CHAPTER III

PAPER I

BUCKLING OF A HEATED CIRCULAR PLATE OF VARIABLE THICKNESS.

Nomenclature :

a	=	radius of the outer boundary of the plate,
b	=	radius of the inner boundary of the plate,
h	=	thickness of the plate at a distance r from the centre,
$D = \frac{Eh^3}{12(1-\sigma^2)}$	=	flexural rigidity of the plate,
E	=	Young's modulus,
σ^2	=	Poisson's ratio,
$r,\ r+dr$	=	radii of two concentric Cylindrical surfaces,

$d\theta$ = a small angle at the centre of the plate between two radial planes,

$\phi = -\frac{dw}{dr}$ = slope at a distance r, w being the corresponding displacement,

$u_1 = u_1(r)$ = the radial displacement in the middle plane of the plate.

We consider here an element of the deflexion surface bounded by two concentric cylindrical surfaces of radii r and $r+dr$ and two radial planes including an angle $d\theta$.

$$N = \frac{Eh}{12(1-\sigma^2)}\left[\frac{du_1}{dr} - \frac{\sigma}{r}u_1\right] - \frac{N_T}{1-\sigma},$$

$$N_T = E\alpha\int_{-\frac{h}{2}}^{\frac{h}{2}} T\cdot dz,$$ $T = T(r,z)$ temperature varies in the directions of the thickness and radius,

$$M_r = D\left[\frac{d\varphi}{dr} + \frac{\sigma}{r}\phi\right] - \frac{M_T}{1-\sigma}$$ = bending moment per unit length of the of the section perpendicular to the radious,

$$M_T = E\alpha \int_{-\frac{h}{2}}^{\frac{h}{2}} z \cdot T \cdot dz \quad ,$$

$$M_\theta = D\left[\sigma \frac{d\varphi}{dr} + \frac{\phi}{r}\right] - \frac{M_T}{1-\sigma}$$ = bending moment per unit length of the section perpendicular to tangent.

1. Introduction

Mansfield [19] has discussed the buckling and curling of a heated thin circular plate, when the temperature varies through the thickness of the plate and the edges are restrained. He has also discussed the post buckling behavior. Klosner and Forray [17] have studied the buckling of simply-supported plates under symmetrical temperature distribution by using the Raleigh-Ritz method. Boley and Weiner [8] have discussed the buckling of rectangular plates under different edge conditions. Sarkar [26] has solved the buckling problem of a heated thin circular plate of isotropic material under uniform compression in the plane of the plate and obtained the critical buckling temperatures for plates under different edge conditions and different temperature distributions.

The object of this paper is to study the behavior of buckled annular plates under uniform compression and particular temperature distribution. The plate thickness is supposed to vary

(A) linearly,

(B) inversely,

as the distance from the centre of the plate. The edge of the plate is restrained in the plane. The critical buckling compression and the critical temperature are obtained in known terms.

2. The Problem and its solution :

An isotropic annular plate occupying the space $b \leq r \leq a \; ; \; -\frac{h}{2} \leq z \leq \frac{h}{2}$ is considered here. The edges of the plate are restrained in the plane, so that the displacement $u_1 = u_1(r)$ in the middle plane of the plate is zero. Also we suppose that the deflections obey the linear theory of thermo-elasticity.

Considering the equilibrium of the surface element and taking moments we have the following differential equation, vide [3],

$$\left(M_r + \frac{dM_r}{dr}\cdot dr\right)(r + dr)\,d\theta - M_r\, r\,d\theta - M_\theta\, dr\,d\theta + (Q_r + Nh\phi)\, r\,d\theta\, dr = 0 \quad \ldots(1)$$

In this case there is no shearing force Q_r and hence the above equation stands as

$$\frac{1}{r}(M_r - M_\theta) + \frac{dM_r}{dr} + Nh\phi = 0. \quad \ldots(2)$$

(A) The thickness of the plate varies linearly as the distance from the Centre.

