Chapter 7

Sagnac Effect in Curved Spacetime

7.1 Introduction

In the last chapter we remarked that introduction of GR in explaining the Sagnac effect from the rotating observer's perspective is misleading from conceptual standpoint since essentially here we are dealing with flat spacetime. However this does not means that there cannot be such an effect in a curved spacetime, and on the contrary the full machinery of GR is required to calculate the effect in presence of a gravitational field. Therefore, apart from the special relativistic discussion of Sagnac effect, there have been several endeavours to discuss this effect in a curved spacetime [1–3]. Ashtekar and Magnon [1] provides an elegant mathematical treatment of this effect in a general co-ordinate systems. In their description which discusses the problem from a geometrical point of view, the path of light beams in the Sagnac experiment travel in a toroidal tube with perfectly reflecting internal walls, called the Sagnac tube. The Sagnac tube is described by a two dimensional timelike submanifold embedded in the usual four dimensional spacetime. In this submanifold the beam splitter is represented by a timelike curve and light rays travel in two null geodesics. The emission of light beams and the superposition of light beams on the mirror after their (single) roundtrip journey are represented by two events on the curve. The events are the first intersections of the null geodesics and the worldline of the mirror. The Sagnac phase shift is given by the distance between the two events along the worldline of the mirror. The authors avoided mathematical complications by assuming that 1) *the submanifold allows a timelike Killing field vector field* and 2) *the Sagnac tube has a stationary motion along the trajectories of the Killing field.* They also contended that general relativistic Sagnac effect represents a gravitational analog of Aharonov-Bohm experiment in electrodynamics and the Sagnac shift "may be regarded as a measure of the flux, through the tube, of the natural magnetic field" associated with the Killing vector. We have discussed the treatment of Ashtekar and Magnon in Sec. 2.6. Anandan [2] provides analyses of the Sagnac effect in 'relativistic and non-relativistic' world and also discusses a group-theoretic treatment.

Although elegant, these analyses seem to be too sophisticated to be used directly in physically interesting situations. From the experimental side, the accuracy of the measurements of the Sagnac effect has reached an unprecedented precision due to the introduction of ring lasers. The great accuracy of these measurements poses the problem of higher order corrections to SD which have been sought usually through special relativistic approach.[1] In view of possibility of achieving greater accuracies, it will be interesting to study the general relativistic corrections to the Sagnac effect performed on rotating disc on the surface of a massive body. After all, all the Sagnac experiments are performed on the surface of the earth which is a massive body. In fact the detail calculation will show the general relativistic corrections is often more than the special relativistic one due to the γ factor or its absence.

Earlier Tartglia [3] has studied the effect in a situation where the observer of the effect is a satellite like object orbiting round a massive body (in one situation the turn table itself is the source of the gravitational field). In the present chapter the effect will be derived for a *disc-type* experiment where the turn table is placed in an axisymmetric gravitational field. Instead of spherical symmetry the term 'axisymmetry' has been chosen to highlight the fact that the earth is a rotating massive body. Primarily this rotation of the source will have influence on the Sagnac effect only through the stationarity of the gravitational field (Kerr field).

[1]From the experimental point of view whether the SD formula contains a γ factor (honouring the Ehrenfest paradox) or not is still an open question. However, from theoretical point of view the issue has been discussed in detail in Chap. 3

However, it will be shown that special cases of the present gedanken set up include a scenario where the beam splitter is placed on the surface of the earth which then acts as a turn table. Other special cases which include observers (beam splitter) moving in natural (geodetic) or unnatural (non-geodetic) orbits will also be taken up. To make the calculations more general, we shall work in Kerr-Newman spacetime. The terrestrial situation can then be understood as a special case. However, in order to not to lose forest in the trees we will do the calculations in the Schwarzschild metric first before making the generalizations. Although the primary intention has been to understand small general relativistic corrections to the disc type[2] Sagnac effect in the far field, one cannot but explore interesting general relativistic effects in the deep field. Some interesting such effects will be discussed in the connection. Although these things may be just academic – indeed nobody is going to perform a Sagnac experiment around neutron star or a black hole, however the effort will give us a lot of insight into the nature of such deep fields in the pre-horizon regime.

