REPRINTS AND LETTERS

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GOVERNING EQUATIONS FOR NON-LINEAR ANALYSIS OF SANDWICH PLATES—A NEW APPROACH

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Abstract—In this paper non-linear static and dynamic behaviours of sandwich plates have been studied by a new set of decoupled differential equations. Numerical results of rectangular sandwich plates under mechanical as well as dynamical loading, has been computed and compared with the other known results.

INTRODUCTION

Investigations on finite deformation of sandwich plates are gaining importance day by day due to its wide applications in modern designs. Outstanding research workers who carried out interesting investigations in this field are Reissner [1], Alwan [2], and Nowinski and Ohnabe [3].

Kamiya [4] has offered a new set of governing equations by using Berger's approximation to study the non-linear static behaviours of sandwich plates. The author has analysed in detail the case of rectangular sandwich plates. The accuracy of his method depends on a correction factor $F(b/a)$. Thus although Kamiya has been able to simplify the coupled differential equations of sandwich plates proposed by earlier authors, his attempt has been restricted to plate geometry due to introduction of this correction factor. Moreover Berger's method fails completely for movable edge conditions.

In this paper a new set of differential equations of sandwich plates in rectangular a Cartesian co-ordinate system, have been derived in a decoupled form. The accuracy of this set does not depend on any correction factor and thus holds good for sandwich plates of different shapes. Moreover results for movable edge conditions can be derived from the same cubic equations. Numerical results of rectangular plates with different side ratios, both under mechanical and dynamical loading have been computed and compared with other known results.

GOVERNING EQUATIONS

First we posit a rectangular co-ordinate system x, y, z; x, y in the middle plane of the core, z thickness direction (positive downwards).

For the sake of simplicity consider a sandwich plate with an isotropic core as well as isotropic upper and lower faces of identical thickness while the faces respond to the bending and membrane actions of the plate, the core is assumed to transfer only shear deformations. Moreover compared with the core thickness h, the face thickness t_1 is supposed to be thin enough to ignore a variation of stress in the thickness direction of the faces.

By virtue of Hook's law for isotropic elastic materials, the strain energy per unit area of both the faces are represented as [4]

$$\bar{V}_0^{\mathrm{f}} = \frac{Et_1}{1-\nu^2}\left[\varepsilon_x^{m^2} + \frac{1}{4}\left(\frac{\partial r}{\partial x}\right)^2 + \varepsilon_y^{m^2} + \frac{1}{4}\left(\frac{\partial s}{\partial y}\right)^2 + 2\nu\left(\varepsilon_x^m\varepsilon_y^m + \frac{1}{4}\frac{\partial r}{\partial x}\cdot\frac{\partial s}{\partial y}\right)\right.$$
$$\left. + \frac{1-\nu}{2}\left\{\gamma_{xy}^{m^2} + \frac{1}{4}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial s}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial r}{\partial y}\frac{\partial s}{\partial x}\right\}\right] \quad (1)$$

Furthermore, if we rewrite equation (1) by introducing two invariants of the averaged strains [4]

$$I_1^m = \varepsilon_x^m + \varepsilon_y^m, \quad I_2^m = \varepsilon_x^m\varepsilon_y^m - \tfrac{1}{4}\gamma_{xy}^{m^2} \quad (2)$$

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we obtain

$$\bar{V}_0^{\mathrm{f}} = \frac{Et_1}{1-\nu^2}\left[I_1^{m^2} - 2(1-\nu)I_2^m + \frac{1}{4}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\nu}{2}\frac{\partial r}{\partial x}\frac{\partial s}{\partial y} + \frac{1-\nu}{8}\left(\frac{\partial r}{\partial y} + \frac{\partial s}{\partial x}\right)^2\right]. \tag{3}$$

Since the shear strains of the core may be expressed as

$$\gamma_{xz} = \frac{1}{h}(U^{\mathrm{l}} - U^{\mathrm{u}}) + \frac{\partial w}{\partial x}; \quad \gamma_{yz} = \frac{1}{h}(V^{\mathrm{l}} - V^{\mathrm{u}}) + \frac{\partial w}{\partial y} \tag{4}$$

the strain energy per unit area of the isotropic core due to the shear becomes [4]

$$\bar{V}_0^{\mathrm{s}} = \frac{1}{2}hG'\left[\left(\frac{r}{h}\right)^2 + \left(\frac{s}{h}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 - \frac{2}{h}\left(r\frac{\partial w}{\partial x} + s\frac{\partial w}{\partial y}\right)\right]. \tag{5}$$

In consequence, the total strain energy per unit area of the sandwich plate is

$$\bar{V}_0 = \bar{V}_0^{\mathrm{f}} + \bar{V}_0^{\mathrm{s}}. \tag{6}$$

Using the modified strain energy expression proposed by Banerjee [5], (3) can be re-written as

$$\bar{V}_0^{\mathrm{f}} = \frac{Et_1}{1-\nu^2}\left[I_1^{1m^2} + \frac{\lambda}{4}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}^2 + \frac{1}{4}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial s}{\partial y}\right)^2 \right.$$
$$\left. + \frac{\nu}{2}\frac{\partial r}{\partial x}\frac{\partial s}{\partial y} + \frac{1-\nu}{8}\left(\frac{\partial r}{\partial y} + \frac{\partial s}{\partial x}\right)^2\right] \tag{7}$$

where

$$I_1^{1m} = \frac{\partial}{\partial x}(u^{\mathrm{u}} + u^{\mathrm{l}}) + \nu\frac{\partial}{\partial y}(V^{\mathrm{u}} + V^{\mathrm{l}}) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{7$'$}$$

and λ is a constant given by [5].

