical considerations leading to Eqs. (2.5.2) and (2.5.1) in Sec. 2.5.1 still remain valid. For example, for the parallel propagation experiment, in order to complete the round trip in time t_1, the light pulse, as viewed from the laboratory, has covered in addition to the distance L, an extra distance $x = vt_1$ because of the to and fro motion (with speed v) of the aircraft. This means $ct_1 = L + vt_1$, or,

$$t_1 = \frac{L}{c - v}.$$

The round trip time for the antiparallel experiment in the S frame is given by

$$t_2 = \frac{L}{c + v},$$

and their difference can be written as

$$\Delta t = t_1 - t_2 = \frac{2Lv}{c^2}\,\gamma^2.$$

If $\delta\tau$ denotes the corresponding time difference by an observer in the first aircraft one obtains, on account of time dilation

$$\delta\tau = \gamma^{-1}\,\Delta t = \frac{2Lv}{c^2}\,\gamma. \tag{3.6.1}$$

Note that $\delta\tau$ is now *not* linearly related to v. This formula (3.6.1) differs from the formula (3.5.6). While the SD for the rigid platform (3.5.6) depends linearly on

v, the SD for unbonded (software bonded) system has a non-linear relation with v.

One may however argue that in Eq. (3.5.6) L_0 (actually $L_0/2$) is the proper length of the rod and in contrast L (actually $L/2$) in Eq. (3.6.1) represents the distance of the aircrafts as measured from the laboratory frame. If the Eq. (3.5.6) were expressed in terms of L the formulae (3.5.6) and (3.6.1) would have agreed.

However, there is subtle point here. Certainly one is at liberty to quote the expressions in terms of proper distance L_0 or of the co-ordinate distance L between the source and the reflector. But if the issue is the question of dependence of SD on the speed of the platform, the formulae (3.5.6) and (3.6.1) predict different results. One must know here, while quoting the results, which of the lengths, L_0 or L is independent of speed of the platform v. In the type I experiment, the source and the reflector maintains a constant separation in its rest frame (S', S'', K). Thus the proper length of the separation is independent in v. In the type II experiment, however, the source and the reflector (mounted on two separate untied aircrafts) maintains their separation by software programme in the laboratory frame S. Thus their proper separation is stretched according to Lorentz formula and depends on v. The difference will be clear if one looks at the graph (Fig. 3.6) where the linear (dashed) one represents Eq. (3.5.6) (all the constants are set to unity).

3.7 Coordinate System of the Unbonded Frame

Consider an one-dimensional array of some software bonded particles that constitutes a frame of reference K and suppose from its state of rest at $t = 0$, the system is set in motion. If the space-time co-ordinates of the laboratory frame S are denoted by x and t, the equation of motion of the particle at the origin of K

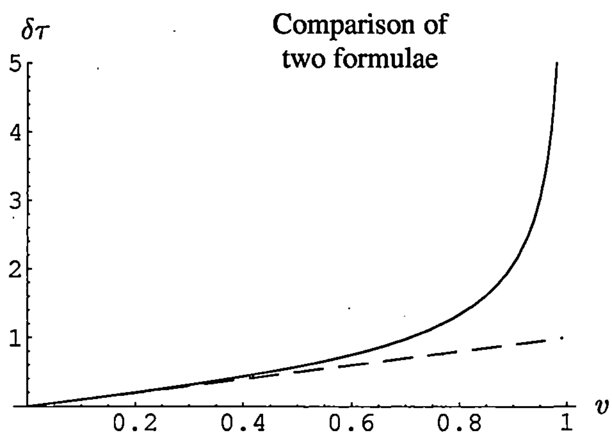

Figure 3.6: Comparison of two formulae for two types of LSE. Dashed one represents LSE1

may be expressed as

$$x = f(t),$$

where $f(t)$ is some function of time which is zero at $t = 0$.

For any particle of the array this may be written as

$$x = x' + f(t),$$

where x' is the spatial coordinate of the particle with respect to S, when it was at rest with the laboratory (at $t = 0$). The variable x' may be used to label the array of points and these may act as spatial coordinates in K. Taking the coordinate time t' of K same as t one may write the following transformations between S and K in terms of the coordinate differentials

$$\mathrm{d}x' = \mathrm{d}x - \dot{f}(t)\mathrm{d}t, \qquad \mathrm{d}t' = \mathrm{d}t, \tag{3.7.1}$$

where $\dot{f}(t) = v = \dfrac{\mathrm{d}f(t)}{\mathrm{d}t}$ is the instantaneous velocity of the aircraft.