Here $$h = h_0 \cdot r \quad \ldots(3)$$

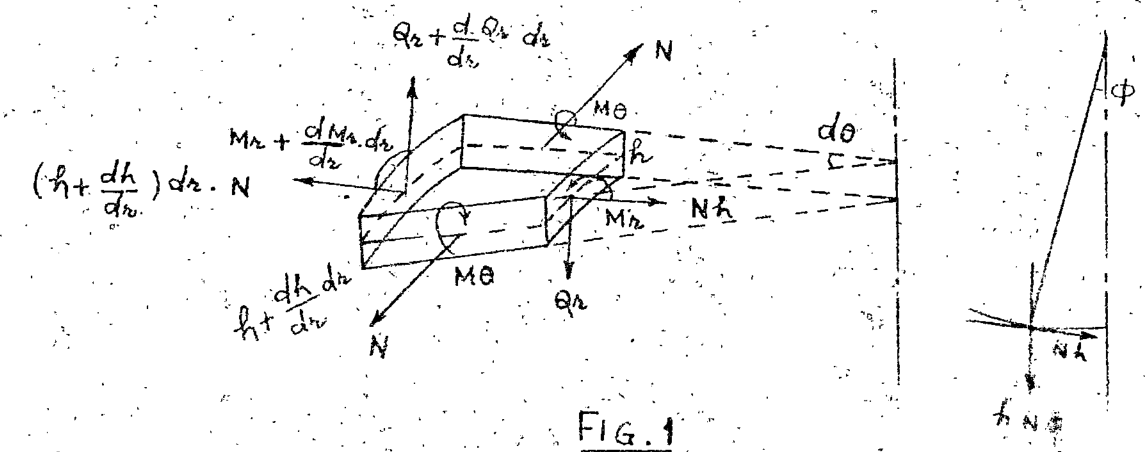
$Q_r + \frac{d}{dr} Q_r \, dr$
N
$M\theta$
$d\theta$
ϕ
$M_r + \frac{dM_r}{dr} \cdot dr$
h
$(h + \frac{dh}{dr}) dr \cdot N$
M_r
Nh
$M\theta$
$h + \frac{dh}{dr} dr$
Q_r
N
Nh
hN

FIG. 1

Using (3) and the expressions for M_r and M_θ in (2) we get

$$r^2 \frac{d^2\phi}{dr^2} + 4 \cdot r \cdot \frac{d\phi}{dr} + (\alpha_1^2 + 3\sigma - 1) \cdot \phi$$

$$= \left[\frac{1}{p(1-\sigma)}\right] \cdot \frac{1}{r} \cdot \frac{dM_T}{dr} \quad , \qquad \ldots(4)$$

where

$$p = \frac{E h_0^3}{12(1-\sigma^2)} \quad \text{and} \quad \alpha_1^2 = \frac{N h_0}{p} \,. \qquad \ldots(5)$$

We choose the temperature distribution $T(r, z)$ such that

$$M_T = M_0 \left\{ 1 - \left(\frac{r}{a}\right)^{m+2} \right\} , \qquad \ldots(6)$$

and N_T becomes independent of r . It is to be noted that according to our assumption

$$N = -\left[\frac{N_T}{(1-\sigma)} \right] . \qquad \ldots(6a)$$

The equation (4) then becomes

$$r^2 \frac{d^2\phi}{dr^2} + 4 \cdot r \cdot \frac{d\phi}{dr} + (\alpha_1^2 + 3\sigma - 1) \cdot \phi = -K \cdot r^m , \qquad \ldots(7)$$

where $$K = \left[\frac{(m+2)\ M_0}{p(1-\sigma)\ a^{m+2}} \right] . \qquad \ldots(8)$$

The solution of equation (7) is

$$\phi = A.\ r^{\left(\frac{2\beta-3}{2}\right)} + B.\ r^{-\left(\frac{2\beta+3}{2}\right)} - \left[\frac{K r^m}{m^2+3m+\left(\frac{9}{4}-\beta^2\right)} \right],$$

$$\ldots(9)$$

where
$$\left. \begin{aligned} 4\beta^2 &= 13-12\sigma-4\alpha_1^2, \\ m &\neq \left[\frac{2\beta-3}{2} \right] \\ &\neq -\left[\frac{2\beta+3}{2} \right] . \end{aligned} \right\} \qquad \ldots(10)$$

Now for clamped edges $\phi = 0$ on $r = a$ and $r = b$.