Addition of equation (5) with equation (7) gives the total potential energy V.

Executing the variational calculus so as to minimize the total potential energy of the present elastic system of the sandwich plate, we arrive at the following differential equations:

$$I_1^{1m} = \frac{\partial f}{\partial x} + \nu\frac{\partial Q}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \text{constant}$$

where

$$P = u^{\mathrm{u}} + u^{\mathrm{l}}, \quad Q = v^{\mathrm{u}} + v^{\mathrm{l}} \tag{8}$$

$$\frac{Et_1}{2(1-\nu^2)}\left[\frac{\partial^2 r}{\partial x^2} + \nu\frac{\partial^2 s}{\partial x \partial y}\right] + \frac{Et_1}{4(1+\nu)}\left[\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 s}{\partial x \partial y}\right] - G'\left[\frac{r}{h} - \frac{\partial w}{\partial x}\right] = 0 \tag{9}$$

$$\frac{Et_1}{2(1-\nu^2)}\left[\frac{\partial^2 s}{\partial y^2} + \nu\frac{\partial^2 r}{\partial x \partial y}\right] + \frac{Et_1}{4(1+\nu)}\left[\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 r}{\partial x \partial y}\right] - G'\left[\frac{s}{h} - \frac{\partial w}{\partial y}\right] = 0 \tag{10}$$

$$\frac{2Et_1}{1-\nu^2}I_1^{1m}\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right] + hG'\nabla^2 w - G'\left(\frac{\partial r}{\partial x} + \frac{\partial s}{\partial y}\right)$$
$$+ \frac{Et_1\lambda}{1-\nu^2}\left[\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}\nabla^2 w + 2\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2}\right.$$
$$\left. + 2\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial y^2} + 4\frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] + q = 0 \tag{11}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

From somewhat re-written equations (9) and (10) it becomes

$$\left[\frac{Et_1}{2(1-\nu^2)}\nabla^2 - \frac{G^1}{h}\right]\left(\frac{\partial r}{\partial x} + \frac{\partial s}{\partial y}\right) + G'\nabla^2 w = 0. \tag{12}$$

Combining equations (11) and (12) we get

$$\left[\frac{Et_1}{2(1-\nu^2)}\nabla^2 - \frac{G'}{h}\right]\left[\frac{2Et_1}{G'(1-\nu^2)}I_1^{1m}\left\{\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right\} + h\nabla^2 w\right.$$
$$+\frac{Et_1\lambda}{G'(1-\nu^2)}\left\{\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right)\nabla^2 w + 2\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2}\right.$$
$$\left.\left.+2\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial y^2} + 4\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\} + \frac{q}{G^1}\right]$$
$$+G^1\nabla^2 w = 0 \tag{13}$$

EXAMPLES OF ANALYSIS

Non-linear static behaviours of sandwich plates

We consider the bending of simply supported rectangular sandwich plates ($a \times b$) with constrained in-plane displacements at the boundaries. The boundary conditions are expressed as follows:

$$\text{at } x = 0 \quad \text{and} \quad a, w = 0, \quad M_x = 0, \quad U^0 + U^1 = 0$$
$$V^u + V^1 = 0; \quad s = 0 \tag{14a}$$
$$\text{at } y = 0 \quad \text{and} \quad b, W = 0, \quad M_y = 0, \quad U^u + U^1 = 0$$
$$V^u + V^1 = 0, \quad r = 0. \tag{14b}$$

Where M_x and M_y denote bending moments.

For simply supported rectangular plate, we assume W in the following form:

$$W = \bar{W}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right). \tag{15}$$

Putting (15) in (8) I_1^{1m} is evaluated through integration over the plate as

$$I_1^{1m} = \frac{\pi^2\bar{w}^2}{8}\left(\frac{1}{a^2} + \frac{\nu}{b^2}\right). \tag{16}$$

We are interested in the normal displacement W only. Therefore the inplane displacement of the upper and lower faces namely u^u, u^1, V^u, V^1, have been eliminated through integration by assuming suitable expressions for these displacements compatible with their boundary conditions.

