C H A P T E R V

PAPER I

TIME-HARDENING AND TIME-SOFTENING ELASTIC SHELLS.

1. Introduction.

The theory of Elasticity in the earlier periods of its development was mainly based on materials of homogeneous elastic properties. The generalization from isotropy to anisotropy was soon realised specially in problems on crystal Elasticity.

A further generalization has been recently made from homogeneity to non-homogeneity of the material in which elastic properties vary from point to point. This has helped much in the improvement of the design of structures.

A still another generalization of the elastic properties has been made by Paria [24]. The Young's modulus has been assumed to be function of time but independent of co-ordinates. According to him, the time-harding material is that in which the Young's modulus increases with increasing time and the time-softening material is that in which the Young's modulus decreases with increasing time. The physical justification for such assumption may be found in concrete, for example, in which the elastic properties vary during its aging periods. The elastic properties of materials may vary from season to season during the year due to variations in temperature, moisture contents and similar other varying factors.

Here, the author has attempted to introduce this new idea to obtain transient displacements and stresses that exist in thick elastic shells under internally and externally applied loads which are special functions of time for the following cases :

(A) radial motion of an infinitely long, circular cylindical shell,

(B) radially symmetric motion of a spherical shell.

2. A New Finite Hankel Transform And An Assumption.

For the sake of completeness of the present problem the properties of the new finite Hankel transform [9] are given here. A bar over a small letter indicates the transform variable, whereas a prime on a capital letter means differentiation with respect to its argument. H denotes the integral operator.

$$\overline{f}(\xi_j) = H[f(r)] = \int_a^b r\, f(r)\, C_m(r, \xi_j)\, dr, \quad a \le r \le b \qquad \ldots(1)$$

$$C_m(r,\xi_j) = \Big\{ J_m(\xi_j r)\left[\xi_j Y_m'(\xi_j a) + h\, Y_m(\xi_j a)\right] - Y_m(\xi_j r)\left[\xi_j J_m'(\xi_j a) + h\, J_m(\xi_j a)\right] \Big\} \qquad \ldots(2)$$

$$f(r) = \frac{\pi^2}{2} \sum_{\xi_j} \xi_j^2 \left[\xi_j J_m'(\xi_j b) + K J_m(\xi_j b)\right]^2 \cdot \left[\bar{f}(\xi_j) \cdot \frac{C_m(r,\xi_j)}{F_m(\xi_j)} \right] \qquad \ldots(3)$$

$$F_m(\xi_j) = \left\{ K^2 + \xi_j^2\left[1 - \left(\frac{m}{\xi_j b}\right)^2\right]\right\}\left[\xi_j J_m'(\xi_j a) + h\, J_m(\xi_j a)\right]^2 - \left\{ h^2 + \xi_j^2\left[1 - \left(\frac{m}{\xi_j a}\right)^2\right]\right\}\left[\xi_j J_m'(\xi_j b) + K J_m(\xi_j b)\right]^2 \qquad \ldots(4)$$

in which ξ_j is a positive root of

$$\left[\xi_j Y_m'(\xi_j a) + h\, Y_m(\xi_j a)\right]\left[\xi_j J_m'(\xi_j b) + K J_m(\xi_j b)\right] = \left[\xi_j Y_m'(\xi_j b) + K Y_m(\xi_j b)\right]\left[\xi_j J_m'(\xi_j a) + h\, J_m(\xi_j a)\right] \qquad \ldots(5)$$

$$H\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{m^2}{r^2}f\right]$$
$$=\frac{2}{\pi}P_{j,m}\left[f'(b)+kf(b)\right]-\frac{2}{\pi}\left[f'(a)+hf(a)\right]$$
$$-\xi_j^2\,\overline{f(\xi_j)}\quad, \qquad \ldots(6)$$

in which

$J_m(\xi_j r)$, $Y_m(\xi_j r)$ = Bessel function of the first and second kind, respectively, and of order m,

h, k = constant coefficient whose value can be positive, negative or zero,

a, b = inner and outer radii of the shells, respectively,

$f(r)$ = arbitrary function in the spatial variable,

$$P_{j,m}=\frac{\xi_j J_m'(\xi_j a)+h\,J_m(\xi_j a)}{\xi_j J_m'(\xi_j b)+k\,J_m(\xi_j b)} = \text{a constant.}$$