7.2 Sagnac effect in a Schwarzschild field

Let us now discuss the general relativistic correction to Sagnac effect. Consider a Sagnac experiment performed in a Schwarzschild field. Approximately this is the terrestrial gravitational field which will influence the light propagation on the Sagnac disc. Expressed in geometrical unit ($G = c = 1$) the metric is given by

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 - r^2\, \mathrm{d}\theta^2 - r^2\, \sin^2\theta\, \mathrm{d}\phi^2, \quad (7.2.1)$$

[2]The term 'disc' includes in its meaning also the 'Sagnac tube' mentioned in the derivation of Ashtekar and Magnaon since here also the light path defines a 2-manifold embedded in the spacetime.

where M is the source mass. For simplicity let us consider the axis of the massless Sagnac disc to coincide with the polar axis of the field.[3] In Sagnac experiment light travels along the periphery of the plane disc. This gives r = constant and θ = constant. Writing $k = \sin\theta$ and $F(r) = \left(1 - \frac{2M}{r}\right)$ in Eq. (7.2.1) we obtain

$$\mathrm{d}s^2 = F(r)\,\mathrm{d}t^2 - r^2k^2\mathrm{d}\phi^2.$$

If we consider the angular momentum of light beam moving in a circular path to be Ω then, $\phi = \Omega t$. Substituting,

$$\mathrm{d}s^2 = \left[F(r) - r^2k^2\Omega^2\right]\mathrm{d}t^2. \tag{7.2.2}$$

For light, $\mathrm{d}s^2 = 0$, giving

$$\Omega_\pm = \pm\frac{\sqrt{F(r)}}{rk}. \tag{7.2.3}$$

The $\pm$ sign in Eq. (7.2.3) indicates two different directions of the motion of light. This is insignificant in the present context because in the Sagnac effect we take the difference of times of reaching of two counter-rotating beams. Thus the angular displacement of light in time t is given by

$$\Omega_+ = \Omega_- = \Omega_\pm = \frac{F(r)^{1/2}}{rk}. \tag{7.2.4}$$

The motive behind still retaining the $\pm$ sign in $\Omega_\pm$ will be evident in the forthcoming sections when we generalize the calculations to Kerr-Newmann field. If the angular velocity of the rotating observer is ω_0, the rotation angle will be

$$\phi_0 = \omega_0 t. \tag{7.2.5}$$

[3]In the Schwarzschild case the choice of the polar axis is arbitrary, hence in the context of a terrestrial experiment the analyses presented in this section is equivalent to assuming the turn table to be horizontal at any point on the earth surface. This conclusion would however not be valid if an axissymetric metric were used since the polar axis in this case would represent that of the rotation of the source producing the field.

Eliminating t from Eqs. (7.2.4) and (7.2.5), we obtain

$$\phi_{\pm} = \Omega_{\pm}\left(\frac{\phi_0}{\omega_0}\right). \qquad (7.2.6)$$

The co-rotating light meets the beam-splitter after traversing an angle $2\pi + \phi_{0+}$ where ϕ_{0+} is the extra path traversed by the co-rotating beam. Thus

$$\Omega_{+}\left(\frac{\phi_{0+}}{\omega_0}\right) = 2\pi + \phi_{0+}.$$

Solving for ϕ_{0+} we obtain

$$\phi_{0+} = \frac{2\pi\omega_0}{\Omega_{+} - \omega_0}.$$

Similar argument for counter-rotating light leads to,

$$\phi_{0-} = \frac{2\pi\omega_0}{\Omega_{-} + \omega_0}.$$

In a compact form, we can write

$$\phi_{0\pm} = \frac{2\pi\omega_0}{\Omega_{\pm} \mp \omega_0} \qquad (7.2.7)$$

where $\Omega_{+} = \Omega_{-} = \dfrac{F(r)^{1/2}}{rk}$.

The proper time of the rotating observer moving with an angular velocity ω_0 in a Schwarzschild metric is given by

$$\mathrm{d}\tau = \left[F(r) - r^2k^2\omega_0^2\right]^{\frac{1}{2}}\frac{\mathrm{d}\phi}{\omega_0}. \qquad (7.2.8)$$

Thus the SD in the frame of rotating observer is found by integrating from ϕ_{0-} to ϕ_{0+}

$$\delta\tau = \left[F(r) - r^2k^2\omega_0^2\right]^{\frac{1}{2}}\frac{\phi_{0+} - \phi_{0-}}{\omega_0}. \qquad (7.2.9)$$

Putting the values. we obtain the SD for Schwarzschild metric for a polar orbit

$$\delta\tau = \frac{4\pi\omega_0\rho^2}{\left(1 - \frac{2M}{r} - \omega_0^2\rho^2\right)^{1/2}} \qquad (7.2.10)$$

where $\rho = rk = r\sin\theta$. Note that ρ is effectively the radius of the polar orbit, *i.e.* the radius of the disc. Also note that this formula reduces to special relativistic one, Eq. (2.5.10) if M is put equal to zero.