Let us now pay our attention to the final equation (13). Putting (15) in this equation, remembering (16) and applying Galerkin procedure we arrive at the following cubic equation determining the deflection $W(x, y)$.

$$\frac{Et_1\pi^4}{1-\nu^2}\left[\frac{1}{4G^1}\left(\frac{1}{a^2} + \frac{\nu}{b^2}\right)^2\left\{\frac{\pi^2 Et_1}{2(1-\nu^2)}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \frac{G^1}{h}\right\}\right.$$
$$+\frac{4\lambda}{ab}\left\{\frac{Et_1\pi^2}{2G^1(1-\nu^2)}\left(\frac{3}{64}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left(\frac{b}{a^3} + \frac{a}{b^3} - \frac{1}{ab}\right)\right.\right.$$
$$\left.+\frac{3}{32}\left(\frac{b}{a^s} + \frac{1}{a^3 b} + \frac{1}{ab^3} + \frac{a}{b^s}\right) - \frac{1}{16a^2b^2}\left(\frac{b}{a} + \frac{a}{b}\right)\right)$$
$$\left.\left.+\frac{1}{64h}\left(3\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left(\frac{b}{a} + \frac{a}{b}\right) + 6\left(\frac{b}{a^3} + \frac{a}{b^3}\right) - \frac{4}{ab}\right)\right]\bar{W}^3$$
$$+\frac{Et_1\pi^4 h}{2(1-\nu^2)}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2\bar{W}$$
$$= \bar{q}\left[\frac{1}{h} + \frac{Et_1\pi^2}{2G^1(1-\nu^2)}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right]. \tag{17}$$

The load q is considered to be sinusoidal given by

$$q = \bar{q} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}.$$

Table 1 shows numerical results of the maximum deflections of rectangular plates obtained by (17), where geometries of plates and the material constants are identical to those utilized in the investigation of Nowinski–Ohnabe [3] namely,

$$a = 10 \text{ in.}, \quad t_1 = 0.025 \text{ in.}, \quad h = 0.6746 \text{ in.}, \quad E = 10.45 \times 10^6 \text{ psi}$$

$$G^1 = 6 \times 10^3 \text{psi}, \quad \nu = 0.03 \quad \text{and} \quad \lambda = 0.09 \text{ [5]}.$$

Vibrations under dynamic loading

Let us now consider free vibrations of sandwich plates.

Adding the potential energy given in (7) to the total kinetic energy of the plate, one may form the Lagrangian function and then applying Hamilton's principle, the following equation is obtained (neglecting in-plane inertia) through Euler's equations

$$\begin{aligned} &\frac{2Et_1}{1-\nu^2} I_1^{1m} \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] + hG' \nabla^2 W - G' \left(\frac{\partial r}{\partial x} + \frac{\partial s}{\partial y} \right) \\ &\quad + \frac{Et_1 \lambda}{1-\nu^2} \left[\left\{ \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right\} \nabla^2 W + 2 \left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^2 w}{\partial x^2} \right. \\ &\quad \left. + 2 \left(\frac{\partial w}{\partial y} \right)^2 \frac{\partial^2 w}{\partial y^2} + 4 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \\ &\quad - (\varphi_1 t_1 + \varphi_2 h) \frac{\partial^2 w}{\partial t^2} = 0 \end{aligned} \tag{18a}$$

where

$$I_1^{1m} = cf(t) = \frac{\partial}{\partial x}(u^u + u^l) + \nu \frac{\partial}{\partial y}(v^u + v^l) = \frac{1}{2}\left(\frac{\partial w}{\partial x} \right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y} \right)^2 \tag{18b}$$

φ_1 and φ_2 are the surface density and core density respectively and $f(t)$ is a function of time.

Let $W = \bar{W} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} F(t)$ for fundamental mode of vibration, where $F^2(t) = f(t)$.

The defining equation (18b) of I_1^{1m} is integrated over the plate and yields the result:

$$I_1^{1m} = \frac{\pi^2 \bar{w}^2}{8} \left(\frac{1}{a^2} + \frac{\nu}{b^2} \right). \tag{18c}$$

Combining equations (12) and (18), inserting the expression of w and remembering (18c), one gets the final equation in the form:

$$\ddot{F} + AF + BF^3 = 0. \tag{19}$$

Table 1. $\frac{\bar{q}a^4}{Eh^4} = 10$. Immovable edge

Value of b/a ratio	$\bar{w}/h$ (calculated value)	$\bar{w}/h$ [3, 4] (known value)
1	1.53	1.30
2	1.822	1.77
3	1.900	1.837

