Chapter 2

Sagnac Effect

2.1 Introduction

As the title suggests, the present thesis concerns different aspects of the Sagnac effect. Thus, it is worthwhile to give an account of the effect from its historical and conceptual perspectives. With this aim we present a brief review of earlier works (both theoretical and experimental) in the subject in this chapter. The discourse presented here is by no means exhaustive, although we have tried our best to make it self-contained so that this article can stand by itself as an introduction to the more profound studies on the Sagnac effect for the readers who are unfamiliar with this very important phenomenon. In particular we have left behind discussions over matter wave Sagnac effect (Sagnac experiment performed with matter waves) since our work involves only optical Sagnac effect.

2.2 Sagnac Effect

There has been a great deal of interest in recent years in the Sagnac effect. This is not only for its practical importance in navigational application for sensing rotation but also for its rich theoretical ramifications. In 1913 Sagnac [1] in his life long quest for ether devised an experiment where he compared round-trip times of two light signals traveling in opposite directions along a closed path on a rotating disc. It was observed that the time required by a light signal to make a close circuit on the plane of the disc differed depending on the sense (direction) of the signal's round-trip with respect to the spin of the disc. The essentials of the experiment consist of a monochromatic source of light, interferometer and a set of mirrors mounted on a turn-table. Light from the source is split into two beams by a beam splitter (half-silvered mirror) allowing them to propagate in opposite directions. These beams then are constrained (by suitably placed mirrors) to make round trips and are then reunited at the beam splitter to produce a fringe pattern. The difference in round-trip times for these counter propagating beams leads to a phase difference with a consequent shift in the fringe pattern when the turn table is put into rotation. This phenomenon is commonly known as the Sagnac effect (see Fig. 2.1).

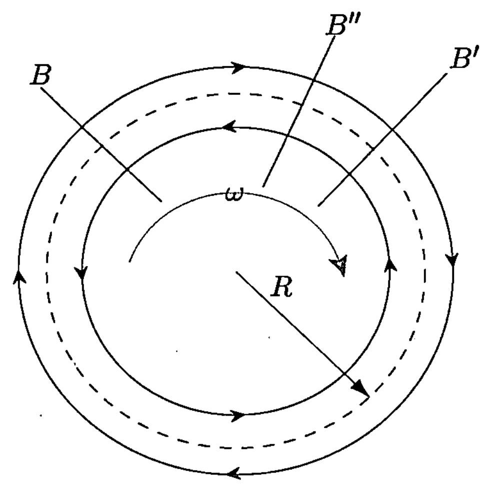


Figure 2.1: A diagram for Sagnac experiment

The Sagnac experiment performed using light beam is called the optical Sagnac effect in contrast with Sagnac experiment performed with other matter waves. The effect is universal and is also manifested for matter waves. Sagnac type experiments were done with electrons [2], neutrons [3], atoms [4], Superperconducting Cooper pairs [5], Radiowaves [6] and X-rays [7].

Accuracy of the result in Sagnac-type experiments is much improved by using laser beams (instead of usual light beams) and fibre optic interferometers using fibre optic cables to guide light beam along the periphery of the disc (instead of mirrors placed on the periphery). Traveling-wave laser in a ring cavity as the sensor is used to detect 'absolute rotation' using the Sagnac effect. This device, known as *ring laser gyroscopes* replaces the spinning-mass gyroscopes with a purely optical sensor in the field of navigation and guidance [8]. Importance of this effect also lies in connection with the question of time keeping clocks of clock-stations around the earth [9]. Here the earth plays the role of rotating platform from which clocks between clock stations are synchronized by sending lights via satellites [10, 11]. Among many applications of Sagnac effect we shall give a very brief account of global positioning system (GPS) in Sec. 2.7.

2.3 Historical Perspective

All the experiments involving interference with the help of rotation is in general termed as 'Sagnac-type experiments' [12]. But Sagnac was not the first one to conceive such an experiment, as recent historical surveys [13] reveal.

Oliver Lodge [14, 15] was possibly the first, as far as present knowledge goes, to suggest a Sagnac-type experiment. His aim was to provide a means to detect the ether drag by heavy bodies by an immobile Fizeau interferometer. An observed band shift would have suggested the ether drag by the Earth. The null result made him to conclude that ether is not dragged by the Earth's rotation and this rotation could not be measured with respect to the immobile ether. Lodge [15] wrote that "if the system as a whole, *i.e.* a lamp, an optical system [interferometer], a telescope [instrumental set up to observe fringe shift] and an observer had been set upon a rotating table and put into rotation, then a relative shift of interfering bands could have been observed at different directions of the rotations."

Albert Michelson [16] suggested construction of large sized circular interferometers with dimension of 1×1km to measure the angular rate of the

Earth's rotation about its axis. The experiment was not done immediately but performed much later (in 1923). In 1925 he, in collaboration with H. Gale and F. Pearson constructed another one of dimension 630 $\times$ 340m and placed in a system of 305mm steel tubes. The air pressure maintained inside the tubes was very low (12mm Hg) because in open air, the fringes were blurred due to airflows. With this they were the first to measure the angular rate of the Earth's rotation. Michelson [16] suggested another set of experiment with dimension 10 $\times$ 10km to measure the angular rate of the Earth's rotation about the Sun. But the project was never realized.

Another important contribution to the Sagnac-type experiments was made by Francis Harres in 1911 although his objective, quite different from Sagnac, was to study dispersion properties of glasses by substituting the linear motion in Fresnel-Fizeau experiment by a circular motion. The dispersion data obtained by Harress did not agree well with data available from others methods. "By substituting a circular motion for a linear motion[1] Harress tacitly assumed the absence of exactly the effect" that Sagnac was looking for later on [12]. Harress' interpretation was that the interference band shifts were solely associated with dragging of light by the rotating glass inside the interferometer rather than with the rotation of the interferometer as a whole [13]. Harress died a few years later (in 1915) and could not do further work towards the solution of discrepancy [12]. Harzer [17] reworked on Harress' data from a different kinematic perspective accounting properly for the rotation [12]. His most important conclusion was that the fringe shift does not depend on the presence of a co-moving (with the table) refracting medium along the path of the light beams. He expressed surprise that his analysis suggests that dispersion had no influence on the final fringe shift.

