CHAPTER - I

ON THE CESÀRO DERIVATIVES

§ 1. Introduction

The concepts of Cesàro-continuity and Cesàro-derivative have been introduced by Burkill [27]. Later on Sargent ([68], [71]) has established several results on these notions.

Lahiri [59] has introduced the concepts of uniform and non-uniform differentiability at a point and on an interval and has shown that for a differentiable function, the set of points of non-uniform differentiability is a set of the first category. Chakrabarty and Bhakta [32] have defined the point of absolute continuity and the point of non-absolute continuity and have proved that for a differentiable function F, the set of points of non-absolute continuity of F on [a,b] is non-dense in [a,b]. A similar result for Schwarz derivative has been proved by Chakrabarty [33].

In this chapter, we introduce the concepts of uniform and non-uniform Cesàro-differentiability on an interval and at a point and establish some results on these concepts. Also we prove the Clarkson's property for the Cesàro-derivative and establish a result on extreme Cesàro-derivates.

§ 2. Preliminaries

Definition 1.2.1 (Chakrabarty and Bhakta [32], p. 115). A point $\eta \in [\, a,b \,]$ is said to be a point of absolute continuity or ac-point of a function F on $[\, a,b \,]$ if there exists a closed neighbourhood of η on which F is absolutely continuous (AC). If there exists no closed neighbourhood of η on which F is absolutely continuous, then η is said to be a point of non-absolute continuity or nac-point of F on $[\, a,b \,]$.

Definition 1.2.2 (Natanson [65] I, p. 216). A real valued function F defined on $[\, a,b \,]$ is said to satisfy a Lipschitz condition if there exists a constant K such that for any two points x and y in $[\, a,b \,]$,

$$| F(x) - F(y) | \leq K | x - y |.$$

Definition 1.2.3 (Kubota [48] I, p. 713). A function F is said to be (ACG) on $[\,a,b\,]$ if $[\,a,b\,]$ is the union of countable number of closed sets E_k on each of which F is AC.

Let the function F be Perron-integrable (Natanson [65] II, p. 159) on $[\,a,b\,]$. We write

$$C(\,F,x,x+h\,) \;=\; \frac{1}{h}\,(P)\int_x^{x+h} F\,.$$

Definition 1.2.4 (Burkill [27], p. 315). The function F is said to be Cesàro-continuous or, C-continuous at $x \in [\,a,b\,]$ if

$$\lim_{h \to 0} C(\,F,x,x+h\,) \;=\; F(x).$$

Definition 1.2.5 (Burkill [27], p. 315). The upper right Cesàro-derivate or, C-derivate of F at x, $CD^{+}F(x)$ is defined by

$$CD^{+}F(x) \;=\; \limsup_{h \to 0+} \frac{2}{h}\,\big[\,C(\,F,x,x+h\,) - F(x)\,\big].$$

The other derivates $CD_{+}F(x)$, $CD^{-}F(x)$, $CD_{-}F(x)$ have corresponding definitions. The upper and lower C-derivates

$\overline{CD}F(x)$, $\underline{CD}F(x)$ are defined to be $\max \{ CD^+F(x), CD^-F(x) \}$ and $\min \{ CD_+F(x), CD_-F(x) \}$ respectively. If

$$\overline{CD}F(x) = \underline{CD}F(x),$$

then F is said to have a C-derivative $CDF(x)$ which is equal to their common value i.e.

$$CDF(x) = \lim_{h \to 0} \frac{2}{h} [C(F,x,x+h) - F(x)].$$

Theorem 1.2.1 (Natanson [65] II, p. 152). A necessary and sufficient condition that a function f be lower semi-continuous on $[a,b]$ is that the set

$$\{ x \in [a,b] : f(x) \leq c \}$$

be closed for arbitrary real number c.

Theorem 1.2.2 (Theorem II, Burkill [27], p. 319). If at all points of $[a,b]$, G has finite C-derivative g, then g is integrable in the Cesàro-Perron sense and

$$(CP) \int_a^b g = G(b) - G(a).$$

Theorem 1.2.3 (Theorem III, Sargent [68], p. 239). If F is C-continuous in $[a,b]$ and $CDF(x)$ exists for $a < x < b$, then there is a point $\xi \in (a,b)$ such that

$$\frac{F(b) - F(a)}{b - a} = CDF(\xi).$$

Theorem 1.2.4 (Theorem 5.2, Bruckner and Ceder [14]). Let the function F be Darboux on $[a,b]$. If F is not monotonic on $[a,b]$, then there exists two points $p,q \in [a,b]$ such that

$$F(p) = F(q).$$

Theorem 1.2.5 (Burkill [26], p. 274). Given a set E of measure zero on $[a,b]$ and any positive number ϵ, there exists a continuous non-decreasing function μ on $[a,b]$ such that

$$\mu(a) = 0, \qquad \mu(b) < \epsilon,$$

$$\mu'(x) = +\infty \qquad \text{for } x \in E.$$

Theorem 1.2.6 (Theorem II, Sargent [70], p. 150). If F is D^*-integrable on $[a,b]$,

$$\liminf_{h \to 0-} C(F,x,x+h,D^*) \leqslant F(x) \leqslant \limsup_{h \to 0+} C(F,x,x+h,D^*)$$

for all $x \in [a,b]$ and $CD^+F(x) \geqslant 0$ n.e., then F is non-decreasing on $[a,b]$ ($CD^+F(x) \geqslant 0$ n.e. means $CD^+F(x) \geqslant 0$ for all $x \in [a,b]$ except a countable set).