Table 2. $\frac{\bar{q}a^4}{Eh^4} = 10$. Movable edge

Value of b/a ratio	1	2	2
Calculated value of $\bar{w}/h$	2.6	3.33	3.60

With the initial conditions $F(0) = 1$, $\dot{F}(0) = 0$, the solution is

$$F(t) = C_n(w_1^* t, K).$$

The ratio of non-linear and linear frequencies is given as

$$\frac{w_1^*}{w_1} = \Bigg[1 + \frac{a^4 b^4 h}{(a^2+b^2)^2}\left(\frac{\bar{w}}{2h_1}\right)^2\left(1+\frac{2t_1}{h}\right)^2\Bigg\{\frac{1}{2}\left(\frac{1}{a^2}+\frac{v}{b^2}\right)^2\left(\frac{Et_1}{2G(1-v^2)}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{h}\right)$$
$$+\frac{8\lambda}{ab}\left(\frac{Et_1\pi^2}{2G'(1-v^2)}\left(\frac{3}{64}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{b}{a^3}-\frac{1}{ab}+\frac{a}{b^3}\right)\right.\right.$$
$$+\frac{3}{32}\left(\frac{b}{a^5}+\frac{1}{a^3 b}+\frac{1}{ab^3}+\frac{a}{b^5}-\frac{1}{16a^2b^2}\left(\frac{b}{a}+\frac{a}{b}\right)\right)$$
$$\left.\left.+\frac{1}{64h}\left(3\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{b}{a}+\frac{a}{b}\right)+6\left(\frac{b}{a^3}+\frac{a}{b^3}-\frac{4}{ab}\right)\right)\right)\Bigg\}\Bigg]^{1/2} \tag{20}$$

where

$$h_1 = t_1 + \frac{h}{2}; \quad w_1 = \sqrt{A}, \quad w_1^* = \sqrt{A+B}.$$

Numerical results of the ratio of non-linear and linear frequencies are shown in Tables 3 and 4.

Table 3. Immovable edge

Value of b/a ratio	Value of $\bar{w}/2h$	w_1^*/w_1 calculated value	w_1^*/w [6] (known value)
	0	1	1
	0.5	1.12	1.14
1	1	1.42	1.48
	1.5	1.82	1.86
	2	2.24	2.38
	0	1	1
	0.5	1.08	1.08
2	1	1.30	1.31
	1.5	1.60	1.63
	2	1.93	1.98
	0	1	1
	0.5	1.06	1.07
3	1	1.25	1.28
	1.5	1.51	1.56
	2	1.81	1.87

Table 4. Movable edge

Value of b/a ratio	Value of $\bar{w}/2h_1$	w_1^*/w_1 calculated value
	0	1
	0.5	1.024
1	1	1.094
	1.5	1.202
	2	1.338
	0	1
	0.5	1.023
2	1.0	1.090
	1.5	1.190
	2	1.320
	0	1
	0.5	1.018
3	1.0	1.072
	1.5	1.156
	2	1.260

OBSERVATIONS AND CONCLUSIONS

(1) From the given tables it is observed that the present study yields greater deflections than those obtained from known theoretical analysis. It is also well-known that experimental results always show greater values than those obtained in the theoretical analysis. Hence the method shown in the present study is more acceptable for the practical purpose.

(2) From the decoupled equations presented in [4] results of immovable edges only can be obtained, but the present study yields accurate results both for movable as well as immovable edge conditions. This is certainly an advantage.

(3) From the same cubic equation, results both for movable as well as immovable edge conditions can be obtained. This is an additional advantage of the present study.

(4) Accuracy of Kamiya's [4] method depends on a correction factor which is a function of the ratio of the sides of plate geometries. So this correction factor will vary according to the plate goemetry. But the present study does not depend on any correction factor. The decoupled differential equations proposed in the present study is simple, accurate and thus able to supply void in the literature of the non-linear theory of sandwich plates.

REFERENCES

1. E. Reissner, Finite deflections of sandwich plates. *J. Aeronaut. Sci.* **15**, 435–440 (July 1948).
2. A. M. Alwan, Large deflection of sandwich plates with orthotropic cores. *AIAA J.* **2**, 1820–1822 (October 1964).
3. J. L. Nowinski and H. Ohnabe, Fundamental equations for large deflections of sandwich plates with orthotropic cores and faces. *Proc. 10th Int. Symp. on Space Technology and Sciences*, Tokyo, pp. 311–318 (1973).
4. N. Kamiya, Governing equations for large deflections of sandwich plates. *AIAA J.* **14**, pp. 250–253 (February 1976).
5. B. Banerjee and S. Dutta, A new approach to an analysis of large deflections of thin elastic plates. *Int. J. Non-Linear Mech.* **16**, 47–52 (1981).
6. B. M. Karmakar, Amplitude frequency characteristics of large amplitude vibrations of sandwich plates. *J. Appl. Mech.* (June 1978).

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VOTRE REFERENCE	VOTRE LETTRE DU	NOTRE REFERENCE	DATE
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Dear Sir,

The text of your paper "Large deflections of sandwich plates with orthotropic core and faces" has been submitted to the Reading Committee.

The comments of the said committee are given as an enclosure to this letter. We dare hope that you will find a suitable solution for these comments and we are awaiting your revised text, based on these comments.

Furthermore you will find a document for the transfer of copyright to be signed and sent back to the BSME secretariat. This copyright transfer is part of the publishing policy of the E.J.M.E., requiring a signed statement by the authors.

We thank you for your collaboration and we hope to hear from you soon.

F. VAN VAERENBERGH
Secretary of the Editorial Board

Encl. : - Comments of the Reading Committee.
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Note : The comments on the paper "Large deflection of thin elastic plates..." (Banerjee, Chanda) will soon follow.