Let us assume that $E = E_0\left(1+\alpha\cdot e^{-\frac{t}{t_0}}\right)$, ...(7)

where E_0 and t_0 are the reference values of the Young's modulus and time, respectively and α is a prescribed parameter. From (7), $E(t)$ runs from $E_0(1+\alpha)$ at $t=0$ to E_0 as t approaches infinity. If α is negative the material is time-hardening and if α is positive it is time-softening. From (7) Lamé's constants may be written as

$$\left.\begin{aligned} \lambda &= \lambda_0\left(1+\alpha . e^{-t/t_0}\right) \\ \mu &= \mu_0\left(1+\alpha . e^{-t/t_0}\right) \end{aligned}\right\} \quad \text{and} \qquad \ldots(8)$$

in which $\lambda_0 = \nu E_0 / (1+\nu)(1-2\nu)$ and $\mu_0 = E_0 / 2(1+\nu)$ ...(9)

where ν is the Poisson's ratio, independent of the spatial and temporal positions.

3. Radial Vibrations of a Spherical Shell.

The differential equation along with the initial and the boundary conditions for the radially symmetric motion of a thick isotropic, homogeneous, spherical shell subjected to loads on both surfaces may be put as, vide [10],

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} - \frac{2}{r^2}U = \left[\frac{1}{c^2\left(1+\alpha . e^{-t/t_0}\right)}\right] . \frac{\partial^2 U}{\partial t^2} ,$$

$$a \leq r \leq b, \quad t \geq 0 \qquad \ldots(10)$$

$$c^2 = \left[\frac{\lambda_0 + 2\mu_0}{\rho} \right] , \qquad \ldots(11)$$

initial conditions

$$\left. \begin{aligned} U &= 0 \\ \frac{\partial U}{\partial t} &= 0 \end{aligned} \right\} t = 0 \, ; \; a \le r \le b \qquad \ldots(12)$$

boundary conditions

$$\left[\sigma_r(r,t) \right]_{r=a} = \left[(\lambda_0 + 2\mu_0) \frac{\partial U}{\partial r} + \frac{2\lambda_0}{r} U \right] \cdot \left(1 + \alpha . e^{-t/t_0} \right)$$

$$= A (e^{-t/2t_0}) , \quad r = a \, ; \; t \geq 0 \qquad \ldots(13)$$

$$\left[\sigma_r(r,t) \right]_{r=b} = \left[(\lambda_0 + 2\mu_0) \frac{\partial U}{\partial r} + \frac{2\lambda_0}{r} U \right] \cdot \left(1 + \alpha . e^{-t/t_0} \right)$$

$$= B (e^{-t/2t_0}) , \quad r = b \, ; \; t \geq 0$$

$$\ldots(14)$$

where U radial displacement of the sphere, ρ density of the medium and A, B radial stresses on inner and outer surfaces, respectively.

Let us substitute $\sqrt{r}\cdot V(r,t) = U(r,t)$ ...(15)

and hence the equations (10), (12), (13) and (14) become

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{\left(\frac{3}{2}\right)^2}{r^2} U$$

$$= \left[\frac{1}{c^2\left(1+\alpha.\bar{e}^{t/t_0}\right)}\right]\cdot\frac{\partial^2 U}{\partial t^2}\ ;\quad a \le r \le b,\ t \geqslant 0$$

...(16)

initial conditions :-

$$\left.\begin{array}{l} U = 0 \\ \dfrac{\partial U}{\partial t} = 0 \end{array}\right\} t = 0\ ;\quad a \le r \le b$$