§ 3. Uniform and non-uniform Cesàro-differentiability

Throughout this section, we assume that the real valued function F is C-differentiable on $[\,a,b\,]$. We introduce the following definitions :

Let

$$(1) \qquad \Psi(\,x,h\,) = \frac{2}{h}\left[\,C(\,F,x,x+h\,) - F(x)\,\right] - CDF(x).$$

The function F is said to be uniformly C-differentiable on $[\,a,b\,]$ if for every $\epsilon > 0$, there exists a positive number δ, independent of $x \in [\,a,b\,]$, such that

$$|\,\Psi(\,x,h\,)\,| < \epsilon$$

whenever $0 < |h| < \delta$, $x+h \in [\,a,b\,]$.

Also since F is C-differentiable on $[\,a,b\,]$, for every $\epsilon > 0$, there is a positive number δ (where δ is a function of $x \in [\,a,b\,]$), such that

$$|\,\Psi(\,x,h\,)\,| < \epsilon$$

whenever $0 < |h| < \delta(x)$ and $x,\ x+h \in [\,a,b\,]$.

If $\epsilon > 0$ be taken to be fixed, then $\delta(x)$ may not have positive lower bound on $[\,a,b\,]$. Then by a theorem of

Weierstrass ([41], p. 284), it is possible to find out at least one point $\eta \in [\, a,b \,]$ and a neighbourhood of η on which $\delta(x)$ has no positive lower bound.

A point η, at which for each $\varepsilon > 0$, there exists a neighbourhood in which $\delta(x)$ has a positive lower bound, is said to be a point of uniform C-differentiability of F. Otherwise, η is said to be a point of non-uniform C-differentiability.

Let $\eta \in [\, a,b \,]$ and d,P be two positive numbers. Let the least upperbound of $| \Psi(x,h) |$ for $x \in [\, \eta - d, \eta + d \,]$ and $|h| \leq P$ be U. The function U is monotonically non-increasing as $d \to 0$ and $P \to 0$. We denote the greatest lower bound of U as $d \to 0$ and $P \to 0$ independently by $g(\eta)$. It is clear that if η is a point of uniform C-differentiability, then $g(\eta) = 0$ and conversely. Otherwise, η is a point of non-uniform C-differentiability.

Lemma 1.3.1 The function g is upper semi-continuous on $[\, a,b \,]$.

Proof. Let $\varepsilon > 0$ be taken to be arbitrary. Suppose, if possible, g is not upper semi-continuous at $x_0 \in [\, a,b \,]$. Then there exist points y in every neighbourhood of x_0, such

that

$$g(y) \geqslant g(x_0) + \epsilon.$$

Then there are points in an arbitrary small neighbourhood N of y such that

$$| \Psi(x,h) | > g(x_0) + \frac{\epsilon}{2}$$

for sufficiently small values of $|h|$. Clearly the neighbourhood N is contained in any preassigned neighbourhood of x_0. Hence in every neighbourhood of x_0, there are points at which

$$| \Psi(x,h) | > g(x_0) + \frac{\epsilon}{2}$$

for sufficiently small values of $|h|$, which contradicts the definition of $g(x)$. Hence the lemma follows.

Theorem 1.3.1 The set of points of non — uniform C — differentiability of the C — differentiable function F on $[a,b]$ is of the first category.

Proof. If S denotes the set of points of non - uniform C — differentiability of F on $[a,b]$, then it is clear that

$$(2) \qquad S = \bigcup_{m=1}^{\infty} \left\{ x : x \in [a,b], \ g(x) \geqslant \beta_m \right\},$$

where $\{ \beta_m \}$ is a positive null sequence.

We now show that each of the sets

$$G_{\beta_m} = \left\{ x : x \in [a,b], \; g(x) \geqslant \beta_m \right\}$$

is non — dense in $[a,b]$.

By Theorem 1.2.1 and Lemma 1.3.1, the set G_{β_m} is closed. Therefore to show the set G_{β_m} is non — dense in $[a,b]$ it is sufficient to show that G_{β_m} has no interior point.