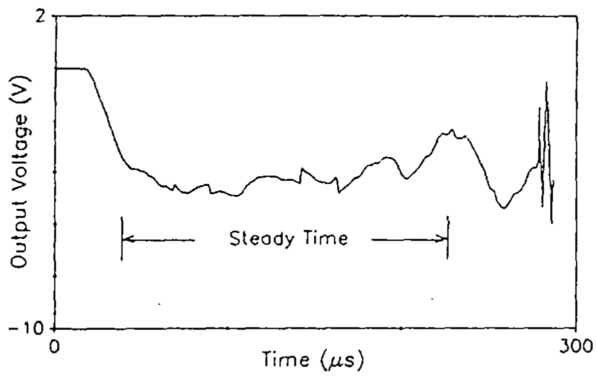

Fig. 2 Uncalibrated time history of filtered gauge output for the following test section conditions: Mach number = 3.1; velocity = 3486 m/s; static pressure = 114 kPa; stagnation enthalpy = 12.1 MJ/kg; temperature = 3383 K.

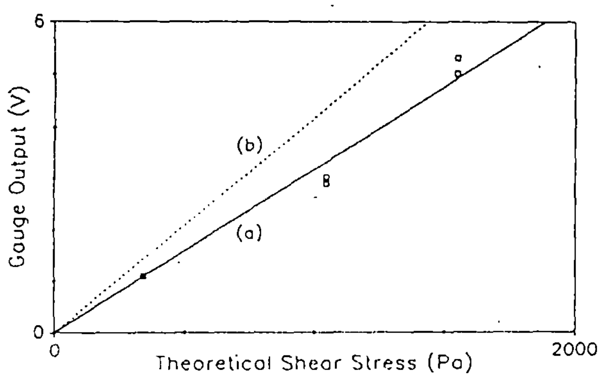

Fig. 3 Measured gauge output in volts (line a) and predicted gauge output (line b) vs theoretical shear stress for test section stagnation pressure from 0.77 to 5.88 MPa; line b is determined from nominal manufacturer's sensitivities for the piezoceramics.

were associated with the flow and not with the gauge, a commercial piezoelectric pressure transducer was mounted in a second flat-plate model to measure static pressure on the surface. The second flat-plate model had substantially different geometry and different flow-induced stress wave reflection times, but the onset of large fluctuations occurred in the unfiltered outputs from both the pressure transducer and one piezoceramic element at the same time, about 250 μs after the start of the flow. We conclude that the large-scale unsteadiness after 250 μs is due either to shock wave reflections resulting from the impulsive flow meeting the back face of the dump tank or to the arrival of the helium driver gas. However, in the available 200 μs of useful test time, the skin-friction gauge indicated steady boundary-layer flow (Fig. 2).

Figure 3 displays the skin-friction gauge averaged output voltage in the 200-μs test time plotted against theoretical shear stress values obtained using the method of van Driest.[2] The test section stagnation pressure ranged from 0.77 to 5.88 MPa. The flow Mach number was nominally 3.2. The relationship is linear, confirming the fact that shear stress has effectively been isolated from the unwanted contributions due to pressure, temperature, and flow-induced vibration. One of the major problems encountered in the development of the gauge was the decoupling of pressure and shear stress. For the range of conditions considered, pressure varies as a nonlinear function of shear stress. Hence, if the gauge were responding to pressure, the linear relationship in Fig. 3 would not occur. Using the nominal manufacturer's sensitivities for the piezoceramic material, calculations indicate that the gauge outputs for the theoretical shear stresses should lie along the second straight line in Fig. 3. This small difference is not surprising and further confirms the proper functioning of the gauge.

Table 1 Four test conditions

	1	2	3	4
Stagnation enthalpy, MJ/kg	12.07	9.507	7.34	5.76
Stagnation temperature, K	6348	5287	4373	3763
Stagnation pressure, MPa	5.88	3.62	1.63	0.77
Temperature, K	3383	2262	1952	1463
Pressure, kPa	114	65.8	27.4	12.1
Density, kg/m^3	0.107	0.079	0.046	0.027
Velocity, m/s	3486	3134	2789	2444
Mach number	3.10	3.13	3.20	3.22

Conclusions

Tests of a prototype skin friction gauge at Mach 3.2 in a small free piston shock tunnel have demonstrated the effectiveness of the design concept and the calibration against theoretical skin-friction values in a simple flow. The gauge has a rise time of about 20 μs, sufficiently short for most shock-tunnel applications and approaching the rise times needed for expansion tube applications.

Acknowledgments

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References

[1] Dunn, M. G., "Current Studies at Calspan Utilizing Short-Duration Flow Techniques," *Proceedings of the 13th International Symposium on Shock Tubes and Waves*, edited by C. E. Treanor and J. G. Hall, State Univ. of New York, Albany, NY, 1981, pp. 32–40.

[2] Van Driest, E. R., "Investigation of Laminar Boundary Layer in Compressible Fluids Using the Crocco Method," NACA TN 2597, Jan. 1952.