[1] *i.e.* Ring interferometer instead of linear Fresnel-Fizeau experiment

Einstein [18] showed that this is not so surprising as the dispersion term in Fresnel-Fizeau drag co-efficient really stems from a Doppler shift due to a motion between source and medium.

Sagnac conceived his experiment well before they were performed. As far as recent knowledge goes, Sagnac discussed the possibility of measurement of angular velocity with a circular interferometer with G. Lipman in 1909 [13]. He published the results of his experiment in 1913 [1, 19, 20].[2] We shall discuss further on Sagnac experiment in the next section.

One of the few text books on relativity which discusses the Sagnac effect is the book by Ugarov [22]. There he commented that he is describing "Garess's experiment (1912), subsequently repeated by Sagnac." [22, page 344]. In the same page the diagram refers to it as Sagnac-Garess experiment. In the index the experiment is also referred to as Sagnac-Garess experiment. The name "Garess" used thrice in this manner carried no further reference in the book, neither this name was found in any of the other literatures quoted earlier. It seems Ugarov has mistaken this name for 'Harress'.

2.4 Sagnac's Experiment

The interferometer in Sagnac's experiment essentially look like Fig. 2.2 Light comes out of a light source, gets reflected by a mirror and falls upon a half silvered mirror. This is the beam splitter. One of the beams goes through the half silvered mirror, gets reflected by four mirrors before finally entering the telescope passing through the beam splitter. The other part of the beam gets reflected by

[2]Unfortunately the present author does not have access to the original works of Sagnac, nor does he have enough knowledge of the French language to know whether Ref. [20] and Ref. [21] are the same or not. These citations are gathered from secondary sources.

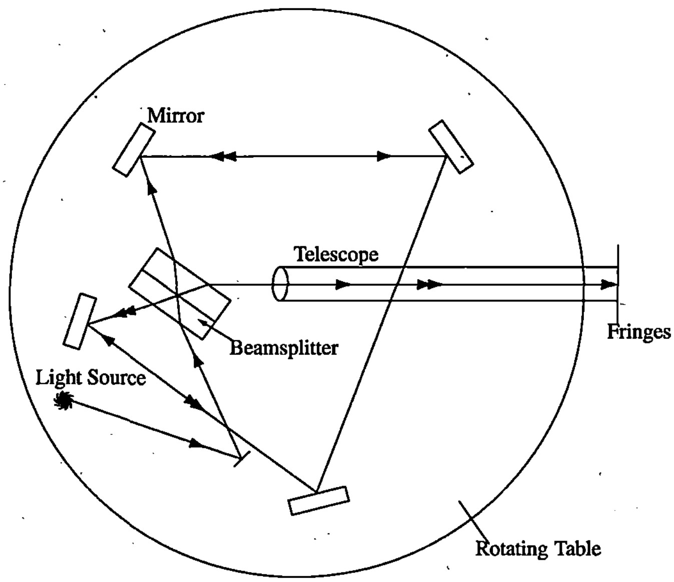


Figure 2.2: A schematic diagram of Sagnac Experiment

the half silvered mirror, goes to one of the four mirrors (the last one for the previous case), the gets reflected by the other three mirrors tracing the same path as the former beam did, but in the opposite direction before finally reaching the telescope. Through the telescope one can observe the interference fringes. If the interferometer is now given a constant angular motion, a fringe shift can be observed. The fringe shift is observed experimentally to be [12]

$$\Delta Z = \frac{4\omega \cdot \mathbf{A}}{\lambda c}, \qquad (2.4.1)$$

where

$\vec{\omega} =$ angular velocity,

$\vec{A} =$ area vector of the disc,

$\lambda =$ wave length of light used,

$c =$ velocity of light.

If the disc is rotated once clockwise and then anticlockwise, then the fringe shift between these will be double of that in each case. After taking the photographs of interference bands upon rotation, the platform is brought to rest, the photo plate is displaced. Then the platform was set in rotation in the opposite direction and another photograph is taken. Thus on one hand a random initial shift of the bands is excluded, on the other hand the fringe shift is doubled. Sagnac used indigo mercury light, plane polarized by Nickol prism to make bands sharper. The interferometer geometry was 866cm^2. Rotation rate was 120rpm or 2Hz and fringe shift observed was 0.07 ± 0.01. Sagnac's instrument was precise enough to measure this fringe shift clearly [23].

Though the increase of rotational speed increases fringe shift imposing less percentage of error, the distortion of the interferometer geometry (change of mirror positions, in particular) due to strong centrifugal force would not give *pure fringe shift*.

Hungarian physicist Bel Pogany [24–26] performed Sagnac experiment with a greater precision in 1926. The parameters were [12]

$$A = 1178 \times \text{cm}^2,$$
$$\omega = 157.43\text{rad/s}(\sim 50\text{Hz or } 3000\text{rpm}),$$
$$\lambda = 5460 \times 10^{-8}.$$

He reproduced the theoretically expected (double) fringe shift $\Delta Z = 0.906$ with an error of 2 per cent. His interferometer was in the form of a square[3]. It consisted of four mirrors and a prism of total internal reflection. The prism was used as the

[3]There is a discrepancy here between the specifications given by Post [12] and Malykin [13]. According to Malykin the sides of the interferometer was 35cm thus making $A = 1225\text{cm}^2$, rotation rate is 1000-2000rpm. Unfortunately the present author does not have access to Pogany's original works to confirm the data.

beam splitter. Light from the source came in and went out as that did in Harress' arrangement.

In 1928 Pogany performed a modified experiment with two 32cm long massive glass rods in the path of the light beam to increase the optical length of the interferometer. With this he demonstrated with high precision that the Sagnac result is independent of the refractive index of the medium in the light path (see also Ref. [6]). Due to the development of wave guide circular interferometer this issue was much discussed in late 70's [27, 28].

Dufour and Prunier [29, 30] showed that the Sagnac effect is practically independent of where the light source is placed – on the platform (Sagnac) or outside the platform (Harress, Pogany). They also showed that if the interferometer rotates but not the medium, then the observed fringe shift increases. Effect of medium vanishes when the relative velocity of medium and the interferometer vanishes [12].

Scheme of a guide circular interferometer was patented by Aaron Wallace on 3 September, 1963 (submitted for consideration on 7 July, 1958). Interestingly, this was first constructed on 1976 [27]. This document first presented the idea of construction of Sagnac experiment using X-rays [7], electrons [2] and γ-rays (the last one is not realized uptill now). But construction of circular lasers [31] made scientists to lose interest in circular interferometers [13].