Evidently, this calculation holds even if the field is the Reissner-Norstrøm one:

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2, \tag{7.2.11}$$

where Q is the charge of the star. The calculations will be unchanged except that in this case $F(r)$ will be given by

$$F(r) = \left(1 - \frac{2M}{r} - \frac{Q^2}{r^2}\right)$$

in all the expression upto Eq. (7.2.9) and the expression will be

$$\delta\tau = \frac{4\pi\omega_0\rho^2}{\left(1 - \frac{2M}{r} - \frac{Q^2}{r^2} - \omega_0^2\rho^2\right)^{1/2}}. \tag{7.2.12}$$

However this exercise is purely academic without any immediate observational importance. As a special case, if this ω_0 is assumed to be the angular velocity of the source the results (Eqs. (7.2.10) and (7.2.12) would correspond to the massive turn table scenario as mentioned in Sec. 7.1 where the beam splitter rotates with the gravitating body.

7.3 Sagnac Effect in an Equatorial Orbit

7.3.1 Experiment with Rockets

One may now put $\theta = \pi/2$ to get the result for an equatorial orbit of light. To simulate this case one has to construct an experiment using a system of rockets which forces the observer to move in a circular path with the massive source as the centre. One orbiting rocket-system will be the observer *i.e.* the beam-splitter.

Light will be forced to follow the same circular orbit by some mirrors, orbiting with other such rocket systems. Alternately one may think that as if the disc is now an annular one which contains the earth at the center. A special case would be that with annular thickness equal to zero and the disc rotating with the planet. The situation is equivalent to that with the observer on the earth observing fringe shift by sending and receiving the counter propagating light beams around the earth.

Coming back to the rocket arrangement, note that, all the calculations done in Sec. 7.2 smoothly go over to the equatorial case with $\theta \to \pi/2$. The result is thus given by

$$\delta\tau = \frac{4\pi\omega_0 r^2}{\left(1 - \frac{2M}{r} - r^2\omega_0^2\right)^{1/2}}. \tag{7.3.1}$$

This, again, reduces to the usual special relativistic Sagnac formula (with a γ factor) in the special relativistic limit, *i.e.* $M = 0$ for unbonded particle system (2.5.10), (vide Ref. [4, 5]). This is expected as rocket type experiment with the orbiting mirrors obviously consists of an unbonded system.

7.3.2 Natural Orbit, the "Free Falling Observer"

In the previous case, both the observer (rocket controlled) and the light beams are forced to move in the same circular orbit by some mechanism. Therefore these are not geodetic or natural orbits. In a geodetic orbit observer follows a particular path as soon as the initial conditions are set. As long as the coordinate radius $r > 3M$, geodetic 'circular' orbits for a test particle (observer) exists – hence in principle a Sagnac experiment can be performed by these free falling observers and mirrors although the circular trajectories of the light beams may no correspond to null geodesics. However, at $r = 3M$, both the light beams and the observer move in a common circular trajectory, *i.e.* both the beam splitter and a photon can (in principle) follow natural orbits without any rocket and mirror arrangements. For equatorial natural orbit, clearly ρ is to be replaced by r in Eq. (7.3.1). Also ω_0 is then not a free parameter, indeed as will be shown below that it is a function of r. Hence the Sagnac formula in this case is a function of the coordinate radius only. Imposition of the circular orbit condition on the geodesic fixes one of the conserved quantities U_3 and U_0 which is obtained as [6, pp. 106]

$$\frac{U_3}{U_0} = \left[\frac{Mr^3}{(r-2M)^2}\right]^{1/2}, \tag{7.3.2}$$

where U_μ = the μ-th component of the covariant velocity, *i.e.* $U^3 = \frac{\mathrm{d}\phi}{\mathrm{d}s}$ and $U^0 = \frac{\mathrm{d}t}{\mathrm{d}s}$, $\mathrm{d}s$ being the arc length.

In this case the coordinate angular speed of the observer can be calculated as [6, 7]

$$\omega_0 = \frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{U_3}{U_0}\frac{1-\dfrac{2M}{r}}{r^2}. \tag{7.3.3}$$

Plugging in Eq. (7.3.2) in Eq. (7.3.3) gives

$$\omega_0 = -\left[\frac{Mr^3}{(r-2M)^2}\right]^{1/2}\frac{1-\dfrac{2M}{r}}{r^2},$$

i.e.

$$\omega_0^2 r^2 = \frac{M}{r}. \tag{7.3.4}$$

Using this, the Sagnac formula reduces to

$$\delta\tau = \frac{4\pi\sqrt{Mr}}{\left(1-\dfrac{3M}{r}\right)^{1/2}} \tag{7.3.5}$$

which is now for a given source a function of coordinate radius alone. An interesting point here is that there is a singularity at $r = 3M$. The equation $r = 3M$ stands for the last circular orbit for a particle, while this is the only natural orbit for light. Another interesting matter concerning the orbit is that if we want to carry on the experiment in this orbit, we do not need any rocket arrangement for the satellite to make it revolve round the gravitating source. Also for light propagation no mirror system is required to guide it. However in this case $\delta\tau$ is arbitrarily large.

