# CHAPTER - III

# PARTICLE TRAJECTORIES IN SRNG

## 3.1 INTRODUCTION

It has been pointed out in the previous chapter that SRNG is a simple minded approach of Special Relativistic extension of Newtonian Gravity. In chapter II an attempt was made to consolidate Special Relativity and gravitation in order to obtain a viable Lorentz covariant theory of gravitation. There we, in one hand, developed tensorial field equations for the gravitational potential $h_{\mu\nu}$ and on the other hand developed heuristically a covariant Lagrangian formulation for the equation of motion. The proposed covariant Lagrangian have been given by [equation (2.15) of chapter II]

$$L' = (a_{\mu\nu}\bar{u}^{\mu}\,\bar{u}^{\nu})^{1/2} \qquad (3.1)$$

where $\bar{u}^{\mu} = dx^{\mu}/ds$ and s may be any scalar parameter of motion.

$$\text{Here,} \quad a_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} \qquad (3.2)$$

where $\eta_{\mu\nu}$ is the flat metric which in spherical coordinate system is taken as $\eta_{\mu\nu}$= diag $(-1,1,r^2,r^2\sin^2\theta)$, and $h_{\mu\nu}$ is the tensor potential of the gravitational field which are obtained from the proposed field equations. The vacuum spherically symmetric solutions for $h_{\mu\nu}$ of the field equations (vide chapter II) when inserted in the Lagrangian [equation (3.1)] via equation (3.2) gives the equation of motion.

$$\frac{1}{2}\,(a_{\mu\lambda,\nu} + a_{\nu\lambda,\mu} + a_{\mu\nu,\lambda})\,\frac{dx^{\mu}}{ds}\,\frac{dx^{\nu}}{ds} + a_{\mu\nu}\,\frac{d^2x^{\mu}}{ds^2} = 0 \qquad (3.3)$$

where the parameter 'ds' is taken to be

$$ds = ( a_{\mu\nu}\, dx^{\mu}\, dx^{\nu} )^{1/2} \tag{3.4}$$

Note once again that the present theory is a flat-space-time one and the last equation and that preceding it have nothing to do with the metric or geodesic of General Relativity (GR). Here raising and lowering of indices are still done by $\eta_{\mu\nu}$ (and not by $a_{\mu\nu}$). We have then shown how the precession of Mercury's orbit, light bending and also the gravitational red-shift can be explained by using explicit forms of $h_{\mu\nu}$ . Our results match with the empirical observations.

It is however well known that all such empirical observations refer to the far field. In the asymtotic region the results of GR and that of Newtonian Gravity (NG) do not qualitatively differ much. For example, both bending of light and red-shift can be predicted (for the latter accurately) by NG. Introducing special relativistic force law in NG one can also explain qualitatively the precession of planetary orbits (Bagge 1981; Phipps 1986; Ghosal et al 1987). The real qualitative difference of these two theories of gravitation lies in the deep field. For example a Newtonian gravity does not predict a characteristic radius like Schwarzschild radius in GR. In GR the Schwarzschild radius defines a surface from which no particle even a photon can escape. (In NG however a characteristic radius can be calculated from the escape velocity considerations, but unlike in GR the term "escape" in NG refers to escape of a particle from the field and not from a characteristic surface). The non-Newtonian features of the gravitation may be best obtained by studying particle trajectories near a massive compact object. In an article Pandey and Gupta (1987) brought out the role of "repulsive force" in GR by studying the equation of motion of a test object in a radial trajectory:

$$\frac{d^2r}{dt^2} = - MG/r^2 \,[1-(2MG/c^2r)]+3MG/c^2r^2[1-(2MG/c^2r)]^{-1}(dr/dt)^2 \quad (3.5)$$

Note that r and t in (3.5) refer to Schwarzschild coordinates and they do not refer to measurements by standard rods and clocks. Thus the above equation which involves coordinate velocity and acceleration does not *directly* give us insight into the true dynamic behaviours of test objects. [For example, due to the presence of "repulsive force" term in (3.5) particle motion is characterised by a terminal velocity; however "proper velocity" (Bose 1980) does not exhibit such an effect]. In SRNG theory there is no such coordinate problem. Since here, r and t in the equation of motion will refer to standard radial length and standard time. This fact will therefore provide a straight forward interpretation of the equation of motion and a lot of insight may be gained by bringing out the real qualitative difference between NG and SRNG in the deep field.

