# CHAPTER - IV

## PROXIMAL PERRON-STIELTJES INTEGRAL

### §1. Introduction

Without the help of the theory of measure, Perron [43] has developed a simple theory of integration which is based on the major and the minor functions introduced by him. Although, in its original form this theory is applied only to bounded functions, later Bauer [2] has shown that it can be extended to unbounded functions. This extension of Perron's method of integration solves completely the problem of reconstructing the primitives of finite derivatives. Also this integral includes both the Lebesgue integral and the Newton integral (Saks [44], p. 186).

An extensive literature has grown on Perron's theory of integration. Most of these works deal with the relaxations of the conditions for the major and the minor functions.

Burkill [17] has defined an approximately continuous Perron integral ($AP_0$-integral) with the help of major and minor functions in terms of approximate continuity and approximate derivates. Then Sunouchi and Utagawa [50] have

generalised this integral by dropping the condition of approximate continuity in the definition of major and minor functions.

Next Sarkhel and De [48] have introduced a theory of proximally continuous Perron integral which generalises Sunouchi and Utagawa's integral (AP-integral).

Many other Perron type integrals have been defined. The first of these are Cesàro-Perron integrals of Burkill ([18], [19]) and symmetrical Cesàro-Perron integral of Burkill [20]. Next Burkill's $AP_0$-integral has been generalised to symmetrical approximate Perron integral (Kubota [35], Bhattacharyya [5], Bhattacharyya and Lahiri [6]), Cesàro-Perron's integral has been generalised to the approximate Cesàro-Perron integrals by Chakrabarti and Mukhopadhyay ([21], [22]). Recently the approximate Cesàro-Perron integral has been generalised to the proximal Cesàro-Perron integral by Bose and Nath [12]. Perron's method is also studied in abstract spaces (see Bullen ([14], [15]), Bullen and Lee [16]).

Generalising the AP-integral, Sarkhel [47] has defined the approximate Perron-Stieltjes integral with respect to a non-decreasing function $\omega$ and generalising the Cesàro-Perron integral, Dutta [28] has introduced the Cesàro-Perron-

Stieltjes integral with respect to $\omega$.

With the help of the concepts developed in the earlier Chapters, we introduce in this Chapter the proximal Perron-Stieltjes integral and study its different properties including an integration by parts formula.

## §2. Preliminaries

Let $\mathcal{U}^{(1)}$ denote the class of functions $f$ of class $\mathcal{U}_0$ for which $f(x) = \frac{1}{2}(f(x+) + f(x-))$ for $x \in D$. For a function $f$ on a closed interval I, we write $\omega_f(x) = f(x)(\omega(x+) - \omega(x-))$ on $I \cap D$ and 0 elsewhere.

Definition 4.2.1(§2, Def.2.1, Dutta [28]). Let $f$ be an extended real valued function on $[a,b]$. A function $M \in \mathcal{U}_0$ is said to be a (PS)major function of $f$ on $[a,b]$ if (i) $M^-(a) = 0$, (ii) $M$ is non-decreasing on each of the open intervals $(a_1,b_1) \subset S_0$, (iii) $D_+M_\omega(x) > -\infty$ for $x \in S_3 \cup S_2^+$, $D_-M_\omega(x) > -\infty$ for $x \in S_3 \cup S_2^-$, (iv) $D_+M_\omega(x) \geq f(x)$ for $x \in S_3 \cup S_2^+$, $D_-M_\omega(x) \geq f(x)$ for $x \in S_3 \cup S_2^-$ and (v) $M(x+) - M(x-) \geq f(x)(\omega(x+) - \omega(x-))$ for $x \in D$. Analogously a (PS)minor function of $f$ on $[a,b]$ is defined. $f$ is said to be Perron-Stieltjes integrable (or, (PS)-integrable) on $[a,b]$ if (i) it has at least one (PS)major function $M$ and at least one (PS)minor function $m$ and (ii) $\inf\{M^+(b)\} = \sup\{m^+(b)\}$. If $f$ is (PS)-integrable on $[a,b]$, then the common value $\inf\{M^+(b)\} =$

$\sup\{m^{+}(b)\}$ is called the (PS)-integral of $f$ on $[a,b]$ and is denoted by

$$(PS)\int_a^b f \, d\omega.$$

Definition 4.2.2(see §1,Bhakta [4]). A function $f \in \mathcal{Y}$ is said to be AC-$\omega$ below on $[a,b]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any finite set of pairwise disjoint open intervals $\{(c_n,d_n)\}$ on $[a,b]$ with

$$\sum_n (\omega(d_n+) - \omega(c_n-)) < \delta ,$$

the relation

$$\sum_n (f^{+}(d_n) - f^{-}(c_n)) > -\varepsilon$$

holds.

Definition 4.2.3(Def.2.5,Bose [7]). A function $f \in \mathcal{U}_0$ is said to be AC*-$\omega$ on a set $E \subset [a,b]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any set of pairwise disjoint open intervals $\{(c_n,d_n)\}$ having end points on $E$ with

$$\sum_n (\omega(d_n+) - \omega(c_n-)) < \delta ,$$

the relations

$$\sum_n \overline{\underset{c_n<x<d_n}{\text{bound}}} |f^+(x) - f^-(c_n)| < \varepsilon$$

and

$$\sum_n \overline{\underset{c_n<x<d_n}{\text{bound}}} |f^+(d_n) - f^-(x)| < \varepsilon$$

hold.

$f$ is said to be ACG*-$\omega$ on $E = [a,b]$ if it is AC*-$\omega$ on each part of a closed $E$-form.