Suppose, if possible, η be an interior point of G_{β_m}. Then there exists an interval $[a',b']$, containing η, which is a subset of $G_{\beta_m} \subset [a,b]$. Since $\eta \in G_{\beta_m}$, $g(\eta) \geqslant \beta_m$. Let $\{\delta_n\}$ be a positive null sequence and k ($< \beta_m$) be a positive number. Since $g(\eta) \geqslant \beta_m$, there exists a point η' in some neighbourhood N of η, contained in $[a',b']$ and a number h_1 with $|h_1| < \delta_1$ such that

$$(3) \qquad \left| \frac{\frac{1}{h_1} (P) \int_{\eta'}^{\eta'+h_1} F - F(\eta')}{\frac{h_1}{2}} - CDF(\eta') \right| > k$$

or,

$$| \Phi (\eta', h_1) - CDF(\eta') | > k,$$

where

$$\Phi(\eta', h_1) = \frac{\frac{1}{h_1}(P)\int_{\eta'}^{\eta'+h_1} F - F(\eta')}{\frac{h_1}{2}}.$$

Since F is C — continuous at η', it follows from (3) that there exists a number h_1' with sign same as h_1 and with $|h_1'| \leqslant |h_1|$ for which

$$|G(\eta', h_1, h_1') - CDF(\eta')| > k,$$

where

$$G(\eta', h_1, h_1') = \frac{\frac{1}{h_1}(P)\int_{\eta'}^{\eta'+h_1} F - \frac{1}{h_1'}(P)\int_{\eta'-h_1'}^{\eta'} F}{\frac{h_1}{2}}.$$

Therefore there exists an h_2 with $|h_2| < |h_1|$ such that

$$|G(\eta', h_1, h_1') - G(\eta', h_2, h_2')| > k,$$

where h_2' has the same sign as h_2 and $|h_2'| \leqslant \frac{1}{2}|h_2|$.

Since the indefinite Perron — integral of F is continuous, $G(x, h_1, h_1')$ and $G(x, h_2, h_2')$ are continuous at η'.

Therefore there exists a neighbourhood N_1 of η', contained in N, such that

$$| G(x,h_1,h_1') - G(x,h_2,h_2') | > k, \quad \forall\ x \in N_1.$$

Again $\eta' \in N_1 \subset N \subset [a',b'] \subset G_{\beta_m}$. So $g(\eta') \geq \beta_m$.

Therefore for the positive number $\delta_2' = \min (\delta_2, |h_2'|)$, there exists a point $\eta'' \in N_1$ and a number h_3, $|h_3| < \delta_2'$ such that

$$| \Phi (\eta'',h_3) - CDF(\eta'') | > k.$$

Proceeding as before, we get a neighbourhood N_2 of η'' where $N_2 \subset N_1 \subset N$, such that

$$| G(x,h_3,h_3') - G(x,h_4,h_4') | > k, \quad \forall\ x \in N_2,$$

$$|h_4| < |h_3|, \quad |h_3'| \leq \frac{1}{3}|h_3|, \quad |h_4'| \leq \frac{1}{4}|h_4|,$$

h_3' and h_4' have same sign as h_3 and h_4 respectively.

Continuing this process, we can find a decreasing sequence $\{ N_n \}$ of neighbourhoods such that

$$| G(x,h_{2n-1},h_{2n-1}') - G(x,h_{2n},h_{2n}') | > k, \quad \forall\ x \in N_n$$

and

$$|h_{2n}| < |h_{2n-1}| < \delta_n' = \min\{\delta_n, |h_{2n-2}'|\},$$

$$|h_{2n-1}'| \leqslant \frac{1}{2n-1}|h_{2n-1}|, \qquad |h_{2n}'| \leqslant \frac{1}{2n}|h_{2n}|,$$

h_{2n-1}' and h_{2n}' have the same sign as h_{2n-1} and h_{2n} respectively.

The neighbourhoods N_n can be chosen in such a way that there exists a point $\xi \in \bigcap_n N_n$, and at this point

$$(4) \qquad |G(\xi, h_{2n-1}, h_{2n-1}') - G(\xi, h_{2n}, h_{2n}')| > k, \quad \forall\, n.$$

As $\{\delta_n\}$ is a positive null sequence, so also $\{h_n\}$ and $\{h_n'\}$. Thus from (4) it follows that

$$\lim_{n\to\infty} G(\xi, h_n, h_n')$$

does not exist. Again

$$\left|\frac{h_{2n}'}{h_{2n}}\right| \leqslant \frac{1}{2n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore

$$\lim_{n\to\infty} \frac{h_{2n}'}{h_{2n}} = 0.$$

Now

$$G(\xi, h_n, h_n') = \frac{\frac{1}{h_n}(P)\int_{\xi}^{\xi+h_n} F - \frac{1}{h_n'}(P)\int_{\xi-h_n'}^{\xi} F}{\frac{h_n}{2}}$$

$$= \frac{\frac{1}{h_n}(P)\int_{\xi}^{\xi+h_n} F - F(\xi)}{\frac{h_n}{2}} - \frac{\frac{1}{h_n'}(P)\int_{\xi-h_n'}^{\xi} F - F(\xi)}{\frac{h_n}{2}}$$

$$= \frac{\frac{1}{h_n}(P)\int_{\xi}^{\xi+h_n} F - F(\xi)}{\frac{h_n}{2}} + \left(\frac{h_n'}{h_n}\right)\frac{\frac{1}{h_n'}(P)\int_{\xi-h_n'}^{\xi} F - F(\xi)}{-\frac{h_n'}{2}}$$

Since F is C — differentiable on $[a,b]$, it follows that

$$\lim_{n \to \infty} G(\xi, h_n, h_n')$$

exists. Thus we arrive at a contradiction. Hence η is not an interior point of G_{β_m} and consequently G_{β_m} has no interior point. Therefore G_{β_m} is non — dense in $[a,b]$.