Large Deflections of Sandwich Plates with Orthotropic Cores— A New Approach

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Introduction

THE problem of large deflections of isotropic sandwich plates has been investigated by several authors.[1-6] Kamiya[5] presented governing equations for large deflections of isotropic sandwich plates following Berger's approximation. Accuracy of his solution depends on a correction factor. Dutta and Banerjee[6] have offered a simplified approach to investigate the nonlinear static as well as dynamic behaviors of sandwich plates. The literature on large deflection analysis of sandwich plates of orthotropic materials is scarce.[7,8] The present study investigates the large deflections of rectangular sandwich plates with a core as an orthotropic honeycomb-type structure. It is felt that this type of core corresponds more

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Table 1 Values of central deflection for various loads

	W_o/h immovable edge		W_o/h movable edge		
Side ratio, b/a	Present study	Obtained from classical equation given in the Appendix	Present study	Obtained from classical equation given in the Appendix	Load function, $q_o a^4/Eh^4$
1	1.548	1.49	2.46	2.48	10
2	1.802	—	3.055	—	10
3	1.855	—	3.18	—	10

exactly to the behavior of actual sandwich construction used in industry. Numerical results obtained from the present study have been compared with those obtained by solving the classical equations given in the Appendix.

Analysis

First, we take a rectangular coordinate system X, Y, Z: X, Y in the middle plane of the core, Z in the thickness direction (positive upwards).

The total potential energy of the system is given by

$$\bar{V}_o = V_o^f + \bar{V}_o^c \tag{1}$$

where V_o^f is the strain energy per unit area of both isotropic faces proposed by Dutta and Banerjee[6] and $\bar{V}_o^c$ is the strain energy per unit area of the orthotropic core due to shear.[8]

Executing the variational calculus so as to minimize the total potential energy in Eq. (1) of the present elastic system of the sandwich plates, we arrive at the following sets of differential equations.

$$I_1^{1m} = \frac{\partial}{\partial x}(u^u + u^l) + \nu\frac{\partial}{\partial y}(v^u + v^l) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \text{const} \tag{2}$$

$$\left[\frac{Et}{2(1-\nu^2)}\left(\frac{\partial^2 r}{\partial x^2} + \nu\frac{\partial^2 s}{\partial x \partial y}\right)\right] + \frac{Et}{4(1+\nu)}\left(\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 s}{\partial x \partial y}\right) + \left(\frac{\partial w}{\partial x} - \frac{p}{h}\right)G_{xz} = 0 \tag{3}$$

$$\left[\frac{Et}{2(1-\nu^2)}\left(\frac{\partial^2 s}{\partial y^2} + \nu\frac{\partial^2 r}{\partial x \partial y}\right)\right] + \frac{Et}{4(1+\nu)}\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 r}{\partial x \partial y}\right) + \left(\frac{\partial w}{\partial y} - \frac{s}{h}\right)G_{yz} = 0 \tag{4}$$

$$\frac{Et}{1-\nu^2}\Bigg[2I_1^{1m}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + \lambda\Bigg\{\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\nabla^2 w + 2\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} + 2\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial y^2} + 4\frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Bigg\}\Bigg] + h\left(G_{xz}\frac{\partial^2 w}{\partial x^2} + G_{yz}\frac{\partial^2 w}{\partial y^2}\right) - \left(G_{xz}\frac{\partial r}{\partial x} + G_{yz}\frac{\partial s}{\partial y}\right) + q = 0 \tag{5}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

For a movable edge, $I_1^{1m} = 0$. Here $r = u^u - u^l$, $s = v^u - V^l$, and G_{xz} are the shear modulii.

We consider the bending of a simply supported rectangular sandwich plate $(a \times b)$ with constraints in-plane displacements at the boundaries.

For a simply supported rectangular plate, we assume

$$W = W_o \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{6}$$

r and s are assumed as follows[5]:

$$r = r_o \cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{7}$$

$$s = s_o \sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \tag{8}$$

Putting Eq. (6) into Eq. (2), I_1^{1m} is evaluated through integration over the plate as

$$I_1^{1m} = \frac{W_o^2\pi^2}{8}\left(\frac{1}{a^2} + \frac{\nu}{b^2}\right) \tag{9}$$

The in-plane displacements u^u, u^l, v^u, v^l of the upper and lower faces have been eliminated through integration by choosing suitable expressions for them, compatible with their boundary conditions. This is because the normal displacement "W" is our primary interest.