The whole world of optical interferometers changed after the invention of laser in 1958. In 1961 Javan, Benett and Herriot produced first gas laser. With this the foundation for the creation of the laser gyroscope was laid. Main advantage of laser is that it is possible to maintain a constant phase difference very precisely between two beams. This can be used to measure the optical beat frequencies at the interference as an alternative to fringe shift experiments. Rosenthal [32] in

1962 first suggested the possibility of making a self oscillations version of Sagnac ring by using lasers. The device was made at Sperry Gyroscope Company by Macek and Davis (1963) [31]. Here a helium-neon discharge as the amplifying medium was used to produce the laser. Its wave length was 1150nm and a 1m square optical cavity was used. This kind of experimental set up is, in general, called *ring lasers* [33].

In ring lasers clockwise and counter clockwise modes of laser beams occur in the same optical cavity. Thus an "unusually stable beat between two optical frequencies" [12] can be generated almost ideally. Due to mechanical instabilities of the frequency the individual modes may fluctuate many MHz. But both of the modes have almost the identical fluctuation, so the frequency difference can be stable to within a few Hz.

At high rotation rates, the linear relationship between the beat frequency and rotation rate is maintained. But the problem begins when the rotation rate is reduced – the response becomes non-linear. The beat disappeared altogether at sufficiently low rates. This is called the 'lock-in' region or 'dead-band' region and surpasses the sensitivity required in typical navigation applications by several orders of magnitude. One of the ways to circumvent this problem is to apply "rotation, real or apparent" to " 'bias' the gyroscope away from the lock-in region" [33]. Most favours to date, is to apply an actual rotation, in the form of a mechanical oscillation or 'dither'.

Ring laser is a subject by its own. We restrain to make further discussion of this because it lies outside the basic purview of this dissertation. The readers are referred to an excellent review article (running 103 pages) under the title '*Ring Lasers*' by J. R. Wilkinson [33].

Sagnac effect was performed using rotating superconducting interferometers where the de Broglie waves of a Cooper pair of superconducting electrons in a metal replaces the optical signal in original Sagnac experiment. Here an annulus of superconducting material interrupted at one point by a Josephson junction acts an interferometer. This is called 'superconducting quantum interference device' or SQUID [5, 34–37]. The SQUID allowed a new demonstration of the Bohm-Aharonov effect.

Sagnac type experiment was performed using neutron interferometer. In this case, thermal neutrons with $\lambda \approx 1.3 \times 10^{-10}m$ and $v/c \cong 1.1 \times 10^{-5}$ was used. Three 'ears' sculptured from a single crystal of silicon act as successive Bragg reflectors [35, 38, 39]. The beams interfere at the third 'ear'. Due to Bragg reflection the beams are aligned to an accuracy of 10^{-6}rad, and imported a coherent length of the order of $10^9\lambda$. According to Greenberger [38] the neutron interferometer forms explicit and striking illustrations; twin paradox, sign change of spinors under a 2π rotation, uncertainty, Schrödinger cat paradox etc. [35].

2.5 Formal Derivations of Sagnac Formula

2.5.1 Classical Derivation of Sagnac Effect

The optical Sagnac effect can be suitably analysed by assuming the light circuits to be circular. This can be achieved by constraining the light beams to propagate tangentially to the internal surface of a cylindrical mirror. Within the framework of Newtonian physics it is quite straightforward to calculate the Sagnac phase-shift in terms of the difference of arrival times of the corotating and counterrotating light signals when they are reunited at the beam splitter. We give here a simple derivation [12, 40]. Let us assume that the circular interferometer experiment ('interferograph' in Sagnac's language [13]) is mounted on a turntable (vide

Fig. 2.1 of radius R. The turntable is rotating with a uniform angular velocity ω. It is possible to compute Sagnac shift for an arbitrary angular velocity, "although the analysis becomes somewhat complicated" [41].

In Sagnac's experiment (Fig. 2.2) four mirrors were used and thus the light beams did not follow a path grazing to the periphery of the disc. But as stated earlier, without any problem one can assume that light is traveling a circumferential path. This experiment can indeed be done (and has been already done) with optical fiber as the light guide, where light macroscopically will follow an average circular path.

Let us suppose that a light pulse leaves the beam splitter at position B' and meets the beam splitter again at time t_1 (Fig. 2.1). During this period the beam splitter has moved to a new position B'; hence light has to travel an extra distance (BB')

$$x = vt_1 \tag{2.5.1}$$

with respect to the inertial frame of the laboratory which is at rest with the rotation. One therefore has

$$L + x = ct_1. \tag{2.5.2}$$

In order to write (2.5.2) it is implicitly assumed that the speed of light is c in the laboratory and in accordance with special relativity it is independent of the motion of the source. There is no harm in assuming this as classically this is equivalent to the hypothesis of ether, through which light propagates and which is believed to be stationary with respect to the inertial frame of the laboratory. Eliminating x

from Eqs. (2.5.1) and (2.5.2) one obtains

$$t_1 = \frac{L}{c - v}. \tag{2.5.3}$$

A similar set of arguments for the counter rotating beam gives its round trip time t_2. In this the beam splitter moves to its new position B'', so that with respect to the laboratory, the counter rotating light pulse travels a path shorter than L by the amount $BB'' = vt_2$. For light propagation one may therefore write

$$l - vt_2 = ct_2,$$

$$t_2 = \frac{L}{c + v}. \tag{2.5.4}$$

Similar set of results (Eqs. (2.5.3) and (2.5.4)) may also be obtained from a classical idea that light travels with velocities $c-v$ and $c+v$ for the co and counter rotating beams. This result gives correct prediction upto the limit $v^2/c^2 \gg 1$. The difference in these times is therefore given by

$$\Delta t = t_1 - t_2 = \frac{2lv}{c^2 - v^2}.$$

Substituting $L = 2\pi R$ and $v = \omega R$ we obtain

$$\Delta t = \frac{4\pi r^2 \omega}{c^2}\gamma^2 \tag{2.5.5}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