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Therefore from (2) it follows that S is a set of the first category.

Theorem 1.3.2 If F is unifomly C — differentiable at $\eta \in [a,b]$, then

$$\lim_{h \to 0} \frac{1}{h} (CP) \int_{\eta}^{\eta+h} CDF = CDF(\eta).$$

Proof. Let $\epsilon > 0$ be chosen arbitrarily. Since F is uniformly C— differentiable at $\eta \in [a,b]$, there exists a $\delta_1 > 0$ and an h_1 such that

$$(5) \qquad \left| \frac{\frac{1}{h_1} (P) \int_{x}^{x+h_1} F - F(x)}{\frac{h_1}{2}} - CDF(x) \right| < \epsilon$$

for all $x \in (\eta - \delta_1, \eta + \delta_1)$.

We write

$$H(x) = \frac{\frac{1}{h_1} (P) \int_{x}^{x+h_1} F - F(x)}{\frac{h_1}{2}}$$

Since the indefinite Perron – integral is continuous and since F is C – continuous, it follows that H is C – continuous. Hence there exists a $\delta > 0$ with $0 < \delta < \delta_1$, such that

$$(6) \qquad \left| \frac{1}{h} (P) \int_{\eta}^{\eta+h} H - H(\eta) \right| < \epsilon$$

for $|h| \leqslant \delta$. By Theorem 1.2.2, CDF is CP – integrable. From (5) we have

$$(7) \qquad -\epsilon < H(x) - CDF(x) < \epsilon,$$

for all $x \in (\eta - \delta_1, \eta + \delta_1)$.

From (7) we have

$$CDF(x) < \epsilon + H(x) \qquad \forall \quad x \in (\eta - \delta_1, \eta + \delta_1)$$

$$\Longrightarrow \qquad \frac{1}{h} (CP) \int_{\eta}^{\eta+h} CDF \leqslant \epsilon + \frac{1}{h} (P) \int_{\eta}^{\eta+h} H$$

for $|h| \leqslant \delta$

$$\Longrightarrow \qquad \frac{1}{h} (CP) \int_{\eta}^{\eta+h} CDF - CDF(\eta) \leqslant 2\epsilon + \frac{1}{h} (P) \int_{\eta}^{\eta+h} H - H(\eta)$$

(by (7))

$$\Longrightarrow \qquad \frac{1}{h} (CP) \int_{\eta}^{\eta+h} CDF - CDF(\eta) < 3\epsilon$$

for $|h| \leqslant \delta$ (by (6)).

Similarly we have

$$-3\epsilon \quad < \quad \frac{1}{h}\,(CP)\int_{\eta}^{\eta+h} CDF \;-\; CDF(\eta) \qquad \text{for} \quad |h| \leqslant \delta.$$

Therefore

$$\left|\, \frac{1}{h}\,(CP)\int_{\eta}^{\eta+h} CDF \;-\; CDF(\eta) \,\right| \quad < \quad 3\epsilon \qquad \text{for} \quad |h| \leqslant \delta.$$

This proves the theorem.

Corollary 1.3.1 Let CDF be Perron — integrable on $[\,a,b\,]$. If F is uniformly C — differentiable at $\eta \in [\,a,b\,]$, then CDF is C — continuous at η.

The corollary follows directly from Theorem 1.3.2.

Theorem 1.3.3 If F is continuous and uniformly C — differentiable at $\eta \in [\,a,b\,]$, then CDF is continuous at η.

Proof. Since F is uniformly C — differentiable at $\eta \in [\,a,b\,]$, $g(\eta) = 0$. Then to every $\epsilon > 0$, there exists a neighbourhood $[\,\eta - \delta_1, \eta + \delta_1\,]$ of η, $\delta_1 > 0$ such that

$$|\,\Psi(\,x,h\,)\,| \;<\; \epsilon$$

for all $x \in [\,\eta - \delta_1, \eta + \delta_1\,]$ and for sufficiently small values

of $|h|$. Therefore for a fixed value h_1 of h and for all $x \in [\eta - \delta_1, \eta + \delta_1]$

$$|\Psi(x,h_1)| < \varepsilon. \tag{8}$$

As the indefinite Perron – integral of F is continuous

$$\Phi(x,h_1) = \frac{\frac{1}{h_1}(P)\int_x^{x+h_1} F - F(x)}{\frac{h_1}{2}}$$

is continuous at η. Hence there exists a neighbourhood $[\eta - \delta_2, \eta + \delta_2]$ of η, $\delta_2 > 0$ such that