Definition 4.2.4(Def.5.1, Rose [7]). Let $f$ be an extended real valued function defined on $[a,b]$. If there exists a function $F \in \mathcal{U}_0$ such that $F$ is ACG*-$\omega$ on $[a,b]$, $F'_\omega(x) = f(x)$ $(\omega)$a.e. on $[a,b] \cap S$ and $F(x+) - F(x-) = f(x)(\omega(x+) - \omega(x-))$ for $x \in [a,b] \cap D$, then $f$ is said to be special Denjoy-Stieltjes integrable or, (D*S)-integrable on $[a,b]$ and the function $F$ is called an indefinite (D*S)-integral of $f$ on $[a,b]$. The difference $F^+(b) - F^-(a)$ is called the definite (D*S)-integral of $f$ on $[a,b]$ and is denoted by

$$(D^*S)\int_a^b f \, d\omega .$$

Theorem 4.2.1(Lemma 2.6, Dutta [28]). If the function $f$ is (LS)-summable on $[a,b]$, then it is (PS)-integrable on $[a,b]$ and

$$(PS)\int_a^b f \, d\omega = (LS)\int_a^b f \, d\omega.$$

Theorem 4.2.2(Th.3, Bhakta [4]). If a function $f \in \mathcal{U}_0$ is AC-$\omega$ below on $[a,b]$, then it is BV-$\omega$ on $[a,b]$.

Theorem 4.2.3(Lemma 2.4, Dutta [28]). If a function $f \in \mathcal{U}_0$ is AC-$\omega$ below on $[a,b]$, then

$$(LS)\int_a^b f_\omega^i \, d\omega \leqslant f^+(b) - f^-(a)$$

where $f_\omega^i$ is defined as in Theorem 3.2.2.

Theorem 4.2.4(Lemma 5.1, Bose [7]). If a function $f \in \mathcal{U}_0$, then $f^{\pm}(x)$ is bounded on $[a,b]$.

Theorem 4.2.5(see Th.3.3, Chakrabarty [24]). If $f \in \mathcal{U}_0$ is AC-$\omega$ on $[a,b]$, then $f(x) = f_1(x) - f_2(x)$, $x \in [a,b] \cap S$ where $f_1$ and $f_2$ are non-decreasing and AC-$\omega$ on $[a,b]$ and belong to the class $\mathcal{U}_0$.

## §3. The (PPS)-integral

Definition 4.3.1. Let $f$ be an extended real valued function on $[a,b]$. A function $M \in \mathcal{Y}$ is said to be a (PPS)major function of $f$ on $[a,b]$ if

(i) $\overline{M}$ is $(\omega)$proximally continuous on $[a,b] \cap S$,

(ii) $M^{-}(a) = 0$,

(iii) $\underline{PDM}_{\omega}(x) > -\infty$ n.e. on $[a,b] \cap S_3$,

(iv) $\underline{PDM}_{\omega}(x) \geqslant f(x)$ $(\omega)$a.e. on $[a,b] \cap S_3$,

(v) $M(x+) - M(x-) \geqslant f(x)(\omega(x+) - \omega(x-))$ for all $x \in [a,b] \cap D$ and

(vi) $M$ is non-decreasing on each of the intervals of $[a,b] \cap (S_0 \cup S_2)$.

A (PPS)minor function $m \in \mathcal{Y}$ of $f$ on $[a,b]$ is defined analogously.

By Theorem 2.3.3, we can prove that for any (PPS)major function $M$ and any (PPS)minor function $m$ of $f$ on $[a,b]$, the function $M - m$ is non-decreasing on $[a,b] \cap S$.

If $f$ has (PPS)major and (PPS)minor functions, then

$I(b) = \inf\{M^+(b)\}$ and $J(b) = \sup\{m^+(b)\}$ are finite and $I(b) \geqslant J(b)$.

Definition 4.3.2. An extended real valued function $f$ on $[a,b]$ is said to be integrable in the proximal Perron-Stieltjes sense or, in short, (PPS)-integrable with respect to the function $\omega$ if

(i) it has at least one (PPS)major function $M$ and at least one (PPS)minor function $m$,

(ii) $I(b) = J(b)$.

If $f$ is (PPS)-integrable on $[a,b]$, then the common value $I(b) = J(b)$ is called the definite proximal Perron-Stieltjes integral or, in short (PPS)-integral of $f$ on $[a,b]$ and is denoted by

$$(PPS)\int_a^b f \, d\omega .$$

If in the definition of the (PPS)major and the (PPS)minor functions, the ($\omega$)proximal continuity is replaced by the ($\omega$)approximate continuity and the ($\omega$)proximal derivates are replaced by the ($\omega$)approximate derivates, then the corresponding integral is called the approximate Perron-Stieltjes integral or, in short, (APS)-integral (see Sarkhel [47]).

Note 4.3.1. It is clear from the definitions that the (PPS)-integral is a generalisation of the (APS)-integral and so of the (PS)-integral. Again since by Theorem 4.2.1, the (PS)-integral is a generalisation of the (LS)-integral, the (PPS)-integral is a generalisation of the (LS)-integral.

The function constructed in the Example 6.1 of Sarkhel and De [48] is not (APS)-integrable with respect to the function $\omega(x) = x$ but it is (PPS)-integrable with respect to the function $\omega(x) = x$. In the following we give another example of a function which is (PPS)-integrable but not (APS)-integrable.

