Chapter 6

Sagnac Formula from the Rotating Frame Perspective and the Absolute Synchrony

6.1 Introduction

In an interesting paper Franco Selleri [1] remarks that the calculations of SD available in the literature is mostly done from the perspective of the laboratory frame, and they "... say nothing about the description of phenomenon given by an observer placed on the rotating platform." Selleri even remarked that standard SR predicts null result for the Sagnac effect from the perspective of rotating frame. While the last statement is not strictly correct as we have discussed in detail in Chap. 5 [2, 3] in connection with the so called Selleri paradox, the former one may prompt one to survey the literature to find if there exists the derivation of Sagnac effect from the standpoint of a rotating observer. [1]

In some of our earlier chapters we presented some of the common derivations of the Sagnac effect. More than one derivations were presented to highlight different aspects of the effect in the appropriate contexts.

For example, in Sec. 2.5.1, classical Sagnac effect was obtained from simple kinematics. Indeed under relevant approximations no distinction between classical Sagnac effect and relativistic Sagnac effect is usually made. In the section following it this treatment is extended to the realm of relativity. However, the result has been obtained essentially in the laboratory frame and then the effect on board the rotating platform is obtained by introducing appropriate relativistic factor to account for time dilation.

A special relativistic metrical treatment of Sagnac effect is given in Sec. 2.5.3

[1]It is often remarked that such a thing is not possible in fully special relativistic framework since the observer is non-inertial, as if the full treatment may require GR. This kind of argument is often given in connection with the calculation of asymmetric aging of the twin paradox from the perspective of the traveler twin. However introduction of GR for these problems in a flat spacetime is decidedly misleading [4].

(some times these types of treatments are erroneously considered as 'general relativistic treatments'). There, a flat spacetime metric in the laboratory (inertial) frame is expressed in terms of coordinates which directly represent measurement by the standard clocks and rods. The calculation of the effect is performed by asserting that both a photon and the observer (beam splitter) rotate in the rest frame along a circular path, although with different angular velocities. Since the spacetime metric is written in terms of standard time and length coordinates of the inertial frame, here also the calculation is essentially done from the perspective of the laboratory.

Dieks [5] offered another calculation of SD which directly makes use of the LT and this has been used by us in Sec. 3.5, but a close scrutiny will reveal that the calculation too is done from the perspective of the laboratory frame.

Observe that all these justify Selleri's contention that the calculations in general (including the textbook ones [6]) are done from the *perspective of the laboratory frame*. Since the Sagnac effect is essentially a phenomenon on a rotating frame, it is proper to think that a calculation should be available which is done from the *perspective of the rotating frame*. To provide one, Selleri proposes a calculation based on *inertial transformation* introduced by him. The transformation contains a free parameter ϵ which, in conformity with the CS thesis, is capable of describing relativistic physics. The author then considers a set of localized inertial frames moving tangentially to the disc which are instantaneously comoving with the small circumferential elements of the disc. The OWS' of light moving parallel and antiparallel to the direction in these frames predicted by the inertial transformation are found out. The difference between the times of flight of the light beams in the two directions to traverse the circumferential length of the disc gives the SD in terms of the ϵ parameter. It comes out that the laboratory and the disc observer would agree on the SD if and only if $\epsilon = 0$, for which the inertial transformation implies TT which corresponds to absolute synchrony. On the contrary the value of ϵ corresponding to LT gives *null result* as already mentioned.

Note that by considering absolute synchrony Selleri allows anisotropy of OWS' of light in a locally chosen instantaneously comoving inertial frame and is thus able to extend this local feature to a global one when he finds correctly the difference of time of flight. This amounts to meshing up all the instantaneously comoving inertial frames without any problem. We believe that Selleri is indeed correct in this regard.[2] Indeed for various reasons we have given a value judgment in favour of the absolute synchrony on a rotating platform (vide Sec. 5.6). We shall show below that an important treatment of the Sagnac effect (by E. J. Post [7]) implicitly assumes absolute synchrony thus confirming Selleri's claim.