The line-element in natural unit ($c = 1$) of the 2-dimensional Minkowskian space in the coordinate system S

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$$

may be transformed accordingly so that with respect to x' and t' one may write

$$\mathrm{d}s^2 = \gamma^{-2}\mathrm{d}t'^2 - 2v\mathrm{d}x'\mathrm{d}t' - \mathrm{d}x'^2 \tag{3.7.2}$$

where $\gamma = (1 - v^2)^{-\frac{1}{2}}$.

Again for any line-element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

the proper distance dl between two points x^i and $x^i + dx^i$ is given by (vide reference [18] for instance)

$$\mathrm{d}l = \sqrt{\left(\frac{g_{oi}g_{ok}}{g_{oo}} - g_{ik}\right)\mathrm{d}x^i\mathrm{d}x^k}, \tag{3.7.3}$$

where $i, k = 1, 2, 3$. For the line element (3.7.2) this is given by

$$\mathrm{d}l = \frac{\mathrm{d}x'}{\sqrt{1 - v^2}} \tag{3.7.4}$$

If the points x' and $x' + \mathrm{d}x'$ refer to the co-ordinates of two neighbouring particles of K, by definition $\mathrm{d}x'$ is an invariant. In that case the proper distance should stretch according to equation (3.7.4) honouring the well-known special relativistic effect of length contraction as the unbounded array of particles is set into motion. Note that expression (3.7.2) represents a perfectly legitimate co-ordinate description of the 2-dimensional Minkowsky space and although the transformations (3.7.1) have Galilean structure, special relativity is taken care of when we used equation (3.7.3) to obtain the proper distance.

3.8 Discussion

Let us now return to the original question as to the correct relativistic formula for the usual Sagnac effect. As pointed out earlier there are two contesting claims for Sagnac delay:

Claim 1: $\Delta\tau \propto \dfrac{\beta}{\sqrt{1-\beta^2}}$,

Claim 2: $\Delta\tau \propto \beta$.

If the special relativistic correction due to time dilatation is incorporated in the classical expression (Eq. (2.5.5)) one obtains Eq. (2.5.10) which corresponds to claim 1. On the other hand if not only time dilatation but also Lorentz contraction of the circumference of the rotating disc is taken into account one gets equation (2.5.12)) and claim 2 appears to be the true one. Barring a few exceptions [7, 19] most authors adhere to claim 1 without stating explicitly the reason behind not considering the length contraction effect. However as we have seen there was no ambiguity as to the correct formula for time delay for the linear Sagnac effect (thought) experiment with rigid rod. There it was evident that the Lorentz contraction of the rigid platform ought to be taken into account in addition to the time dilatation effect.

For the usual Sagnac experiment on a rotating disc, the Lorentz contraction of the disc's circumference is not generally considered perhaps to avoid the Ehrenfest paradox. On the contrary, favouring claim 2, Selleri [19] and Goy [7] assume relativistic contraction of the edge of the disc without addressing any possible paradox that may arise due to such an assumption. Indeed the result is inconsistent unless it is explicitly assumed that the disc does not remain a disc and becomes a non-flat object. It is therefore amply clear that in order to decide between claim 1 and claim 2 it is necessary to understand how the Ehrenfest problem is resolved. Ehrenfest's problem concerns the mechanical behaviour of a material disc set in rotation from rest. The paradox remains a paradox as long as one implicitly assumes that the disc is Born rigid [2, 20]. By definition Born rigid motion of a body leaves the proper lengths of the body unchanged. Grøn [11] showed that the transition of the disc from rest to rotational motion in a Born rigid way is a kinematic impossibility. It is the recognition of this fact which is known as the kinematic resolution of the Ehrenfest paradox [11, 12].

Cavalleri [20] on the contrary observes that the Ehrenfest paradox cannot be solved from a purely kinematical point of view and the solution of the paradox is intrinsically dynamical. This was refuted by Grøn [11] who rather endorsed a remark by Phipps [21] that to think that dynamics can exist "without the foundation of logically consistent kinematics" is an absurdity.