The purpose of the present text is to theoretically probe into the deep field by studying the radial and circular trajectories of a test particle and to study the role of the velocity dependent and repulsive force terms (which have no classical analogue) in SRNG trajectories and finally to compare qualitatively the results with that of GR (in Schwarzschild coordinate). It will be evident that some of the well known general relativistic results which are assumed to be purely general relativistic in origin are also true in SRNG.

We shall work with the solutions of three different field equations. One of which has been proposed by Biswas (1988) [correct solutions obtained by Peters (1990)] where the other two have been suggested in earlier chapter II. Hereafter we refer to these solutions as solutions I,II and III respectively. The non-trivial components of $h_{\mu\nu}$ according to I,II and III are given below

$$h_{00} = \frac{1}{K} \ln \left(1 - \frac{2KGM}{r}\right) \quad \& \quad h_{11} = 0 \qquad \text{I}$$

$$h_{00} = \frac{1}{K} \ln \left(1 - \frac{2KGM}{r}\right) \quad \& \quad h_{11} = -\frac{1}{K} \ln \left(1 + \frac{2KGM}{r}\right) \qquad \text{II}$$

$$h_{00} = h_{11} = \frac{1}{K} \ln \left(1 - \frac{2KGM}{r}\right) \qquad \text{III}$$

where K is the undetermined factor in the non-linear term of the proposed field equations (Biswas 1988; Ghosal & Chakraborty 1991c; Ghosal & Chakraborty 1991d) and M is the mass of the spherically symmetric source. The field equations (in the absence of matter) are given by [eqns. (2.30), (2.42) & (2.36) of chapter II ].

$$h_{\mu\nu;\alpha}{}^{;\alpha} = K\, h_{\mu\alpha;\beta}\, h_{\nu}^{\alpha\;;\beta}$$

$$h_{\mu\nu;\alpha}{}^{;\alpha} - \phi\, \Lambda_{\mu\nu;\alpha}{}^{;\alpha} = K\, (h_{\mu\alpha;\beta}\, h_{\nu}^{\alpha\;;\beta} - \phi\, \Lambda_{\mu\alpha;\beta}\, \Lambda_{\nu}^{\alpha\;;\beta})$$

$$H_{\mu\nu;\alpha}{}^{;\alpha} = K\, H_{\mu\alpha;\beta}\, H_{\nu}^{\alpha\;;\beta}$$

where $\phi = \eta^{\alpha\beta}\, h_{\alpha\gamma}\, \Lambda^{\gamma}_{\beta}$ and $H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} D^{\alpha}_{\beta}\, h^{\beta}_{\alpha}\, \Lambda_{\mu\nu}$
and the non-zero components of the tensors $\Lambda_{\mu\nu}$ and $D^{\mu}_{\nu}$ in spherical polar coordinates are given by

$$\Lambda_{11} = 1, \qquad D^{1}_{1} = -D^{0}_{0} = 1.$$

Note that in chapter II we assumed K=1. Under the far field approximation one can verify that the known general relativistic results are *practically* independent of the sign and value of K. In the deep field also the conclusions regarding the qualitative features of a test particle motion will also not depend on the value of K. In absence of any empirical information in the deep field we assume for simplicity K=1.

Although the results and their interpretations can be obtained by studying the equation of motion numerically it is instructive to obtain the results analytically as far as possible. In section 3.3, in order to study the radial motion, solution III will be used for the analytical treatment and numerical computations for I and II will be given just for a quantitative comparison. However we shall see that for red-shift calculations (sec.3.2) and for the study of circular orbits (sec.3.4) all the solutions (I,II & III) can be treated on equal footing.

## 3.2 THE SURFACE OF INFINITE REDSHIFT

In GR Schwarzschild radius is also characterised by a Surface of Infinite Red-Shift (SIRS) [Bose 1980; Weinberg 1972]. A surface with the similar property is also predicted according to the present theory. However, the characteristic radius ($r_c$) in this case is expected to be different from the Schwarzschild radius ($r_s$). To obtain $r_c$ (if it exists) corresponding to the present SRNG we proceed as follows.

We first recall our red-shift formula (Ghosal & Chakraborty 1991c) [vide eqn. (2.51) of chapter II].

$$\nu = \nu'[1+h_{00}]^{1/2} = \nu'[1+\ln(1-\frac{2GM}{r})]^{1/2} \qquad (3.6)$$

where $\nu'$ denotes the frequency of the emitted photon when the emitter atom is at infinity, and $\nu$ is the redshifted frequency of the photon when the emitter atom is situated at r. Note that the above formula is independent of $h_{11}$ and hence the equation (3.6) and the results that follow from it are equally valid for the solutions I, II & III mentioned in the earlier section.