This is known as classical Sagnac Delay (SD). This is the difference of time of arrival (some times called 'the time of flight') of the co and counter rotating beams at the beam splitter after making a round trip. The phase difference between the two light beams at the beam splitter is thus given by

$$\Delta\phi = \frac{2\pi c}{\lambda}\Delta t, \tag{2.5.6}$$

where λ is the wavelength of used light beam[4]. Explicitly written, this gives [23]

$$\begin{aligned}\Delta\phi &= \frac{2\pi c}{\lambda}\frac{4\pi R^2\omega}{c^2}\gamma^2 \\ &= \frac{8\pi R^2\omega}{\lambda c}\gamma^2.\end{aligned}$$

Thus the fractional fringe shift [33] due to rotation is

$$\Delta Z = \frac{\Delta\phi}{2\pi} = \frac{4\pi R^2\omega}{\lambda c}\gamma^2. \tag{2.5.7}$$

For all practical purposes, $\gamma \approx 1$ and thus the fractional fringe shift reduces to

$$\Delta Z = \frac{4\pi R^2\omega}{\lambda c}. \tag{2.5.8}$$

This is the result in Scalar form given by Post [12].

The result was predicted by Sagnac in some of his earlier papers [23, 42] and some of his papers in 1901. He gave a full account in Ref. [21].

2.5.2 Relativistic Sagnac effect

The theoretical derivation of Sagnac result given earlier (Sec. 2.5.1) is derived from classical mechanics. Also, the whole treatment was done from the perspective of laboratory frame where the disc axis is stationary. Let us now use SR to discuss the time delay. The SD given by Eq. (2.5.5) is obtained from the stationary laboratory frame. As Sagnac effect is essentially an effect on the rotation platform (A linear version of Sagnac effect [40, 43, 44] has been proposed too). The analysis of SD must be done from be view point of an observer on board the platform. The SD obtained in the earlier section is dilated. In $\Delta\tau_{\text{rot}}$ corresponds to the time difference as observed from the rotating frame, this is

[4]The phase difference is given by $2\pi c/\lambda$ (a constant for a given wave length) times the SD. Thus, it is customary to use SD in discussions of Sagnac effect.

related to that on the stationary frame (2.5.5) by the time dilation factor

$$\Delta\tau_{\text{rot}} = \gamma^{-1}\Delta t. \tag{2.5.9}$$

For Eqs. (2.5.5) and (2.5.9) we obtain the expression

$$\Delta\tau_{\text{rot}} = \frac{4\pi\omega R^2}{c^2}\gamma. \tag{2.5.10}$$

This expression is obtained by several authors in a variety of ways. Post deduced this from metrical consideration by using first arbitrary transformation and then specifying it. Dieks and Nienhuis [45] calculated the result by direct use of LT.

An interesting point of the above derivation is that only time dilation effect of SR is used. The authors who derived Eq. (2.5.10) never showed explicitly any cause of not considering the length contraction effect of SR, with the exception of Selleri [46]. The periphery of the disc ($2\pi R$) as seen from the laboratory is the contracted length. Thus if L is the periphery of the disc for the rotating observer (proper length and $L_0 = 2\pi R$ is the length measured by the laboratory observer, then they are related by

$$L_0 = \gamma^{-1}L. \tag{2.5.11}$$

Using this relation with Eq. (2.5.10) we obtain

$$\Delta\tau_{\text{rel}} = \frac{2Lv}{c^2}. \tag{2.5.12}$$

This is the same expression that was experimentally obtained. Thus we obtained two expressions, one is Eq. (2.5.10) and the other one is Eq. (2.5.12) and upto the first order of v/c they agree. The implication of this will be addressed in a separate chapter 3. We refrain from discussing this here any further.

2.5.3 Metrical Treatment of Sagnac Effect

In the previous section a simplistic derivation of relativistic Sagnac effect has been given. In this section we shall give a treatment based on spacetime metric. This formulation will be generalized when we discuss Sagnac effect in curved spacetime in Chap. 7.

The metric of flat spacetime in spherical polar co-ordinate is given by

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\mathrm{d}\phi^2. \tag{2.5.13}$$

For Sagnac experiment

$$r = \text{constant}, \qquad \theta = \text{constant} = \pi/2 \text{ (say)}.$$

With these values, the metric becomes,

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - r^2\mathrm{d}\phi^2. \tag{2.5.14}$$

For the motion of light $\phi = \Omega t$ where Ω is the angular velocity of light. Thus we obtain

$$\mathrm{d}s^2 = c^2\left(1 - \frac{r^2\Omega^2}{c^2}\mathrm{d}t^2\right). \tag{2.5.15}$$

As light follows null geodesics, we may solve Eq. (2.5.15)for Ω and obtain

$$\Omega_\pm = \pm\frac{c}{r}. \tag{2.5.16}$$

The $\pm$ sign here in the right hand side indicates two different directions of light. In Sagnac effect we take the difference of arrival time of two counter rotating beams at the beam splitter. Thus, the $\pm$ sign is unimportant here and ignoring it we obtain

$$\Omega_\pm = \frac{c}{r}. \tag{2.5.17}$$

The angular displacement of light in time t is thus given by

$$\phi_{\pm} = \Omega_{\pm} t. \tag{2.5.18}$$

Now, if the observer is on the platform rotating with an angular velocity ω_0, the rotation angle can be written as

$$\phi_0 = \omega_0 t. \tag{2.5.19}$$

Eliminating t from Eqs. (2.5.18) and (2.5.19) we obtain

$$\phi_{\pm} = \Omega_{\pm} \left(\frac{\phi_0}{\omega_0} \right). \tag{2.5.20}$$

Now, the co-rotating light meets the source after traversing an angle of $2\pi + \phi_{0+}$ angle. Thus

$$\Omega_{+} \left(\frac{\phi_{0+}}{\omega_0} \right) = 2\pi + \phi_{0+}. \tag{2.5.21}$$

or,

$$\phi_{0+} = \frac{2\pi\omega_0}{\Omega_{+} - \omega_0}. \tag{2.5.22}$$

The counterrotating beam meets the beamsplitter after traversing an angle $2\pi - \phi_{0-}$. Thus we can write

$$\phi_{0-} = \frac{2\pi\omega}{\Omega_{-} + \omega}. \tag{2.5.23}$$

These can be written in a combined form

$$\phi_{0\pm} = \frac{2\pi\omega}{\Omega_{\pm} \mp \omega}. \tag{2.5.24}$$

Now the proper time of the observer is given by

$$\mathrm{d}\tau = \sqrt{g_{00}}\, \frac{\mathrm{d}x^0}{c}. \tag{2.5.25}$$

or,

$$d\tau = \sqrt{\left(1 - \frac{r^2\omega^2}{c^2}\right)}\ \frac{d\phi}{\omega}. \tag{2.5.26}$$