The present authors believe that both the viewpoints are correct in the present context. To recognise that Born rigid rotation is an impossibility and an implicit assumption on the contrary is the source of the paradox, may follow from pure kinematics; but if it is asked – "what exactly will happen to the solid disc?" the answer will lie in the realm of dynamics. It appears that there is no unanimity in the literature as to this precise question. Synge [22] and Pounder [23] introduce the concept of superficial rigidity [20] according to which the circumference and radius of the disc when put into rotation, undergo change in accordance with special relativity but suggest that the flat disc changes to a surface of revolution symmetric about the axis of rotation. In this way the possible violation of Euclidean geometry in the inertial system is avoided. In the case of uniform rotation this allows radial contraction without any change of meridian arc-length. Some specific prescriptions were also suggested by a few earlier authors who proposed bending of the surface of the rotating disc in the form of a paraboloid of revolution (vide [20] for further references). The rotating disc or wheel taking the shape of spherical segment when in rotation was suggested by Sokolovsky [24] as a resolution of the 'wheel paradox'.

Eddington [25] also investigated the problem of the rotating disc. He studied the question of alteration of the radius of a disc made of homogeneous incompressible material when caused to rotate with angular velocity ω. He showed that the radius of the disc is a function of the angular velocity ω and is approximately given by

$$a' = a\left(1 - \frac{1}{8}\omega^2 a^2\right)$$

where a is the rest radius of the disc. A similar view has also been expressed by Weinstein [26] who holds that the disc under rotation will be in torsion with a consequent reduction of both the radius and the circumference.

Of recent interest is the so called kinematic resolution of the Ehrenfest paradox as discussed by Grøn [11] and Weber [12]. According to the authors, it follows from purely kinematic considerations, that the radius of the disc remains unaltered but the proper measure of the circumference is increased in such an extent that the Lorentz contraction effect just gets compensated. In other word although there is a Lorentz contraction of the periphery with respect to the laboratory frame it is not visible because of the stretching of the periphery [3]. However, the stretching of the disc's circumference in its proper frame is a dynamical effect (related to the property of the solid material of the disc). How can one hope to get this dynamical effect purely from the kinematic considerations? Clearly the result must have been assumed implicitly. To clarify this let us consider a rotating co-ordinate system which is often discussed in connection

[3]Recently Klauber [9, 10] and Tartaglia [27] based upon different arguments also conclude that there will be no contraction of the circumference of the disc. The authors believe that relativistic contraction effect will not at all take place for rotating discs.

with the Ehrenfest paradox. Suppressing one spatial dimension Grøn considers the following transformation:

$$r' = r, \qquad \theta' = \theta - \omega t, \qquad t' = t \tag{3.8.1}$$

where r and θ refer to the radial and angular co-ordinates and t refers to the time co-ordinate of laboratory frame and the primed quantities refer the same in the rotating system.

The rotating frame of reference is equated with that of the rotating disc. It is precisely this equation where lies the implicit assumption that the disc's periphery is stretched due to rotation. Based on these transformations the line-element may be written as

$$\mathrm{d}s^2 = \mathrm{d}r'^2 + r'^2\mathrm{d}\theta'^2 + 2\omega r'^2\mathrm{d}\theta'\mathrm{d}t' - \left(1 - \frac{\omega^2 r'^2}{c^2}\right) c^2\mathrm{d}t'^2. \tag{3.8.2}$$

Using the formula (3.7.3) for proper spatial distance for the line element (3.8.2) the tangential proper spatial line-element is obtained as

$$\mathrm{d}l = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} r\mathrm{d}\theta'. \tag{3.8.3}$$

Note that while using Eq. (3.7.3) a minus sign under the radical is required since now the metric (3.8.2) has a different signature.

Integrating Eq. (3.8.3) along the whole element one obtains the proper length L_0

$$L_0 = L\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \tag{3.8.4}$$

where $L = 2\pi R$ is the circumference of the disc as observed from the inertial frame of the laboratory. However it will be wrong to assume, from the above argument, that the proper circumference of the disc has increased by a Lorentz factor. To conclude from (3.8.4) that the periphery of the disc is stretched by a

γ-factor due to rotation is to assume that L was also the circumference of the disc when it was at rest and remains the same, as it is brought up to rotation from its state of rest. Any transformation (representing rotation) must reflect the validity of this assumption. Clearly Eq. (3.8.1) does not guarantee this.