The reason for this singularity is not far to seek. One intuitively understands that infinite Sagnac effect can take place if light after its emergence from the source takes arbitrarily large time to reunite with the beam splitter in one of its journeys. Surely in this case the coordinate speed of light must be equal to that of the observer. From the global point of view as if, in this case, the wavefromt of light for the corotatin beam should appear to stand still with respect to the beam splitter. That this stipulation is correct can be shown as follows:

From Eq. (7.2.4), the coordinate speed of a photon is

$$v_{\text{photon}} = \Omega r = \sqrt{1 - \frac{2M}{r}}.$$

For a particle the same is given by (see Eq. (7.3.4))

$$v_{\text{particle}} = \sqrt{\frac{M}{r}}.$$

The equality of these speeds now give the condition

$$\sqrt{1 - \frac{2M}{r}} = \sqrt{M} r$$

which gives $r = 3M$.

Carrying on this sort of argument in the disc type arrangement one also expects a singularity. In this case, as

$$\begin{aligned} v_{\text{particle}} &= \omega_0 \rho \\ v_{\text{photon}} &= \Omega \rho = \sqrt{1 - \frac{2M}{r}} \end{aligned} \tag{7.3.6}$$

(vide Eq. (7.2.4)). The equality of these speeds in this case gives

$$\omega_0 = \frac{1}{\rho}\sqrt{1 - \frac{2M}{r}}. \tag{7.3.7}$$

This condition if put in Eq. (7.2.10) gives infinite value for $\delta\tau$. Thus for a given disc size ρ one expects a singularity if the angular velocity of the disc has the value given by the expression (7.3.7).

Coming back to the free falling situation, arbitrarily large Sagnac effect does not pertain to observation on Earth or on a normal star. In order to access $r = 3M$ (a region where the Schwarzschild exterior solution is valid), one must consider a very compact star or a black hole. We shall hence use the term black hole instead of a "star" in the following discussion.

7.4 Sagnac Effect in Kerr-Newman Field

In the last section we have discussed two types of Sagnac experiment in the field of a Schwarzschild black hole (viz. disc type and sattelite based). The same calculation has been extended easily in the Reissner-Nordstrøm field. Let us now perform these calculations in the field of a Kerr-Newman black hole. At first we shall discuss the disc type experiment.

The Kerr-Newman geometry expressed in Boyer-Lindquist coordinate is given by [8, page 877-878] (in geometrical units)

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - adt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \tag{7.4.1}$$

where

$$\begin{aligned} \Delta &= r^2 - 2Mr + a^2 + Q^2, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \\ a &= J/M = \text{angular momentum per unit mass}, \\ Q &= \text{charge of the source}. \end{aligned}$$

Please note that the special cases are

$Q = 0,$	Kerr geometry,
$J = 0$	Reisner-Nordstrøm geometry,
$Q = J = 0$	Schwarzschild geometry,
$M^2 = Q^2 + a^2$	"Extreme Kerr-Newman geometry."

Also note that in Boyer-Lindquist co-ordinate the black hole rotates in the ϕ direction [8]. In Sagnac experiment light travels along the edge of a plane disc. As before we define the disc

$$\begin{aligned} r &= \text{constant}, \\ \theta &= \text{constant}, \\ \sin\theta &= \text{constant} = k. \end{aligned} \tag{7.4.2}$$

Unlike the Schwarzschild case however the disc orientation (perpendicular to the polar axis) has an absolute significance, since the polar axes represents the axis of rotation. Putting the conditions (7.4.2) in Eq. (7.4.1) we obtain

$$\begin{aligned} \mathrm{d}s^2 = \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2(1-k^2)}(\mathrm{d}t - ak^2\mathrm{d}\phi)^2 \\ - \frac{k^2}{r^2 + a^2(1-k^2)}\left[(r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t\right]^2. \end{aligned} \tag{7.4.3}$$

Let us make the following substitutions to simplify our calculation

$$\begin{aligned} A &= r^2 - 2Mr + a^2 + Q^2, \\ B &= r^2 + a^2(1-k^2), \\ D &= r^2 + a^2. \end{aligned}$$

With this substitutions the metric (7.4.3) becomes

$$\mathrm{d}s^2 = \frac{A}{B}(\mathrm{d}t - ak^2\mathrm{d}\phi)^2 - \frac{k^2}{B}(D\mathrm{d}\phi - a\mathrm{d}t)^2. \tag{7.4.4}$$

If the coordinate angular velocity of light is Ω, then, for light $\phi = \Omega t$. Substituting

$$\mathrm{d}s^2 = \frac{A}{B}(1 - ak^2\Omega)^2\mathrm{d}t^2 - \frac{k^2}{B}(D\Omega - a)^2\mathrm{d}t^2. \tag{7.4.5}$$