To obtain infinite red-shift we put $\nu=0$ in equation (3.6). The corresponding characteristic radius $r_c$ is given by

$$r_c = \frac{2GM}{1-e^{-1}} \tag{3.7}$$

The positive definiteness of $r_c$ provides the proof of existence of the SIRS which according to the present theory is given by equation (3.7). Observe that as an atom, which emits photon, approaches the above radius the red-shift becomes increasingly large and finally becomes infinite at this radius. This radius in the present case is directly measurable by rigid standard rods and $r=r_c$ here therefore defines a "physical" singular surface characterised by the impossibility of a photon being emitted from it. From (3.7), numerically $r_c$ is given by $r_c \simeq 1.58\ r_s$. Loosely speaking the size of a SRNG Black-Hole is slightly more than that of the relativistic one.

## 3.3 Radial motion near SIRS

The essential non-Newtonian features of the SRNG field may be best understood by studying the radial motion of a test particle. Near SIRS the non-Newtonian character of the field is expected to be more pronounced.

We first write down the energy integral for the particle trajectory [eqn. (2.46) of chapter II].

$$E = \frac{m_0 \, (-a_{00})^{1/2}}{[1+(a_{11}/a_{00})v^2]^{1/2}} \tag{3.8}$$

Or, rearranging one obtains

$$g = -\frac{a_{00}}{a_{11}}(1+fa_{00}) \tag{3.9}$$

where $g=v^2$ & $f=m_0^2/E^2$. Note that the constant E can be interpreted as the total energy of the particle in the gravitational field (Ghosal & Chakraborty 1991c). After substituting the values of $a_{00}$ and $a_{11}$ (using solution III) equation (3.9) reduces

$$fY^2 - Y(1+g) + 2g = 0 \tag{3.10}$$

$$\text{where } Y = 1+X \text{ and } X = \ln(1-\frac{2GM}{r})$$

The last equation gives the dependence of velocity of the test particle as a function of its distance from the origin with energy as a parameter. One of the roots of the above quadratic equation is given by

$$Y = \frac{(1+g)-[(1+g)^2-8gf]^{1/2}}{2f} \tag{3.11}$$

With the aid of (3.11) we may now enquire whether we have a unique surface in the deep field (similar to that one finds in GR) where the velocity of a particle irrespective of its initial energy will attain in its free fall a value equal to zero (which corresponds to g=0).

First we consider the case when the test object starts from infinity with zero velocity. Clearly at infinity $a_{00}=-1$ and from equation (3.8) putting v=0, we get $f=m_0^2/E^2=1$. Then by putting g=0 in equation (3.11) one obtains the null velocity surface which comes out to be the same as the SIRS:

$$r = \frac{2GM}{1-e^{-1}} \tag{3.7}$$

The other root of equation (3.11) corresponds to $r=\infty$. This result is of course trivial by virtue of the chosen initial condition. Thus we see that a test particle initially at rest at infinity will come to rest again at the radius $r \simeq 1.58 r_s$ which, as we have already seen, defines the surface of infinite redshift.

Now one may ask whether it is possible for a test particle to be able to cross this surface (i.e. $r \simeq 1.58 r_s$) if its energy is increased arbitrarily. We shall see that this is not possible. To understand this, return again to (3.11) in order to determine the null-velocity-surface (characterised by g=0), but this time we do not put f=1. This gives again two values of r of which one is again $r \simeq 1.58 r_s$ and is independent of energy (f). The other root is given by

$$r = \frac{2GM}{1-e^{n-1}} \tag{3.12}$$

where $n=1/f$.

Now note that r should be positive by definition. To ensure this, as it follows from (3.12), f must be greater than unity. Clearly this is impossible since the maximum value of f ( $= m_0^2/E^2$ ) is unity. Thus the second root of r of (3.11) is to be discarded. It is therefore evident that even if the energy of the particle increases indefinitely it cannot cross the surface $r \simeq 1.58 r_s$. We experience in Schwarzschild coordinates a similar thing to happen in GR too. However, there is a subtle difference. In GR the singular surface characterised by the metric (field) and the surface marked by the peculiarity of the test particle motion coincide. But in SRNG, though the field singularity is still at r=2GM, it is bounded from above by an event-horizon-like-surface $r \simeq 1.58 r_s$.