...(17)

boundary conditions :

$$\sigma_r(a,t) = \left[\frac{\lambda_0 + 2\mu_0}{\sqrt{r}}\frac{\partial U}{\partial r} + \frac{3\lambda_0 + 2\mu_0}{2r\sqrt{r}} U\right]\cdot\left(1+\alpha.\bar{e}^{t/t_0}\right)$$

$$= A\left(\bar{e}^{t/2t_0}\right) \text{ at } r = a\ ;\quad t \geqslant 0 \qquad ...(18)$$

$$\sigma_r(b,t) = \left[\frac{\lambda_0+2\mu_0}{\sqrt{r}}\frac{\partial u}{\partial r} + \frac{3\lambda_0+2\mu_0}{2r\sqrt{r}}u\right]\left(1+\alpha . e^{-\frac{t}{t_0}}\right)$$

$$= B\left(e^{-\frac{t}{2t_0}}\right) \quad at\ r=b;\ t \geqslant 0$$

...(19)

Using the following definitions

$$\left.\begin{aligned} h &= \frac{\left[\frac{3\lambda_0+2\mu_0}{2a\sqrt{a}}\right]}{\left[\frac{\lambda_0+2\mu_0}{\sqrt{a}}\right]} \\ k &= \frac{\left[\frac{3\lambda_0+2\mu_0}{2b\sqrt{b}}\right]}{\left[\frac{\lambda_0+2\mu_0}{\sqrt{b}}\right]} \end{aligned}\right\}$$

...(20)

in the equation (6) and putting $m = (3/2)$ one gets

$$H\left[\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\left(\frac{3}{2}\right)^2}{r^2}f\right]$$

$$= \frac{2}{\pi(\lambda_0+2\mu_0)}\left\{\sqrt{b}\,.\,P_{j,\frac{3}{2}}\left[\frac{\lambda_0+2\mu_0}{\sqrt{b}}f'(b) + \frac{3\lambda_0+2\mu_0}{2b\sqrt{b}}f(b)\right]\right.$$

$$\left. - \sqrt{a}\left[\frac{\lambda_0+2\mu_0}{\sqrt{a}}f'(a) + \frac{3\lambda_0+2\mu_0}{2a\sqrt{a}}f(a)\right]\right\} - \xi_j^2\,f(\xi_j).$$

...(21)

considering the equations (18), (19) and (21) one can easily suggest the appropriate finite Hankel transform for the equation (16) to be

$$\overline{u} = \overline{u(\xi_j, t)} = \int_a^b r\, u(r,t)\, C_{\frac{3}{2}}(r, \xi_j)\, dr \qquad \ldots(22)$$

where $C_{\frac{3}{2}}(r,\xi_j)$ may be obtained from (2) on putting $m = \frac{3}{2}$.

Applying (22) to (16) and using (18), (19) and (21) one finds

$$\left[\frac{1}{c^2(1+\alpha.\bar{e}^{t/t_0})}\right]\cdot\frac{d^2\overline{u}}{dt^2} = \left[\frac{2}{\pi(\lambda_0+2\mu_0)}\right]\cdot\left[\frac{\sqrt{b}\; P_{j,\frac{3}{2}}\,\sigma_r(b,t) - \sqrt{a}\;\sigma_r(a,t)}{(1+\alpha.\,\bar{e}^{t/t_0})}\right] - \xi_j^2\,\overline{u}. \qquad \ldots(23)$$

On using surface loads, the equation (23) may be put as

$$\frac{d^2\overline{u}}{dt^2} + c^2\xi_j^2\left(1+\alpha.\,e^{-\frac{t}{t_0}}\right)\overline{u} = \frac{2}{\pi\rho}\left\{\sqrt{b}\;P_{j,\frac{3}{2}}\,B(\bar{e}^{\frac{t}{2t_0}}) - \sqrt{a}\;A(\bar{e}^{\frac{t}{2t_0}})\right\}. \qquad \ldots(24)$$

Now putting
$$\left.\begin{aligned} t &= 2t_0 \log\frac{1}{\tau} \\ 2\xi_j t_0 C &= K_j \end{aligned}\right\} \quad \text{and using} \qquad \ldots(25)$$

the equation (24) may be transformed to

$$\tau^2 \frac{d^2\bar{u}}{d\tau^2} + \tau\frac{d\bar{u}}{d\tau} + k_j^2(\alpha\tau^2+1)\bar{u}$$
$$= \frac{2}{\pi\rho}\left\{\sqrt{b}\; P_{j,\frac{3}{2}}\; B(\tau) - \sqrt{a}\; A(\tau)\right\}.$$

$$\ldots(26)$$

We choose here the prescribed functions $A(\tau)$, $B(\tau)$ in the following form

$$\left.\begin{aligned} B(\tau) &= \sum_{n=0}^{\infty} b_n \cdot \tau^{n+n_1+1} \\ A(\tau) &= \sum_{n=0}^{\infty} a_n \, \tau^{n+n_2+1} \end{aligned}\right\} \qquad \ldots(27)$$

in which n_1, n_2 may take up any value.