Using the condition $ds = 0$, we may write

$$A(1 - ak^2\Omega)^2 - k^2(D\Omega - a)^2 = 0. \tag{7.4.6}$$

Solving for Ω we obtain

$$\Omega_\pm = \pm(X \pm Y), \tag{7.4.7}$$

where

$$\begin{aligned} X &= \frac{\sqrt{a^2(D-A)^2 - \frac{1}{k^2}(Aa^2k^2 - D^2)(A - k^2a^2)}}{Aa^2k^2 - D^2}, \\ Y &= \frac{a(A-D)}{Aa^2k^2 - D^2}. \end{aligned}$$

As in the Schwarzschild case, the $\pm$ sign at the beginning (outside the bracket) in Eq. (7.4.7) indicates two different directions of the motion of light and this is insignificant because in the Sagnac effect we take essentially the difference of time of reaching of two counter-rotating beams. Note that unlike the Schwarzschild case (Eq. (7.2.4)) here $\Omega_+ \neq \Omega_-$. This is because of the rotating nature of the Kerr black hole. The angular displacement of light in time t is thus given by

$$\phi_\pm = \Omega_\pm t. \tag{7.4.8}$$

If the angular velocity of the observer is ω_0, the rotation angle $\phi_0(t)$ at any time t will be

$$\phi_0 = \omega_0 t. \tag{7.4.9}$$

Eliminating t from Eqs. (7.4.8) and (7.4.9) we obtain (following the notations of Sec. 7.2)

$$\phi_\pm = \Omega_\pm \left(\frac{\phi_0}{\omega_0}\right). \tag{7.4.10}$$

The co-rotating light meets the beam-splitter after traversing an angle equal to $2\pi + \phi_{0+}$, (say), thus

$$\Omega_+ \left(\frac{\phi_{0+}}{\omega_0}\right) = 2\pi + \phi_{0+}.$$

Solving, we obtain

$$\phi_{0+} = \frac{2\pi\omega_0}{\Omega_+ - \omega_0}.$$

Similar reasoning leads to, for counter-rotating light

$$\phi_{0-} = \frac{2\pi\omega_0}{\Omega_- + \omega_0}.$$

Combining, we obtain

$$\phi_{0\pm} = \frac{2\pi\omega_0}{\Omega_\pm \mp \omega_0}. \tag{7.4.11}$$

Now, the proper time of the rotating of observer is, from Eq. (7.4.5)

$$d\tau = \frac{1}{\sqrt{B}} \left[A(1 - ak^2\omega_0)^2 - k^2(D\omega_0 - a)^2\right]^{1/2} \frac{d\phi}{\omega_0}. \tag{7.4.12}$$

The Sagnac delay in the frame of the rotating observer is found by integrating from ϕ_{0-} to ϕ_{0+},

$$\delta\tau = \frac{1}{\sqrt{B}} \left[A(1 - k^2\omega_0)^2 - k^2(D\omega_0 - a)^2\right]^{1/2} \frac{\phi_{0+} - \phi_{0-}}{\omega_0}. \tag{7.4.13}$$

Putting the values of $\Omega_\pm$ from Eq. (7.4.7) into Eq. (7.4.11) and values of $\phi_{0\pm}$ from Eq. (7.4.11) we obtain the Sagnac result in the proper frame of the observer

$$\delta\tau = \frac{4\pi}{\sqrt{B}} \left[A(1 - ak^2\omega_0)^2 - k^2(D\omega_0 - a)^2\right]^{1/2} \times \frac{-a(A - D) + \omega_0(Aa^2k^2 - D^2)}{-\frac{1}{k^2}(A - k^2a^2) + 2\omega_0 a(A - D) - \omega_0^2(Aa^2k^2 - D^2)}. \tag{7.4.14}$$

This is the Sagnac result for disc experiment in the proper frame of the observer in a Kerr-Newmann gravitational field. To our Knowledge this is new result and is not available in the literature to date.

It is interesting to note here that if the disc rotates with an angular velocity

$$\omega_{\rm n}(r,k) = \frac{\left(2Mr - Q^2\right)a}{(r^2+a^2)^2 - (r^2 - 2Mr + a^2 + Q^2)a^2k^2}, \tag{7.4.15}$$

one obtains a null result *i.e.* the Sagnac delay becomes zero. This means that the observer here will receive both the co-rotating and counter-rotating light beams simultaneously. The observer will consider both the directions (of light travel) as equivalent in his local geometry. Thus the disc in this case is *nonrotating relative to the local spacetime geometry* [8]. The observer on board the rotating platform is called 'locally nonrotating observer". Any observer on the disc rotating in Kerr-Newmann field with an angular velocity $\omega_{\rm n}(r,k)$ is a locally nonrotating observer. This is equivalent to a disc with zero angular velocity in Schwarzschild geometry (putting $a = 0$ in Eq. (7.4.15) one obtains $\omega_{\rm n} = 0$) in which case no Sagnac effect will be observed.