Thus, the Sagnac delay in the frame of the observer is found by integrating it from ϕ_{0-} to ϕ_{0+}

$$\delta\tau = \int_{\phi_{0-}}^{\phi_{0+}} d\tau = \left(1 - \frac{r^2\omega^2}{c^2}\right)^{1/2} \frac{\phi_{0+} - \phi_{0-}}{\omega}.$$

Now

$$\phi_{0+} - \phi_{0-} = \frac{4\pi\omega^2 r^2}{c^2}\left(1 - \frac{r^2\omega^2}{c^2}\right)^{-1}.$$

Thus the Sagnac delay is given by

$$\delta\tau = \frac{4\pi\omega r^2}{c^2}\left(1 - \frac{r^2\omega^2}{c^2}\right)^{-1/2} \tag{2.5.27}$$

This is the Sagnac delay in the rotating frame. One must notice here that while we used 'proper time' of the rotating observer, we have not used 'proper length' of the disc. Thus, quite expectedly we obtain the result as in Eq. (2.5.10) with $v = \omega r$. If one now recognizes that the periphery is a contracted one when observed from the laboratory, using Eq. (2.5.11) one obtains the same Eq. (2.5.12).

Thus one may write relativistic SD as follows

$$\Delta\tau_{\text{rel}} = \frac{4\pi\omega r^2}{c^2}\,\Gamma \tag{2.5.28}$$

where $\Gamma = 1$ or $\Gamma = \gamma$. Later we shall see that value of Γ depends on the material of the disc and intimately connected with Ehrenfest paradox. This will be addressed in details in Chapter 3.

2.6 Sagnac Effect in Curved Spacetime : Formal Approach

So far we have reviewed the theoretical aspect of the experiment performed by Sagnac (and others). The experiment is either done in flat spacetime or done in so weak a gravitational field that the correction due to gravitation is negligible. But if the experiment is performed in a strong gravitational field then the correction terms will be considerable. In Chap. 7 we shall address precisely that issue.

In regard to general relativistic Sagnac effect, there are two methods of derivation available. One is metrical derivation first used by Post [12]. This is extended to Kerr field by Tartaglia [47] and we shall discuss a version of this in Chap. 7. Another method of derivation based on differential geometry as mathematical tool has been given by Ashtekar and Magnon [41] and later "elaborated" by Anandan [48]. Here we give a brief review of the treatment given by Ashtekar and Magnon.

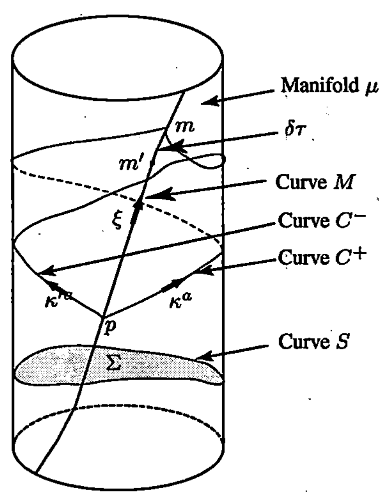


Figure 2.3: Sagnac Experiment in a general spacetime

Ashtekar and Magnaon considers a hollow toroid as the path of light. In practice one may consider the use of optical fibre. At some point there is a half silvered morror as beam splitter. They considered a spacetime $(\mathcal{M}, g_{ab})$ where $\mathcal{M}$ is a 4 manifold and g_{ab} is a metric with signature $(+2)$. In this spacetime, Sagnac tube is represented by a 2D timelike submanifold μ of $\mathcal{M}$. $(G = c = 1)$. The mirror is represented by a timelike curve M. p denotes the event when light is first emitted by the source. Two counter rotating light beams are represented by two null curves C^+ and C^-. ξ^2, κ^a, κ'^a are tangent vectors to the curves M, C^+ and C^- respectively. The rotational frequency of the light rays are thus given by

$$\nu = \kappa_a \xi^a = \kappa'_a \xi^a. \tag{2.6.1}$$

The two events when the light beams meet the mirror again after traversing the round trip are denoted by m and m'. They are represented by first intersecting points of C^+ and C^- with M. Thus, on the spacetime diagram 2.3, SD is the distance mm' measured along the world line M of the mirror.

During the experiment the whole set up should not be deformed (due to centrifugal force). This amounts to demand, in the present context, that the Sagnac tube is rigid, or 4-velocity field ξ^a to be Born rigid on μ (*i.e.* $\mathcal{L}_\xi(g_{ab} + \xi_a \xi_b) = 0$). If now the tube is assumed to be in *stationary* motion (equivalent to *uniform rotation* in Newtonian analysis which simplifies our calculation) and the 2-manifold μ admits a timelike killing vector field and if the tube moves along the trajectories of this Killing field, then the above requirement is naturally satisfied.

The metric induced on μ by the metric g_{ab} on $\mathcal{M}$ is denoted by h_{ab} (signature 0 or $-+$), and the timelike Killing vector field on (μ, h_{ab}) is denoted by t^a. The tube follows the integral curves t^a. The 4-velocity ξ^a of the tube is given by

$$\xi^a = \lambda^{-1/2} t^a \tag{2.6.2}$$

where $t_a t^a = -\lambda$.

μ being a 2 manifold with h_{ab} (sig.0) there are exactly two null directions at each point of ν. Once the event p is fixed, the null curves C^+ and C^-, events m and m' and the Sagnac shift $\Delta\tau$ are completely determined. However, for convenience, the authors introduce two null vector fields κ^a and κ'^a tangent to

C^+ and C^- such that

$$\kappa^a t_a = \kappa'^a t_a = -1 \qquad (2.6.3)$$

From, Eqs. (2.6.2) and (2.6.3) one obtains the value of ν (2.6.1)

$$\nu = -\lambda^{1/2} \qquad (2.6.4)$$

As these vectors are curl free, the integrals

$$I = \oint_c \kappa_a \mathrm{d}s^a \qquad \text{and} \qquad I' = \oint_{c'} \kappa'_a \mathrm{d}s^a$$

are independent of the particular choice of the closed curves c and c' on μ. The curves are defined in the following way (Fig. 2.3)

$c \Rightarrow$ closed curve pmp along C^+.