Now we can have a picture of the radial free fall of a test particle towards a spherically symmetric source. As a particle falls from infinity the velocity is increased due to the gravitational attraction but as the velocity is increased the gravitational acceleration gets reduced and finally the particle gets decelerated and its velocity becomes zero at the SRNG horizon (i.e at $r=r_c$). It is therefore clear that the velocity field (v as a function of r) should display a "maximum" during the free fall.

In order to estimate the position of these maxima (as a function of particle energies) we return to the equation (3.10) and differentiate it with respect to Y.

$$\frac{dg}{dY} = \frac{2fY - (g+1)}{Y-2} \qquad (3.13)$$

which should be equal to zero for the "maximum" to occur. Again from equation (3.10) one may write

$$g = \left[\frac{Y(fY-1)}{Y-2}\right] \qquad (3.14)$$

which when inserted in (3.13) gives

$$\frac{dg}{dY} = \frac{fY^2-4fY+2}{Y-2} \tag{3.15}$$

Now putting dg/dY = 0 in (3.15) one obtains

$$fY^2 - 4fY + 2 = 0 \tag{3.16}$$

the physically meaningful root of which is given by

$$Y_m = \frac{2f-(4f^2-2f)^{1/2}}{f} \tag{3.17}$$

( The other root of (3.16) has been discarded since it corresponds to r<0 ). The value of $Y_m$ obtained from (3.17) as function of f gives the radial position of the maxima under study. The value of $Y_m$, when substituted in equation (3.14) give the maximum velocity that a particle attains. The results of these calculations are given in table I.

TABLE I

| $f(=m_0^2/E^2)$ | The radial position ($r_m$) where the maximum velocity occurs (in unit of 2GM) | Maximum velocity (c=1) |
|---|---|---|
| 0.60 | −4.96 | 0.65 |
| 0.62 | −7.83 | 0.62 |
| 0.64 | −14.99 | 0.60 |
| 0.66 | −65.00 | 0.58 |
| 0.67 | 129.58 | 0.57 |
| 0.77 | 5.98 | 0.51 |
| 0.87 | 3.78 | 0.46 |
| 1.00 | 2.98 | 0.41 |

From the table it is clear that a particle having energy more than a critical value $E_c$ characterised by a value of $f$ which lies between 0.66 & 0.67, does not exhibit velocity maxima in the region $r>0$. Physically this means that the gravitational force on a particle (thinking in Newtonian way) having energy more than $E_c$ is always repulsive. For energies less than $E_c$ however, the maxima begin to occur. From the table it is also evident that as the particle energy decreases the maxima get nearer to the singular surface ($r \simeq 1.58 r_s$). However the value of the maximum velocity becomes lesser. Similar tables may be obtained for solutions I and II also. However, for the purpose of comparison it is better to study the (r–v) diagrams for all the solutions (Figs. I, II & III). In these diagrams r and v have been plotted numerically, with $f$ as the parameter, by directly using eqn. (3.9). The figures I, II & III are basically of the same nature except some minor quantitative variations. The graphs indicate the general features of the test particle's free fall trajectories which may be summarised as follows:

A particle starting from a large distance with a vanishingly small velocity (energy $f=1$) will first be accelerated due to the gravitational attractive force (which is asymtotically Newtonian in nature) and as the velocity is increased the particle gets retarded because of a velocity dependent repulsive force and finally it comes to rest at a point $r=r_c \simeq 1.58 r_s$. Note that this value of $r_c$ not only is independent of energy but also is independent of the solutions considered. For increased particle energies the gravitational attraction gets gradually replaced by repulsion and complete deceleration takes over. This phenomenon is marked by the gradual flattening of (r–v) curves as $f$ decreases.

We have already indicated that the above mentioned qualitative features of radial trajectories though akin to general relativistic predictions in Schwarzschild coordinates, have no classical analogue.

To bring out another non-classical feature let us study the question of existence of circular orbits.