7.5 Experiments with Rockets and Sattelites in Kerr-Newman Field

Let us now discuss the Sagnac experiment with rockets and satellites[4] in a Kerr-Newman field. For simplicity we assume the satellite to move in an equatorial orbit. Thus with the condition

$$\theta = \pi/2 \Rightarrow k = \sin\theta = 1,$$

[4]We use the term *sattelite* when the observer is in natural (free falling) orbits for which there is no human control on its angular velocity; on the contrary the term rocket is used to highlight that ω can be handput.

we obtain

$$A = r^2 - 2Mr + a^2 + Q^2,$$
$$B = r^2,$$
$$D = r^2 + a^2,$$

With these values we obtain the Sagnac result from Eq. (7.4.14) as

$$\delta\tau = 4\pi \left[a\left(-\frac{2M}{r} + \frac{Q^2}{r^2}\right) + \omega_0 \left(r^2 + a^2 + \frac{2M}{r}a^2 - \frac{Q^2}{r^2}a^2\right)\right] \times \left[\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) - 2\omega_0 a \left(-\frac{2M}{r} + \frac{Q^2}{r^2}\right) - \omega_0^2 \left(\frac{2M}{r}a^2 + r^2 + a^2 - \frac{Q^2}{r^2}a^2\right)\right]^{-1/2}. \tag{7.5.1}$$

In ordinary Kerr field however, $Q = 0$ and the corresponding result was deduced by Tartaglia [3] and is given by

$$\delta\tau = 4\pi \frac{\omega_0 \left(\frac{2M}{r}a^2 + a^2 + r^2\right) - \frac{2M}{r}a}{\left[\left(1 - \frac{2M}{r}\right) + 2a\omega_0 \frac{2M}{r} - \omega_0^2 \left(\frac{2M}{r}a^2 + a^2 + r^2\right)\right]^{1/2}}. \tag{7.5.2}$$

Here also the Sagnac delay vanishes for *local non-rotating* observers whose angular velocity is given by

$$\omega_{\rm n} = \frac{2Ma}{2Ma^2 + a^2 r + r^3}, \tag{7.5.3}$$

where we have assumed the Kerr geometry ($Q = 0$).

For a sattelite the observer is moving in a geodetic path and the angular velocity is given by [3]

$$\omega_{\pm} = \frac{2aM \pm \sqrt{3a^2M^2 + Mr^3}}{a^2M - r^3} \tag{7.5.4}$$

Putting Eq. (7.5.4) in Eq. (7.5.2) we obtain

$$\delta\tau = \frac{4\pi}{r} \frac{S_{\pm}(2Ma^2 + a^2 r + r^3) - 2MaT}{Z}, \tag{7.5.5}$$

where

$$\omega_{\pm} = \frac{S}{T}, \qquad S_{\pm} = 2aM \pm \sqrt{3a^2M^2 + Mr^3} \qquad T = a^2M - r^3,$$

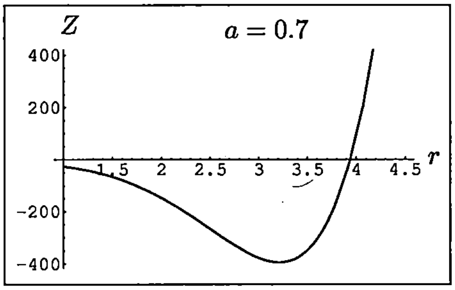
and

$$Z = \left[T^2\left(1 - \frac{2M}{r}\right) + \frac{4Ma}{r}S_{\pm}T - S_{\pm}^2\left(\frac{2M}{r}a^2 + a^2 + r^2\right)\right]^{1/2} \quad (7.5.6)$$

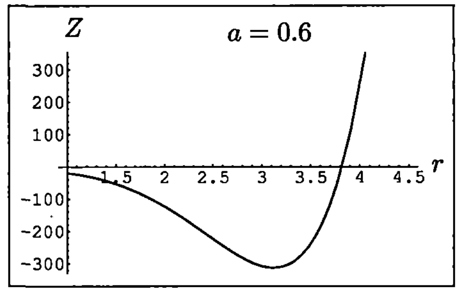
7.6 Arbitrarily Large Sagnac Phase-shift