We use further $\alpha = (\alpha_1)^2$ for $\alpha > 0$ and $\alpha = (-1)(\alpha_2)^2$

when $\alpha < 0$ and apply
$$\left.\begin{aligned} \alpha_1 k_j \tau &= x \\ \text{and} \quad \alpha_2 k_j \tau &= y \end{aligned}\right\} \quad \ldots(28)$$

in the equation (26) to obtain the following equations,

$$x^2 \frac{d^2 \bar{u}}{dx^2} + x \frac{d\bar{u}}{dx} + \left[x^2 - (i k_j)^2\right] \bar{u}$$

$$= \frac{2}{\pi\rho} \sum_{n=0}^{\infty} \left[\sqrt{b}\, P_{j,\frac{3}{2}} \frac{b_n}{(\alpha_1 k_j)^{n+n_1+1}} \cdot x^{n+n_1+1} - \sqrt{a} \frac{a_n}{(\alpha_1 k_j)^{n+n_2+1}} \cdot x^{n+n_2+1} \right]. \quad \ldots(29)$$

$$y^2 \frac{d^2 \bar{u}}{dy^2} + y \frac{d\bar{u}}{dy} + \left[-y^2 - (i k_j)^2\right] \bar{u}$$

$$= \frac{2}{\pi\rho} \sum_{n=0}^{\infty} \left[\sqrt{b}\, P_{j,\frac{3}{2}} \frac{b_n}{(\alpha_2 k_j)^{n+n_1+1}} \cdot y^{n+n_1+1} - \sqrt{a} \frac{a_n}{(\alpha_2 k_j)^{n+n_2+1}} \cdot y^{n+n_2+1} \right]. \quad \ldots(30)$$

When K_j is a non-intiger, the solutions of (29) and (30) can be laid down as, [31],

$$\bar{u} = \beta_j \cdot J_{ik_j}(x) + \gamma_j J_{-ik_j}(x) + \sum_{n=0}^{\infty} R_{j,\frac{3}{2},n}(x) \quad ,\qquad \text{...(31)}$$

in which

$$R_{j,\frac{3}{2},n}(x) = \frac{2}{\pi\rho}\left[\frac{\sqrt{b}\, P_{j,\frac{3}{2}} \cdot b_n}{(\alpha_1 k_j)^{n+n_1+1}} \cdot S_{n+n_1,ik_j}(x) - \frac{\sqrt{a}\; a_n}{(\alpha_1 k_j)^{n+n_2+1}} \cdot S_{n+n_2,ik_j}(x)\right], \qquad \text{...(32)}$$

and $$\bar{u} = \eta_j J_{ik_j}(iy) + \zeta_j J_{-ik_j}(iy) + \sum_{n=0}^{\infty} Q_{j,\frac{3}{2},n}(iy) \quad , \qquad \text{...(33)}$$

where

$$Q_{j,\frac{3}{2},n}(iy) = \frac{2}{\pi\rho}\left[\frac{\sqrt{b}\; P_{j,\frac{3}{2}} \cdot b_n}{(i\alpha_2 k_j)^{n+n_1+1}} \cdot S_{n+n_1,ik_j}(iy) - \frac{\sqrt{a}\; a_n}{(i\alpha_2 k_j)^{n+n_2+1}} \cdot S_{n+n_2,ik_j}(iy)\right] \qquad \text{...(34)}$$

respectively. $S_{a,b}(z)$ is a Lommel's function

Initial conditions may now be used to (31) and (33) to get unknowns β_j, γ_j, η_j and ζ_j in known terms as follows :