Hence

$$A = \begin{vmatrix} a^{-\frac{2\beta+3}{2}} & K' a^m \\ b^{-\frac{2\beta+3}{2}} & K' b^m \end{vmatrix} \div \begin{vmatrix} a^{\frac{2\beta-3}{2}} & a^{-\frac{2\beta+3}{2}} \\ b^{\frac{2\beta-3}{2}} & b^{-\frac{2\beta+3}{2}} \end{vmatrix}$$

and

$$\mathcal{B} = \begin{vmatrix} k'a^{m} & a^{\frac{2\beta-3}{2}} \\ & \\ k'b^{m} & b^{\frac{2\beta-3}{2}} \end{vmatrix} \div \begin{vmatrix} a^{\frac{2\beta-3}{2}} & a^{-\frac{2\beta+3}{2}} \\ & \\ b^{\frac{2\beta-3}{2}} & b^{-\frac{2\beta+3}{2}} \end{vmatrix} \quad \ldots(11)$$

where

$$k' = -\left[\frac{k}{m^2 + 3m + \left(\frac{9}{4} - \beta^2\right)}\right] \quad \ldots(12)$$

Therefore

$$\omega = -\left(\frac{2A}{2\beta-1}\right)\cdot r^{\frac{2\beta-1}{2}} + \left(\frac{2\mathcal{B}}{2\beta+1}\right)\cdot r^{-\frac{2\beta+1}{2}} - \left(\frac{k'}{m+1}\right)\cdot r^{m+1} + C \qquad \text{for } m \neq -1.$$

Now using the clamped edge condition $\omega = 0$ at $r = a$ we finally obtain

$$\omega = -\left(\frac{2A}{2\beta-1}\right)\left(r^{\frac{2\beta-1}{2}} - a^{\frac{2\beta-1}{2}}\right) + \left(\frac{2\mathcal{B}}{2\beta+1}\right)\left(r^{-\frac{2\beta+1}{2}} - a^{-\frac{2\beta+1}{2}}\right) - \left(\frac{k'}{m+1}\right)\cdot\left(r^{m+1} - a^{m+1}\right)$$

$$\ldots(13)$$

From (11) and (13) the deflection becomes infinite when

$$\begin{vmatrix} a^{\frac{2\beta-3}{2}} & \bar{a}^{\frac{2\beta+3}{2}} \\ b^{\frac{2\beta-3}{2}} & \bar{b}^{\frac{2\beta+3}{2}} \end{vmatrix} = 0 \qquad \text{...(14)}$$

which has the lowest root (the only root here)

$$\beta = 0 \qquad \text{...(15)}$$

Equations (5), (10) and (15) lead us to find the critical value for the compression as,

$$(N)_{cr} = \left[\frac{(13-12\sigma) E h_0^2}{48 (1-\sigma^2)} \right] \qquad \text{...(16)}$$

According to our assumption $u_1 = 0$, we have then equation (6a) and hence the critical value for the temperature

$$(N_T)_{cr} = -\left[\frac{(13-12\sigma) E h_0^2}{48 (1+\sigma)} \right] \qquad \text{...(17)}$$

at which the plate buckles. The positive value of this expression gives the critical buckling temperature.

(B) The thickness of the plate varies inversely as the distance from the centre.

Here

$$h = h_0 \cdot r^{-1} \qquad \ldots(18)$$

Using (18) and the expressions for M_r & M_θ in (2) we get

$$r^2 \frac{d^2\phi}{dr^2} - 2r \frac{d\phi}{dr} + \left(\frac{N h_0}{p} \cdot r^4 - 3\sigma - 1 \right) \cdot \phi$$

$$= \left[\frac{1}{p(1-\sigma)} \right] \cdot r^5 \cdot \frac{dM_T}{dr} \qquad \ldots(19)$$