6.2 Post's Calculation

Coming back to the question of the existence of treatments of the Sagnac effect with respect to an observer on board the rotating platform, let us consider the well-known calculation by E. J. Post [7]. Post's approach to finding the SD from the rotating frame perspective is to follow the metrical treatment with a transformed (to rotating frame) flat spacetime metric. A lesson that one learns from GR is that any co-ordinate transformation, even the Galilean one will be able to obtain physical effects provided the co-ordinates used are properly interpreted in the

[2]That LT gives null result in this treatment essentially means that this kind of straightforward extension is not valid while the clocks are Einstein synchronized. The calculation of SD within the standard relativistic frame work is still possible provided the synchrony gap which has been exemplified in Sec. 5.5 is taken into consideration.

context of the rotating frame. Such an approach was intiated by P. Langevin [8, 9]. Using Galilean type transformation he was able to predict the experimentally observable first order effect. However a treatment will be considered to be one done truly from the perspective of rotating frame provided it includes coordinates which directly relates measurements by standard rods and standard clocks and Langevin's treatments fail on that count.

Post in 1967, in a very comprehensive review of Sagnac effect (and perhaps the most cited in this field) tried to obtain the expression for SD from a more suitable metric. Surely for such a calculation one would need a coordinate system which is rotating with respect to the system in the underlying inertial frame. He therefore looks for a suitable transformation which may represent a rotating frame and would have given the correct relativistic Sagnac formula

$$\Delta\tau = \frac{4\pi R^2}{c^2}\gamma. \tag{2.5.10}$$

Starting from the expression of flat line element in the cylindrical co-ordinate in laboratory frame

$$\mathrm{d}s^2 = c^2\mathrm{d}t_0^2 - \mathrm{d}r_0^2 - r_0^2\mathrm{d}\phi_0^2, \tag{6.2.1}$$

Post subjects it to a (somewhat arbitrarily) Galilean-type transformation,

$$\begin{aligned} \mathrm{d}t_0 &= \epsilon\mathrm{d}t, \\ \mathrm{d}r_0 &= \mathrm{d}r, \\ \mathrm{d}\phi_0 &= \mathrm{d}\phi + \epsilon\omega\mathrm{d}t. \end{aligned} \tag{6.2.2}$$

where ϵ is upto now an undetermined factor (our notation is different from that used by Post). The co-ordinates without a suffix represents the coordinates in rotating frame. For circular symmetric path of the light beams $\mathrm{d}r_0 = \mathrm{d}r = 0$. The line element expressed in the rotating frame under this transformation now reads

$$\mathrm{d}s^2 = \epsilon^2c^2\left(1 - \frac{R^2\omega^2}{c^2}\right)\mathrm{d}t^2 - R^2\mathrm{d}\phi - 2R\,\epsilon\,\omega\,\mathrm{d}\phi\mathrm{d}t, \tag{6.2.3}$$

where R is the radius of the disc. Imposition of null geodesic condition $\mathrm{d}s = 0$ for the light beams gives two solutions for round trip times of two counter rotating beams. Their difference gives the SD

$$\Delta\tau = \frac{4\pi R^2\omega}{\epsilon\left(c^2 - \omega^2 R^2\right)} \tag{6.2.4}$$

which still contains the undetermined factor ϵ. $\epsilon = 1$ gives the classical Sagnac formula

$$\Delta\tau = \frac{4\pi R^2\omega}{c^2}\,\gamma^2. \tag{2.5.5}$$

Again if the assumed formula is the relativistic one, already obtained theoretically one gets a value for $\epsilon = \gamma$. Now since in the calculation ϵ remains a free parameter, only to be determined using an assumed Sagnac formula one can hardly call the exercise to be truly a derivation; at least it does not serve our original objective, *i.e.* obtaining the Sagnac effect from the rotating frame perspective.

6.3 Derivation of TT from Post's Transformation

However, Post's approach will constitute a proof if instead of finding out the value ϵ from the known relativistic formula for SD one could somehow guess the expression for ϵ beforehand. We now suppose that intuitively Post postulated (from the consideration of time dilation effect or otherwise) that $\epsilon = \gamma$. In that case Post's derivation is based on the postulate of the coordinate transformation representing the *relativistic* rotation of the disc

$$\mathrm{d}t_0 = \gamma \mathrm{d}t, \tag{6.3.1}$$

$$\mathrm{d}\phi_0 = \mathrm{d}\phi + \gamma\omega \mathrm{d}t. \tag{6.3.2}$$

Multiplying the second equation (6.3.2) by R one may write

$$R\,\mathrm{d}\phi_0 = R\,\mathrm{d}\phi + \gamma v\,\mathrm{d}t, \tag{6.3.3}$$

where we have replaced ωR by v, the speed of an element of the circumference of the disc. Writing $R\,\mathrm{d}\phi_0 = \mathrm{d}x_0$ and $R\,\mathrm{d}\phi = \mathrm{d}x$ one obtains

$$\mathrm{d}x_0 = \mathrm{d}x + v\gamma\,\mathrm{d}t \tag{6.3.4}$$

where the term $\mathrm{d}x_0$ represents the linear length of the element of the disc as seen from the rest frame. Regarding the meaning of the coordinate length we shall return soon.