$$\left.\begin{aligned}
\beta_j &= \frac{i\pi x_0}{2\sinh \pi k_j} \sum_{n=0}^{\infty} \left[J_{-ik_j}(x_0)\, R'_{j,\frac{3}{2},n}(x_0) - J'_{-ik_j}(x_0)\, R_{j,\frac{3}{2},n}(x_0) \right] \\
\gamma_j &= \frac{i\pi x_0}{2\sinh \pi k_j} \sum_{n=0}^{\infty} \left[J'_{ik_j}(x_0)\, R_{j,\frac{3}{2},n}(x_0) - J_{ik_j}(x_0)\, R'_{j,\frac{3}{2},n}(x_0) \right]
\end{aligned}\right\} \quad \ldots(35)$$

$$\left.\begin{aligned}
\eta_j &= -\frac{\pi y_0}{2\sinh \pi k_j} \sum_{n=0}^{\infty} \left[J_{-ik_j}(iy_0)\, Q'_{j,\frac{3}{2},n}(iy_0) - J'_{-ik_j}(iy_0)\, Q_{j,\frac{3}{2},n}(iy_0) \right] \\
\zeta_j &= -\frac{\pi y_0}{2\sinh \pi k_j} \sum_{n=0}^{\infty} \left[J'_{ik_j}(iy_0)\, Q_{j,\frac{3}{2},n}(iy_0) - J_{ik_j}(iy_0)\, Q'_{j,\frac{3}{2},n}(iy_0) \right]
\end{aligned}\right\} \quad \ldots(36)$$

in which

$$\left.\begin{aligned} x_0 &= [x]_{t=0} = [\alpha_1 k_j \tau]_{t=0} = \left[\alpha_1 k_j e^{-\frac{t}{2t_0}}\right]_{t=0} = \alpha_1 k_j \\ y_0 &= [y]_{t=0} = [\alpha_2 k_j \tau]_{t=0} = \left[\alpha_2 k_j e^{-\frac{t}{2t_0}}\right]_{t=0} = \alpha_2 k_j \end{aligned}\right\} \quad \ldots(37)$$

It is to be noted that the formula, [31],

$$\left[J_{\mu_1}(z)\, J'_{-\mu_1}(z) - J'_{\mu_1}(z)\, J_{-\mu_1}(z)\right] = -\left(\frac{2\sin\mu_1\pi}{\pi z}\right) \quad \ldots(38)$$

has been employed in (35) and (36).

From (31) and (35) we get

$$\overline{U(\xi_j,t)} = \frac{i\pi x_0}{2\sinh\pi k_j}\sum_{n=0}^{\infty}\Big[\Big\{J_{-ik_j}(x_0)\, R'_{j,\frac{3}{2},n}(x_0) - J'_{-ik_j}(x_0)\, R_{j,\frac{3}{2},n}(x_0)\Big\} J_{ik_j}(x) + \Big\{J_{ik_j}(x_0)\, R_{j,\frac{3}{2},n}(x_0) - J_{ik_j}(x_0)\, R'_{j,\frac{3}{2},n}(x_0)\Big\} J_{-ik_j}(x)\Big] + \sum_{n=0}^{\infty} R_{j,\frac{3}{2},n}(x) \,. \quad \ldots(39)$$

Similarly (33) and (36) lead to

$$\overline{U(\xi_j,t)} = -\frac{\pi y_0}{2\sinh \pi k_j}\sum_{n=0}^{\infty}\Big[\Big\{ J_{-ik_j}(iy_0)\, Q'_{j,\frac{3}{2},n}(iy_0) - J'_{-ik_j}(iy_0)\, Q_{j,\frac{3}{2},n}(iy_0)\Big\} J_{ik_j}(iy) + \Big\{ J'_{ik_j}(iy_0)\, Q_{j,\frac{3}{2},n}(iy_0) - J_{ik_j}(iy_0)\, Q'_{j,\frac{3}{2},n}(iy_0)\Big\} J_{-ik_j}(iy)\Big] + \sum_{n=0}^{\infty} Q_{j,\frac{3}{2},n}(iy) \quad \ldots(40)$$

The solution of (24) for time-softening material is given by (39) whereas the equation (40) denotes the solution of the equation (24) where the material concerned is time-hardening.

