CHAPTER III

SOME RESULTS ON INTERVAL DIGRAPHS

3.1 Introduction

Interval digraphs have been studied previously more or less widely. Nevertheless it seems a lot more remains to be explored in this area. In this chapter we have attempted some of the problems connected with interval digraphs.Most interesting result of this chapter is given in section 3.2 where we have seen that the concept of an interval digraph is actually a generalization of an interval graph (and not merely an analogue) in the sense that a graph G is an interval graph means that the corresponding symmetric digraph with loop is an interval digraph. Given a graph G, the adjacency matrix of the corresponding "symmetric digraph with loops" is obtained from the adjacency matrix of G by adding 1's on the diagonal.

In section 3.3 we obtain a characterization of an interval digraph in terms of an ordering of the edges, we have called Δ-ordering and then again we have seen how this result generalizes a corresponding result for an interval graph.

Lekkerkerker and Boland [1962] obtained a most fascinating characterization of an interval graph in terms of an asteroidal triple. In section 3.4 we have attemped a similar line of attack in the case of digraphs although we admit that a comprehensive result in this line still remains elusive.

3.2 Interval graph and interval digraphs

In this section we consider relationships between representations of graphs and representations of digraphs. While discussing adjacency and nonadjacency, we find it convenient to write $u \longleftrightarrow v$ and $u \longrightarrow v$ to mean "uv is an edge" (in an undirected or directed graph) and $u \not\longleftrightarrow v$ and $u \not\longrightarrow v$ to mean is not edge".

Theorem 3.1. *A graph G is an interval graph if and only if the corresponding symmetric digraph with loops $D(G)$ is an interval digraph.*

Proof. Necessity is trivial. If G is an interval graph, with interval I_v assigned to vertex v, then setting $S_v = T_v = I_v$ yields an intersection representation of the digraph $D(G)$.

For sufficiency, suppose $\{(S_v, T_v): v \in V(G)\}$ is an interval intersection representation of $D(G)$, where $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$. We claim that setting $I_v = [a_v + c_v, b_v + d_v]$ yields an interval intersection representation of G. The verification depends on the observation that two intervals intersects if and only if each left end point is less than or equal to the other right end point.

For the desired edges, we want $I_u \cap I_v \neq \phi$ if $u \longrightarrow v$ and $v \longrightarrow u$ in $D(G)$. This means $c_v \leq b_u$ and $a_v \leq d_u$, which implies $a_v + c_v \leq b_u + d_u$. Similarly we have $c_u \leq b_v$ and $a_u \leq d_v$, which implies $a_u + c_u \leq b_v + d_v$.

The other possibility is $u \not\longrightarrow v$ and $v \not\longrightarrow u$ in $D(G)$, but also $u \longrightarrow u$ and $v \longrightarrow v$. The non-edges imply $d_v < a_u$ or $b_u < c_v$, and also $d_u < a_v$ or $b_v < c_u$. If we choose the first option in each case the loops give us $d_v < a_u \leq d_u < a_v \leq d_v$. If we choose the second option in each case, we find $b_u < c_v \leq b_v < c_u \leq b_u$.

Hence we must choose first/second or second/first. Summing the resulting inequalities yields $b_v + d_v < a_u + c_u$ or $b_u + d_u < a_v + c_v$. Each of these implies $I_u \cap I_v = \phi$, as desired. Hence the result. ∎

3.3 Interval Digraphs and Edge Ordering.

In this section we introduce an ordering of the edge set E called Δ-ordering and use this notion to obtain a characterization of an interval digraph. Here we write $ab \in E$ for an edge from vertex a to vertex b.

The set E of all edges of a digraph $D(V,E)$ will be said to have a Δ-*ordering* if E has a linear ordering ($<$) such that for $ab, cd, af, eb \in E$

(i) $ab < cd < af$ implies $ad \in E$

and (ii) $ab < cd < eb$ implies $cb \in E$.

Below we shall prove that a digraph is an interval digraph if and only if its edge set has a Δ-ordering.