It is interesting to note that in the above transformation although intuitively somehow time dilation has been taken into consideration (see the appearance of Lorentz factors as coefficients of the time differentials in the TE), the other relativistic effects, *viz* the length contraction has not been considered. It is therefore evident that the transformation takes care of the Ehrenfest paradox by considering the disc to be composed of *unbonded* (software bonded) particles (see Sec. 3.7. Also see Refs. [10, 11].). Indeed we have shown in details in Chap. 3(Ref. [10, 11]) that SD formula with a γ factor (which Post has assumed in the context) pertains to such cases.

Inverting the transformation Eqs. (6.3.1) and (6.3.2) one writes

$$\mathrm{d}t = \gamma^{-1}\mathrm{d}t_0, \tag{6.3.5}$$

$$\mathrm{d}x = \mathrm{d}x_0 - v\mathrm{d}t_0. \tag{6.3.6}$$

We already observed that the absence of any space term in Eq. (6.3.5) implies absolute synchrony!

The physical meaning of $\mathrm{d}x$ is now very clear. Eq. (6.3.6) does not predict any length contraction, this is in conformity with what we have said earlier that the distance between the particles remain the same when the disc picks up its uniform motion from its state of rest. It has already been discussed that in such a situation the rest length of the elements of the disc (material) will have to be *stretched* by a Lorentz factor. Thus $\mathrm{d}x$ is Eq. (6.3.6) represents stretched length. However there is no *stretching* in relativistic transformations in standard coordinates. Therefore, according to the standard coordinate (say $\mathrm{d}x'$), $\mathrm{d}x' = \gamma^{-1}\mathrm{d}x$. The final transformation in standard coordinates can then be written as

$$\begin{aligned} \mathrm{d}x' &= \gamma\left(\mathrm{d}x_0 - v\mathrm{d}t_0\right), \\ \mathrm{d}t' &= \gamma^{-1}\mathrm{d}t_0, \end{aligned} \tag{6.3.7}$$

which is nothing but the Tangherlini transformation. Note that we have changed the notation $\mathrm{d}t \to \mathrm{d}t'$ for symmetry.

6.4 Derivation of TE (Appendix of Post's Paper)?

It seems that Post himself was not satisfied with his intuitive approach. He thus presents a interesting 'derivation' of his TE (Eq. (6.2.2)) in the appendix of the paper. He starts from the generalized LT where the relative motion is along any arbitrary direction:

$$\begin{aligned} t_0 &= \gamma\left(t + \frac{\mathbf{v.r}}{c^2}\right), \\ \mathbf{r}_0 &= \mathbf{r} - \mathbf{v}\left[(1-\gamma)\frac{\mathbf{v.r}}{v^2} - \gamma t\right], \end{aligned} \tag{6.4.1}$$

where, again, the co-ordinates without suffix represent rotating frame. He then sets $\mathbf{v.r} = 0$ on the basis that at every point on a rotating disc $\mathbf{v}\perp\mathbf{r}$ to show that the transformation in differential term assumes the form

$$\begin{aligned} \mathrm{d}t_0 &= \gamma\mathrm{d}t, \\ \mathrm{d}\mathbf{r}_0 &= \mathrm{d}\mathbf{r} - \mathbf{v}\gamma\mathrm{d}t. \end{aligned} \tag{6.4.2}$$

Assuming coordinate differential perpendicular to the radius (along the unit vector $\mathbf{n}$ say)

$$\mathrm{d}\mathbf{r} = r\mathrm{d}\phi\mathbf{n}, \qquad \mathrm{d}\mathbf{r}_0 = r\mathrm{d}\phi_0\mathbf{n}, \tag{6.4.3}$$

and

$$\mathbf{v} = \omega r \mathbf{n}, \tag{6.4.4}$$

he obtains the transformation in his desired form (6.2.2). In this manner *adjusting* a free parameter to obtain the transformation is avoided, as if the transformation is *derivable* for LT!

Before we point out an error in the above derivation we once again note that the absence of any space term in the time transformation equation implies absolute synchrony, where two events simultaneous in one frame are simultaneous in the other frame (vide App. A and App. B). In LT, on the contrary, simultaneity is relative, since in standard SR the clocks are synchronized according to the Einstein procedure. Surprisingly, Post finds a TE obeying absolute synchrony starting from a TE obeying Einstein synchrony which is absurd.