Example 4.3.1. We take $\omega(x) = x$ and take the set

$$E = \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$$

of Example 1.4.1. We define the function $F$ on $[0,1]$ as follows :

$$F(x) = \begin{cases} \sin^2 \pi \dfrac{x - \alpha_n}{\beta_n - \alpha_n} & \text{for } \alpha_n \leq x \leq \beta_n, \quad n \in Z, \\ 0 & \text{for } x \in [0,1] - E. \end{cases}$$

The function $F$ is $(\omega)$proximally continuous (but not $(\omega)$approximately continuous) at $x = 0$ and at other points

F is continuous. Also $PDF_\omega(0) = 0$ and $F'_\omega(x)$ exists finitely for all points $x \in (0,1]$. Let

$$f(x) = \begin{cases} F'_\omega(x) & \text{for } x \in (0,1], \\ 0 & \text{for } x = 0. \end{cases}$$

The function f is obviously (PPS)-integrable on [0,1] but not (APS)-integrable on [0,1].

Therefore it follows that the (PPS)-integral is substantially more general than the (APS)-integral.

We omit the simple proof of the following theorems :

Theorem 4.3.1. Let the functions f and g defined on [a,b] be such that $f(x) = g(x)$ $(\omega)$a.e. on [a,b] and f is (PPS)-integrable on [a,b]. Then g is (PPS)-integrable on [a,b] and

$$(PPS)\int_a^b g \, d\omega = (PPS)\int_a^b f \, d\omega.$$

Theorem 4.3.2. Let $a < c < b$. Then the function f is (PPS)-integrable on [a,b] if and only if it is (PPS)-integrable on both [a,c] and [c,b]. Also when f is (PPS)-

integrable on $[a,b]$, then

$$(\text{PPS})\int_a^b f \, d\omega$$

$$= (\text{PPS})\int_a^c f \, d\omega + (\text{PPS})\int_c^b f \, d\omega = \omega_f(c).$$

Theorem 4.3.3. Let $f$ be a real valued function on $[a,b]$ and let $F \in \mathcal{V}$. If

$$\text{PDF}_\omega(x) = f(x)$$

for all $x \in [a,b] \cap S - S_0$, $F$ is constant on the intervals of $S_0 \cup S_2$ and if

$$F(x+) - F(x-) = f(x)(\omega(x+) - \omega(x-))$$

for all $x \in [a,b] \cap D$, then $f$ is (PPS)-integrable on $[a,b]$ and

$$(\text{PPS})\int_a^b f \, d\omega = F^+(b) - F^-(a).$$

Theorem 4.3.4. If the function $f$ is (PPS)-integrable on $[a,b]$, then it is finite $(\omega)$a.e. on $[a,b]$.

Proof. Suppose that $M$ and $m$ are respectively a

(PPS)major and a (PPS)minor function of $f$ on $[a,b]$. Then the two relations

$$M(x+) - M(x-) \geqslant f(x)(\omega(x+) - \omega(x-)), \quad x \in [a,b] \cap D$$

and

$$m(x+) - m(x-) \leq f(x)(\omega(x+) - \omega(x-)), \quad x \in [a,b] \cap D$$

imply that $f$ is finite at every point of $[a,b] \cap D$. Since $|S_0 \cup S_2|_\omega = 0$, the proof of the theorem will be completed if we show that $f$ is finite $(\omega)$a.e. on $[a,b] \cap S_3$.

We write

$$P(x) = M(x) - m(x),$$

then

(91) $$\underline{PD}P_\omega(x) \geqslant \underline{PD}M_\omega(x) - \overline{PD}m_\omega(x), \quad x \in [a,b] \cap S_3.$$

Since $P$ is non-decreasing on $[a,b] \cap S$ and $\bar{P}$ is $(\omega)$proximally continuous on $[a,b] \cap S$, it follows that $P \in \mathcal{U}_0$ and $P$ is $AC-\omega$ below on $[a,b]$. So by Theorem 4.2.2, $P$ is $BV-\omega$ on $[a,b]$. Hence by Theorem 3.2.1, $P'_\omega(x)$ exists $(\omega)$a.e. on $[a,b] \cap S$ and so if

$$E = \left\{ x : x \in [a,b] \cap S_3 \text{ and } \underline{PD}P_\omega(x) = +\infty \right\},$$

then $|E|_\omega = 0$. Let

$$E_1 = \{ x : x \in [a,b] \cap S_3 \text{ and } f(x) = +\infty \},$$

$$E_2 = \{ x : x \in [a,b] \cap S_3 \text{ and } f(x) = -\infty \},$$

$$F_1 = \{ x : x \in [a,b] \cap S_3 \text{ and } \underline{PDM}_\omega(x) < f(x) \}$$
$$\cup \{ x : x \in [a,b] \cap S_3 \text{ and } \overline{PDm}_\omega(x) = +\infty \}$$

and

$$F_2 = \{ x : x \in [a,b] \cap S_3 \text{ and } \overline{PDm}_\omega(x) > f(x) \}$$
$$\cup \{ x : x \in [a,b] \cap S_3 \text{ and } \underline{PDM}_\omega(x) = -\infty \}.$$

Now for any point $x_0 \in E_1 - F_1$,

$$\underline{PDM}_\omega(x_0) = +\infty$$

and

$$\overline{PDm}_\omega(x_0) < +\infty .$$

Therefore from (91), we get

$$\underline{PDP}_\omega(x_0) = +\infty .$$

Hence $E_1 - F_1 \subset E$ and so $|E_1 - F_1|_\omega = 0$. Similarly $|E_2 - F_2|_\omega = 0$. Since $|F_1|_\omega = 0$ and $|F_2|_\omega = 0$, we have $|E_1 \cup E_2|_\omega = 0$. This completes the proof of the theorem.