$$|\Phi(x_1,h_1) - \Phi(x_2,h_1)| < \varepsilon \tag{9}$$

whenever $x_1, x_2 \in [\eta - \delta_2, \eta + \delta_2]$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Therefore for any two points $y_1, y_2 \in [\eta - \delta, \eta + \delta]$,

$$|CDF(y_1) - CDF(y_2)|$$

$$= |\Phi(y_1,h_1) - \Psi(y_1,h_1) - \Phi(y_2,h_1) + \Psi(y_2,h_1)|$$

(using (1))

$$\leq |\Phi(y_1,h_1) - \Phi(y_2,h_1)| + |\Psi(y_1,h_1)| + |\Psi(y_2,h_1)|$$

$$< \varepsilon + \varepsilon + \varepsilon \qquad \text{(by (8) and (9))}$$

$$= 3\varepsilon$$

$\Longrightarrow$ CDF is continuous at η.

Theorem 1.3.4 If CDF is continuous at $\eta \in [a,b]$, then F is uniformly C — differentiable at η.

Proof. Since CDF is continuous at $\eta \in [a,b]$, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

(10) $$|CDF(y_1) - CDF(y_2)| < \varepsilon$$

for all $y_1, y_2 \in (\eta - \delta, \eta + \delta)$. Now

$$|\Psi(x,h)| = \left| \frac{\frac{1}{h}(P)\int_x^{x+h} F(t)dt - F(x)}{\frac{h}{2}} - CDF(x) \right|$$

$$= \left| \frac{2}{h^2}(P)\int_x^{x+h} [F(t) - F(x)]dt - CDF(x) \right|.$$

By Theorem 1.2.3, there exists a point ξ_t between t and x

such that

$$F(t) - F(x) = (t - x) \, CDF(\xi_t).$$

Hence for $x \in (\eta - \delta/2, \eta + \delta/2)$ and $|h| < \delta/2$,

$$| \Psi(x,h) | = \left| \frac{2}{h^2} (P) \int_x^{x+h} CDF(\xi_t)(t - x)dt - CDF(x) \right|$$

$$= \left| \frac{2}{h^2} (P) \int_x^{x+h} [CDF(\xi_t) - CDF(x)](t - x)dt \right|$$

$$\leqslant \epsilon \left| \frac{2}{h^2} \int_x^{x+h} (t - x)dt \right| \qquad (\text{by (10)})$$

$$= \epsilon.$$

Hence F is uniformly C — differentiable at η.

Theorem 1.3.5 If $\eta \in [a,b]$ is a point of uniform C — differentiability, then F is differentiable at η.

The theorem follows from Theorem 1.3.2 and the following Lemma :

Lemma 1.3.2 If

$$\lim_{h \to 0} \frac{1}{h} (CP) \int_{\eta}^{\eta+h} CDF = CDF(\eta),$$

then F is differentiable at η.

Proof. For every $\epsilon > 0$, there exists a positive number δ, such that

$$\left| \frac{1}{h} (CP) \int_{\eta}^{\eta+h} CDF - CDF(\eta) \right| < \epsilon,$$

when $0 < |h| < \delta$. Since

$$(CP) \int_{a}^{x} CDF = F(x) - F(a),$$

we get from above

$$\left| \frac{F(\eta + h) - F(\eta)}{h} - CDF(\eta) \right| < \epsilon,$$

when $0 < |h| < \delta$. Hence

$$\lim_{h \to 0} \frac{F(\eta + h) - F(\eta)}{h}$$

exists and is equal to $CDF(\eta)$ and the lemma is proved.

Theorem 1.3.6 The set of points of non — differentiability of the C — differentiable function F on $[a,b]$ is of the first category.

Proof. By Theorem 1.3.5, it follows that the set of points of non-differentiability of F on $[a,b]$ is a subset of the set of points of non-uniform C-differentiability of F on $[a,b]$. But by Theorem 1.3.1, the set of points of non-uniform C-differentiability of F is of the first category. Therefore it follows that the set of points of non-differentiability of F is of the first category.

Mukhopadhyay [64] has shown that if F is uniformly Schwarz differentiable on $[a,b]$, then

(i) $F^{(1)}$ is bounded on $[a,b]$ if F is continuous on $[a,b]$,

(ii) F is continuous on $[a,b]$ if $F^{(1)}$ is bounded on $[a,b]$,

where $F^{(1)}(x)$ is the Schwarz derivative of F at $x \in [a,b]$.

In case of C-derivative the result (i) follows from Theorem 1.3.3. We prove below that in case of C-derivatives, the result (ii) is true without assuming the uniformity of the C-derivatives.

Theorem 1.3.7 If CDF is bounded on $[a,b]$, then F is continuous on $[a,b]$.

Proof. As CDF is bounded on $[a,b]$, there exists an $M > 0$ such that

$$| \, CDF(x) \, | \leqslant M$$

for all $x \in [\, a,b \,]$.