## 3.4 The Existence of Circular Orbits

The first integrals that follow from the Lagrangian given in sec. 3.1 (eqn.(3.1)) are given by

$$a_{00}\,\bar{u}^{\,0} = \varepsilon \tag{3.18}$$

$$a_{33}\,\bar{u}^{\,3} = p_{\phi} \tag{3.19}$$

where the constants $\varepsilon$ and $p_{\phi}$ are proportional to energy and angular momentum of the particle respectively. The other integral is provided by the definition of the parameter 'ds' (eqn.(3.4)) which after rearrangement gives

$$a_{00}(\bar{u}^{\,0})^2 + a_{11}(\bar{u}^{\,1})^2 + r^2(\bar{u}^{\,3})^2 = 1 \tag{3.20}$$

where we have assumed $\theta$ = constant as usual. The equationns (3.18), (3.19) & (3.20) together give

$$(dr/dt)^2 = (a_{00}{}^2/a_{11}\varepsilon^2) - (a_{00}/a_{11}) - (p_{\phi}{}^2 a_{00}{}^2/a_{11}r^2\varepsilon^2) \tag{3.21}.$$

The last equation would represent circular orbit provided, the r.h.s and its first derivative with respect to r vanish (Bose 1980). They together give the following expression for the particle energy as a function of the orbit radius R

$$-\frac{1}{\varepsilon^2} = \frac{R}{2}\ \frac{a_{00}^{/}(R)}{a_{00}^{2}(R)} - \frac{1}{a_{00}(R)} - \frac{m_0^2}{E^2} \qquad (3.22)$$

where we have put $-m_0^2\varepsilon^2 = E^2$ (Ghosal & Chakraborty 1991c).

Note that the above condition is also independent of $h_{11}$ and therefore is valid for all the solutions of $h_{\mu\nu}$ mentioned in sec. 3.1.

Written explicitly the equation (3.22) takes the following form

$$\frac{m_0^2}{E^2} = \frac{2R\ \{1-(1/R)\}\ [1 + \ln\ \{1-(1/R)\}] - 1}{2R\ \{1-(1/R)\}\ [1 + \ln\ \{1-(1/R)\}]^2} \qquad (3.23)$$

The existence of circular orbit will be ensured as long as E is real. Numerically the limiting radius turns out to be approximately equal to $2.23r_s$ below which, circular orbits are not possible. This limiting radius is slightly higher than that one would have obtained for GR (Narlikar 1978).

## 3.5 SUMMARY

In the light of a recent Lorentz covariant theory of gravitation, the radial and the circular trajectories of test objects have been studied in a static spherically symmetric situation. It has been found that the gravitational field is characterised by a characteristic radius $r_c = 1.58r_s$ ( $r_s$= Schwarzschild radius ) which defines the surface of infinite red-shift. For a radial free fall it has been shown that a particle coming from a large distance first gets accelerated