Putting Eqs. (6–8) into Eq. (5), recalling the values of r_o and S_o and the value of I_1^{1m} obtained from Eqs. (3), (4), and (9), we arrive at the following cubic equation determining the central deflection $W_o(x,y)$ after applying Galerkin procedure.

$$\frac{4\pi^4 Et}{ab(1-\nu^2)}\left\{\frac{(b^2+\nu a^2)^2}{16a^3b^3} + \lambda\left[\frac{1}{32ab} + \frac{9}{64}\left(\frac{b}{a^3} + \frac{a}{b^3}\right)\right]\right\}W_o^3 + \frac{\pi^2 W_o}{ab}\left[h\left(G_{xz}\frac{b}{a} + G_{yz}\frac{a}{b}\right) + G_{xz}\frac{bk_3}{k_2} - G_{yz}\frac{k_1 a}{k_2 b}\right] = q_o \tag{10}$$

where

$$k_1 = \frac{Et\pi^2(3-\nu)}{4a^2(1-\nu^2)}G_{xz} - G_{yz}\left[\frac{Et\pi^2}{4(1-\nu^2)}\left(\frac{2}{a^2} + \frac{1-\nu}{b^2}\right) + \frac{G_{xz}}{h}\right]$$

$$k_2 = \frac{E^2t^2\pi^4}{16a^2b^2(1-\nu^2)} - \left[\frac{Et\pi^2}{4(1-\nu^2)}\left(\frac{2}{a^2} + \frac{1-\nu}{b^2}\right) + \frac{G_{xz}}{h}\right] \times \left[\frac{Et\pi^2}{4(1-\nu^2)}\left(\frac{2}{b^2} + \frac{1-\nu}{a^2}\right) + \frac{G_{yz}}{h}\right]$$

$$k_3 = \frac{G_{xz}}{a}\left[\frac{Et\pi^2}{4(1-\nu^2)}\left(\frac{2}{b^2} + \frac{1-\nu}{a^2}\right) + \frac{G_{yz}}{h}\right] - G_{yz}\frac{Et\pi^2}{4ab^2(1-\nu)}$$

and the load "q" considered being sinusodial and given by

$$q = q_o \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

Numerical results have been obtained for a rectangular sandwich plate of orthotropic core with the following data[5,7]: $a = 10$ in., $t = 0.025$ in., $h = 0.6746$ in., $E = 10.45 \times 10^6$ psi, $\nu = 0.03$, $\lambda = 0.09$, $G_{xz} = 10^4$ psi, and $G_{yz} = 10^3$ psi.

Table 1 shows the maximum deflection parameter (W_o/h) for simply supported rectangular sandwich plates with different side ratios, for the given load function $(q_o a^4/Eh^4)$.

Conclusions

The results of the present study have been obtained with ease and accuracy whereas the solutions to the classical equation involves mathematical complexity and much computational labor. Thus, the differential equations proposed in the present study are simple because of its decoupled form and seem to predict the nonlinear behaviors of sandwich plates with orthotropic core with much ease and accuracy.

Appendix

Alwan[8] proposed differential equations for large deflection of sandwich plates with orthotropic core in terms of the Airy stress function. Now the displacement formulations of his proposed equation take the following form:

$$\frac{\partial^2 u}{\partial x^2}+\frac{(1-\nu)}{2}\frac{\partial^2 u}{\partial y^2}+\frac{(1+\nu)}{2}\frac{\partial^2 v}{\partial x\partial y} = -\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}-\frac{(1-\nu)}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}-\frac{(1+\nu)}{2}\frac{\partial w}{\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y} \tag{A1}$$

$$\frac{\partial^2 v}{\partial y^2}+\frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2}+\frac{(1+\nu)}{2}\frac{\partial^2 u}{\partial x\partial y} = -\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}-\frac{(1-\nu)}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}-\frac{(1+\nu)}{2}\frac{\partial w}{\partial x}\cdot\frac{\partial^2 w}{\partial x\partial y} \tag{A2}$$

$$\left(1-D_y\frac{\partial^2}{\partial x^2}-D_x\frac{\partial^2}{\partial y^2}\right)\Delta\Delta\, W=\frac{1}{D}\left[1-\left(D_y+\frac{2D_x}{1-\nu}\right)\frac{\partial^2}{\partial x^2} -\left(D_x+\frac{2D_y}{1-\nu}\right)\frac{\partial^2}{\partial y^2}+\frac{2D_xD_y}{1-\nu}\Delta\Delta\right]\left(q+\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 F}{\partial x\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y}\right) \tag{A3}$$

where

$$\frac{\partial^2 F}{\partial y^2}=\frac{2Et}{1-\nu^2}\left\{\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\nu\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\right\} \tag{A3a}$$

$$\frac{\partial^2 F}{\partial x^2}=\frac{2Et}{1-\nu^2}\left\{\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\nu\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\} \tag{A3b}$$

$$\frac{\partial^2 F}{\partial x\partial y}=-\frac{Et}{1+\nu}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \tag{A3c}$$

F being the Airy's stress function.

Here u, v, w are displacement components along the X, Y, Z, directions, respectively, and

$$D_X=\frac{(1-\nu)D}{2hG_{xz}}, \qquad D_Y=\frac{(1-\nu)D}{2hG_{yz}}$$

For homogeneous plates, $D_x = D_y = 0$.

Let us now analyze the case of a simply supported square plate of side a. For this purpose, we solve first the equations for u and v. Let $W = W_o \sin(\pi x/a)\sin(\pi y/b)$, which satisfies the simply supported edge conditions. Putting the value of W in Eqs. (1) and (2), the solutions of u and v can be obtained in the following form:

$$u = Ax + C_1\sin\frac{2\pi x}{a}+C_2\sin\frac{2\pi x}{a}\cos\frac{2\pi y}{a} \tag{A4}$$

$$v = By + C_3\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{a}+C_4\sin\frac{2\pi y}{a} \tag{A5}$$

where A and B are determinded from the boundary conditions.