Let $c \in [\, a,b \,]$. By Theorem 1.2.2, CDF is CP-integrable on $[\, a,b \,]$, and

$$F(x) - F(c) = (CP) \int_c^x CDF(t)dt,$$

and so

$$-M(x - c) \leqslant F(x) - F(c) \leqslant M(x - c).$$

Therefore F is continuous at c. This proves the theorem.

Theorem 1.3.8 If the C-differentiable function F is continuous on $[\, a,b \,]$, then the set of nac-points of F on $[\, a,b \,]$ is non-dense in $[\, a,b \,]$.

Proof. Let N and S denote the set of nac-points of F on $[\, a,b \,]$ and the set of points of non-uniform C-differentiability of F on $[\, a,b \,]$, respectively. By Theorem 1.3.1, S is a set of the first category. Since N is closed and a set of the first category can not contain an interval, the theorem will be proved if we show that $N \subset S$.

Let $\eta \in [\, a,b \,]$ be a point of uniform C-differentiability of F on $[\, a,b \,]$. Then by Theorem 1.3.3, CDF is con-

tinuous at η. So there exists a positive number δ such that CDF is bounded on $[\eta - \delta, \eta + \delta] \subset [a,b]$. Hence there exists a positive number M such that

$$(11) \qquad |\,CDF(x)\,| \leqslant M \qquad \forall \; x \in [\eta - \delta, \eta + \delta].$$

Let $x_1, x_2 \in [\eta - \delta, \eta + \delta]$, $x_2 > x_1$. Then by Theorem 1.2.3 there exists a number ξ, $x_1 < \xi < x_2$, such that

$$F(x_2) - F(x_1) = (x_2 - x_1)\, CDF(\xi),$$

and so by (11)

$$|\,F(x_2) - F(x_1)\,| \leqslant M|\,x_2 - x_1\,|.$$

This shows that F satisfies Lipschitz condition on $[\eta - \delta, \eta + \delta]$ and therefore F is absolutely continuous on $[\eta - \delta, \eta + \delta]$. Hence η is a ac-point of F.

Therefore it follows that $N \subset S$ and the theorem is proved.

§ 4. Clarkson's property for C-derivatives

Clarkson [34] has proved the following theorem :

Theorem Let the function F be differentiable on

$[\, a,b \,]$. Then for any two real numbers α and β, the set

$$E(\alpha,\beta) = \left\{ x \in [\, a,b \,] \;:\; \alpha < F'(x) < \beta \right\}$$

is either empty or of positive measure.

Now we prove the above theorem for C-derivatives. For this we require the following two results :

Theorem 1.4.1 If CDF exists finitely on $[\, a,b \,]$, then CDF is Darboux on $[\, a,b \,]$.

Proof. Let r be a real number such that

$$CDF(\alpha) < r < CDF(\beta), \qquad \alpha,\beta \in [\, a,b \,].$$

Let

$$G(x) = F(x) - rx, \qquad x \in [\, a,b \,].$$

Then $CDG(x)$ exists finitely on $[\, a,b \,]$. Since $CDG(\alpha) < 0$ and $CDG(\beta) > 0$, it follows that G is not monotonic on the closed interval I having α and β as end points.

Let Q be the set of points of I throughout no neighbourhood of which G is monotonic. Then Q is a non - empty closed set. If $(\, c,d \,)$ be a component interval of $I^o - Q$, then G is monotonic and continuous on $c \leqslant x \leqslant d$ (I^o denotes the set of interior points of I). It thus follows that neither α nor β can be an isolated point of Q.

If ξ is an isolated point of Q, then G has an extremum at ξ, and hence $CDG(\xi) = 0$. We now suppose that Q is a perfect set. Since CDG(x) exists finitely, by Theorem I and Lemma IV of Sargent [69], it follows that G is (ACG) i.e there exists a sequence of closed sets $\{Q_n\}$ such that G is AC on each Q_n and $[a,b] = \bigcup_n Q_n$. Since $Q = \bigcup_n (Q \cap Q_n)$, there exists by Baire's theorem, a closed interval $[\ell, m]$ with $(\ell, m) \cap Q \neq \phi$ and a positive integer n such that

$$[\ell, m] \cap Q \subset Q \cap Q_n.$$

Thus G is AC and, hence, continuous on $[\ell, m] \cap Q$. Since G is monotonic and continuous on each of the complementary interval of $[\ell, m] \cap Q$ relative to $[\ell, m]$, it follows that G is continuous on $[\ell, m]$. Therefore there exist two points x_1 and x_2 in $[\ell, m]$ such that $G(x_1) = G(x_2)$ (by Theorem 1.2.4).

Thus it follows that G has an extremum at some point $\xi \in (x_1, x_2)$, and thus

$$CDG(\xi) = 0$$

$$\Longrightarrow \quad CDF(\xi) = r.$$

Thus CDF is Darboux on [a,b].