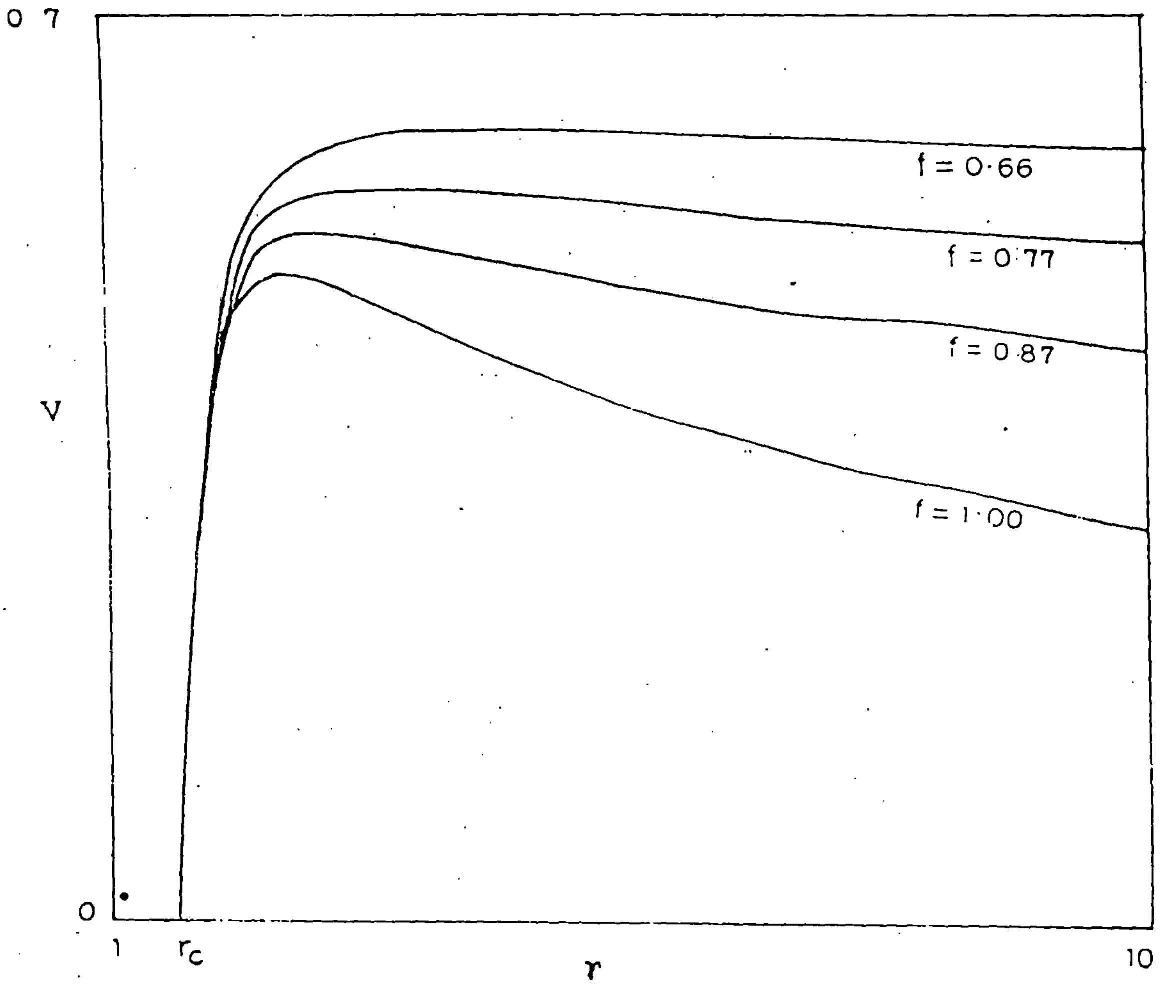


Fig I.

r–v curves for different particle energies ( Solution I ).