Theorem 4.3.5. If the functions $f$ and $g$ are (PPS)-

integrable on [a,b] and if $\alpha, \beta \in R$, then the function $\alpha f + \beta g$ is (PPS)-integrable on [a,b] and

$$(PPS)\int_a^b (\alpha f + \beta g)\, d\omega = \alpha (PPS)\int_a^b f\, d\omega + \beta (PPS)\int_a^b g\, d\omega.$$

The proof of the above theorem is straightforward.

Definition 4.3.3. Let the function f be (PPS)-integrable on [a,b]. We define the function F by

$$F(x) = \begin{cases} (PPS)\int_a^x f\, d\omega & \text{if } a < x \leq b, \\ F(b) & \text{if } x > b, \\ \omega_f(a) & \text{if } x = a, \\ 0 & \text{if } x < a. \end{cases}$$

Then F is called the indefinite (PPS)-integral of f on [a,b].

Theorem 4.3.6. If the function f is (PPS)-integrable on [a,b], then its indefinite (PPS)-integral $F \in \mathcal{V}$. Specifically

$$F(x+) = F(x)$$

and

$$F(x-) = F(x) - f(x)(\omega(x+) - \omega(x-))$$

if $x \in [a,b] \cap D$.

Proof. From the definition of $F$, we have

$$F(a-) = 0 = F(a) - f(a)(\omega(a+) - \omega(a-))$$

and

$$F(b+) = F(b).$$

Let $\varepsilon > 0$ be chosen arbitrarily. Then there exist a (PPS)-major function $M$ and a (PPS)minor function $m$ of $f$ on $[a,b]$ such that

(92) $$M(b+) - \frac{\varepsilon}{2} < F(b) < m(b+) + \frac{\varepsilon}{2}.$$

Suppose $a \in D$. We have

$$m(a+) \leq f(a)(\omega(a+) - \omega(a-)) \leq M(a+).$$

Again

$$m(a+) - \varepsilon < m(x)$$

$$\leq F(x) \leq M(x)$$

$$< M(a+) + \varepsilon$$

for all $x \in S$ sufficiently close to $a$ on the right. Thus for those $x$,

$$|F(x) - f(a)(\omega(a+) - \omega(a-))| < 2\varepsilon$$

and so

$$F(a+) = f(a)(\omega(a+) - \omega(a-)) = F(a).$$

Now suppose $b \in D$. We have

$$m(b+) - m(b-) \leqq f(b)(\omega(b+) - \omega(b-))$$

$$\leqq M(b+) - M(b-). \tag{93}$$

Also

$$m(b-) - \frac{\varepsilon}{2} < m(x)$$

$$\leqq F(x) \leqq M(x)$$

$$< M(b-) + \frac{\varepsilon}{2} \tag{94}$$

for all $x \in S$ sufficiently close to $b$ on the left. From (92),(93) and (94), we get

$$F(b) - \varepsilon < F(x) + f(b)(\omega(b+) - \omega(b-))$$

$$< F(b) + \varepsilon$$

for all $x \in S$ sufficiently close to $b$ on the left. Therefore

$$F(b-) = F(b) - f(b)(\omega(b+) - \omega(b-)).$$

Again let $\alpha \in (a,b) \cap D$. Let $F_1$ and $F_2$ be the indefinite (PPS)-integrals of $f$ on $[a,\alpha]$ and $[\alpha,b]$ respectively. Then

$$F(x) = F_1(x) \quad \text{for} \quad x \leq \alpha$$

and

$$F(x) = F(\alpha) + F_2(x) - f(\alpha)(\omega(\alpha+) - \omega(\alpha-))$$

for $x > \alpha$. Now

$$F(\alpha-) = F_1(\alpha-) = F(\alpha) - f(\alpha)(\omega(\alpha+) - \omega(\alpha-))$$

and

$$F(\alpha+) = F(\alpha) + F_2(\alpha+) - f(\alpha)(\omega(\alpha+) - \omega(\alpha-)) = F(\alpha).$$

This completes the proof of the theorem.

Theorem 4.3.7. Let the function $f$ be (PPS)-integrable on $[a,b]$ and $F$ be its indefinite (PPS)-integral on $[a,b]$. If $M$ and $m$ are a (PPS)major function and a (PPS)minor function respectively of $f$ on $[a,b]$, then the functions $M - F$ and $F - m$ are non-decreasing on $[a,b] \cap S$.

Proof. Suppose $x_1$ and $x_2$ are any two points of $[a,b] \cap S$ with $x_1 < x_2$. The function $M - M(x_1)$ will serve as a (PPS)major function of $f$ on $[x_1,x_2]$. It now follows that

$$F(x_2) - F(x_1) = (PPS)\int_{x_1}^{x_2} f \, d\omega \leq M(x_2) - M(x_1)$$

and so

$$M(x_1) - F(x_1) \leq M(x_2) - F(x_2).$$

Thus the function $M - F$ is non-decreasing on $[a,b] \cap S$. Similarly we can show that $F - m$ is non-decreasing on $[a,b] \cap S$.

Theorem 4.3.8. Let $F$ be the indefinite (PPS)-integral of the function $f$ on $[a,b]$. Then $\overline{F}$ is ($\omega$)proximally continuous on $[a,b] \cap S$.