Theorem 1.4.2 If for the function F on $[a,b]$

(i) $\liminf_{h \to 0-} C(F,x,x+h) \leqslant F(x) \leqslant \limsup_{h \to 0+} C(F,x,x+h)$

for all x,

(ii) $\underline{CD}F(x) \geqslant 0$ a.e. and

(iii) $\underline{CD}F(x) > -\infty$ n.e.,

then F is non-decreasing on $[a,b]$.

Proof. Let α,β $(\alpha < \beta)$ be any two points in $[a,b]$.

From (ii), it follows that there exists a set S such that $|S| = 0$ and

$$\underline{CD}F(x) \geqslant 0 \qquad \forall\ x \in [a,b] - S.$$

Let $\epsilon > 0$ be chosen arbitrary. Then by Theorem 1.2.5, there exists a non-decreasing continuous function μ such that

$$\mu(a) = 0, \qquad \mu(b) < \epsilon,$$

$$\mu'(x) = +\infty \qquad \text{for } x \in S.$$

Let

$$H(x) = F(x) + \mu(x).$$

Then

$$\underline{CD}H(x) \geqslant \underline{CD}F(x) + \underline{CD}\mu(x).$$

Since μ is non-decreasing,

$$\underline{CD}\mu(x) \geqslant 0$$

for all $x \in [a,b]$. Hence

$$\underline{CD}H(x) \geqslant 0$$

for all $x \in [a,b] - S$. Again from (iii),

$$\underline{CD}F(x) > -\infty$$

for all $x \in [a,b] - S_1$, where $S_1 \subset S$ is a countable set. Therefore

$$\underline{CD}H(x) \geqslant 0$$

for all $x \in [a,b] - S_1$.

Since H is C-continuous, by Theorem 1.2.6, H is non-decreasing on $[a,b]$. Therefore it follows that

$$F(\beta) - F(\alpha) + \epsilon > 0.$$

As $\epsilon > 0$ is arbitrary,

$$F(\beta) \geqslant F(\alpha).$$

This proves the theorem.

Now we prove the Clarkson's property for the C-derivatives.

Theorem 1.4.3 For the C-differentiable function F on $[a,b]$, the set

$$S(\alpha,\beta) = \{ x \in [a,b] : \alpha < CDF(x) < \beta \}$$

is either empty or of positive measure (α and β being any two real numbers).

Proof. By Theorem 1.4.2, it follows that if $CDF(x) \geq k$ for almost all $x \in [a,b]$, then $CDF(x) \geq k$ for all $x \in [a,b]$, (where k is a real number). Therefore it follows that the least upper bound and greatest lower bound of CDF on $[a,b]$ are same as those over a subset E of $[a,b]$ where $|E| = b - a$.

Let

$$S = S(\alpha,\beta) = \{ x \in [a,b] : \alpha < CDF(x) < \beta \}$$

$$S_\alpha = \{ x \in [a,b] : CDF(x) \leq \alpha \}$$

$$S_\beta = \{ x \in [a,b] : CDF(x) \geq \beta \}.$$

Then the sets S, S_α, S_β are mutually disjoint and

$$[a,b] = S \bigcup S_\alpha \bigcup S_\beta.$$

If possible, suppose S is not void and $|S| = 0$. Under this condition, we first prove

$$S \subset S'_\alpha \bigcap S'_\beta$$

where A' denotes the derived set of the set A.

Let us assume $y_0 \in S$ but $y_0 \notin S'_\alpha$. Then there exists an interval I such that $y_0 \in I$ and $S_\alpha \bigcap I = \phi$. So $CDF(x) \geqslant \beta$ for almost all $x \in I$. But $S \bigcap I \neq \phi$. So we arrive at a contradiction to the conclusion derived in the first paragraph. Hence

$$S \subset S'_\alpha.$$

Similarly we get

$$S \subset S'_\beta.$$

Therefore

$$S \subset S'_\alpha \bigcap S'_\beta. \tag{12}$$

If $\eta \in S$, then by (12), for any interval J containing η, we have

$$\sup_{x \in J} CDF(x) \geqslant \beta, \qquad \inf_{x \in J} CDF(x) \leqslant \alpha.$$

By Theorem 1.4.1, CDF possesses Darboux property, and hence

$$\sup_{x \in J \cap S} CDF(x) = \beta$$

$$\inf_{x \in J \cap S} CDF(x) = \alpha.$$

Hence it follows that the function CDF considered over $\overline{S} = S \cup S'$ is discontinuous at each point of $\overline{S}$. Since CDF belongs to Baire's first class, the set of points of discontinuity of CDF considered over the subset $\overline{S}$ must form a set of the first category on $\overline{S}$. Since $\overline{S}$ is a set of the second category on itself, the theorem follows.

§ 5. A theorem on extreme C-derivates

Theorem 1.5.1 If the function F on $[a,b]$ is continuous, then the set

$$\{ x \in [a,b] : CD^{-}F(x) \neq CD^{+}F(x) \}$$

is of the first category.