In any case, the solution for $U(r,t)$ may be written from equation (3) as

$$U(r,t) = \frac{\pi^2}{2}\sum_{\xi_j} \xi_j^2\Big[\xi_j\, J'_{\frac{3}{2}}(\xi_j b) + k\, J_{\frac{3}{2}}(\xi_j b)\Big]^2 \Big[\frac{\overline{U(\xi_j,t)}\cdot C_{\frac{3}{2}}(r,\xi_j)}{F_{\frac{3}{2}}(\xi_j)}\Big] \quad \ldots(41)$$

where ξ_j may be obtained from equation (5) for $m = (3/2)$ and $F_{3/2}(\xi_j)$ is given by equation (4) for $m = (3/2)$. The equations (15) and (41) lead to the solution of the original problem in $U(r,t)$ as

$$U(r,t) = \frac{\pi^2}{2} \sum_{\xi_j} \xi_j^2 \left[\xi_j J'_{\frac{3}{2}}(\xi_j b) + K J_{\frac{3}{2}}(\xi_j b)\right]^2 \left[\frac{\overline{u(\xi_j, t)}}{F_{\frac{3}{2}}(\xi_j)} \cdot \frac{C_{\frac{3}{2}}(r, \xi_j)}{r^{\frac{1}{2}}} \right] \qquad \ldots(42)$$

in which $\overline{u(\xi_j, t)}$ is to be replaced either by equation (39) or by equation (40) as the case demands. We know that the stresses are given by the formulas as

$$\sigma_r = (\lambda + 2\mu) \frac{\partial U}{\partial r} + \frac{2\lambda}{r} U, \qquad \ldots(43)$$

$$\sigma_\theta = \sigma_\varphi = \lambda \frac{\partial v}{\partial r} + \frac{2(\lambda+\mu)}{r} v . \qquad \ldots(44)$$

Applying (42) to (43) and (44), these stresses come out to be

$$\sigma_r(r,t) = \frac{\pi^2}{2} \sum_{\xi_j} \xi_j^2 \left[\xi_j \, J'_{\frac{3}{2}}(\xi_j b) + k \, J_{\frac{3}{2}}(\xi_j b) \right]^2 \cdot \left[\frac{\overline{U(\xi_j, t)}}{F_{\frac{3}{2}}(\xi_j)} \left\{ (\lambda + 2\mu) \frac{\partial}{\partial r} \left[\frac{C_{\frac{3}{2}}(r, \xi_j)}{\sqrt{r}} \right] + \frac{2\lambda}{r\sqrt{r}} \, C_{\frac{3}{2}}(r, \xi_j) \right\} \right] . \qquad \ldots(45)$$

$$\sigma_\theta(r,t) = \frac{\pi^2}{2} \sum_{\xi_j} \xi_j^2 \left[\xi_j \, J'_{\frac{3}{2}}(\xi_j b) + k \, J_{3/2}(\xi_j b) \right]^2 \cdot \left[\frac{\overline{U(\xi_j, t)}}{F_{\frac{3}{2}}(\xi_j)} \left\{ \lambda \frac{\partial}{\partial r} \left[\frac{C_{\frac{3}{2}}(r, \xi_j)}{\sqrt{r}} \right] + \frac{2(\lambda+\mu)}{r\sqrt{r}} \, C_{\frac{3}{2}}(r, \xi_j) \right\} \right] = \sigma_\varphi(r,t) \qquad \ldots(46)$$

4. Radial Vibrations of a Cylindrical Shell.

The differential equation and the corresponding initial and boundary conditions for the dynemic radial motion of a thick isotropic, homogeneous, cylindrical shell of infinite length subjected to loads on both surfaces are, vide [10],

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}u = \left[\frac{1}{c^2(1+\alpha . e^{-t/t_0})}\right] . \frac{\partial^2 u}{\partial t^2}$$

$$a \leq r \leq b \; ; \; t \geqslant 0$$

...(47)

initial conditions :

$$\left.\begin{aligned} u &= 0 \\ \frac{\partial u}{\partial t} &= 0 \end{aligned}\right\} t = 0 \; ; \; a \leq r \leq b$$