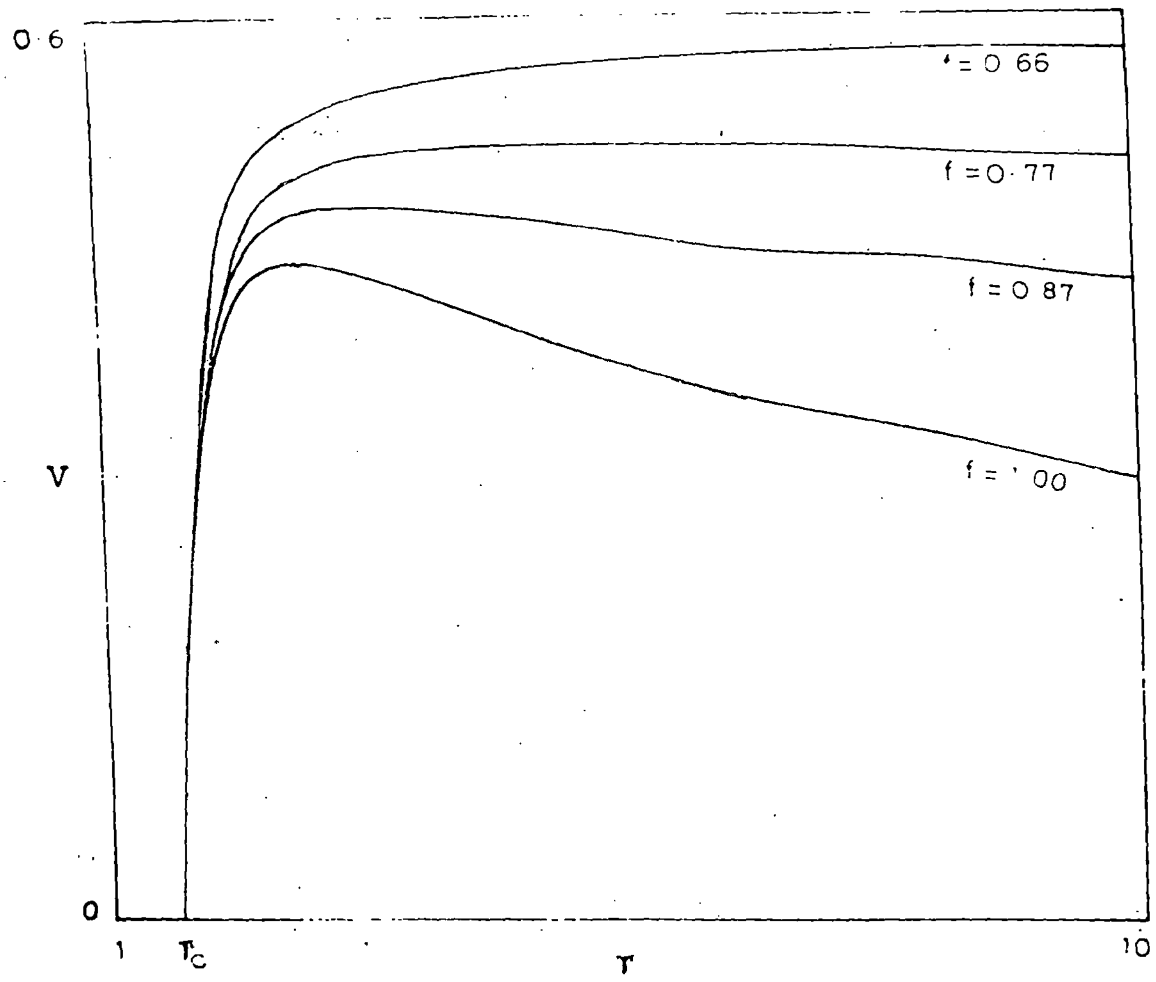


Fig II

r–v curves for different particle energies ( Solution II ).