Proof. Let $c \in [a,b] \cap S$. Then there exist a sequence of (PPS)major functions $\{M_n\}$ and a sequence of (PPS)minor functions $\{m_n\}$ such that

$$(95) \qquad \lim_{n\to\infty} m_n(c) = F(c) = \lim_{n\to\infty} M_n(c).$$

Again for each $n \in Z$, we have

$$m_n^*(x) \leq F(x) \leq M_n^*(x)$$

for all $x \in [a,b]$. Therefore by Theorem 4.3.6, we get for all $n \in Z$,

$$\overline{m}_n(x) \leq \overline{F}(x) \leq \overline{M}_n(x)$$

on $[a,b]$ and so

$$(\underline{p})(\overline{m}_n,c,\omega) \leq (\underline{p})(\overline{F},c,\omega) \leq (\overline{p})(\overline{F},c,\omega) \leq (\overline{p})(\overline{M}_n,c,\omega).$$

Since $\overline{M}_n$ and $\overline{m}_n$ are $(\omega)$proximally continuous at $c$, we get from above

$$m_n(c) \leq (\underline{p})(\overline{F},c,\omega) \leq (\overline{p})(\overline{F},c,\omega) \leq M_n(c).$$

Therefore by (95), we get

$$F(c) = (\underline{p})(\overline{F},c,\omega) = (\overline{p})(\overline{F},c,\omega).$$

Thus $\overline{F}$ is $(\omega)$proximally continuous at $c$.

Theorem 4.3.9. If $F$ is the indefinite (PPS)-integral of $f$ on $[a,b]$, then

$$PDF_\omega(x) = f(x)$$

$(\omega)$a.e. on $[a,b] \cap S$ and

$$F(x+) - F(x-) = f(x)(\omega(x+) - \omega(x-))$$

for all $x \in [a,b] \cap D$.

Proof. First we show that

(96) $$\underline{PDF}_{\omega}(x) \geqslant f(x)$$

($\omega$)a.e. on $[a,b] \cap S$. Assume on the contrary that the relation (96) is not true. Then there exist a positive number $p$ and a set $E \subset [a,b] \cap S_3$ of positive outer ($\omega$)measure $\mu$ such that

(97) $$f(x) - \underline{PDF}_{\omega}(x) > p$$

for all $x \in E$. We choose $\varepsilon$ arbitrarily so that

$$0 < \varepsilon < \tfrac{1}{2}\mu p.$$

Let $M$ be a (PPS)major function of $f$ on $[a,b]$ such that

$$M^{+}(b) - F^{+}(b) < \varepsilon .$$

We write

$$P(x) = M(x) - F(x).$$

Then by Theorem 4.3.7, $P$ is non-decreasing on $[a,b] \cap S$. Again by Theorem 4.3.8, $\overline{F}$ is ($\omega$)proximally continuous on $[a,b] \cap S$. Therefore $\overline{P}$ is continuous on $[a,b] \cap S$. So $P \in \mathcal{U}_0$ and AC-$\omega$ below on $[a,b]$. Hence by Theorem 4.2.3,

$$(LS)\int_a^b P'_{\omega}\, d\omega \leq P^{+}(b) < \varepsilon$$

where $P'_{\omega}(x)$ is equal to $P'_{\omega}(x)$ when it exists for

$x \in [a,b] \cap S$ and is equal to

$$\frac{P(x+) - P(x-)}{\omega(x+) - \omega(x-)}$$

for $x \in [a,b] \cap D$. From the above inequality, it follows that the set in which

$$P^1_\omega(x) > \tfrac{1}{2} p$$

has $(\omega)$measure less than $\mu$. So there is a set $E_1 \subset E$ with $\omega^*(E_1) > 0$ in which $P'_\omega(x)$ exists and lies between 0 and $\frac{1}{2} p$ for $x \in E_1$. Now for $x \in E_1$,

$$f(x) \leq \underline{PD}\, M_\omega(x)$$

$$= \underline{PDF}_\omega(x) + P'_\omega(x)$$

$$< \underline{PDF}_\omega(x) + \tfrac{1}{2}p$$

which contradicts the relation (97). Hence the relation (96) is true. Considering a (PPS)minor function it can similarly be proved that

(98) $$\overline{PDF}_\omega(x) \leq f(x)$$

$(\omega)$a.e. on $[a,b] \cap S$. Combining (96) and (98), the first part of the theorem follows. The second part follows from Theorem 4.3.6.

Theorem 4.3.10. If the function $f$ is (PPS)-integrable

on $[a,b]$, then it is $(\omega)$measurable on $[a,b]$.

Proof. Let $F$ be the indefinite (PPS)-integral of $f$ on $[a,b]$. By Theorem 4.3.9, we get

$$PDF_{\omega}(x) = f(x)$$

$(\omega)$a.e. on $[a,b] \cap S$. Again by Theorem 2.3.4, $PDF_{\omega}$ is $(\omega)$measurable on $[a,b] \cap S$. Since the set $D$ is countable, it follows that $f$ is $(\omega)$measurable on $[a,b]$.

Theorem 4.3.11. If the function $f$ is non-negative and (PPS)-integrable on $[a,b]$, then $f$ is (LS)-summable on $[a,b]$ and

$$(LS)\int_a^b f \, d\omega = (PPS)\int_a^b f \, d\omega.$$

We omit the easy proof of the above theorem.