$c' \Rightarrow$ closed curve $pm'p$ along C^-.

As κ^a and κ'^a are null

$$I = \int_{\substack{m \\ \mathcal{M}M}}^{p} \kappa_a \mathrm{d}s^a \qquad \text{and} \qquad I' = \int_{\substack{p \\ \mathcal{M}M}}^{m'} \kappa'_a \mathrm{d}s^a. \qquad (2.6.5)$$

and the integrals are evaluated along $\mathcal{M}M$. Using Eqs. (2.6.2) and (2.6.3) in Eq. (2.6.5) one may write

$$I + I' = (\lambda_M)^{-1/2}\ \Delta\tau$$

where λ_M is the value of the scalar field along the worldline M of the mirror and $\Delta\tau$ is the distance between the events m and m' evaluated along M.

Now, in place of c in the integrals if any closed curve S on μ is chosen, then

$$I + I' = 2\oint_S \lambda^{-1}\ t_a \mathrm{d}s^a.$$

Thus the Sagnac shift is given by

$$\Delta\tau = 2\,(\lambda_M)^{1/2} \oint_S \lambda^{-1}\, t_a \mathrm{d}s^a, \tag{2.6.6}$$

or alternatively,

$$\Delta\tau = 2\,(\nu_M)^{-1} \oint_S \nu\, \xi_a \mathrm{d}s^a, \tag{2.6.7}$$

where ν (2.6.1) is the frequency of the light as seen by ξ^a and ν_M is that along the world line M of the mirror.

One can notice that the integration does not depend on the choice of S, and SD depends upon ν_M. If the position of the mirror is changed SD will also change unlike the Newtonian case where it is independent of the location of the mirror on the tube. Ashtekar and Magnon comments on this difference –

> Whereas in general relativity the frequency of the light rays remains constant only along the world line of a point on the tube, but changes from one point of the tube to another, in the Newtonian frame work the frequency is constant everywhere on the tube.

Ashtekar and Magnon then considered two cases . For the first, it is stationary object. It is, as usual in GR, expressed by a timelike Killing vector field t^a and its state of rotation by a twist t^a. Using Stoke's law in Eq. (2.6.6), they obtained the SD as

$$\begin{aligned}\Delta\tau &= 2\,(\lambda_M)^{1/2} \int_\Sigma \nabla_{[a}\lambda^{-1} t_{b]} \mathrm{d}s^{ab} \\ &= (\lambda_M)^{1/2} \int_\Sigma \lambda^{-3/2} \omega^a\, \epsilon_{abc}\, \mathrm{d}s^{bc},\end{aligned} \tag{2.6.8}$$

where,

$$\nabla_a = \text{any derivative operator on } \mathcal{M},$$

$$\epsilon_{abc} = \text{alternating tensor},$$

$$\epsilon_{abc} = \epsilon_{abc}\lambda^{-1/2}t^d,$$

$$\omega^a = \epsilon^{abcd}t_b\nabla_c t_d, \text{the twist of Killing field } t^a.$$

Thus Sagnac shift is interpreted as a measure of the flux of ($\lambda^{-3/2}$ times) the twist of Killing field through the tube.

In the second case the object is considered to be a stationary axisymmetric. Such an object in GR is described by two commuting Killing vector fields; one timelike, in some neighbourhood of spatial infinity and one rotational, denoted by T^a and R^a, respectively. The rotation is thus represented by T^aR_a. Thus they obtained, from Eq. (2.6.6) as

$$\Delta\tau = \left[2\lambda^{-1/2}\left(R^aR_a\right)^{-1/2}(\text{length of } S)\right]\left(T^aR_a\right). \qquad (2.6.9)$$

In Chap. 7 we shall however, follow a simpleminded approach by directly using the explicit forms of the metric possessing physically interisting symmetries. The approach will be the extended version of metrical treatment in flat spacetime [12, 49–51]. Tartaglia [52] also followed this approach. Indeed we shall see that Tartaglia's results agree with ours in the same physical situations as special cases.

2.7 GPS and Sagnac Effect

As described earlier, there are several application of Sagnac effect. Here we briefly mention one such application due to its immense importance at the present age. This is called global position system (GPS). A GPS is a system constructed to make navigation on or near the surface of earth, and to provide accurate worldwide clock synchronization and timing system. The whole procedure is achieved by three principal "segments', a space segment, a control segment and a user segment [53].

The space segment is consisted of some satellites carrying atomic clocks. Presently 24 satellites are working. Spare satellites and spare clocks are also provided.

To provide at least four satellites always above the local horizon, "four satellites in each of six orbital planes inclined at 55^o with respect to earth's equatorial plane"[53]. Navigation and timing signals are provided with the clocks.

The data from satellites are collected by several ground-based monitoring stations. From these stations data are then sent to a Master Control station (MCS) situated at Colorado Spring, CO. This MCS then "analyzes the constellation and projects the satellite ephemerides and clock behaviour forward for the next few hours. This information is then uploaded into the satellites for retransmission to users". The whole segment is called the control segment.

The user segment consists of all the users. These users receive signals retransmitted from the satellites and can determine their position, velocity and the time on their local clocks.

In the GPS the primary reference frame is Earth centred, Earth fixed system, the WGS-84 frame, because users are interested to know their positions on the surface (or very near it) of earth. This makes it clear that due to the rotation of earth one must consider the Sagnac effect to do any calculations accurate. As Ashby [53] points out that a very small Sagnac correction of one nanosecond can save a navigational error 30 cm, while Sagnac correction amount to hundreds of nanoseconds. The reader is referred to the article by Neil Ashby [53] for detailed discussion, where he describes Sagnac (or "Sagnac like") corrections "from the point of view of the local inertial frame and also, from the point of view of an earth-centred rotating frame in which Sagnac effect is described by terms in the fundamental scalar invariant that couple space and time.