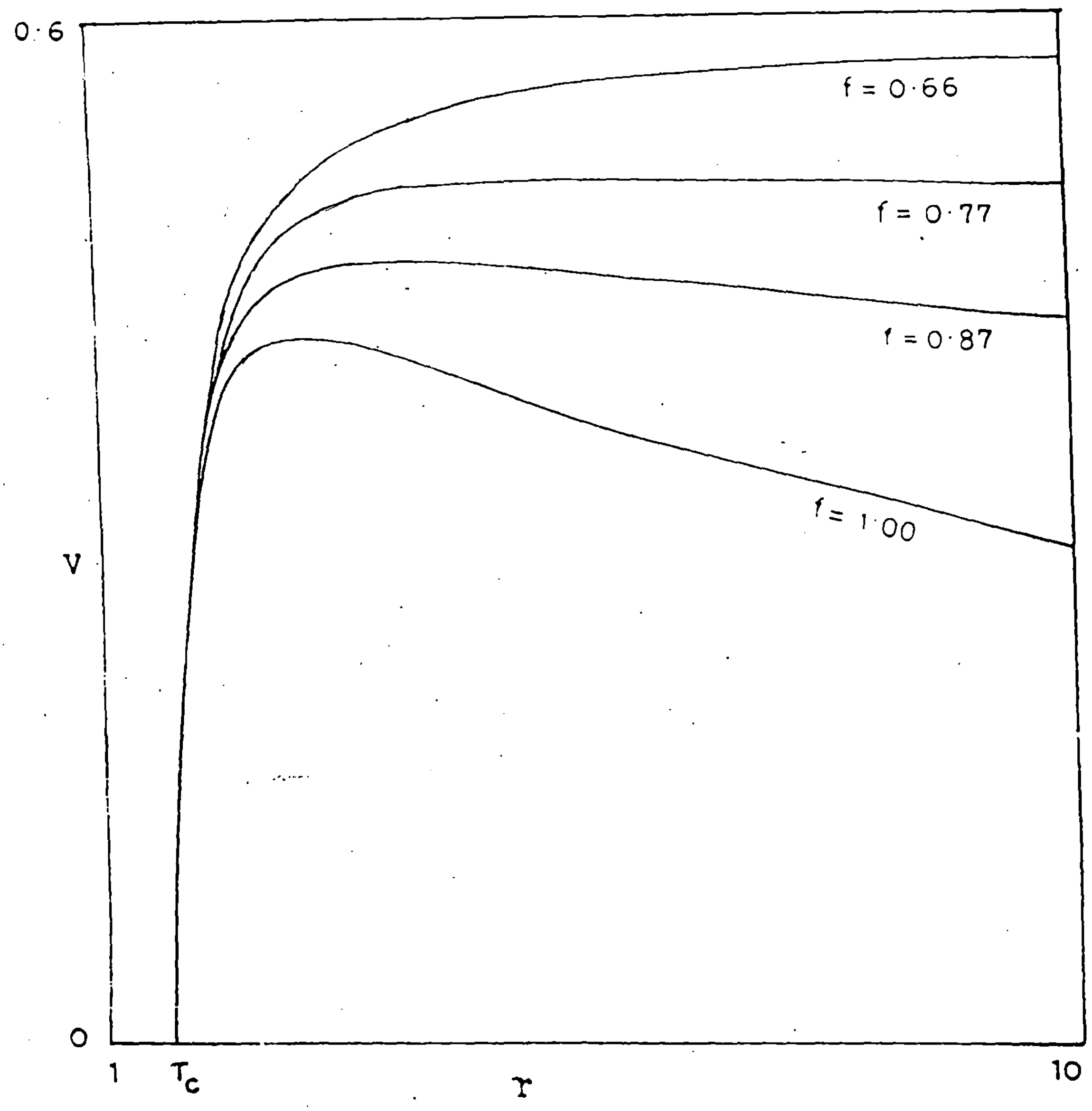


Fig III.

r–v curves for different particle energies ( Solution III ).

towards the source. However, as the velocity increases and the particle penetrates deep into the field, the - non-Newtonian features of gravity begin to show up. From some point along the radial trajectory, depending on the initial energy, the particle starts getting retarded and finally stops at $r_c$. It is therefore observed that the radial fall in general is characterised by a "terminal velocity" in the velocity field. Another non-Newtonian character of the present flat-space-time gravity concerns the question of existence of circular orbits. Calculations revealed that circular orbits cannot exist below a limiting radius which is approximately equal to $2.23r_s$.

## 3.6 EPILOGUE

In chapter II, in connection with discussions on gravitational red-shift we have seen that the concept of global inertial frame is not inconsistent with gravity even though Schield (1960) argued on the contrary. Having been taken care of the arguments against the existence of global inertial frames in a gravitational field, the problem seems to be over and one can happily declare that gravitation and special relativity can (logically at least) go together. But that is not to be. The objection may come from other directions. In a gravitational field light assumes a curved path. Hence one may ask how one is going to synchronize distant clocks in a given inertial frame according to Einstein's prescription (Winnie 1970) which implicitly assumes the rectilinear propagation of light.

Again there is another problem. According to SRNG, even locally the speed of light is not equal to c inside a gravitational field. To understand this quantitatively let us consider for example the radial trajectory of a photon, which can directly be obtained from (2.45) by substituting dt/ds by $(a_{00}\varepsilon')^{-1}$ where the constant of motion $\varepsilon' = 1/\varepsilon$ is defined through the relation (2.21). One thus obtains the following integral of motion for a photon.

$$\varepsilon' = \frac{[1+(a_{11}/a_{00})v^2]^{1/2}}{(-a_{00})^{1/2}} \tag{3.24}$$

Inserting explicitly the expressions for $a_{00}$ & $a_{11}$ one writes

$$\varepsilon' = \frac{[1-\frac{1-\ln(1-2GM/r)}{1+\ln(1-2GM/r)}\, v^2]^{1/2}}{[\,1 + \ln(1-2GM/r)\,]^{1/2}} \tag{3.25}$$

The condition that as $r\to\infty$ (where gravitation is absent), $v\to 1$ (i.e the speed of light equals to c) gives $\varepsilon'=0$. Therefore, from (3.25) one obtains the radial velocity of light at any point r as

$$v^2 = \frac{1+\ln(1-2GM/r)}{1-\ln(1-2GM/r)} \tag{3.26}$$

The relation clearly illustrates that the speed of light is a function of r.

Now is it consistent with special relativity? The answer is apparently no, since it is known that the special relativity rests on the well-known Constancy of Velocity of Light (CVL) assumption!

These two problems to our knowledge are never addressed in the literature. The answer to these problems cann't be fully given till we come to the end of the present volume. However, at this point it will be enough to say that the first problem i.e the problem of synchrony arising out of non-rectilinear propagation of light can be easily dealt with by virtue of the Conventionality of distant Simultaneity thesis of Reichenbach-Grunbaum, which will form the focal theme of the next part of this volume. It will be noted that Einstein's choice of distant clock synchrony is nothing but a convention.

It will be seen that the problem arising out of the non-constancy of the optical speed can also be taken care of. The question of CVL and its role in special relativity will be dealt thoroughly in chapter VII. It will be observed that one can always arrive at the standard form of special relativity even if one starts with the description of space-time in a medium (Ghosal et al 1992) where the optical speed is not equal to c, the speed of light in empty space.

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