To characterize interval digraphs we take the help of vertex-edge incidence matrix. Recall that an n by e matrix $[a_{ij}]$ whose n rows corresponds to the n vertices and the e columns corresponds to the e edges of a digraph is the *vertex-edge incidence matrix* where

$$a_{ij} = \begin{cases} +1\ (-1), & \text{if the } j\text{th edge is incident out of (into) the } i\text{th vertex} \\ \pm 1 & \text{if the } j\text{th edge is a self loop at the } i\text{th vertex} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2 *A digraph $D(V,E)$ is an interval digraph if and only if its edge set E has a Δ-ordering.*

Proof. *(Necessary)* To prove the necessity, we introduce the notion of primitive* intervals for the family $\mathcal{F} = (S_v, T_v)$. An interval which is the intersection of S_u and T_v for some

* Gavril [1973] used the notion of primitive arcs in a different context.

$u, v \in V$, $uv \in E$ is called a primitive interval. Consider the family of primitive intervals corresponding to all the edges. Now order the edges by a relation ($<$) so that $ab < cd$ when the left end point of the primitive interval of ab is less than the left end point of that of cd If however the two left end points coincide then order them according to the precedence of right end points of the corresponding primitive intervals. For edges having identical primitive intervals order them arbitrarily amongst themselves.

Now let $ab < cd < eb$. Then T_b, the terminal interval of vertex b must contain the primitives of both ab and eb and by the above ordering the left end of the primitive interval of cd lies between the left ends of the primitive intervals of ab and eb or is identical with one or both of them. It follows that the source interval S_c of the vertex c must have a nonempty intersection with T_b implying that cb is an edge of D. Thus condition (ii) of Δ-ordering is satisfied.

(Sufficient) Consider the vertex-edge incidence matrix of the digraph D, the ordering of the columns being determined by the given linear ordering of its edges. Now label the edges in increasing order of natural number as they occur in the matrix.

For a vertex u consider the labelling of the edges having entries +1 in u-row. The minimum and maximum labellings of these edges will determine an interval on the real line which will be source interval S_u for our purpose. Similary the interval determined by the minimum and maximum labellings of the edges having entries −1 in the same U-row will determine terminal interval T_u. Evidently, ± 1 is treated as +1 while considering the +1 entries and again as −1 while considering the −1 entries in a row.

Now we prove that these intervals S_v and T_v for $v \in V$ are actually the source and terminal intervals in an interval representation of the digraph D. Every edge in the vertex-edge incidence matrix has in its corresponding column only two entries +1 and −1 corresponding to the initial vertex and terminal vertex (or only one entry ±1 corresponding to a self loop). Whenever $S_u \cap T_v = \phi$ there exists no such column which corresponds to a number belonging to both S_u and T_v. In other words, there is no edge in the digraph whose initial end point is u and terminal end point is v, i.e. $uv \notin E$.

Next let $S_u \cap T_v \neq \emptyset$. We shall show that there is a number such that a column corresponding to this number has entries +1 in row U and −1 in row V and this will prove that uv is an an edge in D.

We write $S_u = [a_u, b_u]$, $T_u = [c_u, d_u]$.

From the nonemptyness of $S_u \cap T_v$ arise two cases, viz.,

(i) $a_u \leq c_v \leq b_u$ or (ii) $c_v \leq a_u \leq d_v$.

Consider case (i). When either $a_u = c_v$ or $c_v = b_u$ the proof is immediate. So let $a_u < c_v < b_u$. The entry +1 which corresponds to a_u determines an edge say ux in digraph D.

Similarly let the entries +1 at b_u and −1 at c_v determine the edges say uy and pv respectively. Since $a_u < c_v < b_u$ it follows that $ux < pv < uy$ and by the condition (ii) of Δ-ordering it follows that $uv \in E$. The other case can be similarly proved. ∎

3.3.1. Interval graphs Edge ordering

Observe that by making minor changes in the definition of Δ-ordering and primitives we can have an analogous result for interval graphs. The set E of all edges of a graph $G(V,E)$ will be said to have a *δ-ordering* if E has a linear ordering $(<)$ such that for $ab,\ cd,\ af \in E$

$$ab < cd < af \Rightarrow ad \in E \text{ and } ac \in E.$$

Let $\mathcal{F} = \{I_v\}$ be a family of intervals on the real line corresponding to vertex set V of an interval graph $G(V,E)$. Consider the family of all the intervals which are the intersections of I_u and I_v for some $u,v \in V$, $uv \in E$.