Theorem 4.3.12. Suppose that $\{f_n\}$ is a sequence of (PPS)-integrable functions on $[a,b]$ such that

$$\lim_{n\to\infty} f_n = f.$$

Suppose, further, that there exist two (PPS)-integrable functions $g$ and $h$ on $[a,b]$ such that

$$g(x) \leq f_n(x) \leq h(x)$$

for all $n \in Z$ and $x \in [a,b]$. Then the function $f$ is (PPS)-integrable on $[a,b]$ and

$$\lim_{n\to\infty} (PPS)\int_a^b f_n \, d\omega = (PPS)\int_a^b f \, d\omega.$$

Proof. We write

$$G_n = f_n - g,$$

$$G = f - g$$

and

$$H = h - g.$$

Then by Theorem 4.3.5, $G_n$ and $H$ are (PPS)-integrable on $[a,b]$. Moreover $G_n$ and $H$ are non-negative on $[a,b]$. So by Theorem 4.3.11, $G_n$ and $H$ are (LS)-summable on $[a,b]$. Since

$$0 \leq G_n(x) \leq H(x)$$

for all $x \in [a,b]$, we have

$$\lim_{n\to\infty} (LS)\int_a^b G_n \, d\omega = (LS)\int_a^b G \, d\omega.$$

Therefore

$$\lim_{n\to\infty} (PPS)\int_a^b G_n \, d\omega = (PPS)\int_a^b G \, d\omega$$

and so

$$\lim_{n\to\infty} (PPS)\int_a^b f_n \, d\omega = (PPS)\int_a^b f \, d\omega .$$

We now prove the Cauchy property of the (PPS)-integral. The (CPS)-integral of Dutta [28] possesses this property as shown by Bose [11].

Theorem 4.3.13. (i) Suppose the function $f$ is (PPS)-integrable on every segment $[a, \beta]$ where $a < \beta < b$. If $f(b)$ is finite for $b \in D$ and if the limits

$$(99) \quad L_1 = (p_\omega) \lim_{\beta \to b} (PPS)\int_a^\beta f \, d\omega \quad \text{if } b \in S,$$

and

$$(100) \quad L_2 = \lim_{\substack{\beta \to b \\ \beta \in S}} (PPS)\int_a^\beta f \, d\omega \quad \text{if } b \in D$$

exist and are finite, then $f$ is (PPS)-integrable on $[a,b]$ and

$$(PPS)\int_a^b f \, d\omega = L_1 \quad \text{if } b \in S,$$

$$(PPS)\int_a^b f \, d\omega = L_2 + f(b)(\omega(b+) - \omega(b-)) \quad \text{if } b \in D.$$

(ii) Suppose the function $f$ is (PPS)-integrable on

every segment $[\alpha, b]$ where $a < \alpha < b$. If $f(a)$ is finite for $a \in D$ and if the limits

$$K_1 = (p_\omega) \lim_{\alpha \to a} (PPS) \int_\alpha^b f \, d\omega \qquad \text{if } a \in S,$$

and

$$K_2 = \lim_{\substack{\alpha \to a \\ \alpha \in S}} (PPS) \int_\alpha^b f \, d\omega \qquad \text{if } a \in D$$

exist and are finite, then $f$ is (PPS)-integrable on $[a,b]$ and

$$(PPS) \int_a^b f \, d\omega = K_1 \qquad \text{if } a \in S,$$

$$(PPS) \int_a^b f \, d\omega = K_2 + f(a)(\omega(a+) - \omega(a-)) \qquad \text{if } a \in D.$$

Proof. (i). If $b \in S_0 \cup S_2^+$, then the theorem is obviously true. So we suppose that $b \notin S_0 \cup S_2^+$. We consider an increasing sequence of points $\{d_n\}$ ($d_n \in S$, $n = 1,2,\ldots$) which converges to $b$ and take $a = d_0$. The function $f$ is (PPS)-integrable on every segment $[d_{n-1}, d_n]$ ($n \geqslant 1$). Let $\varepsilon > 0$ be chosen arbitrarily and $M_n \in \mathcal{M}$ ($n \geqslant 1$) be a (PPS)-major function of $f$ on $[d_{n-1}, d_n]$ such that

$$0 \leqslant M_n(d_n) - (PPS) \int_{d_{n-1}}^{d_n} f \, d\omega < \frac{\varepsilon}{2^n}.$$

We define the function $M$ in the following way :

$$M(x)=\begin{cases} M_1(x) & \text{for } x\in[a,d_1]\cap S \text{ and for } x<a,\\ M_1(d_1)+M_2(x) & \text{for } x\in[d_1,d_2]\cap S,\\ \cdots\cdots\cdots & \\ M_1(d_1)+M_2(d_2)+\cdots\cdots+M_{n-1}(d_{n-1})+M_n(x) & \\ \qquad\qquad \text{for } x\in[d_{n-1},d_n]\cap S. \end{cases}$$

For each $n \geqslant 1$ and for each $x \in [d_{n-1}, d_n] \cap S$,

$$0 \leqslant M(x) - F(x)$$

$$= \sum_{r=1}^{n-1} M_r(d_r) + M_n(x) - F(x)$$

$$= \sum_{r=1}^{n-1} \left(M_r(d_r) - (PPS)\int_{d_{r-1}}^{d_r} f\, d\omega\right) + \sum_{r=1}^{n-1} (PPS)\int_{d_{r-1}}^{d_r} f\, d\omega$$

$$+ M_n(x) - F(x)$$

$$< \sum_{r=1}^{n-1} \frac{\varepsilon}{2^r} + M_n(x) - (PPS)\int_{d_{n-1}}^{x} f\, d\omega$$

$$(101) < \sum_{r=1}^{n} \frac{\varepsilon}{2^r} < \varepsilon$$

where $F$ is the indefinite (PPS)-integral of $f$ on $[a, \beta]$ $(a < \beta < b)$.