As we have already noted in the context of Schwarzschild field that the Sagnac effect in the Kerr spacetime also displays singularities which may be obtained by putting Z of Eq. (7.5.5) equal to zero. Indeed one can readily verify from Eq. (7.5.5) that there is a singularity at $r = 3M$ for $a \to 0$ which is the Schwarzschild case. Incidentally the coordinate sphere $r = 3M$ indeed marks the onset of the pre-horizon regime [9] where the Fermi drag changes sign. We do not however notice any spectacular thing to happen in this region for Sagnac effect. Note that the Sagnac effect is not imaginary in the region $r < M$ as the Eq. (7.3.5) seems to suggest, since the equation is valid only for $r \geq 3M$. Within the pre-horizon regime ($2M < r < 3M$) the angular velocity can be handput and again it can be shown that for a wide range of its values the effect is regular showing singularity only when the observer speed is arbitrarily close to that of the coordinate speed of light. In order to see the occurence of arbitrarily large phase shifts it is instructive to plot Z (Eq. (7.5.6)) against r with different values of a and find the position of the singularity. Alternatively it is possible to solve the equation numerically and obtain the value of r of the expression (7.5.5) where the singularity occurs.

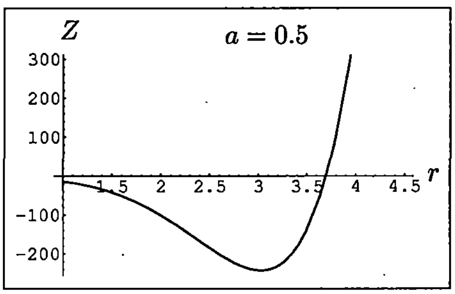
Fig. 7.1(a) through Fig. 7.1(h) represent such plots for retrograde ($\omega = \omega_+$) orbits for $a = 0.7$ to 0.06 ($M = 1$). The zero of Z occurs generally away from