We choose the temperature distribution T as in equation (6). The equation (19) beomes

$$r^2 \frac{d^2\phi}{dr^2} - 2r \frac{d\phi}{dr} + \left(\alpha_1^2 \cdot r^4 - 3\sigma - 1 \right) \cdot \phi$$

$$= \left(\lambda \cdot r^{n+4} \right), \qquad \ldots(20)$$

Where $\lambda = (-K)$ and $(m+2) = n$ ...(21)

The solution of equation (20) is

$$\phi = r^{\frac{3}{2}} \left[A . J_\mu\left(\frac{\alpha_1 r^2}{2}\right) + B . Y_\mu\left(\frac{\alpha_1 r^2}{2}\right) + K_1 . S_{\frac{2n+1}{4},\mu}\left(\frac{\alpha_1 r^2}{2}\right) \right] \quad ...(22)$$

Where

$$\left.\begin{aligned} 16\mu^2 &= 13 + 12\sigma , \\ K_1 &= \lambda . \frac{2^{\frac{2n-11}{4}} . \left\{(2n+5)^2 - 16\mu^2\right\}}{(\alpha_1)^{\frac{2n+5}{4}} . \left\{(n+4)(n+1) - (3\sigma+1)\right\}} , \end{aligned}\right\} \quad ...(23)$$

A, B being constants and $J_\mu\left(\frac{\alpha_1 r^2}{2}\right)$, $Y_\mu\left(\frac{\alpha_1 r^2}{2}\right)$

being the Bessel functions of first and second kind of order μ.

$$S_{\frac{2n+1}{4},\mu}\left(\frac{\alpha_1 r^2}{2}\right)$$

is the Lommel's function.

If the outer boundary be clamped and supported and the inner boundary clamped, then we have,

$$\left.\begin{aligned} \phi &= 0 \quad \text{when } r = a \text{ and } r = b \\ w &= 0 \quad \text{when } r = a. \end{aligned}\right\} \qquad \ldots(24)$$

From equations (8) and (10) we get

$$A = \begin{vmatrix} Y_\mu\left(\frac{\alpha_1 a^2}{2}\right) & K_1 S_{\frac{2n+1}{4},\mu}\left(\frac{\alpha_1 a^2}{2}\right) \\ Y_\mu\left(\frac{\alpha_1 b^2}{2}\right) & K_1 S_{\frac{2n+1}{4},\mu}\left(\frac{\alpha_1 b^2}{2}\right) \end{vmatrix} \div \begin{vmatrix} J_\mu\left(\frac{\alpha_1 a^2}{2}\right) & Y_\mu\left(\frac{\alpha_1 a^2}{2}\right) \\ J_\mu\left(\frac{\alpha_1 b^2}{2}\right) & Y_\mu\left(\frac{\alpha_1 b^2}{2}\right) \end{vmatrix} \qquad \ldots(25)$$

$$B = \begin{vmatrix} K_1 S_{\frac{2n+1}{4},\mu}\left(\frac{\alpha_1 a^2}{2}\right) & J_\mu\left(\frac{\alpha_1 a^2}{2}\right) \\ K_1 S_{\frac{2n+1}{4},\mu}\left(\frac{\alpha_1 b^2}{2}\right) & J_\mu\left(\frac{\alpha_1 b^2}{2}\right) \end{vmatrix} \div \begin{vmatrix} J_\mu\left(\frac{\alpha_1 a^2}{2}\right) & Y_\mu\left(\frac{\alpha_1 a^2}{2}\right) \\ J_\mu\left(\frac{\alpha_1 b^2}{2}\right) & Y_\mu\left(\frac{\alpha_1 b^2}{2}\right) \end{vmatrix} \qquad \ldots(26)$$