Proof. Let us first show that the set

$$S = \{ x \in [a,b] : CD^{-}F(x) < CD^{+}F(x) \}$$

is of the first category. We write

S_{rk}

$$= \left\{ x \in S_r \ : \ \frac{-\frac{1}{h}\int_x^{x-h} F \ - \ F(x)}{-\frac{h}{2}} < r < CD^+F(x), \right.$$

$$\left. \forall \ \ h \in \left(0, \frac{1}{k} \right) \right\},$$

where

$$S_r = \left\{ x \in [\, a,b \,] \ : \ CD^-F(x) < r < CD^+F(x) \right\},$$

r is a rational number and k is a positive integer.

Let

$$p \in \bigcup_k S_{rk}.$$

Then $p \in S_{rk}$ for some k

$$\Longrightarrow \quad \frac{-\frac{1}{h}\int_p^{p-h} F \ - \ F(p)}{-\frac{h}{2}} < r, \quad \forall \ \ h \in \left(0, \frac{1}{k} \right).$$

$$\Longrightarrow \quad CD^-F(p) < r$$

$$\Longrightarrow \quad p \in S_r$$

$$(13) \quad \Longrightarrow \quad S_{rk} \subset S_r.$$

Again let $q \in S_r$. Then

$$CD^{-}F(q) < r.$$

Hence for a positive integer k,

$$\frac{-\frac{1}{h}\int_{q}^{q-h} F - F(q)}{-\frac{h}{2}} < r, \quad \forall \quad h \in (0, \frac{1}{k})$$

$$\Longrightarrow \qquad q \notin S_{rk}$$

$$\Longrightarrow \qquad q \notin \bigcup_{k} S_{rk}$$

$$(14) \Longrightarrow \qquad S_r \subset \bigcup_{k} S_{rk}.$$

From (13) and (14) we get

$$S_r = \bigcup_{k} S_{rk}.$$

Again

$$S = \bigcup_{r} S_r.$$

Therefore

$$S = \bigcup_{r} \bigcup_{k} S_{rk}.$$

Thus it suffices to show that S_{rk} is non-dense for all r and k.

If possible, suppose S_{rk} is dense in some interval $(\alpha,\beta) \subset [a,b]$. Let ϵ be an arbitrarily chosen positive number. We choose two points x', x'' such that

$$x' \in S_{rk} \cap (\alpha,\beta)$$

and

$$x'' \in (\alpha,\beta)$$

with

$$0 < x'' - x' < \frac{1}{k},$$

$$|F(\xi) - F(\eta)| < \epsilon \tag{15}$$

for any $\xi, \eta \in [x',x'']$ and

$$\frac{\frac{1}{x''-x'}\int_{x'}^{x''} F - F(x')}{\frac{(x''-x')}{2}} > r.$$

Again

$$\frac{\frac{1}{x-x'}\int_{x'}^{x} F - F(x')}{\frac{(x-x')}{2}}$$

is continuous at x''. Therefore there exists a $\delta > 0$ such that

$$(x'' - \delta, x'') \subset (\alpha, \beta) \bigcap (x', x' + \frac{1}{k})$$

and

$$\frac{\frac{1}{x - x'} \int_{x'}^{x} F - F(x')}{\frac{(x - x')}{2}} > r$$

for all $x \in (x'' - \delta, x'')$. Since S_{rk} is dense in (α, β), we can choose a point

$$x_1 \in (x'' - \delta, x'') \bigcap S_{rk} .$$

Then

$$\frac{\frac{1}{x_1 - x'} \int_{x'}^{x_1} F - F(x')}{\frac{(x_1 - x')}{2}} > r. \tag{16}$$

Again since

$$x_1 \in S_{rk} \quad \text{and} \quad 0 < x_1 - x' < \frac{1}{k} ,$$

we have

$$(17) \qquad \frac{\dfrac{1}{x' - x_1} \int_{x_1}^{x'} F - F(x_1)}{\dfrac{(x' - x_1)}{2}} < r.$$

Since F is continuous, there is a point $\Upsilon \in (x', x_1)$ such that

$$\int_{x'}^{x_1} F = F(\Upsilon)(x_1 - x').$$

Therefore by (16), we get

$$(18) \qquad \frac{F(\Upsilon) - F(x')}{\dfrac{(x_1 - x')}{2}} > r,$$

and by (17),

$$(19) \qquad \frac{F(\Upsilon) - F(x_1)}{\dfrac{(x' - x_1)}{2}} < r.$$

Therefore from (15) and (18), we get

$$(20) \qquad \frac{F(x_1) - F(x')}{\frac{(x_1 - x')}{2}} > r - \frac{2\epsilon}{x_1 - x'}$$

and from (15) and (19)

$$(21) \qquad \frac{F(x') - F(x_1)}{\frac{(x' - x_1)}{2}} < r - \frac{2\epsilon}{x_1 - x'} .$$

The relations (20) and (21) are contradictory. Therefore S_{rk} is non-dense. This completes the proof of the theorem.