2.8 The Area factor: the LSE and the FOC Experiments

It has long been believed that Sagnac phase shift depends on the projection of the area of the light contour on the plane of the disc. The first order Sagnac phase shift form is usually quoted in terms of that (vide Eq. (2.4.1)). Consequently if a modified Sagnac type experiment is performed in the laboratory which contains the essential elements of Sagnac experiment but have zero projection area, we should expect zero phase. Recent experiments (gedanken and laboratory) suggest that there is no reason for obtaining a zero phase shift in this case. Indeed, it has been shown [40, 43, 44] that the result depends on the length of the light path and not on the projection area, because the thought experiment corresponds to a zero area configuration.

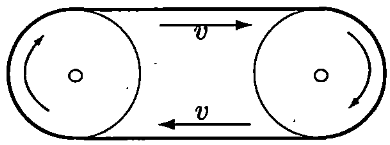
2.8.1 Linear Sagnac Effect (LSE)

The basic ingredients of Sagnac experiment are that the light signals moving parallel and antiparallel to the motion of the platform throughout its journey complete round trip with the help of mirrors on a platform moving with uniform "speed" in the laboratory. Let us now consider a set of two experiments which modifies Sagnac experiment retaining its essential ingredients. We consider a rigid platform with a light source and a facing mirror at the two extreme ends. Light is emitted from the source, reaches the mirror to be reflected back and reaches the source point again. This is essentially the round trip feature of motion of light beams in the Sagnac type experiment. A mechanism is attached to the platform which reverses the journey of it at the instant when light falls on the mirror. For the first experiment, when light is emitted from the source, the platform moves in the same direction as light travels with respect to the laboratory while reverses its direction when light is reflected by the mirror. Thus throughout the journey the motion of light remains parallel to the motion of the platform mimicking the paralel propagation of light beam in Sagnac experiment. For the second experiment the same mechanism is adopted except for the fact that the direction of light is antiparallel to direction of platform throughout its round trip journey mimicking the antiparallel propagation of light in in the Sagnac type experiment. The difference between the round-trip times gives the Sagnac formula as function of the length of the platform, *i.e.* length of the light contour. This is referred to as linear Sagnac effect [40, 43, 44]. This has been discussed in detail in Sec. 3.4. One should note that though the area enclosed by light contour is zero, the Sagnac phase shift is still non-zero and the length of the light contour determines the Sagnac phase shift formula.

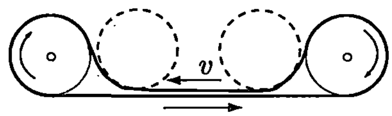
2.8.2 FOC Experiment

An experiment has been recently performed by Wang, Zheng, Yao and Langley [54] and must be referred to in connection with the LSE. The authors modified a Fibre optic gyroscope (FOG) by adding an extra 50m of single mode fibre to the original fibre loop in their experiment with a new device which they call fibre optic conveyor(FOC). They wrapped the extra fibre onto a polyester ribbon that went around two wheels with diameter of 30cm. A mechanical conveyor is used to rotate one of the wheels which makes the fibre move. As an FOG is sensitive to rotational motion, even though the FOG travels with the mechanical conveyor its uniform motion does not cause any phase shift.

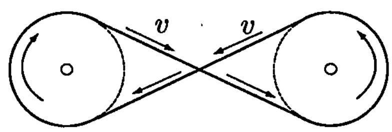
The group performed the experiment with 24 different arrangements of conveyor speeds, fibre lengths and three different configurations as shown in the figures.



(a) Regular configuration



(b) Zero-area configuration



(c) Figure-8 configuration

Figure 2.4: FOC Experiments of Wang, Zheng, Yao and Langley

Fig. 2.4(a) is an FOC experiment in a *regular configuration.* Several fibre lengths were considered. The setup in Fig. 2.4(b) is called the *zero-area configuration* since the area enclosed by the fibre cable (light contour) is almost zero except for the areas enclosed at the two ends because of the wheel. The experimental setup shown in Fig. 2.4(c) is called *figure-8 configuration* for obvious reasons. Their contention is that from rotational view point the two enclosed areas having opposite directions cancel other's effect simulating a zero-area experiment. The whole arrangement and its conclusions fit well with LSE (gedanken) experiment although with minor adjustment in the reasoning.

The group observed the Sagnac effect, obtained the result as first order effect and concluded –

1. Sagnac phase shift depends on the lengths and speed of the moving fibre and these are the fundamental factors, rather than the enclosed area determining

the SD, as customarily quoted (say, in Ref. [12]) – because they observed Sagnac effect even when their FOC has zero-area.

2. Sagnac effect is also observable for uniform motion – as they authors observed Sagnac effect when the interferometer is in linear motion.

We make the following observations regarding their conclusion. First, they rightly observe that Sagnac effect is dependent on the length of the path light travels and the velocity of the platform. The difference of distances traveled by the two counter rotating beams before reaching the beam splitter sets a phase difference between them. This difference depends on the length of the path and the velocity of the platform. Note that in the limit where the radii of the two wheels vanishes, the FOC experiments simulates LSE and becomes ideally a zero-area experiment. The LSE (thought) and FOC (laboratory) experiment prove beyond doubt that Sagnac formula has nothing to do with the area enclosed by the light contour. However, one is certainly free to choose his/her own way of quoting Sagnac formula – in terms of the length of light path or the area enclosed by the light contour – although, in view of these experiments the former is preferable.

Their second conclusion is rather superficial although this is essentially their guiding principle in designing the zero-area experiment. The essential ingredient of the Sagnac effect lies in the facts that 1) Mutually counter rotating light beams complete a round-trip before reunion and 2) the platform suffers acceleration at any point on the path of the light beams. It does not matter if the measuring device is moving in uniform motion at the time of measurement *locally* because *globally* the platform is not an inertial frame. This is evident from our analysis of LSE. In fact, one of the most important contributions of the FOC experiment to the understanding of the Sagnac effect is that rotation of the platform is not an essential ingredient, rather it is a *special case*.

Reference

[1] G. Sagnac, *Comp. Rend. Acad. Sci.*, **157**, (1913) pp. 708–710.

[2] H. Hasselbach and M. Nicklaus, *Phys. Rev. A*, **48**, (1993) pp. 143–151.

[3] R. Colella, A. W. Overhauser, J. L. Staudenmann and S.A. Werner, *Phys. Rev. A*, **21**, (1980) pp. 1419–1438.

[4] F. Riehle, Th. Kister, A. Wittie, J. Helmcke and ch. J. Borde, *Phys. Rev. Lett.*, **67**, (1991) pp. 177–180.