Intigrating both sides of equation (22) with respect to small r we get

$$\omega = -\frac{2^{\frac{1}{4}}}{(\alpha_1)^{5/4}}\Bigg[A\Big\{\left(\tfrac{1}{4}+\mu-1\right)\left(\tfrac{\alpha_1 r^2}{2}\right)J_\mu\left(\tfrac{\alpha_1 r^2}{2}\right)S_{\frac{1}{4}-1,\mu-1}\left(\tfrac{\alpha_1 r^2}{2}\right) - \Big[\left(\tfrac{\alpha_1 r^2}{2}\right)J_{\mu-1}\left(\tfrac{\alpha_1 r^2}{2}\right)S_{\frac{1}{4},\mu}\left(\tfrac{\alpha_1 r^2}{2}\right)\Big]\Big\} + B\Big\{\left(\tfrac{1}{4}+\mu-1\right)\left(\tfrac{\alpha_1 r^2}{2}\right).$$

$$Y_\mu\left(\tfrac{\alpha_1 r^2}{2}\right)S_{\frac{1}{4}-1,\mu-1}\left(\tfrac{\alpha_1 r^2}{2}\right) - \left(\tfrac{\alpha_1 r^2}{2}\right)Y_{\mu-1}\left(\tfrac{\alpha_1 r^2}{2}\right)S_{\frac{1}{4},\mu}\left(\tfrac{\alpha_1 r^2}{2}\right)\Big\} +$$

$$+ K_1\left\{\sum_{m_1=0}^{\infty}\frac{\left[(-1)^{m_1}\left(\frac{\alpha_1 r^2}{2}\right)^{\frac{n+3}{2}+1+2m_1}\right]}{\left(\frac{n+3}{2}+1+m_1\right)\left[\left\{\left(\frac{2n+1}{4}+1\right)^2-\mu^2\right\}\cdots\left\{\left(\frac{2n+1}{4}+1+2m_1\right)^2-\mu^2\right\}\right]}\right\}$$

$$+ C\Bigg] \qquad \ldots(27)$$

From (24) and (27) the integration constant C can be evaluated in known terms.

It is obvious from (25),(26) and (27) that the deflection becomes infinite when

$$J_\mu\left(\frac{\alpha_1 b^2}{2}\right)Y_\mu\left(\frac{\alpha_1 a^2}{2}\right) = J_\mu\left(\frac{\alpha_1 a^2}{2}\right)Y_\mu\left(\frac{\alpha_1 b^2}{2}\right). \qquad \ldots(28)$$

Equation (28) can be put into the form,

$$J_\mu(x)\, Y_\mu(\rho . x) - J_\mu(\rho x)\, Y_\mu(x) = 0$$

...(29)

Where $x = (\alpha_1 b^2/2)$ and $\rho = (a^2/b^2) > 1$ (since $a > b$, chosen) ..(29a)

The lowest root of the equation (29) is given by [14],

$$x_\mu = \delta + \frac{p_1}{\delta} + \frac{q - p_1^2}{\delta^3} + \frac{r - 4p_1 . q + 2p_1^3}{\delta^5} + \ldots,$$

Where ...(30)

$$\left.\begin{aligned} \delta &= \frac{\pi}{\rho - 1}, \quad p_1 = \frac{4\mu^2 - 1}{8\rho}, \\ q &= \frac{4(4\mu^2-1)(4\mu^2-25)(\rho^3-1)}{3(8\rho)^3 . (\rho-1)}, \\ r &= \frac{32(4\mu^2-1)(16\mu^4 - 456\mu^2 + 1073)(\rho^5-1)}{5.(8\rho)^5 . (\rho-1)} \end{aligned}\right\}$$

...(31)

From (29a), (30) and (31) α_1 can be expressed in known terms and hence the critical value for compression i.e. $(N)_{cr}$ as well as the critical value for the temperature $(N_T)_{cr}$ at which the plate buckles can be obtained with the help of the equations (5) and (6a)

In a particular case where $\sigma = .25$ (accordingly $\mu^2 = 1$) and $\rho = 1.8309$ or 2.6549 or 41.2040 we get from (29a), (30) and (31)

$$x = \frac{\alpha_1 b^2}{2} = 3.8317$$

This equation with the help of equation (5) gives the critical value for the compression as

$$(N)_{cr.} = \frac{4p}{h_0 . b^4} (3.8317)^2$$

and hence the critical value for the temperature

$$(N_T)_{cr.} = -\frac{4(1-\sigma)p}{h_0 . b^4} (3.8317)^2 \qquad \ldots(32)$$

at which the plate buckles. The positive value of this expression gives the critical buckling temperature.