Lastly recall that if we disregard the orientations of the edges of a digraph and correspondingly replace the −1's by 1's in the incidence matrix, we can have a (0,1)-matrix which is the incidence matrix of the corresponding graph. We then have the following result.

Theorem 3.3 *A graph $G(V,E)$ is an interval graph if and only if its edge set E has a δ-ordering.*

3.4. Interval digraphs and diasteroidal triple.

In this section, the digraphs considered will be supposed to have a loop at every vertex, but no multiple arcs. We shall search for a result relating an interval digraph with a suitably defined diasteroidal triple, keeping in mind the corresponding result on interval graphs by Lekkerkerker and Boland [1962] and on interval catch digraphs by Prisner [1989].

Theorem. 3.4 (Lekkerkerker and Boland [1962]. *G is an interval graph iff G is chordal and contains no asteroidal triple*

Theorem 3.5 (Prisner [1989]) *A digraph is an interval catch digraph if and only if it has no diasteroidal triple.*

Definition. An x-y chain of a digraph $D(V,E)$ is called *z-avoiding*, if neither any initial end point of an arc of the chain precedes z nor any final end point succeeds z. Three vertices a_1, a_2, a_3 of a digraph $D(V,E)$ are said to form a *diasteroidal triple* in D, if any vertex a_i (i = 1, 2, 3) avoids a chain joining the other two vertices; in other words for any permutation p of {1,2,3} there is a $a_{p(1)}$ avoiding $a_{p(2)}$ – $a_{p(3)}$ chain in D.

Note that the definition of a z-avoiding x-y chain given here is stronger than the corresponding definition given by Prisner. He stipulates the condition that no initial end point of an arc precedes z in the x-y chain, whereas we impose alongwith another condition that no final end point of the arc should succeed z.

Theorem 3.6 *Let D by a (finite) digraph with a loop at every vertex. Then if D is an interval digraph, it has no diasteroidal triple.*

We will prove the theorem by transforming the problem into a problem on undirected graphs. It is known that a digraph $D(V,E)$ is an interval digraph iff there is a covering of D by a family of GBS's that can be indexed so that the one's in the corresponding V,X – and V,Y – matrices appear consecutively.

Let $D = \bigcup_{i=1}^{k} B_i(X_i, Y_i)$, Where B_i's are the GBS's in D.

Let $\bar{V} = \{\bar{x} \mid x \in V\}$ be a disjoint copy of V and. $\bar{Y}$ denote a disjoint copy of a subset of $Y \subset V$.

Associate a vertex ρ_i with each GBS B_i in the above family and denote the set $\{\rho_i,..\rho_k\}$ by P. Construct an undirected graph $G(D)$ whose vertex set is $V(G(D)) = V \cup \overline{V} \cup P$ and whose edge set is given by

$$E(G(D)) = \{xy \mid x \in X_i \cup \bar{y}_i, y \in X_i \cup \bar{Y}_i,\} \cup \{x\rho_i \mid x \in X_i \cup Y_i\}$$

In constructing $G(D)$, we first observe that the number of vertices of $G(D)$ is $2n + k$, where n is the number of vertices of D and k is the number of GBS's in the given covering. We next note that every vertex of $X_i \cup \overline{Y}_i$ is adjacent to one another and also that ρ_i is adjacent to every vertex of $X_i \cup \overline{Y}_i$ (and to no other); accordingly every GBS's B_i in the family $\mathcal{B}$ induces a maximal complete subgraph of $G(D)$.

To prove theorem 3.6, We need the folowing lemma.

Lemma 3.1. Let D be a finite digraph having loop at every vertex. Then D is an interval digraph implies $G(D)$ is an interval graph.

Proof of the Lemma. Let D be an interval digraph, each of its vertices possessing a loop. As already observed the graph $G(D)$ is the union of maximal cliques C_i where

$$v \in X_i \Rightarrow v \in C_i \text{ and } v \in Y_i \Rightarrow \overline{v} \in C_i.$$

Since one's appear consecutively in the $V-X$ and $V-Y$ matrices, we can easily observe that the vertex-maximal clique matrix of $G(D)$ has consecutive property of rows, the ordering $C_1, C_2, \ldots, C_k$ in the columns serving the purpose.