The transition of the disc to its rotational motion from its state of rest can be expressed by writing the transformations (3.8.1) in a slightly modified form

$$r' = r, \qquad \theta' = \theta - f(t), \qquad t' = t \tag{3.8.5}$$

where the function $f(t)$ is assumed to have the following properties: $f(t)$ and $\frac{\mathrm{d}f}{\mathrm{d}t} = \omega(t) = 0$ at $t = 0$ and $\omega(t)$ thereafter increases and finally approaches a constant value. Obviously the transformations (3.8.5) then represent a rotating coordinate system with constant angular velocity after a period of angular acceleration from its state of rest, which in terms of the differentials read

$$\mathrm{d}r' = \mathrm{d}r, \qquad \mathrm{d}\theta' = \mathrm{d}\theta - \omega(t)\mathrm{d}t, \quad \mathrm{d}t' = \mathrm{d}t. \tag{3.8.6}$$

Note that relations (3.8.2) and (3.8.3) still remain valid. Now, if we recall our discussions in Sec. 3 and 4, and the transformations (3.7.1), we see by analogy that the transformations (3.8.6) represent the motion of the disc which is composed of an unbonded arrangement of particles that are programmed to move in a particular way so that their mutual separations with respect to the laboratory frame of reference remain constant. Therefore the constancy of L (and not L_0) in other words, is the outcome of he assumed programme of motion of the particles of the disc governed by equation (3.8.5).

3.9 Summary

We are now in a position to summarize our findings. We have seen that there is a scope for confusion regarding the correct relativistic expression for the Sagnac delay. Although the oft-quoted result is that given by equation (2.5.10), no role of the so-called Ehrenfest paradox in arriving at the result is usually discussed. It is expected that the special relativistic result for the Sagnac effect will differ from its classical counterpart, usually due to two kinematic effects of special relativity – the length contraction and the time dilatation. The inclusion of the length contraction effect in the circumference (and not in the radius) of a rotating disc invites a paradox that there is an apparent violation of the Euclidean geometry in an inertial frame. On the other hand, the non-inclusion of the Lorentz contraction effect will violate special relativity. To understand and clarify these issues a Sagnac-type thought experiment (without rotation) performed on a linear rigid platform has been presented. Since no paradox is associated with this arrangement although kinematically all aspects of the usual Sagnac experiment are incorporated in it, the linear experiment sets the right kind of perspective against which the role of the Ehrenfest paradox in the rotating disc experiment can be discussed.

Although the resolution of the Ehrenfest paradox lies in appreciating the fact that 'Born rigid' rotation of the disc from its state of rest is a kinematic impossibility, people differ when trying to be specific about the exact deformation of the disc brought about by rotation. We give below just two opposite viewpoints that are found in the literature.

According to the so-called kinematic resolution of the paradox there should not be any contraction of the circumference as observed from the inertial frame of the laboratory that is at rest with the axis but, the periphery should stretch in terms of proper measure so that the Lorentz contraction effect of special relativity is automatically taken care of. The conclusion apparently follows from the widely discussed line element representing a rotating co-ordinate system [11, 18, 28]. It has been however shown, by drawing analogy with the version II linear Sagnac type (thought) experiment, that the kinematic resolution presupposes that the disc material is composed of 'unbonded' particles that are programmed to rotate in such a way that the distances between the particles remain fixed with respect to the laboratory as the system passes to a rotational motion from its state of rest. If this happens, the formula for the Sagnac delay will be given by Eq. (2.5.10).

The other view point is to suppose that the disc material obeys the Synge-Pounder criterion of superficial rigidity. In this case the disc should bend and take a shape of a paraboloid so that at any radial point, the circumference is Lorentz contracted but there is no contraction of the meridian. However the distance of the periphery from the centre will be shortened. Therefore the paradox does not exist. In this case too the resolution of the paradox is based on a specific postulate regarding the behaviour of the material of the disc undergoing rotation. As a consequence, the Sagnac delay should be given by equation (2.5.12) that corresponds to the result obtained for the linear Sagnac effect of the first form (Eq. (3.5.6)). Some authors [7, 19] have quoted this result too however not addressing any role of the Ehrenfest paradox in their derivation. If instead of a disc, the rotating platform is assumed to be a massive solid cylinder, the deformations of the kind just mentioned are perhaps excluded and the usual formula (Eq. (2.5.10)) pertains to this case. However, in this case, the constraint imposed on the particles of the cylinder by the form of the solid body would work in such a way, that the particles of the body can be thought of as 'unbonded' as the cylinder is set into rotation (vide Sec. 3.8). Indeed for a disc there cannot be one right formula; for example, the deformation of the kind considered by Eddington [25] as mentioned in Sec. 3.8 would give a result different from Eq. (2.5.10) or Eq. (2.5.12).