It is clear that $M$ is a (PPS)major function of $f$ on $[a, \beta]$ for any $\beta \in (a,b)$. Thus by Theorem 4.3.7, $M - F$ is non-decreasing on $[a,b) \cap S$. Again by (101), $M - F$ is bounded and hence possesses finite ordinary limit over $S$ at $x = b$ on the left. Therefore by (99) and (100), it follows that

$$(p_\omega) \lim_{x \to b-} M(x)$$

exists finitely when $b \in S$ and

$$M(b-) = \lim_{\substack{x \to b- \\ x \in S}} M(x)$$

exists finitely when $b \in D$. If $b \in S$, we take

$$M(b) = (p_\omega) \lim_{x \to b-} M(x)$$

and

$$M(x) = M(b), \quad x > b,$$

then it readily follows that $M$ is a (PPS)major function of $f$ on $[a,b]$ with

$$M(b) < L_1 + \varepsilon .$$

If $b \in D$, we take

$$M(x) = M(b-) + f(b)(\omega(b+) - \omega(b-)), \quad x > b,$$

then $M$ is a (PPS)major function of $f$ on $[a,b]$ with

$$M(b+) < L_2 + f(b)(\omega(b+) - \omega(b-)) + \varepsilon .$$

Similarly we can construct a (PPS)minor function $m$ of $f$ on $[a,b]$ with

$$m(b) > L_1 - \varepsilon \quad \text{if } b \in S$$

and

$$m(b+) > L_2 + f(b)(\omega(b+) - \omega(b-)) - \varepsilon \quad \text{if } b \in D.$$

Thus in any case $f$ is (PPS)-integrable on $[a,b]$ and

$$(PPS)\int_a^b f \, d\omega = L_1 \qquad \text{if } b \in S$$

and

$$(PPS)\int_a^b f \, d\omega = L_2 + f(b)(\omega(b+) - \omega(b-)) \qquad \text{if } b \in D.$$

The proof of the part (ii) can be obtained analogously.

## §4. Integration by parts formula for the (PPS)-integral

To prove an integration by parts formula for the (PPS)-integral, we need an integration by parts formula for the (D*S)-integral which is equivalent to the (PS)-integral. To prove an integration by parts formula for the (D*S)-integral, we require the following :

Lemma 4.4.1. If $F \in \mathcal{U}_0$ is $AC^*-\omega$ on $E \subset [a,b]$ and $g \in \mathcal{U}_0$ is $AC-\omega$ on $E$, then $Fg$ is $AC^*-\omega$ on $E$.

Proof. Let us assume that $g$ is non-decreasing on $[a,b]$ (Theorem 4.2.5). By Theorem 4.2.4, there exist two positive numbers $M_1$ and $M_2$ such that

$$|F^{\pm}(x)| \leq M_1 \quad \text{and} \quad |g^{\pm}(x)| \leq M_2$$

for all $x \in [a,b]$. We write $H = Fg$. Now we show that $H$ is $AC^*-\omega$ on $E$. Let $\varepsilon > 0$ be chosen arbitrarily. Then there exists a $\delta > 0$ such that for any set of non-overlapping open intervals $\{(c_r, d_r)\}$ having end points on $E$ with

$$\sum_r (\omega(d_r+) - \omega(c_r-)) < \delta,$$

we have

$$\sum_r \overline{\operatorname{bound}}_{c_r<x<d_r} |F^+(x) - F^-(c_r)| < \frac{\varepsilon}{2M_2}$$

$$\sum_r \overline{\operatorname{bound}}_{c_r<x<d_r} |F^+(d_r) - F^-(x)| < \frac{\varepsilon}{2M_2}$$

and

$$\sum_r |g^+(d_r) - g^-(c_r)| < \frac{\varepsilon}{2M_1}.$$

Now

$$|H^+(x) - H^-(c_r)|$$

$$\leqslant |F^+(x) - F^-(c_r)||g^+(x)| + |g^+(x) - g^-(c_r)||F^-(c_r)|$$

$$\leqslant M_2|F^+(x) - F^-(c_r)| + M_1|g^+(x) - g^-(c_r)|$$

$$\leqslant M_2|F^+(x) - F^-(c_r)| + M_1|g^+(d_r) - g^-(c_r)|.$$

Therefore

$$\sum_r \overline{\text{bound}}_{c_r<x<d_r} |H^+(x) - H^-(c_r)|$$

$$\leqslant M_2 \sum_r \overline{\text{bound}}_{c_r<x<d_r} |F^+(x) - F^-(c_r)| + M_1 \sum_r |g^+(d_r) - g^-(c_r)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Similarly we can prove that

$$\sum_r \overline{\text{bound}}_{c_r<x<d_r} |H^+(d_r) - H^-(x)| < \varepsilon .$$

Thus $H = Fg$ is $AC^*-\omega$ on $E$.