...(48)

boundary conditions

$$\Big[\sigma_r(r,t)\Big]_{r=a} = \left[(\lambda_0 + 2\mu_0)\frac{\partial u}{\partial r} + \frac{\lambda_0}{r}u\right] . \left(1+\alpha . e^{-\frac{t}{t_0}}\right)$$

$$= A\left(e^{-\frac{t}{2t_0}}\right) \text{ at } r = a \; ; \; t \geqslant 0$$

...(49)

$$\left[\sigma_r(r,t)\right]_{r=b} = \left[(\lambda_0+2\mu_0)\frac{\partial u}{\partial r} + \frac{\lambda_0}{r}u\right]\left(1+\alpha.\, e^{-\frac{t}{t_0}}\right)$$

$$= B\left(e^{-\frac{t}{2t_0}}\right) \text{ at } r=b\ ;\ t \geqslant 0. \qquad \ldots(50)$$

Using the following definitions,

$$\left.\begin{aligned} h &= \left[\frac{\lambda_0}{a}\right] \div \left[\lambda_0+2\mu_0\right] \\ K &= \left[\frac{\lambda_b}{b}\right] \div \left[\lambda_0+2\mu_0\right] \end{aligned}\right\} \qquad \ldots(51)$$

in equation (6) for $m = 1$ we get

$$H\left[\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{1}{r^2}f\right]$$

$$= \frac{2}{\pi(\lambda_0+2\mu_0)}\left\{P_{j>1}\left[(\lambda_0+2\mu_0)\,f'(b) + \frac{\lambda_0}{b}\,f(b)\right]\right.$$

$$\left. - \left[(\lambda_0+2\mu_0)\,f'(a) + \frac{\lambda_0}{a}\,f(a)\right]\right\} - \xi_j^2\,\overline{f(\xi_j)}.$$

$$\ldots(52)$$

Considering the equations (49), (50) and (52) one can rightly suggest that the appropriate finite Hankel transform for the equation (47) is to be

$$\overline{u} = \overline{u(\xi_j, t)} = \int_a^b r\, u(r,t)\, C_1(r, \xi_j)\, dr\ , \qquad \ldots(53)$$

where $C_1(r,\xi_j)$ may be obtained from equation (2) for $m = 1$.

Applying (53) to (47) and using (49), (50) and (52) one gets

$$\left[\frac{1}{c^2\left(1+\alpha . e^{-t/t_0}\right)}\right] . \frac{d^2\overline{u}}{dt^2}$$

$$= \frac{2}{\pi(\lambda_0 + 2\mu_0)} \left[\frac{P_{j,1}\, \sigma_r(b,t) - \sigma_r(a,t)}{\left(1+\alpha . e^{-t/t_0}\right)}\right] - \xi_j^2\, \overline{u}\ .$$

...(54)

The above equation can be rearranged to find

$$\frac{d^2\overline{u}}{dt^2} + c^2\xi_j^2\left(1+\alpha . e^{-\frac{t}{t_0}}\right) . \overline{u}$$

$$= \frac{2}{\pi\rho}\left[P_{j,1}\, B\left(e^{-\frac{t}{2t_0}}\right) - A\left(e^{-\frac{t}{2t_0}}\right)\right] .$$

...(55)

As before, the solution of (55) may be put as

$$\overline{U(\xi_j,t)} = \frac{i\pi x_0}{2\sinh\pi K_j}\sum_{n=0}^{\infty}\Big[\Big\{J_{-iK_j}(x_0)\,R'_{j,1,n}(x_0) - J'_{-iK_j}(x_0)\,R_{j,1,n}(x_0)\Big\}J_{iK_j}(x) + \Big\{J'_{iK_j}(x_0)\,R_{j,1,n}(x_0) - J_{iK_j}(x_0)\,R'_{j,1,n}(x_0)\Big\}J_{-iK_j}(x)\Big] + \sum_{n=0}^{\infty} R_{j,1,n}(x)\,, \quad \ldots(56)$$

