CHAPTER I

PAPER - I

NOTE ON THE DEFLECTION OF A CLAMPED RECTANGULAR PLATE UNDER VARIABLE LOAD.

1. Introduction.

Simply supported rectangular plate problems under different load conditions are vividly discussed by Timoshenko and Woinowsky - Krieger [30]. Plate problems in which all the edges are clamped are disscussed by Weinstein and Rock [32]. But their solutions depend upon prior solutions to the problems in which all the edges are simply supported.

Morley, L.S.D [21] obtains a simple series solution for the bending of a clamped rectangular plate under uniform load.

In this paper, the deflection of a rectangular plate under variable load, in which all the edges are clamped, has been investigated.

2. The Problem and its solution.

We introduce a rectangular cartesian co-ordinate system $O(x, y)$ and consider an isotropic solid rectangular plate of uniform thickness occupying the space $-a \leq x \leq a$, $-b \leq y \leq b$. The differential equation satisfying the lateral displacement ω' is

$$\nabla^4 \omega' = \frac{q}{D} \qquad \ldots(1)$$

where q is the normal load distribution function and D is the flexural rigidity of the plate. For the clamped edges the boundary conditions are

$$\omega' = 0 \quad , \quad \frac{\partial \omega'}{\partial n} = 0 . \qquad \ldots\ldots (2)$$

We assume $$q = q_0 \left(\alpha + \beta x + \gamma x^2 \right) \qquad \ldots\ldots (3)$$

To solve the equation (1) we choose

$$\omega'(x, y) = W(x) + \omega (x, y) \qquad \ldots\ldots (4)$$

where $W(x)$ satisfies the following conditions

$$W(x) = 0 , \ \frac{dW}{dx} = 0 \text{ at } x = \pm a . \qquad \ldots\ldots (5)$$

From (1), (3), (4) and (5) we find

$$W(x) = \frac{q_0 (a^2 - x^2)^2}{D \cdot 16} \left[30\alpha + 6\beta x + \gamma (2a^2 + x^2) \right] \qquad \ldots\ldots (6)$$

Hence the equation (1) reduces to

$$\nabla^4 \omega = 0 , \qquad \ldots\ldots (7)$$

with the boundary conditions

$$\omega = 0, \quad \frac{\partial \omega}{\partial x} = 0 \text{ at } x = \pm a \qquad \text{.... (8)}$$

and

$$\omega = -W(x), \quad \frac{\partial \omega}{\partial y} = 0 \text{ at } y = \pm b. \qquad \text{.... (9)}$$

The solution of the partial differential equation (7) satisfying equations of (8) can be put into the form

$$\omega = \sum_{r} A_r \, \omega_r(x, y), \qquad \text{.... (10)}$$

where $\omega_r(x, y)$ satisfies $\nabla^4 \omega_r = 0$ .... (11) with $\omega_r = 0$, $\frac{\partial \omega_r}{\partial x} = 0$ at $x = \pm a$. To obtain $\omega_r(x, y)$ we put

$$\omega_r(x, y) = f(x) \cosh \lambda_r y, \qquad \text{.... (12)}$$

Where $f(x)$ satisfied the conditions $f(x) = 0$ and $f'(x) = 0$ at $x = \pm a$. Hence

$$f(x) = \frac{q_0}{D}\left(x \sin \lambda_r x \cos \lambda_r a - a \sin \lambda_r a \cos \lambda_r x\right). \qquad \text{.... (13)}$$

The constants λ_r can be had from

$$2 \lambda_r a + \sin 2 \lambda_r a = 0. \qquad \text{.... (14)}$$

Mittelman and Hillman [20] have tabulatated the values of $2\lambda_r a$ of equation (14). The first two values i.e. λ_1 and λ_2 are

$4.2123922 + i\,2.2507286$ and $10.7125374 + i\,3.1031487$.

From (10), (12), (13) we find that

$$\nabla^2 \omega = \frac{2q_0}{D} \sum_r B_r \cos\lambda_r x \cosh\lambda_r y \qquad \ldots\ldots (15)$$

where $$B_r = A_r \lambda_r \cos\lambda_r a.$$

In these, as in the following equations, we like to introduce the convention $\lambda_{-r} = \overline{\lambda_r}$;

$$\omega_{-r}(x,y) = \overline{\omega_r(x,y)} \qquad \text{etc.,} \qquad \ldots\ldots (16)$$

where the bar indicates the conjugate complex value. Hence ω and $\nabla^2\omega$ are real quantities when the summations are taken over positive and negative values of r. The equation (15) gives $\nabla^2\omega$ = constant for $\lambda_0 = 0$ i.e. for $r = 0$.

The value of this constant must be zero, otherwise terms in x^2 and y^2 appear in equation (10), which are inadmissible in virtue of the boundary conditions (8).