The flaw in Post's calculation resides in the way the term $\mathbf{v}.\mathbf{r}$ is set to zero before writing the TE in differential form. One should be very cautious in using LT in the case of rotation since LT is valid here *only locally*. Thus, for a circumferential element of the disc one must write down the transformation in differential form first and the constraint condition $dr = 0$ has to be put afterwards. Indeed the differential of the time transformation would be

$$\mathrm{d}t_0 = \gamma \left(\mathrm{d}t + \frac{\mathbf{r}.\mathrm{d}\mathbf{v}}{c^2} \right), \tag{6.4.5}$$

where we have put $\mathrm{d}\mathbf{r} = 0$ *after* the differention. Note that $\frac{\mathbf{r}.\mathrm{d}\mathbf{v}}{c^2} \neq 0$ since $\mathrm{d}\mathbf{v}$ is *not* perpendicular to $\mathbf{r}$.

The similar mistake has been committed in arriving at the equation

$$\mathrm{d}\phi_0 = \mathrm{d}\phi + \gamma v \, \mathrm{d}t. \tag{6.4.6}$$

That there is a drawback in the arguments can be understood in the following way.

Multiplying Eq. (6.4.6) by R, the radius of the disc, and noting that $R\,\mathrm{d}\phi_0$ and $R\,\mathrm{d}\phi$ are the elements of length of the circumference with respect to the rest frame and the instantaneously comoving inertial frame respectively, one observes that there is no length contraction effect. We have already noted that the time transformation also does not display relativity of simultaneity. It is therefore surprising that starting from LT which generates these important special relativistic effects does not display them when written in terms of time and space differentials.

To our knowledge these drawbacks of Post's treatment almost remain unnoticed. However, Selleri [12], noting that Post's review paper is *very influential*, once perfunctorily remarked that [Post] ... *hastens to make the second term disappear with the (arbitrary) choice of* $\mathbf{r}$ *perpendicular to* $\mathbf{v}$. According to him the time transformation Post used in the main text is arbitrary. However, Selleri fails to recognize that Post's transformation only confirms his own thesis that only TT can explain the Sagnac effect from the *rotation frame perspective*.

Reference

[1] F. Selleri, *Found. Phys.*, **26**, pp. 641–664.

[2] S. K. Ghosal, Biplab Raychaudhuri, Anjan Kumar Chowdhuri and Minakshi Sarker, *Found. Phys. Lett.*, **17**(5), (2004) pp. 457–477.

[3] S. K. Ghosal, Biplab Raychaudhuri, Minakshi Sarker and Saroj Nepal, "On synchronization and desynchronization in rotating frames and a recent paradox", in M. C. Duffy (Ed.), "Physical Interpretation of Relativity Theory (IX)", (2004). Held at Imperial College, London (To be Published).

[4] Talal A. Debs and Michael L. G. Redhead, *Am. J. Physics*, **64**(4), (1996) pp. 384–391.

[5] D. Dieks, *Eur. J. Phys.*, **12**, (1991) pp. 253–259.

[6] V. A. Ugarov, *Special Theory of Relativity* (Mir Publisher, Moscow, 1979). Translated from Russian by Yuri Atanov.

[7] E. J. Post, "Sagnac effect", *Rev. Mod. Phys.*, **39**, (1967) p. 475.

[8] P. Langevin, *Compt. Rend.*, **173**, (1921) p. 831.

[9] P. Langevin, *Compt. Rend.*, **205**, (1937) p. 51.

[10] S. K. Ghosal, Biplab Raychaudhuri, Anjan Kumar Chowdhuri and Minakshi Sarker, "Relativistic sagnac effect and ehrenfest paradox", *Found. Phys.*, **33**(6), (2003) pp. 981–1001.

[11] S. K. Ghosal, Biplab Raychaudhuri, Anjan Kumar Chowdhuri and Minakshi Sarker, "Rotating disc problem and sagnac phase-shift formula", in M. C. Duffy (Ed.), "Physical Interpretations of Relativity Theory, VIII", (P. D. Publication, Liverpool, 2000). Held at Imperial College London.

[12] F. Selleri, "Sagnac effect: End of a mystery", in G. Rizzi and M. L. Ruggiero (Eds.), "Relativity in Rotating Frame", Fundamental Theories of Physics (Kluwer Academic, New York, 2004). Series Editor: A. van der Merwe.