Reference

[1] Paul Ehrenfest, *Phys. Z.*, **10**, (1909) p. 918.

[2] Max Born, *Ann. Physik*, **30**, (1909) p. 840.

[3] V. Varićak, *Phys. Z.*, **12**, (1911) p. 169.

[4] Ø. Grøn, "Space geometry in rotating reference frames: A historical appraisal", in G. Rizzi and M. L. Ruggiero (Eds.), "Relativity in Rotating Frames", Fundamental Theories of Physics (Kluwer, New York, 2004). Series Editor: A. van der Merwe.

[5] J. Statchel, "Einstein and the rigidly rotating disk", in A. Held (Ed.), "General Relativity and Gravitation: One Hundred Years after the Birth of Alber Einstein", volume I, pp. 1–15 (Plenum, New York).

[6] F. Selleri, *Found. Phys.*, **26**, pp. 641–664.

[7] F. Goy, "Non-invariant velocity of light and clock synchronization in accelerated systems", in M. C. Duffy (Ed.), "Physical Interpretation of Relativity Theory (V)", pp. 129–138 (1996). Held at Imperial College, London.

[8] E. J. Post, "Sagnac effect", *Rev. Mod. Phys.*, **39**, (1967) p. 475.

[9] R. D. Klauber, *Found. Phys. Lett.*, **11**, (1998) pp. 405–444.

[10] R. D. Klauber, *Am. J. Phys.*, **67**, (1998) pp. 158–159.

[11] Ø Grøn, *Am. J. Phys.*, **43**, (1975) pp. 869–876.

[12] T. A. Weber, *Am. J. Phys.*, **65**, (1997) pp. 486–487.

[13] S.K. Ghosal and Minakshi Sarker, "Cosmologically preferred frame and sagnac effect", in M. C. Duffy (Ed.), "Physical Interpretation of Relativity Theory (VII)", volume Late Papers (1998). Held at Imperial College, London.

[14] S. K. Ghosal, Biplab Raychaudhuri, Anjan Kumar Chowdhuri and Minakshi Sarker, "Relativistic sagnac effect and ehrenfest paradox", *Found. Phys.*, **33**(6), (2003) pp. 981–1001.

[15] D. Dieks and G. Nienhuis, *Am. J. Phys.*, **58**, (1990) pp. 650–655.

[16] D. Dieks, *Eur. J. Phys.*, **12**, (1991) pp. 253–259.

[17] C. W. Sherwin, "Three theories of relative measurement", (Physical Interpretation of Relativity Theory Meeting (III), Imperial College, London, 1992).

[18] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth Heinemann, Oxford, 2002), 4th revised english edition edition.

[19] F. Selleri, *Found. Phys. Lett.*, **10**, (1997) pp. 73–83.

[20] G. Cavalleri, *Nuovo Cimento B*, **53**, (1968) pp. 415–431.

[21] Jr. T. E. Phipps, *NOLTR*, **47**, (1973) pp. 73–79.

[22] J. L. Synge, "Studies in mathematics and mechanics: Presented to richard von mises by friends, colleagues and pupils", (1954). New York.

[23] J. R. Pounder, *Comm. Dublin Institute for Advanced Studies A*, **11**.

[24] Yu. I. Sokolvsky, *The Special Theory of Relativity* (Hindustan Publishing, New Delhi, 1962).

[25] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, 1965).

[26] D. H. Weinstein, *Nature*, **232**, (1971) p. 548.

[27] A. Tartaglia, *Found. Phys. Lett.*, **12**(1), (1999) pp. 17–28.

[28] C. Møller, *The Theory of Relativity* (Oxford University Press, New Delhi, 1972), second edition edition.