For movable edges,

$$A=B=-\frac{W_o^2\pi}{8a^2}$$

and for immovable edge conditions, $A = B = 0$. Here C_1, C_2, C_3, and C_4 are given by

$$C_1=C_4=-\frac{W_o^2\pi(1-\nu)}{16a}$$

$$C_2=C_3=\frac{W_o^2\pi}{16a}$$

Putting $W = W_o \sin(\pi x/a)\sin(\pi y/b)$, values of u and v obtained from Eqs. (A4) and (A5), using Eqs. (A3a–A3c), we finally obtain the following two cubic equations after applying Galerkin's technique.

For immovable edges:

$$\left\{\frac{E^2t\pi^6h^3}{64a^8(1-\nu^2)}\left[75\left(\frac{1-\nu}{G_{xz}}+\frac{2}{G_{yz}}\right)-(31+20\nu)\left(\frac{1-\nu}{G_{yz}}+\frac{2}{G_{xz}}\right) +\frac{Et(h+t)^2\pi^2(33-16\nu)}{2(1-\nu^2)a^2hG_{xz}G_{yz}}\right]-\frac{Eh^4\pi^4(1+2\nu)}{16a^6(h+t)^2}\right\}W_o^3 +\frac{Eh^4\pi^4}{a^6}\left[1+\frac{Et\pi^2(h+t)^2}{4a^2h(1+\nu)}\left(\frac{1}{G_{xz}}+\frac{1}{G_{yz}}\right)\right]W_o =\frac{h^2q_o}{2a^2}\left[\frac{h^2(1-\nu^2)}{t(h+t)^2}+\frac{Eh\pi^2(3-\nu)}{4a^2}\left(\frac{1}{G_{xz}}+\frac{1}{G_{yz}}\right) +\frac{E^2\pi^4t(h+t)^2}{2a^4(1-\nu^2)G_{xz}G_{yz}}\right] \tag{A6}$$

For immovable edges:

$$\left\{\frac{E^2t\pi^6h^3}{64a^8(1-\nu^2)}\left[75\left(\frac{1-\nu}{G_{xz}}+\frac{2}{G_{yz}}\right)-(31+20\nu)\left(\frac{1-\nu}{G_{yz}}+\frac{2}{G_{xz}}\right) +\frac{Et\pi^2(h+t)^2(33-16\nu)}{2(1-\nu^2)a^2hG_{xz}G_{yz}}\right]+\frac{Eh^4\pi^4(3+2\nu)}{16a^6(h+t)^2}\right\}W_o^3 +\frac{Eh^4\pi^4}{a^6}\left[1+\frac{Et\pi^2(h+t)^2}{4a^2h(1+\nu)}\left(\frac{1}{G_{xz}}+\frac{1}{G_{yz}}\right)\right]W_o =\frac{h^2q_o}{2a^2}\left[\frac{h^2(1-\nu^2)}{t(h+t)^2}+\frac{Eh\pi^2(3-\nu)}{4a^2}\left(\frac{1}{G_{xz}}+\frac{1}{G_{yz}}\right) +\frac{E^2\pi^4t(h+t)^2}{2a^4(1-\nu^2)G_{xz}G_{yz}}\right] \tag{A7}$$

References

[1]Reissner, E., "Finite Deflection of Sandwich Plates," *Journal of Aeronautical Sciences,* Vol. 15, July 1948, pp. 435–440.

[2]Wang, C. T., "Principle and Application of Complementary Energy Method for Thin Homogeneous and Sandwich Plates and Shells with Finite Deflections," NACA IN-2620, 1952.

[3]Hoff, N. J., "Bending and Buckling of Rectangular Sandwich Plates," NACA IN-2225, 1950.

[4]Eringen, A. C., "Bending and Buckling of Sandwich Plates," *Proceedings of the First U.S. National Congress of Applied Mechanics,* American Society of Mechanical Engineers, New York, 1951, pp. 381–390.

[5]Kamiya, N., "Governing Equations for Large Deflections of Sandwich Plates," *AIAA Journal,* Vol. 14, No. 2, 1976, pp. 250–253.

[6]Dutta, S., and Banerjee, B., "Governing Equations for Nonlinear Analysis of Sandwich Plates," *International Journal of Nonlinear Mechanics,* Vol. 26, No. 3/4, 1991, pp. 313–319.

[7]Nowinski, J. L., and Ohnabe, H., "Fundamental Equations for Large Deflections of Sandwich Plates with Orthotropic Core and Faces," *Proceedings of the 10th International Symposium on Space Technology and Sciences,* Tokyo, 1973, pp. 311–318.

[8]Alwan, A. M., "Large Deflection of Sandwich Plates with Orthotropic Cores," *AIAA Journal,* Vol. 2, No. 10, 1964, pp. 1820–1822.