Proof of the Theorem 2.6 Let D be an interval digraph. Let if possible, (x,y,z) be a diasteroidal triple of D. We will observe that the edges $x\bar{x}$, $y\bar{y}$ and $z\bar{z}$ belong to the three distinct maximal cliques of $G(D)$ and if ρ_x, ρ_y, ρ_z denote the vertices corresponding to these cliques in $G(D)$ then we will show that ρ_x, ρ_y, ρ_z form an asteroidal triple, which is contrary to lemma 1.

let $A = (x = x_0, x_1, \ldots x_n = y)$ be an $x-y$ chain in D. Then either (i) z is not incident to or from any vertex x_i; or (ii) z is connected to some vertex x_i of A_n, then either (or both) of the following hold.

1. If $zx_i \in E$ for some i, then $x_i x_{i-1} \in E$ and $x_i x_{i+1} \in E$.

2. If $x_i z \in E$ for some i, then $x_{i-1} x_i \in E$ and $x_{i+1} x_i \in E$.

Note that when i = 0 or n, the above condition may be adjusted accordingly. Also note that since every vertex has a loop, two consecutive vertices may become the same.

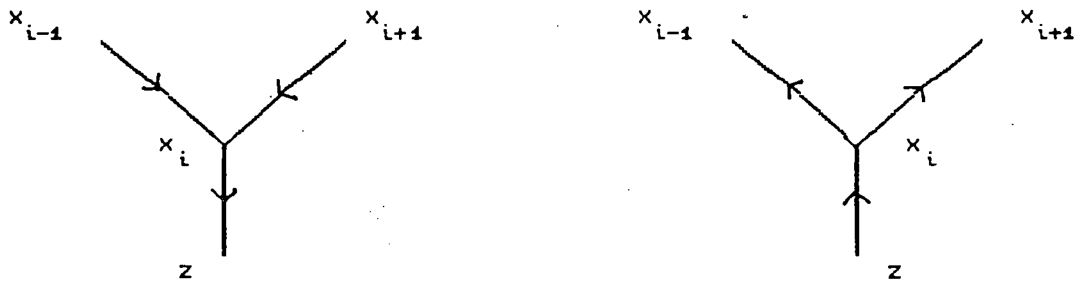

Fig 3.1

First let $zx_i \in E$ for some i. Since every vertex z has a loop in D, $z\bar{z}$. is an edge of $G(D)$ and so it belongs to some clique, say C_z, So that the vertices z, $\bar{z}$, and ρ_z belong to C_z. We claim that $\rho_z x_i \neq E(G(D))$; for otherwise, x_i must belong to the clique C_z (Since ρ_z belongs to the only maximal clique C_z), and since z is also a vertex of C_z, $x_i \bar{z} \in E(G(D))$ so that $x_i z$ becomes an edge of D. But this violates the hypothesis that $\{x,y,z\}$ is a diasteroidal triple of D.

Similarly for the case when $x_i z \in E$, we may prove that $\rho_z x_i \notin E(G(D)$. Then in $G(D)$, the vertex ρ_z avoids the x-y chain in $G(D)$ whose edges are the edges corresponding to the edges of the chain in D, connected by the edge $x_i \bar{x}_i$ where necessary. If ρ_x and ρ_y denote the vertices corresponding to the maximal clique containing $x\bar{x}$ and $y\bar{y}$ respectively, then we can argue as above to see that of the three vertices ρ_x , ρ_y , ρ_z any vertex avoids a path joining the other two in $G(D)$. Hence ρ_x, ρ_y, ρ_z form an asteroidal triple in $G(D)$. ∎

The difficulty in obtaining a sufficient condition for an interval digraph in terms of a diasteroidal triple is that we have been able to obtain $G(D)$ from the covering of a family of GBS's only when D is given to be an interval digraph; while in Prisner's paper on interval catch digraph, $G(D)$ is constructed from D independently of its being an interval catch digraph or not.