The coefficients A_r are to be determined from the principle of least work [28] which requires that

$$\int_{-b}^{b}\int_{-a}^{a} \nabla^2\omega\, \nabla^2\delta\omega \,dx\,dy = 2\int_{-a}^{a} \left[W(x) \frac{\partial}{\partial y} \nabla^2 \delta\omega \right]_{y=b} dx. \qquad \ldots\ldots (17)$$

Where [11], noting that the boundary value problem is completely independent of Poisson's ratio, its value is suitably chosen as $\nu = 1$. Applying the classical Green's identity we may put the equation (17) in the following form

$$\int_{-b}^{b}\int_{-a}^{a} \nabla^2 \omega' \, \nabla^2 \delta\omega' \, dx\, dy = 0. \qquad \text{.... (18)}$$

In finding out the constants A_r we introduce the following notations [21]

$$V(\omega_r, \omega_s) = V(\omega_s, \omega_r) = \int_{-b}^{b} \nabla^2 \omega_r \cdot \nabla^2 \omega_s \, dx\, dy \qquad \text{.... (19)}$$

$$V(\omega_r, \omega) = 2\int_{-a}^{a} \left[W(x) \frac{\partial}{\partial y} \nabla^2 \omega_r \right]_{y=b} dx \qquad \text{.... (20)}$$

Using equation (15) we easily get,

$$V(\omega_r, \omega_s) = \frac{16 q_0^2 a \lambda_r \lambda_s}{D^2 (\lambda_r^2 - \lambda_s^2)^2} (\lambda_r^2 \cos^2 \lambda_s a - \lambda_s^2 \cos^2 \lambda_r a) \times$$
$$(\lambda_s \sinh \lambda_s b \cosh \lambda_r b - \lambda_r \sinh \lambda_r b \cosh \lambda_s b) \text{ for } r \neq s. \qquad \text{.... (21)}$$

Also $V(\omega_r, \omega_r) = 0.$ .... (22)

An important thing we find here $V(\omega_r, \bar{\omega}_s) = \overline{V(\bar{\omega}_r, \omega_s)}$. Also we see that $V(\omega_r, \bar{\omega}_r) \neq 0$; the expression for $V(\omega_r, \bar{\omega}_r)$ can be written on changing λ_s by $\bar{\lambda}_r$ in equation (21).

From (15) and (20) we get

$$V(\omega_r,\omega) = -\frac{4q_0^2 a \sinh \lambda_r b}{15 D^2 \lambda_r^4}\Big[5\left\{\lambda_r^2(6\alpha+\gamma a^2)-6\gamma\right\}\cos^2\lambda_r a$$

$$-\left\{30\gamma-15(2\alpha+\gamma a^2)\lambda_r^2+a^2(10\alpha+\gamma a^2)\lambda_r^4\right\}\Big]$$

..... (23)

In practice, it is sufficient if we consider a few terms of equation (10) say when $-n \leq r \leq n$ instead of the whole series. Equations (10) to (23) provide a set of n equations from which we can determine the n complex quantities A_r,

$$\begin{bmatrix} V(\omega_1,\bar{\omega}_1) & V(\omega_1,\bar{\omega}_2) & \cdots & V(\omega_1,\bar{\omega}_n) \\ V(\omega_2,\bar{\omega}_1) & V(\omega_2,\bar{\omega}_2) & \cdots & V(\omega_2,\bar{\omega}_n) \\ \cdots & \cdots & \cdots & \cdots \\ V(\omega_n,\bar{\omega}_1) & V(\omega_n,\bar{\omega}_2) & \cdots & V(\omega_n,\bar{\omega}_n) \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \cdots \\ \bar{A}_n \end{bmatrix} +$$

$$\begin{bmatrix} 0 & V(\omega_1,\omega_2) & \cdots & V(\omega_1,\omega_n) \\ V(\omega_2,\omega_1) & 0 & \cdots & V(\omega_2,\omega_n) \\ \cdots & \cdots & \cdots & \cdots \\ V(\omega_n,\omega_1) & V(\omega_n,\omega_2) & \cdots & \cancel{V(\omega} \; 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_n \end{bmatrix} = \begin{bmatrix} V(\omega_1,\omega) \\ V(\omega_2,\omega) \\ \cdots \\ V(\omega_n,\omega) \end{bmatrix}$$

..... (24)

3. Other types of Variations.

If we put $\beta = \gamma = 0$ in the above results, we get the same results obtained by Morley for uniform load. For a load function $q = q_0(\alpha + \beta x)$ the lateral displacement ω' can be obtained just by putting $\gamma = 0$ in the above results.

Replacing q by q_0, α by c^2, β by $2cd$, γ by d^2 and $D\left\{= \frac{Eh^3}{12(1-\nu^2)}\right\}$ by $D_0\left\{= \frac{Eh_0^3}{12(1-\nu^2)}\right\}$ in the results obtained in this paper we can get the lateral displacement ω' for the chosen rectangular plate with all its edges clamped under uniform load $q = q_0$ where the thickness of the plate h varies in the direction of x such that $(h_0/h)^3 = (c + d\cdot x)^2$; h_0, c, d being constants.