<u>Theorem 4.4.1.</u> Let the function $f$ be $(D^*S)$-integrable on $[a,b]$, $F$ be its indefinite $(D^*S)$-integral and let $g \in \mathcal{U}_0$

be AC-$\omega$ on $[a,b]$. If $F$ and $g$ belong to $\mathcal{U}^{(1)}$, then $fg$ is (D*S)-integrable on $[a,b]$ and

$$(D^*S)\int_a^b fg\ d\omega = [Fg]_{a-}^{b+} - (LS)\int_a^b Fg_\omega^1\ d\omega$$

where

$$[Fg]_{a-}^{b+} = F^+(b)g^+(b) - F^-(a)g^-(a)$$

and

$$g_\omega^1(x) = \begin{cases} g'_\omega(x) & \text{if } g'_\omega(x) \text{ exists, } x \in [a,b] \cap S, \\ \dfrac{g(x+) - g(x-)}{\omega(x+) - \omega(x-)} & \text{if } x \in [a,b] \cap D, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $F$ is ACG*-$\omega$ on $[a,b]$, it follows from Lemma 4.4.1 that the function $Fg$ is ACG*-$\omega$ on $[a,b]$. Therefore the function

$$H(t) = F(t)g(t) - K(t)$$

where

$$K(t) = (LS)\int_a^t Fg_\omega^1\ d\omega\ ,$$

is ACG*-$\omega$ on $[a,b]$. Again

$$H'_\omega(t) = f(t)g(t)$$

$(\omega)$a.e. on $[a,b] \cap S$ and for all $t \in [a,b] \cap D$,

$$H(t+) - H(t-) = f(t)g(t)(\omega(t+) - \omega(t-)).$$

Therefore $fg$ is $(D^*S)$-integrable on $[a,b]$ and

$$(D^*S)\int_a^b fg \, d\omega = [Fg]_{a-}^{b+} - (LS)\int_a^b Fg_\omega^1 \, d\omega.$$

We now give an example to show that the indefinite (PPS)-integral is not necessarily (PPS)-integrable.

Example 4.4.1. We take $\omega(x) = x$ and take the set

$$E = \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$$

of Example 1.4.1. We define the function $F$ on $[0,1]$ in the following way :

$$F(x) = \begin{cases} \dfrac{\pi}{\beta_n - \alpha_n} \sin^2 \pi \dfrac{x - \alpha_n}{\beta_n - \alpha_n} & \text{for } \alpha_n \le x \le \beta_n, \quad n \in Z, \\ 0 & \text{for } x \in [0,1] - E. \end{cases}$$

The function $F$ is $(\omega)$proximally continuous at $x = 0$ and at other points $F$ is continuous . It is obvious that $F$ has finite $(\omega)$derivative everywhere except at the point

$x = 0$ where it is $(\omega)$proximally differentiable. Therefore $F$ is the indefinite (PPS)-integral of $f(x) = PDF_{\omega}(x)$. Since the function $F$ is non-negative and

$$(LS)\int_0^1 F \, d\omega = \infty ,$$

it follows that $F$ is not (PPS)-integrable.

In the integration by parts formula for the (PPS)-integral on $[a,b]$, we shall suppose that the indefinite (PPS)-integral is (D*S)-integrable on $[a,b]$ and $[a,b] \cap D = \phi$ .

Theorem 4.4.2. Let $[a,b] \cap D = \phi$ and $f$ be (PPS)-integrable on $[a,b]$. Let $g \in \mathcal{U}_o$ be AC-$\omega$ on $[a,b]$ and $G$ be its indefinite (LS)-integral on $[a,b]$. If the indefinite (PPS)-integral $F$ of $f$ is (D*S)-integrable, then $fG$ is (PPS)-integrable on $[a,b]$ and

$$(PPS)\int_a^b fG \, d\omega = [FG]_a^b - (D^*S)\int_a^b Fg \, d\omega$$

where

$$[FG]_a^b = F(b)G(b) - F(a)G(a).$$

Proof. By Theorem 4.2.5, $g$ can be expressed as the difference of two non-negative functions $g_1$ and $g_2$ belonging to $\mathcal{U}_o$. It then follows that $G$ is the difference of two

non-negative functions $G_1$ and $G_2$. Therefore if the theorem is true for the pairs $g_1, G_1$ and $g_2, G_2$, it is also true for the pair $g, G$. So we suppose that $g$ and $G$ both are non-negative. First we prove that $fG + Fg$ is (PPS)-integrable and that

$$(PPS)\int_a^b (fG + Fg)\, d\omega = [FG]_a^b.$$

Let $M$ be a (PPS)major function of $f$ on $[a,b]$. We consider the function $H = MG$. Then $H$ is ($\omega$)proximally continuous on $[a,b]$ and

$$\underline{PDH}_\omega \geqslant G\, \underline{PDM}_\omega + M\, \underline{PDG}_\omega$$

on $[a,b] \cap S_3$. Again since $g \in \mathfrak{U}_0$, it follows that $G'_\omega = g$ on $[a,b] \cap S_3$. It thus follows that

$$\underline{PDH}_\omega \geqslant G\, \underline{PDM}_\omega + Mg$$

on $[a,b] \cap S_3$. Therefore $H$ is a (PPS)major function of $fG + Fg$. If $m$ is a (PPS)minor function of $f$, it can similarly be proved that $h = mG$ is a (PPS)minor function of $fG + Fg$. Since $MG$ and $mG$ can be made to differ arbitrarily little by taking $M$ and $m$ sufficiently near, it follows that $fG + Fg$ is (PPS)-integrable on $[a,b]$ and

$$(\text{PPS})\int_a^b (fG + Fg)\ d\omega = [FG]_a^b.$$

Again by Theorem 4.4.1, it follows that $Fg$ is (D*S)-integrable and hence (PPS)-integrable on $[a,b]$. Therefore $fG$ is (PPS)-integrable on $[a,b]$ and

$$(\text{PPS})\int_a^b fG\ d\omega = [FG]_a^b - (\text{D*S})\int_a^b Fg\ d\omega.$$