[5] D. Fargion, L. Chiatti and A. Aiello, "Quantum mach effect by sagnac phase shift on cooper pair in rf-squid", Preprint astro-ph/9606117, babbage.sissa.it.

[6] I. L. Bershtein, *Dokl. Akad. Nauk. SSSR*, **75**, (1950) p. 635.

[7] V. Vyschtskii *et al.*, *Phys. Usp.*, **62**, (1994) p. 289.

[8] S. Merhab, *Aerospace Sensor Systems and Applications* (Springer, New York, 1996).

[9] D. W. Allan *et al.*, *IEEE Trans. Instrum. Meas.*, **IM-34**(2), (1985) pp. 118–125.

[10] M. Yamamoto N. Saburi and K. Harada, *IEEE Trans. Instr. Meas.*, **IM-25**, (1976) pp. 473–477.

[11] G. Petit and G. Wolf, *Astron. Astrophys.*, **286**, (1994) pp. 971–977.

[12] E. J. Post, "Sagnac effect", *Rev. Mod. Phys.*, **39**, (1967) p. 475.

[13] G. B. Malykin, "Sagnac effect", *Physics-Uspekhi*, **40**(3), (1997) pp. 317–321.

[14] Oliver Lodge, *Philos. Trans. R. Soc. (London)*, **184A**, (1893) p. 727.

[15] Oliver Lodge, *Philos. Trans. R. Soc. (London)*, **189A**, (1897) p. 149.

[16] A. A. Michelson, *Philos. Mag*, **8**(56), (1904) p. 716.

[17] P. Harzer, *Astron. Nach.*, **198**, (1914) p. 378.

[18] A. Einstein, *Astron. Nach.*, **198**, (1914) p. 378.

[19] G. Sagnac, *Compt. Rend.*, **157**, (1913) p. 1410.

[20] G. Sagnac, *J. Phys. (Paris)*, **4**, (1914) p. 177.

[21] G. Sagnac, *J. Phys. Radium Ser.*, **4**, (1914) pp. 177–195.

[22] V. A. Ugarov, *Special Theory of Relativity* (Mir Publisher, Moscow, 1979). Translated from Russian by Yuri Atanov.

[23] R. Anderson, H. R. Bilger and G. E. Stedman, *Am. J. Phys.*, **62**(11), (1994) pp. 975–985.

[24] B. Pogany, *Ann. der Phys.*, **80**, (1926) p. 217.

[25] B. Pogany, *Naturwissenschaften*, **15**, (1927) p. 177.

[26] B. Pogany, *Ann. der Phys.*, **85**, (1928) p. 244.

[27] V. Vali, R. W. Shorthill and M. F. Berg, *Appl. Opt.*, **16**, (1977) p. 2605.

[28] G. Schiffner W. R. Leeb and E. Scheiterer, *Appl. Opt.*, **18**, (1979) p. 1293.

[29] A. Dufour and F. Prunier, *Compt. Rend.*, **204**, (1937) p. 1322.

[30] A. Dufour and F. Prunier, *Compt. Rend.*, **205**, (1937) p. 658.

[31] W. M. Macek and D. T. M Davis., *Appl. Phys. Lett.*, **2**, (1963) p. 67.

[32] A. H. Rosenthal, *J. Opt. Soc. Am.*, **52**, (1962) p. 1143.

[33] J. R. Wilkinson, "Ring lasers", *Prog. Quant. Electr.*, **11**, (1987) pp. 1–103.

[34] B. W. Petley, *Contemp. Phys.*, **21**, (1980) p. 607.

[35] G. E. Stedman, *Contemp. Phys.*, **26**, (1985) p. 311.

[36] J. E. Zimmerman, *Cryogenics*, **20**, (1980) p. 3.

[37] J. E. Zimmerman and J. E. Mercereau, *Phys. Rev. Lett.*, **14**, (1965) p. 887.

[38] D. M. Greenberger, *Rev. Mod. Phys.*, **55**, (1983) p. 875.

[39] S. A. Werner, *Phys. Today*, **33**, (1980) p. 24.

[40] S. K. Ghosal, Biplab Raychaudhuri, Anjan Kumar Chowdhuri and Minakshi Sarker, "Relativistic sagnac effect and ehrenfest paradox", *Found. Phys.*, **33**(6), (2003) pp. 981–1001.

[41] A. Ashtekar and A. Magnon, *J. Math. Phys.*, **16**(2), (1975) pp. 341–344.

[42] G. Sagnac, *Compte. Rendus.*, **141**, (1905) p. 1220.

[43] S. K. Ghosal, Biplab Raychaudhuri, Anjan Kumar Chowdhuri and Minakshi Sarker, "Rotating disc problem and sagnac phase-shift formula", in M. C. Duffy (Ed.), "Physical Interpretations of Relativity Theory, VIII", (P. D. Publication, Liverpool, 2000). Held at Imperial College London.

[44] S.K. Ghosal and Minakshi Sarker, "Cosmologically preferred frame and sagnac effect", in M. C. Duffy (Ed.), "Physical Interpretation of Relativity Theory (VII)", volume Late Papers (1998). Held at Imperial College, London.

[45] D. Dieks and G. Nienhuis, *Am. J. Phys.*, **58**, (1990) pp. 650–655.

[46] F. Selleri, *Found. Phys.*, **26**, pp. 641–664.

[47] A. Tartaglia, *Found. Phys. Lett.*, **12**(1), (1999) pp. 17–28.

[48] J. Anandan, *Phys. Rev. D*, **24**, (1981) pp. 338–346.

[49] A. A. Logunov and Yu. V. Chugree, *Sov. Phys. Uspekhi*, **31**, (1988) p. 861.

[50] P. Langevin, *Compt. Rend.*, **173**, (1921) p. 831.

[51] P. Langevin, *Compt. Rend.*, **205**, (1937) p. 51.

[52] A. Tartaglia, *Phys. Rev. D*, **58**, (1998) p. 064009.

[53] Neil Ashby, "The sagnac effect in the global positioning system", in G. Rizzi and M. L. Ruggiero (Eds.), "Relativity in Rotating Frame", Fundamental Theories of Physics (Kluwer, Dordrecht, 2004).

[54] R. Wang, Yi. Zheng, Aiping Yao and D. Langley, *Phys. Lett. A*, **312**, (2003) pp. 7–10.