$$\overline{U(\xi_j,t)} = -\frac{\pi y_0}{2\sinh\pi K_j}\sum_{n=0}^{\infty}\Big[\Big\{J_{-iK_j}(iy_0)\,Q'_{j,1,n}(iy_0) - J'_{-iK_j}(iy_0)\,Q_{j,1,n}(iy_0)\Big\}J_{iK_j}(iy) + \Big\{J'_{iK_j}(iy_0)\,Q_{j,1,n}(iy_0) - J_{iK_j}(iy_0)\,Q'_{j,1,n}(iy_0)\Big\}J_{-iK_j}(iy)\Big] + \sum_{n=0}^{\infty} Q_{j,1,n}(iy)\,, \quad \ldots(57)$$

in which

$$R^{(x)}_{j,1,n} = \frac{2}{\pi\rho}\Big[\Big\{P_{j,1}\cdot b_n\cdot S^{(x)}_{n+n_1,ik_j}\Big\} \div \Big\{(\alpha_1 k_j)^{n+n_1+1}\Big\} - \Big\{a_n\cdot S^{(x)}_{n+n_2,ik_j}\Big\} \div \Big\{(\alpha_1 k_j)^{n+n_2+1}\Big\}\Big],$$

$$Q^{(iy)}_{j,1,n} = \frac{2}{\pi\rho}\Big[\Big\{P_{j,1}\cdot b_n\cdot S^{(iy)}_{n+n_1,ik_j}\Big\} \div \Big\{(i\alpha_2 k_j)^{n+n_1+1}\Big\} - \Big\{a_n\cdot S^{(iy)}_{n+n_2,ik_j}\Big\} \div \Big\{(i\alpha_2 k_j)^{n+n_2+1}\Big\}\Big].$$

...(58)

The solution of (47) can then be obtained from equation (3) as below

$$U(r,t) = \frac{\pi^2}{2}\sum_{\xi_j}\xi_j^2\left[\xi_j J_1'(\xi_j b) + k J_1(\xi_j b)\right]^2\cdot\left[\frac{\overline{U(\xi_j,t)}\ C_1(r,\xi_j)}{F_1(\xi_j)}\right],$$

...(59)

in which ξ_j may be found from equation (5) for $m = 1$ and $F_1(\xi_j)$ may be obtained from equation (4) for $m = 1$.

The formulas for stresses are

$$\sigma_r(r,t) = \left[(\lambda + 2\mu) \frac{\partial u}{\partial r} + \frac{\lambda}{r} u \right], \qquad \ldots(60)$$

$$\sigma_\theta(r,t) = \left[\lambda \frac{\partial u}{\partial r} + \frac{\lambda + 2\mu}{r} u \right], \qquad \ldots(61)$$

$$\sigma_z(r,t) = \left[\lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) \right]. \qquad \ldots(62)$$

Applying (59) we find the stress components

$$\sigma_r(r,t) = \frac{\pi^2}{2} \sum_{\xi_j} \xi_j^2 \left[\xi_j \, J_1'(\xi_j b) + K \, J_1(\xi_j b) \right]^2 \cdot \left[\frac{\overline{U(\xi_j, t)}}{F_1(\xi_j)} \left\{ (\lambda + 2\mu) \frac{\partial C_1}{\partial r} + \frac{\lambda}{r} C_1 \right\} \right], \qquad \ldots(63)$$

$$\sigma_\theta(r,t) = \frac{\pi^2}{2} \sum_{\xi_j} \xi_j^2 \left[\xi_j\, J_1'(\xi_j b) + K\, J_1(\xi_j b) \right]^2 \cdot \left[\frac{\overline{U(\xi_j, t)}}{F_1(\xi_j)} \left\{ \lambda \frac{\partial c_1}{\partial r} + \frac{\lambda + 2\mu}{r} c_1 \right\} \right], \quad \ldots(64)$$

$$\sigma_z(r,t) = \frac{\pi^2}{2} \sum_{\xi_j} \xi_j^2 \left[\xi_j\, J_1'(\xi_j b) + K\, J_1(\xi_j b) \right]^2 \cdot \left[\frac{\overline{U(\xi_j, t)}}{F_1(\xi_j)} \left\{ \lambda \left(\frac{\partial c_1}{\partial r} + \frac{c_1}{r} \right) \right\} \right]. \quad \ldots(65)$$