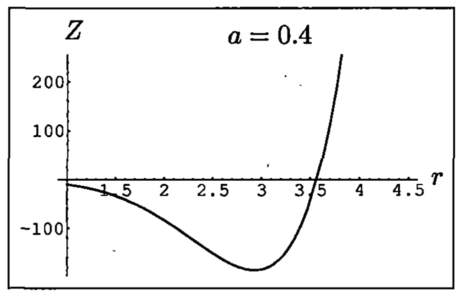
(a) Singularity with $a = 0.7$



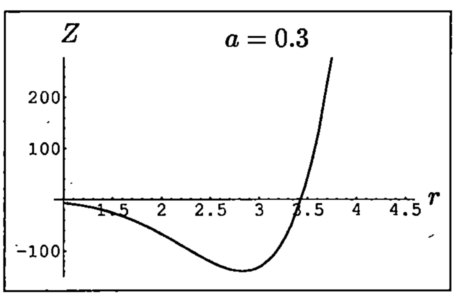
(b) Singularity with $a = 0.6$



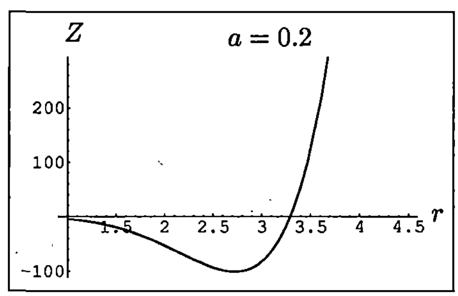
(c) Singularity with $a = 0.5$



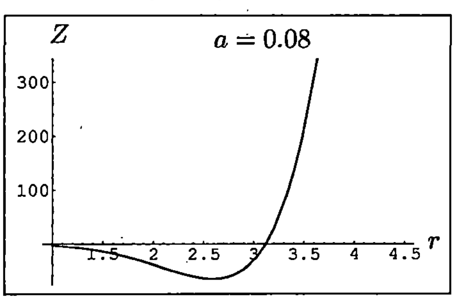
(d) Singularity with $a = 0.4$



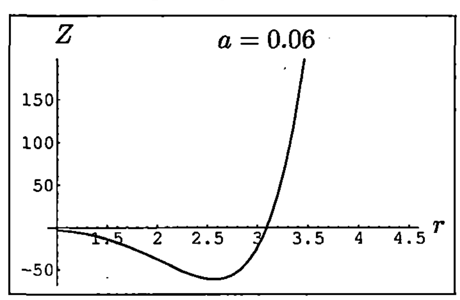
(e) Singularity with $a = 0.3$



(f) Singularity with $a = 0.2$

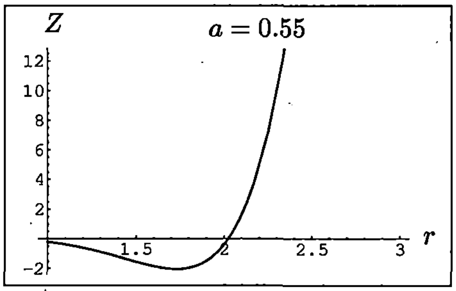


(g) Singularity with $a = 0.08$

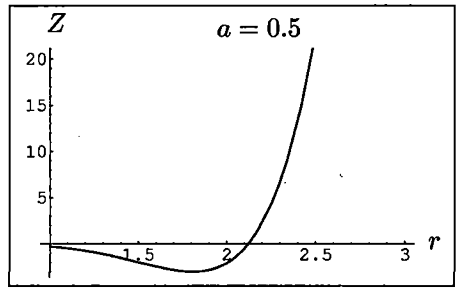


(h) Singularity with $a = 0.06$

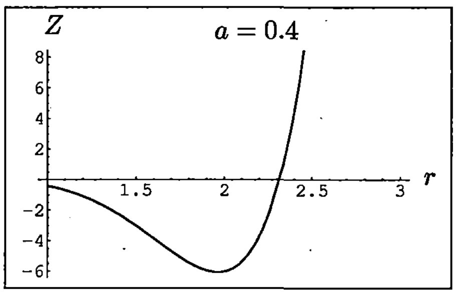
Figure 7.1: Plot of $Z(r) = 0$: Retrograde orbit ($M = 1$)



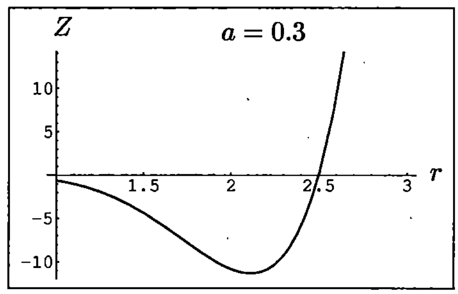
(a) Singularity with $a = 0.55$



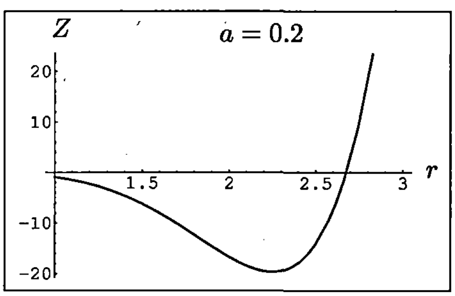
(b) Singularity with $a = 0.5$



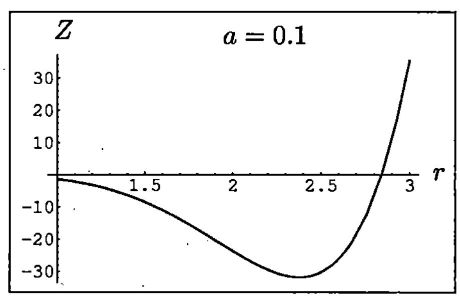
(c) Singularity with $a = 0.4$



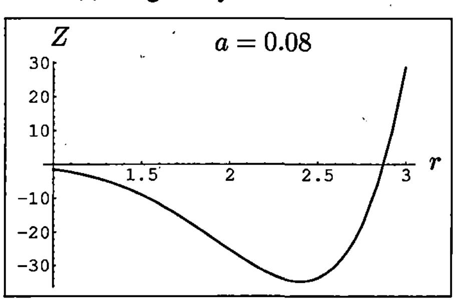
(d) Singularity with $a = 0.3$



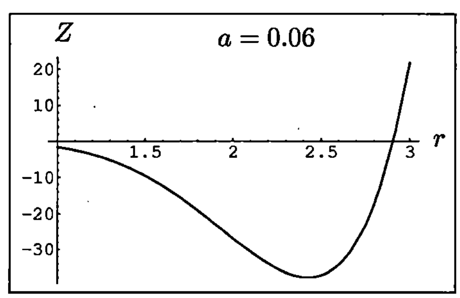
(e) Singularity with $a = 0.2$



(f) Singularity with $a = 0.1$



(g) Singularity with $a = 0.08$



(h) Singularity with $a = 0.06$

Figure 7.2: Plot of $Z(r) = 0$: Direct orbit ($M = 1$)

$r = 3M$ which is obtained only for $a = 0$. Recall that $r = 3M$ represents possible photon orbit in Schwarschild spacetime. For direct orbits ($\omega = \omega_-$) however the singularity occurs below $r = 3M$ and approaches it in zero angular momentum limit. This has been shown in Fig. 7.2(a) through Fig. 7.2(h)

In the literature plots of radii of circular equatorial orbits around a Kerr black hole as functions of the parameter a are available both for direct and retrograde orbits [6, pp.341]. It may be seen that the radii of the photon orbits approach to $r = 3M$ from lower values $r < 3M$ for direct orbits and from higher values $r > 3M$ for retrograde orbits. Our result therefore matches these results.

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